*Submitted exclusively to the London Mathematical Society*
doi:10.1112/0000/000000

# On The Existence Of Exponential Polynomials With Prefixed Gaps

G. Mora, J.M. Sepulcre, T. Vidal

Abstract

This paper shows that the conjecture of Lapidus and Van Frankenhuysen on the set of dimensions of fractality associated with a nonlattice fractal string is true in the important special case of a generic nonlattice self-similar string, but in general is false. The proof and the counterexample of this have been given by virtue of a result on exponential polynomials $P(z)$, with real frequencies linearly independent over the rationals, that establishes a bound for the number of gaps of $R_P$, the closure of the set of the real projections of its zeros, and the reason for which these gaps are produced.

## 1. *Introduction*

In [**1**, **2**, **3**], Lapidus and Van Frankenhuysen consider the functions known as *nonlattice Dirichlet polynomials*, which are exponential polynomials of the form

$$P(z) = 1 - \sum_{j=1}^{M} m_j r_j^z = 1 - \sum_{j=1}^{M} m_j e^{z \log r_j},\ M \geq 2, \tag{1.1}$$

where $m_1, ..., m_M$ are complex numbers (called multiplicities) and $r_1 > ... > r_M > 0$ (called scaling ratios) with some ratio $\dfrac{\log r_j}{\log r_1}$, $j \geq 2$, irrational (so, $\log r_1$ and $\log r_j$ are linearly independent over the rationals). The zeros of the functions $P(z)$ are connected to the concept of *fractal string*: a set that is a disjoint union of open intervals whose lengths form a sequence $\mathcal{L} = l_1, l_2, ...$ of finite total length $\sum_{j=1}^{\infty} l_j$.

These authors also define the *complex dimensions of a fractal string* $\mathcal{L}$ as the poles of the meromorphic extension of the geometric zeta function of $\mathcal{L}$, which is defined by $\zeta_{\mathcal{L}}(z) = \sum_{j=1}^{\infty} l_j^z$. For the case of *self-similar strings* (an important subclass of fractal strings) with scaling ratios $r_1, r_2, ..., r_N$ (repeated according to multiplicity) and gaps $g_1, ..., g_K$ (whose construction is reminiscent of the construction of the Cantor set), with $1 > r_1 \geq r_2 \geq ... \geq r_N > 0$, $g_j > 0$ and $\sum_{j=1}^{N} r_j + \sum_{k=1}^{K} g_k = 1$, it has the form

$$\zeta_{\mathcal{L}}(z) = \frac{L^z \sum_{k=1}^{K} g_k^z}{1 - \sum_{j=1}^{N} r_j^z},$$

where $L$ is the total length of $\mathcal{L}$ [**1**, Theorem 5.2].

---

2000 *Mathematics Subject Classification* 30Axx, 30D20 (primary), 28A80 (secondary).

An important subclass of self-similar strings is provided by the *generic nonlattice case*, which is produced when the scaling ratios generate a multiplicative group of maximal rank, i.e. the logarithms of the underlying scaling ratios are independent over the rationals.

Also, the set of *dimensions of fractality* of a fractal string is defined as the closure of the set of real parts of its complex dimensions. In fact, in [**1**, **2**, **3**], the authors give a conjecture about the density of the real parts of the complex dimensions for the case of nonlattice strings (associated to the nonlattice Dirichlet polynomials (1.1)) which they formulate, respectively, in the following form:

- **([1, Conjecture 8.3])**: *If $\mathcal{L}$ is a generic nonlattice string, the set of dimensions of fractality of $\mathcal{L}$ is equal to the entire interval $[D_l, D]$, where $D_l$ is defined in (2-16) and $D$ is the Minkowski dimension of $\mathcal{L}$.*
- **([2, Conjecture 4.9])**: *Let $\mathcal{L}$ be a nonlattice string. Then the real parts of its complex dimensions form a set that is dense in the connected interval $[\sigma_l, D]$.*
- **([3, Conjecture 3.55])**: *The set of dimensions of fractality of a nonlattice string, as defined above, is a bounded connected interval $[\sigma_l, D]$, where $D$ is the Minkowski dimension of the string; in other words, the set of real parts of the complex dimensions is dense in $[\sigma_l, D]$, for some real number $\sigma_l$. In the generic nonlattice case, $\sigma_l = D_l$.*

They define $D$ (*the Minkowski dimension of the string* $\mathcal{L}$) as the unique real solution of the equation

$$\sum_{j=1}^{M} |m_j| r_j^x = 1$$

[**3**, Remark 3.8 and expression (3.8a)] and $D_l$ as the unique real number such that

$$1 + \sum_{j=1}^{M-1} |m_j| r_j^{D_l} = |m_M| r_M^{D_l}$$

[**1**, expression (2-16)].

Finally, $\sigma_l$ is defined as $\sigma_l = \inf\{\operatorname{Re} w : \sum_{j=1}^{N} r_j^w = 1\}$ which coincides with $D_l$ in the generic nonlattice case [**2**, Theorem 4.2].

In this paper we will prove that this conjecture is true for a particular and important case when the fractal string $\mathcal{L}$ is a generic nonlattice self-similar string, while it is false for the general case. The significance of the conjecture is that the set of dimensions of fractality of a generic nonlattice self-similar string is dense in a single interval.

On the other hand, to our best knowledge, the first work on the existence of zeros of an exponential polynomial arbitrarily close to any line contained in certain substrips of its critical strip was made by Moreno [**5**], whose main result we quote:

MAIN THEOREM (Moreno [**5**, p. 73]). *Assume that* $1, \alpha_1, ..., \alpha_m$ *are real numbers linearly independent over the rationals. Consider the exponential polynomial*

$$\varphi(z) = \sum_{k=1}^{m} A_k e^{\alpha_k z},\ z = \sigma + it,$$

*where the* $A_k$ *are complex numbers. Then a necessary and sufficient condition for* $\varphi(z)$ *to have zeros arbitrarily close to any line parallel to the imaginary axis inside the strip*

$$I = \{\sigma + it : \sigma_0 < \sigma < \sigma_1,\ -\infty < t < \infty\}$$

*is that*

$$|A_j e^{\sigma \alpha_j}| \leq \sum_{k=1,\ k\neq j}^{m} |A_k e^{\sigma \alpha_k}|,\ (j = 1, 2, ..., m)$$

*for any* $\sigma$ *with* $\sigma + it \in I$.

So, in order to prove that the conjecture of Lapidus and Van Frankenhuysen is true when the fractal string $\mathcal{L}$ is a generic nonlattice self-similar string and to give a counterexample to the general case, we need to figure out the maximum number of gaps that the set

$$R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}} \tag{1.2}$$

can have, and likewise to understand the reason why the gaps are produced. Here the exponential polynomial $P(z)$ will be of the form $1 + \sum_{j=1}^{n} m_j e^{w_j z}$, $n \geq 2$, $m_j \in \mathbb{C} \setminus \{0\}$, with positive real frequencies $w_1 < w_2 < ... < w_n$ linearly independent over the rationals. In fact, for this type of exponential polynomial, our paper proves the following:

i) The set $R_P$ is the union of at most $n$ disjoint non-degenerate closed intervals (see Theorem 9). From this result, we can construct examples that point out that the mentioned conjecture fails in general.
ii) $R_P$ is a single interval when $|m_j| = 1$ for all $j = 1, ..., n$ (see Theorem 10), which proves that the mentioned conjecture is true in the important case of a generic nonlattice self-similar string.
iii) If $z_0$ is a zero of $P(z)$ such that its real part is a boundary point of $R_P$, then $z_0$ is a simple zero of $P(z)$ (see Theorem 11).
iv) $P(z)$ can have pair zeros, i.e. zeros having the same imaginary part (see Theorem 12).

## 2. *First Results*

Firstly we point out that Moreno's Main Theorem [**5**, p. 73] holds by assuming only that $\alpha_1, \alpha_2, ..., \alpha_m$ are linearly independent over the rationals and in this way we obtain the second version of Moreno's result. Indeed, by following step by step the proof of Moreno we observe in page 75 of his paper (1973) that inequality (6), crucial in the proof, is the direct application of Kronecker-Weyl theorem, that the author obtains from Cassel's book *An introduction to diophantine approximations, Cambridge (*1957*)*, stated under the form:

*(Kronecker-Weyl). If* 1, $\alpha_1, \alpha_2, ..., \alpha_m$ *are real numbers which are linearly independent over the rational number field,* $\gamma_1, \gamma_2, ..., \gamma_m$ *are arbitrary real numbers, and* $T$ *and* $\epsilon$ *are positive real numbers, then there exist a real number* $t$ *and integers* $p_1$, $p_2, ..., p_m$ *such that* $t > T$ *and*

$$\left| t\alpha_k - p_k - \frac{\gamma_k}{2\pi} \right| < \epsilon$$

*for* $k = 1, 2, ..., m$.

However by substituting the above Kronecker-Weyl theorem by Kronecker theorem, Theorem 444, p. 382 of Hardy-Wright's book *An introduction to the Theory of Numbers (Fifth edition), Clarendon Press (1979),* stated under the form:

*(Kronecker) Let* $\{a_1, a_2, ..., a_m\}$ *be a linearly independent set of non-null real numbers. For arbitrary real numbers* $b_1$, $b_2, ..., b_m$ *and* $T$, $\epsilon > 0$*, there exist a real number* $t > T$ *and integers* $n_1, n_2, ..., n_m$ *such that*

$$|ta_k - n_k - b_k| < \epsilon, \text{ for all } k = 1, 2, ..., m,$$

Moreno's main theorem follows, by assuming only that $\alpha_1, \alpha_2, ..., \alpha_m$ are linearly independent over the rationals.

From this second version of Moreno's result we have the following theorem in terms of the set $R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}}$.

THEOREM 1. *Let*

$$P(z)=1+\sum_{j=1}^{n}m_je^{w_jz},\ n\geq 2,\ m_j\in\mathbb{C}\setminus\{0\} \tag{2.1}$$

*be an exponential polynomial with positive real frequencies $w_1<...<w_n$ linearly independent over the rationals. Then an open interval $(\sigma_0,\sigma_1)$ is contained in $R_P$ if and only if the $n+1$ inequalities*

$$1\leq\sum_{j=1}^{n}|m_j|\,e^{w_j\sigma};\ |m_k|\,e^{w_k\sigma}\leq 1+\sum_{j=1,\ j\neq k}^{n}|m_j|\,e^{w_j\sigma},\ k=1,2,...,n, \tag{2.2}$$

*are satisfied for any $\sigma\in(\sigma_0,\sigma_1)$.*

*Proof.* Given the frequencies $w_1,w_2,...,w_n$, consider in $\mathbb{R}$ the additive subgroup $G:=\left\{\sum_{j=1}^{n}q_jw_j:q_j\in\mathbb{Q}\right\}$. Then as $G$ is countable, there is some real number, say $\alpha$, such that $\alpha\notin G$. By multiplying $P(z)$ by $e^{\alpha z}$, we obtain the exponential polynomial

$$\varphi(z)=e^{\alpha z}+\sum_{j=1}^{n}m_je^{(\alpha+w_j)z}$$

with frequencies $\alpha,\alpha+w_1,...,\alpha+w_n$ which are linearly independent over the rationals by virtue of the linear independence over the rationals of $w_1,...,w_n$. Then by applying the second version of Moreno's result to $\varphi(z)$ we have that a necessary and sufficient condition for $\varphi(z)$ to have zeros arbitrarily close to any line parallel to the imaginary axis inside the strip

$$I=\{\sigma+it:\sigma_0<\sigma<\sigma_1,\ -\infty<t<\infty\}$$

is that the $n+1$ inequalities

$$e^{\alpha\sigma}\leq\sum_{j=1}^{n}|m_j|\,e^{(\alpha+w_j)\sigma};\ |m_k|\,e^{(\alpha+w_k)\sigma}\leq e^{\alpha\sigma}+\sum_{j=1,\ j\neq k}^{n}|m_j|\,e^{(\alpha+w_j)\sigma},\ 1\leq k\leq n,$$

are satisfied for any $\sigma$ with $\sigma+it\in I$, which is equivalent to say that the interval $(\sigma_0,\sigma_1)$ is contained in $R_\varphi:=\overline{\{\operatorname{Re}z:\varphi(z)=0\}}$. Dividing the above inequalities by $e^{\alpha\sigma}$ we obtain the inequalities (2.2) and, noticing that $P(z)$ and $\varphi(z)$ have the same zeros, the theorem follows. □

Given an exponential polynomial $P(z)$ of type (2.1), at any boundary point of the set $R_P$ the equality is attained in only one of inequalities (2.2).

LEMMA 2. *Let*

$$P(z)=1+\sum_{j=1}^{n}m_je^{w_jz},\ n\geq 2,\ m_j\in\mathbb{C}\setminus\{0\}$$

*be an exponential polynomial with positive real frequencies $w_1<...<w_n$ linearly independent over the rationals. If $\sigma_0$ is a boundary point of $R_P$, then it satisfies all the inequalities (2.2) and only one of them is an equality.*

*Proof.* As $R_P$ is closed, the boundary of $R_P$, denoted by $\partial R_P$, is a subset of $R_P$. Then $\sigma_0\in R_P$, hence there exists a sequence of zeros $z_l=\sigma_l+it_l$ of $P(z)$ satisfying $\lim_{l\to\infty}\sigma_l=\sigma_0$. Since $1+\sum_{j=1}^{n}m_je^{w_jz_l}=0$ for any $l=1,2,...$, by taking modulus and applying the triangular property, the inequalities (2.2) are obviously satisfied for any $\sigma_l$. Now by taking the limit when $l \to \infty$ on each inequality, we have

$$1 \leq \sum_{j=1}^{n} |m_j| e^{w_j \sigma_0}; \ |m_k| e^{w_k \sigma_0} \leq 1 + \sum_{j=1,\ j \neq k}^{n} |m_j| e^{w_j \sigma_0},\ k = 1, 2, ..., n. \tag{2.3}$$

If some of the above inequalities is an equality, as any couple of equalities are incompatible, the lemma follows. Otherwise we have $n+1$ strict inequalities and by continuity there are $n+1$ open neighbourhoods $(a_k, b_k)$, $k = 1, 2, ..., n+1$ of $\sigma_0$ verifying strictly those inequalities. Thus any $\sigma \in (a, b) := \bigcap_{k=1}^{n+1} (a_k, b_k)$ satisfies (2.2) and, from Theorem 1, $(a, b) \subset R_P$. But $\sigma_0 \in (a, b)$ and this means that $\sigma_0$ is an interior point of $R_P$, which is a contradiction because $\sigma_0 \in \partial R_P$. The lemma is then proved. □

## 3. *The extremes of the critical interval*

Given an exponential polynomial $P(z)$ of the form (2.1), we define the extreme points of its critical interval [**4**, Lemma 2.5], that is the minimal interval that contains the real projection of its zeros, as

$$a_P := \inf \{\operatorname{Re} z : P(z) = 0\}$$

and

$$b_P := \sup \{\operatorname{Re} z : P(z) = 0\}.$$

Associated with the above bounds we define the numbers $x_P^0$, $x_P^1$ as the unique real solutions (it will be justified in the proof of the next theorem) of the real equations

$$1 = \sum_{j=1}^{n} |m_j| e^{w_j \sigma}$$

and

$$|m_n| e^{w_n \sigma} = 1 + \sum_{j=1}^{n-1} |m_j| e^{w_j \sigma},$$

respectively. These four numbers are related of the following manner.

THEOREM 3. *If $P(z)$ is an exponential polynomial of type (2.1), then $a_P = x_P^0$ and $b_P = x_P^1$. Moreover, there exist $\sigma_1 > a_P$ and $\sigma_2 < b_P$ such that the intervals $[a_P, \sigma_1]$ and $[\sigma_2, b_P]$ are both contained in $R_P$.*

*Proof.* The real function

$$f_0(\sigma) := \sum_{j=1}^{n} |m_j| e^{w_j \sigma}$$

is strictly increasing and satisfies $\lim_{\sigma \to -\infty} f_0(\sigma) = 0$ and $\lim_{\sigma \to \infty} f_0(\sigma) = \infty$. Then the equation $f_0(\sigma) = 1$ has only the solution $\sigma = x_P^0$, so $x_P^0$ is well defined and for all $z = \sigma + it$ with $\sigma < x_P^0$ we have

$$1 > f_0(\sigma) = \sum_{j=1}^{n} |m_j| e^{w_j \sigma} \geq \left| \sum_{j=1}^{n} m_j e^{w_j z} \right|,$$

which implies that $\operatorname{Re} z < x_P^0$ is a zero-free region of $P(z)$. Therefore it follows

$$x_P^0 \leq a_P. \tag{3.1}$$

On the other hand, since $f_0(x_P^0) = 1$, that is, $\sum_{j=1}^{n} |m_j| e^{w_j x_P^0} = 1$, we deduce that $|m_j| e^{w_j x_P^0} < 1$ for all $j$ and then

$$|m_k| e^{w_k x_P^0} < 1 + \sum_{j=1, j\neq k}^{n} |m_j| e^{w_j x_P^0} \text{ for any } k = 1, 2, ..., n.$$

From the fact that $1 < f_0(\sigma)$ for all $\sigma > x_P^0$ and from continuity applied to the above $n$ strict inequalities we can determine $\sigma_1 > x_P^0$ such that any $\sigma$ of the interval $(x_P^0, \sigma_1)$ satisfies the $n+1$ inequalities (2.2). Then Theorem 1 implies that $(x_P^0, \sigma_1) \subset R_P$. Noticing $R_P$ is closed it follows that

$$\left[x_P^0, \sigma_1\right] \subset R_P \tag{3.2}$$

implying that

$$a_P \leq x_P^0. \tag{3.3}$$

From (3.1) and (3.3) we obtain $x_P^0 = a_P$ so, noticing (3.2), the first part of the theorem is then proved.

In order to prove that $x_P^1 = b_P$ we define the real function

$$f_1(\sigma) := |m_n| e^{w_n \sigma} - \sum_{j=1}^{n-1} |m_j| e^{w_j \sigma}.$$

Since $\lim_{\sigma \to -\infty} f_1(\sigma) = 0$ and $\lim_{\sigma \to \infty} f_1(\sigma) = \infty$, there exists at least a real number $\alpha$ such that $f_1(\alpha) = 1$. As the derivative

$$f_1'(\alpha) = w_n |m_n| e^{w_n \alpha} - \sum_{j=1}^{n-1} w_j |m_j| e^{w_j \alpha} > w_{n-1} |m_n| e^{w_n \alpha} - \sum_{j=1}^{n-1} w_j |m_j| e^{w_j \alpha} =$$

$$= w_{n-1} \left(1 + \sum_{j=1}^{n-1} |m_j| e^{w_j \alpha}\right) - \sum_{j=1}^{n-1} w_j |m_j| e^{w_j \alpha} =$$

$$= w_{n-1} + \sum_{j=1}^{n-1} (w_{n-1} - w_j) |m_j| e^{w_j \alpha} \geq w_{n-1} > 0,$$

the function $f_1(\sigma)$ is strictly increasing at the point $\alpha$ and then the equation $f_1(\sigma) = 1$ has only the solution $\sigma = x_P^1$. Therefore, on one hand $x_P^1$ is well defined and, on the other hand,

$$f_1(\sigma) < 1 \text{ for all } \sigma < x_P^1;\ f_1(\sigma) > 1 \text{ for all } \sigma > x_P^1. \tag{3.4}$$

From the last inequality in (3.4) it follows

$$|m_n| e^{w_n \sigma} > 1 + \sum_{j=1}^{n-1} |m_j| e^{w_j \sigma} \geq \left|1 + \sum_{j=1}^{n-1} m_j e^{w_j z}\right| \text{ for all } z = \sigma + it \text{ with } \sigma > x_P^1,$$

which means that $\operatorname{Re} z > x_P^1$ is a zero-free region of $P(z)$ and then

$$b_P \leq x_P^1. \tag{3.5}$$

Now, noticing $f_1(x_P^1) = 1$ and the first inequality of (3.4), we have

$$|m_n|\, e^{w_n x_P^1} = 1 + \sum_{j=1}^{n-1} |m_j|\, e^{w_j x_P^1} \tag{3.6}$$

and

$$|m_n|\, e^{w_n \sigma} < 1 + \sum_{j=1}^{n-1} |m_j|\, e^{w_j \sigma} \text{ for all } \sigma < x_P^1. \tag{3.7}$$

From (3.6) it follows

$$1 < \sum_{j=1}^{n} |m_j|\, e^{w_j x_P^1};\ |m_k|\, e^{w_k x_P^1} < 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j x_P^1},\ k = 1, 2, ..., n-1,$$

and then by continuity we can determine $\sigma_2 < x_P^1$ such that any $\sigma \in \left(\sigma_2, x_P^1\right)$ satisfies the $n$ inequalities

$$1 < \sum_{j=1}^{n} |m_j|\, e^{w_j \sigma};\ |m_k|\, e^{w_k \sigma} < 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma},\ k = 1, 2, ..., n-1. \tag{3.8}$$

Now, (3.7) and (3.8) allows us to apply Theorem 1 and then $\left(\sigma_2, x_P^1\right) \subset R_P$. Noticing $R_P$ is closed we get

$$\left[\sigma_2, x_P^1\right] \subset R_P, \tag{3.9}$$

involving that

$$x_P^1 \leq b_P. \tag{3.10}$$

Then, because of (3.5) and (3.10), we have $x_P^1 = b_P$ and according to (3.9) the second part of the theorem is proved. Hence the theorem follows. □

By the triangle inequality for complex numbers we obtain a result that will be very useful throughout the paper. If a zero $z_0$ of an exponential polynomial $P(z)$ is such that the corresponding term $A_k e^{\alpha_k z_0}$ satisfies (3.11), then that term is in the opposite sense that the rest of the terms of $P(z)$.

LEMMA 4. *Let* $P(z) = \sum_{j=1}^{n+1} A_j e^{\alpha_j z}$, $n \geq 2$, $A_j \in \mathbb{C} \setminus \{0\}$, $\alpha_j \in \mathbb{R}$. *Assume that* $z_0 = \sigma_0 + it_0$ *is a zero of* $P(z)$ *for which there is some* $k \in \{1, 2, ..., n+1\}$ *such that*

$$|A_k|\, e^{\alpha_k \sigma_0} = \sum_{j=1,\ j\neq k}^{n+1} |A_j|\, e^{\alpha_j \sigma_0}. \tag{3.11}$$

*Then the principal argument* $\arg(A_k e^{\alpha_k z_0}) = \arg(A_j e^{\alpha_j z_0}) \pm \pi$ *and* $\arg(A_j e^{\alpha_j z_0})$ *are equal for all* $j \neq k$.

*Proof.* Since $P(z_0) = 0$, one has $|A_k|\, e^{\alpha_k \sigma_0} = \left| - \sum_{j=1,\ j\neq k}^{n+1} A_j e^{\alpha_j z_0} \right|$ and by (3.11),

$$\left| \sum_{j=1,\ j\neq k}^{n+1} A_j e^{\alpha_j z_0} \right| = \sum_{j=1,\ j\neq k}^{n+1} |A_j|\, e^{\alpha_j \sigma_0}. \tag{3.12}$$

Then by using the property that two non-null complex numbers $u$, $v$ verify $|u+v| = |u| + |v|$ iff there is some $\lambda > 0$ such that $v = \lambda u$, from (3.12), one has that $\arg(A_j e^{\alpha_j z_0})$ is equal for all $j \neq k$. Now, according to $A_k e^{\alpha_k z_0}$ is the opposite of $\sum_{j=1,\ j\neq k}^{n+1} A_j e^{\alpha_j z_0}$, we get $\arg(A_k e^{\alpha_k z_0}) = \arg(A_j e^{\alpha_j z_0}) \pm \pi$. This proves the lemma. □

By applying the above lemma to a normalized exponential polynomial $P(z)$ we obtain a result on the order of multiplicity of its zeros.

Corollary 5. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial with $0 < w_1 < \ldots < w_n$ and $z_0 = \sigma_0 + it_0$ a zero of $P(z)$ such that $\sigma_0$ is the unique solution of one equation (3.11) for some $k = 2, \ldots, n$. Then $z_0$ is a zero of second order.*

*Proof.* Since for some $1 < k < n+1$, $\sigma_0$ satisfies (3.11) with $A_1 = 1$ and $A_{j+1} = m_j$ for $j = 1, \ldots, n$, by applying Lemma 4 it follows that $0 = \arg(1) = \arg(m_j e^{w_j z_0})$ for all $j \neq k$ and $\arg(m_k e^{w_k z_0}) = \pi$. Then $m_j e^{w_j z_0} > 0$ for all $j \neq k$ and $m_k e^{w_k z_0} < 0$. Hence $m_j e^{w_j z_0} = |m_j e^{w_j z_0}| = |m_j| e^{w_j \sigma_0}$ for all $j \neq k$ and $m_k e^{w_k z_0} = -|m_k e^{w_k z_0}| = -|m_k| e^{w_k \sigma_0}$. Consequently we can write

$$P(z_0) = 1 + \sum_{j=1}^{k-1} |m_j| e^{w_j \sigma_0} - |m_k| e^{w_k \sigma_0} + \sum_{j=k+1}^{n} |m_j| e^{w_j \sigma_0} = 0.$$

By defining the real function

$$Q(\sigma) := 1 + \sum_{j=1}^{k-1} |m_j| e^{w_j \sigma} - |m_k| e^{w_k \sigma} + \sum_{j=k+1}^{n} |m_j| e^{w_j \sigma},$$

the number of changes of the sign of its coefficients, say $W$, is 2. Then if $N$ is the number of zeros of $Q(\sigma)$, by Pólya's result [**6**, Pg.46], $W - N$ is an even nonnegative integer and, since $\sigma_0$ is by hypothesis the unique solution of equation $Q(\sigma) = 0$, $\sigma_0$ is necessarily a double zero of $Q(\sigma)$. Therefore $z_0$ is a zero of $P(z)$ of second order. □

Apart from the possible zeros on the line $x = a_P$, an exponential polynomial $P(z)$ of type (2.1) with negative coefficients does not have any zero whose real part be a boundary point of $R_P$.

Proposition 6. *Let $P(z) = 1 - \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (2.1) with $m_j > 0$ for all $j = 1, \ldots, n$. If $z_0$ is a zero of $P(z)$ such that $\operatorname{Re} z_0 \in \partial R_P$, then necessarily $\operatorname{Re} z_0 = a_P$.*

*Proof.* Since $\operatorname{Re} z_0 \in \partial R_P$, by Lemma 2, $\sigma_0 := \operatorname{Re} z_0$ satisfies only one of the $n+1$ equalities

$$1 = \sum_{j=1}^{n} m_j e^{w_j \sigma_0};\ m_k e^{w_k \sigma_0} = 1 + \sum_{j=1,\ j\neq k}^{n} m_j e^{w_j \sigma_0},\ k = 1, 2, \ldots, n.$$

If $\sigma_0$ satisfies the first equality, from Theorem 3, $\sigma_0 = a_P$ and then the proposition follows. If $\sigma_0$ satisfies some of the rest of equalities, since $z_0$ is a zero of $P(z)$ one has (3.11) for some $k > 1$ with $A_1 = 1$ and $A_{j+1} = -m_j$ for all $j = 1, \ldots, n$, we apply Lemma 4. Hence, since $\arg(1) = 0$, one has $\arg(-m_j e^{w_j z_0}) = 0$ for all $j \neq k$ and $\arg(m_k e^{w_k z_0}) = 0$, which means, by taking some $j \neq k$, that $e^{w_j z_0} < 0$ and $e^{w_k z_0} > 0$. Then necessarily there exists some odd integer $p$ such that the imaginary part of $z_0$, say $t_0$, verifies $w_j t_0 = p\pi$, so $t_0 \neq 0$ (consequently if $z_0$ is real the proposition follows). Analogously, $w_k t_0 = \pi q$ for some even integer $q$ which will be non-null because $t_0$ does. Now by dividing we obtain $\frac{w_j}{w_k} = \frac{p}{q}$, which is a contradiction because $w_j$ and $w_k$ are linearly independent over the rationals. □

A relevant theorem [**1**, Theorem 8.1] is directly obtained from Proposition 6 and Theorem 3.

THEOREM 7. *The set of the real projections of the zeros of an exponential polynomial* $P(z) = 1 - \sum_{j=1}^{n} m_j e^{w_j z}$ *of type (2.1), with* $m_j > 0$ *for all* $j$*, has no isolated point.*

*Proof.* If the real projection of a zero $z_0$ of $P(z)$, say $\sigma_0$, were an isolated point of $\overline{\text{the set } \{\operatorname{Re} z : P(z) = 0\}}$, necessarily $\sigma_0$ would be a boundary point of the set $R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}}$. Then, by Proposition 6, $\sigma_0 = a_P$. But, from Theorem 3, there exists $\sigma_1 > a_P$ such that the interval $[a_P, \sigma_1] \subset R_P$, which contradicts the fact of that $\sigma_0$ be an isolated point of the set $\{\operatorname{Re} z : P(z) = 0\}$. □

## 4. *The gaps in* $R_P$

The number of gaps that can have the set $R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}}$ associated to an exponential polynomial $P(z)$ of type (2.1) depends on the number of real solutions of the $n-1$ intermediate equations

$$|m_k|\, e^{w_k \sigma} = 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma},\ k = 1, 2, ..., n-1. \tag{4.1}$$

LEMMA 8. *Let* $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ *be an exponential polynomial of type (2.1). Then each equation (4.1) has at most 2 real solutions.*

*Proof.* Fixed $k = 1, 2, ..., n-1$, we define the real function

$$P_k(\sigma) := 1 + \sum_{j=1}^{k-1} |m_j|\, e^{w_j \sigma} - |m_k|\, e^{w_k \sigma} + \sum_{j=k+1}^{n} |m_j|\, e^{w_j \sigma}.$$

Then, since the number $W_k$ of changes of sign of the coefficients of $P_k(\sigma)$ is 2, from Pólya's result [**6**, Pg.46], $W_k - N_k$ is an even nonnegative integer, where $N_k$ is the number of zeros of $P_k(\sigma)$ counting multiplicities. Hence, necessarily $N_k$ is either 0 or 2. If $N_k = 0$, then equation (4.1) has no solution. When $N_k = 2$, equation (4.1) can have either 1 solution whether the zero of $P_k(\sigma)$ is of second order or 2 solutions (distinct) when $P_k(\sigma)$ has two simple zeros. Consequently the corresponding equation (4.1) can have 0, 1 or 2 solutions and then the lemma follows. □

Now we are ready to give the description of the set $R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}}$ associated to an exponential polynomial $P(z)$ of type (2.1).

THEOREM 9. *Given an exponential polynomial $P(z)$ of type (2.1), $R_P$ is either $[a_P, b_P]$ or the union of at most $n$ disjoint non-degenerate closed intervals. In the latter case, the gaps of $R_P$ are exclusively produced by those equations (4.1) having 2 solutions.*

*Proof.* Assume $\sigma_0$ is a boundary point of $R_P$ distinct from the extreme points $a_P$ and $b_P$. Then, by Lemma 2, $\sigma_0$ satisfies only one of $n-1$ equations (4.1) and the rest of inequalities (2.3) are satisfied strictly. From Lemma 8, equation (4.1), for some $k = 1, 2, ..., n-1$, that satisfies $\sigma_0$ has 1 or 2 solutions. Firstly we suppose that it has 2 solutions $\sigma_{01}$, $\sigma_{02}$ with $\sigma_{01} < \sigma_{02}$. Then, if $\sigma_0 = \sigma_{01}$, because the continuity of the real functions $|m_k|\, e^{w_k \sigma}$ and $1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma}$ and taking into account that $w_k < w_n$, there exists some $\sigma_0^- < \sigma_0$ such that any $\sigma$ of the interval $\left(\sigma_0^-, \sigma_0\right)$ satisfies the inequalities (2.3). Then, by Theorem 1, $\left(\sigma_0^-, \sigma_0\right) \subset R_P$ and, noticing $R_P$ is closed, one has

$$\left[\sigma_0^-, \sigma_0\right] \subset R_P. \tag{4.2}$$

Analogously, by supposing that $\sigma_0 = \sigma_{02}$, we obtain

$$\left[\sigma_0, \sigma_0^+\right] \subset R_P. \tag{4.3}$$

Furthermore, an elementary analysis on the above functions proves that

$$|m_k|\, e^{w_k \sigma} > 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma} \text{ for all } \sigma \in (\sigma_{01}, \sigma_{02}),$$

which means that the strip $\{z : \sigma_{01} < \operatorname{Re} z < \sigma_{02}\}$ is a zero-free region of $P(z)$, so the interval $(\sigma_{01}, \sigma_{02})$ is a gap of $R_P$.

If we suppose that the equation (4.1), for some $k = 1, 2, ..., n-1$, has only the solution $\sigma_0$, it follows immediately that

$$|m_k|\, e^{w_k \sigma} < 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma} \text{ for all } \sigma \neq \sigma_0.$$

Then, by repeating verbatim the above reasoning, there exist two numbers $\sigma_0^-$ and $\sigma_0^+$ with $\sigma_0^- < \sigma_0 < \sigma_0^+$ such that $\left[\sigma_0^-, \sigma_0\right]$ and $\left[\sigma_0, \sigma_0^+\right]$ would be contained in $R_P$. That means that $\left[\sigma_0^-, \sigma_0^+\right] \subset R_P$ and consequently $\sigma_0$ would be an interior point of $R_P$ which is a contradiction because we are assuming that $\sigma_0$ is a boundary point of $R_P$. This proves that, apart from the extreme points, the existence of a boundary point of $R_P$ is due to the fact that some equation (4.1) have 2 solutions $\sigma_{01} < \sigma_{02}$. Then, as there are $n-1$ equations, $R_P$ can have at most $n-1$ gaps and, consequently, at most $2(n-1)$ boundary points which are distinct from the extreme points $a_P$, $b_P$. Finally, if no equation (4.1) has 2 solutions, there is no boundary point different from $a_P$, $b_P$. Then, noticing Theorem 3, one has $R_P = [a_P, b_P]$. For those equations (4.1) that have 2 solutions, from (4.2), (4.3) and Theorem 3 again, $R_P$ is a finite union of disjoint closed intervals of positive length. The proof is completed and then the theorem follows. □

From Theorem 9 it follows an easy property on the set $R_P$ of an exponential polynomial $P(z)$ of type (2.1) with $|m_j| = 1$ for all $j = 1, ..., n$.

THEOREM 10. *If $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ is an exponential polynomial of type (2.1) with $|m_j| = 1$ for all $j = 1, ..., n$, then $R_P = [a_P, b_P]$.*

*Proof.* It suffices to check that, as $|m_j| = 1$ for all $j$, any equation (4.1) does not have any solution and to apply Theorem 9. □

Moreno in [**5**, p.77] deduces from his Main Theorem that the polynomial exponential

$$P(z) = \sum_{p \leq n} \frac{1}{p^z},\ p \text{ prime},\ n \geq 5,$$

has zeros near any line contained in the strip $\{z : 0 \leq \operatorname{Re} z \leq 1\}$.

As a consequence of Theorem 10, we are going to obtain an alternative proof to that of Moreno. Indeed, if $p_{k_n}$ is the last prime less than or equal to $n$, then $Q(z) := p_{k_n}^z P(z)$ is an exponential polynomial with positive increasing frequencies linearly independent over the rationals and coefficients 1, having the same zeros that $P(z)$. Then by applying Theorem 10, $R_Q = [a_Q, b_Q]$, so $R_P = [a_P, b_P]$. Now, from Theorem 3, $a_P = x_P^0$ and $b_P = x_P^1$ and after an elementary computation we have that $a_P < 0$ and $b_P > 1$. Therefore $[0, 1] \subset R_P$ and consequently Moreno's example follows.

Another consequence from Theorem 9 is that if an exponential polynomial $P(z)$ of type (2.1) has a zero whose real part is a boundary point of $R_P$ then it is simple.

THEOREM 11. *Let* $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ *be an exponential polynomial of type (2.1) and* $z_0 = \sigma_0 + it_0$ *a zero of* $P(z)$ *such that* $\sigma_0 \in \partial R_P$. *Then* $z_0$ *is a simple zero of* $P(z)$.

*Proof.* From Theorem 9, $\sigma_0$ is an extreme $a_P$, $b_P$ or $\sigma_0$ is a solution of an equation (4.1) having exactly two solutions. If $\sigma_0 = a_P$, from Theorem 3, $\sigma_0$ is the unique solution of the equation

$$1 = \sum_{j=1}^{n} |m_j| \, e^{w_j \sigma}.$$

On the other hand, as $z_0$ is a zero of $P(z)$, from Lemma 4 one has $\arg(m_j e^{w_j z_0}) = \pi$ for all $j = 1, 2, ..., n$. Therefore $m_j e^{w_j z_0} < 0$ for all $j$ and then $m_j e^{w_j z_0} = -|m_j e^{w_j z_0}| = -|m_j| \, e^{w_j \sigma_0}$ for all $j$. Consequently we can write

$$P(z_0) = 1 - \sum_{j=1}^{n} |m_j| \, e^{w_j \sigma_0} = 0.$$

Now we define the real function

$$Q(\sigma) := 1 - \sum_{j=1}^{n} |m_j| \, e^{w_j \sigma}.$$

Since the number of changes of the sign of the coefficients, say $W$, of $Q(\sigma)$ is 1; if $N$ is the number of zeros (counting multiplicities) of $Q(\sigma)$, by Pólya's result [**6**, Pg.46], $W - N$ is an even nonnegative integer which means that $\sigma_0$ is necessarily a simple zero of $Q(\sigma)$. Therefore $z_0$ is a simple zero of $P(z)$ and in this case the theorem follows.

If $\sigma_0 = b_P$, from Theorem 3, $\sigma_0$ is the unique solution of the equation

$$|m_n| \, e^{w_k \sigma} = 1 + \sum_{j=1}^{n-1} |m_j| \, e^{w_j \sigma}.$$

Again Lemma 4 applied to $z_0$ involves that $\arg(m_j e^{w_j z_0}) = 0$ for all $j = 1, 2, ..., n-1$ and $\arg(m_n e^{w_n z_0}) = \pi$. Then $m_j e^{w_j z_0} > 0$ for all $j \neq n$ and $m_n e^{w_n z_0} < 0$. Hence $m_j e^{w_j z_0} =$

$|m_j e^{w_j z_0}| = |m_j| e^{w_j \sigma_0}$ for all $j \neq n$ and $m_n e^{w_n z_0} = -|m_n e^{w_n z_0}| = -|m_n| e^{w_n \sigma_0}$. Consequently, we can write

$$P(z_0) = 1 + \sum_{j=1}^{n-1} |m_j| e^{w_j \sigma_0} - |m_n| e^{w_n \sigma_0} = 0.$$

Now, by defining

$$Q(\sigma) := 1 + \sum_{j=1}^{n-1} |m_j| e^{w_j \sigma} - |m_n| e^{w_n \sigma},$$

since the number of changes of the sign of its coefficients is 1 and $Q(\sigma_0) = P(z_0) = 0$, by Pólya's result [**6**, Pg.46], as above, one has that $\sigma_0$ is a simple zero of $Q(\sigma)$. That means that $z_0$ is a simple zero of $P(z)$ and also in this case the theorem follows.

Finally, if $\sigma_0$ satisfies, for some $k = 1, 2, ..., n-1$, one equation (4.1) having two solutions $\sigma_{01} < \sigma_{02}$, by repeating the above argument, we write

$$P(z_0) = 1 + \sum_{j=1}^{k-1} |m_j| e^{w_j \sigma_0} - |m_k| e^{w_k \sigma_0} + \sum_{j=k+1}^{n} |m_j| e^{w_j \sigma_0} = 0$$

and

$$Q(\sigma) = 1 + \sum_{j=1}^{k-1} |m_j| e^{w_j \sigma} - |m_k| e^{w_k \sigma} + \sum_{j=k+1}^{n} |m_j| e^{w_j \sigma}.$$

Then as $Q(\sigma)$ has two simple zeros at $\sigma_{01}$, $\sigma_{02}$ and $\sigma_0$ can only be equal to some of them, we get $Q'(\sigma_0) \neq 0$. Since it is immediate that $P'(z_0) = Q'(\sigma_0)$, then $P'(z_0) \neq 0$ and so $z_0$ is a simple zero of $P(z)$. The proof is now completed. □

A new property on the zeros of an exponential polynomial $P(z)$ of type (2.1) can be derived from Theorem 9, namely, that $P(z)$ can have pair zeros, that is, zeros having the same imaginary part.

THEOREM 12. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (2.1) and $z_0 = \sigma_0 + it_0$ a zero of $P(z)$ such that $\sigma_0$ is a boundary point of $R_P$, distinct from $a_P$, $b_P$. Then there exists another zero $z_1 = \sigma_1 + it_0$ of $P(z)$, called pair zero of $z_0$.*

*Proof.* From Theorem 9, $\sigma_0$ is a solution of an equation (4.1) having exactly two solutions $\sigma_{01} < \sigma_{02}$. Hence either $\sigma_0 = \sigma_{01}$ or $\sigma_0 = \sigma_{02}$. In the first case, the pair zero of $z_0$ is $z_1 = \sigma_{02} + it_0$ and, if $\sigma_0 = \sigma_{02}$, then the pair zero of $z_0$ is $z_1 = \sigma_{01} + it_0$. We only prove the first case, the other is completely analogous. Indeed, since $z_0$ is a zero of $P(z)$ and $\sigma_0$ satisfies an equation (4.1) for some $k = 1, 2, ..., n-1$, Lemma 4 implies that $\arg(m_j e^{w_j z_0}) = \arg(1) = 0$ for all $j \neq k$ and $\arg(m_k e^{w_k z_0}) = \pi$, which means that $m_j e^{w_j z_0} > 0$ for all $j \neq k$ and $m_k e^{w_k z_0} < 0$. Then, since $z_0 = \sigma_0 + it_0$, it follows that $m_j e^{i w_j t_0} > 0$ and $m_k e^{i w_k t_0} < 0$, so $m_j e^{w_j \sigma_1} e^{i w_j t_0} > 0$ and $m_k e^{w_k \sigma_1} e^{i w_k t_0} < 0$. Therefore, by taking $z_1 = \sigma_{02} + it_0$, we have

$$m_j e^{w_j z_1} = m_j e^{w_j \sigma_{02}} e^{i w_j t_0} = \left| m_j e^{w_j \sigma_{02}} e^{i w_j t_0} \right| = |m_j| e^{w_j \sigma_{02}}$$

and

$$m_k e^{w_k z_1} = m_k e^{w_k \sigma_{02}} e^{i w_k t_0} = -\left| m_k e^{w_k \sigma_{02}} e^{i w_k t_0} \right| = -|m_k| e^{w_k \sigma_{02}}.$$

Then, because $\sigma_{02}$ is the other solution of the same equation, we get

$$P(z_1) = 1 + \sum_{j=1,}^{n} m_j e^{w_j z_1} = 1 + \sum_{j=1,\ j\neq k}^{n} |m_j|\, e^{w_j \sigma_{02}} - |m_k|\, e^{w_k \sigma_{02}} = 0,$$

which proves that $z_1 = \sigma_{02} + it_0$ is the pair zero of $z_0$. The proof is now completed. □

## 5. *Lapidus and Van Frankenhuysen's conjecture*

As it was defined in the introduction, the set of dimensions of fractality of a fractal string associated to an exponential polynomial $P(z) = 1 - \sum_{j=1}^{M} m_j e^{z \log r_j}$ with $\{\log r_M, \log r_{M-1}, ..., \log r_1\}$ linearly independent over the rationals ($r_1 > r_2 > ... > r_M > 0$) and multiplicities $m_j$, is the closure of the set of real parts of its complex dimensions, i.e. this concept coincides with the set $R_P$ defined in (1.2).

Recall also that the authors define the Minkowski dimension, $D$, of a string as the unique real solution of the equation

$$\sum_{j=1}^{M} |m_j| r_j^x = 1$$

and $D_l$ as the unique real number such that

$$1 + \sum_{j=1}^{M-1} |m_j| r_j^{D_l} = |m_M| r_M^{D_l}.$$

Hence, from Theorem 3, observe that $-D$ and $-D_l$ coincides respectively with the minimum and the maximum of the critical interval of $P(-z)$ (which has weights $w_1 < w_2 < ... < w_M$, with $w_j = -\log r_j$).

So, we answer to the conjectures presented in the introduction in the following sense:

THEOREM 13. *The conjectures [**1**, Conjecture 8.3], [**2**, Conjecture 4.9] and [**3**, Conjecture 3.55] are true in the case that $\mathcal{L}$ is a generic nonlattice self-similar string.*

*Proof.* By the given conditions, the scaling ratios $r_1, r_2, ..., r_N$ of $\mathcal{L}$ are distinct and the weights $w_j = -\log r_j$ are linearly independent over the rationals. So, from Theorem 10, as the coefficients $m_j \in \mathbb{C}$ of its associated Dirichlet polynomial are such that $|m_j| = 1$ for each $j = 1, 2, ..., N$, the conjectures are true for this case. □

But, in general, these conjecture are false as we will show through the following example.

COUNTEREXAMPLE 14. *Let $\mathcal{L}$ be the nonlattice self-similar string with scaling ratios $r_1 = \frac{1}{2}$, $r_2 = r_3 = r_4 = \frac{1}{7}$, $r_5 = \frac{1}{23}$ and a simple gap $g_1 = \frac{9}{322}$. Its associated Dirichlet polynomial is*

$$f(z) = 1 - \frac{1}{2^z} - 3\frac{1}{7^z} - \frac{1}{23^z} = 1 - e^{z \log \frac{1}{2}} - 3e^{z \log \frac{1}{7}} - e^{z \log \frac{1}{23}}.$$

*Therefore, $g(z) := f(-z)$ is an exponential polynomial of the form (2.1) with weights $w_1 = \log 2$, $w_2 = \log 7$ and $w_3 = \log 23$ and it occurs that $Z_f = -Z_g$, where $Z_f$ and $Z_g$ denote the sets of zeros of $f$ and $g$ respectively.*

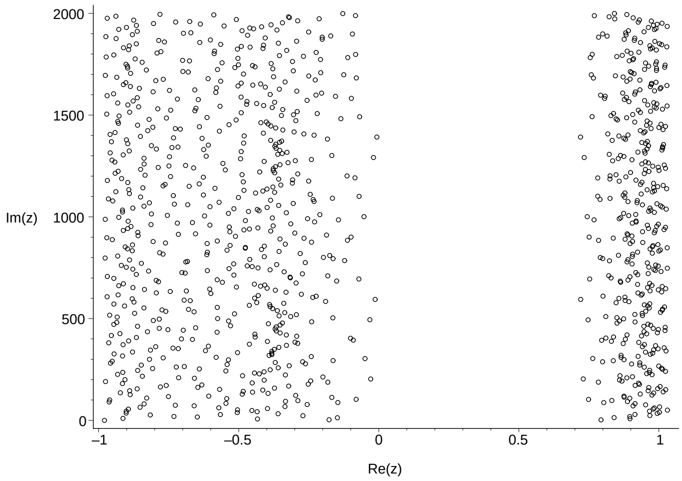


FIGURE 1. *Zeros of $g(z)$ with one gap*

*Let $[a_g, b_g]$ denotes the critical interval of $g(z)$ which, according to Theorem 3, is determined by the unique real numbers that satisfy the equations*

$$2^\sigma + 3 \cdot 7^\sigma + 23^\sigma = 1$$

*and*

$$1 + 2^\sigma + 3 \cdot 7^\sigma = 23^\sigma.$$

*That is, $a_g \approx -0.979$ and $b_g \approx 1.031$.*

*Also, the inequalities (2.2) are given by*

$$\begin{aligned} 1 &\leq 2^\sigma + 3 \cdot 7^\sigma + 23^\sigma, \\ 2^\sigma &\leq 1 + 3 \cdot 7^\sigma + 23^\sigma, \\ 3 \cdot 7^\sigma &\leq 1 + 2^\sigma + 23^\sigma, \\ 23^\sigma &\leq 1 + 2^\sigma + 37^\sigma. \end{aligned}$$

*Now, by using Theorem 9, we can calculate the boundary points of $R_g$ which are produced when we find two solutions in some of the following equations:*

$$2^\sigma = 1 + 3 \cdot 7^\sigma + 23^\sigma \tag{5.1}$$

*and*

$$3 \cdot 7^\sigma = 1 + 2^\sigma + 23^\sigma. \tag{5.2}$$

*Equation (5.1) has no solutions and $\sigma_1 = 0$ and $\sigma_2 \approx 0.71612$ are the solutions of (5.2). Therefore, the two dense intervals are $[a_g, \sigma_1]$ and $[\sigma_2, b_g]$. In fact, we can observe through the Figure 1 the location of its zeros and we check that, in a large interval included in the critical interval $[a_g, b_g]$, specifically $(\sigma_1, \sigma_2)$, the property of density is not accomplished. So, it also occurs for the zeros of $f(z)$.*

*Consequently, we have just proved that the conjecture is not true for this case.*

Nextly, we consider an exponential polynomial of the form (2.1) which has three gaps.

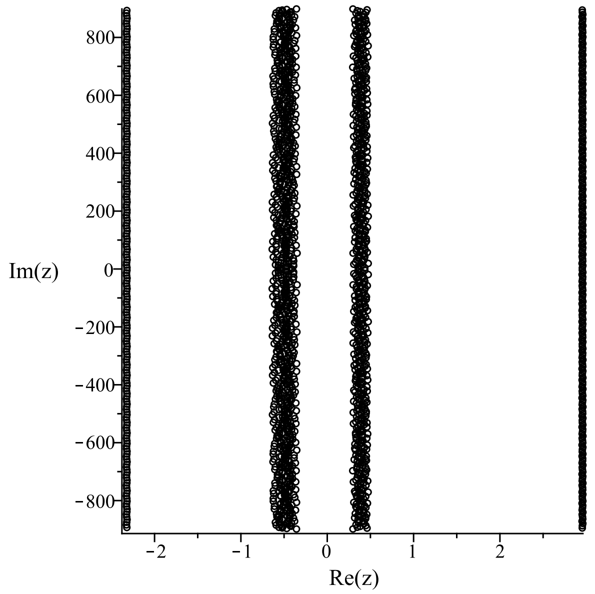


FIGURE 2. *Zeros of $h(z)$ with three gaps*

EXAMPLE 15. *Consider the generic nonlattice case provided by*

$$h(z) = 1 + 5 \cdot 2^z + 25 \cdot 53^z + 11 \cdot 443^z + 997^z$$

*whose distribution of zeros with imaginary part between $-900$ and $900$ can be observed on Figure 2 (observe that $h(-z)$ does not come from any self-similar string because the sum of the associated scaling ratios is greater than 1: $r_1 = ... = r_5 = \frac{1}{2}$, $r_6 = ... = r_{30} = \frac{1}{53}$, $r_{31} = ... = r_{41} = \frac{1}{443}$ and $r_{42} = \frac{1}{997}$).*

*The value $a_h$ is determined by*

$$5 \cdot 2^\sigma + 25 \cdot 53^\sigma + 11 \cdot 443^\sigma + 997^\sigma = 1,$$

*whose solution is $a_h \approx -2.326$.*

*The value $b_h$ is the unique real number that satisfies the equation*

$$1 + 5 \cdot 2^\sigma + 25 \cdot 53^\sigma + 11 \cdot 443^\sigma = 997^\sigma,$$

*whose solution is $b_h \approx 2.96$.*

*Furthermore, the inequalities (2.2) are given in this case by*

$$1 \leq 5 \cdot 2^\sigma + 25 \cdot 53^\sigma + 11 \cdot 443^\sigma + 997^\sigma,$$

$$5 \cdot 2^\sigma \leq 1 + 25 \cdot 53^\sigma + 11 \cdot 443^\sigma + 997^\sigma, \tag{5.3}$$

$$25 \cdot 53^\sigma \leq 1 + 5 \cdot 2^\sigma + 11 \cdot 443^\sigma + 997^\sigma, \tag{5.4}$$

$$11 \cdot 443^\sigma \leq 1 + 5 \cdot 2^\sigma + 25 \cdot 53^\sigma + 997^\sigma, \tag{5.5}$$

$$997^\sigma \leq 1 + 5 \cdot 2^\sigma + 25 \cdot 53^\sigma + 11 \cdot 443^\sigma.$$

*We use Theorem 9 again in order to calculate the large intervals of $R_h$. The equality in (5.3) is reached in $d_1 \approx -2.31829$ and $d_2 \approx -0.639416$. Also, the equality in (5.4) is reached in $d_3 \approx -0.343644$ and $d_4 \approx 0.2911116$. Finally, the equality in (5.5) is attained in $d_5 \approx 0.4763893$ and $d_6 \approx 2.95071$. Therefore, the gaps in $R_h$ are $(d_1, d_2)$, $(d_3, d_4)$ and $(d_5, d_6)$, such as we observe in Figure 2.*

NOTE 16. *During the refereeing process, one referee drew our attention to the fact that the second edition of the book [**3**] has an example (Example 3.55) related to our Counterexample 14. This second edition (New York, 2013) appeared after our paper had been submitted (June, 2012).*

## *References*


**1.** M.L. Lapidus and M. Van Frankenhuysen, *Complex Dimensions of Self-Similar Fractal Strings and Diophantine Approximation*, Experiment. Math. No. 1, **12** (2003), 41-69.

**2.** M.L. Lapidus and M. Van Frankenhuysen, Fractality, Self-Similarity and Complex Dimensions, *Proc. Sympos. Pure Math.,* **72**, Part 1, Amer. Math. Soc., Providence, R.I., (2004), pp. 349-372.

**3.** M.L. Lapidus and M. Van Frankenhuysen, *Fractal geometry, complex dimensions and zeta functions: Geometry and Spectra of Fractal Strings.* (Springer Monographs in Mathematics. Springer, New York, 2006).

**4.** G. Mora and J.M. Sepulcre, *The critical strips of the sums* $1 + 2^z + ... + n^z$, Abstr. Appl. Anal., vol. **2011**, Article ID 909674, 15 pages, (2011). doi:10.1155/2011/909674.

**5.** C.J. Moreno, *The zeros of exponential polynomials (I)*, Compos. Math., tome **26**, n$^o$ 1 (1973), p. 69-78.

**6.** G. Pólya and G. Szegö, *Problems and Theorems in Analysis.* (Springer-Verlag, Vol. II, New York, 1976).



*G. Mora, J.M. Sepulcre, T. Vidal*
*Department of Mathematical Analysis*
*University of Alicante, 03080-Alicante*
*Spain*

gaspar.mora@ua.es
JM.Sepulcre@ua.es
tmvg@alu.ua.es