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Efficient bilateral trade with interdependent values: The use of two-stage mechanisms

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Abstract

As efficient, voluntary bilateral trades are generally not incentive compatible in an interdependent-value environment (Fieseler, Kittsteiner, Moldovanu (2003) and Gresik (1991)), we seek for more positive results by employing *two-stage* mechanisms (Mezzetti (2004)). We say that a two-stage mechanism satisfies incentive compatibility if the truth-telling in both stages constitutes an equilibrium strategy.

First, we show by means of a stylized example that the generalized two-stage Groves mechanism never guarantees voluntary trade, while it satisfies efficiency and incentive compatibility. In a general environment, we next propose Assumption 1 under which there exists a two-stage incentive compatible mechanism implementing an efficient, voluntary trade. Third, within the same example, we confirm that our Assumption 1 is very weak because it holds as long as the buyer's degree of interdependence of preferences is not too high relative to the seller's counterpart. Finally, we show by the same example that if Assumption 1 is violated, our proposed two-stage mechanism fails to achieve voluntary trade.

JEL Classification: C72, D78, D82.

Keywords: bilateral trade, interdependent values, two-stage mechanisms.

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1 Introduction

This paper investigates efficient, voluntary bilateral trades in an interdependent values environment. By "bilateral trade" we mean a simple trading problem in which two individuals, one of whom has a single indivisible object to sell to the other, attempt to agree on exchange of the object for money. So, in this setup, the seller has the full property right for the object to be sold. Efficiency adopted in this paper is an ex post notion, which requires that (i) there be a trade of the good if and only if the buyer's valuation for the good is at least as high as the seller's valuation (decision efficiency) and (ii) whatever the buyer pays is always exactly what the seller receives (budget balance). This paper is mainly concerned with the following normative question: when can an efficient, voluntary trade be implementable in this bilateral trade problem? By the well-known revelation principle, we say that efficient, voluntary trades are possible if there exists a direct revelation mechanism that satisfies Bayesian incentive compatibility (BIC), efficiency (EFF), interim individual rationality (IIR), and ex post budget balance (BB). In the case of private values (i.e., each player is certain of the value of the object at the timing of trade), the celebrated impossibility result of Myerson and Satterthwaite (1983) shows that there are generally no mechanisms satisfying BIC, IIR, EFF, and BB in a bilateral trade setting. On the contrary, Cramton, Gibbons, and Klemperer (1987) show that under the equal-share ownership, there is a mechanism satisfying BIC, IIR, EFF, and BB. Hence, the equal-share partnership is dissolved efficiently.

In many practical instances, however, the assumption of private values is violated. This motivates us to investigate when efficient, voluntary bilateral trades are possible in interdependent values environments, which capture a class of situations in which the payoff of an agent depends not only on his own type, but also on the types (or informational signals) of the other agents. Such interdependence is natural in many trading situations. For instance, we consider a situation in which a seller has private information about the quality of the good which influences the valuations of both the seller and a potential buyer. This type of interdependence is the very situation this paper considers. Once we turn to interdependent values environments, however, we are well aware of bad news. We know from Fiesler, Kittsteiner, and Moldovanu (henceforth, FKM, 2003) and Gresik (1991) that Myerson and Satterthwaite's impossibility result is extended to interdependent values environments. FKM (2003, Theorem 4) also show that the efficient partnership dissolution of Cramton, Gibbons, and Klemperer (1987) cannot be extended to interdependent values environments.

To overcome this negative message in interdependent values environments, we seek for more positive results by looking at *two-stage* generalized revelation mechanisms (Mezzetti (2004)): in the first stage, agents are asked to report their type and the allocation of the good is determined on the type reports; after agents observe their allocation payoff, they are asked to report their realized allocation payoff in the second stage; and finally, the monetary transfers are finalized on the reports of both stages. In his Proposition 1, Mezzetti (2003) establishes the generalized revelation principle, which says that it entails no loss of generality to focus on two-stage generalized revelation mechanisms we briefly described above. By this generalized revelation principle, a two-stage generalized revelation mechanism is simply called a two-stage mechanism in this paper.

The assumption behind the use of two-stage mechanisms can be justified. For example, in the context of a labor market, employers learn the quality of the workers after employing them and after both the employer and the worker find out that the the worker is qualified for the job, the worker's contract is upgraded. We find this type of contracts in a tenure-track contract in academic institutions and consider this as a particular type of two-stage mechanisms. On the contrary, it is sometimes difficult to justify that an agent who obtains the good can experience its quality. To see this, consider a situation in which the object to be traded is some art work and an agent's payoff from obtaining this art work depends on how the other people appreciate it. In this case, the agent will not be able to experience the quality of the object by consuming it. Hence, the power of two-stage mechanisms is sometimes dubious. In any case, we stress that our question here is mainly theoretical. If no two-stage mechanisms implement an efficient, voluntary trade, it is almost impossible to imagine that any mechanism used in a more realistic setup can implement it. In this sense, we are concerned with pushing the boundary between what is implementable and what is not by expanding our scope into two-stage mechanisms.

Considering two-stage mechanisms, we modify the notion of incentive compatibility. Following Mezzetti (2004), we say that a two-stage mechanism satisfies BIC if there exists a perfect Bayesian equilibrium of that two-stage mechanism in which all agents tell the truth in both stages. Here, the main question of our paper is rephrased: "when does there exist a two-stage mechanism satisfying BIC, IIR, EFF, and BB in a bilateral trade model with interdependent values?" In a general mechanism design problem, Mezzetti (2004) proposes the *generalized two-stage Groves* mechanism and shows that it always satisfies BIC, EFF, and BB. When we are concerned with efficient trades, the standard *one-stage* Groves mechanism is shown to be a "canonical" mechanism (See Krishna and Perry (2000) and Williams

(1999) for the case of private values and FKM (2003) for the case of interdependent values). What we mean by "canonical" is that if we are to investigate the existence of the standard one-stage mechanisms satisfying BIC and EFF, we lose nothing to restrict our search to the family of the Groves mechanisms.

This paper considers a bilateral trade model with the following features: (i) each agent's type space constitutes a nonempty closed, bounded interval over the real line; (ii) each agent's type is chosen independently across agents; (iii) each agent's valuation depends on not only his own type but also the type of other agent (i.e., interdependent values); (iv) each agent's valuation for the object is strictly increasing in both his own type and the opponent's type; (v) utilities are quasilinear and so, utilities consist of the sum of a payoff from an outcome decision and a monetary transfer; and (vi) the single crossing property is satisfied. This condition is imposed in FKM (2003). It means that each agent's type must have a greater effect on his own valuation than on that of the other agent.

In Section 3, we confine our attention to a stylized model in which each agent's type is chosen from the uniform distribution over $[0, 1]$ and each agent i's valuation for the object is represented by a linear function, i.e., $\tilde{u}_i(\theta_i, \theta_j) = \theta_i + \gamma_i\theta_j$, where γ_i denotes the degree of interdependence of preferences for agent i. In this context, the single crossing property requires that $\gamma_i < 1$ for each agent i. We find it natural to start our investigation from the generalized two-stage Groves mechanism. We show that the generalized two-stage Groves mechanism never satisfies IIR (Proposition 1). Throughout the paper, we revisit this example multiple times to illustrate the implications of our analysis.

In Section 4, we establish the main result of this paper in a general environment. This section consists of several subsections. In Subsection 4.1, we introduce an additional property imposed on two-stage mechanisms. The property says that if trade does not occur, no payments are made. We call this property the "no-trade-then-no-payments" (henceforth, NTNP) property. In the example in Section 3, we confirm that the generalized two-stage Groves mechanism violates the NTNP property (Claim 3). We impose another additional monotonicity property on two-stage mechanisms. We say that a two-stage mechanism is *monotone* if the buyer's payment is nondecreasing in his own type announcement conditional upon trade occurring. In the example of Section 3, we confirm that the generalized two-stage Groves mechanism is indeed monotone (Claim 4). This suggests that monotonicity is a mild condition. In Subsection 4.2, we propose a two-stage NTNP, monotone mechanism which is used for our main result. Subsection 4.3 introduces Assumption 1 which is needed for our main result. Subsection 4.4 states Theorem 1 as our main result. Theorem 1 of this paper says that if our Assumption 1 is satisfied, the two-stage NTNP, monotone mechanism proposed in Subsection 4.2 satisfies BIC, EFF, BB, and IIR. Thus, the generalized two-stage Groves mechanism turns out to be "not" canonical because the generalized two-stage Groves mechanism does not implement an efficient, voluntary trade, whereas our proposed two-stage mechanism implements it. What distinguishes our proposed two-stage mechanism from the generalized two-stage Groves one is the NTNP property.

Section 5 assesses the restrictiveness of our Assumption 1 using the example in Section 3. We argue that our Assumption 1 is very weak because it is satisfied as long as the buyer's degree of interdependence of preferences (γ_2) is not too high relative to the seller's counterpart γ_1. By a set of simulation results, we conclude that our Assumption 1 is satisfied for a large number of cases.

In Section 6, we compare our results with the results of Galavotti, Muto, and Oyama (henceforth GMO, 2011), who consider the problem of partnership dissolution of Cramton, Gibbons, and Klemperer (1987) in an interdependent values environment. GMO (2011) show in their Theorem 4 that when GMO's Assumption 5.1 is satisfied, for any ownership structure, there exists a two-stage mechanism satisfying BIC, IIR, EFF, and BB.[1] To make our comparison meaningful, we focus on our bilateral trade setup, i.e., there are only two agents and the seller has the full property right over the good. We first show in our Lemma 8 that our Assumption 1 is weaker than GMO's Assumption 5.1. Second, we show in Lemma 9 that in the example in Section 3, GMO's Assumption 5.1 is satisfied if and only if $\gamma_1 = \gamma_2$, i.e., the seller's degree of interdependence of preferences is exactly identical to the buyer's counterpart. This suggests that GMO's Assumption 5.1 is generically violated in our bilateral trade setup. Of course, the advantage of GMO (2011) lies in rather handling any ownership structure, which exhibits a contrast with this paper's focus on a particular ownership structure in which the seller has the full property right over the good.

The rest of the paper is organized as follows. In Section 2, we introduce the general notation and basic concepts for the paper and go over some key important results in the literature to benchmark our paper. Section 3 specializes in a highly stylized but well studied model of bilateral trade with interdependent values. In Section 4, we introduce our Assumption 1 and discuss our main result. Section 5 assesses the restrictiveness of our Assumption 1. In Section 6, we compare the results of our paper with those of GMO (2011). In the Appendix, we provide all

[1] To be precise, their result is stronger than this because GMO (2011) strengthen IIR into ex post individual rationality (EPIR). See Section 6 for the definition of EPIR.

the proofs of the results omitted from the main text of the paper.

2 Preliminaries

The seller (agent 1) has one indivisible object for sale and there is one potential buyer (agent 2). Each agent $i \in \{1, 2\}$ has his type θ_i about the value of the object. The set of possible types for agent i is denoted by Θ_i and we assume that $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$ is a closed, bounded interval over $\mathbb{R}$ with $\underline{\theta}_i < \bar{\theta}_i$. We use the notation convention that $\Theta = \Theta_1 \times \Theta_2$ and $\Theta_{-i} = \Theta_j$ where $j \neq i$ with a generic element θ_{-i}. Types are independently distributed between agents. For each agent $i \in \{1, 2\}$, denote by f_i and F_i the probability density function and cumulative distribution function of θ_i, respectively. We further assume that $f_i(\theta_i) > 0$ for all $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$ and $i \in \{1, 2\}$.

Let $q \in Q = [0, 1]$ be the probability that the good is sold to the buyer, or trading probability for short. Preferences of each agent $i \in \{1, 2\}$ is given by $U_i : Q \times \Theta \times \mathbb{R} \to \mathbb{R}$, which depends on the trading probability q, the type profile θ and his monetary transfer p_i:

$$\begin{aligned} U_1(q, \theta, p_1) &= u_1(q, \theta) + p_1 = (1 - q)\tilde{u}_1(\theta) + p_1; \\ U_2(q, \theta, p_2) &= u_2(q, \theta) + p_2 = q\tilde{u}_2(\theta) + p_2, \end{aligned}$$

where $u_i(q, \theta)$ is agent i's allocation payoff and $\tilde{u}_i(\theta)$ is agent i's valuation for the object in state $\theta \in \Theta$. We assume that for each agent $i \in \{1, 2\}$, $\tilde{u}_i(\theta_i, \theta_{-i})$ is differentiable in both θ_i and θ_{-i} and $\tilde{u}_{i,i} \equiv \partial\tilde{u}_i(\theta_i, \theta_{-i})/\partial\theta_i > 0$ and $\tilde{u}_{i,j} \equiv \partial\tilde{u}_i(\theta_i, \theta_{-i})/\partial\theta_{-i} > 0$ (i.e., strictly increasing in both agents' types).

We further assume the following single crossing condition:

$$\tilde{u}_{i,i} > \tilde{u}_{j,i}, \ \ \forall i, j \in \{1, 2\} \text{ with } i \neq j.$$

When the agents' types are independently distributed, as we assume, Dasgupta and Maskin (2000, footnote 13) argue that in the auction setups, the single crossing property is necessary for the existence of mechanisms satisfying efficiency. This is one of the reasons why we impose the single crossing property. Another reason for this imposition is that we need to rely on Theorem 5 (shown below) of Fieseler, Kittsteiner, and Moldovanu (FKM, 2003) who impose the single crossing condition on their environment. We denote by $\Pi_i = \{\tilde{u}_i(\theta) | \ \theta \in \Theta\}$ the range of agent i's allocation payoff. We assume that for any realization of the type profile $\theta \in \Theta$, if agent i receives the object, he observes his realized allocation payoff $\tilde{u}_i(\theta)$ before final transfers are made.

We first introduce the notion of (one-stage) direct revelation mechanism. A one-stage direct revelation mechanism is defined as a triple (Θ, x, t) in which each agent announces his type and thereafter, the allocation decision is determined by the rule $x : \Theta \to [0,1]$ and the monetary transfer is determined by $t : \Theta \to \mathbb{R}^2$ "simultaneously" based on all agents' type announcement. By the standard revelation principle, we lose nothing to focus on direct revelation mechanisms in which truth-telling each agent's type constitutes a Bayesian Nash equilibrium, which is known as Bayesian incentive compatibility (BIC). In the case of private-value environments, Myerson and Satterthwaite (1983) show that efficiency and voluntary participation are not achieved in an incentive compatible manner. Focusing on the standard one-stage direct mechanisms, Fieseler, Kittsteiner, and Moldovanu (2003) establish the following counterpart of the Myerson and Satterthwaite impossibility result in an interdependent values environment.

Lemma 1 (Theorem 5 of FKM (2003))**.** There exists a (one-stage) mechanism satisfying Bayesian incentive compatibility (BIC), interim individual rationality (IIR), ex post efficiency (EFF), and ex post budget balance (BB) if and only if there is a price p such that $\tilde{u}_1(\underline{\theta}_1, \bar{\theta}_2) \geq p \geq \tilde{u}_2(\underline{\theta}_1, \bar{\theta}_2)$.[2]

Remark: This condition means that there is a price p such that all types of the buyer and seller agree to trade at this price. It is important to note that Gresik (1991) already derived a different condition for the existence of a one-stage mechanism satisfying all the four properties (Theorem 3 of Gresik (1991)). What is useful in the above lemma is that we can only focus on simple price mechanisms to check if an efficient, voluntary trade is possible.

In the example in Section 3, we consider a simple environment in which (i) each agent i's type is chosen independently from the uniform distribution over $[0,1]$; (ii) each agent i's valuation function is $u_i(\theta_i, \theta_{-i}) = \theta_i + \gamma_i \theta_{-i}$; and (iii) $\gamma_i \in (0,1)$. In this environment, the relevant condition becomes $\gamma_1 \geq 1$, which contradicts (iii) $\gamma_i \in (0,1)$. Therefore, within this example, which is, we believe, a representative one, we conclude that there are no "one-stage" mechanisms satisfying all the four properties.

Taking this negative result seriously, we then follow Mezzetti (2004) to define a *two-stage* mechanism as a quadruple (M^1, M^2, δ, τ) such that

[2]To be precise, FKM (2003) only requires ex ante budget surplus rather than ex post budget balance (BB), which we assume. However, Borgers (2015) in Proposition 3.6 and Borgers and Norman (2009) in Proposition 2 show that in the case of independent beliefs, as in our paper, ex ante budget surplus implies ex post budget balance (BB).

- M_i^1 is agent i's message space in the first stage and M_i^2 is agent i's message space in the second stage, respectively;

- $\delta : M^1 \to [0,1]$ is the decision rule specifying the trading probability; and

- $\tau = (\tau[1], \tau[2])$ where $\tau[i] : M^1 \times M_i^2 \to \mathbb{R}^2$ is the transfer rule specifying the monetary transfer for both agents when agent i receives the good at the beginning of the second stage.

In words, in the first stage, after observing his own type, each agent sends a message from M_i^1 and then the good is allocated according to the decision rule δ; in the second stage, after agent i who receives the good (either the seller or the buyer) observes his realized allocation payoff, he is asked to send a message from M_i^2; and finally, the monetary transfers are finalized based on the reports of both stages. We denote by $r_i = (r_i^1, r_i^2)$ agent i's strategy such that $r_i^1 : \Theta_i \to M_i^1$ is his strategy in the first stage and $r_i^2 : Q \times \Theta_i \times \Pi_i \to M_i^2$ is his strategy in the second stage.

In particular, if we set $M_i^1 = \Theta_i$ and $M_i^2 = \Pi_i$, i.e., the agents are asked to report their types in the first stage and realized allocation payoffs in the second stage, then we can construct the corresponding *generalized revelation* mechanism (Θ, Π, x, t) as follows: the decision rule $x : \Theta \to [0,1]$ is given by the composite function $x(\theta) = \delta(r^1(\theta))$ and the transfer rule $t = (t[1], t[2])$ such that $t[i] : \Theta \times \Pi_i \to \mathbb{R}^2$ is given by the composite function $t[i](\theta; u_i) = \tau[i](r^1(\theta), r^2(\delta(r^1(\theta)), \theta, u_i))$. Since each agent i's allocation payoff $\tilde{u}_i(\theta_i, \theta_{-i})$ depends on the whole type profile, then the second-stage reports in the generalized revelation mechanism indeed provide extra information about the type profile, while there is a loss of generality in assuming that the designer only uses the standard "one-stage" revelation mechanisms.

Following Mezzetti (2003), we adopt perfect Bayesian equilibrium as a solution concept and appeal to the following generalized revelation principle, the counterpart of revelation principle in one-stage mechanisms.[3]

Lemma 2 (The Generalized Revelation Principle in Mezzetti (2003))**.** For any perfect Bayesian equilibrium outcome of any two-stage mechanism (M^1, M^2, δ, τ), there exist a generalized revelation mechanism (Θ, Π, x, t) and a perfect Bayesian equilibrium such that, for each agent, reporting his true allocation payoff in the second stage and reporting his true type in the first stage constitute the equilibrium strategy.

[3]For perfect Bayesian equilibrium, for example, the reader is referred to Osborne and Rubinstein (1994, pp.232-233).

From now on, by the generalized revelation principle, we call a generalized revelation mechanisms simply a two-stage mechanism. We now discuss the main properties we want our two-stage mechanisms to satisfy. We denote by (θ_1^r, θ_2^r) the first-stage report and (u_1^r, u_2^r) the second-stage report in a two-stage mechanism, respectively.

Definition 1. A two-stage mechanism (Θ, Π, x, t) satisfies *Bayesian incentive compatibility* (BIC) if truthtelling in both stages constitutes an equilibrium strategy of each agent in a perfect Bayesian equilibrium; that is, for each agent i and each type profile $(\theta_i, \theta_{-i}), (\theta_i^r, \theta_{-i}^r) \in \Theta_i \times \Theta_{-i}$, the equilibrium second-stage report is $u_i^r = u_i(x(\theta_i^r, \theta_{-i}^r), \theta_i, \theta_{-i})$ and the equilibrium first-stage report is $\theta_i^r = \theta_i$.

BIC implies that, given the first-stage report, each agent reports his realized allocation payoff truthfully in the second stage. BIC further implies that, on the equilibrium path, each agent reports his true type in the first stage and for any type profile $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, $u_i(x(\theta_1, \theta_2), \theta_1, \theta_2)$ is agent i's true allocation payoff.

We also assume that each agent has the option of not participating in the two-stage mechanism (Θ, Π, x, t) and let $U_i^O(\theta_i)$ be the expected utility of agent i with type θ_i from non-participation. To be specific,

$$U_1^O(\theta_1) = \int_{\Theta_2} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \text{ for all } \theta_1 \in \Theta_1$$

and

$$U_2^O(\theta_2) = 0 \text{ for all } \theta_2 \in \Theta_2.$$

We introduce the following individual rationality constraint:

Definition 2. A two-stage mechanism (Θ, Π, x, t) satisfies *interim individual rationality* (IIR) if, for all $\theta_1 \in \Theta_1$,

$$\int_{\Theta_2} \left(u_1(x(\theta_1, \theta_2), \theta_1, \theta_2) + t_1(\theta_1, \theta_2; u_1, u_2)\right) dF(\theta_2) \geq U_1^O(\theta_1),$$

and for all $\theta_2 \in \Theta_2$,

$$\int_{\Theta_1} \left(u_2(x(\theta_1, \theta_2), \theta_1, \theta_2) + t_2(\theta_1, \theta_2; u_1, u_2)\right) dF(\theta_1) \geq U_2^O(\theta_2),$$

where $u_1 = u_1(x(\theta_1, \theta_2), \theta_1, \theta_2)$ and $u_2 = u_2(x(\theta_1, \theta_2), \theta_1, \theta_2)$.

Note that this paper's formulation of IIR is the same as the one used by FKM (2003) and Gresik (1991). Next, we require that trade occur if and only if there are gains from trade from ex post point of view.

Definition 3. A two-stage mechanism (Θ, Π, x, t) satisfies *decision efficiency* (EFF) if, for all $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$,

$$x(\theta_1, \theta_2) \in \arg\max_{x \in Q} \left(u_1(x, \theta_1, \theta_2) + u_2(x, \theta_1, \theta_2) \right).$$

In what follows, we denote by x^* the efficient decision rule. We further require that what the seller receives be exactly the same as what the buyer pays.

Definition 4. A two-stage mechanism (Θ, Π, x, t) satisfies *ex post budget balance* (BB) if, for all $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$,

$$t_1(\theta_1, \theta_2; u_1, u_2) + t_2(\theta_1, \theta_2; u_1, u_2) = 0,$$

where $u_1 = u_1(x(\theta_1, \theta_2), \theta_1, \theta_2)$ and $u_2 = u_2(x(\theta_1, \theta_2), \theta_1, \theta_2)$.

Mezzetti (2004) proposes the following generalized two-stage Groves mechanism and shows that it always satisfies BIC, BB and EFF.

Definition 5. A two-stage mechanism (Θ, Π, x^*, t^G) is called the *generalized two-stage Groves mechanism* if, for each agent $i \in \{1, 2\}$, type report $(\theta_i^r, \theta_{-i}^r) \in \Theta_i \times \Theta_{-i}$ and payoff report $(u_i^r, u_{-i}^r) \in \Pi_i \times \Pi_{-i}$,

$$t_i^G(\theta_i^r, \theta_{-i}^r; u_i^r, u_{-i}^r) = u_{-i}^r - h_i(\theta_i^r, \theta_{-i}^r)$$

where

$$\begin{aligned} 2h_i(\theta_i^r, \theta_{-i}^r) \quad = \quad & \sum_{j=1}^{2} u_j\left(x^*(\theta^r), \theta^r\right) - \mathbb{E}_{\theta_{-i}} \left(\sum_{j=1}^{2} u_j\left(x^*(\theta_i^r, \theta_{-i}), \theta_i^r, \theta_{-i}\right) \right) \\ & + \mathbb{E}_{\theta_{-(i+1)}} \left(\sum_{j=1}^{2} u_j\left(x^*(\theta_{i+1}^r, \theta_{-(i+1)}), \theta_{i+1}^r, \theta_{-(i+1)}\right) \right) \end{aligned}$$

with $\mathbb{E}_{\theta_{-i}}$ being the expectation operator over θ_{-i} and $\mathbb{E}_{\theta_{-3}} = \mathbb{E}_{\theta_{-1}}$.

Although the result below is already proved by Mezzetti (2004), we find it instructive to go through its proof to appreciate how the generalized two-stage Groves mechanism works in our bilateral trade setup.

Lemma 3 (Proposition 2 in Mezzetti (2004))**.** The generalized two-stage Groves mechanism always satisfies BIC, EFF, and BB.

Proof. The transfer rule is constructed in such a way that the generalized two-stage Groves mechanism always satisfies BIC and BB. Note that agent i's transfer is independent of his payoff report u_i^r so that he has no incentive to deviate in the second stage. Suppose agent i of type θ_i misreports θ_i^r whereas his opponent always reports the true type θ_{-i} in the first stage. Assume further that both agents report the allocation payoff truthfully in the second stage, i.e., $u_i^r = u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i})$ and $u_{-i}^r = u_{-i}(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i})$. Then, agent i's expected utility is

$$\begin{aligned}
& \mathbb{E}_{\theta_{-i}} \left[u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}) + t_i^G(\theta_i^r, \theta_{-i}; u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}), u_{-i}(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i})) \right] \\
= \; & \mathbb{E}_{\theta_{-i}} \left[u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}) + u_{-i}(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}) - h_i(\theta_i^r, \theta_{-i}) \right] \\
= \; & \mathbb{E}_{\theta_{-i}} \left(\sum_{j=1}^{2} u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}) \right) - \mathbb{E}_{\theta_{-i}} \left(h_i(\theta_i^r, \theta_{-i}) \right) \\
= \; & \mathbb{E}_{\theta_{-i}} \left(\sum_{j=1}^{2} u_i(x^*(\theta_i^r, \theta_{-i}), \theta_i, \theta_{-i}) \right) - \frac{1}{2} \mathbb{E} \left(\sum_{j=1}^{2} u_j(x^*(\theta), \theta) \right) \\
\leq \; & \mathbb{E}_{\theta_{-i}} \left(\sum_{j=1}^{2} u_i(x^*(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \right) - \frac{1}{2} \mathbb{E} \left(\sum_{j=1}^{2} u_j(x^*(\theta), \theta) \right),
\end{aligned}$$

where $\mathbb{E}$ denotes the expectation over (θ_i, θ_{-i}), and the last inequality follows because, by definition, $x^*(\theta_i, \theta_{-i}) \in \arg\max_{x \in Q} \sum_{j=1}^{2} u_j(x, \theta_i, \theta_{-i})$ and the second term is a constant. Hence, agent i achieves the highest expected utility by truth-telling so that BIC is satisfied.

Furthermore, on the equilibrium path where each agent i reports his true type θ_i and true allocation payoff $u_i = u_i(x^*(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})$, the total transfer is computed as follows: for each $(\theta_1, \theta_2) \in \Theta$,

$$\begin{aligned}
& t_1^G(\theta_1, \theta_2; u_1(x^*(\theta_1, \theta_2), \theta_1, \theta_2), u_2(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \\
& + t_2^G(\theta_1, \theta_2; u_1(x^*(\theta_1, \theta_2), \theta_1, \theta_2), u_2(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \\
= \; & u_2(x^*(\theta_1, \theta_2), \theta_1, \theta_2) - h_1(\theta_1, \theta_2) + u_1(x^*(\theta_1, \theta_2), \theta_1, \theta_2) - h_2(\theta_1, \theta_2) \\
= \; & \sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \\
& - \frac{1}{2} \left[\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) - \mathbb{E}_{\theta_2} \left(\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \right) + \mathbb{E}_{\theta_1} \left(\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \right) \right] \\
& - \frac{1}{2} \left[\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) - \mathbb{E}_{\theta_1} \left(\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \right) + \mathbb{E}_{\theta_2} \left(\sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \right) \right] \\
= \; & \sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) - \sum_{j=1}^{2} u_j(x^*(\theta_1, \theta_2), \theta_1, \theta_2) \\
= \; & 0.
\end{aligned}$$

Hence, BB is satisfied. This completes the proof. $\square$

However, it is not clear whether the generalized two-stage Groves mechanism also satisfies IIR or not. We investigate this issue by means of an example in the next section.

3 An Example

In this section, we show by means of an example that the generalized two-stage Groves mechanism with lump-sum transfers always fails IIR.

Both agents' types are uniformly distributed on the unit interval $[0,1]$ and for each type profile $(\theta_1,\theta_2) \in [0,1]^2$, their valuation functions are $\tilde{u}_1(\theta_1,\theta_2) = \theta_1 + \gamma_1\theta_2$ and $\tilde{u}_2(\theta_1,\theta_2) = \theta_2 + \gamma_2\theta_1$ where $\gamma_1, \gamma_2 > 0$. Then,

$$\tilde{u}_2(\theta_1,\theta_2) - \tilde{u}_1(\theta_1,\theta_2) = (\gamma_2\theta_1 + \theta_2) - (\theta_1 + \gamma_1\theta_2) = (1-\gamma_1)\theta_2 - (1-\gamma_2)\theta_1,$$

implying that the efficient decision rule depends on the values of γ_1 and γ_2. We need to satisfy the single crossing condition, which implies that $\gamma_1 < 1$ and $\gamma_2 < 1$. Then, we are left with two cases to consider: (i) $0 < \gamma_2 \leq \gamma_1 < 1$ and (ii) $0 < \gamma_1 < \gamma_2 < 1$.

We first show that the generalized two-stage Groves mechanism violates IIR in both cases.

Proposition 1. The generalized two-stage Groves mechanism (Θ, Π, x^*, t^G) with lump-sum transfers violates IIR in both cases.

Proof. Recall that on the equilibrium path in which both agents' reports are truthful in both stages, agent i of type θ_i receives the following expected utility:

$$\begin{aligned} U_i^G(\theta_i) &= \mathbb{E}_{\theta_{-i}}\left[u_i(x^*(\theta_i,\theta_{-i}),\theta_i,\theta_{-i}) + t_i^G(x^*(\theta_i,\theta_{-i}),u_i,u_{-i})\right] \\ &= \mathbb{E}_{\theta_{-i}}\left(\sum_{j=1}^{2} u_j(x^*(\theta_i,\theta_{-i}),\theta_i,\theta_{-i})\right) - \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right), \end{aligned}$$

where $u_i = u_i(x^*(\theta_i,\theta_{-i}),\theta_i,\theta_{-i})$, $u_{-i} = u_{-i}(x^*(\theta_i,\theta_{-i}),\theta_i,\theta_{-i})$, $\mathbb{E}_{\theta_{-i}}$ denotes the expectation over θ_{-i}, and $\mathbb{E}$ denotes the expectation over (θ_i,θ_{-i}). Then we can derive the worst-off type θ_i^w of each agent i from participating in the generalized two-stage Groves mechanism:

$$\begin{aligned} \theta_i^w &\in \operatorname*{arg\,min}_{\theta_i\in\Theta_i}\left[U_i^G(\theta_i) - U_i^O(\theta_i)\right] \\ &= \operatorname*{arg\,min}_{\theta_i\in\Theta_i}\left[\mathbb{E}_{\theta_{-i}}\left(\sum_{j=1}^{2} u_j(x^*(\theta_i,\theta_{-i}),\theta_i,\theta_{-i})\right) - \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) - U_i^O(\theta_i)\right]. \end{aligned}$$

Since the second term is a constant and hence independent of θ_i, it is equivalent to say

$$\theta_i^w \in \arg\min_{\theta_i \in \Theta_i} \left[\mathbb{E}_{\theta_{-i}} \left(\sum_{j=1}^{2} u_j(x^*(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \right) - U_i^O(\theta_i) \right].$$

Let $L_i \equiv U_i^O(\theta_i^w) - U_i^G(\theta_i^w)$ be the expected loss for agent i's worst-off type. By Proposition 3 of Mezzetti (2003), we know that the generalized two-stage Groves mechanism with lump-sum transfers satisfies IIR without violating BIC, EFF and BB if and only if $L_1 + L_2 \leq 0$. So, it remains to verify whether $L_1 + L_2 \leq 0$ is satisfied in this example. There are two cases we consider: (i) $0 < \gamma_2 \leq \gamma_1 < 1$ and (ii) $0 < \gamma_1 < \gamma_2 < 1$.

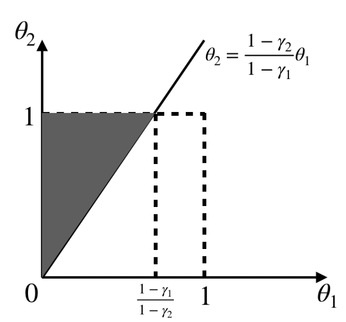
Case (i): $0 < \gamma_2 \leq \gamma_1 < 1$

Since $\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2) = (1 - \gamma_1)\theta_2 - (1 - \gamma_2)\theta_1$ for each $(\theta_1, \theta_2) \in \Theta$, then we have that $\tilde{u}_2(\theta_1, \theta_2) > \tilde{u}_1(\theta_1, \theta_2)$ if and only if $\theta_2 > (1 - \gamma_2)\theta_1/(1 - \gamma_1)$. Hence, the efficient decision rule dictates that, for each $(\theta_1, \theta_2) \in \Theta$,

$$x^*(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 > (1 - \gamma_2)\theta_1/(1 - \gamma_1) \\ 0 & \text{otherwise.} \end{cases}$$

The following figure illustrates the decision at different type profiles in this case; in particular, the shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which exhausts all the type profiles in which trade occurs.

Figure 1: when $0 < \gamma_2 \leq \gamma_1 < 1$



Claim 1. $L_1 + L_2 > 0$ when $0 < \gamma_2 \leq \gamma_1 < 1$.

Proof. The proof is in the Appendix. □

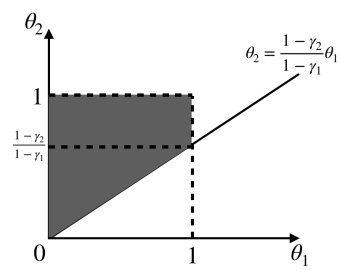
Case (ii): $0 < \gamma_1 < \gamma_2 < 1$

Similar to the previous case, for each $(\theta_1, \theta_2) \in \Theta$, we have that $\tilde{u}_2(\theta_1, \theta_2) > \tilde{u}_1(\theta_1, \theta_2)$ if and only if $\theta_2 > (1-\gamma_2)\theta_1/(1-\gamma_1)$. Hence, the efficient decision rule dictates that, for each $(\theta_1, \theta_2) \in \Theta$,

$$x^*(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 > (1-\gamma_2)\theta_1/(1-\gamma_1) \\ 0 & \text{otherwise.} \end{cases}$$

Figure 2 below illustrates the decision at different type profiles in this case; in particular, the shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles for which it is efficient to trade.

Figure 2: when $0 < \gamma_1 < \gamma_2 < 1$



Claim 2. $L_1 + L_2 > 0$ when $0 < \gamma_1 < \gamma_2 < 1$.

Proof. The proof is in the Appendix. □

By Claims 1 and 2, the generalized two-stage Groves mechanism fails IIR. □

This example will become important for illustrating many of our results, and we shall revisit it in multiple times in due course.

4 The Main Result

This section is organized as follows. In Subsection 4.1, we propose two properties on the class of two stage mechanisms. The first property is called the "no-trade-and-then-no-payment (NTNP) property, which means that when trade does not occur, no agents either receive subsidies or make payments. The second property requires that a two-stage mechanism be "monotone" in the sense that, conditional on the trade occurring, the buyer's payment is nondecreasing in his own type announcement. Subsection 4.2 proposes a NTNP, monotone two-stage mechanism which is used for our main result (Theorem 1). Subsection 4.3 introduces Assumption 1. In the example in Section 3, Assumption 1 loosely says that the buyer's degree of interdependence of preferences is not too high relative to the seller's counterpart. In Subsection 4.4, we show in our Theorem 1 that when Assumption 1 holds, our proposed NTNP, monotone two-stage mechanism satisfies BIC, IIR, EFF, and BB.

4.1 A Class of Two-Stage Mechanisms

Since the generalized two-stage Groves mechanism always fails IIR in the example of Section 3, we propose a new class of two-stage mechanisms which satisfy all the desired properties including IIR. To do so, we first impose the following property on two-stage mechanisms.

Definition 6 (NTNP)**.** A two-stage mechanism (Θ, Π, x, t) satisfies the "no-trade-then-no-payments" (NTNP) property if, for any $(\theta_1, \theta_2) \in \Theta$,

$$x^*(\theta_1, \theta_2) = 0 \Rightarrow t_1(\theta_1, \theta_2; u_1, u_2) = t_2(\theta_1, \theta_2; u_1, u_2) = 0,$$

where $u_1 = u_1(x^*(\theta_1, \theta_2), \theta_1, \theta_2) = \tilde{u}_1(\theta_1, \theta_2)$ and $u_2 = u_2(x^*(\theta_1, \theta_2), \theta_1, \theta_2) = 0$.

This property says that if both agents' reports are truthful in both stages and as a result, no trade occurs, then there are no monetary transfers.[4] In what follows, we call a two-stage mechanism satisfying this property a two-stage NTNP mechanism. We first confirm that in the example of Section 3, the generalized two-stage Groves mechanism fails this property.

Claim 3. In the example of Section 3, the generalized two-stage Groves mechanism (Θ, Π, x^*, t^G) always violates NTNP.

Remark: In the generalized two-stage Groves mechanism, even if trade does not occur in some state, the buyer might receive some positive subsidy from the seller. This is the reason why NTNP is violated.

Proof. The proof is in the Appendix. □

This result suggests that the NTNP property is a defining one that is distinguished from the generalized two-stage Groves mechanism. To propose another property we impose on two-stage mechanisms, we first establish the following useful lemma:

[4]Off the equilibrium path, on the contrary, we impose no restrictions on monetary transfers under no trade so that penalties can be imposed once a deviation is detected.

Lemma 4. Suppose the single crossing condition holds. Then, there exists $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2]$ such that for all $\theta_2 \in \Theta_2$,

$$\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) \begin{cases} < 1 & \text{if } \theta_2 < \theta_2^* \\ = 1 & \text{if } \theta_2 \geq \theta_2^* \end{cases}$$

where θ_2^* denotes the unique cutoff point.

Proof. There are two cases we need to consider. The first case is that $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$. The second case is that there exists $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ such that $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \theta_2^*$ and $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) = 1$ for any $\theta_2 \geq \theta_2^*$. When $\theta_2^* = \bar{\theta}_2$, the event $\{(\theta_1, \theta_2) \in \Theta | \ \theta_2 \geq \theta_2^*\}$ is of measure zero in Θ. Therefore, if $\theta_2^* = \bar{\theta}_2$, the expression $\int_{\Theta_1} x^*(\theta_1, \bar{\theta}_2) dF_1(\theta_1) = 1$ does not affect the calculation of interim expected payoffs of any agent at all so that this requirement is inconsequential. Therefore, the first case can be handled as a special case of the second case by setting $\theta_2^* = \bar{\theta}_2$.

Thus, we assume that $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$. Suppose on the contrary that there exists some $\tilde{\theta}_2 < \hat{\theta}_2$ such that

$$\int_{\Theta_1} x^*(\theta_1, \tilde{\theta}_2) dF_1(\theta_1) = 1,$$

and

$$\int_{\Theta_1} x^*(\theta_1, \hat{\theta}_2) dF_1(\theta_1) < 1.$$

Note that $\int_{\Theta_1} x^*(\theta_1, \tilde{\theta}_2) dF_1(\theta_1) = 1$ implies $\tilde{u}_2(\theta_1, \tilde{\theta}_2) > \tilde{u}_1(\theta_1, \tilde{\theta}_2)$ for all $\theta_1 \in \Theta_1$. By the single crossing condition, for any $\theta_1 \in \Theta_1$, $\tilde{u}_2(\theta_1, \theta_2)$ must grow faster than $\tilde{u}_1(\theta_1, \theta_2)$ as θ_2 increases; since $\hat{\theta}_2 > \tilde{\theta}_2$ and $\tilde{u}_2(\theta_1, \tilde{\theta}_2) > \tilde{u}_1(\theta_1, \tilde{\theta}_2)$ for all $\theta_1 \in \Theta_1$, it follows that

$$\tilde{u}_2(\theta_1, \hat{\theta}_2) > \tilde{u}_1(\theta_1, \hat{\theta}_2)$$

for all $\theta_1 \in \Theta_1$, or equivalently,

$$\int_{\Theta_1} x^*(\theta_1, \hat{\theta}_2) dF_1(\theta_1) = 1,$$

contradicting our hypothesis. This completes the proof. □

To have a better understanding of Lemma 4, we also provide two figures for illustration. The following figures illustrate the allocation decision at different type profiles in general. The shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles for which it is efficient to trade. In the left figure, we have $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all

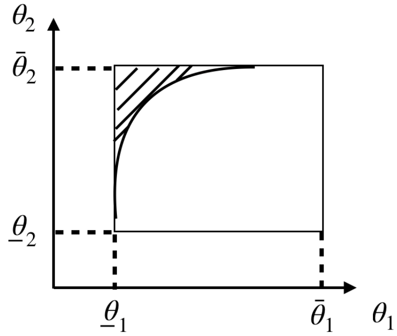


Figure 3

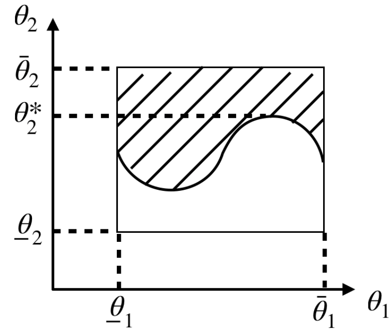


Figure 4

$\theta_2 < \bar{\theta}_2$. In the right figure, it is always efficient to trade when θ_2 is greater than the cutoff type θ_2^*.

We introduce the following monotonicity property on the class of two-stage mechanisms.

Definition 7. Let $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2]$ be the unique cutoff point identified in Lemma 4. A two-stage mechanism (Θ, Π, x^*, t^M) is *monotone* if, for any $\theta_1^r \in \Theta_1$, any $\theta_2^r, \hat{\theta}_2^r \in \Theta_2$, and any $(u_1^r, u_2^r), (\hat{u}_1^r, \hat{u}_2^r) \in \Pi_1 \times \Pi_2$, whenever $\hat{\theta}_2^r > \theta_2^r$ and $x^*(\theta_1^r, \hat{\theta}_2^r) = x^*(\theta_1^r, \theta_2^r) = 1$, then

$$\begin{cases} t_2^M(\theta_1^r, \hat{\theta}_2^r, \hat{u}_1^r, \hat{u}_2^r) < t_2^M(\theta_1^r, \theta_2^r, u_1^r, u_2^r) & \text{if } \hat{\theta}_2^r < \theta_2^* \\ t_2^M(\theta_1^r, \hat{\theta}_2^r, \hat{u}_1^r, \hat{u}_2^r) = t_2^M(\theta_1^r, \theta_2^r, u_1^r, u_2^r) & \text{if } \theta_2^r \geq \theta_2^* \end{cases}$$

In words, a monotone two-stage mechanism has the property that, conditional on the trade occurring, the buyer's payment is strictly increasing in his own type if his type is smaller than θ_2^* and it is constant if his type is at least as high as θ_2^*.

We will show that in the example in Section 3, the generalized two-stage Groves mechanism is monotone.

Claim 4. In the example in Section 3, the generalized two-stage Groves mechanism (Θ, Π, x^*, t^G) is monotone.

Proof. The proof is in the Appendix. □

This suggests that monotonicity is a mild requirement imposed on two-stage mechanisms. On the contrary, as we already argued, the NTNP is rather a stringent requirement.

4.2 The Proposed Two-Stage Mechanism

In this subsection, we propose a two-stage NTNP, monotone mechanism we use for our main result in the next subsection.

Recall that the efficient decision rule dictates that, for each $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$,

$$x^*(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \tilde{u}_2(\theta_1, \theta_2) > \tilde{u}_1(\theta_1, \theta_2) \\ 0 & \text{otherwise.} \end{cases}$$

We propose the following two-stage mechanism (Θ, Π, x^*, t^M) which satisfies BIC, IIR, EFF, and BB.[5] By construction, the proposed two-stage mechanism satisfies EFF. For each type report $(\theta_1^r, \theta_2^r) \in \Theta_1 \times \Theta_2$ and each payoff report $(u_1^r, u_2^r) \in \Pi_1 \times \Pi_2$,

$$t_1(\theta_1^r, \theta_2^r; u_1^r, u_2^r) = \begin{cases} \tilde{u}_2(\theta_1^r, \theta_2^r) & \text{if } \theta_2^r < \theta_2^*, x^*(\theta_1^r, \theta_2^r) = 1, \text{ and } u_2^r = u_2(x^*(\theta_1^r, \theta_2^r), \theta_1^r, \theta_2^r) \\ -g(\theta_1^r) & \text{if } \theta_2^r \geq \theta_2^*, x^*(\theta_1^r, \theta_2^r) = 1, \text{ and } u_2^r = u_2(x^*(\theta_1^r, \theta_2^r), \theta_1^r, \theta_2^r) \\ -\psi & \text{if } x^*(\theta_1^r, \theta_2^r) = 1 \text{ and } u_2^r \neq u_2(x^*(\theta_1^r, \theta_2^r), \theta_1^r, \theta_2^r) \\ 0 & \text{if } x^*(\theta_1^r, \theta_2^r) = 0, \end{cases}$$

and

$$t_2(\theta_1^r, \theta_2^r; u_1^r, u_2^r) = \begin{cases} -\tilde{u}_2(\theta_1^r, \theta_2^r) & \text{if } \theta_2^r < \theta_2^* \text{ and } x^*(\theta_1^r, \theta_2^r) = 1 \\ g(\theta_1^r) & \text{if } \theta_2^r \geq \theta_2^* \text{ and } x^*(\theta_1^r, \theta_2^r) = 1 \\ 0 & \text{if } x^*(\theta_1^r, \theta_2^r) = 0 \text{ and } u_1^r = u_1(x^*(\theta_1^r, \theta_2^r), \theta_1^r, \theta_2^r) \\ -\psi & \text{if } x^*(\theta_1^r, \theta_2^r) = 0 \text{ and } u_1^r \neq u_1(x^*(\theta_1^r, \theta_2^r), \theta_1^r, \theta_2^r), \end{cases}$$

where ψ is a strictly positive constant (which is determined later), $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2]$ is the cutoff point identified in Lemma 4, and

$$g(\theta_1^r) = \begin{cases} -\tilde{u}_2(\theta_1^r, \theta_2^*) & \text{if } \theta_2^* = \bar{\theta}_2 \\ G(\theta_1^r)/\left(1 - F_2(\theta_2^*)\right) & \text{if } \theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2) \end{cases}$$

with

$$\begin{aligned} G(\theta_1^r) \;=\; & \int_{\Theta_2^*(\theta_1^r)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1^r, \theta_2) dF_2(\theta_2) - \int_{\Theta_2^*(\theta_1^r)} \tilde{u}_1(\theta_1^r, \theta_2) dF_2(\theta_2) \\ & - \int_{\Theta_1} \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1) \\ & - \int_{\Theta_1} \int_{\Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2^*) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1), \end{aligned} \tag{1}$$

[5] There are two senses in which the proposed two-stage mechanism is considered a generalized version of the "shoot-the-liar" mechanism in Mezzetti (2007). First, the seller is asked to make a monetary transfer based on the reports. Second, the payment rule below the cutoff θ_2^* shares the same spirit as the "shoot-the-liar" mechanism but the payment above the cutoff θ_2^* is independent of the buyer's type report.

where

$$\Theta_2^*(\theta_1) = \begin{cases} \{\bar{\theta}_2\} & \text{if } \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} = \emptyset \\ \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} & \text{otherwise,} \end{cases}$$

and $\Theta_2^{**} = [\theta_2^*, \bar{\theta}_2]$.

In this mechanism, if each agent i reports his true type θ_i and true allocation payoff $u_i = u_i(x^*(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})$, then the following three properties are confirmed.

1. when $x^*(\theta_1, \theta_2) = 0$, $t_1(\theta_1, \theta_2; u_1, u_2) = t_2(\theta_1, \theta_2; u_1, u_2) = 0$, i.e., when no trade occurs, there are no monetary transfers. Hence, NTNP is satisfied;

2. when $x^*(\theta_1, \theta_2) = 1$ and $\theta_2 < \theta_2^*$, $t_1(\theta_1, \theta_2; u_1, u_2) = -t_2(\theta_1, \theta_2; u_1, u_2) = \tilde{u}_2(\theta_1, \theta_2)$, implying that the buyer's payment is strictly increasing in his type; and

3. when $x^*(\theta_1, \theta_2) = 1$ and $\theta_2 \geq \theta_2^*$, $t_1(\theta_1, \theta_2; u_1, u_2) = -t_2(\theta_1, \theta_2; u_1, u_2) = -g(\theta_1)$ which is independent of the buyer's type.

By construction, the proposed two-stage mechanism is monotone. It also satisfies BB by construction. By contrast, we need break the budget off the equilibrium. If it is efficient not to trade and the seller's payoff report in the second stage is inconsistent with the type reports in the first stage, then the buyer is punished with a penalty ψ. Similarly, if it is efficient to trade and the buyer's payoff report in the second stage is inconsistent with the type reports in the first stage, then the seller is punished with a penalty ψ.

4.3 An Assumption

To state our main result, we introduce the following assumption.

Assumption 1.

$$\begin{aligned} & \int_{\Theta_1} \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1) \\ + & \int_{\Theta_1} \int_{\Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2^*) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1) \geq 0, \end{aligned} \tag{2}$$

where for each $\theta_1 \in \Theta_1$,

$$\Theta_2^*(\theta_1) = \begin{cases} \{\bar{\theta}_2\} & \text{if } \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} = \emptyset \\ \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} & \text{otherwise,} \end{cases}$$

$\Theta_2^{**} = [\theta_2^*, \bar{\theta}_2]$, and $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2]$ is the cutoff point identified in Lemma 4.

Remark: If $\Theta_2^*(\theta_1) = \{\bar{\theta}_2\}$ for some $\theta_1 \in \Theta_1$, then $\Theta_2^*(\theta_1)\backslash\Theta_2^{**}$ is an empty set. In this case, any integration over $\Theta_2^*(\theta_1)\backslash\Theta_2^{**}$ is always zero. Since the first term in the left-hand-side of inequality (2) corresponds to the ex ante gains from trade over $[\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \theta_2^*]$, it is always nonnegative. If $\theta_2^* = \bar{\theta}_2$, the second term in inequality (2) is zero by definition. Therefore, Assumption 1 is automatically satisfied when $\theta_2^* = \bar{\theta}_2$.

To further illustrate this assumption, we first consider Case (i) $0 < \gamma_2 \le \gamma_1 < 1$ in the example of Section 3. In this case, we have $\theta_2^* = \bar{\theta}_2 = 1$. Then, we obtain

$$\Theta_2^*(\theta_1) = \begin{cases} \left[\frac{1-\gamma_2}{1-\gamma_1}\theta_1, 1\right] & \text{if } 0 < \theta_1 < (1-\gamma_1)/(1-\gamma_2) \\ \{1\} & \text{if } (1-\gamma_1)/(1-\gamma_2) \le \theta_1 < 1. \end{cases}$$

We next consider Case (ii) $0 < \gamma_1 < \gamma_2 < 1$. In this case, we obtain $\theta_2^* = (1-\gamma_1)/(1-\gamma_2) < 1 = \bar{\theta}_2$ and $\Theta_2(\theta_1) = [(1-\gamma_2)\theta_1/(1-\gamma_1), 1]$ for any $\theta_1 \in [0, 1]$.

We describe the logic behind why Assumption 1 is needed for the proposed two-stage mechanism to satisfy all the desired properties. First, we let the buyer pay an amount equal to his "reported" valuation when his type report is below the cutoff θ_2^*. Next, we solve the appropriate payment function above the cutoff which satisfies BIC and IIR. It turns out that we can find an upper bound and lower bound on the buyer's payment function above the cutoff θ_2^*. Specifically, the upper bound comes from the seller's IIR constraints and the lower bound comes from the buyer's IIR constraints. Assumption 1 plays a role of ensuring that the upper and lower bound are compatible with each other in the two-stage mechanism constructed in Subsection 4.2.

As we mentioned in the above remark, Assumption 1 is automatically satisfied in Case (i) $0 < \gamma_2 \le \gamma_1 < 1$ in the example of Section 3. To check when Assumption 1 is satisfied even in Case (ii) of the example of Section 3, we are going to use the following result.

Lemma 5. In the example of Section 3, our Assumption 1 is reduced to

$$\frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2 - \gamma_1 - 1) \ge 0.$$

Proof. The proof is in the Appendix. □

The lemma below shows that Assumption 1 sometimes holds in Case (ii) of the example in Section 3.

Lemma 6. Suppose that in the example in Section 3, both agents' valuation functions are $\tilde{u}_1(\theta_1, \theta_2) = \theta_1 + \theta_2/3$ and $\tilde{u}_2(\theta_1, \theta_2) = \theta_2 + \theta_1/2$. That is, $0 < 1/3 = \gamma_1 < 1/2 = \gamma_2 < 1$. Then, Assumption 1 holds.

Proof. Plugging $\gamma_1 = 1/3$ and $\gamma_2 = 1/2$ into the inequality in Lemma 5, we obtain

$$\frac{1}{6}\frac{(1/2)^2}{2/3} + \frac{1/2}{2/3} - \frac{1}{2}\left(\frac{1/2}{2/3}\right)^2 + \frac{1}{2}(1/2 - 1/3 - 1) \quad = \quad \frac{11}{96} > 0.$$

Thus, Assumption 1 is satisfied. $\square$

We can also show in the lemma below that Assumption 1 is sometimes violated in Case (ii) of the example of Section 3.

Lemma 7. Suppose that in the example in Section 3, both agents' valuation functions are $\tilde{u}_1(\theta_1, \theta_2) = \theta_1 + \theta_1/2$ and $\tilde{u}_2(\theta_1, \theta_2) = \theta_2 + 4\theta_1/5$. That is, $0 < 1/2 = \gamma_1 < 4/5 = \gamma_2 < 1$. Then, Assumption 1 fails.

Proof. Plugging $\gamma_1 = 1/2$ and $\gamma_2 = 4/5$ into the inequality in Lemma 5 so that we obtain

$$\frac{1}{6}\frac{0.2^2}{0.5} + \frac{0.2}{0.5} - \frac{1}{2}\left(\frac{0.2}{0.5}\right)^2 + \frac{1}{2}(0.8 - 0.5 - 1) \quad = \quad -\frac{1}{60} < 0.$$

Thus, Assumption 1 is violated. $\square$

4.4 The Theorem

Using the two-stage NTNP, monotone mechanism proposed in Subsection 4.2, we are able to establish the main result of the paper.

Theorem 1. Suppose that Assumption 1 holds. Then, there exists a two-stage NTNP, monotone mechanism (Θ, Π, x^*, t^M) satisfying BIC, IIR, EFF, and BB.

Proof. We make use of the two-stage mechanism constructed in Subsection 4.2.

Since the seller's transfer t_1^M is independent of his payoff report u_1^r and the buyer's transfer t_2^M is independent of u_2^r, then each agent has no incentive to deviate from the truth-telling in their payoff report in the second stage. Given this, it remains to verify that the truth-telling in the first stage constitutes part of a perfect Bayesian equilibrium (Steps 1 and 2) and that IIR is satisfied for both agents (Step 3). The proof is completed by the following three steps.

Step 1: If the buyer always reports the truth in the first stage, the seller has no incentive to tell a lie in the first stage.

Proof. The proof is in the Appendix. $\square$

Step 2: If the seller always reports the truth in the first stage, the buyer has no incentive to tell a lie in the first stage.

Proof. The proof is in the Appendix. □

In Steps 1 and 2, we show that the constructed two-stage mechanism (Θ, Π, x^*, t^M) satisfies BIC.

Step 3: The two-stage mechanism (Θ, Π, x^*, t^M) also satisfies IIR.

Proof. The proof is in the Appendix. □

Taking into account that both EFF and BB are already built in the mechanism, we complete the proof of Theorem 1. □

We record the implications of Theorem 1 as well as the properties of the proposed two-stage mechanism in the context of the example in Section 3.

1. When $0 < \gamma_2 \leq \gamma_1 < 1$, we have $\theta_2^* = 1 = \bar{\theta}_2$. In this case, we get $g(\theta_1^r) = -\tilde{u}_2(\theta_1^r, \bar{\theta}_2)$. Recall $t_2(\theta_1^r, \theta_2^r; u_1^r, u_2^r) = -\tilde{u}_2(\theta_1^r, \theta_2^r)$ when $\theta_2^r < \theta_2^*$ and $x^*(\theta_1^r, \theta_2^r) = 1$. If both agents report truthfully in both stages, the buyer always pays an amount equal to his true valuation to the seller. In other words, the seller extracts the full surplus in this case.

2. When $0 < 1/3 = \gamma_1 < 1/2 = \gamma_2 < 1$, we have $\theta_2^* = 3/4$. In this case, we set $g(\theta_1^r) = 3(\theta_1^r)^2/4 - 5\theta_1^r/2$. If both agents report truthfully in both stages and the buyer's true type is $\theta_2 > \theta_2^*$, the buyer's ex post utility becomes

$$\tilde{u}_2(\theta_1, \theta_2) + g(\theta_1) = \theta_2 + \frac{1}{2}\theta_1 + \frac{3}{4}(\theta_1)^2 - \frac{5}{2}\theta_1 = \theta_2 - \frac{4}{3} + \frac{3}{4}\left(\theta_1 - \frac{4}{3}\right)^2.$$

   To further illustrate the properties of the proposed two-stage mechanism when $\gamma_1 = 1/3$ and $\gamma_2 = 1/2$, we consider the following subcases:

   (a) when $\theta_1 = 0$, we have $g(\theta_1) = 0$. This means that the buyer receives the good without making any payment. Hence, the buyer receives the full surplus.

   (b) when $\theta_1 = 1$, we have that $\tilde{u}_2(\theta_1, \theta_2) + g(\theta_1) = \theta_2 - 5/4 < 0$, implying that the buyer's ex post utility is always negative because $\theta_2 \leq 1$. Thus, the ex post individual rationality (EPIR) is violated. Nonetheless, since our Assumption 1 holds, the proposed two-stage mechanism satisfies IIR (as opposed to EPIR) together with BIC, EFF, and BB. This exhibits a contrast with the analysis of GMO (2011) which maintains EPIR throughout.

Moreover, from Step 3 in the proof of Theorem 1 (see the Appendix (Section 8.8) for the details), we know that if $\theta_2 \geq \theta_2^*$, the expected utility of the buyer of type θ_2 after participation is

$$\int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) + \int_{\Theta_1} g(\theta_1) dF_1(\theta_1) = \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) - \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2^*) dF_1(\theta_1) \geq 0,$$

where the weak inequality follows because $\tilde{u}_2(\cdot)$ is strictly increasing in θ_2. Therefore, if $\theta_2 \leq \theta_2^*$, the buyer of type θ_2 is always left with zero expected surplus; if $\theta_2 > \theta_2^*$, the buyer receives a positive expected surplus.

These features we described above distinguishes our proposed two-stage mechanism from the "shoot-the-liar" mechanism proposed by Mezzetti (2007). By contrast, GMO (2011, Section 5) apply the "shoot-the-liar mechanism" without modifications to their partnership dissolution problem.

5 Simulation

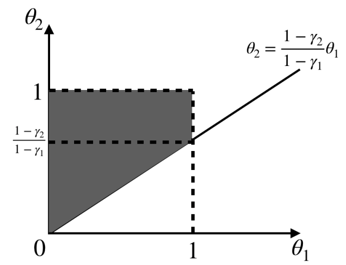
To assess the permissiveness and restrictiveness of Assumption 1, we provide a set of simulation results based on the example in Section 3. Both agents' types are uniformly distributed on the unit interval $[0, 1]$ and for each type profile $(\theta_1, \theta_2) \in [0, 1]^2$, their valuation functions are $\tilde{u}_1(\theta_1, \theta_2) = \theta_1 + \gamma_1\theta_2$ and $\tilde{u}_2(\theta_1, \theta_2) = \theta_2 + \gamma_2\theta_1$ where $\gamma_1 \in \{0.01, 0.02, \cdots, 0.98\}$ and $\gamma_2 \in \{\gamma_1 + 0.01, \gamma_1 + 0.02, \cdots, 0.99\}$ for each γ_1. As we discuss in the previous section, Assumption 1 is always satisfied when $0 < \gamma_2 \leq \gamma_1 < 1$, which is called Case (i) in the example of Section 3. Then, by our Theorem 1, we know that there exists a two-stage NTNP, monotone mechanism satisfying BIC, EFF, BB, and IIR. Thus, what remains to investigate is the extent to which there exists a two-stage NTNP, monotone mechanism satisfying all the desired properties in Case (ii) $0 < \gamma_1 < \gamma_2 < 1$. In the simulation, we select finitely many values of γ_1 and γ_2 satisfying this inequality.

We note that $\tilde{u}_2(\theta_1, \theta_2) > \tilde{u}_1(\theta_1, \theta_2)$ if and only if $\theta_2 > (1-\gamma_2)\theta_1/(1-\gamma_1)$. Since we assume $\gamma_2 > \gamma_1$, the slope of the efficient frontier is $(1-\gamma_2)/(1-\gamma_1) < 1$. The efficient decision rule dictates that, for each $(\theta_1, \theta_2) \in [0, 1]^2$,

$$x^*(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 > (1-\gamma_2)\theta_1/(1-\gamma_1) \\ 0 & \text{otherwise.} \end{cases}$$

The shaded region in Figure 2 (which is reproduced below) represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles for which it is efficient to trade.

Figure 2: when $0 < \gamma_1 < \gamma_2 < 1$



Recall that Lemma 5 allows us to translate our Assumption 1 into the following inequality:

$$\frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2 - \gamma_1 - 1) \geq 0. \tag{3}$$

Observe that Assumption 1 becomes an inequality about γ_1 and γ_2. Then, for each pair (γ_1, γ_2) satisfying $1 > \gamma_2 > \gamma_1 > 0$, we check whether or not inequality (3) is satisfied. Here is a summary of the simulation results. There are two possible scenarios:

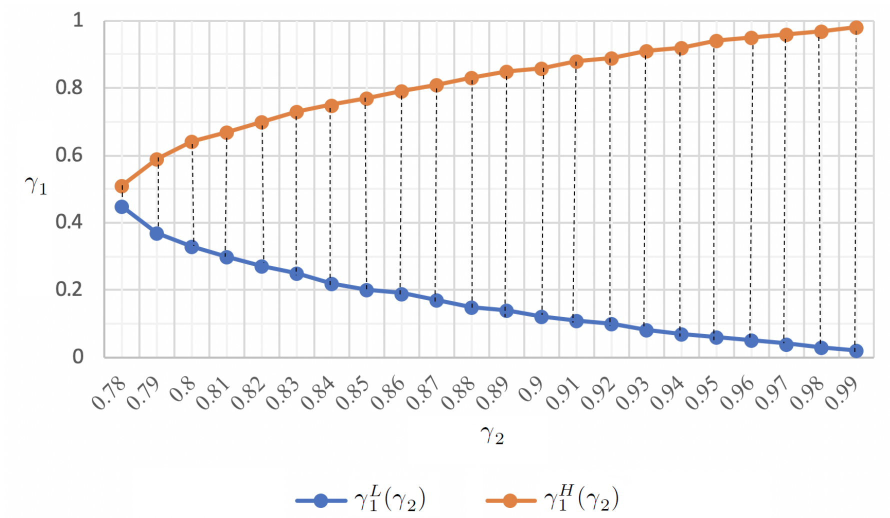
1. If $\gamma_2 \leq 0.77$, inequality (3) is always satisfied for all $\gamma_1, \gamma_2 \in (0, 1)$ satisfying $\gamma_1 < \gamma_2$;

2. For each $\gamma_2 > 0.77$, there exist $\gamma_1^L(\gamma_2), \gamma_1^H(\gamma_2) \in (0, 1)$ such that $\gamma_1^L(\gamma_2) < \gamma_1^H(\gamma_2)$ and inequality (3) is violated whenever $\gamma_1^L(\gamma_2) < \gamma_1 < \gamma_1^H(\gamma_2)$ and it is satisfied otherwise.

We illustrate the second scenario in Figure 5 below. For each $\gamma_2 > 0.77$, there are a corresponding point on the upper curve indicating $\gamma_1^H(\gamma_2)$ and another corresponding point on the lower curve indicating $\gamma_1^L(\gamma_2)$. Then, if $\gamma_1^L(\gamma_2) < \gamma_1 < \gamma_1^H(\gamma_2)$, inequality (3) is violated. The region where inequality (3) is violated is represented by the dotted region in Figure 5. The region outside the dotted region dictates the case in which inequality (3) is satisfied.

In Figure 6, we track all possible pairs of $(\gamma_1, \gamma_2) \in (0, 1)^2$ satisfying inequality (3). In particular, the upper triangle in $[0, 1]^2$, i.e., the region where $\gamma_2 > \gamma_1$ corresponds to Case (ii) of the example in Section 3. The lightly shaded region describes all pairs of (γ_1, γ_2) within this upper triangle for which our Assumption 1 is satisfied.

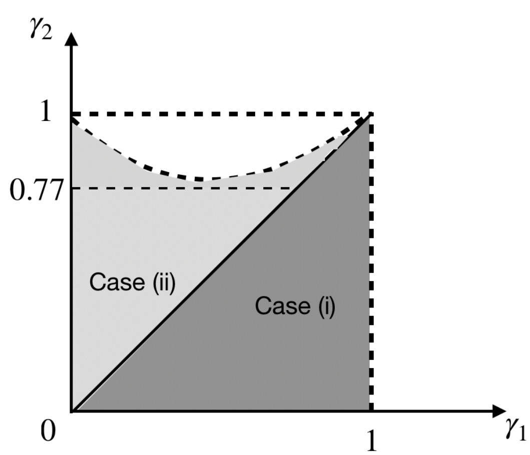
Figure 5: When $\gamma_2 > 0.77$



On the other hand, the lower triangle in the unit square, i.e., the region where $\gamma_2 < \gamma_1$ corresponds to Case (i) of the example in Section 3. Then, by our Theorem 1, we can always find a two-stage NTNP, monotone mechanism satisfying BIC, IIR, EFF, and BB within this region. Hence, the heavily shaded region describes all pairs of (γ_1, γ_2) within the lower triangle for which our Assumption 1 is satisfied.

Therefore, the lightly and heavily shaded regions together indicate the set of (γ_1, γ_2) for which our Assumption 1 is satisfied. Since the whole shaded (regardless of whether lightly or heavily) region spans quite a large part of the unit square, we conclude that our Assumption 1 can be satisfied in many cases in the example of Section 3.

Figure 6: Summary of Simulation



We can also verify that if Assumption 1 is violated, then the two-stage mechanism we propose in Section 4.2 violates the seller's IIR constraint.[6] We make this point by the following claim:

Claim 5. If $\gamma_1 = 1/2$ and $\gamma_2 = 4/5$ in the example of Section 3, the seller's IIR constraint is violated in our two-stage NTNP, monotone mechanism constructed in Subsection 4.2.

Proof. The proof is in the Appendix. $\square$

By this claim, we loosely say that our Assumption 1 is violated when the degree of interdependence of preferences of the buyer is too high relative to that of the seller.

6 The Relation with Galavotti, Muto, and Oyama (2011)

In this section, we will discuss the relation between this paper and Galavotti, Muto, and Oyama (2011) (hereafter, GMO). GMO (2011) consider the problem of partnership dissolution in a model with interdependent values where there are one asset, and n risk-neutral agents indexed by $i \in \{1, \ldots, n\}$ where $n \geq 2$. Each agent i owns a share α_i of the asset such that $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$. In private values environments, Cramton, Gibbons, and Klemperer (1987) show that the equal-share ownership $(\alpha_1, \ldots, \alpha_n) = (1/n, \ldots, 1/n)$ allows us to construct a mechanism satisfying BIC, EFF, IIR, and BB, which exhibits a contrast with this paper's extreme ownership structure where the seller has the full property right over the good. However, FKM (2003) show that this positive result of Cramton, Gibbons, and Klemperer (1987) cannot be extended to a model with interdependent values. This explains why GMO (2011, Section 5) also resort to the use of two-stage mechanisms in order to obtain more positive results.

To make the comparison between GMO (2011) and our paper, we assume that there are only two agents, i.e., $n = 2$. By an *ownership structure* (α_1, α_2) where each $\alpha_i \in [0, 1]$ and $\alpha_1 + \alpha_2 = 1$, we mean that agent 1 (the seller) has the property right over α_1 fraction of the asset and agent 2 (the buyer) has the property right over α_2 fraction of the asset. To discuss the contribution of GMO (2001), we first strengthen our IIR constraint into its ex post counterpart.

[6]By the very proof of Theorem 1, Assumption 1 has a bite exactly when the seller's IIR constraint has a bite. In other words, if inequality (2) is violated, it is the seller's IIR constraint that is violated.

Definition 8. Let (α_1, α_2) be an ownership structure. A two-stage mechanism (Θ, Π, x, t) satisfies *ex post individual rationality* (EPIR) if, for all $(\theta_1, \theta_2) \in \Theta$ and $(u_1, u_2) \in \Pi$,

$$u_1(x(\theta_1, \theta_2), \theta_1, \theta_2) + t_1(\theta_1, \theta_2; u_1, u_2) \geq \alpha_1 \tilde{u}_1(\theta_1, \theta_2),$$

and

$$u_2(x(\theta_1, \theta_2), \theta_1, \theta_2) + t_2(\theta_1, \theta_2; u_1, u_2) \geq \alpha_2 \tilde{u}_2(\theta_1, \theta_2).$$

GMO (2011) provide the following sufficient condition (called Assumption 5.1 on p.14) under which the "shoot-the-liar" mechanism satisfies BIC, EPIR, EFF, and BB for any ownership structure. We formally state GMO's Assumption 5.1.

GMO's Assumption 5.1: There exist $M_1, M_2 \geq 0$ such that for all $i \in \{1, \ldots, n\}$, all $\theta_i, \hat{\theta}_i \in \Theta_i$ with $\hat{\theta}_i \neq \theta_i$,

$$\begin{aligned} & \mathbb{E}_{\theta_{-i}} \left[\mathbb{1}_{\{\theta_{-i} | i = m(\hat{\theta}_i, \theta_{-i})\}}(\theta_{-i}) \left(\tilde{u}_i(\bar{\theta}_i, \theta_{-i}) - \tilde{u}_i(\hat{\theta}_i, \theta_{-i}) \right) \right] \\ \leq \quad & M_1 \sum_{j \neq i} \mathbb{E}_{\theta_{-i}} \left[\mathbb{1}_{\{\theta_{-i} | j = m(\hat{\theta}_i, \theta_{-i}), \tilde{u}_j(\theta_i, \theta_{-i}) \neq \tilde{u}_j(\hat{\theta}_i, \theta_{-i})\}}(\theta_{-i}) \right], \end{aligned} \tag{4}$$

and

$$\begin{aligned} & \sum_{j \neq i} \mathbb{E}_{\theta_{-i}} \left[\mathbb{1}_{\{\theta_{-i} | j = m(\hat{\theta}_i, \theta_{-i}), \tilde{u}_j(\theta_i, \theta_{-i}) = \tilde{u}_j(\hat{\theta}_i, \theta_{-i})\}}(\theta_{-i}) \right] \\ \leq \quad & M_2 \sum_{j \neq i} \mathbb{E}_{\theta_{-i}} \left[\mathbb{1}_{\{\theta_{-i} | j = m(\hat{\theta}_i, \theta_{-i}), \tilde{u}_j(\theta_i, \theta_{-i}) \neq \tilde{u}_j(\hat{\theta}_i, \theta_{-i})\}}(\theta_{-i}) \right], \end{aligned} \tag{5}$$

where $\mathbb{1}_X(x)$ is the index function such that $\mathbb{1}_X(x) = 1$ if $x \in X$ and 0 if $x \notin X$, and $m(\theta) = \max(\arg\max_j \tilde{u}_j(\theta))$.

In our bilateral trade setup, we always have $(\alpha_1, \alpha_2) = (1, 0)$, i.e., the seller has the property right over the good, while the buyer has no property right over it. We know from our Lemma 4 that there are generally two cases: (i) $\theta_2^* = \bar{\theta}_2$ and (ii) $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ where θ_2^* is the cutoff point identified in Lemma 4. In Case (i) $\theta_2^* = \bar{\theta}_2$, which corresponds to the case that $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$, we can use our proposed two-stage mechanism and show that it satisfies BIC, IIR, EFF, and BB. As in GMO (2011), we can strengthen IIR into EPIR for this result.

In what follows, we will focus on the bilateral trade model and then compare GMO's Assumption 5.1 with our Assumption 1. We obtain the following claim:

Lemma 8. The relation between Assumption 5.1 in GMO (2011) and our Assumption 1 is summarized as follows:

1. Inequality (4) in GMO's Assumption 5.1 implies our Assumption 1;

2. Inequality (5) in GMO's Assumption 5.1 is automatically satisfied under the bilateral trade model in our paper.

Proof. The proof is in the Appendix. □

Intuitively, inequality (4) requires that each agent's deviation be detected by the other agent with strictly positive probability, Case (i) $\theta_2^* = \bar{\theta}_2$, i.e., $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1)$ for all $\theta_2 < \bar{\theta}_2$ requires that only the buyer's deviation be detected by the seller with strictly positive probability. Therefore, the condition that $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1)$ for all $\theta_2 < \bar{\theta}_2$ is weaker than inequality (4).[7]

To further illustrate the stringent nature of inequality (4) relative to our Assumption 1, we revisit the example in Section 3. We obtain the following claim:

Lemma 9. In the example in Section 3, GMO's Assumption 5.1 is satisfied if and only if $\gamma_1 = \gamma_2$.

Proof. The proof is in the Appendix. □

This suggests that GMO's Assumption 5.1 is generically violated in the bilateral trade model.

7 Conclusion

This paper characterizes when efficient, voluntary bilateral trades are incentive compatible in an environment with interdependent values. Acknowledging some existing impossibility results by Gresik (1991) and FKM (2003), we obtain more positive results by looking at two-stage mechanisms proposed by Mezzetti (2004). We show by means of an example that the generalized two-stage Groves mechanism never satisfies IIR. If our Assumption 1 is satisfied in a general environment, we next show that there exists a two-stage mechanism satisfying BIC, IIR, EFF, and BB. In the context of the example in Section 3, our Assumption 1 roughly says that the buyer's degree of interdependence of preferences is not too high relative to the seller's counterpart. In Section 5, we also argue by the same example that our Assumption 1 can be satisfied for a large number of cases. The property that distinguishes our proposed two-stage mechanism from the generalized two-stage

[7] The logic behind our first general case is that even if the seller's deviation is not detected by the buyer, this is not a profitable deviation because in this case, the seller keeps the good without receiving any monetary transfer.

Groves mechanism is the "no-trade-then-no-payments" (NTNP) property, which means that if trade does not occur, no payments are made. Indeed, the generalized two-stage Groves mechanism does not satisfy the NTNP property. By expanding our scope into two-stage mechanisms, we consider our paper as the first attempt to further pushing the boundary between when efficient, voluntary bilateral trades are implementable and when they are not.

8 Appendix

In the Appendix, we provide all the proofs omitted from the main text of the paper.

8.1 Proof of Claim 1

Proof. We first identify the worst-off type for each agent by checking the following cases.

Case 1: $0 \leq \theta_1 \leq (1-\gamma_1)/(1-\gamma_2)$

We compute the following.

$$\begin{aligned}
&\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
=\ & \int_0^{\frac{1-\gamma_2}{1-\gamma_1}\theta_1} \tilde{u}_1(\theta_1,\theta_2)d\theta_2 + \int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^{1} \tilde{u}_2(\theta_1,\theta_2)d\theta_2 \\
=\ & \int_0^{\frac{1-\gamma_2}{1-\gamma_1}\theta_1} (\theta_1+\gamma_1\theta_2)d\theta_2 + \int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^{1} (\gamma_2\theta_1+\theta_2)d\theta_2 \\
=\ & \frac{1-\gamma_2}{1-\gamma_1}(\theta_1)^2 + \frac{1}{2}\frac{\gamma_1(1-\gamma_2)^2}{(1-\gamma_1)^2}(\theta_1)^2 + \gamma_2\theta_1\left(1-\frac{1-\gamma_2}{1-\gamma_1}\theta_1\right) + \frac{1}{2}\left[1-\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2(\theta_1)^2\right] \\
=\ & \frac{1}{2} + \gamma_2\theta_1 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2. \qquad (6)
\end{aligned}$$

Hence, the objective function becomes

$$\begin{aligned}
\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) - U_1^O(\theta_1) &= \frac{1}{2} + \gamma_2\theta_1 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2 - \int_0^1 \tilde{u}_1(\theta_1,\theta_2)d\theta_2 \\
&= \frac{1}{2} + \gamma_2\theta_1 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2 - \theta_1 - \frac{1}{2}\gamma_1 \\
&= \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\left(\theta_1 - \frac{1-\gamma_1}{1-\gamma_2}\right)^2.
\end{aligned}$$

So, when $\theta_1 = (1-\gamma_1)/(1-\gamma_2)$, the objective function attains its minimum, which is zero. So, in this case, the seller's worst-off type is $\theta_1^w = (1-\gamma_1)/(1-\gamma_2)$.

Case 2: $(1-\gamma_1)/(1-\gamma_2) < \theta_1 \leq 1$

We compute the following:

$$\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) - U_1^O(\theta_1) = \int_0^1 \tilde{u}_1(\theta_1,\theta_2)d\theta_2 - \int_0^1 \tilde{u}_1(\theta_1,\theta_2)d\theta_2 = 0.$$

Therefore, in this case, the seller's worst-off type is $\theta_1^w = 1$.

We compute the expected loss for his worst-off type θ_1^w below:

$$\begin{aligned}
L_1 &\equiv U_1^O(\theta_1^w) - U_1^G(\theta_1^w) \\
&= -\left[\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1^w,\theta_2),\theta_1^w,\theta_2)\right) - U_1^O(\theta_1^w)\right] + \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
&= 0 + \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
&= \frac{1}{2}\mathbb{E}_{\theta_2}\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)\right],
\end{aligned}$$

where $\mathbb{E}_{\theta_1}$ denotes the expectation operator over Θ_1. Note that for each $\theta_2 \in \Theta_2$,

$$\begin{aligned}
&\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
= &\int_0^{\frac{1-\gamma_1}{1-\gamma_2}\theta_2} \tilde{u}_2(\theta_1,\theta_2)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\theta_2}^1 \tilde{u}_1(\theta_1,\theta_2)d\theta_1 \\
= &\int_0^{\frac{1-\gamma_1}{1-\gamma_2}\theta_2} (\gamma_2\theta_1+\theta_2)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\theta_2}^1 (\theta_1+\gamma_1\theta_2)d\theta_1 \\
= &\frac{1}{2}\gamma_2\left(\frac{1-\gamma_1}{1-\gamma_2}\theta_2\right)^2 + \frac{1-\gamma_1}{1-\gamma_2}(\theta_2)^2 + \frac{1}{2}\left(1-\left(\frac{1-\gamma_1}{1-\gamma_2}\theta_2\right)^2\right) + \gamma_1\theta_2\left(1-\frac{1-\gamma_1}{1-\gamma_2}\theta_2\right) \\
= &\frac{1}{2} + \gamma_1\theta_2 + \frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}(\theta_2)^2. \qquad (7)
\end{aligned}$$

Therefore, we compute the expected loss for the seller's worst-off type θ_1^w:

$$\begin{aligned} L_1 &= \frac{1}{2}\mathbb{E}_{\theta_2}\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)\right] \\ &= \frac{1}{2}\int_0^1 \left[\frac{1}{2}+\gamma_1\theta_2+\frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}(\theta_2)^2\right] d\theta_2 \\ &= \frac{1}{4}+\frac{1}{4}\gamma_1+\frac{1}{12}\frac{(1-\gamma_1)^2}{1-\gamma_2}. \end{aligned} \tag{8}$$

Since $\gamma_1 > 0$ and $\gamma_2 < 1$, we obtain $L_1 > 0$, which implies that the seller is worse off after participating in the mechanism. On the other hand, we obtain the buyer's worst-off type θ_2^w from participating in the generalized two-stage Groves mechanism:

$$\begin{aligned} \theta_2^w &\in \arg\min_{\theta_2\in\Theta_2}\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) - U_2^O(\theta_2)\right] \\ &= \arg\min_{\theta_2\in\Theta_2}\left[\frac{1}{2}+\gamma_1\theta_2+\frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}(\theta_2)^2 - 0\right], \end{aligned}$$

where the equality follows from (7). It is easy to see that the buyer's worst-off type is $\theta_2^w = 0$. So, we compute the expected loss for his worst-off type $\theta_2^w = 0$ as follows:

$$\begin{aligned} L_2 &\equiv U_2^O(\theta_2^w) - U_2^G(\theta_2^w) \\ &= -\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2^w),\theta_1,\theta_2^w)\right) - U_2^O(\theta_2^w)\right] + \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\ &= -\frac{1}{2}+\frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\ &= -\frac{1}{4}+\frac{1}{4}\gamma_1+\frac{1}{12}\frac{(1-\gamma_1)^2}{1-\gamma_2} \quad \text{(recall (8))} \end{aligned}$$

So, the total expected loss is

$$\begin{aligned} L_1 + L_2 &= \frac{1}{4}+\frac{1}{4}\gamma_1+\frac{1}{12}\frac{(1-\gamma_1)^2}{1-\gamma_2} - \frac{1}{4}+\frac{1}{4}\gamma_1+\frac{1}{12}\frac{(1-\gamma_1)^2}{1-\gamma_2} \\ &= \frac{1}{2}\gamma_1+\frac{1}{6}\frac{(1-\gamma_1)^2}{1-\gamma_2} \\ &> 0, \end{aligned}$$

where the last inequality follows because $\gamma_1 > 0$ and $0 < \gamma_2 < 1$. This completes the proof of Claim 1. □

8.2 Proof of Claim 2

Proof. We compute the seller's worst-off type from participating in the generalized two-stage Groves mechanism.

$$\begin{aligned}
\theta_1^w &\in \arg\min_{\theta_1\in\Theta_1}\left[\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)-U_1^O(\theta_1)\right]\\
&= \arg\min_{\theta_1\in\Theta_1}\left[\int_0^{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}\tilde{u}_1(\theta_1,\theta_2)d\theta_2+\int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^{1}\tilde{u}_2(\theta_1,\theta_2)d\theta_2-\int_0^1\tilde{u}_1(\theta_1,\theta_2)d\theta_2\right]\\
&= \arg\min_{\theta_1\in\Theta_1}\left[\frac{1}{2}+\gamma_2\theta_1+\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2-\theta_1-\frac{1}{2}\gamma_1\right]\ \text{(recall (6))}\\
&= \arg\min_{\theta_1\in\Theta_1}\left[\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\left(\theta_1-\frac{1-\gamma_1}{1-\gamma_2}\right)^2\right].
\end{aligned}$$

Note that $0<\gamma_1<\gamma_2<1$ implies $(1-\gamma_1)/(1-\gamma_2)>1$. Hence, the seller's worst-off type is $\theta_1^w=1$. We compute the expected loss for his worst-off type as follows:

$$\begin{aligned}
L_1 &\equiv U_1^O(\theta_1^w)-U_1^G(\theta_1^w)\\
&= -\left[\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1^w,\theta_2),\theta_1^w,\theta_2)\right)-U_1^O(\theta_1^w)\right]+\frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)\\
&= -\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\left(1-\frac{1-\gamma_1}{1-\gamma_2}\right)^2+\frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right).
\end{aligned}$$

We further compute the following:

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) &= \frac{1}{2}\mathbb{E}_{\theta_1}\left[\mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)\right]\\
&= \frac{1}{2}\int_0^1\left[\int_0^{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}\tilde{u}_1(\theta_1,\theta_2)d\theta_2+\int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^{1}\tilde{u}_2(\theta_1,\theta_2)d\theta_2\right]d\theta_1\\
&= \frac{1}{2}\int_0^1\left[\frac{1}{2}+\gamma_2\theta_1+\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2\right]d\theta_1\ \text{(recall (6))}\\
&= \frac{1}{4}+\frac{1}{4}\gamma_2+\frac{1}{12}\frac{(1-\gamma_2)^2}{1-\gamma_1}. \qquad (9)
\end{aligned}$$

Therefore, the expected loss for the seller's worst-off type is

$$\begin{aligned}
L_1 &= -\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\left(1-\frac{1-\gamma_1}{1-\gamma_2}\right)^2+\frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2}u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right)\\
&= -\frac{1}{2}\frac{(\gamma_2-\gamma_1)^2}{1-\gamma_1}+\frac{1}{4}+\frac{1}{4}\gamma_2+\frac{1}{12}\frac{(1-\gamma_2)^2}{1-\gamma_1}.
\end{aligned}$$

On the other hand, the buyer's worst-off type from participating in the generalized two-stage Groves mechanism is given as follows:

$$\theta_2^w \in \arg\min_{\theta_2\in\Theta_2}\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) - U_2^O(\theta_2)\right] = \arg\min_{\theta_2\in\Theta_2}\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right).$$

We identify the worst-off type for each agent by the following cases.

Case 1: $0 < \theta_2 < (1-\gamma_2)/(1-\gamma_1)$

then

$$\begin{aligned}
\mathbb{E}_{\theta_1}\left[\sum_{j=1}^{2} u_j(x^*(\theta_2,\theta_1),\theta_2,\theta_1)\right] &= \int_0^{\frac{1-\gamma_1}{1-\gamma_2}\theta_2}\tilde{u}_2(\theta_1,\theta_2)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\theta_2}^{1}\tilde{u}_1(\theta_1,\theta_2)d\theta_1 \\
&= \frac{1}{2} + \gamma_1\theta_2 + \frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}(\theta_2)^2 \text{ (recall (7)).}
\end{aligned}$$

It is easy to see that $\theta_2 = 0$ achieves its minimum, which is $1/2$.

Case 2: $(1-\gamma_2)/(1-\gamma_1) \le \theta_2 \le 1$

then

$$\mathbb{E}_{\theta_1}\left[\sum_{j=1}^{2} u_j(x^*(\theta_2,\theta_1),\theta_2,\theta_1)\right] = \int_0^1 \tilde{u}_2(\theta_1,\theta_2)d\theta_1 = \int_0^1(\gamma_2\theta_1+\theta_2)d\theta_1 = \frac{1}{2}\gamma_2 + \theta_2.$$

Clearly, $\theta_2 = (1-\gamma_2)/(1-\gamma_1)$ achieves its minimum, which is $\gamma_2/2+(1-\gamma_2)/(1-\gamma_1)$.

Since

$$\frac{1}{2} - \left[\frac{1}{2}\gamma_2 + \frac{1-\gamma_2}{1-\gamma_1}\right] = -\frac{(1-\gamma_2)(1+\gamma_1)}{2(1-\gamma_1)} < 0,$$

the buyer's worst-off type is $\theta_2^w = 0$. We compute the expected loss for his worst-off type.

$$\begin{aligned}
L_2 &\equiv U_2^O(\theta_2^w) - U_2^G(\theta_2^w) \\
&= -\left[\mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2^w),\theta_1,\theta_2^w)\right) - U_2^O(\theta_2^w)\right] + \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
&= -\frac{1}{2} + \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{2} u_j(x^*(\theta_1,\theta_2),\theta_1,\theta_2)\right) \\
&= -\frac{1}{2} + \frac{1}{4} + \frac{1}{4}\gamma_2 + \frac{1}{12}\frac{(1-\gamma_2)^2}{1-\gamma_1} \text{ (recall (9)).}
\end{aligned}$$

Therefore, the total expected loss is

$$\begin{aligned}
L_1 + L_2 &= -\frac{1}{2}\frac{(\gamma_2-\gamma_1)^2}{1-\gamma_1} + \frac{1}{4} + \frac{1}{4}\gamma_2 + \frac{1}{12}\frac{(1-\gamma_2)^2}{1-\gamma_1} - \frac{1}{4} + \frac{1}{4}\gamma_2 + \frac{1}{12}\frac{(1-\gamma_2)^2}{1-\gamma_1} \\
&= \frac{1}{2(1-\gamma_1)}\left[\gamma_2(1-\gamma_1) - (\gamma_2-\gamma_1)^2\right] + \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1}.
\end{aligned}$$

Since $\gamma_2 > \gamma_2 - \gamma_1$ and $1 - \gamma_1 > \gamma_2 - \gamma_1$, we obtain $L_1 + L_2 > 0$. This completes the proof of Claim 2. $\square$

8.3 Proof of Claim 3

Proof. We divide our argument into the following two cases.

Case (i): $0 < \gamma_2 \leq \gamma_1 < 1$

From Figure 1 (p.13), we know that if $\theta_1 = 1$ and $\theta_2 = 0$, it is efficient not to trade and the buyer's transfer in the generalized two-stage Groves mechanism is given as follows:

$$\begin{aligned}
& t_2^G(1,0;u_1,u_2) \\
= \ & u_1 - \frac{1}{2}h_2(1,0) \\
= \ & u_1 - \frac{1}{2}\left[\sum_{j=1}^{2} u_j\left(x^*(1,0),1,0\right) - \mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j\left(x^*(\theta_1,0),\theta_1,0\right)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j\left(x^*(1,\theta_2),1,\theta_2\right)\right)\right] \\
= \ & \tilde{u}_1(1,0) - \frac{1}{2}\left(\tilde{u}_1(1,0) - \int_0^1 \tilde{u}_1(\theta_1,0)d\theta_1 + \int_0^1 \tilde{u}_1(1,\theta_2)d\theta_2\right),
\end{aligned}$$

where the third equality follows because $u_1 = u_1(x^*(1,0),1,0) = \tilde{u}_1(1,0)$, $u_2 = u_2(x^*(1,0),1,0) = 0$ and $x^*(\theta_1,0) = x^*(1,\theta_2) = 0$ for any $\theta_1, \theta_2 \in [0,1]$. Plugging the linear valuations in $t_2^G(1,0;u_1,u_2)$ above, we obtain

$$t_2^G(1,0;u_1,u_2) = 1 - \frac{1}{2}\left(1 - \int_0^1 \theta_1 d\theta_1 + \int_0^1 (1+\gamma_1\theta_2)d\theta_2\right) = \frac{1}{4}(1-\gamma_1) > 0,$$

where the last strict inequality follows because $1 > \gamma_1$ in Case (i). Hence, in the type profile $(\theta_1,\theta_2) = (1,0)$, the buyer receives positive subsidy under no trade, contradicting NTNP.

Case (ii): $0 < \gamma_1 < \gamma_2 < 1$

From Figure 2 (p.14), we know that if $\theta_1 = 1$ and $\theta_2 = 0$, it is efficient not to trade. We then compute the buyer's transfer in the generalized two-stage Groves mechanism:

$$\begin{aligned}
& t_2^G(1,0;u_1,u_2) \\
= \ & u_1 - \frac{1}{2}\left[\sum_{j=1}^{2} u_j\left(x^*(1,0),1,0\right) - \mathbb{E}_{\theta_1}\left(\sum_{j=1}^{2} u_j\left(x^*(\theta_1,0),\theta_1,0\right)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j\left(x^*(1,\theta_2),1,\theta_2\right)\right)\right] \\
= \ & \tilde{u}_1(1,0) - \frac{1}{2}\left(\tilde{u}_1(1,0) - \int_0^1 \tilde{u}_1(\theta_1,0)d\theta_1 + \int_0^{\frac{1-\gamma_2}{1-\gamma_1}} \tilde{u}_1(1,\theta_2)d\theta_2 + \int_{\frac{1-\gamma_2}{1-\gamma_1}}^1 \tilde{u}_2(1,\theta_2)d\theta_2\right),
\end{aligned}$$

where the last equality follows because $u_1 = u_1(x^*(1,0),1,0) = \tilde{u}_1(1,0)$, $u_2 = u_2(x^*(1,0),1,0) = 0$, $x^*(\theta_1,0) = 0$ for any $\theta_1 \in [0,1]$, $x^*(1,\theta_2) = 0$ if $\theta_2 < (1-\gamma_2)/(1-\gamma_1)$ and $x^*(1,\theta_2) = 1$ otherwise. Plugging the linear valuations in $t_2^G(1,0;u_1,u_2)$, we obtain

$$\begin{aligned} t_2^G(1,0;u_1,u_2) &= 1 - \frac{1}{2}\left(1 - \int_0^1 \theta_1 d\theta_1 + \int_0^{\frac{1-\gamma_2}{1-\gamma_1}} (1+\gamma_1\theta_2) d\theta_2 + \int_{\frac{1-\gamma_2}{1-\gamma_1}}^1 (\theta_2+\gamma_2) d\theta_2\right) \\ &= 1 - \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1-\gamma_2}{1-\gamma_1} + \frac{\gamma_1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2} - \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \gamma_2 - \gamma_2\frac{1-\gamma_2}{1-\gamma_1}\right]. \end{aligned}$$

After rearranging the terms above, we simplify its expression:

$$t_2^G(1,0;u_1,u_2) = 1 - \frac{1}{2}\left(1 + \gamma_2 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\right),$$

which is strictly decreasing in γ_1. Since $\gamma_1 < \gamma_2$, then for any $\gamma_2 \in (0,1)$, $t_2^G(1,0;u_1,u_2)$ reaches its greatest lower bound when $\gamma_1 = \gamma_2$, i.e.,

$$t_2^G(1,0;u_1,u_2) > 1 - \frac{1}{2}\left(1 + \gamma_2 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_2}\right) = \frac{1}{4}(1-\gamma_2) > 0,$$

where the last strict inequality holds because $1 > \gamma_2$ in Case (ii). Therefore, we conclude $t_2^G(1,0;u_1,u_2) > 0$, implying that, in the type profile $(\theta_1,\theta_2) = (1,0)$, the buyer receives positive subsidies under no trade. Hence, NTNP is violated. This completes the proof. □

8.4 Proof of Claim 4

Proof. We divide our argument into two cases.

Case (i): $0 < \gamma_2 \leq \gamma_1 < 1$

Fix $\hat{\theta}_1 \in [0,1]$ and let $\alpha, \beta \in [0,1]$ be two distinct types of the buyer such that $\alpha > \beta$ and $x^*(\hat{\theta}_1,\alpha) = x^*(\hat{\theta}_1,\beta) = 1$. Then, the difference between the buyer's transfer under $(\hat{\theta}_1,\alpha)$ and that under $(\hat{\theta}_1,\beta)$ is computed below:

$$\begin{aligned} & t_2^G(\hat{\theta}_1,\alpha;u_1^\alpha,u_2^\alpha) - t_2^G(\hat{\theta}_1,\beta;u_1^\beta,u_2^\beta) \\ = \;& u_1^\alpha - \frac{1}{2}\left[\tilde{u}_2(\hat{\theta}_1,\alpha) - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^2 u_j(x^*(\theta_1,\alpha),\theta_1,\alpha)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^2 u_j(x^*(\hat{\theta}_1,\theta_2),\hat{\theta}_1,\theta_2)\right)\right] \\ & -u_1^\beta + \frac{1}{2}\left[\tilde{u}_2(\hat{\theta}_1,\beta) - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^2 u_j(x^*(\theta_1,\beta),\theta_1,\beta)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^2 u_j(x^*(\hat{\theta}_1,\theta_2),\hat{\theta}_1,\theta_2)\right)\right] \\ = \;& -\frac{1}{2}\left[\alpha - \beta - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^2 u_j(x^*(\theta_1,\alpha),\theta_1,\alpha)\right) + \mathbb{E}_{\theta_1}\left(\sum_{i=1}^2 u_j(x^*(\theta_1,\beta),\theta_1,\beta)\right)\right], \qquad (10) \end{aligned}$$

where the second equality follows because $x^*(\hat{\theta}_1, \alpha) = x^*(\hat{\theta}_1, \beta) = 1$ implies $u_1^\alpha = u_1^\beta = 0$ and $\tilde{u}_2(\hat{\theta}_1, \alpha) - \tilde{u}_2(\hat{\theta}_1, \beta) = \alpha + \gamma_2\hat{\theta}_1 - \beta - \gamma_2\hat{\theta}_1 = \alpha - \beta$.

Moreover, we compute the following term in the above expression:

$$
\begin{aligned}
& -\mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1,\alpha),\theta_1,\alpha)\right) + \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1,\beta),\theta_1,\beta)\right) \\
= & -\left(\int_0^{\frac{1-\gamma_1}{1-\gamma_2}\alpha} \tilde{u}_2(\theta_1,\alpha)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\alpha}^{1} \tilde{u}_1(\theta_1,\alpha)d\theta_1\right) + \left(\int_0^{\frac{1-\gamma_1}{1-\gamma_2}\beta} \tilde{u}_2(\theta_1,\beta)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\beta}^{1} \tilde{u}_1(\theta_1,\beta)d\theta_1\right) \\
= & -\int_0^{\frac{1-\gamma_1}{1-\gamma_2}\alpha} (\alpha+\gamma_2\theta_1)d\theta_1 - \int_{\frac{1-\gamma_1}{1-\gamma_2}\alpha}^{1} (\theta_1+\gamma_1\alpha)d\theta_1 + \int_0^{\frac{1-\gamma_1}{1-\gamma_2}\beta} (\beta+\gamma_2\theta_1)d\theta_1 + \int_{\frac{1-\gamma_1}{1-\gamma_2}\beta}^{1} (\theta_1+\gamma_1\beta)d\theta_1 \\
= & -\frac{1-\gamma_1}{1-\gamma_2}\alpha^2 - \frac{1}{2}\gamma_2\left(\frac{1-\gamma_1}{1-\gamma_2}\alpha\right)^2 - \frac{1}{2}\left(1-\left(\frac{1-\gamma_1}{1-\gamma_2}\alpha\right)^2\right) - \gamma_1\alpha\left(1-\frac{1-\gamma_1}{1-\gamma_2}\alpha\right) \\
& +\frac{1-\gamma_1}{1-\gamma_2}\beta^2 + \frac{1}{2}\gamma_2\left(\frac{1-\gamma_1}{1-\gamma_2}\beta\right)^2 + \frac{1}{2}\left(1-\left(\frac{1-\gamma_1}{1-\gamma_2}\beta\right)^2\right) + \gamma_1\beta\left(1-\frac{1-\gamma_1}{1-\gamma_2}\beta\right).
\end{aligned}
$$

After making a further rearrangement of the above expression, we obtain

$$
\begin{aligned}
& -\mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1,\alpha),\theta_1,\alpha)\right) + \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1,\beta),\theta_1,\beta)\right) \\
= & -\gamma_1(\alpha-\beta) - \frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}\left(\alpha^2-\beta^2\right).
\end{aligned}
$$

Plugging this back into (10), we obtain

$$
\begin{aligned}
t_2^G(\hat{\theta}_1,\alpha;u_1^\alpha,u_2^\alpha) - t_2^G(\hat{\theta}_1,\beta;u_1^\beta,u_2^\beta) &= -\frac{1}{2}\left[\alpha-\beta-\gamma_1(\alpha-\beta) - \frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}\left(\alpha^2-\beta^2\right)\right] \\
&= -\frac{1}{4}\frac{1-\gamma_1}{1-\gamma_2}(\alpha-\beta)\left[2(1-\gamma_2)-(1-\gamma_1)(\alpha+\beta)\right] \\
&< 0,
\end{aligned}
$$

where the last strict inequality above follows because $\alpha > \beta$ and $2(1-\gamma_2) - (1-\gamma_1)(\alpha+\beta) > 0$, which is followed by the assumption that $2 > \alpha + \beta$ and $1 - \gamma_2 \geq 1 - \gamma_1 > 0$. Therefore, we show that the generalized two-stage Groves mechanism is monotone in this case.

Case (ii): $0 < \gamma_1 < \gamma_2 < 1$

Fix $\hat{\theta}_1 \in [0,1]$ and let $\alpha, \beta \in [0,1]$ be two distinct types of the buyer such that $\alpha > \beta$ and $x^*(\hat{\theta}_1, \alpha) = x^*(\hat{\theta}_1, \beta) = 1$. Let $\theta_2^* \in (\underline{\theta}_1, \bar{\theta}_2]$ be the unique cutoff point identified in Lemma 4. There are two subcases:

Case 1: $\alpha \leq \theta_2^*$

In this subcase, we can apply here the same argument in Case (i) and the buyer's payment is strictly increasing in his type report.

Case 2: $\beta \geq \theta_2^*$

The difference between the buyer's transfer under $(\hat{\theta}_1, \alpha)$ and $(\hat{\theta}_1, \beta)$ is computed below:

$$\begin{aligned}
& t_2^G(\hat{\theta}_1, \alpha; u_1^\alpha, u_2^\alpha) - t_2^G(\hat{\theta}_1, \beta; u_1^\beta, u_2^\beta) \\
= \;& u_1^\alpha - \frac{1}{2}\left[\tilde{u}_2(\hat{\theta}_1, \alpha) - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \alpha), \theta_1, \alpha)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\hat{\theta}_1, \theta_2), \hat{\theta}_1, \theta_2)\right)\right] \\
& -u_1^\beta + \frac{1}{2}\left[\tilde{u}_2(\hat{\theta}_1, \beta) - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \beta), \theta_1, \beta)\right) + \mathbb{E}_{\theta_2}\left(\sum_{j=1}^{2} u_j(x^*(\hat{\theta}_1, \theta_2), \hat{\theta}_1, \theta_2)\right)\right] \\
= \;& -\frac{1}{2}\left[\alpha - \beta - \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \alpha), \theta_1, \alpha)\right) + \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \beta), \theta_1, \beta)\right)\right], \qquad (11)
\end{aligned}$$

where the second equality follows because $x^*(\hat{\theta}_1, \alpha) = x^*(\hat{\theta}_1, \beta) = 1$ implies $u_1^\alpha = u_1^\beta = 0$ and $\tilde{u}_2(\hat{\theta}_1, \alpha) - \tilde{u}_2(\hat{\theta}_1, \beta) = \alpha + \gamma_2\hat{\theta}_1 - \beta - \gamma_2\hat{\theta}_1 = \alpha - \beta$.

Moreover, we compute the following term in the above expression:

$$\begin{aligned}
& -\mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \alpha), \theta_1, \alpha)\right) + \mathbb{E}_{\theta_1}\left(\sum_{i=1}^{2} u_j(x^*(\theta_1, \beta), \theta_1, \beta)\right) \\
= \;& -\int_0^1 \tilde{u}_2(\theta_1, \alpha) d\theta_1 + \int_0^1 \tilde{u}_2(\theta_1, \beta) d\theta_1 \\
= \;& -\int_0^1 (\alpha + \gamma_2\theta_1) d\theta_1 + \int_0^1 (\beta + \gamma_2\theta_1) d\theta_1 \\
= \;& -\alpha - \frac{1}{2}\gamma_2 + \beta + \frac{1}{2}\gamma_2 = -\alpha + \beta.
\end{aligned}$$

Plugging this back into (11), we obtain

$$t_2^G(\hat{\theta}_1, \alpha; u_1^\alpha, u_2^\alpha) - t_2^G(\hat{\theta}_1, \beta; u_1^\beta, u_2^\beta) = -\frac{1}{2}\left[\alpha - \beta - \alpha + \beta\right] = 0.$$

Therefore, the generalized two-stage Groves mechanism is monotone in this subcase. This completes the proof of the claim. □

8.5 Proof of Lemma 5

Proof. Recall our Assumption 1 says that

$$\begin{aligned}
& \int_{\Theta_1} \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} (\tilde{u}_2(\theta_1,\theta_2) - \tilde{u}_1(\theta_1,\theta_2))\, dF_2(\theta_2) dF_1(\theta_1) \\
& + \int_{\Theta_1} \int_{\Theta_2^{**}} (\tilde{u}_2(\theta_1,\theta_2^*) - \tilde{u}_1(\theta_1,\theta_2))\, dF_2(\theta_2) dF_1(\theta_1) \\
\geq\ & 0,
\end{aligned}$$

where $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2]$ is the cutoff point identified in Lemma 4, $\Theta_2^{**} = [\theta_2^*, \bar{\theta}_2]$, and for each $\theta_1 \in \Theta_1$,

$$\Theta_2^*(\theta_1) = \begin{cases} \{\bar{\theta}_2\} & \text{if } \{\theta_2 \in \Theta_2 : x^*(\theta_1,\theta_2) = 1\} = \emptyset \\ \{\theta_2 \in \Theta_2 : x^*(\theta_1,\theta_2) = 1\} & \text{otherwise,} \end{cases}$$

Here, the cutoff point θ_2^* is equal to $\min\{(1-\gamma_2)/(1-\gamma_1), 1\}$ above which it is always efficient to trade and below which it is efficient not to trade for some $\theta_1 \in \Theta_1$. Moreover, we have $\Theta_2^*(\theta_1) = [\min\{(1-\gamma_2)\theta_1/(1-\gamma_1), 1\}, 1]$ and $\Theta_2^{**} = [\min\{(1-\gamma_2)/(1-\gamma_1), 1\}, 1]$. So, $\Theta_2^*(\theta_1)\setminus\Theta_2^{**} = [\min\{(1-\gamma_2)\theta_1/(1-\gamma_1), 1\}, \min\{(1-\gamma_2)/(1-\gamma_1), 1\}]$.

We divide our argument into the following two cases:

Case (i): $0 < \gamma_2 \leq \gamma_1 < 1$

From Figure 1 (p.13), we know that if $\theta_1 > (1-\gamma_1)/(1-\gamma_2)$, then $(1-\gamma_2)\theta_1/(1-\gamma_1) > 1$; hence,

$$\Theta_2^*(\theta_1) = \left[\min\left\{\frac{1-\gamma_2}{1-\gamma_1}\theta_1, 1\right\}, 1\right] = \begin{cases} \{1\} & \text{if } \theta_1 > (1-\gamma_1)/(1-\gamma_2) \\ \left[\frac{1-\gamma_2}{1-\gamma_1}\theta_1, 1\right] & \text{otherwise.} \end{cases}$$

Moreover, in Case (i), we know $(1-\gamma_2)/(1-\gamma_1) > 1$; hence,

$$\Theta_2^{**} = \left[\min\left\{\frac{1-\gamma_2}{1-\gamma_1}, 1\right\}, 1\right] = \{1\}.$$

As a result,

$$\Theta_2^*(\theta_1)\setminus\Theta_2^{**} = \left[\min\left\{\frac{1-\gamma_2}{1-\gamma_1}\theta_1, 1\right\}, 1\right] = \begin{cases} \emptyset & \text{if } \theta_1 > (1-\gamma_1)/(1-\gamma_2) \\ \left[\frac{1-\gamma_2}{1-\gamma_1}\theta_1, 1\right) & \text{otherwise.} \end{cases}$$

Reflecting the type space $\Theta = [0,1]^2$ and each agent i's valuation function $\tilde{u}_i(\theta_i, \theta_{-i}) = \theta_i + \gamma_i\theta_{-i}$ in Assumption 1, we obtain

$$\int_0^{\frac{1-\gamma_1}{1-\gamma_2}} \int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^1 ((1-\gamma_1)\theta_2 - (1-\gamma_2)\theta_1)\, d\theta_2 d\theta_1 \geq 0.$$

We compute the left-hand side of the above inequality:

$$\begin{aligned}
&\int_0^{\frac{1-\gamma_1}{1-\gamma_2}} \left[\frac{1}{2}(1-\gamma_1)\left(1-\left(\frac{1-\gamma_2}{1-\gamma_1}\theta_1\right)^2\right)-(1-\gamma_2)\theta_1\left(1-\frac{1-\gamma_2}{1-\gamma_1}\theta_1\right)\right]d\theta_1\\
=\ &\int_0^{\frac{1-\gamma_1}{1-\gamma_2}} \left[\frac{1}{2}(1-\gamma_1)-(1-\gamma_2)\theta_1+\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2\right]d\theta_1.
\end{aligned}$$

We continue our computation below:

$$\begin{aligned}
\int_0^{\frac{1-\gamma_1}{1-\gamma_2}} \left[\frac{1}{2}(1-\gamma_1)-(1-\gamma_2)\theta_1+\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2\right]d\theta_1 &= \frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}-\frac{1}{2}\frac{(1-\gamma_1)^2}{1-\gamma_2}+\frac{1}{6}\frac{(1-\gamma_1)^2}{1-\gamma_2}\\
&= \frac{1}{6}\frac{(1-\gamma_1)^2}{1-\gamma_2},
\end{aligned}$$

which is strictly positive. Therefore, Assumtpion 1 is satisfied in Case (i).

Case (ii): $0<\gamma_1<\gamma_2<1$

From Figure 2 (p.14), we know that $(1-\gamma_2)\theta_1/(1-\gamma_1)<1$ for all $\theta_1\in[0,1]$; hence,

$$\Theta_2^*(\theta_1)=\left[\min\left\{\frac{1-\gamma_2}{1-\gamma_1}\theta_1,1\right\},1\right]=\left[\frac{1-\gamma_2}{1-\gamma_1}\theta_1,1\right]$$

for all $\theta_1\in[0,1]$. Moreover, in Case (ii), we know $(1-\gamma_2)/(1-\gamma_1)<1$; hence,

$$\Theta_2^{**}=\left[\min\left\{\frac{1-\gamma_2}{1-\gamma_1},1\right\},1\right]=\left[\frac{1-\gamma_2}{1-\gamma_1},1\right].$$

As a result, we have that for all $\theta_1\in\Theta_1$,

$$\Theta_2^*(\theta_1)\backslash\Theta_2^{**}=\left[\frac{1-\gamma_2}{1-\gamma_1}\theta_1,\frac{1-\gamma_2}{1-\gamma_1}\right).$$

Reflecting the type space $\Theta=[0,1]^2$ and each agent i's valuation function $\tilde{u}_i(\theta_i,\theta_{-i})=\theta_i+\gamma_i\theta_{-i}$ in Assumption 1, we obtain

$$\int_0^1\int_{\frac{1-\gamma_2}{1-\gamma_1}\theta_1}^{\frac{1-\gamma_2}{1-\gamma_1}}((1-\gamma_1)\theta_2-(1-\gamma_2)\theta_1)\,d\theta_2 d\theta_1+\int_0^1\int_{\frac{1-\gamma_2}{1-\gamma_1}}^1\left(\frac{1-\gamma_2}{1-\gamma_1}-(1-\gamma_2)\theta_1-\gamma_1\theta_2\right)d\theta_2 d\theta_1\geq 0.$$

We compute the left-hand side of the above inequality:

$$\begin{aligned}
&\int_0^1\left[\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}\left(1-(\theta_1)^2\right)-\frac{(1-\gamma_2)^2}{1-\gamma_1}\theta_1(1-\theta_1)\right]d\theta_1\\
&+\int_0^1\left[\frac{1-\gamma_2}{1-\gamma_1}\left(1-\frac{1-\gamma_2}{1-\gamma_1}\right)-(1-\gamma_2)\left(1-\frac{1-\gamma_2}{1-\gamma_1}\right)\theta_1-\frac{1}{2}\gamma_1\left(1-\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2\right)\right]d\theta_1\\
=\ &\int_0^1\left[\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}-\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2-\frac{(1-\gamma_2)^2}{1-\gamma_1}\theta_1+\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2\right]d\theta_1\\
&+\int_0^1\left[\frac{1-\gamma_2}{1-\gamma_1}-\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2-(1-\gamma_2)\theta_1+\frac{(1-\gamma_2)^2}{1-\gamma_1}\theta_1-\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_1\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2\right]d\theta_1.
\end{aligned}$$

We continue our computation below:

$$
\begin{aligned}
& \int_0^1 \left[\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1} - \frac{(1-\gamma_2)^2}{1-\gamma_1}\theta_1 + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1}(\theta_1)^2\right] d\theta_1 \\
& + \int_0^1 \left[\frac{1-\gamma_2}{1-\gamma_1} - \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 - (1-\gamma_2)\theta_1 + \frac{(1-\gamma_2)^2}{1-\gamma_1}\theta_1 - \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_1\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2\right] d\theta_1 \\
= \;& \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1} - \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} \\
& + \frac{1-\gamma_2}{1-\gamma_1} - \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 - \frac{1}{2}(1-\gamma_2) + \frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1} - \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_1\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2.
\end{aligned}
$$

Rearranging the terms above, we obtain

$$
\begin{aligned}
& \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2-\gamma_1-1) + \left[\frac{1}{2}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1}{2}\gamma_1\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2\right] \\
= \;& \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2-\gamma_1-1) + \left[\frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2(1-\gamma_1+\gamma_1)\right] \\
= \;& \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2-\gamma_1-1) + \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 \\
= \;& \frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2-\gamma_1-1).
\end{aligned}
$$

Therefore, our Assumption 1 is reduced to

$$
\frac{1}{6}\frac{(1-\gamma_2)^2}{1-\gamma_1} + \frac{1-\gamma_2}{1-\gamma_1} - \frac{1}{2}\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2 + \frac{1}{2}(\gamma_2-\gamma_1-1) \geq 0.
$$

□

8.6 Proof of Step 1 in the Proof of Theorem 1

Step 1: If the buyer always reports the truth in the first stage, the seller has no incentive to tell a lie in the first stage.

Proof. Consider the seller of type θ_1. Then, the expected utility of the seller of type θ_1 under truth-telling is

$$
\int_{\Theta_2\setminus\Theta_2^*(\theta_1)} (\tilde{u}_1(\theta_1,\theta_2)+0)\,dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} (0+\tilde{u}_2(\theta_1,\theta_2))\,dF_2(\theta_2) + \int_{\Theta_2^{**}} (0-g(\theta_1))\,dF_2(\theta_2).
$$

On the other hand, if the seller deviates to $\theta_1^r \neq \theta_1$ and trade occurs, the second-stage report by the buyer of type θ_2 becomes $u_2^r = u_2^r(x^*(\theta_1^r,\theta_2),\theta_1,\theta_2) = \tilde{u}_2(\theta_1,\theta_2)$.

Since $\tilde{u}_2(\cdot)$ is strictly increasing in θ_1, then $u_2^r = \tilde{u}_2(\theta_1, \theta_2) \neq \tilde{u}_2(\theta_1^r, \theta_2)$ and the seller must pay a penalty ψ according to the transfer rule t_1^M. Therefore, the expected utility of the seller of type θ_1 becomes

$$\int_{\Theta_2 \backslash \Theta_2^*(\theta_1^r)} (\tilde{u}_1(\theta_1, \theta_2) + 0)\, dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1^r)} (0 - \psi) dF_2(\theta_2).$$

By Lemma 4, we divide our argument into the following two cases:

Case 1: $\theta_2^* = \bar{\theta}_2$, i.e., $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$.

Then, the expected utility of the seller of type θ_1 becomes

$$\int_{\Theta_2 \backslash \Theta_2^*(\theta_1)} (\tilde{u}_1(\theta_1, \theta_2) + 0)\, dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} (0 + \tilde{u}_2(\theta_1, \theta_2))\, dF_2(\theta_2),$$

where

$$\Theta_2^*(\theta_1) = \begin{cases} \{\bar{\theta}_2\} & \text{if } \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} = \emptyset \\ \{\theta_2 \in \Theta_2 : x^*(\theta_1, \theta_2) = 1\} & \text{otherwise.} \end{cases}$$

Since $\psi > 0$, the best possible deviation the seller of type θ_1 can achieve is to announce θ_1^r such that $\Theta_2^*(\theta_1^r) = \emptyset$. This implies that the seller keeps the good so that the seller's expected payoff becomes $\int_{\Theta_2} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2)$. However, we claim that this expected utility is at most the same as that under truth-telling. To see this, we compute the difference between the seller's expected utility under truth-telling and that under the best deviation:

$$\begin{aligned} & \int_{\Theta_2 \backslash \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) - \int_{\Theta_2} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \\ = & \int_{\Theta_2 \backslash \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) \\ & - \left[\int_{\Theta_2 \backslash \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \right] \\ = & \int_{\Theta_2^*(\theta_1)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2))\, dF_2(\theta_2) \\ \geq & \ 0, \end{aligned}$$

where the weak inequality follows because whenever $\theta_2 \in \Theta_2^*(\theta_1)$, it is efficient to trade, implying that $\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2) > 0$. So, the seller will never be better off after such a deviation so that he has no incentive to deviate from truth-telling.

Case 2: $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ such that for any $\theta_2 \in \Theta_2$,

$$\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) \begin{cases} < 1 & \text{if } \theta_2 < \theta_2^* \\ = 1 & \text{if } \theta_2 \geq \theta_2^* \end{cases}$$

To stop the seller of type θ_1 from deviating to θ_1^r, the penalty ψ must be large enough so that the seller always receives at most the same expected utility as that under truth-telling whenever he deviates. That is, what we want is that for any $\theta_1^r \in \Theta_1$,

$$\begin{aligned}
&\int_{\Theta_2\setminus\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2) - \int_{\Theta_2^{**}} g(\theta_1)dF_2(\theta_2) \\
\geq\ & \int_{\Theta_2\setminus\Theta_2^*(\theta_1^r)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) - \psi \int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2),
\end{aligned}$$

where

$$\Theta_2^*(\theta_1) = \begin{cases} \{\bar{\theta}_2\} & \text{if } \{\theta_2 \in \Theta_2 : x^*(\theta_1,\theta_2) = 1\} = \emptyset \\ \{\theta_2 \in \Theta_2 : x^*(\theta_1,\theta_2) = 1\} & \text{otherwise,} \end{cases}$$

and $\Theta_2^{**} = [\theta_2^*, \bar{\theta}_2]$. After rearranging the terms for ψ, we obtain

$$\begin{aligned}
\psi \ \geq\ & \frac{1}{\int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2)} \left(\int_{\Theta_2\setminus\Theta_2^*(\theta_1^r)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) - \int_{\Theta_2\setminus\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) \right) \\
& + \frac{1}{\int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2)} \left(- \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2) + \int_{\Theta_2^{**}} g(\theta_1)dF_2(\theta_2) \right).
\end{aligned}$$

Then, it remains to find an upper bound of the right-hand side of the above inequality. We obtain this upper bound as follows:

$$\begin{aligned}
\psi \ \geq\ & \sup_{\substack{\theta_1\in\Theta_1 \\ \theta_1^r\in\Theta_1}} \left[\frac{1}{\int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2)} \left(\int_{\Theta_2\setminus\Theta_2^*(\theta_1^r)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) - \int_{\Theta_2\setminus\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2) \right) \right. \\
& \left. + \frac{1}{\int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2)} \left(- \int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2) + \int_{\Theta_2^{**}} g(\theta_1)dF_2(\theta_2) \right) \right].
\end{aligned}$$

In this case, we know $[\theta_2^*, \bar{\theta}_2] \subseteq \Theta_2^*(\theta_1^r)$ for all $\theta_1^r \in \Theta_1$.[8] Therefore, $\Theta_2^*(\theta_1^r)$ is nonempty and carries positive measure under $F_2(\cdot)$ so that the denominator $\int_{\Theta_2^*(\theta_1^r)} dF_2(\theta_2)$ is strictly positive. Moreover, since the type space Θ is bounded and each valuation function $\tilde{u}_i(\cdot,\theta)$ is bounded, the right-hand side of the above inequality is bounded and we denote it by A_1. So, if

$$\psi \geq A_1,$$

the seller will never be better off after such a deviation so that he has no incentive to deviate from truth-telling. This completes the proof of Step 1.

$\square$

[8]In the first general case, the denominator may be zero because $\Theta_2^*(\theta_1)$ may be a singleton for some $\theta_1 \in \Theta_1$.

8.7 Proof of Step 2 in the Proof of Theorem 1

Step 2: If the seller always reports the truth in the first stage, the buyer has no incentive to tell a lie in the first stage.

Proof. Consider the buyer of type $\theta_2 < \theta_2^*$. Then, the buyer's expected utility under truth-telling, denoted by $U_2(\theta_2)$, is

$$U_2(\theta_2) = \int_{\Theta_1^*(\theta_2)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2))\, dF_1(\theta_1) = 0,$$

where $\Theta_1^*(\theta_2) = \{\theta_1 \in \Theta_1 : x^*(\theta_1, \theta_2) = 1\}$. On the other hand, if the buyer of type θ_2 deviates to $\theta_2^r \neq \theta_2$ such that $\theta_2^r < \theta_2^*$ and no trade occurs, the second-stage report of the seller of type θ_1 becomes $u_1^r = u_1^r(x^*(\theta_1, \theta_2^r), \theta_1, \theta_2) = \tilde{u}_1(\theta_1, \theta_2)$. Since the seller's utility function $\tilde{u}_1(\cdot)$ is strictly increasing in θ_2, then $u_1^r = \tilde{u}_1(\theta_1, \theta_2) \neq \tilde{u}_1(\theta_1, \theta_2^r)$ and the buyer must pay a penalty ψ according to the transfer rule t_2^M. Therefore, the expected utility of the buyer of type θ_2 when announcing θ_2^r becomes

$$\int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2^r))\, dF_1(\theta_1) + \int_{\Theta_1 \setminus \Theta_1^*(\theta_2^r)} (0 - \psi) dF_1(\theta_1).$$

To stop the buyer from deviating, the penalty ψ must be large enough so that the buyer always receives at most the same expected utility as that under truth-telling whenever he deviates. That is, for any $\theta_2 < \theta_2^*$ and $\theta_2^r < \theta_2^*$,

$$0 \geq \int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2^r))\, dF_1(\theta_1) - \psi \int_{\Theta_1 \setminus \Theta_1^*(\theta_2^r)} dF_1(\theta_1).$$

After rearranging the terms for ψ in the above inequality, we obtain

$$\psi \geq \frac{\int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2^r))\, dF_1(\theta_1)}{\int_{\Theta_1 \setminus \Theta_1^*(\theta_2^r)} dF_1(\theta_1)},$$

where $\Theta_1 \setminus \Theta_1^*(\theta_2^r) = \{\theta_1 \in \Theta_1 : x^*(\theta_1, \theta_2^r) = 0\}$. We know that for any $\theta_2^r < \theta_2^*$, there must exist some $\theta_1 \in \Theta_1$ such that $x^*(\theta_1, \theta_2^r) = 0$. Therefore, $\Theta_1 \setminus \Theta_1^*(\theta_2^r)$ is nonempty and carries positive measure under $F_1(\cdot)$ so that the denominator is strictly positive. Then, it remains to find an upper bound of the right-hand side of the above inequality. Such an upper bound can be found as follows:

$$\psi \geq \sup_{\substack{\theta_2 \in [\underline{\theta}_2, \theta_2^*) \\ \theta_2^r \in [\underline{\theta}_2, \theta_2^*)}} \frac{\int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2^r))\, dF_1(\theta_1)}{\int_{\Theta_1 \setminus \Theta_1^*(\theta_2^r)} dF_1(\theta_1)}.$$

since $[\underline{\theta}_2, \theta_2^*)$ is bounded and each $\tilde{u}_2(\cdot, \theta)$ is bounded, the numerator is also bounded. Therefore, the right-hand side of the above inequality is bounded and we denote its upper bound by A_2. So, if

$$\psi \geq A_2,$$

the buyer of type $\theta_2 < \theta_2^*$ will never be better off after such a deviation so that he has no incentive to deviate to $\theta_2^r < \theta_2^*$ from truth-telling.

Moreover, if the buyer deviates to $\theta_2^r \geq \theta_2^*$, it is always efficient to trade and the expected utility of the buyer of type θ_2 when announcing θ_2^r, denoted by $U_2(\theta_2, \theta_2^r)$, becomes

$$U_2(\theta_2, \theta_2^r) = \int_{\Theta_1} (\tilde{u}_2(\theta_1, \theta_2) + g(\theta_1))\, dF_1(\theta_1).$$

Then, the difference between the expected utility of the buyer of type θ_2 under truth-telling and that under deviation to θ_2^r is

$$\begin{aligned} U_2(\theta_2) - U_2(\theta_2, \theta_2^r) &= 0 - \int_{\Theta_1} (\tilde{u}_2(\theta_1, \theta_2) + g(\theta_1))\, dF_1(\theta_1) \\ &= -\int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) - \int_{\Theta_1} g(\theta_1) dF_1(\theta_1). \end{aligned}$$

By Lemma 4, we divide our argument into the following two cases:

Case 1: $\theta_2^* = \bar{\theta}_2$, i.e., $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$.

Recall that in this case, $g(\theta_1) = -\tilde{u}_2(\theta_1, \bar{\theta}_2)$. Then, we evaluate the utility difference.

$$\begin{aligned} U_2(\theta_2) - U_2(\theta_2, \theta_2^r) &= -\int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1 + \int_{\Theta_1} \tilde{u}_2(\theta_1, \bar{\theta}_2) dF_1(\theta_1) \\ &= \int_{\Theta_1} \big(\tilde{u}_2(\theta_1, \bar{\theta}_2) - \tilde{u}_2(\theta_1, \theta_2)\big)\, dF_1(\theta_1) \\ &\geq 0, \end{aligned}$$

where the last inequality follows because $\theta_2 \leq \bar{\theta}_2$ and $\tilde{u}_2(\cdot)$ is strictly increasing in θ_2. Therefore, the buyer is never better off after a deviation to $\theta_2^r \geq \theta_2^*$ so that he has no incentive to deviate from truth-telling to $\theta_2^r \geq \theta_2^*$ in this case.

Case 2: $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ such that for any $\theta_2 \in \Theta_2$,

$$\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) \begin{cases} < 1 & \text{if } \theta_2 < \theta_2^* \\ = 1 & \text{if } \theta_2 \geq \theta_2^* \end{cases}$$

Recalling the definition of $g(\theta_1)$, we obtain

$$
\begin{aligned}
&(1-F_2(\theta_2^*))\int_{\Theta_1} g(\theta_1)dF_1(\theta_1) \\
= \; & \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) - \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) \\
& - \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \left(\tilde{u}_2(\theta_1,\theta_2) - \tilde{u}_1(\theta_1,\theta_2)\right) dF_2(\theta_2)dF_1(\theta_1) \\
& - \int_{\Theta_1}\int_{\Theta_2^{**}} \left(\tilde{u}_2(\theta_1,\theta_2^*) - \tilde{u}_1(\theta_1,\theta_2)\right) dF_2(\theta_2)dF_1(\theta_1) \\
= \; & \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) - \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) \\
& - \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)\setminus\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) + \int_{\Theta_1}\int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1,\theta_2)dF_2(\theta_2)dF_1(\theta_1) \\
& - \int_{\Theta_1}\int_{\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2^*)dF_2(\theta_2)dF_1(\theta_1).
\end{aligned}
$$

Noticing that the first four terms are cancelled out, we obtain

$$
\begin{aligned}
(1-F_2(\theta_2^*))\int_{\Theta_1} g(\theta_1)dF_1(\theta_1) &= -\int_{\Theta_1}\int_{\Theta_2^{**}} \tilde{u}_2(\theta_1,\theta_2^*)dF_2(\theta_2)dF_1(\theta_1) \\
&= -(1-F_2(\theta_2^*))\int_{\Theta_1} \tilde{u}_2(\theta_1,\theta_2^*)dF_1(\theta_1)
\end{aligned}
$$

Therefore, we obtain

$$
\int_{\Theta_1} g(\theta_1)dF_1(\theta_1) = -\int_{\Theta_1} \tilde{u}_2(\theta_1,\theta_2^*)dF_1(\theta_1).
$$

Plugging this back into the utility difference, we obtain

$$
\begin{aligned}
U_2(\theta_2) - U_2(\theta_2,\theta_2^r) &= -\int_{\Theta_1} \tilde{u}_2(\theta_1,\theta_2)dF_1(\theta_1) + \int_{\Theta_1} \tilde{u}_2(\theta_1,\theta_2^*)dF_1(\theta_1) \\
&= \int_{\Theta_1} \left(\tilde{u}_2(\theta_1,\theta_2^*) - \tilde{u}_2(\theta_1,\theta_2)\right) dF_1(\theta_1) \\
&> 0,
\end{aligned}
$$

where the last strict inequality follows because $\theta_2 < \theta_2^*$ and $\tilde{u}_2(\cdot)$ is strictly increasing in θ_2. Therefore, the buyer is never better off after a deviation to $\theta_2^r \geq \theta_2^*$ so that he has no incentive to deviate from truth-telling to $\theta_2^r \geq \theta_2^*$.

Consider the buyer of type $\theta_2 \geq \theta_2^*$. In this case, it is always efficient to trade the good regardless of the seller's type. Therefore, the expected utility of the buyer of type θ_2 under truth-telling is

$$
\int_{\Theta_1} \left(\tilde{u}_2(\theta_1,\theta_2) + g(\theta_1)\right) dF_1(\theta_1).
$$

On the other hand, if the buyer deviates to $\theta_2^r \neq \theta_2$ such that $\theta_2^r \geq \theta_2^*$, then it is still always efficient to trade regardless of the seller's type. Thus, the expected utility of the buyer of type θ_2 under the deviation to θ_2^r is

$$\int_{\Theta_1} (\tilde{u}_2(\theta_1,\theta_2) + g(\theta_1))\, dF_1(\theta_1),$$

which is the same as the expected utility under truth-telling. Therefore, the buyer of type $\theta_2 \geq \theta_2^*$ has no incentive to deviate to $\theta_2^r \geq \theta_2^*$.

Moreover, if the buyer of type θ_2 deviates to $\theta_2^r < \theta_2^*$ and trade does not occur, the second-stage report of the seller of type θ_1 becomes $u_1^r = u_1^r(x^*(\theta_1,\theta_2^r),\theta_1,\theta_2) = \tilde{u}_1(\theta_1,\theta_2)$. Since $\tilde{u}_1(\cdot)$ is strictly increasing in θ_2, we have that $u_1^r = \tilde{u}_1(\theta_1,\theta_2) \neq \tilde{u}_1(\theta_1,\theta_2^r)$ so that the buyer must pay a penalty ψ according to the transfer rule t_2^M. Therefore, the expected utility of the buyer of type θ_2 when announcing θ_2^r becomes

$$\int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1,\theta_2) - \tilde{u}_2(\theta_1,\theta_2^r))\, dF_1(\theta_1) + \int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} (0-\psi) dF_1(\theta_1).$$

To stop the buyer from deviating, the penality ψ must be large enough so that the buyer always receives at most the same expected utility as that under truth-telling whenever he deviates. That is, what we want to have is that for any $\theta_2 \geq \theta_2^*$ and $\theta_2^r < \theta_2^*$,

$$\int_{\Theta_1} (\tilde{u}_2(\theta_1,\theta_2) + g(\theta_1))\, dF_1(\theta_1) \quad \geq \quad \int_{\Theta_1^*(\theta_2^r)} (\tilde{u}_2(\theta_1,\theta_2) - \tilde{u}_2(\theta_1,\theta_2^r))\, dF_1(\theta_1) - \psi \int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} dF_1(\theta_1).$$

After rearranging the terms above for ψ in the above inequality, we obtain

$$\psi \geq \frac{1}{\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} dF_1(\theta_1)} \left[-\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} \tilde{u}_2(\theta_1,\theta_2) dF_1(\theta_1) - \int_{\Theta_1^*(\theta_2^r)} \tilde{u}_2(\theta_1,\theta_2^r) dF_1(\theta_1) - \int_{\Theta_1} g(\theta_1) dF_1(\theta_1) \right].$$

Then, it remains to find an upper bound of the right-hand side of the above inequality. Therefore, we want to satisfy the following inequality:

$$\begin{aligned}\psi \quad \geq \quad \sup_{\substack{\theta_2 \in [\theta_2^*,\bar{\theta}_2] \\ \theta_2^r \in [\underline{\theta}_2,\theta_2^*)}} & \left[\frac{1}{\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} dF_1(\theta_1)} \left(-\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} \tilde{u}_2(\theta_1,\theta_2) dF_1(\theta_1) - \int_{\Theta_1^*(\theta_2^r)} \tilde{u}_2(\theta_1,\theta_2^r) dF_1(\theta_1) \right) \right. \\ & \left. - \frac{1}{\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} dF_1(\theta_1)} \int_{\Theta_1} g(\theta_1) dF_1(\theta_1) \right].\end{aligned}$$

Previously, we have argued that if $\theta_2^r < \theta_2^*$, then $\Theta_1 \backslash \Theta_1^*(\theta_2^r)$ carries positive measure under $F_1(\cdot)$, that is, $\int_{\Theta_1 \backslash \Theta_1^*(\theta_2^r)} dF_1(\theta_1) > 0$. Moreover, since $[\theta_2^*,\bar{\theta}_2]$ and $[\underline{\theta}_2,\theta_2^*)$

are bounded and $\tilde{u}_2(\cdot)$ is bounded, the right-hand side of the above inequality is bounded. We denote this upper bound by A_3. So, if

$$\psi \geq A_3,$$

the buyer will never be better off after such a deviation so that he has no incentive to deviate to $\theta_2^r < \theta_2^*$. This completes the proof of Step 2. $\square$

8.8 Proof of Step 3 in the Proof of Theorem 1

Step 3: The two-stage mechanism (Θ, Π, x^*, t) also satisfies IIR.

Proof. By Steps 1 and 2, we set $\psi = \max\{A_1, A_2, A_3\}$. We first show that IIR is satisfied for the seller. Consider the seller of type θ_1. Recall that if both agents report truthfully in both stages, the expected utility of the seller of type θ_1 after participating in the mechanism, denoted by $U_1(\theta_1)$, is

$$\begin{aligned} U_1(\theta_1) &= \int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} (\tilde{u}_1(\theta_1, \theta_2) + 0)\, dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} (0 + \tilde{u}_2(\theta_1, \theta_2))\, dF_2(\theta_2) \\ &\quad + \int_{\Theta_2^{**}} (0 - g(\theta_1))\, dF_2(\theta_2). \end{aligned}$$

By Lemma 4, we continue our discussion by considering the following two cases:

Case 1: $\theta_2^* = \bar{\theta}_2$. That is, $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$.

Then, the expected utility of the seller of type θ_1 after participating in the mechanism becomes

$$U_1(\theta_1) = \int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2).$$

Then, we compute the difference between the expected utility of the seller of type θ_1 after participating in the mechanism and θ_1's outside option utility:

$$\begin{aligned} & U_1(\theta_1) - U_1^O(\theta_1) \\ = & \int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) - \int_{\Theta_2} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \\ = & \int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) \\ & - \left[\int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \right] \\ = & \int_{\Theta_2^*(\theta_1)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2))\, dF_2(\theta_2) \\ \geq & \ 0, \end{aligned}$$

where the weak inequality follows because whenever $\theta_2 \in \Theta_2^*(\theta_1)$, it is efficient to trade, implying $\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2) > 0$.

Case 2: $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ such that

$$\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) \begin{cases} < 1 & \text{if } \theta_2 < \theta_2^* \\ = 1 & \text{if } \theta_2 \geq \theta_2^*. \end{cases}$$

We compute the difference between the expected utility of the seller of type θ_1 after participating in the mechanism and θ_1's outside option utility:

$$\begin{aligned} & U_1(\theta_1) - U_1^O(\theta_1) \\ = & \int_{\Theta_2 \setminus \Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) - \int_{\Theta_2^{**}} g(\theta_1) dF_2(\theta_2) \\ & - \int_{\Theta_2} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \\ = & - \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) - \int_{\Theta_2^{**}} g(\theta_1) dF_2(\theta_2) \\ = & - \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) - g(\theta_1) \left(1 - F_2(\theta_2^*)\right). \end{aligned}$$

Plugging the formula of $g(\theta_1)\left(1 - F_2(\theta_2^*)\right)$ in the above expression, we obtain

$$\begin{aligned} U_1(\theta_1) - U_1^O(\theta_1) = & - \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) \\ & - \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \tilde{u}_2(\theta_1, \theta_2) dF_2(\theta_2) + \int_{\Theta_2^*(\theta_1)} \tilde{u}_1(\theta_1, \theta_2) dF_2(\theta_2) \\ & + \int_{\Theta_1} \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1) \\ & + \int_{\Theta_1} \int_{\Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2^*) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1). \end{aligned}$$

Further rearranging the terms above, we obtain

$$\begin{aligned} U_1(\theta_1) - U_1^O(\theta_1) = & \int_{\Theta_1} \int_{\Theta_2^*(\theta_1) \setminus \Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1) \\ & + \int_{\Theta_1} \int_{\Theta_2^{**}} \left(\tilde{u}_2(\theta_1, \theta_2^*) - \tilde{u}_1(\theta_1, \theta_2)\right) dF_2(\theta_2) dF_1(\theta_1). \end{aligned}$$

Then, by our Assumption 1, we conclude

$$U_1(\theta_1) - U_1^O(\theta_1) \geq 0.$$

Therefore, in both cases, the seller's expected utility by participating in the mechanism is at least as high as that from the outside option. This implies that IIR is satisfied for the seller.

Consider the buyer of type θ_2. If $\theta_2 < \theta_2^*$ and both agents report truthfully in both stages, the expected utility of the buyer of type θ_2 after participating in the mechanism is

$$\int_{\Theta_1^*(\theta_2)} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2))\, dF_1(\theta_1) = 0 = U_2^O(\theta_2).$$

Hence, if $\theta_2 < \theta_2^*$, by participating in the mechanism, the buyer receives exactly the same expected utility as his outside option utility.

If $\theta_2 \geq \theta_2^*$, the expected utility of the buyer of type θ_2 after participating in the mechanism is

$$\int_{\Theta_1} (\tilde{u}_2(\theta_1, \theta_2) + g(\theta_1))\, dF_1(\theta_1) = \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) + \int_{\Theta_1} g(\theta_1) dF_1(\theta_1).$$

By Lemma 4, we divide our argument into the following two cases:

Case 1: $\theta_2^* = \bar{\theta}_2$, i.e., $\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) < 1$ for all $\theta_2 < \bar{\theta}_2$.

Recall that in this case, $g(\theta_1) = -\tilde{u}_2(\theta_1, \bar{\theta}_2)$. Then, the expected utility of the buyer of type $\theta_2 = \bar{\theta}_2$ after participation is

$$\int_{\Theta_1} \tilde{u}_2(\theta_1, \bar{\theta}_2) dF_1(\theta_1) + \int_{\Theta_1} g(\theta_1) dF_1(\theta_1) = \int_{\Theta_1} \tilde{u}_2(\theta_1, \bar{\theta}_2) dF_1(\theta_1) - \int_{\Theta_1} \tilde{u}_2(\theta_1, \bar{\theta}_2) dF_1(\theta_1) = 0.$$

Therefore, if $\theta_2 \geq \theta_2^*$, by participating in the mechanism, the buyer of type θ_2 receives exactly the same expected utility as his outside option utility.

Case 2: $\theta_2^* \in (\underline{\theta}_2, \bar{\theta}_2)$ such that for any $\theta_2 \in \Theta_2$,

$$\int_{\Theta_1} x^*(\theta_1, \theta_2) dF_1(\theta_1) \begin{cases} < 1 & \text{if } \theta_2 < \theta_2^* \\ = 1 & \text{if } \theta_2 \geq \theta_2^*. \end{cases}$$

As we argued previously, we know that

$$\int_{\Theta_1} g(\theta_1) dF_1(\theta_1) = -\int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2^*) dF_1(\theta_1).$$

Hence, if $\theta_2 \geq \theta_2^*$, the expected utility of the buyer of type θ_2 after participating in the mechanism is

$$\begin{aligned} \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) + \int_{\Theta_1} g(\theta_1) dF_1(\theta_1) &= \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2) dF_1(\theta_1) - \int_{\Theta_1} \tilde{u}_2(\theta_1, \theta_2^*) dF_1(\theta_1) \\ &= \int_{\Theta_1} (\tilde{u}_2(\theta_1, \theta_2) - \tilde{u}_2(\theta_1, \theta_2^*))\, dF_1(\theta_1) \\ &\geq 0 = U_2^O(\theta_2), \end{aligned}$$

where the weak inequality follows because $\theta_2 \geq \theta_2^*$ and $\tilde{u}_2(\cdot)$ is strictly increasing in θ_2. Therefore, if $\theta_2 \geq \theta_2^*$, by participating in the mechanism, the buyer of type θ_2 receives at least the same expected utility as his outside option utility. We thus conclude that IIR is satisfied for the buyer. This completes the proof. $\square$

8.9 Proof of Claim 5

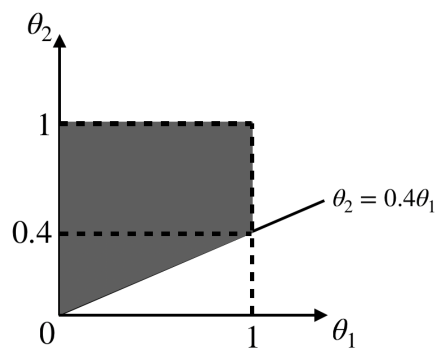
Proof. We show that the two-stage mechanism we propose in Subsection 4.2 violates the seller's IIR constraint.

In this case, $\tilde{u}_2(\theta_1, \theta_2) > \tilde{u}_1(\theta_1, \theta_2)$ if and only if $\theta_2 > (1-\gamma_2)\theta_1/(1-\gamma_1) = 0.4\theta_1$. Hence, the efficient decision rule dictates that, for each $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$,

$$x^*(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 > 0.4\theta_1 \\ 0 & \text{otherwise.} \end{cases}$$

Figure 7 below illustrates the decision at different type profiles in this case. In particular, the shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles for which it is efficient to trade. Note that if $\theta_2 \geq 0.4$, it is always efficient to trade regardless of the seller's type.

Figure 7



Observe that in this case, $\theta_2^* = 0.4$ and $\Theta_2^{**} = [0.4, 1]$. Moreover, the sum of the last two terms in expression (1) in the definition of $G(\theta_1^r)$ (See Subsection 4.2) is exactly the negative of the left-hand side of inequality (2) in our Assumption 1, which is equal to 1/60. Then, expression (1) can be rewritten as the following: for each $\theta_1^r \in [0, 1]$,

$$\begin{aligned} G(\theta_1^r) &= \int_{0.4\theta_1^r}^{0.4} (\theta_2 + 0.8\theta_1^r) d\theta_2 - \int_{0.4\theta_1^r}^{1} (\theta_1^r + 0.5\theta_2) d\theta_2 + \frac{1}{60} \\ &= 0.08\left(1 - (\theta_1^r)^2\right) + 0.32\theta_1^r(1 - \theta_1^r) - \theta_1^r(1 - 0.4\theta_1^r) - 0.25\left(1 - (0.4\theta_1^r)^2\right) + \frac{1}{60}. \end{aligned}$$

Rearranging the terms, we obtain: for each $\theta_1^r \in [0, 1]$,

$$G(\theta_1^r) \;=\; -0.68\theta_1^r - 0.17 + \frac{1}{60} + 0.04(\theta_1^r)^2.$$

Then, we have

$$
\begin{aligned}
g(\theta_1^r) &= \left(-0.68\theta_1^r - 0.17 + \frac{1}{60} + 0.04(\theta_1^r)^2\right)/(1-0.4) \\
&= -\frac{17}{15}\theta_1^r - \frac{23}{90} + \frac{1}{15}(\theta_1^r)^2.
\end{aligned}
$$

Consider the IIR constraint for the seller of type θ_1. If both agents report truthfully in both stages, the seller's expected utility after participation in the mechanism, denoted by $U_1(\theta_1)$, is

$$
U_1(\theta_1) = \int_0^{0.4\theta_1} \tilde{u}_1(\theta_1,\theta_2)d\theta_2 + \int_{0.4\theta_1}^{0.4} \tilde{u}_2(\theta_1,\theta_2)d\theta_2 - \int_{0.4}^{1} g(\theta_1)d\theta_2.
$$

Then, the difference between the seller's expected utility after participation in the mechanism and his outside option utility is computed as follows: for any $\theta_1 \in [0,1]$,

$$
\begin{aligned}
U_1(\theta_1) - U_1^O(\theta_1) &= \int_0^{0.4\theta_1} \tilde{u}_1(\theta_1,\theta_2)d\theta_2 + \int_{0.4\theta_1}^{0.4} \tilde{u}_2(\theta_1,\theta_2)d\theta_2 - \int_{0.4}^{1} g(\theta_1)d\theta_2 - \int_0^1 \tilde{u}_1(\theta_1,\theta_2)d\theta_2 \\
&= \int_{0.4\theta_1}^{0.4} \tilde{u}_2(\theta_1,\theta_2)d\theta_2 - \int_{0.4}^{1} g(\theta_1)d\theta_2 - \int_{0.4\theta_1}^{1} \tilde{u}_1(\theta_1,\theta_2)d\theta_2.
\end{aligned}
$$

Plugging the specific valuation functions and $g(\cdot)$ function into the above equation, we obtain

$$
\begin{aligned}
U_1(\theta_1) - U_1^O(\theta_1) &= \int_{0.4\theta_1}^{0.4} (\theta_2 + 0.8\theta_1)d\theta_2 - \int_{0.4}^{1} \left(-\frac{17}{15}\theta_1 - \frac{23}{90} + \frac{1}{15}(\theta_1)^2\right) d\theta_2 - \int_{0.4\theta_1}^{1} (\theta_1 + 0.5\theta_2)d\theta_2 \\
&= 0.08\left(1-(\theta_1)^2\right) + 0.32\theta_1(1-\theta_1) - 0.6\left(-\frac{17}{15}\theta_1 - \frac{23}{90} + \frac{1}{15}(\theta_1)^2\right) \\
&\quad -\theta_1(1-0.4\theta_1) - 0.25\left(1-(0.4\theta_1)^2\right).
\end{aligned}
$$

Rearranging the terms above further, we obtain

$$
U_1(\theta_1) - U_1^O(\theta_1) = -\frac{1}{60} < 0,
$$

implying that the seller's IIR constraint is violated. □

8.10 Proof of Lemma 8

Proof. We will first show that if inequality (4) is satisfied, our Assumption 1 is satisfied. In our bilateral trade model, inequality (4) becomes the following condition: for all $\hat{\theta}_1 \neq \theta_1$, there exists $M_1 > 0$ such that

$$
\mathbb{E}_{\theta_2}\left[\mathbb{1}_{\{\theta_2|x^*(\hat{\theta}_1,\theta_2)=0\}}(\theta_2)\left(\tilde{u}_1(\bar{\theta}_1,\theta_2) - \tilde{u}_1(\hat{\theta}_1,\theta_2)\right)\right] \leq M_1\mathbb{E}_{\theta_2}\left[\mathbb{1}_{\{\theta_2|x^*(\hat{\theta}_1,\theta_2)=1,\tilde{u}_2(\theta_1,\theta_2)\neq\tilde{u}_2(\hat{\theta}_1,\theta_2)\}}(\theta_2)\right], \tag{12}
$$

and for all $\hat{\theta}_2 \neq \theta_2$, there exists $\tilde{M}_1 > 0$ such that

$$\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1|x^*(\theta_1,\hat{\theta}_2)=1\}}(\theta_1)\left(\tilde{u}_2(\theta_1,\bar{\theta}_2)-\tilde{u}_2(\theta_1,\hat{\theta}_2)\right)\right] \leq \tilde{M}_1\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1|x^*(\theta_1,\hat{\theta}_2)=0,\tilde{u}_1(\theta_1,\theta_2)\neq\tilde{u}_1(\theta_1,\hat{\theta}_2)\}}(\theta_1)\right]. \tag{13}$$

Since $\tilde{u}_1(\cdot)$ is assumed to be strictly increasing in θ_2 in our paper, we have that $\tilde{u}_1(\theta_1,\theta_2) \neq \tilde{u}_1(\theta_1,\hat{\theta}_2)$ for all $\theta_1 \in \Theta_1$ and all $\hat{\theta}_2 \neq \theta_2$. Then, inequality (13) can slightly be simplified as follows: for all $\hat{\theta}_2 \neq \theta_2$, there exists $\tilde{M}_1 > 0$ such that

$$\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1|x^*(\theta_1,\hat{\theta}_2)=1\}}(\theta_1)\left(\tilde{u}_2(\theta_1,\bar{\theta}_2)-\tilde{u}_2(\theta_1,\hat{\theta}_2)\right)\right] \quad\leq\quad \tilde{M}_1\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1|x^*(\theta_1,\hat{\theta}_2)=0\}}(\theta_1)\right].$$

Suppose on the contrary that our Assumption 1 is violated. Then, there exists $\hat{\theta}_2 < \bar{\theta}_2$ such that $\int_{\Theta_1} x^*(\theta_1,\hat{\theta}_2)dF_1(\theta_1) = 1$, or equivalently, $x^*(\theta_1,\hat{\theta}_2) = 1$ for all $\theta_1 \in \Theta_1$. As a result, the above inequality becomes

$$\mathbb{E}_{\theta_1}\left[\tilde{u}_2(\theta_1,\bar{\theta}_2)-\tilde{u}_2(\theta_1,\hat{\theta}_2)\right] \leq \tilde{M}_1\mathbb{E}_{\theta_1}[0] = 0.$$

Since $\bar{\theta}_2 > \hat{\theta}_2$, by the strict increasingness of $\tilde{u}_2(\cdot)$ in θ_2, we have $\tilde{u}_2(\theta_1,\bar{\theta}_2) - \tilde{u}_2(\theta_1,\hat{\theta}_2) > 0$ for all $\theta_1 \in \Theta_1$. Thus, the left-hand side of the above inequality is strictly positive, leading to a contradiction. So, if inequality (4) is satisfied, then our Assumption 1 is also satisfied.

Second, we will show that inequality (5) is automatically satisfied in our bilateral trade model. First we reproduce inequality (5): there exists $M_2 \geq 0$ such that for all $i \in \{1,2\}$, all $\theta_i,\hat{\theta}_i \in \Theta_i$ with $\hat{\theta}_i \neq \theta_i$,

$$\sum_{j\neq i}\mathbb{E}_{\theta_{-i}}\left[\mathbb{1}_{\{\theta_{-i}|j=m(\hat{\theta}_i,\theta_{-i}),\tilde{u}_j(\theta_i,\theta_{-i})=\tilde{u}_j(\hat{\theta}_i,\theta_{-i})\}}(\theta_{-i})\right] \leq M_2\sum_{j\neq i}\mathbb{E}_{\theta_{-i}}\left[\mathbb{1}_{\{\theta_{-i}|j=m(\hat{\theta}_i,\theta_{-i}),\tilde{u}_j(\theta_i,\theta_{-i})\neq\tilde{u}_j(\hat{\theta}_i,\theta_{-i})\}}(\theta_{-i})\right],$$

where $m(\theta) = \max\{\arg\max_j \tilde{u}_j(\theta)\}$. We assume throughout that each agent's valuation is strictly increasing in the other agent's type. Thus, for all $j \neq i$ and all $\hat{\theta}_i \neq \theta_i$, it is impossible to have $\tilde{u}_j(\theta_i,\theta_{-i}) = \tilde{u}_j(\hat{\theta}_i,\theta_{-i})$ so that the left-hand side of the above inequality is zero. On the other hand, the right-hand side of the above inequality is always nonnegative. Therefore, the above inequality is automatically satisfied in our bilateral trade model. This completes the proof. $\square$

8.11 Proof of Lemma 9

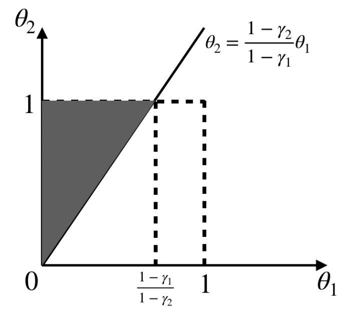
Proof. We revisit the example in Section 3. We divide our argument into the following three cases.

Case 1: $0 < \gamma_2 < \gamma_1 < 1$

Recall that Figure 8 illustrates the decision at different type profiles when $\gamma_2 < \gamma_1$. In particular, the shaded region in the figure represents $\Theta^* = \{(\theta_1,\theta_2) \in$

$\Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles under which it is efficient to trade. We will show that inequality (4) in GMO's Assumption 5.1 is violated in this case.

Figure 8: $0 < \gamma_2 < \gamma_1 < 1$



If the seller's true type is $\theta_1 = 1$ and he deviates to report $\hat{\theta}_1 = (1-\gamma_1)/(1-\gamma_2)$, then it is always efficient not to trade under $\hat{\theta}_1$, i.e., $x^*(\hat{\theta}_1, \theta_2) = 0$ for any $\theta_2 \in \Theta_2$. As a result, inequality (4) becomes

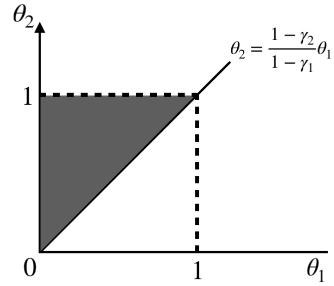
$$\mathbb{E}_{\theta_2}\left[\tilde{u}_1(\bar{\theta}_1, \theta_2) - \tilde{u}_1(\hat{\theta}_1, \theta_2)\right] \quad \leq \quad 0.$$

However, the left-hand side of the above inequality is strictly positive because $\bar{\theta}_1 > \hat{\theta}_1$ implies $\tilde{u}_1(\bar{\theta}_1, \theta_2) - \tilde{u}_1(\hat{\theta}_1, \theta_2) > 0$ by strict increasingness of $\tilde{u}_1(\cdot)$ in θ_1. This is a contradiction. Therefore, inequality (4) in GMO's Assumption 5.1 is violated in this case.

Case 2: $0 < \gamma_2 = \gamma_1 < 1$

Figure 9 illustrates the decision at different type profiles when $\gamma_1 = \gamma_2$. In particular, the shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles under which it is efficient to trade. We will show that inequality (4) in GMO's Assumption 5.1 is satisfied in this case.

Figure 9: $0 < \gamma_1 = \gamma_2 < 1$



We first consider the seller. In this case, we know that for any $\hat{\theta}_1 < \bar{\theta}_1$, there exists $\theta_2 \in \Theta_2$ such that $x^*(\hat{\theta}_1, \theta_2) = 1$. Thus, inequality (4) can be rewritten as

$$M_1 \geq \frac{\mathbb{E}_{\theta_2}\left[\mathbb{1}_{\{\theta_2 | x^*(\hat{\theta}_1, \theta_2)=0\}}(\theta_2)\left(\tilde{u}_1(\bar{\theta}_1, \theta_2) - \tilde{u}_1(\hat{\theta}_1, \theta_2)\right)\right]}{\mathbb{E}_{\theta_2}\left[\mathbb{1}_{\{\theta_2 | x^*(\hat{\theta}_1, \theta_2)=1\}}(\theta_2)\right]}.$$

Since its denominator is positive and its numerator is bounded, the right-hand side of the above inequality is well defined so that we can choose M_1 appropriately. Moreover, if $\hat{\theta}_1 = \bar{\theta}_1$, then $\tilde{u}_1(\bar{\theta}_1, \theta_2) - \tilde{u}_1(\hat{\theta}_1, \theta_2) = 0$ so that the left-hand side of inequality (4) is zero. Since the right-hand side of inequality (4) is always nonnegative, there exists $M_1 > 0$ such that inequality (4) is satisfied.

Next consider the buyer. In this case, we know that for any $\hat{\theta}_2 < \bar{\theta}_2$, there exists some $\theta_1 \in \Theta_1$ such that $x^*(\hat{\theta}_1, \theta_2) = 0$. Thus, inequality (4) can be rewritten as

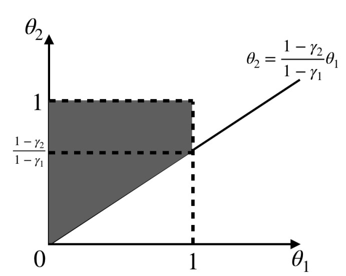
$$\tilde{M}_1 \geq \frac{\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1 | x^*(\theta_1, \hat{\theta}_2)=1\}}(\theta_1)\left(\tilde{u}_2(\theta_1, \bar{\theta}_2) - \tilde{u}_2(\theta_1, \hat{\theta}_2)\right)\right]}{\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{\theta_1 | x^*(\theta_1, \hat{\theta}_2)=0\}}(\theta_1)\right]}.$$

Since its denominator is positive and its numerator is bounded, the right-hand side is well defined so that we can choose $\tilde{M}_1$ appropriately. Moreover, if $\hat{\theta}_2 = \bar{\theta}_2$, then $\tilde{u}_2(\theta_1, \bar{\theta}_2) - \tilde{u}_2(\theta_1, \hat{\theta}_2) = 0$ so that the left-hand side of inequality (4) is zero. Since the right-hand side of the above inequality is always nonnegative, there always exists $\tilde{M}_1 > 0$ such that inequality (4) is satisfied in this case.

Case 3: $0 < \gamma_1 < \gamma_2 < 1$

Figure 10 illustrates the decision at different type profiles when $\gamma_1 < \gamma_2$. In particular, the shaded region represents $\Theta^* = \{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 : x^*(\theta_1, \theta_2) = 1\}$, which describes the set of possible type profiles for which it is efficient to trade. We will show that inequality (4) is violated in this case.

Figure 10: $0 < \gamma_1 < \gamma_2 < 1$



If the buyer's true type is $\theta_2 = 1$ and he deviates to report $\hat{\theta}_2 = (1-\gamma_2)/(1-\gamma_1)$, then it is always efficient to trade under $\hat{\theta}_2$, i.e., $x^*(\theta_1, \hat{\theta}_2) = 1$ for any $\theta_1 \in \Theta_1$. As a result, inequality (4) becomes

$$\mathbb{E}_{\theta_1}\left[\mathbb{1}_{\{x^*(\theta_1,\hat{\theta}_2)=1\}}\left(\tilde{u}_2(\theta_1,\bar{\theta}_2)-\tilde{u}_2(\theta_1,\hat{\theta}_2)\right)\right] \;\leq\; 0.$$

However, the left-hand side of the above inequality is strictly positive because $\bar{\theta}_2 > \hat{\theta}_2$ implies $\tilde{u}_2(\theta_1,\bar{\theta}_2)-\tilde{u}_2(\theta_1,\hat{\theta}_2) > 0$ by the strict increasingness of $\tilde{u}_2(\cdot)$ in θ_2. This is a contradiction. Therefore, inequality (4) is violated in this case.

In the example in Section 3, we conclude that inequality (4) is satisfied if and only if $\gamma_1 = \gamma_2$. This completes the proof. □

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