A new numerical method for a class of Volterra and Fredholm integral equations

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Abstract

In the present work, we introduce a new numerical method based on a strong version of the mean-value theorem for integrals to solve quadratic Volterra integral equations and Fredholm integral equations of the second kind, for which there are theoretical monotonic non-negative solutions. By means of an equality theorem, the integral that appears in the aforementioned equations is transformed into one that enables a more accurate numerical solution with fewer calculations than other previously described methods. Convergence analysis is given.

Keywords: quadratic Volterra integral equations, Fredholm integral equations, monotonic solutions, mean-value theorem.

1 Introduction

Fredholm and Volterra integral equations have practical applications in many fields, including engineering, biology, medicine and economics. Although various methods are available to find analytical solutions for Fredholm and Volterra integral equations, in most cases, finding a closed-form solution is not practical. Several numerical methods have been developed in recent years. Aziz et al. [1] proposed a new algorithm to solve non-linear Fredholm and Volterra integral equations of the second kind using Haar wavelets. Moreover, Aziz et al. [2] proposed a method based on Haar wavelet for the numerical solution of two-dimensional non-linear integral equations. Siraj-ul-Islam et al. [17] suggested a novel technique based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders.

Furthermore, Doucet et al. [10] considered a standard Von Neumann expansion of the solution of a Fredholm integral equation of the second kind approximated by using the Markov chain Monte Carlo methods. In addition, several numerical methods have been developed to deal with Fredholm integral equations, including the Runge-Kutta method, the successive approximations

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method, the Laplace transform method and the Adomian decomposition method (see [5] and [18]).

Some authors have investigated the existence of solutions for non-linear integral equations satisfying specific properties. For example, Banaś et al. [4] established some properties of the superposition operator, which is associated with monotonicity and an application to the study of the solvability of a quadratic Volterra integral equation. Meehan et al. [16], instead, investigated the existence of multiple non-negative solutions of non-linear integral equations on compact and semi-infinite intervals. Finally, Horvart-Marc et al. [12] focused their interest on the existence of non-negative solutions of non-linear integral equations on ordered Banach spaces. Quadratic Volterra integral equations arise in some problems considered in the vehicular traffic theory, biology and queuing theory (see Deimling [9]).

In the present work, we introduce a new numerical method based on a strong version of the mean-value theorem for integrals to solve linear and non-linear quadratic Volterra integral equations and Fredholm integral equations of the second kind for which the are theoretical monotonic non-negative solutions. More specifically, based on the existing result by Banaś et al. [4], it was possible to verify whether a non-linear quadratic Volterra integral equation allowed monotonic non-decreasing solutions. In addition, we found it was possible to verify whether a Fredholm integral equation of the second kind allowed a unique non-decreasing non-negative solution. We were thus able to apply a numerical result that was very simple and give, in comparison to others, very accurate numerical solutions. To test the fitness of our method, we applied it to examples with known solutions. Particular advantages of our method are its simplicity, flexibility and ease of implementation, thus making the method applicable to quadratic Volterra integral equations and Fredholm integral equations of the second kind with unknown non-decreasing and non-negative closed-form solutions.

The paper is organised as follows. In Section 2, we give some results that are the core of the numerical method presented in our work. In Section 3, after reminding the reader of certain theoretical results, we apply them in practice, using our novel numerical method to solve quadratic Volterra integral equations. Convergence analysis is given. In Section 4, after recalling some theoretical results and giving an existence result theorem, Fredholm integral equations of the second kind are solved numerically using our method. In Section 5, we present numerical results that confirm the accuracy of the proposed model. Finally, Section 6 concludes the paper.

2 The mean-value theorem

In this section, we describe some theorems that are at the core of the numerical method we give in the present work.

Theorem 2.1. *Let $\psi : [a,b] \to [0,\infty)$ be a monotonic function and $\phi : [a,b] \to \mathbb{R}$ a Lebesgue integrable function. Then, there exists $\xi \in [a,b]$ such that*

$$\int_a^b \phi(x)\psi(x)\,dx = \psi(a+)\int_a^\xi \phi(x)\,dx + \psi(b-)\int_\xi^b \phi(x)\,dx, \tag{1}$$

where $\psi(a+) := \lim_{x\to a^+}\psi(x)$ and $\psi(b-) := \lim_{x\to b^-}\psi(x)$

Proof. See Witula et al. [19]. ■

Proposition 2.1. *Let $\psi : [0, \infty) \to \mathbb{R}$ be a continuous monotonic function and $\zeta : [a, b] \to [0, \infty)$ a non-negative continuous monotonic function and $\phi : [a, b] \to \mathbb{R}$ a Lebesgue integrable function. Then, there exists $\xi \in [a, b]$ such that*

$$\int_a^b \phi(x)\psi(\zeta(x))\, dx = \psi(\zeta(a)) \int_a^\xi \phi(x)\, dx + \psi(\zeta(b)) \int_\xi^b \phi(x)\, dx. \tag{2}$$

Proof. The result easily comes directly from Theorem 2.1 and results on composition of functions. ■

3 The case of quadratic Volterra integral equations

Preliminarily, let us recall some important definitions and results together with their references from the literature.

Let us fix the interval $I = [0, 1]$ and $C = C(I)$ as the Banach space of all real valued and continuous functions, with I characterised by the maximum norm $\|x\| = max\{|x(t)| : t \in I\}$.

Let us consider the following non-linear quadratic Volterra integral equation:

$$\phi(x) = g(x) + f(x, \phi(x)) \int_0^x v(x, t, \phi(t))\, dt, \tag{3}$$

with $x \in I = [0, 1]$.

Following Banaś et al. [4], we report some assumptions that the components of equation (3) must satisfy for it to have at least one monotonic and non-negative solution on the interval I.

Let I be the set defined above and J be an arbitrary real interval.

Let us consider the function $f(x, y)$ defined on the interval $I \times J$ and let the function f verifies the following assumptions:

(α) f is continuous on the set $I \times J$;

(β) the function $x \to f(x, y)$ is non-decreasing for any fixed $y \in J$;

(γ) the function $y \to f(x, y)$ is non-decreasing on J for any fixed $x \in I$;

(δ) the function $f(x, y)$ satisfies the Lipschitz condition with respect to the variable y, i.e. a constant $N_1 > 0$ exists such that for any $x \in I$ and for $y_1,\ y_2 \in J$ the following inequality holds

$$|f(x, y_2) - f(x, y_1)| \le N_1 |y_2 - y_1|. \tag{4}$$

Furthermore, suppose that the following assumptions are satisfied:

(a) $g \in C(I)$ and g is non-decreasing and non-negative on the interval I;

(b) the function $f : I \times J \to \mathbb{R}$ satisfies the previous conditions $(\alpha) - (\delta)$, where J is an unbounded interval such that $J \subset \mathbb{R}^+$, and $g_0 \in J$, with $g_0 = g(0) = \min\{g(x) : x \in I\}$. Moreover, f is non-negative on $I \times J$;

(c) a non-decreasing function $h(r) = h : [g_0, \infty) \to \mathbb{R}^+$ exists such that

$$|f(x, y_2) - f(x, y_1)| \leq h(r)|y_2 - y_1| \tag{5}$$

for any $t \in I$ and for all $y_1,\ y_2 \in [g_0, r]$;

(d) $v : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v : I \times I \times \mathbb{R}^+ \to \mathbb{R}^+$, and for arbitrarily fixed $t \in I$ and $y \in \mathbb{R}^+$ the function $x \to v(x, t, y)$ is non-decreasing on I;

(e) a non-decreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ exists such that $v(x, t, y) \leq p(y)$, for $x, t \in I$ and $y \geq 0$;

(f) a positive solution r_0 exists for the inequality

$$\|g\| + (rh(r) + F_1)p(r) \leq r, \tag{6}$$

where $F_1 = \sup\{f(x, 0) : x \in I\}$. Moreover, $k(r_0)p(r_0) < 1$.

Theorem 3.1. *Under the assumptions (a)−(f), equation (3) has at least one solution $\phi = \phi(x)$, which belongs to the space $C(I)$ and is non-decreasing and non-negative on the interval I.*

Proof. See Banás et al. [4]. ∎

Let us define the sets $S = \{\phi \in C(I)\ :\ \phi(x) \geq g_0 \text{ for } x \in I\}$ and $S_{r_0} = \{\phi \in S\ :\ \|\phi\| \leq r_0\}$.

Maleknejad et al. [14], following the assumptions and results of Theorem 3.1, offer the theoretical result below that ensures the uniqueness of the solution of equation (3).

Theorem 3.2. *Let T be the operator*

$$T(\phi)(x) = g(x) + f(x, \phi(x)) \int_0^x v(x, t, \phi(t))\, dt \tag{7}$$

that satisfies the following assumptions:

(1) the function $v(x, t, \phi)$ satisfies the Lipschitz condition with respect to variable ϕ and with constant $N_2 > 0$, i.e. for any $x,\ t \in I$ and for $\phi_1,\ \phi_2 \in \mathbb{R}$

$$|v(x, t, \phi_2) - v(x, t, \phi_1)| \leq N_2|\phi_2 - \phi_1|;$$

(2) r_0 satisfies the assumption (f) of Theorem 3.1, as well as the following inequality

$$p(r_0)h(r_0) + (r_0h(r_0) + F_1)N_2 < 1.$$

This relation implies $h(r_0)p(r_0) < 1$ automatically.

Then the operator T is a contractive mapping in S_{r_0}, so it has exactly one fixed point.

Proof. See Maleknejad et al. [14]. ∎

Let $L_1 = L[I]$ be the space of Lebesgue integrable functions on the interval I.

Let us suppose that the assumptions of Theorem 3.1 are satisfied. Under this, let us consider a quadratic Volterra non-linear integral equation of the following form:

$$\phi(x) = g(x) + f(\phi(x)) \int_0^x v(x, t, \phi(t))dt, \tag{8}$$

where $v(x, t, \phi(t)) = k(x, t)\psi(\phi(t))$. We assume $\psi(\cdot)$ is a monotonic continuous function in $\mathbb{R}^+$. The linear case may be considered as a special non-linear one if $\psi(\cdot)$ is the identity function (i.e., $\psi(\phi(\cdot)) = \phi(\cdot)$). The kernel function $k : I \times I \to \mathbb{R}^+$ is assumed to be continuous on $I \times I$.

3.1 The mean-theorem value approach and the convergence analysis

Now we can use the results in Section 2 to build our numerical method.

Let n be a positive integer. Let us consider the following partition Γ of the interval $[0,T]$ into n intervals of equal length $\Delta = T/n$:

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = T. \tag{9}$$

Let $\xi_m \in [x_{m-1}, x_m]$, for $m = 1, 2, \ldots, n$ and $x \equiv x_h$ be one of the points $x_1, x_2, \ldots, x_n$ defined in the partition (9), that is the index h is $1, 2, \ldots, n$.

By means of additive properties for integrals and by applying Proposition 2.1, we can transform, for each $x \equiv x_h$, equation (8) as follows

$$\phi(x) = g(x) + f(\phi(x))\Big[\sum_{m=1}^{h} \psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\xi_m} k(x,t)\,dt + \psi(\phi(x_m))\int_{\xi_m}^{x_m} k(x,t)\,dt\Big]. \tag{10}$$

Clearly, these numbers ξ_m depend on x_{m-1}, x_m and on the unknown function ϕ, with $x_{m-1} \le \xi_m(x_m) \le x_m$. Considering the problem treated in this section, it is very difficult to know the exact value for each of them. As we show in Proposition 3.1 below, it is not restrictive to assume $\xi_m(x_m) = \widetilde{\xi}_m$, where $\widetilde{\xi}_m$ are constants such that $x_{m-1} \le \tilde{\xi}_m \le x_m$.

For the sake of simplicity, we define the following operators for each fixed $x \equiv x_h$:

$$(K\phi)(x) = \sum_{m=1}^{h} \Big[\psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\xi_m} k(x,t)\,dt + \psi(\phi(x_m))\int_{\xi_m}^{x_m} k(x,t)\,dt\Big],$$

$$(\widetilde{K\phi})(x) = \sum_{m=1}^{h} \Big[\psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\widetilde{\xi}_m} k(x,t)\,dt + \psi(\phi(x_m))\int_{\widetilde{\xi}_m}^{x_m} k(x,t)\,dt\Big].$$

It follows

Proposition 3.1. *Let the kernel function $k(x,t)$ be continuous in $I \times I$, the functions $x \to k(x,t)$ and $t \to k(x,t)$ be monotonic and non-negative for any fixed $t \in I$ and $x \in I$, respectively. Let $L > 0$ be a constant such that $|k(x,t)| \le L$ for each $(x,t) \in I \times I$. Then, it follows*

$$|(\widetilde{K\phi})(x) - (K\phi)(x)| \to 0 \qquad \text{when} \qquad n \to \infty. \tag{11}$$

Proof.

$$\Big|(K\phi)(x) - (\widetilde{K\phi})(x)\Big| \le \Big|\sum_{m=1}^{h} \Big[\psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\xi_m} k(x,t)\,dt + \psi(\phi(x_m))\int_{\xi_m}^{x_m} k(x,t)\,dt$$

$$- \psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\widetilde{\xi}_m} k(x,t)\,dt - \psi(\phi(x_m))\int_{\widetilde{\xi}_m}^{x_m} k(x,t)\,dt\Big]\Big| =$$

$$\Big|\sum_{m=1}^{h} \Big[\psi(\phi(x_{m-1}))\Big(\int_{x_{m-1}}^{\xi_m} k(x,t)\,dt - \int_{x_{m-1}}^{\widetilde{\xi}_m} k(x,t)\,dt\Big) +$$

$$+ \psi(\phi(x_m))\Big(\int_{\xi_m}^{x_m} k(x,t)\,dt - \int_{\widetilde{\xi}_m}^{x_m} k(x_i,t)\,dt\Big)\Big]\Big| =$$

$$
= \left| \sum_{m=1}^{h} \left[\psi(\phi(x_{m-1})) \int_{\widetilde{\xi}_m}^{\xi_m} k(x,t)\,dt + \psi(\phi(x_m)) \int_{\xi_m}^{\widetilde{\xi}_m} k(x,t)\,dt \right] \right| =
$$

$$
= \left| \sum_{m=1}^{h} \Big(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \Big) \int_{\widetilde{\xi}_m}^{\xi_m} k(x,t)\,dt \right| \leq
$$

$$
\leq \sum_{m=1}^{h} \left| \Big(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \Big) \right| \cdot \left| \int_{\widetilde{\xi}_m}^{\xi_m} k(x,t)\,dt \right| \leq
$$

$$
\leq \sum_{m=1}^{h} \left| \Big(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \Big) \right| \cdot \left| \int_{x_{m-1}}^{x_m} k(x,t)\,dt \right| \leq
$$

$$
\leq \sum_{m=1}^{h} \left| \Big(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \Big) \right| \cdot \int_{x_{m-1}}^{x_m} |k(x,t)|\,dt \leq
$$

$$
\leq L \sum_{m=1}^{h} \left| \Big(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \Big) \right| \cdot \Delta =
$$

$$
= \frac{L}{n} \sum_{m=1}^{h} | \left(\psi(\phi(x_{m-1})) - \psi(\phi(x_m)) \right) | = \frac{H}{n} \,,
$$

where H is a constant because the sum, for $x \equiv x_h$, is a number.

As $n \to \infty$, it follows $|(K\phi)(x) - \widetilde{K\phi}(x)| \to 0$. We thus conclude the proof. ■

Corollary 3.1. *Under the hypotheses of Proposition 3.1, it follows that*

$$
|\widetilde{\phi(x)} - \phi(x)| \to 0 \qquad \textit{when} \qquad n \to \infty. \tag{12}
$$

Proof. The result comes immediately from Proposition 3.1 and considering the expression (8). ■

If, by means of Theorems 3.1 and 3.2, it results that equation (8) admits a unique non-negative monotonic solution, by applying Proposition 2.1 the equation can be solved in the few steps described below. In fact, we provide the following algorithm in order to find the numerical solution.

Step 3.1. *For each x_i, with $i = 0, 1, 2, \ldots, n$, by means of additive properties for integrals and by applying Proposition 2.1, we can rewrite equation* (8) *in this way*

$$
\phi(x_i) = g(x_i) + f(\phi(x_i)) \left[\sum_{m=1}^{i} \psi(\phi(x_{m-1})) \int_{x_{m-1}}^{\xi_m} k(x_i,t)\,dt + \psi(\phi(x_m)) \int_{\xi_m}^{x_m} k(x_i,t)\,dt \right], \tag{13}
$$

where $\xi_m \in [x_{m-1}, x_m]$, for $m = 1, 2, \ldots, i$.

Step 3.2. *We observe that the value of $\phi(0)=\phi(x_0)$ may be easily computed. In fact, if we consider $x_0=0$ and the following equation*

$$\phi(0)=g(0)+f(\phi(0))\int_0^0 k(0,t)\psi(\phi(t))\,dt, \tag{14}$$

then $\phi(0)=g(0)$. Then, we randomly choose $\tilde{\xi}_m\in(x_{m-1},x_m)$, for $m=1,2,\dots,n$, and insert the n-dimensional random vector $\{\tilde{\xi}_1,\tilde{\xi}_2,\dots,\tilde{\xi}_n\}$ into the following system.

$$\begin{cases} \phi(x_0)=g(x_0) \\ \phi(x_1)=g(x_1)+f(\phi(x_1))\Big[\psi(\phi(x_0))\int_{x_0}^{\tilde{\xi}_1}k(x_1,t)\,dt+\psi(\phi(x_1))\int_{\tilde{\xi}_1}^{x_1}k(x_1,t)\,dt\Big] \\ \phi(x_2)=g(x_2)+f(\phi(x_2))\Big[\sum_{m=1}^{2}\psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\tilde{\xi}_m}k(x_2,t)\,dt+\psi(\phi(x_m))\int_{\tilde{\xi}_m}^{x_m}k(x_2,t)\,dt\Big] \\ \vdots \\ \phi(x_n)=g(x_n)+f(\phi(x_n))\Big[\sum_{m=1}^{n}\psi(\phi(x_{m-1}))\int_{x_{m-1}}^{\tilde{\xi}_m}k(x_n,t)\,dt+\psi(\phi(x_m))\int_{\tilde{\xi}_m}^{x_m}k(x_n,t)\,dt\Big]. \end{cases}$$

The above non-linear system is solved by means of a numerical method, which gives the multivariate $(n+1)$-dimensional vector $\{\tilde{\phi}(x_0),\tilde{\phi}(x_2),\dots,\tilde{\phi}(x_n)\}$.

Step 3.3. *We choose a positive integer q, and repeating Step 3.2 q times we obtain the following $q\times(n+1)$-matrix in which each row represents a possible approximation of the solution:*

$$\begin{bmatrix} \tilde{\phi}_1(x_0) & \tilde{\phi}_1(x_1) & \cdots & \tilde{\phi}_1(x_n) \\ \tilde{\phi}_2(x_0) & \tilde{\phi}_2(x_1) & \cdots & \tilde{\phi}_2(x_n) \\ \vdots & \cdots & \ddots & \vdots \\ \tilde{\phi}_q(x_0) & \tilde{\phi}_q(x_1) & \cdots & \tilde{\phi}_q(x_n) \\ . & & & \end{bmatrix}. \tag{15}$$

Because by virtue of (14), $\phi(x_0)$ is exactly known, it results that $\phi(x_0)=\tilde{\phi}_1(x_0)=\tilde{\phi}_2(x_0)=\cdots=\tilde{\phi}_q(x_0)$. The final approximated solution, for each x_i, for $i=1,2,\dots,n$, is obtained, starting from the second, by computing the mean value of each column of matrix (15):

$$\phi(x_i)_{approx}=\frac{\sum_{j=1}^{q}\phi_j(x_i)}{q} \tag{16}$$

for $i=1,2,\dots,n$, according to the weak law of large numbers.

4 The case of Fredholm-Hammerstein integral equations

Let us consider the following Fredholm-Hammerstein integral equation of the second kind:

$$\phi(x)=f(x)+\int_0^1 k(x,t)\psi(t,\phi(t))\,dt. \tag{17}$$

Let $L_1=L[I]$ be the space of Lebesgue integrable functions on the interval $I=[0,1]$ equipped with the usual norm.

Let $I=[0,1]$.

Definition 4.1. *Let $I = [a, b] \subset \mathbb{R}$ and $J \subset \mathbb{R}^n$ be an open set. The function $f : I \times J \to \mathbb{R}$ satisfies the Carathéodory conditions if:*

(I) $f(\cdot, z) : I \to \mathbb{R}$ is measurable for every $z \in J$;

(II) $f(s, \cdot) : J \to \mathbb{R}$ is continuous for every $s \in I$.

Let $\psi : (0, 1) \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory conditions. We may define the following superposition operator

$$(\Psi\phi)(x) = \psi(x, \phi(x)) \tag{18}$$

for $x \in (0, 1)$. It has been demonstrated (see Krasnosel'skii et al. [13]) that the superposition operator (18) satisfies the following result:

Theorem 4.1. *The superposition operator Ψ maps L_1 into itself if and only if*

$$|\psi(x, s)| \leq c(x) + N_3|s| \tag{19}$$

for all $x \in (0, 1)$, $s \in \mathbb{R}$, $c(x)$ is a function from L_1 and N_3 is a non-negative constant.

Some definitions concerning the Hausdorff and De Blasi measures of non-compactness follows (see, respectively, [3] and [8]).

Definition 4.2. *Let E be a Banach space. The Haursdorff measure of non-compactness of a non-empty and bounded subset Q of E is defined in the following way:*

$$\chi(Q) = \inf\left\{\epsilon > 0 \ : \ Q \subset S + \epsilon B_X, \ S \subset E, \ S \text{ is finite}\right\}. \tag{20}$$

B_X indicates the unit ball in E.

Definition 4.3. *Let E be a Banach space. The De Blasi measure of non-compactness of a non-empty and bounded subset Q of E is defined in the following way:*

$$\beta(Q) = \inf\left\{\epsilon > 0 \ : \ Q \subset S + \epsilon B_X, \ S \ \text{ is weakly compact}\right\}. \tag{21}$$

B_X indicates the unit ball in E.

In addition, for our purposes we report the following two fundamental theorems:

Theorem 4.2. *Let X be an arbitrary non-empty and bounded subset of L_1. If X is compact in measure, then $\beta(X) = \chi(X)$.*

Proof. See [3]. ■

Theorem 4.3. *Let Q be a non-empty, bounded, closed and convex subset of E and let $H : Q \to Q$ be a continuous transformation, which is a contraction with respect to the Hausdorff measure of non-compactness χ, i.e. a constant $\alpha \in [0, 1)$ exists such that $\chi(HX) \leq \alpha\chi(X)$ for any non-empty subset X of Q. Thus, H has at least one fixed point in the set Q.*

Proof. See [7]. ■

Definition 4.4. *Let us define the following operator:*

$$(A\phi)(x) = f(x) + \int_0^1 k(x,t)\psi(t,\phi(t))\,dt. \tag{22}$$

Definition 4.5. *Let us consider a Banach space E. With $B_r = B(\theta, r)$ we describe the closed ball in E centred in θ, the zero element of E and radius $r > 0$.*

The next theorem we give proof of originates from an idea of El-Sayed et al. [11].
Let us assume the following assumptions hold true:

(a1) $f : I \to \mathbb{R}^+$ is integrable and monotonic non-decreasing (on I);

(b1) the function $\psi : I \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Carathéodory conditions and a function $g \in L_1$ and a constant b exist such that

$$\psi(x,s) \le g(x) + b|s|$$

and, in addition:

- the function $t \to \psi(t,y)$ is non-decreasing for any fixed $y \in \mathbb{R}^+$;
- the function $y \to \psi(t,y)$ is non-decreasing for any fixed $t \in I$;

(c1) the function $k : I \times I \to \mathbb{R}^+$ is measurable in both variables and the operator $(K\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$, for $x \in [0,1]$, maps L_1 into itself;

(d1) $x \to k(x,t)$ is a.e. non-decreasing on I for almost all fixed $t \in [0,1]$;

(e1) a positive constant M exists such that $\int_0^1 k(x,t)\,dt \le M$, for $x \in [0,1]$;

As a result, the following theorem holds,

Theorem 4.4. *Under the assumptions* $(a1) - (e1)$*, if $Mb < 1$, equation (17) has at least one solution $\phi = \phi(x) \in L_1$, which is positive and a.e. non-decreasing on I.*

Proof. From the aforementioned assumptions we obtain

$$|(A\phi)(x)| \le |f(x)| + \int_0^1 k(x,t)(g(x) + b|\phi(t)|)\,dt.$$

It follows that

$$\begin{aligned}\int_0^1 |(A\phi)(x)|\,dx &\le \int_0^1 |f(x)|\,dx + \int_0^1\int_0^1 k(x,t)(g(t) + b|\phi(t)|)\,dt\,dx \le \\ &\le \int_0^1 |f(x)|\,dx + M\int_0^1 g(t)\,dt + Mb\int_0^1 |\phi(t)|\,dt \le \\ &\le \int_0^1 |f(x)|\,dx + M\int_0^1 |g(t)|\,dt + Mb\int_0^1 |\phi(t)|\,dt.\end{aligned}$$

Is also follows that

$$\|A\phi\| \le \|f\| + M\|g\| + Mb\|\phi\|,$$

with the operator A that maps the ball B_r into itself and where $r = (\|f(x)\| + M\|g\|)(1 - Mb)^{-1}$.

Recall that $B_r \subset L_1$. Let us consider $Q_r \subset B_r$, the subset of B_r consisting of all functions that are a.e. non-decreasing on $[0, 1]$. The set Q_r is non-empty, closed, convex and compact in measure (see Banaś [3]). Based on assumptions $(a1)$, $(b1)$ and $(d1)$, it follows that the operator A maps Q_r into itself and the operator $K\phi$ is continuous. The continuity of operator A on Q_r follows. Let $\epsilon > 0$, and take $D \subset I$ such that $mes(D) \leq \epsilon$ and X is a non-empty subset of Q_r. Thus, for any $\phi \in X$, it follows that

$$\|A\phi\|_{L_1(D)} = \int_D |(A\phi)(x)|\, dx \leq$$

$$\leq \int_D |f(x)|\, dx + \int_D \int_0^1 k(x,t)(g(t) + b|\phi(t)|)\, dt\, dx \leq$$

$$\leq \int_D |f(x)|\, dx + M \int_D (g(t) + b|\phi(t)|)\, dt \leq$$

$$\leq \|f(x)\|_{L_1(D)} + M\|g\|_{L_1(D)} + Mb \int_D |\phi(t)|\, dt.$$

Because

$$\lim_{\epsilon \to 0} \{\sup\{\int_D f(x)\, dx : D \subset I,\ mes(D) < \epsilon\}\} = 0$$

and

$$\lim_{\epsilon \to 0} \{\sup\{\int_D g(x)\, dx : D \subset I,\ mes(D) < \epsilon\}\} = 0.$$

It follows that

$$\beta(A\phi(x)) \leq Mb\beta\phi(x),$$

which implies

$$\chi(AX) \leq Mb\chi(X).$$

.

From the assumption $Mb < 1$ and Theorem 4.3, it follows that A is a contraction and has at least one fixed point in Q_r.

■

Now we demonstrate the uniqueness of the solution under certain assumptions.

Proposition 4.1. *Under the assumptions of Theorem 4.4 as well as the following assumptions:*

(i) the function $v(t, s, y) = k(t, s)\psi(y)$ *satisfies the Lipschitz condition, i.e.* $|v(t, s, y_2) - v(t, s, y_1)| \leq N_4|y_2 - y_1|$ *for any* $t,\ s \in I$*, a constant* $N_4 > 0$ *and for* $y_1,\ y_2 \in Q_r$*;*

(ii) a positive constant M *exists such that* $k(x, t) \leq M$*, for* $(x, t) \in [0, 1] \times [0, 1]$*,*

equation (17) admits a unique solution if $0 \leq N_4 < 1$.

Proof. Let y_1, $y_2 \in Q_r$. It follows that

$$|(Ay_2)(x) - (Ay_1)(x)| \leq \Big| \int_0^1 k(s,t)\psi(y_2(t))\,dt - \int_0^1 k(s,t)\psi(y_1(t))\,dt \Big| \leq$$

$$\leq \int_0^1 |k(x,t)||\psi(y_2(t)) - \psi(y_1(t))|\,dt \leq$$

$$\leq \int_0^1 M|g(t) + by_2(t) - g(t) - by_1(t)|\,dt \leq$$

$$\leq Mb\|y_2(t) - y_1(t)]\|.$$

Satisfying the requirements described in Theorem 4.4 that $Mb < 1$, it follows that the operator A satisfies the Lipschitz conditions $\|Ay_2 - Ay_1\| \leq Mb\|y_2 - y_1\|$. If $y_1,\ y_2 \in Q_r$ are fixed points of operator A, it follows that $\|y_2 - y_1\| = \|Ay_2 - Ay_1\| \leq Mb\|y_2 - y_1\|$. By means of the assumptions that $Mb < 1$, it follows that $y_1 \equiv y_2$ according to the Banach-Caccioppoli fixed-point theorem. Here $Mb = N_4$. ∎

4.1 The mean-value theorem approach

Let us assume that in equation (17) the kernel function $k(x,t)$ has the form $k(x,t) = \alpha(x)\beta(t)$, where $\alpha(x)$ and $\beta(t)$ are functions such that the properties of the kernel function described in the previous paragraph are preserved. Here we consider the equation

$$\phi(x) = f(x) + \int_0^1 \alpha(x)\beta(t)\psi(\phi(t))\,dt. \tag{23}$$

Let $\psi(\cdot)$ be non-decreasing.

If, by means of Theorem 4.4 and Proposition 4.1, it results that equation (23) admits a unique non-negative monotonic solution, by applying Proposition 2.1, the equation can be solved numerically using the following steps.

Step 4.1. *By means of Proposition 2.1, we can transform equation* (23) *into the following,*

$$\phi(x) = f(x) + \psi(\phi(0)) \int_0^\xi \alpha(x)\beta(t)\,dt + \psi(\phi(1)) \int_\xi^1 \alpha(x)\beta(t)\,dt. \tag{24}$$

Then, equation (24) *becomes*

$$\phi(x) = f(x) + \alpha(x) \left[\psi(\phi(0)) \int_0^\xi \beta(t)\,dt + \psi(\phi(1)) \int_\xi^1 \beta(t)\,dt \right]. \tag{25}$$

It is clear that ξ does not depend on x, where $x \in [0,1]$.

Step 4.2. *We insert the expression of $\phi(x)$ given by equation* (25) *in the integral appearing in equation* (23)*. The following equation results,*

$$\phi(x) = f(x) + \int_0^1 \alpha(x)\beta(t)\psi\Big(f(t) + \alpha(t) \left[\psi(\phi(0)) \int_0^\xi \beta(s)\,ds + \psi(\phi(1)) \int_\xi^1 \beta(s)\,ds \right]\Big) dt. \tag{26}$$

Step 4.3. *In order to find the approximated value of the unknowns $\phi(0)$, $\phi(1)$ and ξ, we build and solve numerically the following non-linear system:*

$$\begin{cases} \phi(0) - f(0) - \alpha(0)\left[\psi(\phi(0))\int_0^\xi \beta(t)\,dt + \psi(\phi(1))\int_\xi^1 \beta(t)\,dt\right] = 0 \\ \phi(1) - f(1) - \alpha(1)\left[\psi(\phi(0))\int_0^\xi \beta(t)\,dt + \psi(\phi(1))\int_\xi^1 \beta(t)\,dt\right] = 0 \\ \phi(1) - f(1) - \alpha(1)\int_0^1 \beta(t)\psi\Big(f(t) + \alpha(t)\left[\psi(\phi(0))\int_0^\xi \beta(s)\,ds + \psi(\phi(1))\int_\xi^1 \beta(s)\,ds\right]\Big)\,dt = 0. \end{cases}$$

Step 4.4. *With the approximated values of $\phi(0)$, $\phi(1)$ and ξ computed in step 4.3, we can compute the approximated values of $\phi(x)$, namely, $\phi(x)_{approx}$, in this way,*

$$\phi(x)_{approx} \approx f(x) + \alpha(x)\left[\psi(\phi(0))\int_0^\xi \beta(t)\,dt + \psi(\phi(1))\int_\xi^1 \beta(t)\,dt\right]. \tag{27}$$

5 Numerical results

In this section, we present some examples of the methods we implemented. In the first two examples, we considered quadratic Volterra non-linear integral equations of the second kind, while in the third example we considered Fredholm non-linear integral equation of the second kind.

Example 5.1. *Let us consider the following quadratic Volterra non-linear integral equation:*

$$\phi(x) = e^x + \frac{x\sqrt{e^x}}{10}(\ln(e^{-x}+1) - \ln(2)) + \sqrt{\phi(x)}\int_0^x \frac{0.1x}{1+\phi(t)}\,dt, \tag{28}$$

where $x \in [0,1]$. The kernel is $k(x,t) = 0.1x$. As reported in [14], equation (28) verifies the assumptions of Theorems 3.1 and 3.2. It follows that the equation admits a unique continuous, non-negative and non-decreasing solution on $I = [0,1]$. In fact, the exact solution is given by $\phi(x) = e^x$. Because $|k(x,t)| \leq 0.1 = L$ in $I \times I$, Proposition 3.1 is verified.

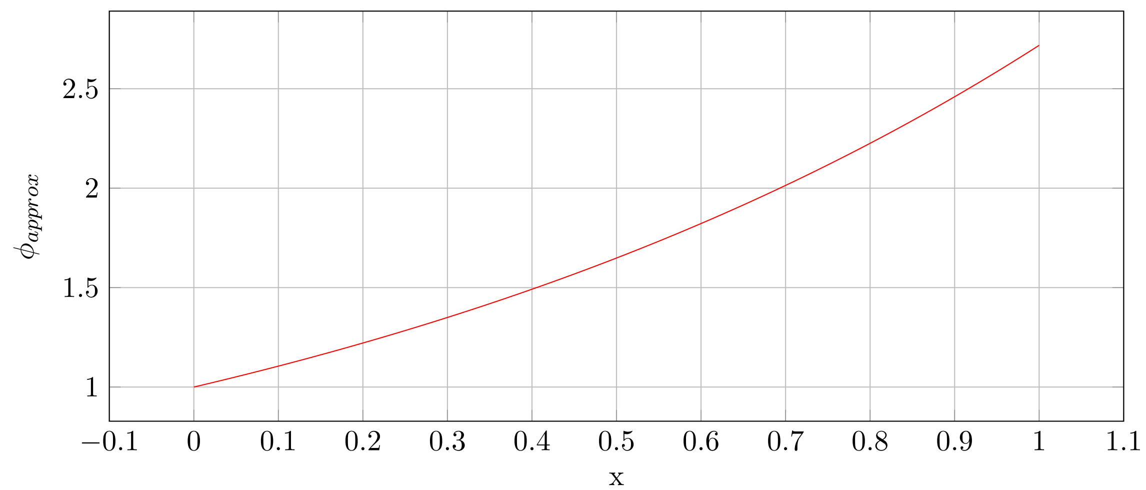

Figure 1: The numerical approximation of $\phi(x)$ in equation (28).

We solved equation (28) *numerically by applying the algorithm described in paragraph 3.1. We considered* $n = 100$ *and* $q = 1000$*. The obtained approximated value is graphed in Figure 1. Meanwhile, Figure 2 shows, instead, the absolute error between the exact and the approximated solution. Coherently with Theorem 3.1, we found our numerical solution to be monotonic, non-negative and non-decreasing in the interval* I*.*

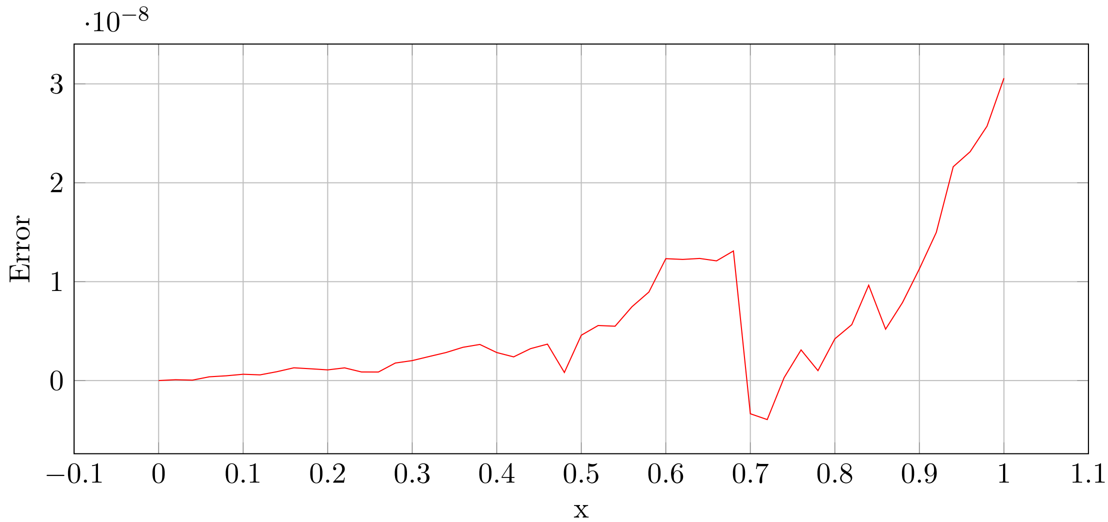

Figure 2: The absolute error for equation (28).

In Maleknejad et al. [14], equation (28) was solved by means of a numerical technique based on the fixed point method and quadrature rules. After choosing an initial function $\phi_0(t)$*, with some properties, Maleknejad et al. in their paper proposed applying a numerical approximating scheme to approximate the solution for the integral equation (3), producing the sequence* $\{\phi_j(x)\}_{j=0}^{\infty}$ *as follows:*

$$\phi_{j+1}(x) = g(x) + f(\phi_j(x)) \int_0^x v(x, t, \phi_j(t))\, dt, \tag{29}$$

with $x \in I$ *and* $n \geq 0$*. Subsequently, the continuous integral, appearing into equation (29), was turned into a discrete equation by means of trapezoidal, Simpson and Sinc quadrature rules. For details, please refer to the original paper [14]. A double approximation resulted: the first corresponding to the transformation of the integral and the second to the iteration due to the approximating scheme (29). The suitability of the methods already reported was compared with the one proposed in this paper.*

Considering $n = 100$ *points in the interval* $[0, 1]$ *(together with trapezoidal, Simpson and Sinc quadrature rules), Maleknejad et al. obtained the best error* $\|\mathrm{e}\|_\infty = \max|x_{n+1} - x_n| = 1.166E{-}7$*. Considering* $n = 50$ *and* 100 *points in the same interval, respectively, with our method we obtained the errors shown in Table 1.*

Table 1: Errors for equation (28). MVT indicates the numerical method proposed in the present paper and MAL the numerical method proposed in [14].

n	(MAL)	(MVT)
	$\|e\|_\infty$	$\|e\|_\infty$
50	$2.435E-7$	$1.669E-07$
100	$1.166E-7$	$3.0574E-08$

As Table 1 suggests, the error obtained by means of our method was comparable in the case $n = 50$ and better when $n = 100$ to that obtained by Maleknejad et al. [14].

Example 5.2. *Let us consider the following quadratic Volterra integral equation considered in [15]:*

$$\phi(x) = \frac{x^{10}}{10} - \frac{x}{x+1}\ln(1+x^{10}/10)(x^3/2 + x^{11}/110) + \frac{x}{x+1}\ln(1+\phi(x))\int_0^x (xt+\phi(t))\,dt, \quad (30)$$

where $x \in [0,1]$. The kernel is a constant i.e. $k(x,t) = 1$.

As reported by Maleknejad et al. [15], equation (30) verifies the assumptions of Theorems 3.1 and 3.2 and admits a unique continuous, non-negative and non-decreasing solution on $I = [0,1]$. In fact, the exact solution is given by $\phi(x) = \frac{x^{10}}{10}$. It is easy to check that Proposition 3.1 is verified.

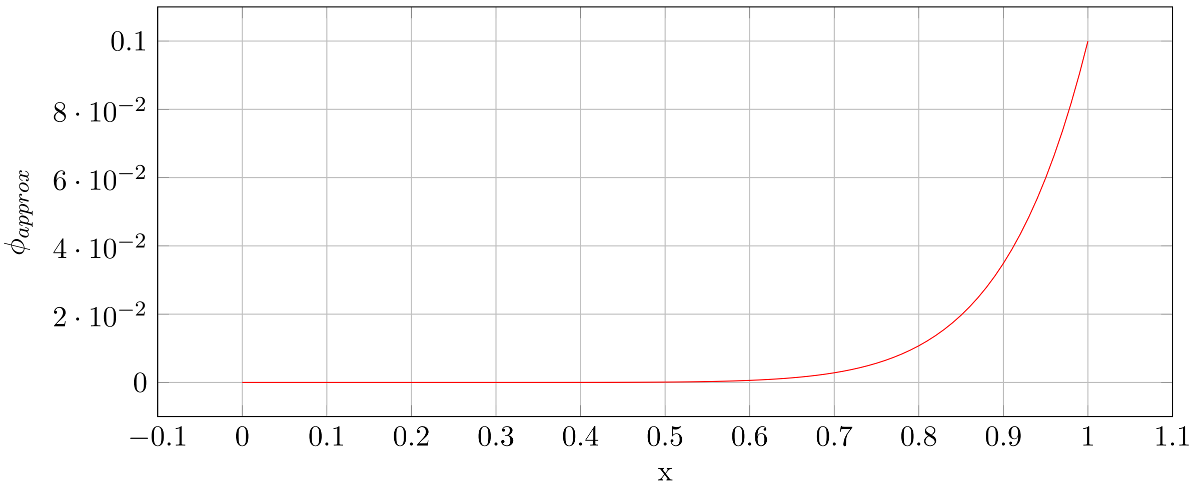

Figure 3: The numerical approximation of $\phi(x)$ in equation (30).

We solved the equation (30) numerically by applying the algorithm described in paragraph 3.1. We considered $n = 100$ and $q = 1000$. The obtained approximated value is graphed in Figure 3. Figure 4 shows, instead, the absolute error between the exact and the approximated solution. Coherently with Theorem 3.1, we found our numerical solution to be monotonic, non-negative and non-decreasing in the interval I.

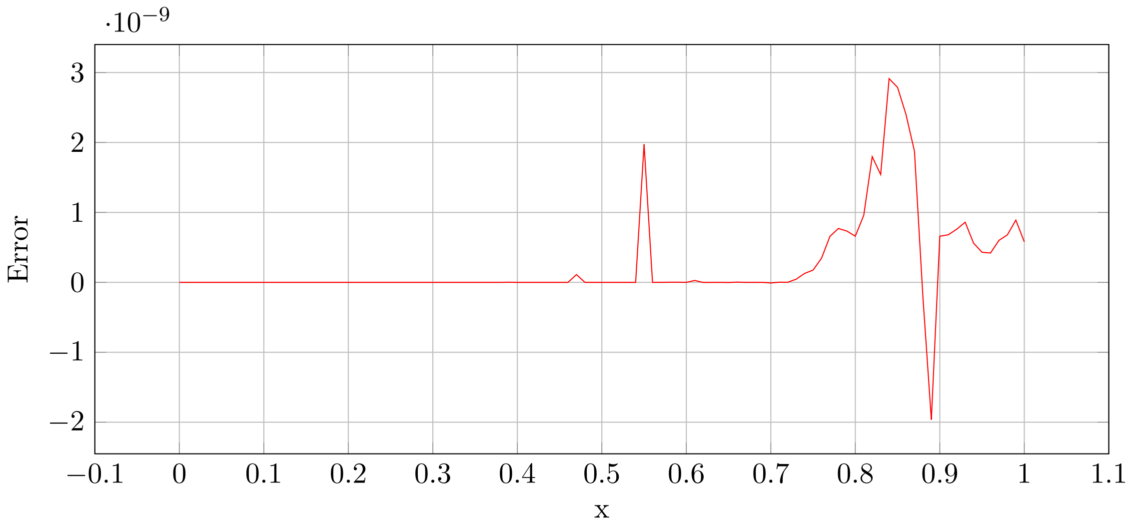

Figure 4: The absolute error for equation (30).

Equation (30) *was considered in [15], where a numerical approach based on a successive approximation technique was applied to solve non-linear integral equations. In more detail, a sequence of functions, that converged to the solution, was produced. The methods included a fixed point method, a quadrature rule and an interpolation method. The suitability of the past methods was compared with the one proposed in this paper. The errors between the exact and the approximated value obtained by means of Maleknejad's method (MAL2) in [15] and the one proposed in the present paper based on a mean-value theorem (MVT) are compared in Table 2. More specifically, similar to previous authors [15], we computed the absolute error in the whole interval* $I = [0, 1]$ *and then compared it according to the number of points considered in interval* I*, as shown in* Table 2.

Table 2: Errors for equation (30). MVT indicates the numerical method proposed in the present paper and $MAL2$ the numerical method proposed in [15].

n	(MAL2)	(MVT)
	$\|e\|_\infty$	$\|e\|_\infty$
110	$10E-08$	$2.912E-9$
350	$10E-10$	$5.1026E-11$

As Table 2 suggests, the errors obtained by means of our method were better than to that obtained by Maleknejad et al. [15].

Example 5.3. *Let us consider the following non-linear Fredholm-Hammerstein integral equation considered in [6]:*

$$\phi(x) = 1 + x + \left(1 - \frac{3}{2}\log(3) + \frac{\sqrt{3}}{6}\pi\right) x^2 + \int_0^1 2x^2 t \log(\phi(t))\, dt, \tag{31}$$

where $x \in [0, 1]$*. Here* $k(x, t) = 2x^2 t = 2\alpha(x)\beta(t)$ *with* $\alpha(x) = x^2$ *and* $\beta(t) = t$*.*

It is easy to check that for equation (31) the assumptions of Theorem 4.4 and Proposition 4.1 are satisfied. It follows that a unique non-negative non-decreasing solution exists on $I = [0, 1]$.

In fact, the exact solution is given by $\phi(x) = 1 + x + x^2$. *If we consider* $x = 0$ *in equation (31), we obtain* $\phi(0) = 1$. *It is well known that log*(x) *is Lipschitzian for* $x \in (1, \infty)$*, in which it is also positive and increasing. It is easy to demonstrate that for* $y > 1$ *the following inequality holds true: log*$(y) \leq 1 + by$ *for* $b = \frac{1}{e^2}$. *In addition, because* $k(x,t) = 2x^2t$*, it follows that* $M = 2$. *It results that Proposition 4.1 may by applied with* $N_4 = Mb = \frac{2}{e^2}$.

We solved equation (31) *numerically by applying the algorithm described in paragraph 4.1. We considered* $n = 100$ *points. The obtained approximated value is graphed in Figure 5. Figure 6 shows the absolute error between the exact and the approximated solution. Coherently with Theorem 3.1, we found our numerical solution to be monotonic, non-negative and non-decreasing in the interval* I.

The equation in this example is considered in [6], where it is solved by means of a numerical method, which is a combination of the variational iteration method and the spectral collocation method. Under certain conditions (see [6] for details), this combined method works after choosing a starting function $\phi_0(x)$. *A family of successive approximations is built to compute* $\phi(x)$. *In addition, the integral appearing in the equation to be solved is substituted with an interpolating polynomial. It results that a double approximation is needed: the first by means of the interaction given by the variational iteration method and the spectral collocation method.*

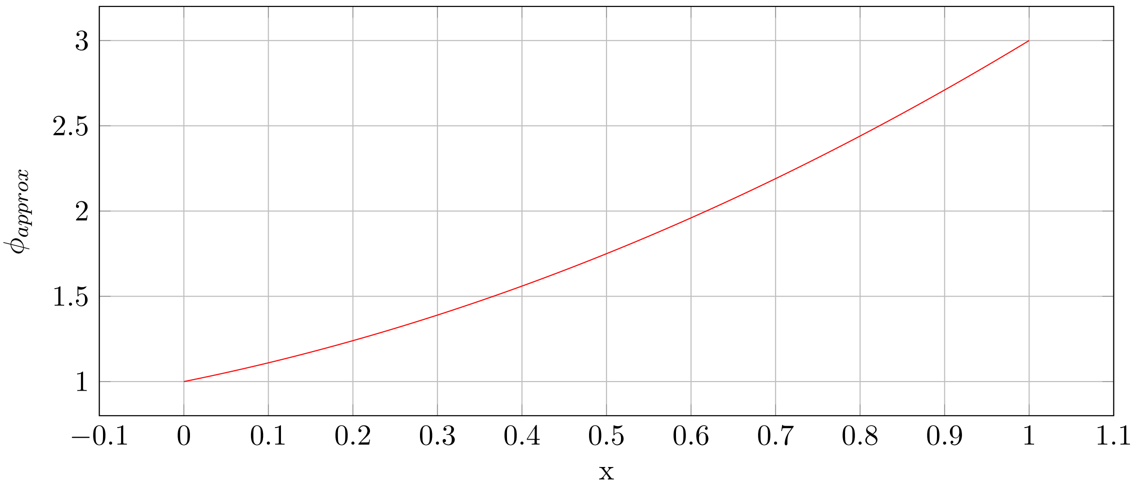

Figure 5: The numerical approximation of $\phi(x)$ in equation (31).

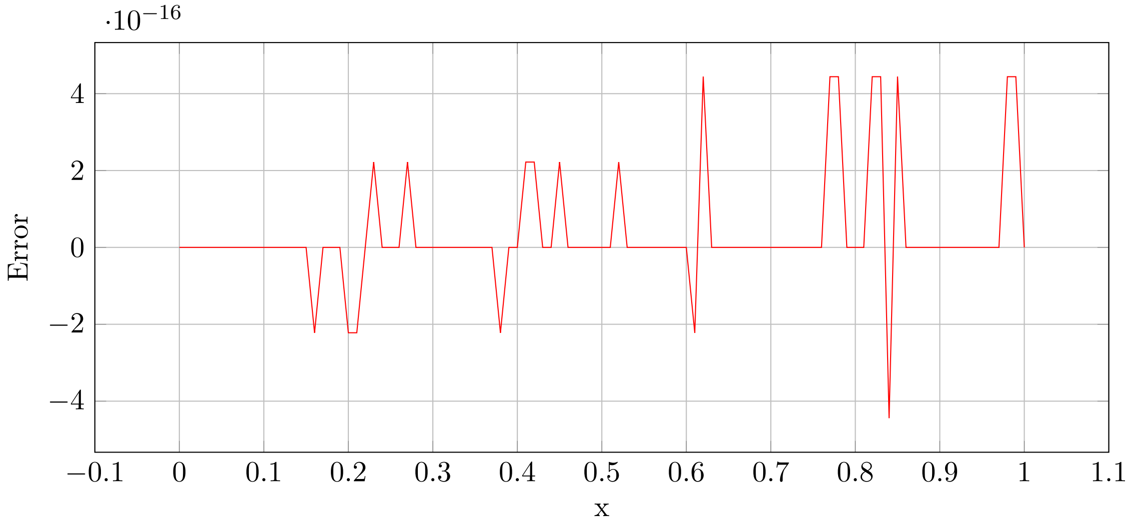

Figure 6: The error for equation (31).

Table 3: Errors for equation (31) considering $n = 16$ points.

n	(DAL)	(MVT)
$\|e\|_2$	$9.6786E-16$	$5.4389E-16$
$\|e\|_\infty$	$4.440E-16$	$4.4409E-16$

The error obtained by Daliri and Saberi-Nadjaf [6] (DAL) considering $n = 16$ points in the interval I was compared with the one obtained for the same interval by applying our method (MVT; see Table 3 for details). Our method produced a very accurate solution. In fact, we considered more points; for example, for $n = 100$, $\|e\|_\infty = 4.4409E-16$. Table 3 shows that our error was comparable to that obtained by Daliri and Saberi-Nadjaf [6], but, as mentioned above, their method required more calculations. With our method we only needed to solve one non-linear system of two equations in two unknowns and to use the related solution to accurately approximate the solution of equation (31). In this example, we obtained $\xi = 0.57052227412994327426787322110612876713275909423828125$.

The computational time needed to compute $n = 1000000$ points was about 0.06 seconds. Calculations were made by means of the Matlab software and were run on a MacBook Pro with a processor 2.6 GHz Intel Core i7.

As the Figures and Tables presented in the paper suggest, the absolute error produced by our method can be considered negligible.

More generally, because Proposition 2.1 induces an equality in the integral that appears in equation (8) or equation (23), in our method we consider directly it without any transformation in contrast with the algorithms proposed in [14], [15] and [6]. Thanks to this, we think that our method can be applied to more complicated cases.

6 Conclusions

This paper introduce a computational method to obtain numerical solutions to linear and non-linear quadratic Volterra integral equations and Fredholm integral equations of the second kind, whose solution - as known from the theory- has to be positive and non-decreasing. The approach proposed is able to numerically solve integral equations with unknown closed-form solutions in a way that is simple, flexible and easy to implement. The examples given in order to compare the method with others mentioned in this paper highlights the effectiveness of the method proposed. Future research should be focused on extending these results to solve particular problems in economic and social fields, for instance in Risk Theory to evaluate the dynamic over time of the capital requirements for solvency purposes of financial institutions or in Collectivity Theory to measure movements of individuals among different groups or states.

References

[1] I. Aziz and Siraj-ul-Islam, *New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets*, J. Comput. Appl. Math., 239 (2013) 333–345.

[2] I. Aziz, Siraj-ul-Islam, F. Khan, *A new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations*, J. Comput. Appl. Math., 272 (2014) 70–80.

[3] J. Banas, *On the superposition operator and integrable solutions of some functional equations*, Nonlin. Analysis T.M.A. Vol. 12 (1988) 777-784.

[4] J. Banas and K. Sadarangani, *Monotonicity properties of the superposition operator and their application* J. Math. Anal. App. 340 (2008) 1385-1394.

[5] H. Brunner, *Volterra Integral Equations: An Introduction to Theory and Applications*, Cambridge Monographs on Applied and Computational Mathematics, No. 30, Cambridge University Press, Cambridge/UK, January 2017.

[6] M. H. Daliri and J. Saberi-Nadjaf, *Improved variational iteration method for solving a class of nonlinear Fredholm integral equations*, J. SeMA (2018). https://doi.org/10.1007/s40324-018-0161-5.

[7] G. Darbo, *Punti uniti in tranzformazioni a codominio non compatto*, Rend. Sere. Mat. Univ. Padova 24,84-92, (1955).

[8] F.S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R.S. Roumanie, 21 (1977), 259–262.

[9] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[10] A. Doucet, A. M. Johansen and V. B. Tadic, *On solving integral equations using Markov chain Monte Carlo methods*, Applied Mathematics and Computations 216 (2010) 2869-2880.

[11] El-Sayed and H. H. G. Hashem, *Integrable solutions for quadratic Hammerstein and quadratic Urysohn functional integral equations*, Commentationes Mathematicae, vol. 48, no. 2, pp. 199–207, 2008.

[12] A. Horvat-Marc and R. Precup, *Nonnegative solutions of nonlinear integral equations in ordered Banach spaces*, Fixed Point Theory 5 (2004), 65-70.

[13] M.A. Krasnosel'skii, P.P. Zabreyko , E.I. Pustylnik et al., *Integral operators in spaces of summable functions.*, J Eng Math (1976) 10: 190. https://doi.org/10.1007/BF01535663.

[14] K. Maleknejad, P. Torabi and R. Mollapourasl, *Fixed point method for solving nonlinear quadratic Volterra integral equations*, Comput. Math. Appl. 62, 2555–2566 (2011).

[15] K. Maleknejad, P. Torabi and S.A. Sauter, *Numerical solution of a non-linear Volterra integral equation*, Vietnam J. Math.., 44, pp. 5-28, 2016.

[16] M. Meehan and D. O'Regan, *Multiple nonnegative solutions of nonlinear integral equations on compact and semi-infinite intervals*, Appl. Anal. 74 (2000) 413-427.

[17] Siraj-ul-Islam, I. Aziz, A.S. Al-Fhaid, *An improved method based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders*, J. Comput. Appl. Math., 260 (2014) 449-469.

[18] A.M. Wazwaz, *Linear and nonlinear integral equations* Springer, Berlin, (2011).

[19] R. Witula, E. Hetmaniok and D. Slota, *A stronger version of the second mean value theorem for integrals*, Comp. Math. Appl., 64(6):1612-1615, 2012.