Quotients for sheets of conjugacy classes

Quotients for sheets of conjugacy classes

Giovanna Carnovale, Francesco Esposito
Dipartimento di Matematica “Tullio Levi-Civita”
Torre Archimede - via Trieste 63 - 35121 Padova - Italy
email: carnoval@math.unipd.it, esposito@math.unipd.it

Abstract

We provide a description of the orbit space of a sheet S for the conjugation action of a complex simple simply connected algebraic group G. This is obtained by means of a bijection between S/G and the quotient of a shifted torus modulo the action of a subgroup of the Weyl group and it is the group analogue of a result due to Borho and Kraft. We also describe the normalisation of the categorical quotient $\overline{S}//G$ for arbitrary simple G and give a necessary and sufficient condition for $\overline{S}//G$ to be normal in analogy to results of Borho, Kraft and Richardson. The example of G_2 is worked out in detail.

1 Introduction

Sheets for the action of a connected algebraic group G on a variety X have their origin in the work of Kostant [16], who studied the union of regular orbits for the adjoint action on a semisimple Lie algebra, and in the work of Dixmier [10]. Sheets are the irreducible components of the level sets of X consisting of points whose orbits have the same dimension. In a sense they provide a natural way to collect orbits in families in order to study properties of one orbit by looking at others in its family. For the adjoint action of a complex semisimple algebraic group G on its Lie algebra they were deeply and systematically studied in [2, 4]. They were described as sets, their closure was well-understood, they were classified in terms of pairs consisting of a Levi subalgebra and suitable nilpotent orbit therein, and they were used to answer affirmatively to a question posed by Dixmier on the multiplicities in the module decomposition of the ring of regular functions of an adjoint orbit in $\mathfrak{sl}(n, \mathbb{C})$. If G is classical then all sheets are smooth [14, 24]. The study of sheets in positive characteristic has appeared more recently in [26].

In analogy to this construction, sheets of primitive ideals were introduced and studied by W. Borho and A. Joseph in [3], in order to describe the set of primitive ideals in a universal enveloping algebra as a countable union of maximal varieties. More recently, Losev in [18] has introduced the notion of birational sheet in a semisimple Lie algebra, he has shown that birational sheets form a partition of the Lie algebra and has applied this result in order to establish a version of the orbit method for semisimple Lie algebras. Sheets were also used in [25] in order to parametrise the set of 1-dimensional representations of finite W-algebras, with some applications also to the theory of primitive ideals. Closures of sheets appear as associated varieties of affine vertex algebras, [1].

In characterisitc zero, several results on quotients S/G and $\overline{S}//G$, for a sheet S were addressed: Katsylo has shown in [15] that S/G has the structure of a quotient and is isomorphic to the quotient of an affine variety by the action of a finite group [15]; Borho has explicitly described the normalisation of $\overline{S}//G$ and Richardson, Broer, Douglass-Röhrle in [27, 6, 11] have provided the list of the quotients $\overline{S}//G$ that are normal.

Sheets for the conjugation action of G on itself were studied in [8] in the spirit of [4]. If G is semisimple, they are parametrized in terms of pairs consisting of a Levi subgroup of parabolic subgroups and a suitable isolated conjugacy class therein. Here isolated means that the connected centraliser of the semisimple part of a representative is semisimple. An alternative parametrisation can be given in terms of triples consisting of a pseudo-Levi subgroup M of G, a coset in $Z(M)/Z(M)^\circ$ and a suitable unipotent class in M. Pseudo-Levi subgroups are, in good characteristic, centralisers of semisimple elements and up to conjugation they are subroot subgroups whose root system has a base in the extended Dynkin diagram of G [22]. It is also shown in [7] that sheets in G are the irreducible components of the parts in Lusztig's partition introduced in [19], whose construction is given in terms of Springer's correspondence.

Also in the group case one wants to reach a good understanding of quotients of sheets. An analogue of Katsylo's theorem was obtained for sheets containing spherical conjugacy classes and all such sheets are shown to be smooth [9]. The proof in this case relies on specific properties of the intersection of spherical conjugacy classes with Bruhat double cosets, which do not hold for general classes. Therefore, a straightforward generalization to arbitrary sheets is not immediate. Even in absence of a Katsylo type theorem, it is of interest to understand the orbit space S/G. In this paper we address the question for G simple provided G is simply connected if the root system is of type C or D. We give a bijection between the orbit space S/G and a quotient of a shifted torus of the form $Z(M)^\circ s$ by the action of a subgroup $W(S)$ of the Weyl group, giving a group analogue of [17, Theorem 3.6],[2, Satz 5.6]. In most cases the group $W(S)$ does not depend on the unipotent part of the triple corresponding to the given sheet although it may depend on the isogeny type of G. This is one of the difficulties when passing from the Lie algebra case to the group case. The restriction on G needed for the bijection depends on the symmetry of the extended Dynkin diagram in this case: type C and D are the only two situations in which two distinct subsets of the extended Dynkin diagram can be equivalent even if they are not of type A. We illustrate by an example in $\mathrm{HSpin}_{10}(\mathbb{C})$ that the restriction we put is necessary in order to have injectivity so our theorem is somehow optimal.

We also address some questions related to the categorial quotient $\overline{S}//G$, for a sheet in G. We obtain group analogues of the description of the normalisation of $\overline{S}//G$ from [2] and of a necessary and sufficient condition on $\overline{S}//G$ to be normal from [27]. Finally we apply our results to compute the quotients S/G of all sheets in G of type G_2 and verify which of the quotients $\overline{S}//G$ are normal. This example will serve as a toy example for a forthcoming paper in which we will list all normal quotients for G simple.

2 Basic notions

In this paper G is a complex *simple* algebraic group with maximal torus T, root system Φ, weight lattice Λ, set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, Weyl group $W = N(T)/T$ and corresponding Borel subgroup B. The numbering of simple roots is as in [5]. Root subgroups are denoted by X_α for $\alpha \in \Phi$ and their elements have the form $x_\alpha(\xi)$ for $\xi \in \mathbb{C}$. Let $-\alpha_0$ be the highest root and let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. The centraliser of an element h in a closed group $H \leq G$ will be denoted by H^h and the identity component of H will be indicated by H°. If $\Pi \subset \tilde{\Delta}$ we set

$$G_\Pi := \langle T, X_{\pm\alpha} \mid \alpha \in \Pi \rangle.$$

Conjugates of such groups are called pseudo-Levi subgroups. We recall from [22, §6] that if $s \in T$ then its connected centraliser $G^{s\circ}$ is conjugated to G_Π for some Π by means of an element in $N(T)$. By [13, 2.2] we have $G^s = \langle G^{s\circ}, N(T)^s \rangle$. W_Π indicates the subgroup of W generated by the simple reflections with respect to roots in Π and it is the Weyl group of G_Π.

We realize the groups $\mathrm{Sp}_{2\ell}(\mathbb{C})$, $\mathrm{SO}_{2\ell}(\mathbb{C})$ and $\mathrm{SO}_{2\ell+1}(\mathbb{C})$, respectively, as the groups of matrices of determinant 1 preserving the bilinear forms: $\left(\begin{smallmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & I_\ell \\ I_\ell & 0 \end{smallmatrix}\right)$ and $\begin{pmatrix} 1 & & \\ & & I_\ell \\ & I_\ell & \end{pmatrix}$, respectively.

If G acts on a variety X, the action of $g \in G$ on $x \in X$ will be indicated by $(g, x) \mapsto g \cdot x$. If $X = G$ with adjoint action we thus have $g \cdot h = ghg^{-1}$. For $n \geq 0$ we shall denote by $X_{(n)}$ the union of orbits of dimension n. The nonempty sets $X_{(n)}$ are locally closed and a sheet S for the action of G on X is an irreducible component of any of these. For any $Y \subset X$ we set Y^{reg} to be the set of points of Y whose orbit has maximal dimension. We recall the parametrisation and description of sheets for the action of G on itself by conjugation and provide the necessary background material.

A Jordan class in G is an equivalence class with respect to the equivalence relation: $g, h \in G$ with Jordan decomposition $g = su$, $h = rv$ are equivalent if up to conjugation $G^{s\circ} = G^{r\circ}$, $r \in Z(G^{s\circ})^\circ s$ and $G^{s\circ} \cdot u = G^{s\circ} \cdot v$. As a set, the Jordan class of $g = su$ is thus $J(su) = G \cdot ((Z(G^{s\circ})^\circ s)^{reg} u)$ and it is contained in some $G_{(n)}$. Jordan classes are parametrised by G-conjugacy classes of triples $(M, Z(M)^\circ s, M \cdot u)$ where M is a pseudo-Levi subgroup, $Z(M)^\circ s$ is a coset in $Z(M)/Z(M)^\circ$ such that $(Z(M)^\circ s)^{reg} \subset Z(M)^{reg}$ and $M \cdot u$ is a unipotent conjugacy class in M. They are finitely many, locally closed, G-stable, smooth, see [20, 3.1] and [8, §4] for further details.

Every sheet $S \subset G$ contains a unique dense Jordan class, so sheets are parametrised by conjugacy classes of a subset of the triples above mentioned. More precisely, a Jordan class $J = J(su)$ is dense in a sheet if and only if it is not contained in $(\overline{J'})^{reg}$ for any Jordan class J' different from J. We recall from [8, Proposition 4.8] that

$$\overline{J(su)}^{reg} = \bigcup_{z \in Z(G^{s\circ})^\circ} G \cdot (s\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ} \cdot u)), \tag{2.1}$$

where $\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ} \cdot u)$ is Lusztig-Spaltenstein's induction from the Levi subgroup $G^{s\circ}$ of a parabolic subgroup of $G^{zs\circ}$ of the class of u in $G^{s\circ}$, see [21]. So, Jordan classes that are dense in a sheet correspond to triples where u is a rigid orbit in $G^{s\circ}$, i.e., such that its class in $G^{s\circ}$ is not induced from a conjugacy class in a proper Levi subgroup of a parabolic subgroup of $G^{s\circ}$.

A sheet consists of a single conjugacy class if and only if $\overline{S} = \overline{J(su)} = \overline{G \cdot su}$ where u is rigid in $G^{s\circ}$ and $G^{s\circ}$ is semisimple, i.e., if and only if s is isolated and u is rigid in $G^{s\circ}$. Any sheet S in G is the image through the isogeny map π of a sheet S' in the simply-connected cover G_{sc} of G, where S' is defined up to multiplication by an element in $\mathrm{Ker}(\pi)$. Also, $Z(G^{\pi(s)\circ}) = \pi(Z(G_{sc}^{s\circ}))$ and $Z(G^{\pi(s)\circ})^\circ = \pi(Z(G_{sc}^{s\circ})^\circ) = Z(G_{sc}^{s\circ})^\circ\mathrm{Ker}(\pi)$.

3 A parametrization of orbits in a sheet

In this section we parametrize the set S/G of conjugacy classes in a given sheet. Let $S = \overline{J(su)}^{reg}$ with $s \in T$ and $u \in U \cap G^{s\circ}$. Let $Z = Z(G^{s\circ})$ and $L = C_G(Z^\circ)$. The latter is always a Levi subgroup of a parabolic subgroup of G, [29, Proposition 8.4.5, Theorem 13.4.2] and if Ψ_s is the root system of $G^{s\circ}$ with respect to T, then L has root system $\Psi := \mathbb{Q}\Psi_s \cap \Phi$.

Let

$$W(S) = \{w \in W \mid w(Z^\circ s) = Z^\circ s\}. \tag{3.2}$$

We recall that $C_G(Z(G^{s\circ})^\circ s)^\circ = G^{s\circ}$. Thus, for any lift $\dot{w}$ of $w \in W(S)$ we have $\dot{w} \cdot G^{s\circ} = G^{s\circ}$, so $\dot{w} \cdot Z^\circ = Z^\circ$ and therefore $\dot{w} \cdot L = L$. Thus, any $w \in W(S)$ determines an automorphism of Ψ_s and Ψ. Let $\mathcal{O} = G^{s\circ} \cdot u$. We set:

$$W(S)^u = \{w \in W(S) \mid \dot{w} \cdot \mathcal{O} = \mathcal{O}\}. \tag{3.3}$$

The definition is independent of the choice of the representative of each w because $T \subset L$.

Lemma 3.1 *Let Ψ_s be the root system of $G^{s\circ}$ with respect to T, with basis $\Pi \subset \Delta \cup \{-\alpha_0\}$. Let W_Π be the Weyl group of $G^{s\circ}$ and let $W^\Pi = \{w \in W \mid w\Pi = \Pi\}$. Then*

$$W(S) = W_\Pi \rtimes (W^\Pi)_{Z^\circ s} = \{w \in W_\Pi W^\Pi \mid wZ^\circ s = Z^\circ s\}.$$

In particular, if $G^{s\circ}$ is a Levi subgroup of a parabolic subgroup of G, then $W(S) = W_\Pi \rtimes W^\Pi = N_W(W_\Pi)$ and it is independent of the isogeny class of G.

Proof. Let W_X denote the stabilizer of X in W for $X = Z^\circ s, G^{s\circ}, Z, Z^\circ$. We have the following chain of inclusions:

$$W(S) = W_{Z^\circ s} \leq W_{G^{s\circ}} \leq W_Z \leq W_{Z^\circ}.$$

We claim that $W_{G^{s\circ}} = W_\Pi \rtimes W^\Pi$. Indeed, $W_\Pi W^\Pi \leq W_{G^{s\circ}}$ is immediate and if $w \in W_{G^{s\circ}}$ then $w\Psi_s = \Psi_s$ and $w\Pi$ is a basis for Ψ_s. Hence, there is some $\sigma \in W_\Pi$ such that $\sigma w \in W^\Pi$. By construction W^Π normalises W_Π. The elements of $W_{G^{s\circ}}$ permute the connected components of $Z = Z(G^{s\circ})$ and $W_{Z^\circ s}$ is precisely the stabilizer of $Z^\circ s$ in there. Since the elements of W_Π fix the elements of $Z(G^{s\circ})$ pointwise, they stabilize $Z^\circ s$, whence the statement. The last statement follows from the equality $W_\Pi \ltimes W^\Pi = N_W(W_\Pi)$ in [12, Corollary3] and [22, Lemma 33] because in this case $Z^\circ s = zZ^\circ$ for some $z \in Z(G)$, so $W_{Z^\circ s} = W_{Z^\circ}$. □

Remark 3.2 *If $G^{s\circ}$ is not a Levi subgroup of a parabolic subgroup of G, then $W(S)$ might depend on the isogeny type of G. For instance, if Φ is of type C_5 and $s = \mathrm{diag}(-I_2, x, I_2, -I_2, x^{-1}, I_2) \in \mathrm{Sp}_{10}(\mathbb{C})$ for $x^2 \neq 1$, then:*

$$\begin{aligned}
&\Pi = \{\alpha_0, \alpha_1, \alpha_4, \alpha_5\}\\
&Z = Z(G^{s\circ}) = \{\mathrm{diag}(\epsilon I_2, y, \eta I_2, \epsilon I_2, y^{-1}, \eta I_2),\ y \in \mathbb{C}^*, \epsilon^2 = \eta^2 = 1\},\\
&Z^\circ s = \{\mathrm{diag}(-I_2, I_2, y, -I_2, I_2, y^{-1}),\ y \in \mathbb{C}^*\},
\end{aligned}$$

and $W^\Pi = \langle s_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} s_{\alpha_2+\alpha_3}\rangle$. Since $s_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} s_{\alpha_2+\alpha_3}(Z^\circ s) = -Z^\circ s$ we have $W(S) = W_\Pi$. However, if $\pi\colon \mathrm{Sp}_{10}(\mathbb{C}) \to \mathrm{PSp}_{10}(\mathbb{C})$ is the isogeny map, then W^Π preserves $\pi(Z^\circ s)$ so $W(\pi(S)) = W_\Pi \rtimes W^\Pi$. Taking $u = 1$ have an example in which also $W(S)^u$ depends on the isogeny type.

Table 1: Kernel of the isogeny map; Φ of type B_ℓ, C_ℓ or D_ℓ

type	parity of ℓ	group	$\mathrm{Ker}\pi$
B_ℓ	any	$\mathrm{SO}_{2\ell+1}(\mathbb{C})$	$\langle \alpha_\ell^\vee(-1)\rangle$
C_ℓ	any	$\mathrm{PSp}_{2\ell}(\mathbb{C})$	$\left\langle \prod_{j \text{ odd}} \alpha_j^\vee(-1)\right\rangle = \langle -I_{2\ell}\rangle$
D_ℓ	even	$\mathrm{PSO}_{2\ell}(\mathbb{C})$	$\left\langle \prod_{j \text{ odd}} \alpha_j^\vee(-1),\ \alpha_{\ell-1}^\vee(-1)\alpha_\ell^\vee(-1)\right\rangle$
D_ℓ	odd	$\mathrm{PSO}_{2\ell}(\mathbb{C})$	$\left\langle \prod_{j \text{ odd} \leq \ell-2} \alpha_j^\vee(-1)\alpha_{\ell-1}^\vee(i)\alpha_\ell^\vee(i^3)\right\rangle$
D_ℓ	any	$\mathrm{SO}_{2\ell}(\mathbb{C})$	$\langle \alpha_{\ell-1}^\vee(-1)\alpha_\ell^\vee(-1)\rangle$
D_ℓ	even	$\mathrm{HSpin}_{2\ell}(\mathbb{C})$	$\left\langle \prod_{j \text{ odd}} \alpha_j^\vee(-1))\right\rangle$

Next Lemma shows that in most cases $W(S)^u$ can be determined without any knowledge of u.

Lemma 3.3 *Suppose G and $S = \overline{J(su)}^{reg}$ are* **not** *in the following situation:*

"G is either $\mathrm{PSp}_{2\ell}(\mathbb{C})$, $\mathrm{HSpin}_{2\ell}(\mathbb{C})$, or $\mathrm{PSO}_{2\ell}(\mathbb{C})$;
$[G^{s\circ},\, G^{s\circ}]$ has two isomorphic simple factors G_1 and G_2 that are not of type A;
the components of u in G_1 and G_2 do not correspond to the same partition."

Then, $W(S) = W(S)^u$.

Proof. The element u is rigid in $[G^{s\circ}, G^{s\circ}] \leq G^{s\circ}$ and this happens if and only if each of its components in the corresponding simple factor of $[G^{s\circ}, G^{s\circ}]$ is rigid. Rigid unipotent elements in type A are trivial [28, Proposition 5.14], therefore what matters are only the components of u in the simple factors of type different from A. In addition, rigid unipotent classes are characteristic in simple groups, [2, Lemma 3.9, Korollar 3.10]. For all Φ different from C and D, simple factors that are not of type A are never isomorphic. Therefore the statement certainly holds in all cases with a possible exception when: Φ is of type C_ℓ or D_ℓ; $[G^{s\circ}, G^{s\circ}]$ has two isomorphic factors of type different from A; and the components of u in those two factors, that are of type C_m or D_m, respectively, correspond to different partitions.

Let us assume that we are in this situation. Then, $W(S) = W(S)^u$ if and only if the elements of $W(S)$, acting as automorphisms of Ψ_s, do not interchange the two isomorphic factors in question. We have 2 isogeny classes in type C_ℓ, 3 in type D_ℓ for ℓ odd, and 4 (up to isomorphism) in type D_ℓ for ℓ even.

If Φ is of type C_ℓ and $G = \mathrm{Sp}_{2\ell}(\mathbb{C})$ up to a central factor s can be chosen to be of the form:

$$s = \mathrm{diag}(I_m, t, -I_m, I_m, t^{-1}, -I_m) \tag{3.4}$$

where t is a diagonal matrix in $\mathrm{GL}_{\ell-2m}(\mathbb{C})$ with eigenvalues different from ± 1. Then Π is the union of $\{\alpha_0,\, \ldots,\, \alpha_{m-1}\}$, $\{\alpha_\ell, \alpha_{\ell-1}, \ldots,\, \alpha_{\ell-m+1}\}$ and possibly other subsets of simple roots orthogonal to these. Then W^Π is the direct product of terms permuting isomorphic components of type A with the subgroup generated by $\sigma = \prod_{j=1}^m s_{\alpha_j+\cdots+\alpha_{\ell-j}}$. In this case the elements of $Z^\circ s$ are of the form $\mathrm{diag}(I_m, r, -I_m, I_m, r^{-1}, -I_m)$, where r has the same shape as t and $\sigma(Z^\circ s) = -Z^\circ s$. Thus, W^Π does not permute the two factors of type C_m and $W(S) = W(S)^u$.

If, instead, $G = \mathrm{PSp}_{2\ell}(\mathbb{C})$ and the sheet is $\pi(S)$, we may take $J = J(\pi(su))$ where s is as in (3.4). Then, σ preserves $\pi(Z^\circ s)$ and therefore $W(\pi(S)) \neq W(\pi(S))^{\pi(u)}$.

Let now Φ be of type D_ℓ and $G = Spin_{2\ell}(\mathbb{C})$. With notation as in [29], we may take

$$s = \left(\prod_{j=1}^{m} \alpha_j^\vee(\epsilon^j)\right) \left(\prod_{i=m+1}^{l-m-1} \alpha_i^\vee(c_i)\right) \left(\prod_{b=2}^{m} \alpha_{\ell-b}^\vee(d^2\eta^b)\right) \alpha_{\ell-1}^\vee(\eta d)\alpha_\ell(d) \tag{3.5}$$

with $\epsilon^2 = \eta^2 = 1$, $\epsilon \neq \eta$, and d, $c_i \in \mathbb{C}^*$.

Here Π is the union of $\{\alpha_0, \ldots, \alpha_{m-1}\}$, $\{\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_{\ell-m+1}\}$ and possibly other subsets of simple roots orthogonal to these. Then W^Π is the direct product of terms permuting isomorphic components of type A and $\langle\sigma\rangle$ where $\sigma = \prod_{j=1}^m s_{\alpha_j+\cdots+\alpha_{\ell-j+1}}$. The coset $Z^\circ s = Z_{\epsilon,\eta}$ consists of elements of the same form as (3.5) with constant value of ϵ and η, and $Z^\circ = Z_{1,1}$ consists of the elements of similar shape with $\eta = \epsilon = 1$. Then $\sigma(Z_{\epsilon,\eta}) = Z_{\eta,\epsilon}$, hence $\sigma \notin W(S)$, so $W(S)$ preserves the components of Ψ_s of type D and $W(S) = W(S)^u$.

If $\ell = 2q$ and $G = \mathrm{HSpin}_{2\ell}(\mathbb{C})$ and $\pi\colon \mathrm{Spin}_{2\ell}(\mathbb{C}) \to \mathrm{HSpin}_{2\ell}(\mathbb{C})$ is the isogeny map we see from Table 1 that $\mathrm{Ker}(\pi)$ is generated by an element k such that $kZ_{\epsilon,\eta} = Z_{-\epsilon,\eta}$, so σ as above preserves $\pi(Z^\circ s)$ whereas it does not preserve the conjugacy class of $\pi(u)$. Therefore $\sigma \in W(\pi(S)) \neq W(\pi(S))^u$.

If $G = \mathrm{SO}_{2\ell}(\mathbb{C})$ and $\pi\colon \mathrm{Spin}_{2\ell}(\mathbb{C}) \to \mathrm{SO}_{2\ell}(\mathbb{C})$ is the isogeny map, then $\mathrm{Ker}(\pi)$ is generated by an element k such that $kZ_{\epsilon,\eta} = Z_{\epsilon,\eta}$. In this case σ does not preserve $\pi(Z^\circ s)$, whence $\sigma \notin W(\pi(S)) = W(\pi(S))^u$.

If $G = \mathrm{PSO}_{2\ell}(\mathbb{C})$ and $\pi\colon \mathrm{Spin}_{2\ell}(\mathbb{C}) \to \mathrm{PSO}_{2\ell}(\mathbb{C})$, then by the discussion of the previous isogeny types we see that $\sigma(Z_{\epsilon,\eta}) \subset \mathrm{Ker}(\pi)Z_{\epsilon,\eta}$, so σ preserves $\pi(Z^\circ s)$ whence $\sigma \in W(\pi(S)) \neq W(\pi(S))^u$. □

Following [2, §5] and according to [8, Proposition 4.7] we define the map

$$\begin{aligned} \theta\colon Z^\circ s &\to S/G \\ zs &\mapsto \mathrm{Ind}_L^G(L \cdot szu) \end{aligned}$$

where $L = C_G(Z(G^{s\circ})^\circ)$.

Lemma 3.4 *With the above notation, $\theta(zs) = \theta(w \cdot (zs))$ for every $w \in W(S)^u$.*

Proof. Let us observe that, since $z \in Z(L)$ and $G^{s\circ} \subset L$ there holds $L^{zs\circ} = G^{s\circ}$. In particular, $G^{s\circ}$ is a Levi subgroup of a parabolic subgroup of $G^{zs\circ}$. Let U_P be the unipotent radical of a parabolic subgroup of G with Levi factor L and let $\dot{w}$ be a representative of w in $N_G(T)$. By [8, Proposition 4.6] we have

$$\begin{aligned}
\mathrm{Ind}_L^G(L\cdot(w\cdot zs)u) &= G\cdot(w\cdot(zs)uU_P)^{reg}\\
&= G\cdot(zs(\dot{w}^{-1}\cdot u)U_{\dot{w}^{-1}.P})^{reg}\\
&= \mathrm{Ind}_L^G(L\cdot(zs(\dot{w}^{-1}\cdot u)))\\
&= G\cdot\big(zs\,\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(\dot{w}^{-1}\cdot(G^{s\circ}\cdot u))\big)\\
&= G\cdot\big(zs\,\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ}\cdot u)\big)\\
&= \mathrm{Ind}_L^G(L\cdot(zsu))
\end{aligned}$$

where we have used that $L=\dot{w}\cdot L$ for every $w\in W(S)^u\leq W(S)$ and independence of the choice of the parabolic subgroup with Levi factor L, [8, Proposition 4.5]. $\square$

Remark 3.5 *The requirement that w lies in $W(S)^u$ rather than in $W(S)$ is necessary. For instance, we consider $G=\mathrm{PSp}_{2\ell}(\mathbb{C})$ with $\ell=2m+1$ and s the class of $\mathrm{diag}(I_m,\lambda,-I_m,I_m,\lambda^{-1},-I_m)$ with $\lambda^4\neq 1$ and u rigid with non-trivial component only in the subgroup $H=\langle X_{\pm\alpha_j},\,j=0,\ldots m-1\rangle$ of $G^{s\circ}$. The element $\sigma=\prod_{j=1}^m s_{\alpha_j+\cdots+\alpha_{\ell-j}}$ lies in $W(S)\setminus W(S)^u$. Taking $\theta(s)$ we have*

$$\mathrm{Ind}_L^G(L\cdot su)=G\cdot su$$

whereas

$$\mathrm{Ind}_L^G(L\cdot w(s)u)=\mathrm{Ind}_L^G(L\cdot s(\dot{w}\cdot u))=G\cdot(s(\dot{w}\cdot u)),$$

where $\dot{w}$ is any representative of w in $N_G(T)$. These classes would coincide only if u and $\dot{w}\cdot u$ were conjugate in G^s. They are not conjugate in $G^{s\circ}$ because they lie in different simple components. Moreover, G^s is generated by $G^{s\circ}$ and the lifts of elements in the centraliser W^s of s in W [13, 2.2], which is contained in $W(S)$. Since $\lambda^4\neq 1$ we see that the elements of W^s cannot interchange the two components of type C_m in $G^{s\circ}$. Hence,

$$\theta(s)=\mathrm{Ind}_L^G(L\cdot su)\neq\mathrm{Ind}_L^G(L\cdot w(s)u)=\theta(w(s)).$$

In analogy with the Lie algebra case we formulate the following theorem. The proof follows the lines of [2, Satz 5.6] but a more detailed analysis is necessary because the naive generalization of statement [2, Lemma 5.4] from Levi subalgebras in a Levi subalgebra to Levi subgroups in a pseudo-Levi subgroup does not hold.

Theorem 3.6 *Assume G is simple and different from $\mathrm{PSO}_{2\ell}(\mathbb{C})$, $\mathrm{HSpin}_{2\ell}(\mathbb{C})$ and $\mathrm{PSp}_{2\ell}(\mathbb{C})$, $\ell \geq 5$. Let $S = \overline{J(su)}^{reg}$, with $s \in T$, $Z = Z(G^{s\circ})$ and let $W(S)$ be as in* (3.2). *The map θ induces a bijection $\overline{\theta}$ between $Z^\circ s/W(S)$ and S/G.*

Proof. Recall that under our assumptions Lemma 3.3 gives $W(S) = W(S)^u$. By Lemma 3.4, θ induces a well-defined map $\overline{\theta}\colon Z^\circ s/W(S) \to S/G$. It is surjective by [8, Proposition 4.7]. We prove injectivity.

Let us assume that $\theta(zs) = \theta(z's)$ for some $z, z' \in Z^\circ$. By construction, $Z^\circ \subset T$. By [8, Proposition 4.5] we have

$$G \cdot \left(zs\left(\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ}\cdot u)\right)\right) = G \cdot \left(z's\left(\mathrm{Ind}_{G^{s\circ}}^{G^{z's\circ}}(G^{s\circ}\cdot u)\right)\right).$$

This implies that $z's = \sigma\cdot(zs)$ for some $\sigma \in W$. Let $\dot{\sigma} \in N(T)$ be a representative of σ. Then

$$\begin{aligned}
\theta(zs) = \theta(z's) &= G \cdot \left((\sigma\cdot zs)(\mathrm{Ind}_{G^{s\circ}}^{G^{z's\circ}}(G^{s\circ}\cdot u))\right)\\
&= G \cdot \left(zs\left(\mathrm{Ind}_{\dot{\sigma}^{-1}\cdot(G^{s\circ})}^{\dot{\sigma}^{-1}\cdot G^{z's\circ}}\left(\dot{\sigma}^{-1}\cdot(G^{s\circ}\cdot u)\right)\right)\right)\\
&= G \cdot \left(zs\left(\mathrm{Ind}_{\dot{\sigma}^{-1}\cdot(G^{s\circ})}^{G^{zs\circ}}\left(\dot{\sigma}^{-1}\cdot(G^{s\circ}\cdot u)\right)\right)\right).
\end{aligned}$$

Since the unipotent parts of $\theta(zs)$ and $\theta(z's)$ coincide, for some $x \in G^{zs}$ we have

$$x \cdot (\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ}\cdot u)) = \mathrm{Ind}_{\dot{\sigma}^{-1}\cdot(G^{s\circ})}^{G^{zs\circ}}\left(\dot{\sigma}^{-1}\cdot(G^{s\circ}\cdot u)\right).$$

The element x may be written as $\dot{w}g$ for some $\dot{w} \in N(T)\cap G^{zs}$ and some $g \in G^{zs\circ}$ [13, §2.2]. Hence,

$$\begin{aligned}
\mathrm{Ind}_{G^{s\circ}}^{G^{zs\circ}}(G^{s\circ}\cdot u) &= \dot{w}^{-1}\cdot\left(\mathrm{Ind}_{\dot{\sigma}^{-1}\cdot(G^{s\circ})}^{G^{zs\circ}}\left(\dot{\sigma}^{-1}\cdot(G^{s\circ}\cdot u)\right)\right)\\
&= \mathrm{Ind}_{\dot{w}^{-1}\dot{\sigma}^{-1}\cdot(G^{s\circ})}^{G^{zs\circ}}\left((\dot{w}^{-1}\dot{\sigma}^{-1})\cdot(G^{s\circ}\cdot u)\right).
\end{aligned}$$

Let us put

$$M := G^{zs\circ} = \langle T, X_\alpha, \alpha \in \Phi_M\rangle, \quad L_1 := G^{s\circ} = \langle T, X_\alpha, \alpha\in\Psi\rangle$$

with $\Phi_M = \bigcup_{j=1}^{l}\Phi_j$ and $\Psi = \bigcup_{i=1}^{m}\Psi_i$ the decompositions in irreducible root subsystems. We recall that L_1 and $L_2 := (\dot{w}^{-1}\dot{\sigma}^{-1})\cdot L_1$ are Levi subgroups of some parabolic subgroups of M. We claim that if L_1 and L_2 are conjugate in M, then zs and $z's$ are $W(S)$-conjugate. Indeed, under this assumption, since L_1 and

L_2 contain T, there is $\dot{\tau} \in N_M(T)$ such that $L_1 = \dot{\tau} \cdot L_2 = \dot{\tau}\dot{w}^{-1}\dot{\sigma}^{-1} \cdot L_1$, so $\tau w^{-1}\sigma^{-1}(Z^\circ) = Z^\circ$. Then, $\tau w^{-1}\sigma^{-1}(z's) = zs$ and therefore

$$\tau w^{-1}\sigma^{-1}(Z^\circ s) = \tau w^{-1}\sigma^{-1}(Z^\circ z's) = Z^\circ zs = Z^\circ s.$$

Hence zs and $z's$ are $W(S)$-conjugate. By Lemma 3.3, we have the claim. We show that if Φ_M has at most one component different from type A, then L_1 is always conjugate to L_2 in M. We analyse two possibilities.

<u>Φ_j is of type A for every j.</u> In this case the same holds for Ψ_i and $u = 1$. We recall that in type A induction from the trivial orbit in a Levi subgroup corresponding to a partition λ yields the unipotent class corresponding to the dual partition [28, 7.1]. Hence, equivalence of the induced orbits in each simple factor M_i of M forces $\Phi_j \cap \Psi \cong \Phi_j \cap w^{-1}\sigma^{-1}\Psi$ for every j. Invoking [2, Lemma 5.5], in each component M_i we deduce that L_1 and L_2 are M-conjugate.
<u>There is exactly one component in Φ_M which is not of type A.</u> We set it to be Φ_1. Then, there is at most one Ψ_j, say Ψ_1, which is not of type A, and $\Psi_1 \subset \Phi_1$. In this case, $w^{-1}\sigma^{-1}\Phi_1 \subset \Psi_1$. Equivalence of the induced orbits in each simple factor M_j of M forces $\Phi_j \cap \Psi \cong \Phi_j \cap w^{-1}\sigma^{-1}\Psi$ for every $j > 1$. By exclusion, the same isomorphism holds for $j = 1$. Invoking once more [2, Lemma 5.5] for each simple component, we deduce that L_1 and L_2 are M-conjugate.

Assume now that there are exactly two components of Φ_M which are not of type A. This situation can only occur if Φ is of type B_ℓ for $\ell \geq 6$, C_ℓ for $\ell \geq 4$ or D_ℓ for $\ell \geq 8$ (we recall that $D_2 = A_1 \times A_1$ and $D_3 = A_3$). By a case-by-case analysis we directly show that σ can be taken in $W(S)$.

If $G = \mathrm{Sp}_{2\ell}(\mathbb{C})$ we may assume that

$$s = \mathrm{diag}(I_m, t, -I_p, I_m, t^{-1}, -I_p)$$

with p, $m \geq 2$ and t a diagonal matrix with eigenvalues different from 0 and ± 1. Then $Z^\circ s$ consists of matrices in this form, so zs and $z's$ are of the form $zs = \mathrm{diag}(I_m, h, -I_p, I_m, h^{-1}, -I_p)$ and $z's = \mathrm{diag}(I_m, g, -I_p, I_m, g^{-1}, -I_p)$, where h and g are invertible diagonal matrices. The elements zs and $z's$ are conjugate in G if and only if $\mathrm{diag}(h,\, h^{-1})$ and $\mathrm{diag}(g,\, g^{-1})$ are conjugate in $G' = \mathrm{Sp}_{2(\ell-p-m)}(\mathbb{C})$. This is the case if and only if they are conjugate in the normaliser of the torus $T' = G' \cap T$. The natural embedding $G' \to G$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} I_m & & & & \\ & A & & B & \\ & & I_{p+m} & & \\ & C & & D & \\ & & & & I_p \end{pmatrix}$$

gives an embedding of $N_{G'}(T') \leq N_G(T)$ whose image lies in $W(S)$. Hence, zs and $z's$ are necessarily $W(S)$-conjugate. This concludes the proof of injectivity for $G = \mathrm{Sp}_{2\ell}(\mathbb{C})$.

If $G = \mathrm{Spin}_{2\ell+1}(\mathbb{C})$, then we may assume that

$$s = \left(\prod_{j=1}^{m} \alpha_j^\vee((-1)^j)\right)\left(\prod_{b=m+1}^{\ell-p-1} \alpha_b^\vee(c_b)\right)\left(\prod_{q=1}^{p} \alpha_{\ell-q}^\vee(c^2)\right)\alpha_\ell^\vee(c)$$

where $m \geq 4$, $p \geq 2$, c, $c_b \in \mathbb{C}^*$ are generic. Here $Z^\circ s$ consists of elements of the form

$$\left(\prod_{j=1}^{m} \alpha_j^\vee((-1)^j)\right)\left(\prod_{b=m+1}^{\ell-p-1} \alpha_b^\vee(d_b)\right)\left(\prod_{q=1}^{p} \alpha_{\ell-q}^\vee(d^2)\right)\alpha_\ell^\vee(d)$$

with d_b, $d \in \mathbb{C}^*$. The reflection $s_{\alpha_1+\cdots+\alpha_\ell} = s_{\varepsilon_1}$ maps any $y \in Z^\circ s$ to $y\alpha_\ell^\vee(-1) \in Z(G)Z^\circ s = Z^\circ s$.

Let us consider the natural isogeny $\pi\colon G \to G_{ad} = \mathrm{SO}_{2\ell+1}(\mathbb{C})$. Then

$$\pi(s) = \mathrm{diag}(1, -I_m, t, I_p, -I_m, t^{-1}, I_p)$$

where t is a diagonal matrix with eigenvalues different from 0 and ± 1. A similar calculation as in the case of $\mathrm{Sp}_{2\ell}(\mathbb{C})$ shows that $\pi(zs)$ is conjugate to $\pi(z's)$ by an element $\sigma_1 \in W(\pi(S)) = W(\pi(S))^u$. Then, $\sigma_1(zs) = kz's$, where $k \in Z(G)$. If $k = 1$, then we set $\sigma = \sigma_1$ whereas if $k = \alpha_\ell^\vee(-1)$ we set $\sigma = s_{\alpha_1+\cdots+\alpha_\ell}\sigma_1$. Then $\sigma(zs) = z's$ and $\sigma(Z^\circ s) = Z(G)Z^\circ s = Z^\circ s$. This concludes the proof for $\mathrm{Spin}_{2\ell+1}(\mathbb{C})$ and $\mathrm{SO}_{2\ell+1}(\mathbb{C})$.

If $G = \mathrm{Spin}_{2\ell}(\mathbb{C})$, up to multiplication by a central element we may assume that

$$s = \left(\prod_{j=m+1}^{\ell-p-1} \alpha_j^\vee(c_j)\right)\left(\prod_{q=2}^{p} \alpha_{\ell-q}^\vee((-1)^q c^2)\right)\alpha_{\ell-1}^\vee(-c)\alpha_\ell^\vee(c)$$

where $m,\, p \geq 4$, $c,\, c_j \in \mathbb{C}^*$ are generic. The elements in $Z^\circ s$ are of the form

$$\left(\prod_{j=m+1}^{\ell-p-1} \alpha_j^\vee(d_j)\right)\left(\prod_{q=2}^{p} \alpha_{\ell-q}^\vee((-1)^q d^2)\right)\alpha_{\ell-1}^\vee(-d)\alpha_\ell^\vee(d)$$

with $d_j,\, d \in \mathbb{C}^*$. We argue as we did for type B_ℓ, considering the isogeny $\pi\colon G \to \mathrm{SO}_{2\ell}(\mathbb{C})$.The Weyl group element $s_{\alpha_\ell}s_{\alpha_{\ell-1}}$ maps any $y \in Z^\circ s$ to $y\alpha_{\ell-1}^\vee(-1)\alpha_\ell^\vee(-1) \in \mathrm{Ker}(\pi)Z^\circ s = Z^\circ s$. The group $\pi(Z^\circ s)$ consists of elements of the form

$$\mathrm{diag}(I_m, t, -I_p, I_m, t^{-1}, -I_p)$$

where t is a diagonal matrix in $\mathrm{GL}_{2(\ell-m-p)}(\mathbb{C})$. Two elements

$$\begin{aligned}\pi(zs) &= \mathrm{diag}(I_m, h, -I_p, I_m, h^{-1}, -I_p),\\ \pi(z's) &= \mathrm{diag}(I_m, g, -I_p, I_m, g^{-1}, -I_p)\end{aligned}$$

therein are W-conjugate if and only if $\mathrm{diag}(1, h, 1, h^{-1})$ and $(1, g, 1, g^{-1})$ are conjugate by an element σ_1 of the Weyl group W' of $G' = \mathrm{SO}_{2(\ell-m-p+1)}(\mathbb{C})$. More precisely, even if h and g may have eigenvalues equal to 1, we may choose σ_1 in the subgroup of W' that either fixes the first and the $(\ell - m - p + 2)$-th eigenvalues or interchanges them. Considering the natural embedding of G' into $\mathrm{SO}_{2\ell}(\mathbb{C})$ in a similar fashion as we did for $\mathrm{SO}_{2\ell}(\mathbb{C})$, we show that $\sigma_1 \in W(\pi(S))$. This proves injectivity for $\mathrm{SO}_{2\ell}(\mathbb{C})$. Arguing as we did for $\mathrm{Spin}_{2\ell+1}(\mathbb{C})$ using $s_{\alpha_\ell}s_{\alpha_{\ell-1}}$ concludes the proof of injectivity for $\mathrm{Spin}_{2\ell}(\mathbb{C})$. $\square$

The translation isomorphism $Z^\circ s \to Z^\circ$ determines a $W(S)$-equivariant map where Z° is endowed with the action $w \bullet z = (w \cdot zs)s^{-1}$, which is in general not an action by automorphisms on Z°. Hence, S/G is in bijection with the quotient $Z^\circ/W(S)$ of the torus Z° where the quotient is with respect to the $\bullet$ action.

Remark 3.7 Injectivity of $\overline{\theta}$ does not necessarily hold for the adjoint groups $G = \mathrm{PSp}_{2\ell}(\mathbb{C})$, $\mathrm{PSO}_{2\ell}(\mathbb{C})$ and for $G = \mathrm{HSpin}_{2\ell}(\mathbb{C})$. We give an example for $G = \mathrm{HSpin}_{20}(\mathbb{C})$, in which $W(S) = W(S)^u$ and $G^{s\circ}$ is a Levi subgroup of a parabolic subgroup of G. Let $\pi\colon \mathrm{Spin}_{20}(\mathbb{C}) \to G$ be the central isogeny with kernel K as in Table 1. Let $u = 1$ and

$$s = \alpha_1^\vee(a)\alpha_2^\vee(a^2)\alpha_3^\vee(a^3)\alpha_4^\vee(b)\alpha_5^\vee(c)\alpha_6^\vee(d^{-2}e^2)\alpha_7^\vee(e)\alpha_8^\vee(d^2)\alpha_9^\vee(d)\alpha_{10}^\vee(-d)K$$

with $a,\ b,\ c,\ d,\ e \in \mathbb{C}^*$ sufficiently generic. Then, $G^{s\circ}$ is generated by T and the root subgroups of the subsystem with basis indexed by the following subset of the extended Dynkin diagram:

$$\begin{array}{ccccccccccccccccc} \bullet & - & \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \bullet & - & \circ & - & \bullet \\ & & | & & & & & & & & & & & & | & & \\ & & \circ & & & & & & & & & & & & \bullet & & \end{array}$$

Here Z° is given by elements of shape:

$$\alpha_1^\vee(a_1)\alpha_2^\vee(a_1^2)\alpha_3^\vee(a_1^3)\alpha_4^\vee(b_1)\alpha_5^\vee(c_1)\alpha_6^\vee(d_1^{-2}e_1^2)\alpha_7^\vee(e_1)\alpha_8^\vee(d_1^2)\alpha_9^\vee(d_1)\alpha_{10}^\vee(-d_1)K$$

with $a_1,\ b_1,\ c_1,\ d_1,\ e_1 \in \mathbb{C}^*$. Let

$$zs = \alpha_5^\vee(c)\alpha_6^\vee(d^2)\alpha_7^\vee(-d^2)\alpha_8^\vee(d^2)\alpha_9^\vee(d)\alpha_{10}^\vee(-d)K \in Z^\circ sK$$

obtained by setting $a_1 = b_1 = 1$, $c_1 = c$, $d_1 = d$ and $e_1 = -d^2$, and

$$z's = \alpha_5^\vee(-c)\alpha_6^\vee(d^2)\alpha_7^\vee(-d^2)\alpha_8^\vee(d^2)\alpha_9^\vee(d)\alpha_{10}^\vee(-d)K \in Z^\circ sK,$$

obtained by setting $a_1 = b_1 = 1$, $c_1 = -c$, $d_1 = d$ and $e_1 = -d^2$. The subgroup $M := G^{zs\circ} = G^{z's\circ}$ is generated by T and the root subgroups of the subsystem with basis indexed by the following subset of the extended Dynkin diagram:

$$\begin{array}{ccccccccccccccccc} \bullet & - & \bullet & - & \bullet & - & \circ & - & \circ & - & \circ & - & \bullet & - & \bullet & - & \bullet \\ & & | & & & & & & & & & & & & | & & \\ & & \bullet & & & & & & & & & & & & \bullet & & \end{array}$$

For $\sigma = \prod_{j=1}^4 s_{\alpha_j+\cdots+\alpha_{10-j}}$ we have $\sigma \cdot zs = z's$. We claim that zs and $z's$ are not $W(S)$-conjugate. Equivalently, we show that $\sigma W^{zsK} \cap W(S) = \emptyset$, where W^{szK} is the stabiliser of zs in W. Let σw be an element lying in such an intersection. We observe that if $\sigma w \in W(S)$, then $\sigma w(G^{s\circ}) = G^{s\circ}$ hence σw cannot interchange the component of type $3A_1$ with the component of type A_2 therein. Thus, it cannot interchange the two components of type D_4 in M. However, by looking at the projection π' onto $G/Z(G) = \mathrm{PSO}_{10}(\mathbb{C})$, we see that $zsZ(G)$ is the class of the matrix

$$\mathrm{diag}(I_4, c, c^{-1}d^2, -I_4, I_4, d^{-2}c, c^{-1}, -I_4)$$

which cannot be centralized by a Weyl group element interchanging these two factors if c and d are sufficiently generic. A fortiori, this cannot happen for the class zsK. Hence, zs and $z's$ are not $W(S)$-conjugate.

Let now M_1 and M_2 be the simple factors of M corresponding respectively to the roots $\{\alpha_j,\ 0 \leq j \leq 3\}$, and $\{\alpha_k,\ 7 \leq k \leq 10\}$, let $L_1 = M_1 \cap G^{s\circ}$ and $L_2 = M_2 \cap G^{s\circ}$. Then,

$$\theta(zs) = \mathrm{Ind}_L^G(L \cdot zs) = G \cdot \left(zs(\mathrm{Ind}_{G^{s\circ}}^M(1))\right) = G \cdot (zs(\mathrm{Ind}_{L_1}^{M_1}(1))(\mathrm{Ind}_{L_2}^{M_2}(1)))$$

and

$$\theta(z's) = \mathrm{Ind}_L^G(L \cdot z's) = G \cdot \left(z's(\mathrm{Ind}_{G^{s\circ}}^M(1)\right) = G \cdot (z's(\mathrm{Ind}_{L_1}^{M_1}(1))(\mathrm{Ind}_{L_2}^{M_2}(1))).$$

Since $\sigma(zs) = z's$ we have, for some representative $\dot\sigma \in N(T)$:

$$\begin{aligned}\theta(z's) \ &= G \cdot \left(zs(\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_1}^{\dot\sigma^{-1}\cdot M_1}(1))(\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_2}^{\dot\sigma^{-1}\cdot M_2}(1)))\right)\\ &= G \cdot \left(zs(\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_1}^{M_2}(1))(\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_2}^{M_1}(1)))\right).\end{aligned}$$

By [23, Example 3.1] we have $\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_1}^{M_2}(1) = \mathrm{Ind}_{L_1}^{M_2}(1)$ and $\mathrm{Ind}_{\dot\sigma^{-1}\cdot L_2}^{M_1}(1) = \mathrm{Ind}_{L_1}^{M_1}(1)$ so $\theta(zs) = \theta(z's)$.

Remark 3.8 *The parametrisation in Theorem 3.6 cannot be directly generalised to arbitrary Jordan classes. Indeed, if $u \in L$ is not rigid, then $L \cdot u$ is not necessarily characteristic and it may happen that for some external automorphism τ of L, the class $\tau(L \cdot u)$ differs from $L \cdot u$ even if they induce the same G-orbit. Then the map $\overline{\theta}$ is not necessarily injective.*

4 The quotient $\overline{S}//G$

In this section we discuss some properties of the categorical quotient $\overline{S}//G = \mathrm{Spec}(\mathbb{C}[\overline{S}])^G$ for G simple in any isogeny class. Since $\overline{S}//G$ parametrises only semisimple conjugacy classes it is enough to look at the so-called Dixmier sheets, i.e., the sheets containing a dense Jordan class consisting of semisimple elements. In addition, since every such Jordan class is dense in some sheet, studying the collection of $\overline{S}//G$ for S a sheet in G is the same as studying the collection of $\overline{J(s)}//G$ for $J(s)$ a semisimple Jordan class in G.

The following Theorem is a group version of [2, Satz 6.3], [17, Theorem 3.6(c)] and [27, Theorem A].

Theorem 4.1 *Let $S = \overline{J(s)}^{reg} \subset G$.*

1. *The normalisation of $\overline{S}//G$ is $Z(G^{s\circ})^\circ s/W(S)$.*

2. *The variety $\overline{S}//G$ is normal if and only if the natural map*

$$\rho\colon \mathbb{C}[T]^W \to \mathbb{C}[Z(G^{s\circ})^\circ s]^{W(S)} \tag{4.6}$$

induced from the restriction map $\mathbb{C}[T] \to \mathbb{C}[Z(G^{s\circ})^\circ s]$ is surjective.

Proof. 1. The variety $Z(G^{s\circ})^\circ s/W(S)$ is the quotient of a smooth variety (a shifted torus) by the action of a finite group, hence it is normal. Every class in $\overline{J(s)}$ meets T and $T \cap \overline{J(s)} = W \cdot (Z(G^{s\circ})^\circ s)$. Also, two elements in T are G-conjugate if and only if they are W-conjugate, hence we have an isomorphism $\overline{J(s)}//G \simeq W \cdot (Z(G^{s\circ})^\circ s)/W$ induced from the isomorphism $G//G \simeq T/W$.

We consider the morphism $\gamma\colon Z(G^{s\circ})^\circ s/W(S) \to W \cdot (Z(G^{s\circ})^\circ s)/W$ induced by $zs \mapsto W \cdot (zs)$. It is surjective by construction, bijective on the dense subset $(Z(G^{s\circ})^\circ s)^{reg}/W(S)$ and finite, since the intersection of $W \cdot (zs)$ with $Z(G^{s\circ})^\circ s$ is finite. Hence γ is a normalisation morphism.

2. The variety $\overline{S}//G$ is normal if and only if the normalisation morphism is an isomorphism. This happens if and only if the composition

$$Z(G^{s\circ})^\circ s/W(S) \simeq \overline{S}//G \subseteq G//G \simeq T/W$$

is a closed embedding, i.e., if and only if the corresponding algebra map between the rings of regular functions is surjective. $\square$

5 An example: sheets and their quotients in type G_2

We list here the sheets in G of type G_2 and all the conjugacy classes they contain. We shall denote by α and β, respectively, the short and the long simple roots. Since G is adjoint, by [7, Theorem 4.1] the sheets in G are in bijection with G-conjugacy classes of pairs (M, u) where M is a pseudo-Levi subgroup of G and u is a rigid unipotent element in M. The corresponding sheet is $\overline{J(su)}^{reg}$ where s is a semisimple element whose connected centralizer is M. The conjugacy classes of pseudo-Levi subgroups of G are those corresponding to the following subsets Π of the extended Dynkin diagram:

1. $\Pi = \emptyset$, so $M = T$, $u = 1$, s is a regular semisimple element and S consists of all regular conjugacy classes;

2. $\Pi = \{\alpha\}$. Here $[M, M]$ is of type $\tilde{A}_1$, so $u = 1$ and $s = \alpha^\vee(\zeta)\beta^\vee(t^2) = (3\alpha + 2\beta)^\vee(\zeta^{-1})$ for $\zeta \neq 0, \pm 1$;

3. $\Pi = \{\beta\}$. Here $[M, M]$ is of type A_1 so $u = 1$ and $s = \alpha^\vee(\zeta^2)\beta^\vee(\zeta^3) = (2\alpha + \beta)^\vee(\zeta)$ for $\zeta \neq 0,\ 1\, e^{2\pi i/3},\ e^{-2\pi i/3}$;

4. $\Pi = \{\alpha_0, \beta\}$. Here $[M, M]$ is of type A_2 so $u = 1$; the corresponding $s = (2\alpha + \beta)^\vee(e^{2\pi i/3})$ is isolated and $S = G \cdot s$;

5. $\Pi = \{\alpha_0, \alpha\}$. Here $[M, M]$ is of type $\tilde{A}_1 \times A_1$ so $u = 1$, the corresponding $s = (3\alpha + 2\beta)^\vee(-1)$ is isolated and $S = G \cdot s$;

6. $\Pi = \{\alpha,\ \beta\}$ so $L = G$. In this case we have three possible choices for u rigid unipotent, namely 1, $x_\alpha(1)$ or $x_\beta(1)$ (cfr. [28]). Each of these classes is a sheet on its own.

The only sheets containing more than one conjugacy classes are the regular one $S_0 = G^{reg}$ corresponding to $\Pi = \emptyset$ and the two subregular ones, corresponding to $\Pi_1 = \{\alpha\}$ and $\Pi_2 = \{\beta\}$. For S_0 we have $Z^\circ s = T$, $W(S) = W$ so S_0/G is in bijection with T/W and $\overline{S_0}//G \simeq G//G$ which is normal. For S_1 and S_2 we have:

$$\begin{aligned}
S_1 &= \overline{J((3\alpha+2\beta)^\vee(\zeta_0))}^{reg} \\
&= \left(\bigcup\nolimits_{\zeta^2\neq 0,1} G\cdot(3\alpha+2\beta)^\vee(\zeta)\right) \cup \mathrm{Ind}_{\tilde{A}_1}^G(1) \cup G\cdot\left((3\alpha+2\beta)^\vee(-1)\mathrm{Ind}_{\tilde{A}_1}^{A_1\times\tilde{A}_1}(1)\right) \\
&= \left(\bigcup\nolimits_{\zeta^2\neq 0,1} G\cdot(3\alpha+2\beta)^\vee(\zeta)\right) \cup G\cdot((x_\beta(1)x_{\alpha_0}(1)) \cup G\cdot(3\alpha+2\beta)^\vee(-1)x_{\alpha_0}(1))
\end{aligned}$$

for $\zeta_0 \neq 0, \pm 1$ and

$$\begin{aligned}
S_2 &= \overline{J((2\alpha+\beta)^\vee(\xi_0))}^{reg} \\
&= \left(\bigcup\nolimits_{\xi^3\neq 0,1} G\cdot(2\alpha+\beta)^\vee(\xi)\right) \cup \mathrm{Ind}_{A_1}^G(1) \cup G\cdot\left((2\alpha+\beta)^\vee(e^{2\pi i/3})\mathrm{Ind}_{A_1}^{A_2}(1)\right) \\
&= \left(\bigcup\nolimits_{\xi^3\neq 0,1} G\cdot(2\alpha+\beta)^\vee(\xi)\right) \cup G\cdot(x_\beta(1)x_{\alpha_0}(1)) \cup G\cdot\left((2\alpha+\beta)^\vee(e^{2\pi i/3})x_{\alpha_0}(1)\right)
\end{aligned}$$

for some $\xi_0 \neq 0, 1, e^{\pm 2\pi i/3}$.

In both cases M is a Levi subgroup of a parabolic subgroup of G. By Lemmata 3.1 and 3.3 we have $W(S_1) = W(S_1)^u = \langle s_\alpha, s_{3\alpha+2\beta}\rangle$ and $W(S_2) = W(S_1)^u = \langle s_\beta, s_{2\alpha+\beta}\rangle$. Also $Z(M)^\circ = Z(M)^\circ s$ in both cases, so

$$\begin{aligned}
S_1/G &\simeq (3\alpha+2\beta)^\vee(\mathbb{C}^\times)/\langle s_\alpha, s_{3\alpha+2\beta}\rangle \simeq (3\alpha+2\beta)^\vee(\mathbb{C}^\times)/\langle s_{3\alpha+2\beta}\rangle \\
S_2/G &\simeq (2\alpha+\beta)^\vee(\mathbb{C}^\times)/\langle s_\beta, s_{2\alpha+\beta}\rangle \simeq (2\alpha+\beta)^\vee(\mathbb{C}^\times)/\langle s_{2\alpha+\beta}\rangle,
\end{aligned}$$

where the $\simeq$ symbols stand for the bijection $\overline{\theta}$.

Let us analyze normality of $\overline{S_1}//G$. Here, $Z(M)^\circ = (3\alpha+2\beta)^\vee(\mathbb{C}^*) \simeq \mathbb{C}^*$, so $\mathbb{C}[Z(M)^\circ]^{W(S)} = \mathbb{C}[\zeta+\zeta^{-1}]$. On the other hand, since G is simply connected, $\mathbb{C}[T]^W = (\mathbb{C}\Lambda)^W$ is the polynomial algebra generated by $f_1 = \sum_{\substack{\gamma\in\Phi \\ \gamma \text{ short}}} e^\gamma$ and $f_2 = \sum_{\substack{\gamma\in\Phi \\ \gamma \text{ long}}} e^\gamma$, [5, Ch.VI, §4, Théorème 1] Then,

$$\rho(f_1)((3\alpha+2\beta)^\vee(\zeta)) = f_1((3\alpha+2\beta)^\vee(\zeta)) = \sum_{\substack{\gamma\in\Phi \\ \gamma \text{ short}}} \zeta^{(\gamma,(3\alpha+2\beta)^\vee)} = 2+2\zeta+2\zeta^{-1}$$

so the restriction map is surjective and $\overline{S_1}//G$ is normal.

Let us consider normality of $\overline{S_2}//G$. Here, $Z(M)^\circ = (2\alpha+\beta)^\vee(\mathbb{C}^*) \simeq \mathbb{C}^*$, so $\mathbb{C}[Z]^\Gamma = \mathbb{C}[\zeta+\zeta^{-1}]$. Then,

$$\rho(f_1)(2\alpha+\beta)^\vee(\zeta) = f_1((2\alpha+\beta)^\vee(\zeta)) = \sum_{\substack{\gamma\in\Phi \\ \gamma \text{ short}}} \zeta^{(\gamma,(2\alpha+\beta)^\vee)} = \zeta^2+\zeta^{-2}+2(\zeta+\zeta^{-1})$$

whereas

$$\rho(f_2)(2\alpha+\beta)^\vee(\zeta) = f_2((2\alpha+\beta)^\vee(\zeta)) = \sum_{\substack{\gamma\in\Phi \\ \gamma \text{ long}}} \zeta^{(\gamma,(2\alpha+\beta)^\vee)} = 2+2\zeta^3+2\zeta^{-3}.$$

Let us write $y = \zeta+\zeta^{-1}$. Then, $(\zeta^2+\zeta^{-2}) = y^2-2$ and $\zeta^3+\zeta^{-3} = y^3-3y$ so $\mathrm{Im}(\rho) = \mathbb{C}[y^2+2y, y^3-3y] = \mathbb{C}[(y+1)^2, y^3+3y^2+6y+3-3y] = \mathbb{C}[(y+1)^2,(y+1)^3]$. Hence, ρ is not surjective and $\overline{S_2}//G$ is not normal.

We observe that $\mathrm{Im}(\rho)$ is precisely the identification of the coordinate ring of $\overline{S_2}//G$ in $\mathbb{C}[T]^W$. We may thus see where this variety is not normal. We have: $\mathrm{Im}(\rho) = \mathbb{C}[(y+1)^2,(y+1)^3] \cong \mathbb{C}[Y,Z]/(Y^3-Z^2)$ so this variety is not normal at $y+1=0$, that is, for $\zeta+\zeta^{-1}+1=0$. This corresponds precisely to the closed, isolated orbit $G\cdot((2\alpha+\beta)^\vee(e^{2\pi i/3}))x_{\alpha_0}(1) = G\cdot((2\alpha+\beta)^\vee(e^{-2\pi i/3}))x_{\alpha_0}(1)$. This example shows two phenomena: the first is that even if the sheet corresponsing to the set Π_2 in $\mathrm{Lie}(G)$ has a normal quotient [6, Theorem 3.1], the same does not hold in the group counterpart. The second phenomenon is that the non-normality locus corresponds to an isolated class in $\overline{S}_2$. In a forthcoming paper we will address the general problem of normality of $\overline{S}//G$ and we will prove and make use of the fact that if the categorical quotient of the closure a sheet in G is not normal, then it is certainly not normal at some isolated class.

References

[1] T. ARAKAWA, A. MOREAU, *Sheets and associated varieties of affine vertex algebras,* Adv. Math. 320, , 157–209, (2017).

[2] W. BORHO, *Über Schichten halbeinfacher Lie-Algebren,* Invent. Math., 65, 283–317 (1981/82).

[3] W. BORHO, A. JOSEPH, *Sheets and topology of primitive spectra for semisimple Lie algebras,* J. Algebra 244, 76–167, (2001). Corrigendum 259, 310–311 (2003).

[4] W. BORHO, H. KRAFT, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen,* Comment. Math. Helvetici, 54, 61–104 (1979).

[5] N. BOURBAKI, *Éléments de Mathématique. Groupes et Algèbres de Lie, Chapitres 4,5, et 6,* Masson, Paris (1981).

[6] A. BROER, *Decomposition varieties in semisimple Lie algebras,* Can. J. Math. 50(5), 929–971 (1998).

[7] G. CARNOVALE, *Lusztig's partition and sheets, with an appendix by M. Bulois,* Mathematical Research Letters, 22(3), 645-664, (2015).

[8] G. CARNOVALE, F. ESPOSITO, *On sheets of conjugacy classes in good characteristic,* IMRN 2012(4) , 810–828, (2012).

[9] G. CARNOVALE, F. ESPOSITO, *A Katsylo theorem for sheets of spherical conjugacy classes,* Representation Theory, 19, 263-280 (2015).

[10] J. DIXMIER, *Polarisations dans les algèbres de Lie semi-simples complexes* Bull. Sci. Math. 99, 45-63, (1975).

[11] J.M. DOUGLASS, G. RÖHRLE, *Invariants of reflection groups, arrangements, and normality of decomposition classes in Lie algebras,* Compos. Math. 148, 921–930, (2012) .

[12] R. B. HOWLETT, *Normalizers of parabolic subgroups of reflection groups,* J. London Math. Soc. 21, 62–80 (1980).

[13] J. HUMPHREYS, *Conjugacy Classes in Semisimple Algebraic Groups,* AMS, Providence, Rhode Island (1995).

[14] A. E. IM HOF, *The sheets in a classical Lie algebra,* PhD thesis, Basel, http://edoc.unibas.ch/257/ (2005).

[15] P.I. KATSYLO, *Sections of sheets in a reductive algebraic Lie algebra,* Math. USSR Izvestyia 20(3), 449–458 (1983).

[16] B. KOSTANT, *Lie group representations on polynomial rings,* Amer. J. Math. 85, 327-404 (1963).

[17] H. KRAFT, *Parametrisierung von Konjugationsklassen in sln,* Math. Ann. 234, 209-220 (1978).

[18] I. LOSEV, *Deformation of symplectic singularities and orbit method for semisimple Lie algebras,* arXiv:1605.00592v1, (2016).

[19] G. LUSZTIG, *On conjugacy classes in a reductive group.* In: Representations of reductive groups, 333–363, Progr. Math., 312, Birkhäuser/Springer (2015).

[20] G. LUSZTIG, *Intersection cohomology complexes on a reductive group,* Invent. Math. 75, 205–272 (1984).

[21] G. LUSZTIG, N. SPALTENSTEIN, *Induced unipotent classes,* J. London Math. Soc. (2), 19, 41–52 (1979).

[22] G. MCNINCH, E. SOMMERS, *Component groups of unipotent centralizers in good characteristic,* J. Algebra 270(1), 288–306 (2003).

[23] A. MOREAU, *Corrigendum to: On the dimension of the sheets of a reductive Lie algebra,* Journal of Lie Theory 23(4), 1075–1083 (2013).

[24] D. PETERSON, *Geometry of the Adjoint Representation of a Complex Semisimple Lie Algebra* PhD Thesis, Harvard University, Cambridge, Massachussetts, (1978).

[25] A. PREMET, L. TOPLEY, *Derived subalgebras of centralisers and finite W-algebras,* Compos. Math. 150(9),1485–1548, (2014).

[26] A. PREMET, D. STEWART, *Rigid orbits and sheets in reductive Lie algebras over fields of prime characteristic,* Journal of the Institute of Mathematics of Jussieu, 1–31, (2016).

[27] R. W. RICHARDSON, *Normality of G-stable subvarieties of a semisimple Lie algebra,* In: Cohen *et al.*, Algebraic Groups, Utrecht 1986, Lecture Notes in Math. 1271, Springer-Verlag, New York, (1987).

[28] N. SPALTENSTEIN, *Classes Unipotentes et Sous-Groupes de Borel,* Springer-Verlag, Berlin (1982).

[29] T.A. SPRINGER, *Linear Algebraic Groups, Second Edition* Progress in Mathematics 9, Birkhäuser (1998).