

FROM THE FARKAS LEMMA TO THE HAHN–BANACH THEOREM*

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Abstract. This paper provides new versions of the Farkas lemma characterizing those inequalities of the form $f(x) \geq 0$ which are consequences of a composite convex inequality $(S \circ g)(x) \leq 0$ on a closed convex subset of a given locally convex topological vector space X , where f is a proper lower semicontinuous convex function defined on X , S is an extended sublinear function, and g is a vector-valued S -convex function. In parallel, associated versions of a stable Farkas lemma, considering arbitrary linear perturbations of f , are also given. These new versions of the Farkas lemma, and their corresponding stable forms, are established under the weakest constraint qualification conditions (the so-called closedness conditions), and they are actually equivalent to each other, as well as equivalent to an extended version of the so-called Hahn–Banach–Lagrange theorem, and its stable version, correspondingly. It is shown that any of them implies analytic and algebraic versions of the Hahn–Banach theorem and the Mazur–Orlicz theorem for extended sublinear functions.

Key words. Farkas lemma, Hahn–Banach theorem, Hahn–Banach–Lagrange theorem, Mazur–Orlicz theorem

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1. Introduction. In 1894 the physicist Gyula Farkas, interested in equilibrium problems in mechanics, observed the necessity of characterizing the inclusion of a given polyhedral convex cone in a closed half-space whose boundary contains the origin or, in algebraic terms, when a linear inequality is the consequence of a homogeneous linear system. After several failed attempts, Farkas proved his characterization in 1902, while the nonhomogeneous version (for polyhedral convex sets and arbitrary closed half-spaces, i.e., for affine functions) was proved by Hermann Minkowski in 1911. The latter version became a very popular tool in applied mathematics in the mid 1900s, after its successful application in linear programming (e.g., in the proof of the duality theorem by Gale, Kuhn, and Tucker in 1951), in nonlinear programming (e.g., the necessary optimality conditions stated by Kuhn and Tucker the same year), in mathematical economics and finance [9], and in moment problems and other fields (see, e.g., [11]). Since then, many extensions have been proposed, most of them in order to get duality theorems and optimality conditions in different branches of mathematical programming and abstract optimization (see, e.g., the review paper [12]). Motivated by the concept of a stable minimax theorem (guaranteeing that a minimax equality holds for each linear perturbation of the function involved), Jeyakumar and Lee [14] introduced in 2008 the concept of a stable Farkas lemma for those situations in which

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the ordinary Farkas lemma holds for each linear perturbation of the function involved. To each version of the stable Farkas lemma corresponds a stable (strong) duality theorem showing that strong duality still holds whenever perturbing the objective function of the primal problem with a linear continuous functional (see, e.g., [2], [14]). Some of the recent versions of the Farkas lemma (see, e.g., [3], [5], [6], [7], [10], [15]) are so general that the following question arises in a natural way: is it possible to approach the fundamentals of mathematics from suitable generalized versions of the Farkas lemma?

The present paper gives an affirmative answer to this question by proving that sufficiently general versions of the Farkas lemma imply the Hahn–Banach and the Mazur–Orlicz theorems. The review paper [20] traces the history of the Hahn–Banach theorem, from the seminal work of Helly to recent extensions, passing through its independent proof by Hahn (1927) and by Banach (1929), who was a pioneer in using the axiom of choice. Narici [20] also presents applications to a variety of fields as probability theory (where expectations can be interpreted as linear functionals on spaces of random variables) or economics and finance (where prices can be viewed as linear functionals too). There exists also a wide literature on the extensions and applications of the Mazur–Orlicz theorem (1953). For instance, [17] reformulates this result as a “max–min” equality from which the author derives the famous von Neumann’s min–max equality for semi-infinite matrices (in finite dimensions), as well as some classical results on reflexive Banach spaces.

In more detail, we consider in this paper two locally convex topological vector spaces X and Y , a nonempty closed convex set $C \subset X$, a proper lower semicontinuous (lsc) convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, an lsc sublinear map $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$, and an S -convex function $g : X \rightarrow Y^\bullet$ satisfying certain additional conditions. (Y^\bullet denotes the extension of Y with an element called “infinite.”) The theorems labeled as the Farkas lemma state that

$$(b_{k+1}) \underset{\text{Ordinary Farkas lemma}}{\Leftrightarrow} (a_{k+1}) \Leftarrow (a_k) \underset{\text{Stable Farkas lemma}}{\Leftrightarrow} (b_k),$$

where (a_k) and (a_{k+1}) assert the closedness of two sets associated with the data (the set C and the maps f , S , and g), (b_k) characterizes those inequalities of the form $f(x) \geq 0$ which are the consequence of a composite inequality $(S \circ g)(x) \leq 0$ on C in terms of the existence of a continuous linear functional satisfying suitable conditions, and (b_{k+1}) , in turn, characterizes in similar terms the situations in which $f(x) \geq x^*(x) + \beta$ is the consequence of $(S \circ g)(x) \leq 0$ on C for any continuous affine functional $x^* + \beta$. Since the statements of type (b) can be seen as new versions of the (ordinary or stable) Farkas lemma, the mentioned theorems are actually characterizations of Farkas’s lemma (in the sense of Jeyakumar, Kum, and Lee [13]). From the new versions of the Farkas lemma we derive analytic versions of the so-called Hahn–Banach–Lagrange theorem (the name given by S. Simons in [22] to an algebraic version of the Hahn–Banach theorem that allows us to cope with many problems of Lagrange type), the Hahn–Banach theorem, and the Mazur–Orlicz theorem. These results also establish the equivalence between a closedness condition (a) involving the data and a statement (b) characterizing the boundedness below of $f + S \circ g$ on C , the existence of a continuous linear minorant of S on X extending a given minorant of S on a linear subspace, and the existence of a continuous linear minorant of S on X which has the same infimum as S on C . In the algebraic versions of the latter results the spaces X and Y are just assumed to be linear, the data f , S , g are not required to satisfy topological assumptions, and our results establish that statement (b) always holds.

The paper is organized as follows. Section 3 provides the new versions of the Farkas lemma and the corresponding stable forms of these results. Section 4 shows that these new versions of the Farkas lemma imply an analytic version of the Hahn–Banach theorem and an algebraic version of the latter result for sublinear functions which may take the value $+\infty$ (as in [19]), which is obviously stronger than the famous Hahn–Banach theorem for finite-valued sublinear functions. Section 5 provides an analytic version of the so-called Hahn–Banach–Lagrange theorem from which we derive analytic and algebraic versions of the well-known Mazur–Orlicz theorem. Finally, section 6 shows that the new versions of the Farkas lemma and the Hahn–Banach–Lagrange theorem established in this paper are equivalent to each other. It is worth mentioning that these new versions of the Farkas lemma, the Hahn–Banach–Lagrange theorem, and their corresponding stable forms are established under the weakest constraint qualification conditions, the so-called closedness conditions.

2. Preliminaries. Consider the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with the following conventions: $\infty + \alpha = \alpha + \infty = \infty$ for all $\alpha \in \overline{\mathbb{R}}$, and $\alpha\infty = \infty$ for all $\alpha \in \mathbb{R}_+ \cup \{\infty\}$. We also extend the usual order \leq in the real numbers set with $\alpha \leq \infty$ for all $\alpha \in \overline{\mathbb{R}}$.

Consider a nontrivial vector space Y which is partially ordered by a convex cone K containing the origin of Y ($0_Y \in K$), i.e.,

$$y_1 \leq_K y_2 \text{ if } y_1 - y_2 \in -K.$$

We add to Y a *greatest element* with respect to \leq_K , denoted by ∞_K , i.e., in the enlarged space $Y^\bullet = Y \cup \{\infty_K\}$ we have

$$y \leq_K \infty_K \text{ for every } y \in Y^\bullet,$$

and we also adopt the following conventions with respect to the operations in Y^\bullet : $y + \infty_K = \infty_K + y = \infty_K$ for all $y \in Y^\bullet$, and $\alpha\infty_K = \infty_K$ if $\alpha \geq 0$.

If Y is a separated locally convex space, with topological dual space Y^* , we also assume $\langle y^*, \infty_K \rangle = \infty$ for all $y^* \in K^+$, where K^+ is the *dual cone* of K , i.e.,

$$K^+ := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0 \text{ for all } y \in K\}.$$

Given two vector spaces X and Y , and a function $h : X \rightarrow Y^\bullet$, we call *domain* of h the set $\text{dom } h = \{x \in X : h(x) \in Y\}$, and we say that h is *proper* if $\text{dom } h \neq \emptyset$. The *K-epigraph* of h is the set

$$\text{epi}_K h := \{(x, y) \in X \times Y : y \in h(x) + K\}.$$

DEFINITION 2.1. *The function $h : X \rightarrow Y^\bullet$ is said to be K-convex if*

$$x_1, x_2 \in X, \mu \in [0, 1] \Rightarrow h((1 - \mu)x_1 + \mu x_2) \leq_K (1 - \mu)h(x_1) + \mu h(x_2),$$

where \leq_K is the binary relation extended to Y^\bullet .

It is obvious that h is K -convex if and only if $\text{epi}_K h$ is convex. If Y is a separated locally convex space, for any $y^* \in K^+$ we shall make use of the function $y^* \circ h : X \rightarrow \overline{\mathbb{R}}$ defined by $(y^* \circ h)(x) := \langle y^*, h(x) \rangle$ (remember that $\langle y^*, \infty_K \rangle = \infty$).

DEFINITION 2.2 (see [18]). *If X and Y are separated locally convex spaces, $h : X \rightarrow Y^\bullet$ is said to be K-epi closed if $\text{epi}_K h$ is a closed set in the product space. Then, the cone K and the set $h^{-1}(-K)$ are both closed as $h^{-1}(-K) \times \{0_Y\} = (\text{epi}_K h) \cap (X \times \{0_Y\})$.*

DEFINITION 2.3. *The function $S : Y \rightarrow \overline{\mathbb{R}}$ is (extended) sublinear if*

$$(a) \quad y_1, y_2 \in Y \Rightarrow S(y_1 + y_2) \leq S(y_1) + S(y_2)$$

and

$$(b) \quad y \in Y \text{ and } \alpha > 0 \Rightarrow S(\alpha y) = \alpha S(y).$$

Taking $y = 0_Y$, the last implication entails $S(0_Y) \in \{0, \infty\}$, but we shall assume throughout the paper that $S(0_Y) = 0$. This assumption holds when S is a proper lsc function, i.e., when $\text{epi } S := \{(y, \gamma) \in Y \times \mathbb{R} : S(y) \leq \gamma\}$ is closed for some topology in Y compatible with the vector space structure.

The sublinear function $S : Y \rightarrow \overline{\mathbb{R}}$ allows us to introduce in Y^\bullet the following partial order:

$$(2.1) \quad y_1 \leq_S y_2 \text{ if } y_1 \leq_K y_2, \text{ where } K := \{y \in Y : S(-y) \leq 0\}.$$

DEFINITION 2.4. *Let X, Y be nontrivial vector spaces, and consider mappings $h : X \rightarrow Y^\bullet$ and $S : Y \rightarrow \overline{\mathbb{R}}$, the last one being sublinear. We say that h is (extended) S -convex if*

$$x_1, x_2 \in X, \mu \in [0, 1] \Rightarrow h((1 - \mu)x_1 + \mu x_2) \leq_S (1 - \mu)h(x_1) + \mu h(x_2),$$

where \leq_S is the binary relation defined in (2.1).

Observe that if h is K -convex, where K is a convex cone containing the origin, and one takes $S = i_{-K}$, then h is S -convex. Here, i_A is the *indicator function* of $A \subset X$ which is given by $i_A(x) = 0$ if $x \in A$ and by $i_A(x) = +\infty$ if $x \in X \setminus A$.

In this paper we are mainly dealing with two separated locally convex spaces X and Y with topological dual spaces X^* and Y^* . The only topology we consider on the dual spaces is the weak*-topology. Given a set A in one of the considered spaces, we denote by $\text{co } A$, $\text{cone } A$, and $\text{cl } A$ the convex hull, the conical convex hull, and the closure of A , respectively. If A is a linear subspace of X , the orthogonal subspace to A is $A^\perp = \{x^* \in X^* : \langle x^*, x \rangle = 0 \text{ for all } x \in A\}$.

Given $f : X \rightarrow \overline{\mathbb{R}}$, we represent by $\text{dom } f$ the effective domain of f , i.e., $\text{dom } f := \{x \in X : f(x) < +\infty\}$, and say that f is proper if $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The epigraph of f is $\text{epi } f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$. The function f is a proper lsc convex function if $\text{epi } f$ is a nonempty closed convex set. The set of proper lsc convex functions defined on X is denoted by $\Gamma(X)$. The *Legendre–Fenchel conjugate* of $f \in \Gamma(X)$ is the function $f^* : X^* \rightarrow \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle - f(x)) \quad \forall x^* \in X^*.$$

A similar notion holds for any $\varphi \in \Gamma(X^*)$:

$$\varphi^*(x) = \sup_{x^* \in X^*} (\langle x^*, x \rangle - \varphi(x^*)) \quad \forall x \in X.$$

The conjugate of i_A is the *support function* of A , i.e., the function $i_A^* : X^* \rightarrow \overline{\mathbb{R}}$ such that $i_A^*(x^*) = \sup_{x \in A} \langle x^*, x \rangle$ for any $x^* \in X^*$. For any proper $f : X \rightarrow \overline{\mathbb{R}}$ one has

$$f \in \Gamma(X) \Leftrightarrow f = f^{**}.$$

Associated with the pair of functions $f : X \rightarrow \overline{\mathbb{R}}$ and $\psi : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, we define the function $\tilde{f} : X \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$

$$(2.2) \quad \tilde{f}(x, \alpha) := f(x) + \psi(\alpha) \quad \forall (x, \alpha) \in X \times \mathbb{R}.$$

Then, if f and ψ are proper,

$$\tilde{f}^*(x^*, \gamma) = f^*(x^*) + \psi^*(\gamma) \quad \forall (x^*, \gamma) \in X^* \times \mathbb{R}$$

and

$$(2.3) \quad \text{epi } \tilde{f}^* = \{(x^*, 0, r) : (x^*, r) \in \text{epi } f^*\} + \{(0_{X^*}, \gamma, r) : (\gamma, r) \in \text{epi } \psi^*\}.$$

In general if $f, g \in \Gamma(X)$ and $(\text{dom } f) \cap (\text{dom } g) \neq \emptyset$, then one has (see, e.g., [4, Theorem 2.1])

$$(2.4) \quad \text{epi}(f + g)^* = \text{cl}(\text{epi } f^* + \text{epi } g^*),$$

where, as already stated, cl represents the closure with respect to the weak*-topology. If one of the functions f or g is continuous at a point of the domain of the other, then the closure cl can be removed from the right-hand side of (2.4) (see, e.g., [25, Theorem 2.8.7]).

Given $a \in f^{-1}(\mathbb{R})$, the subdifferential of f at the point a is defined by

$$\partial f(a) = \{x^* \in X^* : f(x) - f(a) \geq \langle x^*, x - a \rangle \quad \forall x \in X\}.$$

3. New versions of the Farkas lemma. The possibility of switching between S -convexity (convexity w.r.t. a sublinear function) and K -convexity (convexity w.r.t. a convex cone) enables us to establish additional new versions of the Farkas lemma for convex systems. It is worth emphasizing that these results actually provide characterizations (i.e., necessary and sufficient conditions) for these new versions of the Farkas lemma.

Let X and Y be separated locally convex spaces, $C \subset X$ be a nonempty closed convex set, $K \subset Y$ be a convex cone, $g : X \rightarrow Y^\bullet$ be a K -convex and K -epi closed function, and $f \in \Gamma(X)$. Assume that

$$(3.1) \quad \text{dom } f \cap C \cap g^{-1}(-K) \neq \emptyset,$$

and consider, associated with each $x^* \in X^*$, the optimization problem

$$(P_{x^*}) \quad \text{Min } \{f(x) - \langle x^*, x \rangle\} \text{ subject to } x \in C, g(x) \in -K,$$

whose optimal value is denoted by $\inf(P_{x^*})$. In particular, we define

$$(P) := (P_{0_{X^*}}) \quad \text{Min } f(x) \text{ subject to } x \in C, g(x) \in -K,$$

with optimal value $\inf(P)$. Problem (P_{x^*}) comes from (P) after a *linear perturbation* of the objective function. Thanks to (3.1) one has

$$-\infty \leq \inf(P_{x^*}) < +\infty.$$

If we introduce the *perturbation function* $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$,

$$(3.2) \quad \Phi(x, y) := \begin{cases} f(x) & \text{if } x \in C, g(x) \in y - K, \\ +\infty & \text{otherwise,} \end{cases}$$

we see in [2, (3.1)] that its conjugate is

$$\Phi^*(x^*, y^*) := \begin{cases} (f + y^* \circ g + i_C)^*(x^*) & \text{if } y^* \in K^+, \\ +\infty & \text{otherwise} \end{cases}$$

for $(x^*, y^*) \in X^* \times Y^*$.

Like in [8], we consider the function $p : X^* \rightarrow \overline{\mathbb{R}}$,

$$p := \inf_{y^* \in K^+} (f + y^* \circ g + i_C)^*.$$

Obviously,

$$p = \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*).$$

The function Φ is proper, convex, and lsc, and [2, (5.1)] applies and allows us to establish

$$(\Phi(\cdot, 0_Y))^* = \text{cl } p,$$

and so

$$p^* = (\text{cl } p)^* = (\Phi(\cdot, 0_Y))^{**} = \Phi(\cdot, 0_Y) = f + i_C + i_{g^{-1}(-K)}.$$

The function p^* is proper, and so

$$p^{**} = \text{cl } p.$$

Moreover,

$$\inf(P_{x^*}) = -p^{**}(x^*),$$

and, in particular, $\inf(P) = -p^{**}(0_{X^*})$.

It is not difficult to prove (see, e.g., [8]) that

$$\text{epi } p^{**} = \text{cl } \mathfrak{C},$$

where

$$(3.3) \quad \mathfrak{C} := \bigcup_{y^* \in K^+} \text{epi}(f + y^* \circ g + i_C)^*.$$

Consequently, for any $\beta \in \mathbb{R}$, one has

$$\inf(P_{x^*}) \geq \beta \Leftrightarrow p^{**}(x^*) \leq -\beta \Leftrightarrow (x^*, -\beta) \in \text{epi } p^{**} = \text{cl } \mathfrak{C},$$

or equivalently,

$$(3.4) \quad \begin{aligned} & \left(x \in C, g(x) \in -K \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta \right) \\ & \quad \quad \quad \updownarrow \\ & (x^*, -\beta) \in \text{cl } \mathfrak{C}. \end{aligned}$$

The following result is crucial in the rest of the paper.

THEOREM 3.1. *Let X and Y be separated locally convex spaces, $C \subset X$ be a nonempty closed convex set, $K \subset Y$ be a convex cone, $g : X \rightarrow Y^\bullet$ be K -convex*

and K -epi closed, and $f \in \Gamma(X)$. Assume also that $\text{dom } f \cap C \cap g^{-1}(-K) \neq \emptyset$, and consider the following statements:

(a₁) \mathfrak{C} is weak*-closed in $X^* \times \mathbb{R}$.

(a₂)

$$(3.5) \quad (\text{cl } \mathfrak{C}) \cap (\{0_{X^*}\} \times \mathbb{R}) = \mathfrak{C} \cap (\{0_{X^*}\} \times \mathbb{R}).$$

(b₁) For any $x^* \in X^*$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, g(x) \in -K \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta \right) \\ & \quad \Downarrow \\ & \exists y^* \in K^+ \text{ such that } f - x^* + y^* \circ g \geq \beta \text{ on } C. \end{aligned}$$

(b₂) For any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, g(x) \in -K \Rightarrow f(x) \geq \beta \right) \\ & \quad \Downarrow \\ & \exists y^* \in K^+ \text{ such that } f + y^* \circ g \geq \beta \text{ on } C. \end{aligned}$$

Then we have

$$(3.6) \quad (b_2) \underset{\text{Farkas lemma } 1-K}{\Leftrightarrow} (a_2) \Leftarrow (a_1) \underset{\text{Stable Farkas lemma } 1-K}{\Leftrightarrow} (b_1).$$

Proof. The proof follows from (3.4). \square

It is well known that the Farkas lemma has a great deal of applications in optimization (see [3], [5], [7], and references therein). Observe that Theorem 3.1 yields directly the *strong Lagrange duality* and *stable strong Lagrange duality* for the problem

$$(P) \quad \inf f(x) \text{ subject to } x \in C, g(x) \in -K,$$

and its associated *Lagrange dual* problem

$$(D) \quad \sup_{y^* \in K^+} \inf_{x \in C} \{f(x) + y^* \circ g\}.$$

It is easy to see that the *weak duality* holds, i.e., $\inf(P) \geq \sup(D)$. Then the equivalence $[(a_2) \Leftrightarrow (b_2)]$, which we call *Farkas lemma 1-K*, means that $\inf(P) = \sup(D)$ and the problem (D) has at least a solution, namely, $y^* \in K^+$ that appears in (b₂). Shortly, we get from this equivalence the characterization for strong duality of (P) and (D). In other words, condition (a₂) holds if and only if the *strong duality* between (P) and (D) is satisfied.

Similarly, the equivalence $[(a_1) \Leftrightarrow (b_1)]$, which we call *stable Farkas lemma 1-K*, yields the characterization of stable strong Lagrange duality under linear perturbation of the objective function of (P). Formally, (a₁) holds if and only if the *stable strong Lagrange duality* between (P) and (D) is satisfied; i.e., for any $x^* \in X^*$,

$$\inf_{x \in C, g(x) \in -K} [f(x) - \langle x^*, x \rangle] = \sup_{y^* \in K^+} \inf_{x \in C} \{f(x) - \langle x^*, x \rangle + (y^* \circ g)(x)\}.$$

Remarks.

(1) The equivalence of $(a_2) \Leftrightarrow (b_2)$ is [8, Corollary 5.1], and it can also be derived as an elementary consequence of [2, Theorem 9.1] by choosing $V = \{0_{X^*}\}$ and using the function Φ in (3.2).

(2) The equivalence $(a_1) \Leftrightarrow (b_1)$ is [2, Theorem 8.3], it extends [14, Theorem 3.1], and it can also be straightforwardly derived from [10, Theorem 6.7]. Here we have chosen the perturbational approach in order to introduce the *linearly perturbed* problem (P_{x^*}) and to emphasize the geometrical meaning of the set \mathfrak{C} via the relations $\text{inf}(P_{x^*}) = -p^{**}(x^*)$ and $\text{epi } p^{**} = \text{cl } \mathfrak{C}$.

(3) Condition (a_2) accounts for the closedness of $\mathfrak{C} \cap (\{0_{X^*}\} \times \mathbb{R})$ in the topology in $\{0_{X^*}\} \times \mathbb{R}$ induced by the weak*-topology. (It is also said that \mathfrak{C} is *closed regarding* $\{0_{X^*}\} \times \mathbb{R}$ [2, p. 56].) Moreover, if (a_2) holds, we have

$$\mathfrak{C} \cap (\{0_{X^*}\} \times \mathbb{R}) = \{0_{X^*}\} \times I,$$

where $I = [\mathbf{a}, +\infty[$, $\mathbf{a} \in \mathbb{R}$, if $\text{inf}(P) \in \mathbb{R}$, or $I = \emptyset$, if $\text{inf}(P) = -\infty$.

(4) It is worth noticing that if f is continuous at a point belonging to $\text{dom } f \cap C \cap g^{-1}(-K)$, and since

$$\text{dom } f \cap C \cap g^{-1}(-K) \subset \text{dom } f \cap \text{dom}(y^* \circ g + i_C) \quad \forall y^* \in K^+,$$

by (2.4) condition (a_1) turns out to be equivalent to

$$(3.7) \quad \text{epi } f^* + \bigcup_{y^* \in K^+} \text{epi}(y^* \circ g + i_C)^* \text{ is weak}^* \text{-closed.}$$

This fact is observed in [2, Remark 8.14]. There, it is also stated that under this assumption of the continuity of f at a point in $\text{dom } f \cap C \cap g^{-1}(-K)$, the Lagrange and the Fenchel-Lagrange dual problems of (P) coincide.

(5) Notice also that the *Slater-type condition*

$$\exists \bar{x} \in \text{dom } f \cap C \text{ such that } g(\bar{x}) \in -\text{int } K$$

also implies (a_1) although it is strictly weaker (see [2, Theorem 3.4 and Example 8.5]). Other alternative constraint qualifications are given in [2, Chapter I, section 3].

COROLLARY 3.2. *Let X and Y be locally convex spaces, $C \subset X$ be a nonempty closed convex subset, $S : Y \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, and $g : X \rightarrow Y^\bullet$ be an S -convex function such that the set*

$$(3.8) \quad \{(x, y) \in X \times Y : S(g(x) - y) \leq 0\}$$

is closed in the product space $X \times Y$. Let further $f \in \Gamma(X)$, and assume that

$$(3.9) \quad \text{dom } f \cap C \cap \{x \in X : (S \circ g)(x) \leq 0\} \neq \emptyset.$$

Consider the following statements:

(a₃) The set

$$(3.10) \quad \mathfrak{C} := \bigcup_{y^* \in \text{cl}(\mathbb{R}_+ \partial S(0_Y))} \text{epi}(f + y^* \circ g + i_C)^*$$

is weak-closed.*

(a₄) \mathfrak{C} is closed regarding $\{0_{X^*}\} \times \mathbb{R}$, i.e., \mathfrak{C} satisfies (3.5).

(b₃) For any $x^* \in X^*$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, (S \circ g)(x) \leq 0 \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta \right) \\ & \quad \Downarrow \\ & \left(\exists y^* \in \text{cl}(\mathbb{R}_+ \partial S(0_Y)) \text{ such that } f - x^* + y^* \circ g \geq \beta \text{ on } C \right). \end{aligned}$$

(b₄) For any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, (S \circ g)(x) \leq 0 \Rightarrow f(x) \geq \beta \right) \\ & \quad \Downarrow \\ & \left(\exists y^* \in \text{cl}(\mathbb{R}_+ \partial S(0_Y)) \text{ such that } f + y^* \circ g \geq \beta \text{ on } C \right). \end{aligned}$$

Then we have

$$(b_4) \underset{\text{Farkas lemma 1-S}}{\Leftrightarrow} (a_4) \Leftarrow (a_3) \underset{\text{Stable Farkas lemma 1-S}}{\Leftrightarrow} (b_3).$$

Proof. Let us consider the closed convex cone $K := \{y \in Y : S(-y) \leq 0\}$. Since g is S -convex, we know that g is K -convex, and

$$g(x) \in -K \Leftrightarrow (S \circ g)(x) \leq 0.$$

The closedness of the set in (3.8) entails that g is K -epi closed.

Moreover, one has

$$K^+ = \text{cl}(\mathbb{R}_+ \partial S(0_Y)).$$

In fact, we know (see, e.g., [25, Theorem 2.4.14(i)]) that $S^* = i_{\partial S(0_Y)}$ and, since $S \in \Gamma(Y)$, we have $S = S^{**} = i_{\partial S(0_Y)}^*$. This entails $-K = \{z \in Y : i_{\partial S(0_Y)}^*(z) \leq 0\} = -\partial S(0_Y)^+$ and then $K^+ = \partial S(0_Y)^{++} = \text{cl}(\mathbb{R}_+ \partial S(0_Y))$. The result now comes straightforwardly from Theorem 3.1. \square

Along the same lines as Theorem 3.1, Corollary 3.2 yields characterizations of strong duality and stable strong duality for a class of convex problems involving composite functions. In fact, if the spaces X, Y , the subset $C \subset X$, the sublinear function $S : Y \rightarrow \overline{\mathbb{R}}$, and the S -convex function $g : X \rightarrow Y^\bullet$ are as in Corollary 3.2, and we consider the problem

$$(Q) \quad \inf f(x) \text{ subject to } x \in C, S \circ g(x) \leq 0,$$

and its associated Lagrange dual problem

$$(D_Q) \quad \sup_{y^* \in \text{cl}(\mathbb{R}_+ \partial S(0_Y))} \inf_{x \in C} \{f(x) + y^* \circ g\},$$

Corollary 3.2 provides characterizations of strong and stable strong duality for the pair (Q) and (D_Q) .

Remark 3.3. Observe that if $y^* = 0_{Y^*}$ in the second statement in (b_3) and (b_4) the implication (\uparrow) is trivial.

The following theorem constitutes an extension of Corollary 3.2. It is one of the main results in this paper.

THEOREM 3.4 (Farkas lemma 2). *Let X and Y be locally convex spaces, $C \subset X$ be a nonempty closed convex subset, $S : Y \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, and $g : X \rightarrow Y^\bullet$ be an S -convex function such that the set*

$$(3.11) \quad \{(x, y, \lambda) \in X \times Y \times \mathbb{R} : S(g(x) - y) \leq \lambda\}$$

is closed in the product space $X \times Y \times \mathbb{R}$. Let us consider $f \in \Gamma(X)$ and a nonconstant function $\psi \in \Gamma(\mathbb{R})$ and assume the existence of $\bar{x} \in C \cap \text{dom } f$ and $\bar{\alpha} \in \text{dom } \psi$ such that

$$(3.12) \quad (S \circ g)(\bar{x}) \leq \bar{\alpha}.$$

Let further

$$(3.13) \quad D := \{(0_{X^*}, \gamma, r) : (\gamma, r) \in \text{epi } \psi^*\}$$

and

$$(3.14) \quad E := \bigcup_{\substack{y^* \in Y^*, \mu \geq 0 \\ y^* \leq \mu S}} \{(u^*, -\mu, r) : (u^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*\}.$$

Consider the following statements:

(a₅) $D + E$ is weak*-closed in the product space $X^* \times \mathbb{R} \times \mathbb{R}$.

(a₆) $D + E$ is closed regarding $\{0_{X^*}\} \times \{0\} \times \mathbb{R}$.

(b₅) For any $\tilde{x}^* = (x^*, \gamma) \in X^* \times \mathbb{R}$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \Rightarrow f(x) + \psi(\alpha) - \langle x^*, x \rangle - \gamma \alpha \geq \beta \right) \\ & \quad \Updownarrow \\ & \left(\exists \mu \geq 0 \text{ and } y^* \in Y^* \text{ such that } \mu + \gamma \in \text{dom } \psi^*, y^* \leq \mu S \text{ on } Y, \text{ and} \right. \\ & \quad \left. f - x^* + y^* \circ g \geq \psi^*(\gamma + \mu) + \beta \text{ on } C \right). \end{aligned}$$

(b₆) For any $\beta \in \mathbb{R}$,

$$\begin{aligned} & \left(x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \Rightarrow f(x) + \psi(\alpha) \geq \beta \right) \\ & \quad \Updownarrow \\ & \left(\exists \mu \geq 0 \text{ and } y^* \in Y^* \text{ such that } \mu \in \text{dom } \psi^*, y^* \leq \mu S \text{ on } Y, \text{ and} \right. \\ & \quad \left. f + y^* \circ g \geq \psi^*(\mu) + \beta \text{ on } C \right). \end{aligned}$$

Then we have

$$(3.15) \quad (b_6) \underset{\text{Farkas lemma 2-S}}{\Leftrightarrow} (a_6) \Leftarrow (a_5) \underset{\text{Stable Farkas lemma 2-S}}{\Leftrightarrow} (b_5).$$

Before the proof, we introduce some notation:

$$(3.16) \quad \begin{aligned} \tilde{X} &:= X \times \mathbb{R}, & \tilde{C} &:= C \times \mathbb{R}, \\ \tilde{Y} &:= Y \times \mathbb{R}, & \tilde{Y}^\bullet &:= Y^\bullet \times \mathbb{R}, \\ \tilde{S} : \tilde{Y} &\rightarrow \overline{\mathbb{R}} & \text{with } \tilde{S}(y, \lambda) &= S(y) - \lambda. \end{aligned}$$

Then \tilde{S} is an lsc sublinear function, for which $\partial \tilde{S}(0_Y, 0) = \partial S(0_Y) \times \{-1\}$.

Let also $\tilde{g} : \tilde{X} \rightarrow \tilde{Y}^\bullet$ be the mapping defined by

$$(3.17) \quad \tilde{g}(x, \alpha) := (g(x), \alpha) \quad \forall (x, \alpha) \in \tilde{X},$$

as well as, for any $\tilde{y}^* = (y^*, \lambda) \in \tilde{Y}^* = Y^* \times \mathbb{R}$, the function $\tilde{y}^* \circ \tilde{g}$ defined on \tilde{X} as follows:

$$(3.18) \quad (\tilde{y}^* \circ \tilde{g})(x, \alpha) := (y^* \circ g)(x) + \lambda\alpha = \begin{cases} \langle y^*, g(x) \rangle + \lambda\alpha & \text{if } x \in \text{dom } g, \\ +\infty & \text{otherwise.} \end{cases}$$

Now let \tilde{K} be the convex cone defined by

$$\tilde{K} := \{(y, \lambda) \in \tilde{Y} : \tilde{S}(-y, -\lambda) \leq 0\} = -\text{epi } S.$$

We have already established that

$$\tilde{K}^+ = \text{cl}(\mathbb{R}_+ \partial \tilde{S}(0_Y, 0)) = \text{cl}(\mathbb{R}_+(\partial S(0_Y) \times \{-1\})).$$

The following lemma gives another characterization of the dual cone \tilde{K}^+ .

LEMMA 3.5.

$$(y^*, -\mu) \in \tilde{K}^+ \Leftrightarrow \mu \geq 0 \text{ and } y^* \leq \mu S \text{ on } Y.$$

Proof. $[\Rightarrow]$ Since $(y^*, -\mu) \in \tilde{K}^+$ and $(0_Y, -1) \in \tilde{K}$ (as $\tilde{S}(0_Y, 1) = S(0_Y) - 1 = -1 \leq 0$), we have

$$\langle (y^*, -\mu), (0_Y, -1) \rangle = \mu \geq 0.$$

Since $(-y, -S(y)) \in \tilde{K}$,

$$\langle (y^*, -\mu), (-y, -S(y)) \rangle = \langle y^*, -y \rangle + \mu S(y) \geq 0,$$

and $y^* \leq \mu S$ on $\text{dom } S$ and, a fortiori, on Y .

$[\Leftarrow]$ Assume that $\mu \geq 0$ and $y^* \leq \mu S$ on Y . For any $(-y, -\lambda) \in \tilde{K}$,

$$\tilde{S}(y, \lambda) = S(y) - \lambda \leq 0,$$

and by assumption,

$$\langle y^*, y \rangle \leq \mu S(y) \leq \mu\lambda.$$

So,

$$\langle (y^*, -\mu), (-y, -\lambda) \rangle = -\langle y^*, y \rangle + \mu\lambda \geq 0,$$

and we proved that $(y^*, -\mu) \in \tilde{K}^+$. \square

In the proof of Theorem 3.4 we shall apply the following result involving the function $\tilde{f} \in \Gamma(\tilde{X})$ defined by

$$(3.19) \quad \tilde{f}(x, \alpha) = f(x) + \psi(\alpha).$$

LEMMA 3.6. *The following equation holds:*

$$D + E = \bigcup_{\tilde{y}^* \in \tilde{K}^+} \text{epi}(\tilde{f} + \tilde{y}^* \circ \tilde{g} + i_{\tilde{C}})^*,$$

where \tilde{f} is the function in (3.19).

Proof. We observe that for any $\tilde{y}^* = (y^*, -\mu) \in \tilde{Y}^*$,

$$\begin{aligned} (\tilde{f} + \tilde{y}^* \circ \tilde{g} + i_{\tilde{C}})(x, \alpha) &= f(x) + (y^* \circ g)(x) + i_C(x) - \mu\alpha + \psi(\alpha) \\ &= f_1(x) + \psi_1(\alpha) \quad \forall (x, \alpha) \in X \times \mathbb{R}, \end{aligned}$$

where $f_1(x) = f(x) + (y^* \circ g)(x) + i_C(x)$ and $\psi_1(\alpha) = \psi(\alpha) - \mu\alpha$. Applying (2.3) to the proper functions f_1 and ψ_1 , and taking into account that $\psi_1^*(\rho) = \psi^*(\rho + \mu)$, and accordingly

$$\text{epi } \psi_1^* = \{(\rho, r) : \psi_1^*(\rho) \leq r\} = \{(\gamma, r) : \psi^*(\gamma) \leq r\} + \{(-\mu, 0)\},$$

one gets

$$\begin{aligned} \text{epi}(\tilde{f} + \tilde{y}^* \circ \tilde{g} + i_{\tilde{C}})^* &= \{(u^*, -\mu, r) : (u^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*\} \\ &\quad + \{(0_{X^*}, \gamma, r) : (\gamma, r) \in \text{epi } \psi^*\}. \end{aligned}$$

Thanks to Lemma 3.5 we are done. \square

Proof of Theorem 3.4. Let \tilde{X} , \tilde{Y} , \tilde{C} , \tilde{g} , \tilde{S} , and \tilde{f} be defined as in (3.16), (3.17), and (3.19). Since \tilde{S} is an lsc sublinear function and g is S -convex, \tilde{g} is \tilde{S} -convex. Now we apply a similar argument to the one used in the proof of Corollary 3.2 with \tilde{X} , \tilde{Y} , \tilde{C} , \tilde{g} , \tilde{f} , and \tilde{S} playing the roles of X , Y , C , g , f , and S , respectively. So, we must first check the assumptions:

- (i) The assumption (3.11) accounts for (3.8) in Corollary 3.2.
- (ii) By the assumption posed on f and ψ , and by (3.12), we have

$$(\bar{x}, \bar{\alpha}) \in \text{dom } \tilde{f} \cap \tilde{C} \cap \{(x, \alpha) \in \tilde{X} : (\tilde{S} \circ \tilde{g})(x, \alpha) \leq 0\}.$$

It follows from Lemma 3.6 that (a_5) in Theorem 3.4 is equivalent to (a_3) in Corollary 3.2, applied to \tilde{X} , \tilde{Y} , \tilde{C} , \tilde{g} , \tilde{f} , and \tilde{S} . Now we are using the characterization of \tilde{K}^+ provided by Lemma 3.5.

$[(a_5) \Rightarrow (b_5)]$ Assume that (a_5) holds, which means that (a_3) in Corollary 3.2 holds as well. For any $\tilde{x}^* = (x^*, \gamma) \in X^* \times \mathbb{R}$, $\langle \tilde{x}^*, (x, \alpha) \rangle = \langle x^*, x \rangle + \gamma\alpha$ for all $(x, \alpha) \in X \times \mathbb{R}$. From the implication

$$(x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha) \Rightarrow f(x) + \psi(\alpha) - \langle x^*, x \rangle - \gamma\alpha \geq \beta,$$

and the definitions of \tilde{f} and \tilde{g} , one gets

$$((x, \alpha) \in \tilde{C}, (\tilde{S} \circ \tilde{g})(x, \alpha) \leq 0) \Rightarrow \tilde{f}(x, \alpha) - \langle \tilde{x}^*, (x, \alpha) \rangle \geq \beta.$$

The equivalence $[(a_3) \Leftrightarrow (b_3)]$ in Corollary 3.2 applies to ensure the existence of $\tilde{y}^* \in \tilde{K}^+$ satisfying

$$(3.20) \quad \tilde{f} - \tilde{x}^* + \tilde{y}^* \circ \tilde{g} \geq \beta \quad \text{on } \tilde{C}.$$

In other words, by Lemma 3.5, there will exist $y^* \in Y^*$ and $\mu \geq 0$ such that $\tilde{y}^* = (y^*, -\mu)$, $y^* \leq \mu S$, and

$$\tilde{f}(x, \alpha) - \langle \tilde{x}^*, (x, \alpha) \rangle + (\tilde{y}^* \circ \tilde{g})(x, \alpha) \geq \beta \quad \forall x \in C \text{ and } \alpha \in \mathbb{R},$$

giving rise to

$$(3.21) \quad f(x) - \langle x^*, x \rangle + (y^* \circ g)(x) \geq (\mu + \gamma)\alpha - \psi(\alpha) + \beta \quad \forall x \in C, \forall \alpha \in \mathbb{R}.$$

(Note that this inequality holds even when $\alpha \notin \text{dom } \psi$.) Taking the supremum over all $\alpha \in \mathbb{R}$ in (3.21), we get

$$f(x) - \langle x^*, x \rangle + (y^* \circ g)(x) \geq \psi^*(\mu + \gamma) + \beta \quad \forall x \in C.$$

It follows from (3.12) that $C \cap \text{dom } f \cap \text{dom } g \neq \emptyset$, which ensures that $\mu + \gamma \in \text{dom } \psi^*$. Together with $y^* \leq \mu S$, the downward implication in (b_5) has been proved.

For the converse implication in (b_5) , assume now that there are $\mu \geq 0$ and $y^* \in Y^*$ such that $y^* \leq \mu S$ on Y and $f - x^* + y^* \circ g \geq \psi^*(\mu + \gamma) + \beta$ on C . Then for all $x \in C$ and $\alpha \in \mathbb{R}$,

$$(3.22) \quad f(x) - \langle x^*, x \rangle + (y^* \circ g)(x) \geq \psi^*(\mu + \gamma) + \beta \geq (\mu + \gamma)\alpha - \psi(\alpha) + \beta.$$

Since $y^* \leq \mu S$ on Y , $(y^* \circ g)(x) \leq \mu S(g(x))$ for all $x \in C$. Now if $S(g(x)) \leq \alpha$, then $(y^* \circ g)(x) \leq \mu S(g(x)) \leq \mu\alpha$ since $\mu \geq 0$, and it then follows from (3.22) that

$$f(x) - \langle x^*, x \rangle + \mu\alpha \geq (\mu + \gamma)\alpha - \psi(\alpha) + \beta$$

or

$$f(x) + \psi(\alpha) - \langle x^*, x \rangle - \gamma\alpha \geq \beta,$$

and we are done.

$[(b_5) \Rightarrow (a_5)]$ By the previous proof, we see that if (b_5) holds, then (b_3) in Corollary 3.2 holds with \tilde{X} , \tilde{Y} , \tilde{C} , \tilde{g} , \tilde{f} , and \tilde{S} playing the roles of X , Y , C , g , f , and S . Then by this theorem, (a_3) holds with the new sets and functions, which is nothing but (a_5) , thanks to Lemma 3.5.

$[(a_6) \iff (b_6)]$. This is the same as the proof of $[(a_5) \iff (b_5)]$, taking $\tilde{x}^* = (0_{X^*}, 0)$ and using the assertion $(a_4) \iff (b_4)$ in Corollary 3.2. \square

The following result is a consequence of Theorem 3.4, just taking $\psi(\alpha) = \alpha$ for $\alpha \in \mathbb{R}$.

COROLLARY 3.7 (Farkas lemma 3). *Let X and Y be locally convex spaces, $C \subset X$ be a nonempty closed convex subset, $S : Y \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, and $g : X \rightarrow Y^\bullet$ be an S -convex function such that the set in (3.11) is closed. Further let $f \in \Gamma(X)$, and assume that*

$$(3.23) \quad C \cap \text{dom } f \cap \text{dom}(S \circ g) \neq \emptyset.$$

Set

$$F := \bigcup_{\substack{y^* \in Y^*, \mu \geq 0 \\ y^* \leq \mu S}} \{(u^*, 1 - \mu, r) : (u^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*\}.$$

Consider the following statements:

(a_7) F is weak*-closed in the product space $X^* \times \mathbb{R} \times \mathbb{R}$.

(a_8) F is closed regarding $\{0_{X^*}\} \times \{0\} \times \mathbb{R}$.

(b_7) For any $(x^*, \eta) \in X^* \times \mathbb{R}_+$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} (x \in C, \alpha \in \mathbb{R}, ((\eta S) \circ g)(x) \leq \alpha \Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq \beta) \\ \Downarrow \\ (\exists y^* \in Y^* \text{ such that } y^* \leq \eta S \text{ on } Y \text{ and } f - x^* + y^* \circ g \geq \beta \text{ on } C). \end{aligned}$$

(b₈) For any $\beta \in \mathbb{R}$,

$$\begin{aligned} (x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \Rightarrow f(x) + \alpha \geq \beta) \\ \Downarrow \\ (\exists y^* \in Y^* \text{ such that } y^* \leq S \text{ on } Y \text{ and } f + y^* \circ g \geq \beta \text{ on } C). \end{aligned}$$

Then we have

$$(b_8) \underset{\text{Farkas lemma 3-S}}{\Leftrightarrow} (a_8) \Leftarrow (a_7) \underset{\text{Stable Farkas lemma 3-S}}{\Leftrightarrow} (b_7).$$

Proof. Take $\psi(\alpha) = \alpha$. Then $\psi^* = i_{\{1\}}$. By applying Theorem 3.4, and taking into account that the second sentence in (b₅) implies $\mu + \gamma = 1$ and $\mu = 1 - \gamma \geq 0$, which forces $\gamma \leq 1$, we see that (a₇) is equivalent to the following statement:

(b_{7₁}) For any $(x^*, \gamma) \in X^* \times]-\infty, 1]$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} (3.24) \quad (x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \Rightarrow f(x) - \langle x^*, x \rangle + (1 - \gamma)\alpha \geq \beta) \\ \Downarrow \\ (\exists y^* \in Y^* \text{ such that } y^* \leq (1 - \gamma)S \text{ on } Y \text{ and } f - x^* + y^* \circ g \geq \beta \text{ on } C). \end{aligned}$$

Now let $\eta := 1 - \gamma \geq 0$. Then (b_{7₁}) is equivalent to the following:

(b_{7₂}) For any $(x^*, \eta) \in X^* \times \mathbb{R}_+$ and any $\beta \in \mathbb{R}$,

$$\begin{aligned} (3.25) \quad (x \in C, \eta\alpha \in \mathbb{R}, ((\eta S) \circ g)(x) \leq \eta\alpha \Rightarrow f(x) - \langle x^*, x \rangle + \eta\alpha \geq \beta) \\ \Downarrow \\ (\exists y^* \in Y^* \text{ such that } y^* \leq \eta S \text{ on } Y \text{ and } f - x^* + y^* \circ g \geq \beta \text{ on } C). \end{aligned}$$

Actually, for $\eta > 0$ (3.25) comes straightforwardly from (3.24) by multiplying by η . If $\eta = 0$, (3.25) becomes

$$\begin{aligned} (3.26) \quad (x \in C \text{ and } [(0S) \circ g](x) \leq 0 \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta) \\ \Downarrow \\ (\exists y^* \in Y^* \text{ such that } y^* \leq 0S \text{ on } Y \text{ and } f - x^* + y^* \circ g \geq \beta \text{ on } C), \end{aligned}$$

which is satisfied. In fact for the implication (\Downarrow) we take $y^* = 0_{Y^*}$, and for the implication (\Uparrow) it is enough to observe that, recalling that $0\infty = \infty$,

$$[(0S) \circ g](x) \leq 0 \Leftrightarrow g(x) \in \text{dom } S$$

and

$$y^* \leq 0 S \text{ on } Y \Rightarrow \langle y^*, g(x) \rangle \leq 0,$$

entailing

$$f(x) - \langle x^*, x \rangle \geq f(x) - \langle x^*, x \rangle + (y^* \circ g)(x) \geq \beta.$$

Observe also that (b_{7₂}) is then equivalent to (b₇), and we proved the equivalence between (a₇) and (b₇).

Finally, the equivalence (b₈) \Leftrightarrow (a₈) comes also from Theorem 3.4 by observing that the second sentence in (b₆) entails $\mu = 1$. \square

4. From the Farkas lemma to the Hahn–Banach theorem. It is well known that the celebrated Hahn–Banach theorem fails in the case where the sublinear function (the function S in Theorem 4.1 below) takes the value $+\infty$, as shown by a simple example (even in finite dimensional space) given in [23, Remark 2.3]. As a consequence of Corollary 3.7, we now extend the Hahn–Banach theorem to the mentioned situation in a locally convex space. Here we give a closedness condition, namely, (a_9) , which is both necessary and sufficient for the preservation of the Hahn–Banach theorem under linear perturbations of a fixed continuous linear function on M , ℓ_0 .

THEOREM 4.1 (analytic Hahn–Banach theorem). *Let X be a locally convex space, $S : X \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, M be a closed subspace of X such that $M \cap \text{dom } S \neq \emptyset$, and ℓ_0 be a continuous linear function on M . Set*

$$(4.1) \quad G := \tilde{K}^+ + M^\perp \times \{0\}$$

with

$$\tilde{K}^+ = \text{cl}(\mathbb{R}_+(\partial S(0_X) \times \{-1\})) = \{(x^*, -\mu) : \mu \geq 0 \text{ and } x^* \leq \mu S \text{ on } X\}.$$

Consider the following statements:

(a_9) G is weak* -closed in the product space $X^* \times \mathbb{R}$.

(a_{10}) For any $\ell_0^* \in X^*$ such that $(\ell_0^*)|_M = \ell_0$, we have

$$(\ell_0^*, -1) \in \text{cl } G \Rightarrow (\ell_0^*, -1) \in G.$$

(b_9) For any $x^* \in X^*$ and any $\eta \geq 0$ satisfying $\ell_0 + (x^*)|_M \leq (\eta S)|_M$, there exists $y^* \in X^*$ such that $\ell_0 + (x^*)|_M = (y^*)|_M$ and $y^* \leq \eta S$ on X .

(b_{10}) If $\ell_0 \leq S|_M$, then there exists $y^* \in X^*$ such that $\ell_0 = (y^*)|_M$ and $y^* \leq S$.

Then we have

$$(b_{10}) \underset{\text{Analytic HB theorem}}{\Leftrightarrow} (a_{10}) \Leftarrow (a_9) \underset{\text{Stable Analytic HB theorem}}{\Leftrightarrow} (b_9).$$

Moreover, if S is continuous at 0_X , then statement (a_9) holds automatically and hence the remaining sentences also hold.

Remark (before the proof). We begin with the claim that for any continuous linear function ℓ on M , there exists $\ell^* \in X^*$ such that $\ell^*(x) = \ell(x)$ for all $x \in M$. In fact, defining the function $f : X \rightarrow \overline{\mathbb{R}}$ such that

$$(4.2) \quad f(x) := \begin{cases} -\ell(x) & \text{if } x \in M, \\ +\infty & \text{otherwise,} \end{cases}$$

one has $f \in \Gamma(X)$ (as M is closed), and its conjugate $f^*(x^*) = \sup_{x \in M} \{x^*(x) + \ell(x)\}$ is proper, i.e., there exists $x^* \in X^*$ such that $(x^*)|_M = -\ell$. So we can take $\ell^* = -x^*$.

Proof. Take $Y \equiv X$, $C = M$, f the function defined in (4.2) with ℓ_0 playing the role of ℓ , and $g(x) = x$ for all $x \in X$, and apply Corollary 3.7, where β must be zero because here C is a subspace on which f is linear. To this aim, we first observe that since S is lsc, the set in (3.11) is closed.

Second, for any $y^* \in X^*$, one has $y^* \circ g = y^*$, and

$$\text{epi}(y^* \circ g)^* = \{y^*\} \times \mathbb{R}_+, \quad \text{epi } i_C^* = \text{epi } i_M^* = M^\perp \times \mathbb{R}_+.$$

Moreover, if ℓ_0^* is an extension of ℓ_0 to X , then we have

$$\text{epi } f^* = (-\ell_0^* + M^\perp) \times \mathbb{R}_+.$$

Therefore, by (2.4),

$$\begin{aligned} \text{epi}(f + y^* \circ g + i_C)^* &= \text{cl} \left(\text{epi } f^* + \text{epi } y^* + \text{epi } i_M^* \right) \\ &= \text{cl} \left((-\ell_0^* + y^* + M^\perp) \times \mathbb{R}_+ \right) \\ &= (-\ell_0^* + y^* + M^\perp) \times \mathbb{R}_+, \end{aligned}$$

as the last set is weak*-closed. Consequently, in our setting, the set F in Corollary 3.7 becomes

$$\left[(-\ell_0^*, 1) + \tilde{K}^+ + M^\perp \times \{0\} \right] \times \mathbb{R}_+,$$

whose weak*-closedness is the same as that of the set G . This means that (a_9) holds if and only if (a_7) in Corollary 3.7 holds.

[(a_9) \Rightarrow (b_9)] Assume that (a_9) holds. Observe that for any $x^* \in X^*$ and any $\eta \geq 0$, one has

$$\begin{aligned} \ell_0 + x^* &\leq \eta S \quad \text{on } M \\ &\Downarrow \\ f - x^* + (\eta S) \circ g &\geq 0 \quad \text{on } M \\ &\Downarrow \\ \left(x \in M, \alpha \in \mathbb{R}, ((\eta S) \circ g)(x) \leq \alpha \right) &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0. \end{aligned}$$

Since (a_9) holds, (a_7) holds by the previous observation. It now follows from Corollary 3.7 that there exists $y^* \in X^*$ such that $y^* \leq \eta S$ on X and $f - x^* + y^* \geq 0$ on M . Since M is a subspace, we get $\ell_0 + x^* = y^*$ on M and (b_9) holds.

[(b_9) \Rightarrow (a_9)] It is a reformulation of [(b_7) \Rightarrow (a_7)] in Corollary 3.7.

[(a_{10}) \Leftrightarrow (b_{10})] It is a reformulation of [(a_8) \Leftrightarrow (b_8)] in Corollary 3.7.

Concerning the last assertion, if S is continuous at 0_X , it is bounded above on a certain neighborhood of 0_X , and since this neighborhood contains an absorbing set, S is finite valued and continuous on X . Then, by [1, Theorem 7.52], $S = i_K^*$, where K is a nonempty weak*-compact convex subset of X^* . Now we apply [16, section 25.4.2] to conclude that

$$\mathbb{R}_+(\partial S(0_X) \times \{-1\}) = \mathbb{R}_+(K \times \{-1\})$$

is weak*-locally compact and so weak*-closed. Then, the Dieudonné theorem (see, e.g., [25, Theorem 1.1.8]) applies to establish that

$$\tilde{K}^+ + M^\perp \times \{0\} = \mathbb{R}_+(\partial S(0_X) \times \{-1\}) + M^\perp \times \{0\}$$

is weak*-closed. Thus, G is weak*-closed and (a_9) holds. □

The following theorem provides an extension of the analytic Hahn–Banach theorem. Here, we introduce a pair of equivalent new conditions, (a_{9_1}) and (a_{10_1}) , which are both necessary and sufficient for such an extension.

THEOREM 4.2. *Let X be a locally convex space, $S : X \rightarrow \overline{\mathbb{R}}$ an lsc sublinear function, M a closed subspace of X such that $M \cap \text{dom } S \neq \emptyset$, and G the set defined in (4.1). Then, the following statements are mutually equivalent, and they are also equivalent to (b_9) when it is asserted for every continuous linear function on M (namely, ℓ_0):*

(a_{9_1}) *The set $\partial S(0_X) + M^\perp$ is weak*-closed in X^* .*

(a_{10_1}) *For any $x^* \in X^*$, $(x^*, -1) \in \text{cl } G \Rightarrow (x^*, -1) \in G$.*

(b_{10_1}) *For any continuous linear function ℓ on M such that $\ell \leq S|_M$, there exists $y^* \in X^*$ such that $\ell = (y^*)|_M$ and $y^* \leq S$.*

Proof. $[(a_{10_1}) \Rightarrow (a_{9_1})]$ Consider nets $\{u_i^*\}_{i \in I} \subset \partial S(0_X)$, $\{v_i^*\}_{i \in I} \subset M^\perp$ such that $u_i^* + v_i^*$ weak*-converges to $x^* \in X^*$. Then, $(u_i^*, -1) + (v_i^*, 0)$ weak*-converges to $(x^*, -1)$; i.e., $(x^*, -1) \in \text{cl } G$. By (a_{10_1}) we have that $(x^*, -1) \in G$ and there will exist $u^* \in \partial S(0_X)$, $v^* \in M^\perp$, and $\delta \geq 0$ such that $(x^*, -1) = \delta(u^*, -1) + (v^*, 0)$, which entails $\delta = 1$ and $x^* = u^* + v^* \in \partial S(0_X) + M^\perp$, i.e., this set is weak*-closed.

$[(a_{9_1}) \Rightarrow (a_{10_1})]$ Suppose that $\partial S(0_X) + M^\perp$ is weak*-closed and take any $x^* \in X^*$ such that $(x^*, -1) \in \text{cl } G$. This entails the existence of nets $\{u_i^*\}_{i \in I} \subset \partial S(0_X)$, $\{v_i^*\}_{i \in I} \subset M^\perp$, and $\{\delta_i\}_{i \in I} \subset \mathbb{R}_+$ such that

$$(x^*, -1) = \lim_{i \in I} \{\delta_i(u_i^*, -1) + (v_i^*, 0)\}.$$

This itself implies

$$\lim_{i \in I} \delta_i = 1$$

(without loss of generality we can assume $\delta_i > 0$ for all $i \in I$), and

$$\lim_{i \in I} \{\delta_i u_i^* + v_i^*\} = \lim_{i \in I} \delta_i \{u_i^* + (v_i^*/\delta_i)\} = x^*.$$

Since $u_i^* + (v_i^*/\delta_i) \in \partial S(0_X) + M^\perp$ for all $i \in I$, one has

$$x^* \in \text{cl}(\partial S(0_X) + M^\perp) = \partial S(0_X) + M^\perp,$$

and there must exist $u^* \in \partial S(0_X)$ and $v^* \in M^\perp$ such that $x^* = u^* + v^*$, and this yields

$$(x^*, -1) \in (u^*, -1) + (v^*, 0) \in G,$$

and we are done.

$[(a_{10_1}) \Rightarrow (b_{10_1})]$ It is a consequence of Theorem 4.1, with x^* being an extension of ℓ .

$[(b_{10_1}) \Rightarrow (a_{9_1})]$ Take an arbitrary $x^* \in \text{cl}(\partial S(0_X) + M^\perp)$. By Theorem 2.4.14 in [25],

$$\partial(S + i_M)(0_X) = \text{cl}(\partial S(0_X) + \partial i_M(0_X)) = \text{cl}(\partial S(0_X) + M^\perp).$$

The inclusion $x^* \in \partial(S + i_M)(0_X)$ entails that $\ell := (x^*)|_M \leq S|_M$, and (b_{10_1}) yields the existence of $y^* \in X^*$ such that $(x^*)|_M = (y^*)|_M$ and $y^* \leq S$ on X . Consequently,

$$x^* = y^* + (x^* - y^*) \in \partial S(0_X) + M^\perp,$$

and this set is certainly weak*-closed.

Finally, for any continuous linear function on M , ℓ_0 , we have $(b_9) \Leftrightarrow (a_9) \Rightarrow (a_{10_1}) \Leftrightarrow (b_{10_1})$. Moreover, (b_{10_1}) implies (b_9) when asserted for every continuous linear function on M , ℓ_0 . To see this implication, for any $x^* \in X^*$ apply (b_{10_1}) to $\ell := \ell_0 + (x^*)|_M$. \square

Remark. The equivalence between (a_{9_1}) and (b_{10_1}) was proved by M. Volle [24].

Next we recover the algebraic version of the celebrated Hahn–Banach theorem. To this aim, given a vector space X , we shall equip X with the *finest locally convex topology* on this space, which is represented by τ_X^f . This is the weakest locally convex topology on X for which all the seminorms are continuous. The family of sets

$$\{x \in X : p_i(x) < \varepsilon, i \in I\}$$

for any finite set I , any $\varepsilon > 0$, and any arbitrary collection of seminorms $\{p_i, i \in I\}$ is a neighborhood basis of 0_X for τ_X^f . The seminorms on X are the Minkowski gauges of absolutely convex absorbing subsets of X (which are barrels). If B is a set of this type and p_B is the Minkowski gauge of B , $\{x \in X : p_B(x) < 1\}$ is an open set for this finest topology which is contained in B and therefore B is a neighborhood of 0_X . For this reason, this topology is said to be barreled, and it has some appealing properties. For instance, every linear subspace is closed and every linear function on X is continuous [21, Exercise 7, p. 69]. If $X^\#$ is the algebraic dual of X (the space of all the linear maps $x^\# : X \rightarrow \mathbb{R}$), the mentioned property yields $X^\# = X^*$, and the finest locally convex topology on X is nothing but the Mackey topology consistent with the dual pair $\langle X, X^\# \rangle$.

COROLLARY 4.3 (algebraic Hahn–Banach theorem). *Let X be a vector space, $S : X \rightarrow \mathbb{R}$ be a sublinear function, $M \subset X$ be a subspace, and ℓ_0 be a linear function on M satisfying $\ell_0 \leq S|_M$. Then there exists a linear function $x^\#$ on X such that $(x^\#)|_M = \ell_0$ and $x^\# \leq S$ on X .*

Proof. Let us equip X with the finest locally convex topology τ_X^f . With this topology, M is a closed subspace.

Next, let us prove that the finest locally convex topology defined on the space M , denoted by τ_M^f , is the topology induced (relative) by τ_X^f on M . Then, the linearity of ℓ_0 on M will imply its continuity.

We know that the family of sets

$$U := \{y \in M : q_i(y) < \varepsilon, i \in I\}$$

for any finite set I , any $\varepsilon > 0$, and any arbitrary collection of seminorms on M , $\{q_i, i \in I\}$, is a neighborhood basis of 0_X for τ_M^f . Associated with $q_i, i \in I$, we define the following seminorm on X :

$$p_i(x) = q_i(x_M), i \in I,$$

where $x = x_M + x_N$, with $x_M \in M$ and $x_N \in N$, N being an algebraic complement of M in X , i.e., a linear subspace such that $X = M \oplus N$. The conclusion comes from the fact that

$$U = \{x \in X : p_i(x) < \varepsilon, i \in I\} \cap M,$$

which is a neighborhood of 0_X for the topology induced by τ_X^f on M .

Finally, associated with the given sublinear function S , define the function on X

$$p(x) := \max\{S(x), S(-x)\}.$$

It is easy to verify that p is a seminorm and so it is continuous with respect to τ_X^f . Consequently, p is bounded above on a neighborhood of 0_X , and a fortiori, S is also bounded above in this neighborhood as $S \leq p$. Hence, S is continuous at 0_X , and Theorem 4.1 applies, leading us to the conclusion. \square

5. From the Farkas lemma to the Hahn–Banach–Lagrange theorem.

In this section we introduce a new version of the so-called Hahn–Banach–Lagrange theorem which can be considered as an extension of [23, Theorem 1.11] to extended-valued functions. Moreover, we also establish necessary and sufficient conditions for the validity of such a version in $[(a_8) \Leftrightarrow (b_{12})]$. Even better, in the counterpart of the next theorem, namely, $[(a_7) \Leftrightarrow (b_{11})]$, we establish a characterization for the stability of such an extended version of the Hahn–Banach–Lagrange theorem under linear perturbations.

THEOREM 5.1 (Hahn–Banach–Lagrange theorem). *Let X and Y be locally convex spaces, $C \subset X$ be a nonempty closed convex subset, $S : Y \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, and $g : X \rightarrow Y^\bullet$ be an S -convex function such that the set in (3.11) is closed. Let further $f \in \Gamma(X)$, and assume that (3.23) holds.*

Consider the following statements:

(a_7) F is weak*-closed in the product space $X^* \times \mathbb{R} \times \mathbb{R}$.

(a_8) F is closed regarding $\{0_{X^*}\} \times \{0\} \times \mathbb{R}$.

(b_{11}) For any $x^* \in X^*$ and any $\eta \geq 0$,

$$\left(\inf_C [f - x^* + (\eta S) \circ g] \in \mathbb{R} \right)$$

\Updownarrow

$$\left(\exists y^* \in Y^* \text{ such that } y^* \leq \eta S \text{ on } Y \text{ and } \inf_C [f - x^* + y^* \circ g] = \inf_C [f - x^* + (\eta S) \circ g] \in \mathbb{R} \right).$$

$$(b_{12}) \quad \left(\inf_C [f + S \circ g] \in \mathbb{R} \right)$$

\Updownarrow

$$\left(\exists y^* \in Y^* \text{ such that } y^* \leq S \text{ on } Y \text{ and } \inf_C [f + y^* \circ g] = \inf_C [f + S \circ g] \in \mathbb{R} \right).$$

Then we have

$$(b_{12}) \quad \underset{\text{Analytic HBL theorem}}{\Leftrightarrow} \quad (a_8) \quad \Leftarrow \quad (a_7) \quad \underset{\text{Stable Analytic HBL theorem}}{\Leftrightarrow} \quad (b_{11}).$$

Proof. $[(a_7) \Rightarrow (b_{11})]$ Assume that (a_7) holds. Take arbitrary $x^* \in X^*$, $\eta \geq 0$, let $\beta = \inf_C [f - x^* + (\eta S) \circ g]$, and assume that $\beta \in \mathbb{R}$. Then

$$(5.1) \quad f(x) - \langle x^*, x \rangle + ((\eta S) \circ g)(x) \geq \beta \quad \forall x \in C.$$

It is easy to see that the last inequality is equivalent to

$$(5.2) \quad (x \in C, \alpha \in \mathbb{R}, ((\eta S) \circ g)(x) \leq \alpha) \Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq \beta.$$

Since (a_7) holds, it follows from Corollary 3.7 that there exists $y^* \in Y^*$ such that $y^* \leq \eta S$ on Y and

$$f(x) - \langle x^*, x \rangle + (y^* \circ g)(x) \geq \beta \quad \forall x \in C,$$

which leads to

$$\inf_C [f - x^* + y^* \circ g] \geq \beta = \inf_C [f - x^* + (\eta S) \circ g].$$

Since $y^* \leq \eta S$ on Y , the converse inequality holds trivially. Hence,

$$(5.3) \quad \inf_C [f - x^* + y^* \circ g] = \inf_C [f - x^* + (\eta S) \circ g] = \beta.$$

Conversely, it is obvious that if (5.3) holds, then (5.1) also holds, and hence the first statement in (b_{11}) holds as well. Consequently, (b_{11}) is satisfied.

$[(b_{11}) \Rightarrow (a_7)]$ We have seen that (5.1) is equivalent to (5.2). So if (5.3) and (5.1) are equivalent, then all three statements are equivalent to each other, and hence for any $\beta \in \mathbb{R}$ and any $\eta \geq 0$ such that $\inf_C [f - x^* + (\eta S) \circ g] \geq \beta$, (b_7) holds in this case. Thus, the condition (a_7) follows from Corollary 3.7.

$[(a_8) \iff (b_{12})]$ This is the same as the proof of $[(a_7) \iff (b_{11})]$, taking $x^* = 0_{X^*}$ and using the assertion $(a_8) \iff (b_8)$ in Corollary 3.7. \square

We now derive two corollaries from Theorem 5.1. The first characterizes an extended version of the Mazur-Orlicz theorem together with its stable form concerning extended sublinear functions.

THEOREM 5.2 (analytic Mazur-Orlicz theorem). *Let X be a locally convex space, $S : X \rightarrow \overline{\mathbb{R}}$ be an lsc sublinear function, and let C be a closed convex subset of X with $C \cap \text{dom}S \neq \emptyset$. Set*

$$H := \widetilde{K}^+ \times \{0\} + \{(u^*, 1, r) : (u^*, r) \in \text{epi } i_C^*\}.$$

Consider the following statements:

(a_{13}) H is weak*-closed in the product space $X^* \times \mathbb{R} \times \mathbb{R}$.

(a_{14}) H is closed regarding $\{0_{X^*}\} \times \{0\} \times \mathbb{R}$.

(b_{13}) For any $x^* \in X^*$ and any $\eta \geq 0$

$$\begin{aligned} & \left(\inf_C [-x^* + \eta S] \in \mathbb{R} \right) \\ & \quad \Updownarrow \\ & \left(\exists y^* \in X^* \text{ such that } y^* \leq \eta S \text{ on } X \text{ and} \right. \\ & \quad \left. \inf_C [-x^* + y^*] = \inf_C [-x^* + \eta S] \in \mathbb{R} \right). \end{aligned}$$

(b_{14})

$$\begin{aligned} & \left(\inf_C S \in \mathbb{R} \right) \\ & \quad \Updownarrow \\ & \left(\exists y^* \in X^* \text{ such that } y^* \leq S \text{ on } X \text{ and } \inf_C y^* = \inf_C S \in \mathbb{R} \right). \end{aligned}$$

Then we have

$$(b_{14}) \quad \text{Analytic MO theorem} \quad \Leftrightarrow \quad (a_{14}) \quad \Leftarrow \quad (a_{13}) \quad \text{Stable Analytic MO theorem} \quad \Leftrightarrow \quad (b_{13}).$$

If in addition S is continuous at 0_X , then (a₁₃) holds automatically and hence the remaining sentences also hold.

Proof. In the notation of Theorem 5.1, take $Y \equiv X$, $g : X \rightarrow X$ with $g(x) = x$ for all $x \in X$, and $f \equiv 0$. Then by assumption, $C \cap \text{dom } f \cap \text{dom}(S \circ g) \neq \emptyset$ and the set $\{(x, y, \lambda) \in X \times Y \times \mathbb{R} : S(g(x) - y) \leq \lambda\}$ is closed as S is lsc on X .

Now, for any $y^* \in X^*$, one has

$$\begin{aligned} (f + y^* \circ g + i_C)^*(x^*) &= \sup_{x \in X} [\langle x^*, x \rangle - \langle y^*, x \rangle - i_C(x)] \\ &= i_C^*(x^* - y^*). \end{aligned}$$

Hence, $(x^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*$ if and only if $(x^*, r) \in (y^*, 0) + \text{epi } i_C^*$. It follows that the set F in Corollary 3.7 and Theorem 5.1 collapses to

$$\bigcup_{(y^*, -\mu) \in \tilde{K}^+} \{(y^*, -\mu, 0)\} + \{(u^*, 1, r) : (u^*, r) \in \text{epi } i_C^*\},$$

which is exactly the set H . Thus, the conclusion follows from Theorem 5.1.

In the case where S is continuous at 0_X , the set H is weak*-closed by an argument similar to the one of Theorem 4.1, using the Dieudonné theorem [25, Theorem 1.1.8]. \square

The following algebraic version of Mazur–Orlicz (see, e.g., [22, Corollary 3.3]) comes as a consequence of Theorem 5.2.

COROLLARY 5.3 (algebraic Mazur–Orlicz theorem). *Let X be a vector space, $S : X \rightarrow \mathbb{R}$ be a sublinear function, and C be a nonempty convex subset of X . Then there exists a linear function $y^\# \in X^\#$ satisfying $y^\# \leq S$ on X and*

$$\inf_C y^\# = \inf_C S.$$

Proof. Let us equip X with the finest locally convex topology. According to the argument in the last paragraph of the proof of Corollary 4.3, S is continuous at 0_X , and hence, by [25, Theorem 2.2.9], S is continuous on X .

If $\inf_C S = -\infty$, Corollary 4.3 (applied to $M = \{0_X\}$) guarantees the existence of $y^\# \in X^\#$ such that $y^\# \leq S$ on X , so that $\inf_C y^\# = -\infty$ too.

If $\inf_C S > -\infty$, Theorem 5.2 (applied to $\text{cl } C$) ensures the existence of a linear function $y^\# \in X^\#$ such that $y^\# \leq S$ on X and

$$\inf_{\text{cl } C} y^\# = \inf_{\text{cl } C} S.$$

The conclusion follows as both $y^\#$ and S are continuous on X . \square

6. The equivalence between the new versions of the Farkas lemma and the Hahn–Banach–Lagrange theorem. It was shown in the previous sections that

Theorem 3.1 \Rightarrow Corollary 3.2 \Rightarrow Theorem 3.4 \Rightarrow Corollary 3.7 \Rightarrow Theorem 5.1.

Next we prove the equivalence of these results by showing that

$$\text{Theorem 5.1} \Rightarrow \text{Theorem 3.1}.$$

Proof of Theorem 5.1 \Rightarrow Theorem 3.1. Let X , Y , C , K , f , and g be as in Theorem 3.1 and assume that

$$(\text{dom } f) \cap \{x \in C : g(x) \in -K\} \neq \emptyset.$$

We first observe that K is closed by the assumption that g is K -epi closed, and hence, if we set $S := i_{-K}$, then S is an lsc sublinear function, $\partial S(0_Y) = K^+$, and g is S -convex.

Second, since S is the indicator function of K , we get

$$\{(x, y, \lambda) \in X \times Y \times \mathbb{R} : S(g(x) - y) \leq \lambda\} = \text{epi}_K g \times [0, +\infty],$$

which is closed in $X \times Y \times \mathbb{R}$ by the K -epi closedness of the mapping g .

Third, note that we also have

$$C \cap \text{dom } f \cap \text{dom}(S \circ g) = \text{dom } f \cap \{x \in C : g(x) \in -K\} \neq \emptyset.$$

(If $g(x) \in -K$, $S \circ g(x) = 0$; otherwise, i.e., if $g(x) \notin -K$, then $S \circ g(x) = i_{-K}(g(x)) = +\infty$.)

In order to apply Theorem 5.1, we observe that for any $y^* \in Y^*$ and $\mu \geq 0$, one has

$$(6.1) \quad y^* \leq \mu S = \mu i_{-K} = i_{-K} \Leftrightarrow y^* \in K^+.$$

Therefore, in our notation, the set F in Theorem 5.1 (see also Corollary 3.7) becomes

$$I = \bigcup_{y^* \in K^+, \mu \in \mathbb{R}_+} \{(u^*, 1 - \mu, r) : (u^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*\},$$

while the set \mathfrak{C} in Theorem 3.1 is

$$\mathfrak{C} := \bigcup_{y^* \in K^+} \text{epi}(f + y^* \circ g + i_C)^*.$$

So, in the next step, we will show that the set I is weak*-closed if and only if the set \mathfrak{C} enjoys the same property. Indeed, assume that \mathfrak{C} is weak*-closed and take a net $\{(x_i^*, 1 - \mu_i, r_i), i \in I\}$ contained in I , which is weak*-converging to $(x^*, 1 - \mu, r)$. Then there is a net $\{y_i^*, i \in I\}$ such that

$$\begin{aligned} \{y_i^*, i \in I\} &\subset K^+, \quad \{\mu_i, i \in I\} \subset \mathbb{R}_+, \\ \text{and for each } i \in I, &(x_i^*, r_i) \in \text{epi}(f + y_i^* \circ g + i_C)^* \subset \mathfrak{C}. \end{aligned}$$

Since (x_i^*, r_i) weak*-converges to (x^*, r) and \mathfrak{C} is weak*-closed, we get $(x^*, r) \in \mathfrak{C}$, which means that there exist $y^* \in K^+$, $(x^*, r) \in \text{epi}(f + y^* \circ g + i_C)^*$, and hence, $(x^*, 1 - \mu, r) \in I$. Therefore, I is weak*-closed. In a similar way we can show that the weak*-closedness of I implies the weak*-closedness of \mathfrak{C} .

It is now clear that (a_1) is equivalent to (a_7) and, by Theorem 5.1, (a_7) is equivalent to (b_{11}) . So, to prove that (a_1) is equivalent to (b_1) , it is sufficient to show that (b_1) holds if and only if (b_{11}) holds.

Assume that (b_1) holds, i.e., for any $x^* \in X^*$ and any $\beta' \in \mathbb{R}$,

$$\begin{aligned} (x \in C, g(x) \in -K \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta') \\ \Updownarrow \\ (\exists y^* \in K^+ \text{ such that } f - x^* + y^* \circ g \geq \beta' \text{ on } C). \end{aligned}$$

Now, take arbitrary $x^* \in X^*$ and $\eta \geq 0$. Assume that

$$(6.2) \quad \inf_C [f - x^* + (\eta S) \circ g] =: \beta \in \mathbb{R}.$$

Then, as $S = i_{-K}$ (and also $\eta S = i_{-K}$ for every $\eta \geq 0$), (6.2) implies

$$x \in C, g(x) \in -K \Rightarrow f(x) - \langle x^*, x \rangle \geq \beta.$$

By (b_1) , with $\beta' = \beta$, this is equivalent to

$$(6.3) \quad \exists y^* \in K^+ \text{ such that } f - x^* + y^* \circ g \geq \beta \text{ on } C.$$

In turn, (6.3) is equivalent to the fact that there is $y^* \in Y^*$ such that $y^* \leq \eta S = S$ and

$$\inf_C [f - x^* + y^* \circ g] \geq \beta = \inf_C [f - x^* + (\eta S) \circ g].$$

The converse inequality holds also as $y^* \leq \eta S$, that is,

$$(6.4) \quad \inf_C [f - x^* + y^* \circ g] = \inf_C [f - x^* + (\eta S) \circ g] = \beta \in \mathbb{R},$$

which means that (b_{11}) holds.

Note that with $y^* \in K^+$ and taking (6.1) into account, we have that (6.2)–(6.4) are all equivalent together, which means that the implication $(b_{11}) \Rightarrow (b_1)$ has been proved as well. Consequently, $(b_{11}) \Leftrightarrow (b_1)$.

The remaining part of the conclusion can be proved in a similar way. \square

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