ORIGINAL ARTICLE

Mediation in claims problems

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Abstract Mediation is a dispute resolution process whereby agents reach a mutually acceptable agreement among different proposals that satisfy a set of principles. This paper provides a natural way of coming to such agreements in claims problems. In our approach, *mediation* combines (i) a set of *fair* properties (*legitimate principles*); and (ii) a *criterion* for delimiting the admissible manners of distributing the endowment, that is determined by the *mediator* expressing the two (dual) points of view to face such problems: awards and losses. These dual views define a lower and an upper bounds on awards, which are used to implement the so-called *Double Recursive Process*. We find that this process concludes at the midpoint between the two *dual* points of view. Finally, we argue that the criterion of the mediator could be established throughout *Lorenz domination*. In so doing, we retrieve the average of old and well-known rules.

1 Introduction

How should scarce resources be allocated among claimants? Such problems, in which the available quantity of a perfectly divisible good (or *endowment*) is not sufficient to

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satisfy the aggregate claim against the good, are known as *claims problems*. The formal analysis of these situations, which originates in a seminal paper by O'Neill (1982), proposes a number of well-behaved rules recommending a division of the resources among agents that depends both on the endowment and on the agents' claims. The term "well-behaved" indicates that these rules may satisfy principles of *fairness* or other appealing properties. In this context, how can claimants commit to a distribution of resources? How can claimants agree on the properties that must be applied to solve such a problem?

Nowadays, conflicts in which two or more agents cannot decide the allocation have been resolved by a trial. In such cases, a neutral third agent (*the mediator*) dictates *the manner* by which the endowment is distributed. This is the philosophy behind mediation, as used in legal, commercial, family and public policy disputes. More precisely, the World Mediation Forum defines mediation "*as a process by which a neutral third agent, called a mediator, helps agents in conflict to negotiate a mutually acceptable agreement*."

Accordingly, we analyse the figure of a mediator taking as our starting point *claims problems with legitimate principles*. To this respect, Giménez-Gómez and Marco-Gil (2014) synthesize earlier work to thoroughly analyse the consequences of enriching the classical model of rationing by adding an additional element, $\mathcal{P}$, that represents the *legitimate principles* set. This set consists of a group of some *fair* axioms or properties that are commonly accepted by the agents, or that are imposed by law and tradition. Nevertheless, *fairness* hardly leads to a unique commitment: the same claims problem faced by two different agents may lead to the use of two different points of view (Moulin 1988; Schokkaert and Overlaet 1989; Young 1987, among others). Particularly, two natural *dual* points of view appear in claims problems: awards and losses. In the former case, each agent is worried about "what she receives", while in the latter one, she cares about "what is not satisfied of her claims".

In the current approach, mediation combines these two dual points of view with the imposition of *fair* principles. Specifically, a *mediator* proposes a way of distributing the endowment, F, among those that satisfy the *legitimate principles* $\mathcal{P}$ on which the society has agreed, so that the *admissible* rules should fulfil $\mathcal{P}$ and propose an allocation between F and its dual rule, F^d.

Then, we study the consequences of establishing a warranty in awards (lower bound on awards), and the obligation of incurring minimum losses (lower bound on losses, or, equivalently, an upper bound on awards). The idea of establishing a lower bound on awards can be found in many papers on claims problems (O'Neill 1982; Herrero and Villar 2001, 2002; Moulin 2002; Moreno-Ternero and Villar 2004; Giménez-Gómez and Marco-Gil 2014; Dominguez 2013). In fact, the formal definition of a rule already includes both upper and lower bounds on awards by requiring that no agent receives more than her claim or less than zero. Actually, there are many situations where this warranties appear: bankruptcy law, inheritance, divorce, the universal basic income (Tobin et al. 1967; Noguera 2010). On the other hand, since claims problems can be faced from the point of view of the incurred losses, the notion of an upper bound on awards naturally arises.

From this reasoning, the so-called *Double Recursive Process* provides a *manner* to solve claims problems as the recursive application of both:[1]

- a lower bound, which guarantees that each agent should receive at least the smallest amount recommended by all the *admissible* allocations; and
- an upper bound, which limits the awards an agent should receive by the highest amount provided by these allocations.

Our process can be summarized in the following way: Given a claims problem (E, c), a *mediator* determines a set of *legitimate principles* $\mathcal{P}$ and a *criterion* $\mathcal{D}$ to choose a particular rule F that delimits the set of admissible proposals. Accordingly, the agents compute their minimal expected award (that we call *F-right*) among all admissible proposals; and accept this award. At the same time, they also compute their maximal expected award (that we call *F-utopia*); and give up any claims greater than their *utopia*. If there are available resources after distributing the *rights*, the process continues until the endowment is exhausted. We show that this process ends at the midpoint of the two dual proposals, F and F^d. This result provides a new justification of the average of two extreme and dual ways of distributing the endowment.

The paper is organized as follows: Sect. 2 presents the preliminaries, and Sect. 3 provides our approach. Sections 4 and 5 contain our main definitions and results, respectively. Section 6 applies them to different *legitimate principles* sets, and Sect. 7 summarizes our final remarks. Finally, the Appendices gather the technical proofs.

2 Preliminaries

Given a set of agents $N = \{1, 2, \ldots, n\}$, **a claims problem** is a pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ such that $\sum_{i \in N} c_i \geq E$, where E is the endowment, that represents the quantity of a perfectly divisible good to be allocated to the agents, and for each $i \in N$, c_i is the claim that agent i has on the endowment E. We will denote by $C = \sum_{i \in N} c_i$ the aggregate claim. $\mathfrak{B}$ represents the set of all claims problems.

A **rule** (claims problems solution) is a function, $\varphi : \mathfrak{B} \to \mathbb{R}_+^n$, such that for each $(E, c) \in \mathfrak{B}$, satisfies:

1. $\sum_{i \in N} \varphi_i(E, c) = E$ (*efficiency*); and
2. $0 \leq \varphi_i(E, c) \leq c_i, \forall i \in N$ (*non-negativity* and *claim-boundedness*).

Following Thomson (2013), *in claims problems, no claimant should be asked to pay* (non-negativity), *no claimant should be awarded more than his claim* (claim-boundedness) *and the distributions must be balanced: the sum of the awards should be equal to the endowment* (efficiency).

Each claims problem can be faced from two points of view: those of awards and losses. Thus, we have two *dual* positions depending on whether we are focused on awards or on the unsatisfied demand, $L = \sum_{i \in N} c_i - E$. Given a rule φ, its dual rule distributes "what is missing" in the same way that φ divides "what is available" (Aumann and Maschler 1985). Formally,

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[1] The recursive application of a lower bound has been used by Alcalde et al. (2005), Dominguez and Thomson (2006) and Dominguez (2013).

The **dual** of a rule φ, denoted by φ^d, is defined by setting for each $(E, c) \in \mathfrak{B}$ and each $i \in N$, $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$.

It is straightforward to check that the duality operator is well defined, since for each $(E, c) \in \mathfrak{B}$, $(L, c) \in \mathfrak{B}$ and if φ satisfies *efficiency*, *non-negativity* and *claim-boundedness*, so does φ^d. It is also clear that $(\varphi^d)^d = \varphi$.

2.1 Rules

In this paper we will focus on some particular rules: the *Proportional*, the *Constrained Equal Awards*, and the *Constrained Equal Losses* rules.

The ***Proportional*** (*P*) rule recommends a division of the endowment proportionally to the claims: for each $(E, c) \in \mathfrak{B}$ and each $i \in N$,

$$P_i(E, c) = \frac{c_i}{C} E.$$

The ***Constrained Equal Awards*** (*CEA*) rule (Maimonides, twelfth century, among others) recommends equal awards to all agents, subject to no one receiving more than her claim: for each $(E, c) \in \mathfrak{B}$ and each $i \in N$,

$$CEA_i(E, c) \equiv \min\{c_i, \mu\},$$

where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

The ***Constrained Equal Losses*** rule (*CEL*), discussed by Maimonides (Aumann and Maschler 1985), is the dual of the *Constrained Equal Awards* (Herrero and Villar 2001). Specifically, it chooses the awards vector at which all agents incur equal losses, subject to no one receiving a negative amount: for each $(E, c) \in \mathfrak{B}$ and each $i \in N$,

$$CEL_i(E, c) \equiv \max\{0, c_i - \mu\},$$

where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

Note that the *dual* of the *Proportional* rule is itself (this property is known as *self-duality*, see Sect. 2.2).

For the sake of facilitate the understanding of the mentioned rules, we provide the graphical representation (in the two agents case) of these rules, and a simple numerical example.

Example 1 Consider now the claims problem $(E, c) = (15, (8, 12))$. Table 1 shows the proposals of the different rules we have introduced.

Table 1 Proposals given by rules, $(E, c) = (15, (8, 12))$

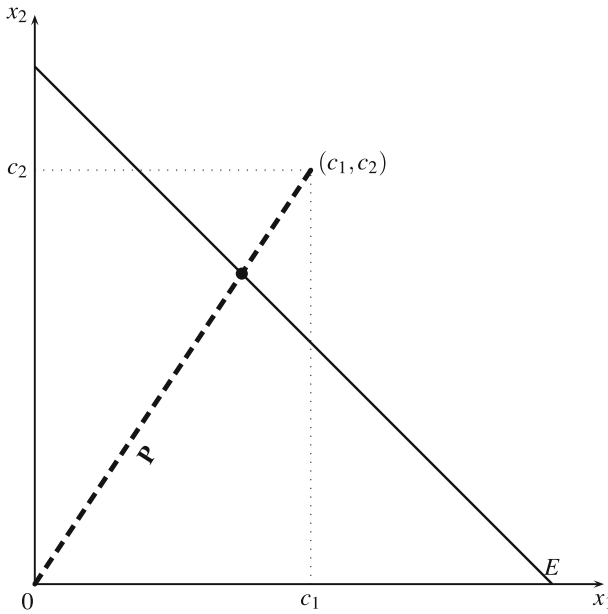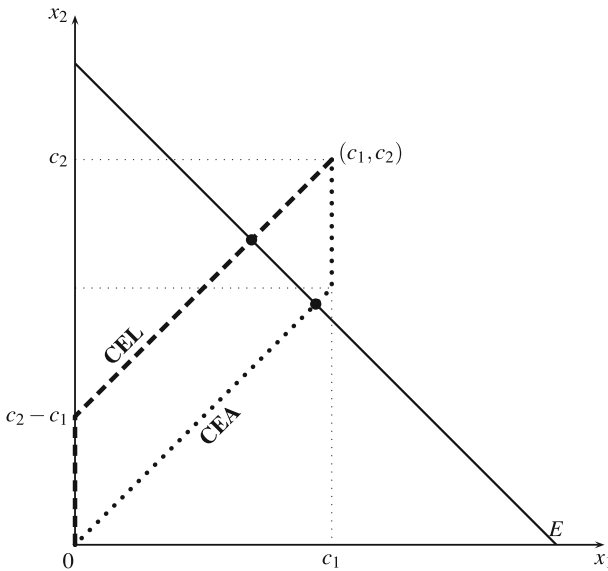| P	CEA	CEL
(6,9)	(7.5,7.5)	(5.5,9.5)

Fig. 1 The Proportional rule

Fig. 2 The CEA and CEL rules

In Figs. 1 and 2 we show the path defined by the above mentioned rules. Then, for a given claims problem $(E, c) \in \mathcal{B}$ with two agents, the proposed solution is obtained where this path intersects the efficient boundary $x_1 + x_2 = E$.

Note that the more properties are required by a society, the fewer the number of *admissible* rules.

2.2 Properties

We present in this section some properties on the rules that will be useful in our discussion.

Resource monotonicity (Curiel et al. 1987; Young 1987) demands that if the endowment increases, then all individuals should get at least what they received initially. Note that, no rule violating this property has been proposed in claims problems literature. Otherwise we could have situations where an increment of the endowment would cause some disadvantages to certain agents. This property is stated as follows: for each $(E, c) \in \mathfrak{B}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then, for each $i \in N$,

$$\varphi_i(E', c) \geq \varphi_i(E, c).$$

Order preservation (Aumann and Maschler 1985) requires that agents with higher claims should receive, and lose, at least as much as agents with lower claims do: for each $(E, c) \in \mathfrak{B}$, and each $i, j \in N$, such that $c_i \geq c_j$, then

$$\varphi_i(E, c) \geq \varphi_j(E, c), \quad \text{and} \quad c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c).$$

Composition up (Moulin 2000) states that if after having distributed the endowment, this increases, then a rule recommends the same if (i) we cancel the initial distribution and apply the rule to the revised problem, or (ii) we let agents keep their initial awards, adjust their claims down by these amounts, and re-apply the rule to allocate the increment of the endowment. For each $(E, c) \in \mathfrak{B}$, each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then

$$\varphi(E', c) = \varphi(E, c) + \varphi(E' - E, c - \varphi(E, c)).$$

Self-duality (Aumann and Maschler 1985) demands that a rule recommends the same allocation for a claims problem (E, c) when dividing "what is available" (awards) or "what is missing" (losses). That is, the rule coincides with its dual: for each $(E, c) \in \mathfrak{B}$,

$$\varphi(E, c) = \varphi^d(E, c).$$

Table 2 summarizes the properties that fulfil the solutions we work on.

Table 2 Properties and rules

	CEA	CEL	P
Resource monotonicity	Yes	Yes	Yes
Order preservation	Yes	Yes	Yes
Composition up	Yes	Yes	Yes
Self-duality	No	No	Yes

3 Mediation

Our notion of **mediation** is given by both a set of properties and an *admissible zone* that selects some of the rules that satisfy those properties. First, a group of properties are accepted by a particular society (either these properties can be assumed by the agents involved in the problem, or they can be established by law).[2] Then, among the rules fulfilling these properties, a reference rule F and its *dual* rule F^d delimit the set of proposals that are admissible by a particular mediator. In this sense, mediation involves:

1. a set of *legitimate principles* (properties) $\mathcal{P}$;
2. a *criterion* $\mathcal{D}$ that selects a *reference rule* F, satisfying the properties in $\mathcal{P}$; and
3. an *admissible zone* delimited by rule F and its dual F^d.

In Sect. 6 we propose the *Lorenz domination* as the criterion to be used in order to select the reference rule F. So, we consider first a society in which the distribution of the endowment must be based on the selected reference rule, that delimits a set of admissible allocations for a specific mediator. Next, we formally present our model.

Definition 1 A **claims problem with mediation** is a triplet (E, c, F), wherein $(E, c) \in \mathfrak{B}$, and F is a fixed rule determined from a set of principles $\mathcal{P}$ upon which a particular society has agreed. Let $\mathfrak{B}_F$ be the set of all *claims problems with mediation*.

Then, we identify the *mediator* with the rule F, that will delimit the *admissible zone*. Although F depends on the set $\mathcal{P}$ of legitimate principles, and the criterion $\mathcal{D}$ used, $F = F(\mathcal{P}, \mathcal{D})$, we omit this fact in the notation, unless needed. We now formally define the admissible rules for a given mediator.

Definition 2 A rule $\varphi : \mathfrak{B} \to \mathbb{R}_+^n$ is said to be **admissible** for mediator $F = F(\mathcal{P}, \mathcal{D})$ if φ satisfies all properties in $\mathcal{P}$ and for all $i \in N$,

$$\min\{F_i(E, c), F_i^d(E, c)\} \le \varphi_i (E, c) \le \max\{F_i(E, c), F_i^d(E, c)\}.$$

Let $\Phi(F)$ denote the set of *admissible* rules for mediator F.

Example 2 (*Proportional mediator*) Consider a mediator such that she asks the rules to fulfill *self-duality* and *composition-up*. In this case, the only rule that satisfies these properties is the *Proportional*. So that, $F = P$. Since this solution is *self-dual*, $F = F^d$, the only admissible rule is the *Proportional* one. In this case, the criterion of the mediator determines directly the solution.

Example 3 (*Egalitarian mediator*) Consider a mediator such that every rule is accepted and assume that the criterion to select one of the rules (see Sect. 6) determines $F = CEA$. Then, the admissible rules are those whose proposals for each agent i are between the values $CEA_i(E, c)$ and $CEL_i(E, c)$.

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[2] For instance, "any bankruptcy rule that is in line with the existing laws as described in the USA Bankruptcy Code is necessarily *population monotonic*" (Grahn and Voorneveld 2002).

For instance, if $(E, c) = (15, (8, 12))$, the *admissible* rules consist of the proposals within the set $\{(a, b) \in \mathbb{R}^2 : a + b = 15 : 5.5 \leq a \leq 7.5\}$, see Fig. 2. In this case the distribution of the endowment is undetermined and a process is needed to obtain a solution for the claims problem.

4 The double recursive process

Given the set of *admissible* rules for mediator F, $\Phi(F)$, we can obtain a minimal and a maximal level of awards that each agent should receive on the basis of the ordinary meaning of a guarantee. The maximal level on awards implies a limit on the losses the agent incurs. In this sense, we can associate to every *claims problem with mediation* two bounds on the awards each agent should receive:

- the *F-rights* value, which is a lower bound on awards that provides each agent the smallest amount recommended to her by all *admissible* rules (see Giménez-Gómez and Marco-Gil 2014); and
- the *F-utopia* value, which is an upper bound that limits the awards that each agent can receive to the highest amount recommended to her by all *admissible* rules.

Definition 3 Given $(E, c, F) \in \mathfrak{B}_F$, the **F-rights** value, r, is for each $i \in N$,

$$r_i(E, c, F) = \min_{\varphi \in \Phi(F)} \{\varphi_i(E, c)\}.$$

Definition 4 Given $(E, c, F) \in \mathfrak{B}_F$, the **F-utopia** value, u, is for each $i \in N$,

$$u_i(E, c, F) = \max_{\varphi \in \Phi(F)} \{\varphi_i(E, c)\}.$$

Remark 1 Given a particular claims problem (E, c) the *admissible* rule that provides the *F-rights* (*F-utopia*) for agent i, could be different from the *admissible* rule that provides the *F-rights* (*F-utopia*) for agent $j \neq i$.

The *F-rights* can be understood as the part of the claims that are not disputed (an idea that has been already used in claims problems (see, for instance, Aumann and Maschler 1985)), so that it should be ensured to each agent. Furthermore, assigning to each agent her *F-rights* (as we have defined it) is compatible with the existing endowment E, that is $\sum_{i \in N} r_i \leq E$. Moreover, all the *admissible* rules recommend to each agent at least this amount. As, in general, the sum of the agents' *F-rights* does not exhaust the endowment, a recursive process arises in a natural way in order to obtain an efficient allocation: we can allocate her *F-rights* to any agent, and then proceed again in the new problem defined by the remaining estate (E minus the allocated amounts), and the claims resulting from subtracting to the initial ones the amount which has already been granted ($c_i - r_i$). As we will see (Example 5) this process may end up at one of the extremes of the admissible zone, defined by the reference rule F and its dual.

By applying the idea of guaranteeing a minimum amount on losses rather than awards, we propose truncate the claims by the maximum amount that each agent can

expect to receive, the *F-utopia*. The idea of truncate the claims is not new and, in fact, it is also commonly assumed that the part of the claims that are not feasible should not to be considered in the problem (see, for instance, Aumann and Maschler 1985).

Therefore, we define an iterative process, that we call the *Double Recursive Process* in which, at any step, each agent receives her *F-rights* and, at the same time, her claim is truncated by her *F-utopia*.

Definition 5 The **Double Recursive Process**, DRP, associates for each $(E, c, F) \in \mathfrak{B}_F$ and each $i \in N$,

$$DRP_i(E, c) = \sum_{m=1}^{\infty} r_i(E^m, c^m, F),$$

where for each step $m \in \mathbb{N}$, $r_i(E^m, c^m, F)$ is defined in the following way:
step 1, for $m = 1$, define $(E^1, c^1) = (E, c)$ and compute

$$r_i(E^1, c^1, F) = \min_{\varphi \in \Phi(F)} \{\varphi_i(E^1, c^1)\},$$
$$u_i(E^1, c^1, F) = \max_{\varphi \in \Phi(F)} \{\varphi_i(E^1, c^1)\}.$$

step m, for $m \geq 2$, define

$$E^m = E^{m-1} - \sum_{i \in N} r_i(E^{m-1}, c^{m-1}, F),$$
$$c_i^m = u_i(E^{m-1}, c^{m-1}, F) - r_i(E^{m-1}, c^{m-1}, F),$$

and compute

$$r_i(E^m, c^m, F) = \min_{\varphi \in \Phi(F)} \{\varphi_i(E^m, c^m)\},$$
$$u_i(E^m, c^m, F) = \max_{\varphi \in \Phi(F)} \{\varphi_i(E^m, c^m)\}.$$

As shown in Theorem 1 this process completely exhausts the endowment E, so it provides an efficient allocation. In Example 4 we show this process in a particular claims problem with mediation.

Example 4 Consider the Egalitarian mediator as in Example 3, and the claims problem $(E, c) = (15, (8, 12))$. We know that the reference rule is $F = CEA$, so $F^d = CEL$. Then,

$$m = 1 : E^1 = 15; \ c^1 = (8, 12);$$

$$CEA(E^1, c^1) = (7.5, 7.5); \quad CEL(E^1, c^1) = (5.5, 9.5);$$
$$r(E^1, c^1, F) = (5.5, 7.5); \quad u(E^1, c^1, F) = (7.5, 9.5).$$

$m = 2 : E^2 = 2; c^2 = (2, 2);$

$$CEA(E^2, c^2) = (1, 1) = CEL(E^2, c^2);$$
$$r(E^2, c^2, F) = (1, 1) = u(E^2, c^2, F).$$

$m = 3 : E^3 = 0; \quad c^3 = (0, 0).$

Therefore, $DRP(E, c) = (5.5, 7.5) + (1, 1) = (6.5, 8.5).$

5 Main results

Our first result proves that the *Double Recursive Process* ends at the midpoint of the allocations provided by F and F^d.

Theorem 1 *For each* $(E, c, F) \in \mathfrak{B}_F$

$$DRP(E, c) = \frac{F(E, c) + F^d(E, c)}{2}$$

Proof See Appendix A. □

The next theorem establishes that the final allocation provided by the *Double Recursive Process* corresponds with the average of the *F-rights* and the *F-utopia*. The proof is obtained as part of the proof of Theorem 1.

Theorem 2 *For each* $(E, c, F) \in \mathfrak{B}_F$

$$DRP(E, c) = \frac{r(E, c, F) + u(E, c, F)}{2}$$

Theorem 1 shows that the *Double Recursive Process* proposes the midpoint between the two rules which represent extreme and opposite ways of sharing awards among claimants. As a consequence, it can be stated that the rationing of the endowment obtained by the recursive double imposition of the *F-rights* and *F-utopia*, neither favours nor hurts any particular agent. Following Thomson and Yeh (2008),

> *"When two rules express opposite points of view on how to solve a bankruptcy problem, it is natural to compromise between them by averaging".*

Note that, from Theorem 1, we obtain that the *Double Recursive Process* satisfies *efficiency*, *non-negativity* and *claim-boundedness*. Furthermore, the following result shows that it also fulfils *self-duality*, whatever the *legitimate principles* set, $\mathcal{P}$, and the reference rule, F, are.

Proposition 1 *For each* $(E, c, F) \in \mathfrak{B}_F, DRP(E, c) = c - DRP(L, c).$

Proof See Appendix B. □

In our context of *claims problems with mediation*, we understand by a *process* any algorithm that associates to each problem $(E, c, F) \in \mathfrak{B}_F$ an allocation satisfying efficiency, non-negativity and claim-boundedness. For instance, apart from our *Double Recursive Process*, we may define a process that in each step assigns the *F-rights* value, and solves the residual problem, as shown by next example.

Example 5 Consider the Egalitarian mediator as in Example 3, and the claims problem $(E, c) = (15, (8, 12))$. We know that the reference rule is $F = CEA$, so $F^d = CEL$. Then, we define a recursive process M_r in the following way:

$$(M_r)_i = \sum_{m=1}^{\infty} r_i(E^m, c^m, F)$$

where E^m is as in Definition 5, and $c^m = c^{m-1} - r^{m-1}$ for all m. Then

$m = 1 : E^1 = 15; c^1 = (8, 12);$

$$CEA(E^1, c^1) = (7.5, 7.5); \quad CEL(E^1, c^1) = (5.5, 9.5);$$
$$r^1 = r(E^1, c^1, F) = (5.5, 7.5).$$

$m = 2 : E^2 = 2; c^2 = (2.5, 4.5);$

$$CEA(E^2, c^2) = (1, 1); \quad CEL(E^2, c^2) = (0, 2);$$
$$r^2 = r(E^2, c^2, F) = (0, 1)$$

$m = 3 : E^3 = 1; c^3 = (2.5, 3.5);$

$$CEA(E^3, c^3) = (0.5, 0.5); \quad CEL(E^3, c^3) = (0, 1);$$
$$r^3 = r(E^3, c^3, F) = (0, 0.5)$$

$m = k :$

$$r^k = r(E^k, c^k, F) = (0, (1/2)^{k-2})$$

Therefore, $M_r(E, c) = (5.5, 9.5)$, that does not coincide with the allocation provided by the *Double Recursive Process*, $DRP(E, c) = (6.5, 8.5)$.

Now, let us consider the "extra awards" each agent receives from a process, i.e., the amount of awards each agent gets apart from what she has guaranteed, $r_i(E, c)$. Given that the process we are looking for is self dual, we ask that the extra amount each individual receives after rights should be the same in a problem (E, c) and in its dual problem (L, c). We call this property **dual right-egalitarian**:

$$M_i(E, c) - r_i(E, c) = M_i(L, c) - r_i(L, c).$$

Now, by combining *self-duality* and *dual right-egalitarian*, we obtain a characterization of the *Double Recursive Process*.

Theorem 3 *If M is a process such that for each $(E, c, F) \in \mathfrak{B}_F$, satisfies self-duality and dual right-egalitarian, then $M(E, c) = DRP(E, c)$.*

Proof See Appendix C. □

Finally, note that there are some properties that are always satisfied by the result of this process independently of the *legitimate principles* set, $\mathcal{P}$, and the reference rule, F. Among the properties we have introduced in Sect. 2.2, the *Double Recursive Process* fulfils: *resource monotonicity*, *order preservation* and *self-duality*. Nonetheless, other properties depend on $\mathcal{P}$ and/or on F, as the next example shows with respect to the *composition-up* property.

Example 6 Consider $F = CEA$ (the Egalitarian mediator) and the problems $(E, c) = (100, (80, 120))$ and $(E', c) = (120, (80, 120))$. Then,

$$DRP(E, c) = (40, 60), \quad DRP(E', c) = (50, 70)$$
$$\text{and} \quad DRP(20, (40, 60)) = (5, 15).$$

So *composition-up* is not fulfilled. Now, consider $F = P$ (the Proportional mediator). In this case, DRP provides an allocation that always coincides with the proportional one, so it satisfies *composition-up*.

6 The mediator: reference rules

In Sect. 3 we have introduced the notion of mediation as a specific rule F, that fulfills a set of *legitimate principles*, $\mathcal{P}$, and determines the set of *admissible* rules. Now, we provide some rationale about the selection of the reference rule, F, by the mediator. As we have mentioned in Sect. 1, the *Lorenz criterion* is a usual equity manner of comparing rules (see Thomson 2012; Bosmans and Lauwers 2011). Under some usual conditions, this criterion provides a comparison of rules so that if a rule φ *Lorenz dominates* a rule ψ, this means that φ favours smaller claimants relative to larger ones. Under such a perspective, the *Lorenz maximal* rule can be considered as the more equitable one. Then, once the legitimate principles have been defined by society, the Lorenz-optimality can be the criterion the mediator uses to select the reference rule F. We call this *Lorenz mediation*, which involves: (1) a set of properties, $\mathcal{P}$ (*legitimate principles*); and (2) an *admissible zone* delimited by the *Lorenz maximal* rule in $\mathcal{P}$, $F = \mathcal{L}$, and its dual rule, $F^d = \mathcal{L}^d$. Therefore, as a consequence of Theorem 1 we obtain that the *Double Recursive Process* coincides with the average of the *Lorenz maximal* and its dual.

Then, by varying the set of *legitimate principles*, the reference rule may change and so the result of the DRP process. The following example and results show this fact.[3]

Example 7 Let us consider the following sets of legitimate principles, and the Lorenz-maximal in each one of them:

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[3] The results about Lorenz maximal rules provided in this example can be found in Bosmans and Lauwers (2011) and Thomson (2012).

- If $\mathcal{P}_1 = \{efficiency,\ claim\text{-}boundedness,\ non\text{-}negativity\}$, $\mathcal{L} = CEA$.
- If $\mathcal{P}_2 = \{efficiency,\ claim\text{-}boundedness,\ non\text{-}negativity,\ composition\text{-}up,\ self\text{-}duality\}$, $\mathcal{L} = P$.

Proposition 2 *Given a society with legitimate principles $\mathcal{P}$, such that a Lorenz maximal $\mathcal{L}$ exists, if $F = \mathcal{L}$, then for each $(E, c, F) \in \mathfrak{B}_F$,*

$$DRP(E, c) = \frac{\mathcal{L}(E, c) + \mathcal{L}^d(E, c)}{2}.$$

By applying the result in Proposition 2 to the different sets of *legitimate principles* introduced in previous example, we obtain the following:

Corollary 1 *We consider the sets of legitimate principles defined in Example 7. Then,*

(a) *If $\mathcal{P} = \mathcal{P}_1$, $\quad DRP(E, c) = \frac{CEA(E,c) + CEL(E,c)}{2}$*
(b) *If $\mathcal{P} = \mathcal{P}_2$, $\quad DRP(E, c) = P(E, c)$*

7 Final remarks

In this paper, we have established that the application of a recursive process, in contexts where two *dual* reference criterion F and F^d appear, retrieves the midpoint between these two dual positions. This fact, apart from its own logic, allows us to anticipate the result of such negotiations. Moreover, we also find that this process concludes at the midpoint between the *F-rights* and the *F-utopia*—that is, the midpoint between the worst and the best position, among the proposals given by the *admissible* rules.

Two interesting questions remain open. On the one hand, what happens if instead of considering an *admissible zone* delimited by a reference function, F, and its dual, F^d, we consider that this zone can be delimited by two any reference functions, F and G, satisfying $\mathcal{P}$. In this case the *admissible* rules are those satisfying $\mathcal{P}$ such that:

$$\min\{F_i(E, c), G_i(E, c)\} \le \varphi_i(E, c) \le \max\{F_i(E, c), G_i(E, c)\}.$$

The *Double Recursive Process* may be adapted to this situation merely by changing F^d by G in Definition 5. We denote by $DRP2$ this new process. Next proposition tells us that the result in Theorem 1 remains true if we consider *two-agent* problems. Nevertheless, we cannot generalize this result to *n-agent* problems, as shown in Example 8.

Proposition 3 *For each two-agent claims problems $(E, c) \in \mathfrak{B}$, and given two any reference rules, F and G,*

$$DRP2(E, c) = \frac{F(E, c) + G(E, c)}{2}$$

Proof See Appendix D. □

Example 8 Consider that the *admissible zone* is delimited by the Constrained Equal Awards, CEA, and Proportional, P, rules. Let us consider the claims problem $(E, c) = (100, (50, 70, 80))$. Then, we obtain that

$$DRP2 = \left(28 + \frac{3}{4}, 34 + \frac{7}{12}, 36 + \frac{2}{3}\right),$$

which does not coincide with the average of $CEA(E, c) = (\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$, and $P(E, c) = (25, 35, 40)$.

On the other hand, as suggested by a referee "sometimes, the lower bound $r(E, c, F)$ may be considered as too low (or even to high), we would like to impose an *adjusted F-right* $r^\alpha(E, c, P) = \alpha \cdot r(E, c, F)$." So, what happens if we consider some *speed* in the recursive process by means of a parameter α? This is an interesting question since it seems that some relationship between the end of the recursive process and some convex combination of the rules F and F^d should exist. Nevertheless, no general relationship is fulfilled.

Finally, in Sect. 6 we have provided a way of selecting the reference rule F, that coincides with the *Lorenz maximal* rule in a specific set of *legitimate principles*. However, there are other possibilities for comparing rules and, so that, defining such a rule (see, for instance, the *Rawls criterion* Rawls 1971). In any case, the *Double Recursive Process* ends at the midpoint of any two dual reference rules, F and F^d.

Appendix A: Proof of Theorem 1

The proof of this result is based on a fact and two lemmas. Throughout the appendix, we will consider $(E, c, F) \in \mathfrak{B}_F$. For notational convenience, for all $m \in \mathbb{N}$, we denote $r^m = r(E^m, c^m, F)$ and $u^m = u(E^m, c^m, F)$.

Fact 1 *For all $m \in \mathbb{N}$, the sum of F-rights plus F-utopia values equals two times the endowment,*

$$\sum_{i \in N} r_i^m + \sum_{i \in N} u_i^m = 2E^m.$$

Lemma 1 *For all $m \in \mathbb{N}, m > 1$,*

$$\sum_{i \in N} [u_i^m + r_i^m] = \sum_{i \in N} c_i^m = C^m.$$

Proof From Fact 1, the next expression follows

$$\sum_{i \in N} \left[\frac{u_i^m + r_i^m}{2} \right] = E^m.$$

By the definition of the *Double Recursive Process*, we know that

$$E^m = E^{m-1} - \sum_{i \in N} r_i^{m-1}$$

$$= \sum_{i \in N} \left[\frac{u_i^{m-1} + r_i^{m-1}}{2} \right] - \sum_{i \in N} r_i^{m-1}$$

$$= \sum_{i \in N} \left[\frac{u_i^{m-1} - r_i^{m-1}}{2} \right] = \frac{\sum_{i \in N} c_i^m}{2} = \frac{C^m}{2}.$$

$\square$

As a direct consequence of Lemma 1, we find that at every step of the *Double Recursive Process*, $m \in \mathbb{N}$, $m > 1$, the half of the claims sum coincides with both the endowment and the total loss incurred at this step of the process,

$$\sum_{i \in N} c_i^m - E^m = E^m = \frac{C^m}{2}$$

Finally, the next lemma states that each agent's claim at every step different from the initial one, coincides with sum of the rights and the utopia.

Lemma 2 *For all $m \in \mathbb{N}$, $m > 1$*

$$c_i^m = u_i^m + r_i^m.$$

Proof We know that

$$\sum_{i \in N} c_i^m - E^m = E^m,$$

so

$$r_i^m = r_i \left(\sum_{i \in N} c_i^m - E^m, c^m, F \right) = \min_{\varphi \in \Phi(F)} \left\{ \varphi_i \left(\sum_{i \in N} c_i^m - E^m, c^m, F \right) \right\}$$

$$= \min_{\varphi \in \Phi(F)} \left\{ c_i^m - \varphi_i^d \left(\sum_{i \in N} c_i^m - E^m, c^m, F \right) \right\}$$

$$= c_i^m - \max_{\varphi \in \Phi(F)} \left\{ \varphi_i^d \left(\sum_{i \in N} c_i^m - E^m, c^m, F \right) \right\}$$

$$= c_i^m - u_i^m \left(\sum_{i \in N} c_i^m - E^m, c^m, F \right) = c_i^m - u_i^m,$$

and the result holds.

□

A.1 Proof of Theorem 1

Proof By the definition of the *Double Recursive Process*,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} [u_i^{m-1} - r_i^{m-1}]$$

$$= u_i^1 + \sum_{m=2}^{\infty} u_i^m - r_i^1 - \sum_{m=2}^{\infty} r_i^m.$$

By Lemma 2,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} [u_i^m + r_i^m].$$

So,

$$u_i^1 + \sum_{m=2}^{\infty} u_i^m - r_i^1 - \sum_{m=2}^{\infty} r_i^m = \sum_{m=2}^{\infty} [u_i^m + r_i^m].$$

Thus,

$$\sum_{m=2}^{\infty} r_i^m = \frac{u_i^1 - r_i^1}{2}.$$

Therefore,

$$DRP_i(E, c) = r_i^1 + \frac{u_i^1 - r_i^1}{2} = \frac{u_i(E, c, F) + r_i(E, c, F)}{2}.$$

Now, from definition of F-rights and F-utopia values, $u_i(E, c, F) + r_i(E, c, F) =$ $\max\{F_i(E, c), F_i^d(E, c)\} + \min\{F_i(E, c), F_i^d(E, c)\} = F_i(E, c) + F_i^d(E, c)$ and the required result holds.

□

Appendix B: Proof of Proposition 1

Let $(E, c, F) \in \mathfrak{B}_F$ a *claims problem with mediation*. By duality,

$$F(L, c) = c - F^d(E, c) \quad \text{and} \quad F^d(L, c) = c - F(E, c).$$

Since

$$DRP(E, c) = \frac{F(E, c) + F^d(E, c)}{2},$$

then

$$DRP^d(E, c) = c - \frac{F(L, c) + F^d(L, c)}{2} = c - \frac{c - F^d(E, c) + c - F(E, c)}{2}$$
$$= \frac{F(E, c) + F^d(E, c)}{2} = DRP(E, c).$$

$\square$

Appendix C: Proof of Theorem 3

Let $(E, c, F) \in \mathfrak{B}_F$ a *claims problem with mediation*. If M is a process that fulfils *self-duality*, we know that

$$M_i(E, c) = M_i^d(E, c) = c_i - M_i(L, c).$$

On the other hand, by applying the *dual right-egalitarian* property

$$M_i(E, c) - r_i(E, c) = M_i(L, c) - r_i(L, c).$$

By adding both equalities, we obtain

$$2M_i(E, c) = c_i + r_i(E, c) - r_i(L, c) = c_i + r_i(E, c) - c_i + u_i(E, c)$$
$$= r_i(E, c) + u_i(E, c),$$

so,

$$M(E, c) = \frac{r_i(E, c) + u_i(E, c)}{2} = DRP(E, c).$$

$\square$

Appendix D: Proof of Proposition 3

Let $(E, c) \in \mathfrak{B}$ be a two-agent *claims problem*, and consider that there exist two reference functions F and G that delimit the admissible zone of proposals. That is, a rule φ is admissible if satisfies the properties in $\mathcal{P}$ and

$$\min\{F_i(E, c), G_i(E, c)\} \leq \varphi_i(E, c) \leq \max\{F_i(E, c), G_i(E, c)\}.$$

Suppose, without loss of generality that $c_1 \leq c_2$ and that $F_1(E, c) < G_1(E, c)$. For notational convenience, for all $m \in \mathbb{N}$, we write $r^m = pr(E^m, c^m, F, G)$, $u^m = u(E^m, c^m, F, G)$, $F^m = F(E^m, c^m)$, and $G^m = G(E^m, c^m)$. By the definition of the *Double Recursive Process*,

- $m = 1$, $E^1 = E$; $c^1 = c$.
  Since $F_1^1 < G_1^1$, $\quad r^1 = (F_1^1, G_2^1)$; $\quad u^1 = (G_1^1, F_2^1)$.
- $m = 2$, $E^2 = E - F_1^1 - G_2^1$; $c^2 = (G_1^1 - F_1^1, F_2^1 - G_2^1)$.
  Note that, $F_1^1 + F_2^1 = G_1^1 + G_2^1$. So,
  $G_1^1 - F_1^1 = F_2^1 - G_2^1$, and $c_1^2 = c_2^2$.
  Hence,

$$F_1^2 = F_1^2 = G_1^2 = G_1^2 = \frac{E - F_1^1 - G_1^2}{2}.$$

Therefore, since $E = G_1^1 + G_1^2$,

$$DRP2_1(E, c) = F_1^1 + \frac{E - F_1^1 - G_1^2}{2} = F_1^1 + \frac{G_1^1 - F_1^1}{2} = \frac{F_1 + G_1}{2},$$

and, by efficiency,

$$DRP2_2(E, c) = E - \frac{F_1 + G_1}{2} = \frac{F_2 + G_2}{2}.$$

$\square$

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