

REPRESENTING DEPENDENCIES IN EVENT STRUCTURES

G. MICHELE PINNA

Dipartimento di Matematica e Informatica, Università di Cagliari, Cagliari, Italy
e-mail address: gmpinna@unica.it

ABSTRACT. Event structures where the causality may explicitly change during a computation have recently gained the stage. In this kind of event structures the changes in the set of the causes of an event are triggered by modifiers that may add or remove dependencies, thus making the happening of an event contextual. Still the focus is always on the dependencies of the event. In this paper we promote the idea that the *context* determined by the modifiers plays a major role, and the context itself determines not only the causes but also what causality should be. Modifiers are then used to understand when an event (or a set of events) can be added to a configuration, together with a set of events modeling dependencies, which will play a less important role. We show that most of the notions of Event Structure presented in literature can be translated into this new kind of event structure, preserving the main notion, namely the one of configuration.

1. INTRODUCTION

The notion of causality is an intriguing one. In the sequential case, the intuition behind it is almost trivial: if the activity e depends on the activity e' , then to happen the activity e needs that e' has already happened. This is easily represented in Petri nets ([Rei85]), the transition e' *produces* a token that is *consumed* by the transition e (the net N' in Figure 1). The dependency is testified by the observation that the activity e' always precedes the activity e . However this intuition does not reflect other possibilities. If we abandon the sequential case and move toward possibly loosely cooperating system the notion of causality become involved. Consider the case of a Petri net with inhibitor arcs ([JK95]) where the

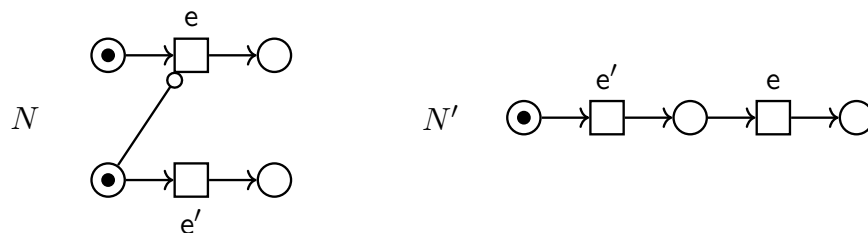


Figure 1: Two Petri nets fostering different interpretations of the dependencies among events

precondition of the transition e' inhibits the transition e (the net N in Figure 1). The latter to happens needs that the transition e' happens first, and the *observation* testifies that the activity e needs that e' has already happened, though resources are not exchanged between e' and e , as it is done in the net N' in Figure 1. In both cases the observation that the event

e' must happen first leads to state that e' precedes e and this can be well represented with a partial order relation among events.

This quite simple discussion suggests that the notion of causality may have many facets. In fact, if the dependencies are modeled just with a well founded partial order, inhibitor arcs can be used to model these dependencies, but the notion of partial order does not capture precisely the subtleties that are connected to the notion of causality.

To represent the semantics of concurrent and distributed systems the notion of *event structure* plays a prominent role. Event structures have been introduced in [NPW81] and [Win87] and since then have been considered as a cornerstone. The idea is simple: the activities of a system are the *events* and their relationships are specified somehow, *e.g.* with a partial order modeling the *enabling* and a predicate expressing when activities are *coherent* or not. Starting from this idea many authors have faced the problem of adapting this notion to many different situations which have as a target the attempt to represent faithfully various situations. This has triggered many diverse approaches. In [Gun91] and [Gun92] *causal automata* are discussed, with the idea that the conditions under which an event may happen are specified by a suitable logic formula, in [GP87] and [Gai89] it is argued that a partial order may be not enough or may be, in some situation, a too rigid notion, and this idea is used also in [PP92] and [PP95] where the notion of *event automata* is introduced, and it is used also in [Pin06] where an enabling/disabling relation for event automata is discussed. Looking at the enabling relation, both *bundle event structures* ([Lan92]) and *dual event structures* ([LBK97]) provide a notion of enabling capturing *or-causality* (the former exclusive *or-causality* and the latter non exclusive *or-causality*). *Asymmetric event structures* ([BCM01]) introduces a weaker notion of causality which models contextual arcs in Petri nets, or in the case *circular event structures* ([BCPZ14]) the enabling notion is tailored to model also *circular dependencies*. In *flow event structures* ([Bou90]) the partial order is required to hold only in configurations. An approach where dependencies may change according to the presence of *inhibitions* is proposed in [BBCP00] and [BBCP04], where *inhibitor event structures* are studied. This short and incomplete discussion (the event structures spectrum is rather broad) should point out the variety of approaches present in literature. It should be also observed that the majority of the approaches model causality with a relation that can be reduced to a partial order, hence causality is represented stating what are the events that should have happened before.

In this paper we introduce yet another notion of event structure. Triggered by recent works on *adding* or *subtracting* dependencies among events based on the fact that apparently unrelated events have happened ([AKPN15a, AKPN18]), we argue that rather than focussing on how to model these enrichment or/and impoverishment, it is much more natural to focus on the context on which an event takes place. In fact it is the context that can determine the proper dependencies that are applicable at the state where the event should take place and the context holds, and the context can also be used as well to forbid that the event is added to the state. This new relation resembles the one used in inhibitor event structures, but it differs in the way the contexts are determined. In the case of inhibitor event structures the presence of a certain event (the inhibiting context) was used to require that another one was present as well (representing the trigger able to remove the inhibition). Here the flavour is different as it is more prescriptive: it is required that exactly a set of events is present and if this happens then also another one should be present as well. It should be stressed that triggers and contexts may exchange their role. Consider again the two nets depicted in Figure 1, we may have that in both cases the trigger is determined by the happening of the

event e' and the context is the empty set, but we can consider as context the event e' and the trigger as the empty set. This simple relation, which we will call *context-dependency* relation, suffices to cover the aforementioned notions. It is worth observing that determining the context and the triggers associated to it is quite similar to trying to understand the dependencies. Consider the net N'' in Figure 2. Here e may be added either to the empty

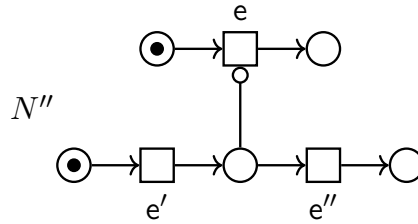


Figure 2: A net where the inhibition can be removed, and the dependency among events is contextual

set or to a set containing both e' and e'' . The context containing e' only leads to require that the event e'' is present (in the spirit of the relation for inhibitor event structures), making e dependent on e'' . However we could also have chosen to focus on contexts only and in this case the context containing just e' is ruled out among the contexts in which, together with some others dependencies, e may take place, and in this case the two contexts are \emptyset and $\{e', e''\}$. As hinted above, it will turn out that the context plays a more relevant role with respect to the dependencies, as the context can be seen positively (it specifies under which conditions an event can be added, together with the dependency) or negatively (it specifies under which conditions an event cannot be added, and in this case the event cannot be added simply stipulating that it depends on itself).

In this paper we will focus on event structures where the change of state is always triggered by the happening of a single event, hence we will not consider steps (*i.e.* non empty and finite subsets of events). However the generalization to steps is straightforward. We will also assume that in the event structures considered, each configuration can be reached by the initial one.

This paper is an extended and revised version of [Pin19]. We have added some examples and further discussed the notion of context-dependent event structure. We also have made more precise its relationship with various kind of event structures, among them flow event structures and inhibitor event structures, and we have better detailed the characteristics of the considered event automata.

Organization of the paper and contributions: In the next section we will introduce and discuss the new brand of event structure, *context-dependent* event structure, which is the main contribution of this paper and we will provide various examples to show how the new relation is used to model different situations. In Section 3 we review and briefly analyze some notions of event structures presented in literature, namely *prime* event structure ([Win87]), *flow* event structure ([Bou90]), *relaxed prime* event structure and *dynamic causality* event structure ([AKPN15a]), *inhibitor* event structure ([BBCP04]) and event structure for *resolvable conflicts* ([vGP04]), and, in section 4, we show that the each event structure presented in section 3 can be translated into this new kind of event structure. This

will be obtained coding each event structure into an *event automaton* ([PP95]) and then showing how to associate a context-dependent event structure to each event automaton. We will also briefly discuss the expressivity of inhibitor event structure with respect to the event structure where dependencies may grow, adding a tiny to information on how the various kind of event structures are related. We will end the paper with some conclusions and we will give some hints for further developments.

Notation: Let A be a set, with let ρ we denote a sequence of elements belonging to A , and with ϵ we denote the empty sequence. With $\bar{\rho}$ we denote the set of elements of A appearing in ρ . Thus $\bar{\rho} = \emptyset$ if $\rho = \epsilon$ and $\overline{\mathbf{a}\rho'} = \{\mathbf{a}\} \cup \bar{\rho}'$ if $\rho = \mathbf{a}\rho'$. Given a sequence $\rho = \mathbf{a}_1 \cdots \mathbf{a}_n \cdots$ with $\text{len}(\rho)$ we denote its length. With ρ_0 we sometimes denote sequence ϵ and, if $\text{len}(\rho) \geq 1$, for each $1 \leq i \leq \text{len}(\rho)$ with ρ_i we denote the sequence $\mathbf{a}_1 \cdots \mathbf{a}_i$. A sequence ρ can be infinite (and then it can be seen as a mapping from \mathbb{N} to A). Let A be a set, with $\mathbf{2}^A$ we denote the subsets of A and with $\mathbf{2}_{fin}^A$ the finite subsets of A .

2. CONTEXT-DEPENDENT EVENT STRUCTURE

We introduce yet another notion of event structure, which is the main contribution of the paper.

We recall what an *event* is. An event is an *atomic* and *individual* action which is able to change the state of a system. Though this may appear totally obvious, we believe it is better to stress these characteristics, namely atomicity, individuality and the ability to change the state of the system, which means that we can observe events, and their happening as a whole.

Event structures are intended to model *concurrent* and *distributed* systems where the activities are represented by events by defining relationships among these events such as *causality* and *conflict*, and establishing the conditions on which a certain event can be added to a state. The *state* of a system modeled by an event structure is a *subset* of events (those happened so far, representing the accomplished activities), and this set of events is often called *configuration*. States can be enriched by adding other information beside the one represented by the events that have determined the state, either adding information on the relationship among the various events in the state, *e.g.* adding dependencies among them (the state is then a partial order, [Ren92] or [AKPN18]) or adding suitable information to the whole state.

We pursue the idea that the happening of an event depends on a set of modifiers (the *context*) and on a set of *real* dependencies, which are activated by the set of modifiers.

We recall that in this paper we will consider only *unlabelled* event structures. To simplify the presentation we retain a binary conflict relation. We first define what a *conflict free* subset of events is.

Definition 2.1. *Let E be a set of events and let $\# \subseteq E \times E$ be an irreflexive and symmetric relation, called *conflict relation*. Let $X \subseteq E$ be a subset of events, we say that X is *conflict free*, denoted with $\text{CF}(X)$, iff $\forall e, e' \in X$ it holds that $\neg(e \# e')$.*

We introduce the notion of *context-dependent* event structure.

Definition 2.2. *A context-dependent event structure (CDES) is a triple $E = (E, \#, \gg)$ where*

- E is a set of events,
- $\# \subseteq E \times E$ is an irreflexive and symmetric relation, called conflict relation, and
- $\gg \subseteq \mathbf{2}^A \times E$, where $A \subseteq \mathbf{2}_{fin}^E \times \mathbf{2}_{fin}^E$, is a relation, called the context-dependency relation (CD-relation), which is such that for each $Z \gg e$ it holds that
 - $Z \neq \emptyset$,
 - for each $(X, Y) \in Z$ it holds that $CF(X)$ and $CF(Y)$, and
 - for each $(X, Y), (X', Y') \in Z$ if $X = X'$ then $Y = Y'$.

Each element of the CD-relation \gg will be called *entry*, and for each entry $Z \gg e$, we will call $(X, Y) \in Z$ the *element* of the entry. Finally, for each element (X, Y) of each entry, X is the set of *modifiers* and Y is the set of *dependencies*. The CD-relation models, for each event, which are the possible contexts in which the event may happen (the modifiers of each element) and for each context which are the events that have to be occurred (the dependencies). We stipulate that dependencies and modifiers are formed by non conflicting events, though this is not strictly needed, as the relation can model also conflicts. Conflicts can be modeled making the event dependent on itself in the presence of a suitable set of modifiers, as we will point out later. We also require that, for each entry of the CD-relation, two elements of the entry cannot have the same modifiers.

The first step toward showing how this relation is used is formalized in the notion of enabling of an event. We have to determine, for each $Z \gg e$, which of the set of modifiers X_i should be considered, *i.e.* which element of the entry should be used. To do so we define the *context* associated to each entry of the CD-relation.

Definition 2.3. Let $E = (E, \#, \gg)$ be a CDES. Let $Z \gg e$ be an entry of \gg , with $Z = \{(X_1, Y_1), (X_2, Y_2), \dots\}$, with $CXT(Z \gg e)$ we denote the set of events $\bigcup_{i=1}^{|Z|} X_i$.

The context associated to the entry $Z \gg e$ is the union of all the set of modifiers for the given entry.

In the definition of *enabling* of an event e at a subset of events C it will be clear what is the role of a context.

Definition 2.4. Let $E = (E, \#, \gg)$ be a CDES and let $C \subseteq E$ be a subset of events. The event $e \notin C$ is enabled at C , denoted with $C[e]$, if for each entry $Z \gg e$, with $Z = \{(X_1, Y_1), (X_n, Y_n), \dots\}$, there is an element $(X_i, Y_i) \in Z$ such that

- $CXT(Z \gg e) \cap C = X_i$, and
- $Y_i \subseteq C$.

An event e , not present in a subset of event, is enabled at this subset whenever, for each entry of the CD-relation, there exists an element such that the set of modifiers is equal to the intersection of the context of this entry and the subset of events. Observe that requiring the non emptiness of the set Z in $Z \gg e$ guarantees that an event e may be enabled at some subset of events. The requirement that two different elements of an entry must have different sets of modifiers is used to rule out the possibility that the same event is enabled by one element in more than one different ways. We do not make any further assumption on the elements of an entry: the sets of modifiers must be different but they can be in any other relation among sets, in particular a set of modifiers can be contained into another. Consider the events involved in net depicted in Figure 2, we have that the event (transition) e can happen at the beginning (hence with \emptyset as set of modifiers) or when e' and e'' have happened, and in any way this will be modeled, the set of modifiers would be non empty

(either containing just e' or also e''), containing obviously the previous set of modifiers. As we have already pointed out, the CD-relation could be used to express conflicts: $e \# e'$ could be modeled by adding $\{(\emptyset, \emptyset), (\{e\}, \{e'\})\} \gg e'$ and $\{(\emptyset, \emptyset), (\{e'\}, \{e\})\} \gg e$ to the \gg relation, and the presence of just one of them would model the asymmetric conflict. The conflicts modeled in this way are *persistent*, namely they cannot be changed.

We define now what is the state of system modeled by a CDES.

Definition 2.5. *Let $E = (E, \#, \gg)$ be a CDES. Let C be a subset of E . We say that C is a configuration of the CDES E iff there exists a sequence of distinct events $\rho = e_1 \cdots e_n \cdots$ over E such that*

- $\bar{\rho} = C$,
- $\text{CF}(\bar{\rho})$, and
- $\forall 1 \leq i \leq \text{len}(\rho). \bar{\rho}_{i-1}[e_i]$.

With $\text{Conf}_{\text{CDES}}(E)$ we denote the set of configurations of the CDES E .

This definition is standard, and it has an operational flavour as an ordering in executing the events should be found such that at each step one of the events can be added as it is enabled. The sequence of distinct events $\rho = e_1 \cdots e_n \cdots$ will be sometimes called *event trace*. In a CDES the empty set is always a configuration, as it is conflict free and the empty sequence of events gives the proper events trace.

The following definition introduces a relation \mapsto_{CDES} among configurations stating when a configuration can be *reached* from another one by adding an event.

Definition 2.6. *Let $E = (E, \#, \gg)$ be a CDES. Given two configurations $C, C' \in \text{Conf}_{\text{CDES}}(E)$ such that $C' \setminus C = \{e\}$, we stipulate that $C \mapsto_{\text{CDES}} C'$ iff $C[e]$.*

We illustrate this new kind of event structure with some examples, that should clarify how the CD-relation is used and give some hints on its *expressivity*.

Example 2.7 (Resolvable conflicts). *Consider three events a, b and c . All the events are singularly enabled but a and b are in conflict unless c has not happened (this are called resolvable conflicts as the conflict between a and b is resolved by the execution of c). For the event a we stipulate*

$$\{(\emptyset, \emptyset), (\{c\}, \emptyset), (\{b\}, \{c\})\} \gg a$$

that should be interpreted as follows: if the context is \emptyset or $\{c\}$ then a is enabled without any further condition (the Y are the empty set), if the context is $\{b\}$ then also $\{c\}$ should be present. The set $\text{CXT}(\{(\emptyset, \emptyset), (\{c\}, \emptyset), (\{b\}, \{c\})\} \gg a)$ is $\{b, c\}$. Similarly, for the event b we stipulate

$$\{(\emptyset, \emptyset), (\{c\}, \emptyset), (\{a\}, \{c\})\} \gg b$$

which is justified as above, and such that $\text{CXT}(\{(\emptyset, \emptyset), (\{c\}, \emptyset), (\{a\}, \{c\})\} \gg b)$ is $\{a, c\}$. Finally, for the event c , we stipulate

$$\{(\emptyset, \emptyset), (\{a\}, \emptyset), (\{b\}, \emptyset)\} \gg c$$

namely any context allows to add the event. In Figure 3(a) the configurations of this CDES and how they are related are depicted. At the configuration $\{a\}$ it is not possible to add the event b as the entry for b has as context $\{a, c\}$, hence the pair selected is $(\{a\}, \{c\})$ but $\{c\} \not\subseteq \{a\}$: the event b is not enabled, but it will eventually be enabled again when c will be executed (at the configuration $\{a, c\}$).

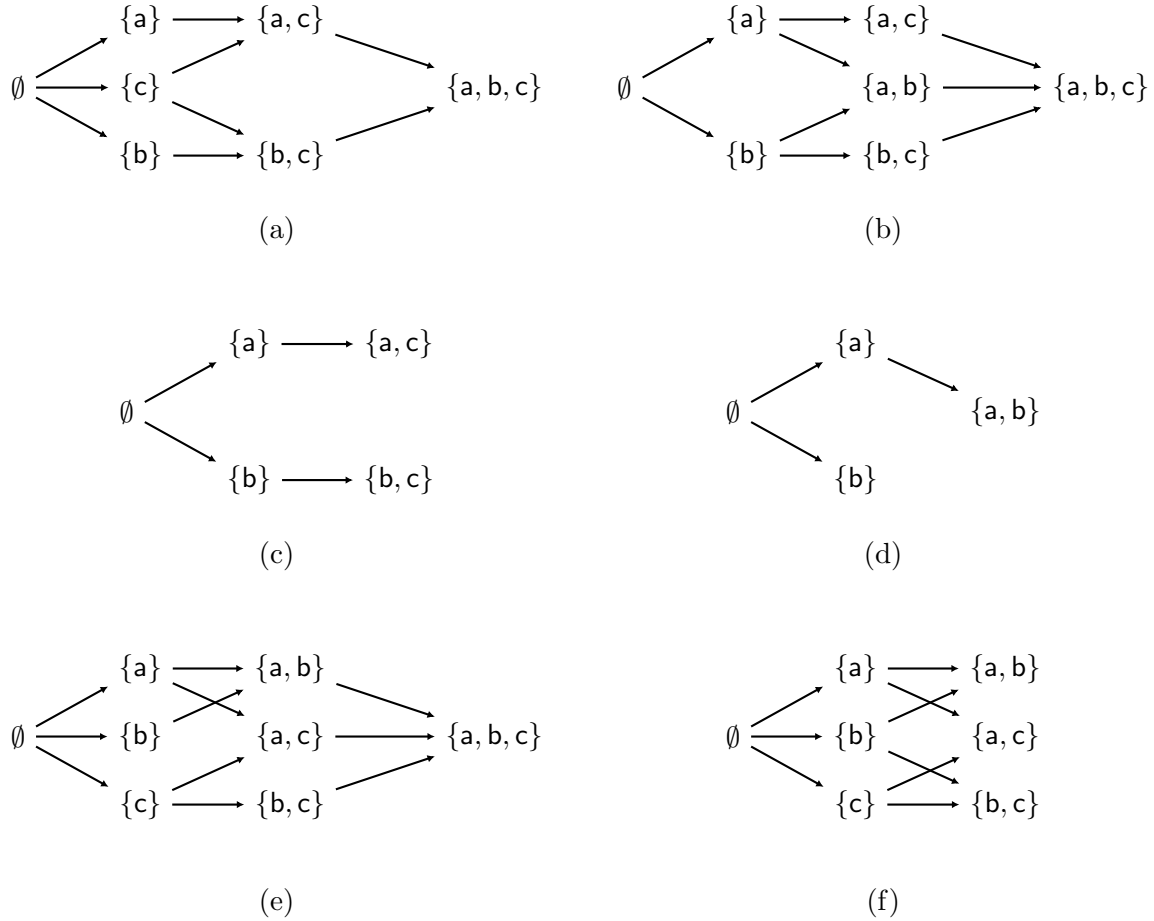


Figure 3: The configurations of the CDES of the Examples 2.7 (a), 2.8 (b), 2.9 (c), 2.10 (d), 2.11 (e) and 2.12 (f)

Example 2.8 (Removing dependencies). Consider three events a, b and c , and assume that c depends on a unless the event b has occurred, and in this case this dependency is removed. Thus there is a classic causality between a and c , but it can be dropped if b occurs. Clearly a and b are always enabled. The entries of the CD-relation are $\{(\emptyset, \emptyset)\} \gg a$, $\{(\emptyset, \emptyset)\} \gg b$ and $\{(\emptyset, \{a\}), (\{b\}, \emptyset)\} \gg c$. The event c is enabled at the configuration $\{b\}$ as the entry for c has as context $\{b\}$ and the element selected of this entry is $(\{b\}, \emptyset)$. At the configuration \emptyset the event c is not enabled as the pair selected would be $(\emptyset, \{a\})$ and $\{a\} \not\subseteq \emptyset$. The configurations of this CDES are in Figure 3(b).

Example 2.9 (Or causality). Consider three events a, b and c , assume that a and b are in conflict and that c depends on either a or b . The CD-relation is $\{(\emptyset, \emptyset)\} \gg a$, $\{(\emptyset, \emptyset)\} \gg b$, and $\{(\{a\}, \emptyset), (\{b\}, \emptyset)\} \gg c$. The configurations are those shown in Figure 3(c).

Example 2.10 (Asymmetric conflict). Consider two events a and b , and assume that a is in asymmetric conflict with b , i.e. once that b has happened, then event a cannot be added, but the vice versa is allowed. The CD-relation is $\{(\emptyset, \emptyset), (\{b\}, \{a\})\} \gg a$ and $\{(\emptyset, \emptyset)\} \gg b$. The configurations are those shown in Figure 3(d). From the configuration $\{b\}$ it not is

possible to add the event a as the entry for a has as context $\{b\}$, the pair selected is $(\{b\}, \{a\})$ but at this configuration the event a is not present.

Example 2.11 (Adding dependencies). Consider three events a, b and c , and assume that c depends on a just when the event b has occurred, and in this case this dependency is added, otherwise c may happen without depending on any other event. Thus the dependency between a and c is added if b occurs. Again a and b are always enabled. The CD-relation is $\{(\emptyset, \emptyset)\} \gg a$, $\{(\emptyset, \emptyset)\} \gg b$ and $\{(\emptyset, \emptyset), (\{b\}, \{a\})\} \gg c$. The configurations are those displayed in Figure 3(e). At the configuration $\{b\}$ it not is possible to add the event c as the entry for c has as context $\{b\}$, hence the element selected at the configuration $\{b\}$ is $(\{b\}, \{a\})$ and the event a is not present as it should.

Example 2.12 (Ternary conflict). Consider three events a, b and c . All the events are singularly enabled and they are in a ternary conflict, not a binary one. For the events a, b and c we stipulate $\{(\emptyset, \emptyset), (\{b, c\}, \{a\})\} \gg a$ $\{(\emptyset, \emptyset), (\{a, c\}, \{b\})\} \gg b$ and $\{(\emptyset, \emptyset), (\{a, b\}, \{c\})\} \gg c$ that should be interpreted as follows: if the context is \emptyset then the event is enabled without any further condition (the Y are the empty set), if the context is a subset of events with two elements then the event is not enabled. In Figure 3(f) the configurations of this CDES and how they are related are depicted.

Remark 2.13. The way context and dependencies are represented in many case is not unique, as we will also see in the remaining of the paper. Consider for instance the example on the Or causality. The entry for c is $\{(\{a\}, \emptyset), (\{b\}, \emptyset)\} \gg c$, but here the events on which c depends are used as context. An alternative could have been $\{(\{a\}, \{a\}), (\{b\}, \{b\})\} \gg c$ where the fact that c depends either on a or b is made more explicit.

We end the presentation of this new kind of event structure adapting the notion of *fullness* and *faithfulness* of an event structure with a conflict relation ([Bou90, AKPN18]). Given a CDES $E = (E, \#, \gg)$, and two events e and e' , we say that that they are in *semantic conflict* if there is no configuration containing both, *i.e.* $\forall C \in \text{Conf}_{\text{CDES}}(E). \{e, e'\} \not\subseteq C$. This will denoted with $e \# e'$.

Definition 2.14. Let $E = (E, \#, \gg)$ be a CDES. Then

- (1) E is said to be *full* if $\forall e \in E. \exists C \in \text{Conf}_{\text{CDES}}(E)$ such that $e \in C$, and
- (2) E is said to be *faithful* if $\forall e, e' \in E. \text{if } e \hat{\#} e' \text{ then } e \# e'$.

Thus fullness means that each event is possible, namely it will executed, whereas faithfulness implies that each symmetric conflict among two events is represented by the conflict relation.

Proposition 2.15. There exists a CDES E that is not full or faithful.

Proof. For fullness consider the CDES $E = (\{e\}, \emptyset, \emptyset)$, The unique configuration is \emptyset . For faithfulness consider $E = (\{e, e', e''\}, e \# e', \{(\emptyset, \emptyset)\} \gg e, \{(\emptyset, \emptyset)\} \gg e', \{(\emptyset, \{e\})\} \gg e'')$. Clearly e' and e'' are in conflict but this does not appear in the conflict relation. \square

The notions of fullness can be adapted to each kind of event structure, the one of faithfulness just for those where a conflict relation is defined.

3. EVENT STRUCTURES

We have introduced a new notion of event structure that we should confront with the others presented in literature (or at least some of them). Therefore we review some of the various definitions of event structures that appeared in literature.

3.1. Prime event structures: Prime event structures are one among the first proposed and the most widely studied ([Win87]), especially for the connections with *prime algebraic domains* and *occurrence nets*. The dependencies among events are modeled using a *partial order* relation, the incompatibility among events is modeled using a symmetric and irreflexive relation, the conflict relation, and it is required that the conflict relation is inherited along the partial order.

Definition 3.1. A prime event structure (PES) is a triple $P = (\mathbf{E}, \leq, \#)$, where

- \mathbf{E} is a set of events,
- $\leq \subseteq \mathbf{E} \times \mathbf{E}$ is a well founded partial order called causality relation,
- $\forall e \in \mathbf{E}$ the set $[e] = \{e' \mid e' \leq e\}$ is finite, and
- $\# \subseteq \mathbf{E} \times \mathbf{E}$ is an irreflexive and symmetric relation, called conflict relation, such that
 - $e \# e' \leq e'' \Rightarrow e \# e''$, and
 - $\leq \cap \# = \emptyset$.

Observe that the finiteness of $[e]$ for each event $e \in \mathbf{E}$ implies well foundedness of the partial order relation.

Definition 3.2. Let $P = (\mathbf{E}, \leq, \#)$ be a PES and $C \subseteq \mathbf{E}$ be a subset of events. C is a configuration of P iff $\text{CF}(C)$ and for each $e \in C$ it holds that $[e] \subseteq C$.

The set of configuration of a PES is denoted with $\text{Conf}_{\text{PES}}(P)$. Clearly $(\text{Conf}_{\text{PES}}(P), \subseteq)$ is a partial order.

Definition 3.3. Let $P = (\mathbf{E}, \leq, \#)$ be a PES. With \mapsto_{PES} we denote the relation over $\text{Conf}_{\text{PES}}(P) \times \text{Conf}_{\text{PES}}(P)$ defined as $C \mapsto_{\text{PES}} C'$ iff $C \subset C'$ and $C' = C \cup \{e\}$ for some $e \in \mathbf{E}$.

3.2. Flow event structure: In flow event structures, introduced in [Bou90], the dependency and the conflict relations have little constraints.

Definition 3.4. A flow prime event structure (FES) is a triple $F = (\mathbf{E}, \prec, \#)$, where

- \mathbf{E} is a set of events,
- $\prec \subseteq \mathbf{E} \times \mathbf{E}$ is an irreflexive relation called the flow relation, and
- $\# \subseteq \mathbf{E} \times \mathbf{E}$ is a symmetric conflict relation.

The flow relation is the one representing the dependencies among events, whereas the conflict relation is required to be just symmetric, which means that an event e could be in conflict with itself, meaning that it cannot be executed. The dependencies are not so *prescriptive* as in the case of PES, as we will see in the definition of configuration. Indeed, the notion of configuration plays a major role, as it is on this level that the conditions on the relations are placed.

Definition 3.5. Let $F = (\mathbf{E}, \prec, \#)$ be a FES and $C \subseteq \mathbf{E}$ be a subset of events. C is a configuration of F iff

- $\text{CF}(C)$,
- for each $e \in C$, for each $e' \prec e$ either $e' \in C$ or there exists $e'' \in C$ such that $e'' \prec e$ and $e' \# e''$, and
- $\prec^* \cap C \times C$ is a partial order.

The intuition is that, for each event e in the configuration, if an event which should be present in the configuration because it is *suggested* by the flow relation is not present then there is another event suggested by the flow relation which conflicts with the absent one. The set of configurations of a FES F is denoted with $\text{Conf}_{\text{FES}}(F)$. Again in [Bou90] it is shown that $(\text{Conf}_{\text{FES}}(F), \subseteq)$ is not only a partial order but also a prime algebraic domain, which establish a clear connection between FES and PES.

Also on these configurations we can state how to reach a configuration from another just adding an event.

Definition 3.6. Let $F = (\mathbf{E}, \prec, \#)$ be a FES. With \mapsto_{FES} we denote the relation over $\text{Conf}_{\text{FES}}(F) \times \text{Conf}_{\text{FES}}(F)$ defined as $C \mapsto_{\text{FES}} C'$ iff $C \subset C'$ and $C' = C \cup \{e\}$ for some $e \in \mathbf{E}$.

3.3. Relaxed prime event structures: Some of the requirements of a PES, the one on the dependencies among events (here called enabling) and the one regarding the conflicts among events (which does not need to be saturated), can be relaxed yielding a *relaxed* prime event structure ([AKPN15a, AKPN18]). This notion is introduced to allow a clear notion of dynamic causality and the idea is to focus on the dependencies and conflicts that are somehow the *generators* of the causal dependency relation and of the conflict relation of a PES. In this definition the events that must be *present* in a state to allow the execution of another one are the events in a (finite) subset called *immediate causes* and often denoted with ic .

Definition 3.7. A relaxed prime event structure (rPES) is a triple $(\mathbf{E}, \rightarrow, \#)$, where

- \mathbf{E} is a set of events,
- $\rightarrow \subseteq \mathbf{E} \times \mathbf{E}$ is the enabling relation such that $\forall e \in \mathbf{E}$ the set $\text{ic}(e) = \{e' \mid e' \rightarrow e\}$ is finite,
- for every $e \in \mathbf{E}$. $\rightarrow^+ \cap \text{ic}(e) \times \text{ic}(e)$ is irreflexive, and
- $\# \subseteq \mathbf{E} \times \mathbf{E}$ is an irreflexive and symmetric conflict relation.

The intuition is that the \rightarrow relation plays the role of the causality relation and the conflict relation models conflicts among events, as before. The immediate causes can be seen as a mapping $\text{ic}: \mathbf{E} \rightarrow \mathbf{2}_{\text{fin}}^{\mathbf{E}}$.

The above definition of relaxed prime event structure is slightly different from the one in [AKPN15a] and [AKPN18]. There \rightarrow is a relation without any further requirement as it is done here. The requirement that $\rightarrow^* \cap \text{ic}(e) \times \text{ic}(e)$ is there moved to the definition of configuration whereas the one that an event should have finite causes is here introduced in the definition as we consider configurations without further constraint.

Definition 3.8. Let $T = (\mathbf{E}, \rightarrow, \#)$ be a rPES and let $C \subseteq \mathbf{E}$ be a subset of events. We say that C is a configuration of the rPES T iff

- $\text{CF}(C)$, and
- for each $e \in C$. $\text{ic}(e) \subseteq C$.

The set of configuration of a rPES is denoted with $\text{Conf}_{\text{rPES}}(T)$. We prove that this definition of configuration implies that there is no causality cycle.

Proposition 3.9. *Let $T = (\mathbf{E}, \rightarrow, \#)$ be a rPES and consider $C \in \text{Conf}_{\text{rPES}}(T)$. Then $\rightarrow^* \cap C \times C$ is a partial order.*

Proof. Assume it is not the case. Then, for $e, e' \in C$, we have $e \rightarrow^* e'$ and $e' \rightarrow^* e$ with $e \neq e'$. But on each $e \in C$ we have that \rightarrow^+ is irreflexive on $\text{ic}(e)$ which contradicts the fact that \rightarrow^* is a partial order on the configuration C . \square

Similarly to PES and FES, we have that also $(\text{Conf}_{\text{rPES}}(T), \subseteq)$ is a partial order.

As it should be expected, PES and rPES are related. A PES is also a rPES: the causality relation is the enabling relation and the conflict relation is the same one. e is added to a configuration C when its causes are in C and no conflict arises. For the vice versa, given a full rPES $T = (\mathbf{E}, \rightarrow, \#)$, it is not difficult to see that $(\mathbf{E}, \rightarrow^*, \hat{\#})$ is a PES, where \rightarrow^* is the reflexive and transitive closure of \rightarrow and $\hat{\#}$ is obtained by $\#$ stipulating that $\# \subseteq \hat{\#}$ and it is closed with respect to \rightarrow^* , i.e. if $e \hat{\#} e' \rightarrow^* e''$ then $e \hat{\#} e''$. Indeed, the fact that \rightarrow^* is a partial order is guaranteed by the fact that each event is executable, that \rightarrow^* is well founded is implied by the finiteness of causes for each event $e \in \mathbf{E}$ and $\hat{\#}$ is the semantic closure of $\#$: no new conflict is introduced. In case the rPES is not full, we have to focus on the events appearing in a configuration.

Definition 3.10. *Let $T = (\mathbf{E}, \rightarrow, \#)$ be a rPES. With \mapsto_{rPES} we denote the relation over $\text{Conf}_{\text{rPES}}(T) \times \text{Conf}_{\text{rPES}}(T)$ defined as $C \mapsto_{\text{rPES}} C'$ iff $C \subset C'$ and $C' = C \cup \{e\}$ for some $e \in \mathbf{E}$.*

3.4. Dynamic causality event structures: We now review a notion of event structure where causality may change ([AKPN15a, AKPN18]). The idea is to enrich a rPES with two relations, one modeling the shrinking causality (some dependencies are dropped) and the other the growing causality (some dependencies are added). The shrinking and the growing causality relations are ternary relations stipulating that the happening of a specific event (the *modifier*) allows to drop or add a specific cause (the *contribution*) for another event (the *target*).

Definition 3.11. *Let $T = (\mathbf{E}, \rightarrow, \#)$ be a rPES. A shrinking causality relation is a ternary relation $\triangleleft \subseteq \mathbf{E} \times \mathbf{E} \times \mathbf{E}$, whose elements are denoted with $e' \triangleleft [e \rightarrow e'']$, such that*

- $\{e'\} \cap \{e, e''\} = \emptyset$, and
- $e \rightarrow e''$.

We say that \triangleleft is a shrinking relation with respect to the enabling relation \rightarrow .

Given $e' \triangleleft [e \rightarrow e'']$, e' is called *modifier*, e'' *target* and e *contribution*. To drop an enabling we require that the enabling is present.

Associated to this relation we introduce a number of auxiliary subsets of events.

Definition 3.12. *Let $\triangleleft \subseteq \mathbf{E} \times \mathbf{E} \times \mathbf{E}$ be a shrinking causality relation with respect to an enabling relation \rightarrow . Then*

- (1) $\text{ShrMod}(e'') = \{e' \mid e' \triangleleft [e \rightarrow e'']\}$ is the set of modifiers for a given target e'' ,
- (2) $\text{Drop}(e', e'') = \{e \mid e' \triangleleft [e \rightarrow e'']\}$ is the set of contributions for a given modifier e' and a given target e'' , and
- (3) $\text{dc}(H, e'') = \bigcup_{e' \in H \cap \text{ShrMod}(e'')} \text{Drop}(e', e'')$ is the set of dropped causes, with respect to a subset of events $H \subseteq \mathbf{E}$, for the event e .

$\text{ShrMod}(e'')$ is the set of all possible modifiers associated to the event e'' , thus it contains the set of events that may change the dependencies for it, and for each of them the set $\text{Drop}(e', e'')$ contains the *dropped* causes. Finally, for a given event e'' , the set $\text{dc}(H, e)$ contains the dropped causes provided that the events in H are already been executed.

Definition 3.13. Let $T = (E, \rightarrow, \#)$ be a rPES. A growing causality relation is a ternary relation $\blacktriangleright \subseteq E \times E \times E$, whose elements are denoted as $e' \blacktriangleright [e \rightarrow e'']$, such that

- $\{e'\} \cap \{e, e''\} = \emptyset$, and
- $\neg(e \rightarrow e'')$.

We say that \blacktriangleright is a growing relation with respect to the enabling relation \rightarrow .

Given $e' \blacktriangleright [e \rightarrow e'']$, e' is called *modifier*, e'' *target* and e *contribution*. To add a dependency among two events this dependency must be absent.

Definition 3.14. Let $\blacktriangleright \subseteq E \times E \times E$ be a growing causality relation with respect to an enabling relation \rightarrow . Then

- (1) $\text{GroMod}(e'') = \{e' \mid e' \blacktriangleright [e \rightarrow e'']\}$ is the set of modifiers for a given target e'' ,
- (2) $\text{Add}(e', e'') = \{e \mid e' \blacktriangleright [e \rightarrow e'']\}$ is the set of contributions for a given modifier e' and a given target e'' , and
- (3) $\text{ac}(H, e) = \bigcup_{e' \in H \cap \text{GroMod}(e)} \text{Add}(e', e)$ is the set of added causes, with respect to a subset of events $H \subseteq E$, for the event e .

The two relations of shrinking and growing causality give the functions $\text{dc}: \mathbf{2}_{\text{fin}}^E \times E \rightarrow \mathbf{2}_{\text{fin}}^E$ and $\text{ac}: \mathbf{2}_{\text{fin}}^E \times E \rightarrow \mathbf{2}_{\text{fin}}^E$.

Definition 3.15. A dynamic causality event structure (DCES) is a quintuple $D = (E, \rightarrow, \#, \triangleleft, \blacktriangleright)$, where $(E, \rightarrow, \#)$ is a rPES, $\triangleleft \subseteq E \times E \times E$ is the shrinking causality relation, $\blacktriangleright \subseteq E \times E \times E$ is the growing causality relation, and are such that for all $e, e', e'' \in E$

- (1) $e' \triangleleft [e \rightarrow e''] \wedge \nexists e''' \in E. e''' \blacktriangleright [e \rightarrow e''] \implies e \rightarrow e''$,
- (2) $e' \blacktriangleright [e \rightarrow e''] \wedge \nexists e''' \in E. e''' \triangleleft [e \rightarrow e''] \implies \neg(e \rightarrow e'')$,
- (3) $e' \blacktriangleright [e \rightarrow e''] \implies \neg(e' \triangleleft [e \rightarrow e''])$, and
- (4) $\forall e, e' \in E. \nexists e'', e''' \in E. e'' \triangleleft [e \rightarrow e']$ and $e''' \blacktriangleright [e \rightarrow e']$.

For further comments on this definition we refer to [AKPN15a] and [AKPN18]. It should be observed, however, that the definition we consider here is slightly less general of the one presented there, as we add a further condition, the last one, which is defined in [AKPN15b] and [AKPN18], and does not allow that the same contribution can be added and removed by two different modifiers. These are called in [AKPN15b] *single state dynamic causality event structures* and rule out the fact that some causality (or absence of) depends on the order of modifiers. Conditions 1 and 2 simply state that in the case of the shrinking relation the dependency should be present, and in the case of the growing the dependency should be absent; condition 3 says that if a dependency is added then it cannot be removed, or a removed dependency cannot be added, and the final condition express the fact that two modifiers, one growing and the other shrinking, cannot act on the same dependency. Clearly a DCES where \triangleleft and \blacktriangleright are empty is a rPES.

Definition 3.16. Let $D = (E, \rightarrow, \#, \triangleleft, \blacktriangleright)$ be a DCES. Let C be a subset of E . We say that C is a configuration of the DCES iff there exists a sequence of distinct events $\rho = e_1 \cdots e_n \cdots$ over E such that

- $\bar{\rho} = C$,

- $\text{CF}(\bar{\rho})$, and
- $\forall 1 \leq i \leq \text{len}(\rho). ((\text{ic}(\mathbf{e}_i) \cup \text{ac}(\bar{\rho}_{i-1}, \mathbf{e}_i)) \setminus \text{dc}(\bar{\rho}_{i-1}, \mathbf{e}_i)) \subseteq \bar{\rho}_{i-1}$.

The set of configuration of a DCES is denoted with $\text{Conf}_{\text{DCES}}(D)$.

To add an event to a subset of events we have to check that the added causes are present ($\text{dc}(\bar{\rho}_{i-1}, \mathbf{e}_i)$) whereas the dropped causes ($\text{ac}(\bar{\rho}_{i-1}, \mathbf{e}_i)$) can be ignored, *i.e.* they do not have to be present.

The following definition introduce the obvious relation between configurations of a DCES.

Definition 3.17. Let $D = (\mathbf{E}, \rightarrow, \#, \triangleleft, \blacktriangleright)$ be a DCES. With \mapsto_{DCES} we denote the relation over $\text{Conf}_{\text{DCES}}(D) \times \text{Conf}_{\text{DCES}}(D)$ defined as $C \mapsto_{\text{DCES}} C'$ iff $C \subset C'$, $C' = C \cup \{\mathbf{e}\}$ for some $\mathbf{e} \in \mathbf{E}$ and $((\text{ic}(\mathbf{e}) \cup \text{ac}(C, \mathbf{e})) \setminus \text{dc}(C, \mathbf{e})) \subseteq C$.

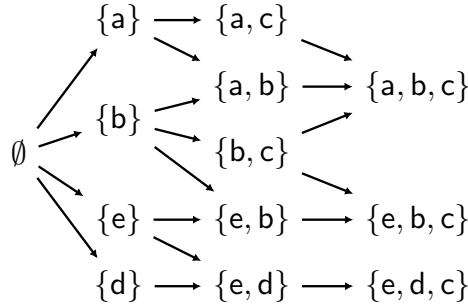


Figure 4: The configurations of the DCES of the Example 3.18

Example 3.18. Consider the set of events $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$, with $\mathbf{b} \rightarrow \mathbf{c}$, $\mathbf{a} \triangleleft [\mathbf{b} \rightarrow \mathbf{c}]$, $\mathbf{d} \blacktriangleright [\mathbf{e} \rightarrow \mathbf{c}]$, $\mathbf{a} \# \mathbf{e}$ and $\mathbf{d} \# \mathbf{b}$. \mathbf{a} and \mathbf{d} are the modifiers for the target \mathbf{c} , the happening of \mathbf{a} has the effect that the cause \mathbf{b} may be dropped, and the one of \mathbf{d} that the cause \mathbf{e} should be added for \mathbf{c} . If the prefix of the trace is \mathbf{bc} (the target \mathbf{c} is executed before of one of its modifiers \mathbf{a} and \mathbf{d}) then the final part of the trace is any either \mathbf{a} or \mathbf{e} , and as $\mathbf{d} \# \mathbf{b}$ we have that \mathbf{d} cannot be added. If the modifier \mathbf{a} is executed before \mathbf{c} then we have the traces \mathbf{ac} (as the immediate cause \mathbf{b} of \mathbf{c} is dropped by \mathbf{a}) followed by \mathbf{b} or \mathbf{d} , and if the modifier \mathbf{d} is executed, then before adding \mathbf{c} , we need \mathbf{e} (the modifier \mathbf{d} add the immediate cause \mathbf{e} for \mathbf{c}), and in this case we cannot add \mathbf{b} for sure as it is in conflict with \mathbf{d} or \mathbf{a} as it is in conflict with \mathbf{e} . If both modifiers \mathbf{a} and \mathbf{d} happen, then the event \mathbf{c} is permanently disabled, as it needs the contribution \mathbf{e} (growing cause) which is in conflict with \mathbf{a} . In Figure 4 are shown the configurations of this DCES and the \mapsto_{DCES} relation.

A shrinking event structure (SES) is a DCES where the \blacktriangleright relation is empty and a growing event structure (GES) is a DCES where the \triangleleft relation is empty.

In [AKPN18] it is shown that many kind of event structures can be seen as DCES, like bundle event structure [Lan92] or dual event structure [LBK97].

3.5. Inhibitor event structures: Inhibitor event structures ([BBCP04]) are equipped with a relation $\vdash^\circ \subseteq 2_1^E \times E \times 2_{fin}^E$ allowing to model conflicts (even asymmetric) as well as temporary inhibitions. With 2_1^E we denote the subsets of events with cardinality at most one (the empty set or singletons). The intuition behind this relation is the following: given $\vdash^\circ(a, e, A)$, the event e is enabled at a configuration a whenever the configuration contains the set a , then its intersection with A is non empty. Hence the event in a non empty a *inhibits* the happening of e unless some event in A has happened as well. We stipulate that given $\vdash^\circ(a, e, A)$ the events in A are pairwise conflicting (denoted with $\#(A)$). Two events e and e' are in conflict if $\vdash^\circ(\{e'\}, e, \emptyset)$ and $\vdash^\circ(\{e\}, e', \emptyset)$. An *or*-causality relation $<$ is definable stipulating that $A < e$ if $\vdash^\circ(\emptyset, e, A)$, and that if $A < e$ and $B < e'$ for some $e' \in A$ then also $B < e$. This relation should be interpreted as follows: $A < e$ means that if e is present, then also an event in A should be present.

Definition 3.19. An inhibitor event structure (IES) is a pair $I = (E, \vdash^\circ)$, where

- E is a set of events and
- $\vdash^\circ \subseteq 2_1^E \times E \times 2_{fin}^E$ is a relation such that for each $\vdash^\circ(a, e, A)$ it holds that $\#(A)$ and $a \cup A \neq \emptyset$.

We briefly recall the intuition: consider an event e and a triple in the \vdash° relation $\vdash^\circ(a, e, A)$. Then e can be added provided that if the event in a is present also one in A should be present.

Definition 3.20. Let $I = (E, \vdash^\circ)$ be an IES. Let C be a subset of E . We say that C is a configuration of the IES I iff there exists a sequence of distinct events $\rho = e_1 \cdots e_n \cdots$ over E such that

- $\bar{\rho} = C$, and
- for each $i \leq \text{len}(\rho)$, for each $\vdash^\circ(a, e_i, A)$, it holds that $a \subseteq \overline{\rho_{i-1}} \Rightarrow \overline{\rho_{i-1}} \cap A \neq \emptyset$.

The set of configuration of a IES is denoted with $\text{Conf}_{\text{IES}}(I)$.

Definition 3.21. Let $I = (E, \vdash^\circ)$ be an IES. With \mapsto_{IES} we denote the relation over $\text{Conf}_{\text{IES}}(I) \times \text{Conf}_{\text{IES}}(I)$ defined as $C \mapsto_{\text{IES}} C'$ iff $C \subset C'$ and $C' = C \cup \{e\}$ for some $e \in E$.

Example 3.22. Consider three events a, b and c , $\vdash^\circ(\{a\}, c, \{b\})$ and $\vdash^\circ(\emptyset, b, \{a\})$. The maximal event traces are cab and abc . The event c is inhibited when the event a has occurred unless the event b has occurred as well. The configurations are $\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}$ and $\{a, b, c\}$ and are reached as follows: $\emptyset \mapsto_{\text{IES}} \{a\}, \emptyset \mapsto_{\text{IES}} \{c\}, \{a\} \mapsto_{\text{IES}} \{a, b\}, \{c\} \mapsto_{\text{IES}} \{a, c\}, \{a, b\} \mapsto_{\text{IES}} \{a, b, c\}$ and $\{a, c\} \mapsto_{\text{IES}} \{a, b, c\}$.

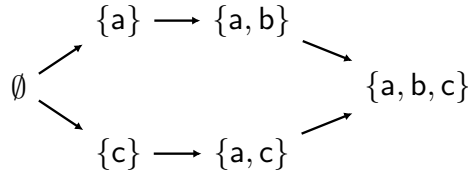


Figure 5: The configurations of the IES in the Example 3.22

3.6. Event structures with resolvable conflicts: We finally recall the notion of event structure with resolvable conflicts ([vGP04]).

Definition 3.23. An event structure with resolvable conflicts (RCES) is the pair $R = (\mathbf{E}, \vdash)$ where \mathbf{E} is a set of events and $\vdash \subseteq \mathbf{2}^{\mathbf{E}} \times \mathbf{2}^{\mathbf{E}}$ is the enabling relation.

No restriction is posed on the enabling relation. The intuition is that stipulating $X \vdash Y$ one state that for all the events in Y to occur, also the events in the set X should have occurred first.

Definition 3.24. The single event transition relation $\rightsquigarrow \subseteq \mathbf{2}^{\mathbf{E}} \times \mathbf{2}^{\mathbf{E}}$ of a RCES $R = (\mathbf{E}, \vdash)$ is given by $X \rightsquigarrow Y \Leftrightarrow (X \subseteq Y \wedge |Y \setminus X| \leq 1 \wedge \forall Z \subseteq Y. \exists W \subseteq X. W \vdash Z)$.

With this notion it is possible to define what a configuration is: it is a subset X of events such that $X \rightsquigarrow X$. The requirement that $X \rightsquigarrow X$ implies that each subset of events is enabled in the configuration.

Definition 3.25. Let $R = (\mathbf{E}, \vdash)$ be a RCES. Let C be a subset of \mathbf{E} . We say that C is a configuration of the RCES R iff there exists a sequence of distinct events $\rho = e_1 \cdots e_n \cdots$ over \mathbf{E} such that for each $1 \leq i \leq \text{len}(\rho)$ it holds that

- $\overline{\rho_{i-1}}$ and $\overline{\rho_i}$ are configurations, and
- $\overline{\rho_{i-1}} \rightsquigarrow \overline{\rho_i}$.

The set of configuration of a RCES is denoted with $\text{Conf}_{\text{RCES}}(R)$.

The enabling relation \vdash is used not only to state under which condition an event may happen, but also to stipulate when an event is *deducible* from a set of events, and this justifies why the deduction symbol is used for this relation.

Definition 3.26. Let $R = (\mathbf{E}, \vdash)$ be a RCES. With \mapsto_{RCES} we denote the relation over $\text{Conf}_{\text{RCES}}(R) \times \text{Conf}_{\text{RCES}}(R)$ defined as $C \mapsto_{\text{RCES}} C'$ iff $C \subset C'$ and $C' = C \cup \{e\}$ for some $e \in \mathbf{E}$ and $C \rightsquigarrow C'$.

Observe that \mapsto_{RCES} is, in this case, essentially \rightsquigarrow .

Example 3.27. Consider three events \mathbf{a}, \mathbf{b} and \mathbf{c} , and $\emptyset \vdash X$ where $X \subseteq \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ with $X \neq \{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{c}\} \vdash \{\mathbf{a}, \mathbf{b}\}$. The intuition is that all the events are singularly enabled but \mathbf{a} and \mathbf{b} are in conflict unless \mathbf{c} has not happened. In fact $\{\mathbf{a}, \mathbf{b}\}$ is not a configuration as taking $\{\mathbf{a}, \mathbf{b}\}$ as the $Z \subseteq \{\mathbf{a}, \mathbf{b}\}$ of the notion of single event transition relation, there is no subset of $\{\mathbf{a}, \mathbf{b}\}$ enabling these two events.

The configurations and how they are reached are those of the Example 2.7 depicted in Figure 3(a).

4. EMBEDDING AND COMPARING EVENT STRUCTURES

We now show that each of the event structure we have seen so far can be seen as a CDES, and also how to compare them. For the sake of simplicity, we will consider full event structures, *i.e.* each event e of the event structure is *executable*, namely that there is at least a configuration containing it.

4.1. Comparing Event Structures: We start by devising how we can compare two event structures of any kind. The intuition is obvious: two event structures are equivalent iff they have the same configurations and the \mapsto relations defined on configurations coincide. We recall the notion of *event automaton* ([PP95]).

Definition 4.1. Let E be a set of events. An event automaton over E (ea) is the tuple $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ such that

- $S \subseteq 2^E$, and
- $\mapsto \subseteq S \times S$ is such that $s \mapsto s'$ implies that $s \subset s'$.

$s_0 \in S$ is the initial state.

An event automaton is just a set of subsets of events and a reachability relation \mapsto with the minimal requirement that if two states s, s' are related by the \mapsto relation, namely $s \mapsto s'$, then s' is reached by s adding at least one event.

Definition 4.2. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea. We say that \mathcal{E} is simple if $\forall e \in E. \exists s \in S$ such that $s \cup \{e\} \in S$, $s \mapsto s \cup \{e\}$ and $s \in 2_{fin}^E$.

In a simple event automaton, for each event, there is a finite state such that this can be reached by adding just this event.

Definition 4.3. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea, and $s \in S$ be a state. With $\text{reach}(s) = \{s' \in S \mid s \mapsto s'\}$ we denote the subsets of states are reached from s in \mathcal{E} .

Definition 4.4. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea. We say that the event automaton \mathcal{E} is complete iff each state $s \in S$ can be reached by s_0 .

A complete event automaton is an event automaton where each states can be reached by the initial state. Observe that $\text{reach}(\cdot)$ of Definition 4.3 can be extended to an operator on subsets of states, and the operator defined in this way is clearly a monotone and continuous one. We can therefore calculate the *least fixed point* of the operator $\text{reach}(\cdot)$, which will be denoted with $\text{lfp}(\text{reach}(\cdot))$, and completeness reduces to require that $\text{lfp}(\text{reach}(\{s_0\})) = S$.

Event automata can easily express configurations of any kind of event structure, provided that for each kind a way to reach a configuration from another is given. The kind of event structure is ranged over by $\mu, \mu' \in \{\text{PES}, \text{FES}, \text{rPES}, \text{DCES}, \text{IES}, \text{RCES}, \text{CDES}\}$.

We first state the following theorem, which proof is almost straightforward.

Theorem 4.5. Let X be an event structure of kind μ over the set of events E . Then $\mathcal{G}_\mu(X) = \langle E, \text{Conf}_\mu(X), \mapsto_\mu, \emptyset \rangle$ is an event automaton.

Proof. First of all, we observe that for each event structure X of kind μ , with $\mu \in \{\text{PES}, \text{FES}, \text{rPES}, \text{DCES}, \text{IES}, \text{RCES}, \text{CDES}\}$, we have that the empty set belongs to $\text{Conf}_\mu(X)$. Then, due to Definitions 2.6, 3.3, 3.6, 3.10, 3.17, 3.21 and 3.26 we have that $s \mapsto_\mu s'$ implies that $s \subset s'$ as required by the Definition 4.1. Hence the thesis follows. \square

The event automata obtained by the configurations of each kind of event structure are simple and complete.

Proposition 4.6. Let X be an event structure of kind μ over the set of events E and $\mathcal{G}_\mu(X) = \langle E, \text{Conf}_\mu(X), \mapsto_\mu, \emptyset \rangle$ the associated event automaton. Then $\mathcal{G}_\mu(X)$ is complete and simple.

Proof. Simplicity is implied by how the \mapsto_μ is defined for the various kind of event structures and by the fact that in each of the considered kind of event structures an event is added at a finite configuration. Completeness is a consequence that each configuration of the considered event structures can be reached by the initial configuration. \square

Using event automata we can decide when two event structures are equivalent.

Definition 4.7. *Let X and Y be event structures over the same set of events \mathbf{E} of kind μ and μ' respectively. We say that X and Y are equivalent, denoted with $X \equiv Y$, iff $\mathcal{G}_\mu(X) = \mathcal{G}_{\mu'}(Y)$.*

The relative expressivity among event structures is explicitly studied in [AKPN15a] and [AKPN18]. Informally a kind of event structure is more expressive with respect to another, when there is a configuration of the former that cannot be a configuration of the latter, whatever is done with the various relations among events. Incomparable means that neither one is more expressive than the other or the vice versa. We shortly summarize part of these findings, when considering finite configurations. PES and rPES are equally expressive, whereas SES and GES are strictly more expressive than rPES, and are incomparable one with respect to the other. These two are both less expressive than DCES and RCES, which are incomparable. The relative expressivity of other kinds of event structure has not been investigated in that paper.

Here we show that a GES can be seen as an IES, adding a tiny piece of information to the expressivity spectrum of [AKPN18].

Proposition 4.8. *Let $D = (\mathbf{E}, \#, \rightarrow, \blacktriangleright)$ be a GES. Let $\mathcal{I}(D) = (\mathbf{E}, \vdash^\circ)$ be an IES where the relation \vdash° is defined as follows:*

- $\forall e, e' \in \mathbf{E}$ such that $e \# e'$ we have $\vdash^\circ(\{e\}, e', \emptyset)$ and $\vdash^\circ(\{e'\}, e, \emptyset)$,
- $\forall e, e' \in \mathbf{E}$ such that $e \rightarrow e'$ we have $\vdash^\circ(\emptyset, e', \{e\})$, and
- $\forall e, e', e'' \in \mathbf{E}$ such that $e \blacktriangleright [e' \rightarrow e'']$ we have $\vdash^\circ(\{e\}, e'', \{e'\})$.

Then $\mathcal{I}(D)$ in an IES and $\mathcal{I}(D) \equiv D$.

Proof. Recall that a GES is a DCES where the shrinking causality relation is empty. The enabling and all the related notions are those of a DCES without the part concerning the shrinking causality relation.

The fact that $\mathcal{I}(D)$ is indeed an IES derives from the fact that the growing causality is a kind a weak causality, which is captured by an IES, thus the \vdash° relation of $\mathcal{I}(D)$ obeys to the requirements posed, as each A is either the empty set or a singleton, hence $\#A$.

We now prove that the sets of configurations coincide. Take $C \in \text{Conf}_{\text{GES}}(D)$, then there is a sequence of events $\rho = e_1 e_2 \cdots e_n \dots$ such that $\bar{\rho} = C$, $\text{CF}(C)$ and for each $i \geq 1$ $\bar{\rho}_{i-1} [e_i]$. We first show that C is conflict free in the IES interpretation. Consider $e \in C$, then $e = e_j$ for some $j \leq \text{len}(\rho)$, and take any e' in conflict with e in D , then we have $\vdash^\circ(\{e\}, e', \emptyset)$ and $\vdash^\circ(\{e'\}, e, \emptyset)$ and then if $e \in C$ it is impossible that e' is in C as well. $\bar{\rho}_{i-1} [e_i]$ in the case $\triangleleft = \emptyset$ reduces to prove that for each $e \blacktriangleright [e' \rightarrow e_i]$ if $e \in \bar{\rho}_{i-1}$ then also $e' \in \bar{\rho}_{i-1}$, but this is exactly what prescribed by $\vdash^\circ(\{e\}, e_i, \{e'\})$. This means that $C \in \text{Conf}_{\text{IES}}(\mathcal{I}(D))$.

Analogously take $C \in \text{Conf}_{\text{IES}}(\mathcal{I}(D))$. It is conflict free and it is easy to see that, given the sequence $\rho = e_1 e_2 \cdots e_n \dots$ such that $\bar{\rho} = C$, for all i we have that $\bar{\rho}_{i-1} [e_i]$ as $\text{ic}(e_i) \cup \text{ac}(\bar{\rho}_{i-1}, e_i) \subseteq \bar{\rho}_{i-1}$.

Finally consider \mapsto_{IES} and \mapsto_{GES} as defined in Definitions 3.21 and 3.17. They clearly coincides, as they are defined starting from the sets of configurations. But this implies that $\mathcal{I}(D) \equiv D$. \square

We observe that IES are more expressive than GES, as it is impossible to find a GES such that the configurations are the one depicted in Figure 6. In fact a GES cannot model this kind of or-causality, as shown also in [AKPN18], whereas an IES can.

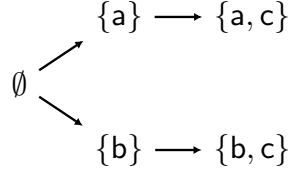


Figure 6: The configuration of the IES with three events $\{a, b, c\}$ and where the \vdash° relation is $\vdash^\circ(\emptyset, c, \{a, b\})$, $\vdash^\circ(\{a\}, b, \emptyset)$ and $\vdash^\circ(\{b\}, a, \emptyset)$

4.2. Embedding event structures into cdes: We prove now a more general result, namely that given any *event automaton* \mathcal{E} , which is obtained by the configurations of any kind of event structure, it is possible to obtain a CDES whose configurations are precisely the ones of the event automaton \mathcal{E} . We start identifying, in an ea, the events that are in *conflict*. The conflict relation we obtain is a *semantic* conflict relation: two events are in conflict iff they never appear together in a state.

Definition 4.9. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea. We define a symmetric and irreflexive conflict relation $\#_{\text{ea}}$ as follows: $e \#_{\text{ea}} e'$ iff for each $s \in S$. $\{e, e'\} \not\subseteq s$.

In order to obtain the CD-relation we need some further definitions. Fixed an event e , the first one identifies the states where this event can be added, and the second one identifies the states where the event cannot be added.

Definition 4.10. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea. To each event $e \in E$ we associate the set of states $\{s \in S \mid s \cup \{e\} \in S \wedge s \in \mathbf{2}_{\text{fin}}^E \wedge s \mapsto s \cup \{e\}\}$, which we denote with $\mathcal{C}(\mathcal{E}, e)$.

Definition 4.11. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be an ea. To each event $e \in E$ we associate the set of states $\{s \in S \mid s \cup \{e\} \notin S \wedge s \in \mathbf{2}_{\text{fin}}^E\}$, which we denote with $\mathcal{I}(\mathcal{E}, e)$.

Definition 4.10 characterizes when an event is enabled giving the *allowing* context, whereas the Definition 4.11 gives the context where the event cannot be added, and it is called *negative* context. These two sets are used to obtain the CD-relation.

Theorem 4.12. Let $\mathcal{E} = \langle E, S, \mapsto, s_0 \rangle$ be a simple and complete ea such that $E = \bigcup_{s \in S} s$. Then $\mathcal{F}_{\text{ea}}(\mathcal{E}) = (E, \#, \gg)$ is a CDES, where $\#$ is the relation $\#_{\text{ea}}$ of Definition 4.9, and for each $e \in E$ we have $\{(X, \emptyset) \mid X \in \mathcal{C}(\mathcal{E}, e)\} \cup \{(X, \{e\}) \mid X \in \mathcal{I}(\mathcal{E}, e)\} \gg e$. Furthermore $\mathcal{E} \equiv \mathcal{G}_{\text{CDES}}(\mathcal{F}_{\text{ea}}(\mathcal{E}))$.

Proof. $\mathcal{F}_{\text{ea}}(\mathcal{E})$ is clearly a CDES: the conflict relation obtained using the Definition 4.9 is trivially symmetric and it is clearly irreflexive as each event appear in a state and the event automaton is simple and complete. For what concern the \gg relation, given $Z \gg e$, clearly $Z \neq \emptyset$, furthermore for each $(X, Y) \in Z$ we have that $\text{CF}(X)$, as X is a finite state of the ea, and also $\text{CF}(Y)$ holds, as Y is either the empty set or e singleton, and finally, given two elements (X, Y) and (X', Y') of the entry $Z \gg e$, we have that always $X \neq X'$, and then also the last condition for the \gg holds.

We show now that, given a state $s \in \mathbf{S}$, s is also a configuration of the CDES $\mathcal{F}_{\text{ea}}(\mathcal{E})$. As \mathcal{E} is simple and complete, there exists a sequence of events $\rho = \mathbf{e}_1 \cdots \mathbf{e}_n \cdots$ such that $\forall i \geq 1$ $\bar{\rho}_i \in \mathbf{S}$ and $\bar{\rho}_i \mapsto \bar{\rho}_i \cup \{\mathbf{e}_{i+1}\}$. For each i , we have that $\text{CXT}(\mathbf{Z} \gg \mathbf{e}_{i+1}) \cap \bar{\rho}$ is exactly $\bar{\rho}_i$, and the element $(\bar{\rho}_i, \emptyset)$ belongs to the entry $\mathbf{Z} \gg \mathbf{e}_{i+1}$, hence $\bar{\rho}_i [\mathbf{e}_{i+1}]$ as required. Clearly the $\mapsto_{\mathcal{F}_{\text{ea}}(\mathcal{E})}$ coincides with \mapsto . Finally we observe that the CDES $\mathcal{F}_{\text{ea}}(\mathcal{E})$ is full and faithful.

The vice versa follows the same line. Consider a configuration $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{ea}}(\mathcal{E}))$, then there exists a sequence $\rho = \mathbf{e}_1 \cdots \mathbf{e}_n \cdots$ of events such that $\forall i \geq 1$ it holds that $\bar{\rho}_i [\mathbf{e}_{i+1}]$. We prove something stronger, namely that each $\bar{\rho}_i$ is a state, and we do by induction on the length of the sequence. The basis is trivial, assume then it hold for n , and we have that $\bar{\rho}_n [\mathbf{e}_{n+1}]$. But the $\text{ea } \mathcal{E}$ is simple and complete, and by the way the $\mathbf{Z} \gg \mathbf{e}_{n+1}$ is constructed, we have that $\bar{\rho}_n \cup \{\mathbf{e}_{n+1}\}$ is a state and the thesis follows. Again $\mapsto_{\mathcal{F}_{\text{ea}}(\mathcal{E})}$ coincides with \mapsto , thus $\mathcal{E} \equiv \mathcal{G}_{\text{CDES}}(\mathcal{F}_{\text{ea}}(\mathcal{E}))$. \square

The theorem has a main consequence, namely that event automata and CDES are equally expressive.

Example 4.13. *Consider the RCES of the Example 3.27. The associated event automaton is the one depicted in the Example 2.7. It has no conflict as all the three events are present in a configuration together. The associated CD-relation, obtained using Definition 4.10 and Definition 4.11, is the following one, which is a little different from the one devised in the Example 2.7 as here it is obtained from an event automaton. We synthesize*

$$\{(\emptyset, \emptyset), (\{\mathbf{c}\}, \emptyset), (\{\mathbf{c}, \mathbf{b}\}, \emptyset), (\{\mathbf{b}\}, \{\mathbf{a}\})\} \gg \mathbf{a}$$

because the set $\mathcal{C}(\text{Conf}_{\text{RCES}}(R), \mathbf{a})$ contains the sets \emptyset , $\{\mathbf{c}\}$ and $\{\mathbf{c}, \mathbf{b}\}$, whereas the set of the negative context $\mathcal{I}(\text{Conf}_{\text{RCES}}(R), \mathbf{a})$ contains just $\{\mathbf{b}\}$,

$$\{(\emptyset, \emptyset), (\{\mathbf{c}\}, \emptyset), (\{\mathbf{a}, \mathbf{c}\}, \emptyset), (\{\mathbf{a}\}, \{\mathbf{b}\})\} \gg \mathbf{b}$$

as $\mathcal{C}(\text{Conf}_{\text{RCES}}(R), \mathbf{b})$ contains the sets \emptyset , $\{\mathbf{c}\}$ and $\{\mathbf{a}, \mathbf{c}\}$, $\mathcal{I}(\text{Conf}_{\text{RCES}}(R), \mathbf{b})$ contains $\{\mathbf{a}\}$, and finally

$$\{(\emptyset, \emptyset), (\{\mathbf{a}\}, \emptyset), (\{\mathbf{b}\}, \emptyset)\} \gg \mathbf{c}$$

as $\mathcal{C}(\text{Conf}_{\text{RCES}}(R), \mathbf{c})$ contains the sets \emptyset , $\{\mathbf{a}\}$ and $\{\mathbf{b}\}$, and $\mathcal{I}(\text{Conf}_{\text{RCES}}(R), \mathbf{c})$ is the empty set.

As a consequence of the Theorem 4.12 we have the following result.

Corollary 4.14. *Let X be an event structure of type μ and let $\mathcal{G}_\mu(X)$ be the associated ea . Then $\mathcal{F}_{\text{ea}}(\mathcal{G}_\mu(X))$ is CDES, and $X \equiv \mathcal{F}_{\text{ea}}(\mathcal{G}_\mu(X))$.*

The construction identifies properly the *context* in which an event is allowed to happen, and this context becomes the main ingredient of the CD-relation, as the construction does not give the *causes* but just the context. If on the one hand this suggests that the context, rather than the causal dependencies, is the relevant ingredient, on the other hand it is less informative with respect to the usual causality definitions.

We review some kind of event structures, showing that a more informative CD-relation can be indeed obtained. We will focus only on few of them.

4.3. Prime event structures: In this case the idea is that causes of an event are just the set of events that should be present in the configuration.

Proposition 4.15. *Let $P = (\mathbf{E}, \leq, \#)$ be a PES. Then $\mathcal{F}_{\text{PES}}(P) = (\mathbf{E}, \#, \gg)$ is a CDES, where $\{(\emptyset, [e] \setminus \{e\})\} \gg e$ for each $e \in \mathbf{E}$. Furthermore $P \equiv \mathcal{F}_{\text{PES}}(P)$.*

Proof. $\mathcal{F}_{\text{PES}}(P)$ is trivially a CDES. We show that $\text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{PES}}(P)) = \text{Conf}_{\text{PES}}(P)$. First we observe that, for each event e , we have that $[e] \in \text{Conf}_{\text{PES}}(P)$. Clearly to $[e]$ we can associate a sequence $\rho = e_1 \cdots e_n$ with $e_n = e$ and for each $1 \leq i \leq n$ we have that $\bar{\rho}_i \in \text{Conf}_{\text{PES}}(P)$. This sequence can be used to show that also each $\bar{\rho}_i \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{PES}}(P))$. Given a configuration $C \in \text{Conf}_{\text{PES}}(P)$ we can associate to it a sequence $\rho = e_1 \cdots e_n \cdots$ and for each i we have again that $\bar{\rho}_i \in \text{Conf}_{\text{PES}}(P)$. Now for each e_{i+1} we have that $\bar{\rho}_i \cap \text{CXT}(\{(\emptyset, [e_{i+1}] \setminus \{e_{i+1}\})\}) \gg e_{i+1} = \emptyset$ and clearly $[e_{i+1}] \setminus \{e_{i+1}\} \subseteq \bar{\rho}_i$ which means that e_{i+1} is enabled at $\bar{\rho}_i$, hence $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{PES}}(P))$. This proves that each configuration of a PES P is also a configuration of the CDES $\mathcal{F}_{\text{PES}}(P)$. The vice versa holds as well observing that, given a configuration $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{PES}}(P))$ and the associated $\rho = e_1 \cdots e_n \cdots$, the fact that $\bar{\rho}_i [e_{i+1}]$ means that $[e_{i+1}] \subseteq \bar{\rho}_{i+1} \subseteq C$. The \mapsto_{PES} and \mapsto_{CDES} associated at these event structures clearly coincide. Hence $P \equiv \mathcal{F}_{\text{PES}}(P)$. \square

The CD-relation defined in the previous proposition is a bit more informative with respect to the one defined in Theorem 4.12. In fact it captures the intuition that in a PES there are not modifiers. We stress that is not the unique way to associate to the causality relation \leq of a PES the \gg relation: one alternative would have been to add $\{(\emptyset, \{e'\})\} \gg e$ for each $e' < e$ and another one would be $\{([e] \setminus \{e\}, \emptyset)\} \gg e$ showing that the events causally before e are indeed the context allowing the event e to happen.

Example 4.16. *Consider the PES $(\{a, b, c\}, \leq, \#)$ where $a \leq b$ (we omit the reflexive part of the \leq relation), $a \# c$ and $b \# c$. The event traces are ϵ , a , ab and c , and the associated configurations are \emptyset , $\{a\}$, $\{a, b\}$ and $\{c\}$ (the \mapsto_{PES} relation is obvious). The conflict relation is the same and the CD-relation is $\{(\emptyset, \emptyset)\} \gg a$, $\{(\emptyset, \emptyset)\} \gg c$ and $\{(\emptyset, \{a\})\} \gg b$. As noticed before we could have stipulated also $\{(\{a\}, \emptyset)\} \gg b$ instead of $\{(\emptyset, \{a\})\} \gg b$ obtaining the same set of configurations and the same transition graph.*

4.4. Relaxed prime event structures: Also in this case the idea is similar to PES, as the set of immediate causes of an event e can be considered akin to the local configuration of e .

Proposition 4.17. *Let $T = (\mathbf{E}, \#, \rightarrow)$ be a rPES. Then $\mathcal{F}_{\text{rPES}}(T) = (\mathbf{E}, \#, \gg)$ is a CDES, where $\{(\emptyset, \text{ic}(e))\} \gg e$ for each $e \in \mathbf{E}$. Furthermore $T \equiv \mathcal{F}_{\text{rPES}}(T)$.*

Proof. Like the one of Proposition 4.15, observing that $\text{ic}(e)$ plays the same role as $[e]$. \square

Observe that in case an event e has no causes, the $\text{ic}(e)$ set is empty, as expected. Again this is not the unique way to associate to the causality relation \rightarrow of a rPES the \gg relation. We could have defined $\{(\emptyset, \{e'\})\} \gg e$ for each $e' \rightarrow e$, or $\{(\text{ic}(e), \emptyset)\} \gg e$.

4.5. Flow event structure: To give a direct translation of a FES into a CDES we need some auxiliary notation. Let $F = (\mathbf{E}, \prec, \#)$, with $\text{fl}(\mathbf{e}) = \{\mathbf{e}' \in \mathbf{E} \mid \mathbf{e}' \prec \mathbf{e}\}$ we denote the set of events *preceding* \mathbf{e} . Observe that this set may contains conflicting events. Starting from the set $\text{fl}(\mathbf{e})$ we identifies the subsets of events that are conflict free and maximal. Thus we have the set $\text{maxfl}(\mathbf{e}) = \{X \subseteq \text{fl}(\mathbf{e}) \mid \text{CF}(X) \wedge \neg\text{CF}(X \cup \{\mathbf{e}'\}) \text{ with } \mathbf{e}' \in \text{fl}(\mathbf{e}) \setminus X\}$ which contains all subsets X of $\text{fl}(\mathbf{e})$ which are conflict free ($\text{CF}(X)$) and maximal ($\neg\text{CF}(X \cup \{\mathbf{e}'\})$ where $\mathbf{e}' \in \text{fl}(\mathbf{e}) \setminus X$). These subsets will play the same role played by the immediate causes for an event.

Proposition 4.18. *Let $F = (\mathbf{E}, \prec, \#)$ be a full and faithful FES. Then $\mathcal{F}_{\text{FES}}(F) = (\mathbf{E}, \#, \gg)$ is a CDES, where for each $\mathbf{e} \in \mathbf{E}$ we have $\{(X, \emptyset) \mid X \in \text{maxfl}(\mathbf{e})\} \gg \mathbf{e}$. Furthermore $F \equiv \mathcal{F}_{\text{FES}}(F)$.*

Proof. Clearly $\mathcal{F}_{\text{FES}}(F)$ is a CDES. The conflict relation is irreflexive, and this is a consequence that F is full and faithful, and each event \mathbf{e} is such that there is an entry $Z \gg \mathbf{e}$, as in the case $\text{fl}(\mathbf{e}) = \emptyset$ we have that $\text{maxfl}(\mathbf{e})$ contains just the emptyset.

Now we prove that $\text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{FES}}(F)) = \text{Conf}_{\text{FES}}(F)$. Consider $C \in \text{Conf}_{\text{FES}}(F)$. It is conflict free and the transitive and reflexive closure of \prec is a partial ordering on C . Consider then the sequence $\rho = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n \cdots$ compatible with the \prec^* , we have to show that for each $i \geq 1$ we have that $\overline{\rho}_i[\mathbf{e}_{i+1}]$. Consider $\mathbf{e} \prec \mathbf{e}_{i+1}$ and assume that $\mathbf{e} \notin \overline{\rho}_i$, then there must be a $\mathbf{e}' \# \mathbf{e}$ such that $\mathbf{e}' \prec \mathbf{e}_{i+1}$ and $\mathbf{e}' \in \overline{\rho}_i$, and this for all the events in $\text{fl}(\mathbf{e}_{i+1})$, which means that there must be a maximal subset of non conflicting events that are in the flow relation with \mathbf{e}_{i+1} , but this implies that $\text{CXT}(Z \gg \mathbf{e}_{i+1}) \cap \overline{\rho}_i = X \in \text{maxfl}(\mathbf{e}_{i+1})$, and we can conclude that C is a configuration of $\mathcal{F}_{\text{FES}}(F)$.

Consider $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{FES}}(F))$. Then there exists a sequence $\rho = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n \cdots$ such that for each $i \geq 1$, $\overline{\rho_{i-1}}[\mathbf{e}_i]$. This means that $\text{CXT}(Z \gg \mathbf{e}_i) \cap \overline{\rho_{i-1}} = X \in \text{maxfl}(\mathbf{e}_i)$. Now consider $\mathbf{e} \prec \mathbf{e}_i$ and assume that $\mathbf{e} \notin \overline{\rho_{i-1}}$, as $X \in \text{maxfl}(\mathbf{e}_i)$ by maximality of X we have that there is an event $\mathbf{e}_j \in \overline{\rho_{i-1}}$ with $j < i$ such that $\mathbf{e}_j \# \mathbf{e}$. The fact that the reflexive and transitive closure of \prec is a partial order on C is trivial. We can conclude that $C \in \text{Conf}_{\text{FES}}(F)$.

The \mapsto relations induced by the two event structure trivially coincide, and the thesis $F \equiv \mathcal{F}_{\text{FES}}(F)$ follows. \square

Example 4.19. *Consider the flow event structure F with 4 events $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and \mathbf{e} and where $\mathbf{e}_i \prec \mathbf{e}$, for $1 \leq i \leq 3$, and $\mathbf{e}_1 \# \mathbf{e}_2$. We have that $\text{maxfl}(\mathbf{e}_i) = \{\emptyset\}$ for $1 \leq i \leq 3$ and $\text{maxfl}(\mathbf{e}) = \{\{\mathbf{e}_1, \mathbf{e}_3\}, \{\mathbf{e}_2, \mathbf{e}_3\}\}$ and then we have $\{(\emptyset, \emptyset)\} \gg \mathbf{e}_i$ for $1 \leq i \leq 3$ and $\{(\{\mathbf{e}_1, \mathbf{e}_3\}, \emptyset), (\{\mathbf{e}_2, \mathbf{e}_3\}, \emptyset)\} \gg \mathbf{e}$, and the conflict relation is $\mathbf{e}_1 \# \mathbf{e}_2$. The configurations of this FES (and of the CDES $\mathcal{F}_{\text{FES}}(F)$) are depicted in Figure 7.*

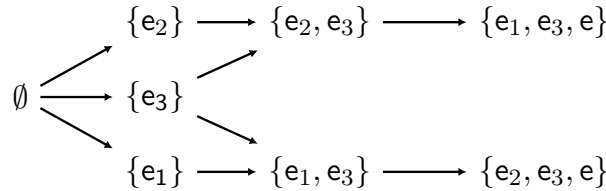


Figure 7: The configurations of the FES and CDES in the Example 4.19

4.6. Dynamic causality event structures: The intuition in this case is to code all the possible subsets of modifiers for a given target, and for each subset of modifiers, determine what is the set of enabling events. In this way the \gg relation can be easily obtained.

Proposition 4.20. *Let $D = (\mathbf{E}, \#, \rightarrow, \triangleleft, \blacktriangleright)$ be a DCES. $\mathcal{F}_{\text{DCES}}(D) = (\mathbf{E}, \#, \gg)$ is a CDES where the relation \gg is defined as $\{(X, (\text{ic}(\mathbf{e}) \setminus \bigcup_{\mathbf{e}' \in X} \text{Drop}(\mathbf{e}', \mathbf{e})) \cup \bigcup_{\mathbf{e}' \in X} \text{Add}(\mathbf{e}', \mathbf{e})) \mid X \subseteq \text{GroMod}(\mathbf{e}) \cup \text{ShrMod}(\mathbf{e})\} \gg \mathbf{e}$ for each $\mathbf{e} \in \mathbf{E}$. Furthermore $D \equiv \mathcal{F}_{\text{DCES}}(D)$.*

Proof. $\mathcal{F}_{\text{DCES}}(D)$ is clearly a CDES. We show that the sets of configurations coincide, which is enough to show that $D \equiv \mathcal{F}_{\text{DCES}}(D)$. Take $C \in \text{Conf}_{\text{DCES}}(D)$, then there is sequence of events $\rho = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n \cdots$ and it is such that for each $i \geq 1$ we have that $((\text{ic}(\mathbf{e}_i) \cup \text{ac}(\overline{\rho_{i-1}}, \mathbf{e}_i)) \setminus \text{dc}(\overline{\rho_{i-1}}, \mathbf{e}_i)) \subseteq \overline{\rho_{i-1}}$, now the entry associated to \mathbf{e}_i is $Z_i \gg \mathbf{e}_i$ and each element (X, Y) of this entry is such that X is subset of modifiers. Consider the X such that $\text{CXT}(Z_i \gg \mathbf{e}_i) \cap \overline{\rho_{i-1}} = X$, then $Y = ((\text{ic}(\mathbf{e}_i) \cup \text{ac}(\overline{\rho_{i-1}}, \mathbf{e}_i)) \setminus \text{dc}(\overline{\rho_{i-1}}, \mathbf{e}_i)) \subseteq \overline{\rho_{i-1}}$ which is exactly what is required, hence $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{DCES}}(D))$ as well.

The vice versa, namely that each configuration $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{DCES}}(D))$ is a configuration of D , is analogous. \square

Example 4.21. *Concerning the DCES of the example 3.18, the conflict relation is the one of the DCES whereas the CD-relation is $\{(\emptyset, \emptyset)\} \gg \mathbf{a}$, $\{(\emptyset, \emptyset)\} \gg \mathbf{b}$, $\{(\emptyset, \emptyset)\} \gg \mathbf{e}$, $\{(\emptyset, \emptyset)\} \gg \mathbf{d}$ and for \mathbf{c} we have $\{(\emptyset, \{\mathbf{b}\}), (\{\mathbf{a}\}, \emptyset), (\{\mathbf{d}\}, \{\mathbf{b}, \mathbf{e}\}), (\{\mathbf{a}, \mathbf{d}\}, \{\mathbf{e}\})\} \gg \mathbf{c}$.*

4.7. Inhibitor event structures: In the case of IES there are two main observations to be done: one, there is no conflict relation, and second, though there is some similarity between the \vdash° relation and the \gg relation, there is also a quite subtle difference. When adding an event \mathbf{e} to a configuration of an IES, and we have $\vdash^\circ(a, \mathbf{e}, A)$, one would simply add the pairs $(a, \{\mathbf{e}'\})$ for each $\mathbf{e}' \in A$ (as the events in A are pairwise conflicting) but this does not work in the case A is the empty set, as it has a different meaning in the \vdash° relation with respect to the \gg relation. In the former, it means that the event in a inhibits the event \mathbf{e} , whereas in the latter the pair (a, \emptyset) simply says that if the context a is present then there is no further event needed. Taking into account these differences, the translation is fairly simple.

Proposition 4.22. *Let $I = (\mathbf{E}, \vdash^\circ)$ be an IES. $\mathcal{F}_{\text{IES}}(I) = (\mathbf{E}, \#, \gg)$ is a CDES, where*

- $\mathbf{e} \# \mathbf{e}'$ iff $\vdash^\circ(\{\mathbf{e}\}, \mathbf{e}', \emptyset)$ and $\vdash^\circ(\{\mathbf{e}'\}, \mathbf{e}, \emptyset)$, and
- for each $\mathbf{e} \in \mathbf{E}$, if $\vdash^\circ(a, \mathbf{e}, A)$ and $A \neq \emptyset$ then $\{(\emptyset, \emptyset) \mid a \neq \emptyset\} \cup \{(a \cup \{\mathbf{e}'\}, \emptyset) \mid \mathbf{e}' \in A\} \gg \mathbf{e}$, if $\vdash^\circ(a, \mathbf{e}, A)$ and $A = \emptyset$ then $\{(a, \{\mathbf{e}\})\} \gg \mathbf{e}$.

Furthermore $I \equiv \mathcal{F}_{\text{IES}}(I)$.

Proof. The \gg relation of $\mathcal{F}_{\text{IES}}(I)$ obeys to the requirements of Definition 2.2 and the conflict relation $\#$ is a symmetric and irreflexive relation.

We prove that $\text{Conf}_{\text{IES}}(I) = \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{IES}}(I))$. Consider $C \in \text{Conf}_{\text{IES}}(I)$, then we have that there exists a sequence of events $\rho = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n \cdots$ such that for each \mathbf{e}_i and $\vdash^\circ(a, \mathbf{e}_i, A)$ we have that $a \subseteq \overline{\rho_{i-1}} \Rightarrow \overline{\rho_{i-1}} \cap A \neq \emptyset$. Assume that $a \subseteq \overline{\rho_{i-1}}$ holds, and $A \neq \emptyset$, then we have the entry $Z \gg \mathbf{e}_i$, where $Z = \{(\emptyset, \emptyset) \mid a \neq \emptyset\} \cup \{(a \cup \{\mathbf{e}'\}, \emptyset) \mid \mathbf{e}' \in A\}$, and then also the enabling condition of the CDES is satisfied, for each entry obtained for the event \mathbf{e}_i , as in the case $a \neq \emptyset$ and $\text{CXT}Z \gg \mathbf{e}_i \cap \overline{\rho_{i-1}} = \emptyset$ we have that the event can be added, and in the other case, if $a \neq \emptyset$, then one of the event in A must be present in

$\overline{\rho_{i-1}}$, which is captured by requiring that the context is a together with this event. We can conclude that C is also a configuration in $\text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{IES}}(I))$.

For the vice versa, assume that $C \in \text{Conf}_{\text{CDES}}(\mathcal{F}_{\text{IES}}(I))$, hence we have $\rho = e_1 e_2 \cdots e_n \cdots$ and for each $i \geq 1$ it holds that $\overline{\rho_{i-1}} [e_i]$. This means that for each entry $Z \gg e_i$, there is an element $(X, \emptyset) \in Z$ such that $\text{CXT}(Z \gg e_i) \cap \vdash^\circ(a, e, A) = X$. But if $X \neq \emptyset$ then $X = a \cup \{e\}$ for some $e \in A$ for a $\vdash^\circ(a, e, A)$, and this implies that $a \subset \overline{\rho_{i-1}} \Rightarrow \overline{\rho_{i-1}} \cap A \neq \emptyset$, hence C is also a configuration in $\text{Conf}_{\text{IES}}(I)$.

The \mapsto in both event structures coincide, hence $I \equiv \mathcal{F}_{\text{IES}}(I)$. \square

Example 4.23. *The IES of the example 3.22 induces the empty conflict relation, and the CD-relation is $\{(\emptyset, \emptyset)\} \gg a$, $\{(\emptyset, \{a\})\} \gg b$ and $\{(\emptyset, \emptyset), (\{a\}, \{b\})\} \gg c$.*

4.8. Higher order causality event structures: The comparison with event structures with higher-order dynamics of [KN15] is done indirectly, as these are equivalent to event structures with resolvable conflicts. In this approach the relations \triangleleft and \blacktriangleright are generalized to take into account set of modifiers, targets and contributions. The drawback is that the happening of an event implies a recalculation of these relation, similarly to what it is done in causal automata. In fact it is fairly obvious that given one simple step transition graph (meaning that a configuration is reached by another one adding just one event), it is always possible to obtain a CDES.

5. CONCLUSION

In this paper we have introduced a new brand of event structure where the main relation, the CD-relation, models the various conditions under which an event can be added to a subset of events. The relation is now defined as $\gg \subseteq \mathbf{2}^A \times \mathbf{E}$, where $A \subseteq \mathbf{2}_{fin}^E \times \mathbf{2}_{fin}^E$, thus it stipulates for each event which are the context-dependency pairs, but it can be easily generalized to subsets of events modeling precisely, when events happen together (as it is done in [PS12] or [vGP04]). The focus is on the contexts in which an event can be added, which may change, rather than modeling the dependencies and how these may change. Here the choice is whether it is better to focus on dependencies (and how they may change) or on the context. The advantage of the latter is its generality, whereas the former may be useful in pointing out relations among events.

It should be clear that this kind of event structures is capable of modeling the same enabling situation for an event in various way, and it could be interesting to understand if there could be an informative way canonically. In fact, the canonical relation just focus on all the contexts in which an event can be added, and the dependency set is less informative. Thus finding a way to identify minimal contexts together with a set of dependencies may be useful, similarly to what it has been discussed when associating PES to CDES.

It remains to stress that CDES can be generalized not only allowing steps but also representing contexts in a richer way. Here we have considered contexts as subset of events, but they can have a richer structure. This would allow to characterize more precisely contexts, allowing, for instance, to drop the last requirement we have placed on DCES, as in this case the order in which the modifiers appear may influence the dependencies. Finally we observe that the idea of context is not new, for instance they have been considered in [LM06] or in [BBB07], and a comparison with these should be considered.

In this paper we have considered various event structures, still some interesting notions remained out of the scope of this paper, like reversible event structures [PU15], but we are confident that our approach can be used also in the reversibility setting, clearly by introducing a suitable relation \ll for the reversing events and upon the identification of the context in which the reversing event can be *performed*. We do not have considered event structures with circular dependencies ([BCPZ14]) basically because in this case the configurations would not give an *ea* like those used here. In fact the configurations of an event structure with circular dependencies are pair of subsets of events, those actually happened and those that must happen to guarantee that the circular dependencies are fulfilled. It is however interesting to understand how a context in this way can be used, the intuition being that somehow it should be taken into account those events that have still to be performed as required by some circular dependency.

Finally we would like to mention that two interesting research issues regard the categorical treatment of this new brand of event structure and their relation to Petri nets. The categorical treatment could be inferred from the categorical treatment of event automata and the enabling/disabling relation studied in [Pin06], whereas for the second one we can follow the lines introduced in [CP17], where Petri nets are related to dynamic event structures.

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