Expressivity of Possibilistic Preference Networks with Constraints

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Abstract. Among several graphical models for preferences, CP-nets are often used for learning and representation purposes. They rely on a simple preference independence property known as the *ceteris paribus independence*. Our paper uses a recent symbolic graphical model, based on possibilistic networks, that induces a preference ordering on configurations consistent with the ordering induced by CP-nets. Ceteris paribus preferences in the latter can be retrieved by adding suitable constraints between products of symbolic weights. This connection between possibilistic networks and CP-nets allows for an extension of the expressive power of the latter while maintaining its qualitative nature. Elicitation complexity is thus kept stable, while the complexity of dominance and optimization queries is cut down.

1 Introduction

Various graphical models have been proposed in the literature in order to represent preferences in an intuitive manner. A survey of such approaches is in [3]. We may roughly distinguish between (i) quantitative models such as GAI networks [15] that use numerical utility functions (ii) qualitative models where preferences are contextually expressed by local comparisons between attribute values. The latter request less assessment effort from the user.

Among qualitative models, CP-nets [7] are the most popular. They provide a well-developed compact representation setting for preference modeling. The CP-net representation consists in a directed graph expressing conditional preference statements, interpreted under the *ceteris paribus* assumption. As an effect of the systematic application of this assumption, it has been observed that priority in the network is given to parent decision variables over children ones, a feature not deliberately required.

The more recently introduced π-pref nets [2,4] may be also classified as qualitative models. Indeed, similarly to CP-nets, this model represents local preferences in terms of conditional comparisons between variable assignments. π-pref nets are inspired by product-based numerical possibilistic networks [14] but they use symbolic (non-instantiated) possibility weights to model conditional preference tables. Additional information about the relative strength of preferences can be taken into account by adding constraints between these weights.

The paper proves that a π-pref net is able to capture ceteris paribus preferences between solutions induced by a CP-net, if suitable constraints between products of symbolic weights are added. These constraints explicitly express the higher importance of parent decision variables over their children nodes in the π-pref net. In [4], it was proved that π-pref nets orderings exactly correspond to a Pareto ordering over vectors expressing levels of satisfaction for each variable. We show that this ordering of configurations is refined by the ordering obtained by comparing sets of satisfied preference tables. These results show that the setting of π-pref nets is closely related to CP-nets since *ceteris paribus* constraints can be expressed by specific inequality constraints between products of symbolic weights.

The paper is organized as follows. Section 2 provides a brief background on CP-nets, while Sect. 3 introduces π-pref nets based on possibilistic networks with symbolic weights. Section 4 investigates conditions that enable preferences expressed by π-pref nets to get closer to CP-nets orderings. Section 5 presents related work, especially CP-theories [16], and the conclusion briefly compares the formalisms in terms of expressive power and query complexity.

2 CP-nets

Let $\mathcal{V} = \{A_1, \ldots, A_n\}$ be a set of Boolean variables, each taking values denoted, e.g., by a_i or $\neg a_i$. Each variable A_i has a value domain D_{A_i}. Ω denotes the universe of discourse, which is the Cartesian product of all variable domains in $\mathcal{V}$. Each element ω_i of Ω is called a *configuration*.

The user is assumed to express preferences under the form of comparisons between values of each variable, conditioned on some other instantiated variables. CP-nets deal with strict *preference statements*. Unconditional statements are of the form: "I prefer a^+ to a^-", where $a^+, a^- \in \{a, \neg a\}$ and $a^- = \neg a^+$, and we denote them by $a^+ \succ a^-$. When $A = a^+$, we say that the *quality* of the choice for A is good, and is bad otherwise. If the preference on A depends on other variables $\mathcal{P}(A)$ called the *parents* of A, and $p(A)$ is an instantiation of $\mathcal{P}(A)$, conditional preference statements are of the form "in the context $p(A)$, I prefer a^+ to a^-", denoted by $p(A) : a^+ \succ a^-$. To each variable we associate a table representing the local preferences on its domain values in each parent context (the value of a^+, respectively a^-, depends on the parents context).

Example 1. *Consider a preference specification about a holiday house in terms of 4 decision variables $\mathcal{V} = \{T, S, P, C\}$ standing for type, size, place and car park respectively, with values $T \in \{flat\ (t_1),\ house\ (t_2)\}$, $S \in \{big\ (s_1),\ small\ (s_2)\}$, $P \in \{downtown\ (p_1),\ outskirt\ (p_2)\}$ and $C \in \{car\ (c_1),\ nocar\ (c_2)\}$. Preference on T is unconditional, while all the other preferences are conditional as follows: $t_1 \succ t_2$, $t_1 : p_1 \succ p_2$, $t_2 : p_2 \succ p_1$, $p_1 : c_1 \succ c_2$, $p_2 : c_2 \succ c_1$, $t_1 : s_2 \succ s_1$, $t_2 : s_1 \succ s_2$.*

Definition 1. *A (conditional)* preference network *is a directed acyclic graph with nodes $A_i, A_j \in \mathcal{V}$, s.t. each arc from A_j to A_i expresses that the preference about A_i depends on A_j. Each node A_i is associated with a preference table CPT_i that associates strict preference statements $p(A_i) : a_i^+ \succ a_i^-$ between the two values of A_i conditional to each possible instantiation $p(A_i)$ of the parents $\mathcal{P}(A_i)$ of A_i.*

The preference statements of Example 1 correspond to the CP-net of Fig. 1.

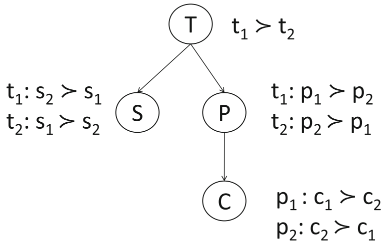

Fig. 1. Preference network for Example 1

Preference networks can be viewed as a qualitative counterpart of Bayesian nets. CP-nets [7,8] are preference networks relying on the *ceteris paribus* preferential independence assumption. Namely, a CP-net induces a partial order $\succ_{CP}$ between configurations, based on this preferential independence assumption: a value is preferred to another in a given context, everything else being equal. Given $\mathcal{U} \subseteq \mathcal{V}$ and $\omega \in \Omega$, $\omega_{\mathcal{U}}$ denotes the restriction of ω to variables in $\mathcal{U}$.

Definition 2 (***Ceteris Paribus***)**.** *Each strict preference statement $p(A_i)$: $a_i^+ \succ a_i^-$, is translated into $\omega \succ_{CP} \omega'$, whenever $\omega_{\{A_i\}} = a_i^+, \omega'_{\{A_i\}} = a_i^-$, and $\omega_{\mathcal{V}\setminus\{A_i\}} = \omega'_{\mathcal{V}\setminus\{A_i\}}$, and $\omega_{\mathcal{P}(A_i)} = \omega'_{\mathcal{P}(A_i)} = p(A_i)$.*

Due to the ceteris paribus assumption, configurations compared in the preference statements differ by a single flip, and switching A_i from a_i^+ to a_i^- is called a *worsening flip*. We get a directed acyclic graph of configurations (the *configuration graph*) with a unique top corresponding to the best configuration ($A_i = a_i^+, \forall i$) and a unique bottom corresponding to the worst one ($A_i = a_i^-, \forall i$). The worsening flip graph for Example 1 is represented in Fig. 2.

The configuration graphs induced by CP-nets are partial in general, and many configurations remain incomparable, for instance $t_1p_1c_1s_1$ and $t_1p_1c_2s_2$ are not comparable in the worsening graph of Fig. 2. Moreover, in the CP-nets semantics, parent preferences look more important than children ones, for example, the preferences of the node P are more important than C and the preferences of the root T are more important than all the other nodes.

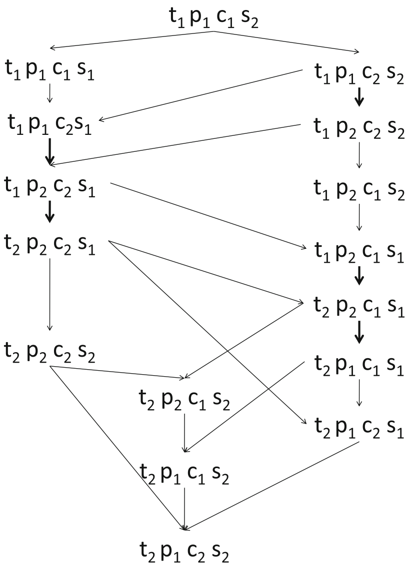

Fig. 2. CP-net preferences for Example 1 up to transitive closure (5 bold arrows represent ceteris paribus preference relations that are not recovered by π-pref net, 8 one-flip comparisons over 32 can be recovered by transitivity, e.g. from $t_1p_1c_1s_2$ to $t_2p_1c_1s_2$).

3 π-Pref nets

Possibility theory [11] can be used for representing preferences. It relies on the idea of a possibility distribution π, i.e., a mapping from a universe of discourse Ω to the unit interval $[0, 1]$. Possibility degrees $\pi(\omega)$ estimate to what extent the configuration ω is not unsatisfactory. π-pref nets are based on possibilistic networks [5], using conditional possibilities of the form $\pi(a_i|p(A_i)) = \frac{\Pi(a_i \wedge p(A_i))}{\Pi(p(A_i))}$, where $\Pi(\varphi) = \max_{\omega \models \varphi} \pi(\omega)$.

Definition 3 ([2,4]). *A* possibilistic preference network *(*π-pref net) *is a preference network in the sense of Definition 1, where each preference statement $p(A_i) : a_i^+ \succ a_i^-$ is associated to a conditional possibility distribution such that $\pi(a_i^+|p(A_i)) = 1 > \pi(a_i^-|p(A_i)) = \alpha_{A_i|p(A_i)}$, and $\alpha_{A_i|p(A_i)}$ is a non-instantiated variable on* $[0, 1)$ *we call* symbolic weight*.*

One may also have indifference statements $p(A_i) : a_i \sim \neg a_i$, expressed by $\pi(a_i|p(A_i)) = \pi(\neg a_i|p(A_i)) = 1$.

On top of the preferences encoded by a π-pref net, a set $\mathcal{C}$ of additional equality or inequality constraints between symbolic weights or products of symbolic weights can be provided by the user. Such constraints may represent, for instance, the relative strength of preferences associated to different instantiations of parent variables of the same variable.

π-pref nets induce a partial ordering between configurations based on the comparison of their degrees of possibility in the sense of a joint possibility distribution computed using the product-based chain rule, expressing a satisfaction erosion effect:

$$\pi(A_i, \ldots, A_n) = \prod_{i=1,\ldots,n} \pi(A_i|p(A_i)) \tag{1}$$

The preferences in the obtained configuration graph are of the form $\omega \succ_\pi \omega'$ if and only if $\pi(\omega) > \pi(\omega')$ *for all instantiations* of the symbolic weights.

Example 2. *Consider preference statements in Example 1. Conditional possibility distributions are as follows:* $\pi(t_1) = 1$, $\pi(t_2) = \alpha$, $\pi(p_1|t_1) = \pi(p_2|t_2) = 1$, $\pi(p_2|t_1) = \beta_1$, $\pi(p_1|t_2) = \beta_2$, $\pi(s_1|t_1) = \gamma_1$, $\pi(s_2|t_2) = \gamma_2$, $\pi(s_2|t_1) = \pi(s_1|t_2) = 1$, $\pi(c_1|p_1) = \pi(c_2|p_2) = 1$, $\pi(c_2|p_1) = \delta_1$ *and* $\pi(c_1|p_2) = \delta_2$. *Applying the product-based chain rule, we can compute the joint possibility distribution relative to T, P, C and S. Figure 3 represents with thin arrows the configuration graph induced from the joint possibility distribution. Clearly, the configuration* $t_1p_1c_1s_2$ *is the root (since it is the unique one with degree* $\pi(t_1p_1c_1s_2) = 1$*).*

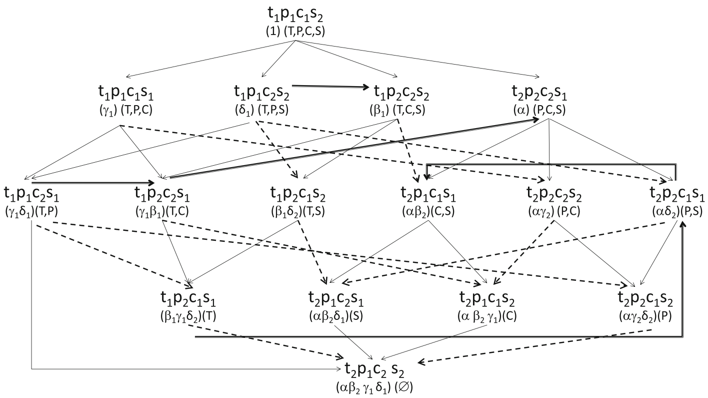

Fig. 3. Configuration graph of Example 1. Thin arrows reflect $\succ_\pi$, dotted arrows compare sets $\mathcal{S}(\omega_i)$ of Sect. 4, and bold arrows reflect additional ceteris paribus comparisons recovered by the constraints, also in bold on Fig. 2. Values under 1st (resp. 2nd) brackets correspond to joint possibility degrees (resp. sets $\mathcal{S}(\omega_i)$)

In the following, we compare configuration graphs induced by both CP-nets and π-pref nets. Clearly, they are different when no additional constraints are assumed between symbolic weights. For instance, if we consider the previous example we can check that in contrast with the CP-net of Fig. 2 where $t_1p_1c_2s_1 \succ_{CP} t_1p_2c_2s_1$, the π-pref net fails to compare them since there is no inequality constraint between $\pi(t_1p_1c_2s_1) = \gamma_1\delta_1$ and $\pi(t_1p_2c_2s_1) = \gamma_1\beta_1$.

4 Main Results

In this section, we show that the configuration graph of any CP-net is consistent with the configuration graph of the π-pref net without local indifference, based on the same preference network, provided that some constraints on products of symbolic weights are added to the π-pref net, in order to restore the ceteris paribus priorities. Precisely, the added constraints reflect the higher importance of parent nodes with respect to their children. Under an additional property whose validity can only be conjectured at this point, π-pref net would capture CP-nets exactly.

4.1 Consistency Between CP-nets and π-pref nets

In the following, we first recall that the ordering between configurations induced by a π-pref net corresponds to the Pareto ordering between the vectors $\boldsymbol{\omega} = (\theta_1(\omega), \ldots, \theta_n(\omega))$ where $\theta_i(\omega) = \pi(\omega_{A_i}|\omega_{\mathcal{P}(A_i)}), i = 1, \ldots, n$. The Pareto ordering is defined by

$$\boldsymbol{\omega} \succ_{Pareto} \boldsymbol{\omega'} \text{ iff } \forall\, k,\ \theta_k(\omega) \geq \theta_k(\omega') \text{ and } \theta_i(\omega) > \theta_i(\omega') \text{ for some } i.$$

It is easy to see that $\theta_i(\omega) \in \{1, \alpha_{A_i|p(A_i)}\}$ where $\alpha_{A_i|p(A_i)}$ is the symbolic weight that appears in the preference table for variable A_i in the context $\omega_{\mathcal{P}(A_i)}$. It is easy to see that $\theta_k(\omega) > \theta_k(\omega')$ if and only if $\theta_k(\omega) = 1$ and $\theta_k(\omega')$ is a symbolic value. But it may be that $\theta_k(\omega)$ and $\theta_k(\omega')$ are distinct symbolic values, hence making ω and ω' incomparable. In particular, there are as many different symbolic weights $\alpha_{A|p(A)}$ pertaining to a boolean variable A as instantiations of parents of A. As symbolic weights are not comparable across variables, it is easy to see that the only way to have $\pi(\omega) \geq \pi(\omega')$ is to have $\theta_k(\omega) \geq \theta_k(\omega')$ in each component k of $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$. Otherwise the products will be incomparable due to the presence of distinct symbolic variables on each side. So,

$$\omega \succ_\pi \omega' \text{ if and only if } \boldsymbol{\omega} \succ_{Pareto} \boldsymbol{\omega'}$$

Given the ordinal nature of preference tables of CP-nets, it also makes sense to characterize the quality of ω using the set $\mathcal{S}(\omega) = \{A_i : \theta_i(\omega) = 1\}$ of satisfied preference statements (one per variable). It is then clear that the Pareto ordering between configurations induced by the preference tables is refined by comparing these satisfaction sets:

$$\boldsymbol{\omega} \succ_{Pareto} \boldsymbol{\omega'} \Rightarrow \mathcal{S}(\omega') \subset \mathcal{S}(\omega) \tag{2}$$

since if two configurations contain variables having bad assignments in the sense of the preference tables, the corresponding symbolic values may differ if the contexts for assigning a value to this variable differ.

Example 3. *To see that this inclusion-based ordering is stronger than the π-pref net ordering, consider Fig. 3 where $\pi(t_1p_2c_1s_2) = \beta_1\delta_2$ with $\mathcal{S}(t_1p_2c_1s_2) = \{T, S\}$ and $\pi(t_2p_1c_2s_1) = \alpha\beta_2\delta_1$ with $\mathcal{S}(t_2p_1c_2s_1) = \{S\}$. We do have that $\mathcal{S}(t_1p_2c_1s_2) \supset \mathcal{S}(t_2p_1c_2s_1)$, but $\beta_1\delta_2$ is not comparable with $\alpha\beta_2\delta_1$. Dotted and thin arrows of Fig. 3 represent the configuration graph induced by comparing sets $\mathcal{S}(\omega)$.*

It is noticeable that if the weights $\alpha_{A_i|p(A_i)}$ reflecting the satisfaction level due to assigning the bad value to A_i in the context $p(A_i)$ do not depend on the context, then we have an equivalence in Eq. (2):

Proposition 1. *If* $\forall i = 1, \ldots, n, \alpha_{A_i|p(A_i)} = \alpha_i, \forall p(A_i) \in \mathcal{P}(A_i)$, *then*

$$\boldsymbol{\omega} \succ_{Pareto} \boldsymbol{\omega'} \iff \mathcal{S}(\omega') \subset \mathcal{S}(\omega).$$

Proof: Suppose $\mathcal{S}(\omega') \subset \mathcal{S}(\omega)$ then if $A \in \mathcal{S}(\omega')$ we have $\theta_i(\omega) = \theta_i(\omega') = 1$; if $A \in \mathcal{S}(\omega) \setminus \mathcal{S}(\omega')$, then $\theta_i(\omega') = \alpha_i, \theta_i(\omega) = 1$ and $\theta_i(\omega') = \alpha_i = \theta_i(\omega)$ otherwise. This implies $\boldsymbol{\omega} \succ_{Pareto} \boldsymbol{\omega'}$.

The inclusion-based ordering $\mathcal{S}(\omega') \subset \mathcal{S}(\omega)$ does not depend on the parent variables context but only on the fact that a variable has a good or a bad value. Similarly, when the symbolic weights no longer depend on parents instantiations, there is only one symbolic weight per variable. So, the above result is not surprising.

Example 4. *Using the same nodes as in Example 3, the unique weight assumption enforces* $\beta_1 = \beta_2 = \beta$ *and* $\delta_1 = \delta_2 = \delta$, *which yields* $\pi(t_1p_2c_1s_2) = \beta\delta > \pi(t_2p_1c_2s_1) = \alpha\beta\delta$.

In the following, we assume that the components of vector $\boldsymbol{\omega}$ are linearly ordered in agreement with the partial ordering of variables in the symbolic preference network, namely, if $i < j$ then A_i is not a descendant of A_j in the preference net (i.e. topological ordering). For instance in the preference net of Fig. 1, we can use the ordering (T, P, C, S).

Let us first prove that, in the configuration graphs induced by a CP-net and the corresponding π-pref net, there cannot be any preference reversals between configurations. Let $\mathcal{C}h(A)$ denote the children set of $A \in \mathcal{V}$.

Lemma 1. *Let* ω *and* ω' *be two configurations such that* $\omega \succ_{CP} \omega'$ *and* ω *and* ω' *differ by one flip of a variable* A_i *then* $\mathcal{S}(\omega) \subset \mathcal{S}(\omega')$ *is not possible.*

Proof: Compare $\mathcal{S}(\omega)$ and $\mathcal{S}(\omega')$. It is clear that $A_i \notin \mathcal{S}(\omega')$ (otherwise the flip would not be improving) and $\mathcal{S}(\omega) = (\mathcal{S}(\omega') \cup \{A_i\} \cup \mathcal{C}h^+_-(A_i)) \setminus \mathcal{C}h^-_+(A_i)$, where $\mathcal{C}h^+_-(A_i)$ is the set of variables that switch from a bad to a good value when going from ω' to ω, and $\mathcal{C}h^-_+(A_i)$ is the set of variables that switch from a good to a bad value when going from ω' to ω. It is clear that it can never be the case that $\mathcal{S}(\omega) \subset \mathcal{S}(\omega')$, indeed A_i is in $\mathcal{S}(\omega)$ and not in $\mathcal{S}(\omega')$ by construction. But $\mathcal{S}(\omega')$ may contain variables not in $\mathcal{S}(\omega)$ (those in $\mathcal{C}h^-_+(A_i)$ if not empty). So either $\mathcal{S}(\omega') \subset \mathcal{S}(\omega)$ or the two configurations are not Pareto-comparable. □

In the following, given two configurations ω and ω', let $\mathcal{D}^{\omega,\omega'}$ be the set of variables which bear different values in ω and ω'.

Proposition 2. *If* $\omega \succ_{CP} \omega'$ *then* $\mathcal{S}(\omega) \subset \mathcal{S}(\omega')$ *is not possible.*

Proof: If $\omega \succ_{CP} \omega'$, then there is a chain of improving flips $\omega_0 = \omega' \prec_{CP} \omega_1 \prec_{CP} \cdots \prec_{CP} \omega_k = \omega$. Applying the above Lemma, $\mathcal{S}(\omega_i) = (\mathcal{S}(\omega_{i-1}) \cup \{V_{i-1}\} \cup \mathcal{C}h^+_-(V_{i-1})) \setminus (\mathcal{C}h^-_+(V_{i-1})$ for some variable $V_{i-1} = A_j$. By the above Lemma, we cannot have $\mathcal{S}(\omega_{i-1}) \subset \mathcal{S}(\omega_i)$. Suppose we choose the chain of improving flips by flipping at each step a top variable A_j in the preference net, among the ones to be flipped, i.e. $j = \min\{\ell : A_\ell \in \mathcal{D}^{\omega_{i-1},\omega}\}$. It means that when following the chain of improving flips, the status of each flipped variable will not be questioned by later flips, as no flipped variable will be a child of variables flipped later on. So $\mathcal{S}(\omega)$ will contain some variables not in $\mathcal{S}(\omega')$, so $\mathcal{S}(\omega) \subset \mathcal{S}(\omega')$ is not possible.

The previous results show that it is impossible to have a preference reversal between the CP-net ordering and the inclusion ordering, which implies that no preference reversal is possible between CP-net ordering and the π-pref net ordering. It suggests that we can try to add ceteris paribus constraints to a π-pref net and so as to capture the preferences expressed by a CP-net.

As previously noticed, in CP-nets, parent preferences look more important than children ones. This property is not ensured by π-pref nets where all violations are considered having the same importance. Indeed, we can check from Figs. 2 and 3 that the two configuration graphs built from the same preference statements of Example 1 are different. In the following, we lay bare local constraints between each node and its children that enable ceteris paribus to be simulated. Let $D_{\mathcal{P}(A)} = \times_{A_i \in \mathcal{P}(A)} D_{A_i}$ denote the Cartesian product of domains of variables in $\mathcal{P}(A)$, $\alpha_{A|p(A)} = \pi(a^-|p(A))$ and $\gamma_{C|p(C)} = \pi(c^-|p(C))$.

Proposition 3. *Suppose a CP-net and a π-pref net built from the same preference statements. Let us add to the latter all constraints induced by the condition: $\forall\ A \in \mathcal{V}$ s.t. $\mathcal{C}h(A) \neq \emptyset$:*

$$\max_{p(A) \in D_{\mathcal{P}(A)}} \alpha_{A|p(A)} < \prod_{C \in \mathcal{C}h(A)} \min_{p(C) \in D_{\mathcal{P}(C)}} \gamma_{C|p(C)} \tag{3}$$

Let $\succ^+_\pi$ be the resulting preference ordering built from the preference tables and applying constraints between symbolic weights of the form of Eq. 3, then, $\omega \succ_{CP} \omega' \Rightarrow \omega \succ^+_\pi \omega'$.

Proof: The relation $\succ_{CP}$ is determined by comparing configurations ω, ω' of the form $\omega = a^+ \wedge p(A) \wedge r$ and $\omega' = a^- \wedge p(A) \wedge r$ (where $R = \mathcal{V} \setminus (A \cup \mathcal{P}(A))$), that differ by one flip of variable A. So the local preference $p(A) : a^+ \succ a^-$ is equivalent to have $\omega \succ_{CP} \omega'$ under ceteris paribus assumption, and also equivalent to have $\pi(a^-|p(A)) < \pi(a^+|p(A)) = 1$ in the corresponding π-pref net based on the same preference tables.

Now let us show that $\forall A \in \mathcal{V}$, $\forall p(A) \in D_{\mathcal{P}(A)}$ and every instantiation r of the variables in $\mathcal{V} \setminus (\{A\} \cup \mathcal{P}(A))$, the local preference $\pi(a^-|p(A)) < \pi(a^+|p(A))$ implies $\pi(a^- \wedge p(A) \wedge r) < \pi(a^+ \wedge p(A) \wedge r)$ under the condition expressed by Eq. (3). Consider the instantiation $ch(A) = \wedge_{C \in \mathcal{C}h(A)} \omega_C$, where $\omega_C \in \{c, \neg c\}$, of the children of A such that $ch(A) \wedge o = r$ (i.e. $O = \mathcal{V} \setminus (A \cup \mathcal{P}(\mathcal{A}) \cup \mathcal{C}h(A))$.

The chain rule states (Eq. (1)):
$\pi(\omega') = \prod_{B\in\mathcal{V}} \pi(\omega'_B|\omega'_{\mathcal{P}(B)}) =$
$\pi(a^-|p(A))\cdot\prod_{C\in\mathcal{C}h(A)}\pi(\omega'_C|p'(C))\cdot\prod_{B\notin\{A\}\cup\mathcal{C}h(A)} \pi(\omega'_B|\omega'_{\mathcal{P}(B)})$.

Clearly the last term does not depend on A and is thus a constant β. So $\pi(\omega') = \beta\cdot\alpha_{A|p(A)}\cdot\prod_{C\in\mathcal{C}h(A)} \pi(\omega'_C|p'(C))$.
Likewise, since $\omega = a^+ \wedge p(A) \wedge ch(A) \wedge o$, we have
$\pi(\omega) = \beta\cdot\prod_{C\in\mathcal{C}h(A)} \pi(\omega_C|p(C))$, since $\pi(a^+|p(A)) = 1$. Note that while $p'(C)$ is of the form $a^- \wedge p_{-A}(C)$ where $\mathcal{P}_{-A}(C)$ is the set of parents of C but for A, $p(C)$ is of the form $a^+ \wedge p_{-A}(C)$.

So the inequality $\pi(\omega) > \pi(\omega')$, present in the CP-net, requires:

$$\prod_{C\in\mathcal{C}h(A)} \pi(\omega_C|p(C)) > \alpha_{A|p(A)}\cdot\prod_{C\in\mathcal{C}h(A)} \pi(\omega'_C|p'(C)).$$

Condition (3) implies $\alpha_{A|p(A)} < \prod_{C\in\mathcal{C}h(A)} \pi(\omega_C|p(C))$, which implies the above inequality. It proves that, under Condition (3), $\omega \succ_{CP} \omega'$ implies $\omega \succ^+_\pi \omega'$. □

This proposition ensures that the ordering induced by the joint possibility distribution of a π-pref net enhanced by constraints of the form (3) can refine the CP-net ordering having the same preference tables, provided that suitable constraints are added at each node $A\in\mathcal{V}$ between the local conditional possibility distribution at this node and the product of possibility degrees of the children of A. It comes down to constraints between each symbolic weight and a product of other ones. Indeed the less preferred value, $\min(\pi(a|p(A), \pi(\neg a|p(A))$, of A in the context of the parents $p(A)$ of A is a symbolic weight (non instantiated possibility degree). In other words, the inequality ensures that the less preferred value of each A given $p(A)$ is strictly less preferred than the product of the less preferred values of the children of A. This result is the symbolic counterpart of the one in [13], using preference networks with numerical ranking functions.

Example 5. *In the graph of Fig. 3 induced by the π-pref net of Example 2, Proposition 3 leads us to add conditions $\alpha < \min(\gamma_1,\gamma_2)\cdot\min(\beta_1,\beta_2)$ and $\max(\beta_1,\beta_2) < \min(\delta_1,\delta_2)$. Clearly these conditions are too strong here. First some of the products like $\gamma_1\beta_2$ never appear in Fig. 3. Moreover, the reader can check that adding constraints $\beta_1\gamma_1 > \alpha$ and $\beta_i < \delta_i, (i = 1, 2)$ turns the configuration graph of Fig. 3 into the CP-net-induced configuration graph of Fig. 2.*

Instead of imposing priority of parents over children, we can also add the ceteris paribus constraints to the π-pref net directly, considering only worsening flips. Let ω,ω' differ by one flip, and such that none of $\omega \succ_\pi \omega'$, $\omega' \succ_\pi \omega$ holds, and moreover, $\omega \succ_{CP} \omega'$. We must enforce the condition $\pi(\omega) > \pi(\omega')$. Suppose the flipping variable is A. Clearly, $A\in\mathcal{S}(\omega)$, but $A\notin\mathcal{S}(\omega')$. Let α be the possibility degree of A when it takes the bad value in context $\omega_{p(A)}$ (it is 1 when it takes the good value). When flipping A from a good to a bad value, only the quality of the children variables $\mathcal{C}h(A)$ of A may change. $\mathcal{C}h(A)$ can be partitioned into at most 4 sets, $\mathcal{C}h^-_-(A)$ (resp. $\mathcal{C}h^-_+(A), \mathcal{C}h^+_-(A), \mathcal{C}h^+_+(A)$), which represents the set of children of A whose values remain bad (resp. change from good to bad, from bad to good, and stay good) when flipping A from a^+ to a^-. Strictly speaking these sets depend upon ω. Then it can be easily checked that:

$$\pi(\omega) = 1 \cdot \prod_{C_i \in \mathcal{C}h_-^+(A)} \gamma_i \cdot \prod_{C_j \in \mathcal{C}h_-^-(A)} \gamma_j \cdot \beta$$

$$\pi(\omega') = \alpha \cdot \prod_{C_k \in \mathcal{C}h_+^-(A)} \gamma_k \cdot \prod_{C_j \in \mathcal{C}h_-^-(A)} \gamma_j \cdot \beta$$

where β is a product of symbols, pertaining to nodes other than A and its children, that remain unchanged by the flip of A. Then the constraint $\pi(\omega) > \pi(\omega')$ comes down to the inequality:

$$\prod_{C_i \in \mathcal{C}h_-^+(A)} \gamma_i > \alpha \cdot \prod_{C_k \in \mathcal{C}h_+^-(A)} \gamma_k \tag{4}$$

where symbols appearing on one side do not appear on the other side. Such constraints are clearly weaker than Condition (3) but are sufficient to retrieve all the preferences of the CP-net. Note that the preferences $\omega \succ_\pi \omega'$ and $\omega \succ_{CP} \omega'$ conjointly hold in both approaches whenever A has no child node, and more generally whenever the worsening flip on A corresponds to no child variable moving from a bad to a good state, i.e. $\mathcal{C}h_-^+(A) = \emptyset$. In fact, condition (4) holds for all preference arcs in the configuration graph of the CP-net, whether this preference appears in the π-pref net or not. We get the following result.

Proposition 4. *Consider a CP-net and the preference relation $\succ_\pi^+$ on configurations built from the same preference tables by adding all constraints of the form (4) between configurations differing by one flip to the preferences of the form $\omega \succ_\pi \omega'$. Then:*

$$\omega \succ_{CP} \omega' \Rightarrow \omega \succ_\pi^+ \omega'$$

Proof: Indeed, first the preferences according to $\succ_{CP}$ and $\succ_\pi$ do not contradict each other, per Proposition 2. Then we add constraints to the π-pref net for all CP-net worsening flips that are not captured by $\succ_\pi$, using constraints (4). So we have then captured the whole preference graph of the CP-net, plus possibly other preferences between configurations.

In the transformation of a CP-net into a π-pref net, we keep the same graphical structure and the tables are filled directly from the preference statements of the CP-net. Besides, we must point out that, when mimicking CP-nets, constraints are not elicited from the user but computed directly from the graph structure.

Example 6. *The above constraints (4) that must be added to the π-pref configuration graph of Fig. 3 are precisely those found to be necessary and sufficient in*

Example 5 to recover the CP-net ordering, i.e., $\alpha < \beta_1\gamma_1$, $\beta_1 < \delta_1$ *and* $\beta_2 < \delta_2$*. Note that the number of additional constraints to be added to capture the CP-net comparisons missed by the* π*-pref net is quite small. For instance, the number of constraints here is 4 against 120 potential comparisons.*

So, in the example, we exactly capture the preference graph of a CP-net using additional constraints between products of symbolic weights. The above considerations thus encourage us to study whether π-pref nets without constraints are *refined* by CP-nets, namely if the configuration graph of the former contains less strict preferences between configurations than the one of the latter, so that adding the constraints (4) are enough to simulate a CP-net by a π-pref net with constraints. Note that if it were not the case, it would mean that CP-nets do not respect Pareto-ordering.

4.2 Towards Exact Representations of CP-nets by π-pref nets

In this subsection, we consider the inclusion-ordering. One may wonder if there may exist some configurations that can be compared by the inclusion-based ordering, while they remain incomparable for CP-nets. This is not the case in our running example.

Example 7. *Consider the top configuration* $\omega' = t_1p_1c_1s_2$ *which inclusion-dominates* $\omega = t_2p_2c_2s_1$ *in the* π*-pref net configuration graph in Fig. 3, since the former has good values for all variables and only the value* t_2 *is bad in the latter, i.e.,* $\mathcal{S}(\omega') = \{T, P, C, S\}$ *and* $\mathcal{S}(\omega) = \{P, C, S\}$*. But the two configurations are far away in terms of flips since* $\mathcal{D}^{\omega,\omega'} = \{T, P, C, S\}$*. They can, however, be related by a chain of worsening flips. Namely, as* $\mathcal{S}(\omega') \setminus \mathcal{S}(\omega) = \{T\}$*, we must flip* T *first, and* $\omega_1 = t_1p_2c_2s_1$*, with* $\mathcal{S}(\omega_1) = \{T, C\}$ *so* $\mathcal{S}(\omega') \setminus \mathcal{S}(\omega_1) = \{P, S\}$ *and* $\mathcal{D}^{\omega_1,\omega'} = \{P, C, S\}$*. We now must flip* P *and get* $\omega_2 = t_1p_1c_2s_1$ *with* $\mathcal{S}(\omega_2) = \{T, P\} = \mathcal{D}^{\omega_2,\omega'}$*. As* $\mathcal{S}(\omega') \setminus \mathcal{S}(\omega_2) = \{C, S\}$*, we must flip* C*, and* $\omega_3 = t_1p_1c_1s_1$*, with* $\mathcal{S}(\omega_3) = \{T, P, C\}$ *so* $\mathcal{S}(\omega') \setminus \mathcal{S}(\omega_3) = \{S\} = \mathcal{D}^{\omega_3,\omega'}$*. We now must flip* S *and get* $\omega_4 = t_1p_1c_1s_2 = \omega'$*.*

The question whether the preference ordering of configurations induced by CP-nets is consistent with the ordering between the sets of variables that take good values in agreement with the preference tables seems to have been overlooked so far in the CP-net literature. The inclusion ordering between sets of variables with satisfactory values is intuitive in the sense that if a configuration ω violates all the preference statements violated by another configuration ω' plus some other(s), then ω' should indeed be strictly preferred to ω. The consistency of CP-nets with inclusion, namely the property

$$\mathcal{S}(\omega_1) \subset \mathcal{S}(\omega_2) \Rightarrow \omega_2 \succ_{CP} \omega_2 \qquad (*)$$

can be naturally conjectured since the opposite case would cast a doubt on the rationality of such networks. Proposition 2 proves a weak consistency between them. However, at this stage providing a formal complete proof looks tricky and besides, is not directly related to the expressivity of π-pref nets, the very topic of this paper. The results in the following are conditioned by the truth of the conjecture, or are restricted to those CP-nets that agree with the inclusion-based orderings. Based on this assumption, Proposition 5 indicates that the CP-net ordering refines, hence is consistent with, the ordering induced by a π-pref net built from the same preference specification. This is because the inclusion-ordering refines the Pareto (or π-pref net) ordering.

Proposition 5. *Consider a CP-net that refines the inclusion-based ordering and a π-pref net built from the same preference statements, we have:*

$$\omega' \succ_\pi \omega \Rightarrow \omega' \succ_{CP} \omega$$

Let us now prove that, if the conjecture (*) is valid, we are able to *exactly* induce the CP-net ordering from the π-pref net ordering by adding suitable constraints between symbolic weights or their products. First, we have seen that we can add to the π-pref net configuration graph all missing preference statements induced by the CP-net and not already present in the π-pref net configuration graph. These statements concern all pairs (ω, ω') that differ by one flip and such that $\pi(\omega)$ and $\pi(\omega')$ are not comparable. Note that adding such preference statements to the Pareto configuration graph in case of Pareto-incomparability yields the CP-net configuration graph (up to transitive closure).

The question remains whether we can express the latter in terms of additional constraints between symbolic weights or products thereof.

Proposition 6. *Consider a CP-net that refines the inclusion-based ordering and the preference relation $\succ_\pi^+$ on configurations built from the same preference tables by enforcing all constraints of the form (4) between configurations differing by one flip. Then:*

$$\omega \succ_{CP} \omega' \Leftrightarrow \omega \succ_\pi^+ \omega'$$

Proof: ($\Rightarrow$) This direction is proved by Proposition 4. ($\Leftarrow$) As $\omega \succ_\pi \omega' \Rightarrow \omega \succ_{CP} \omega'$ by assumption, adding ceteris paribus constraints corresponding to worsening flips to $\succ_\pi$ will not produce by transitivity any preference relation not in $\succ_{CP}$. □

It is clear that, beside ceteris paribus constraints, other constraints could be added to a π-pref net, that cannot be expressed by a CP-net, and that account for different types of preference information. This fact suggests that π-pref nets with constraints have a better expressive power and are more flexible than CP-nets (known as a powerful qualitative model), and provide a general class of qualitative graphical models where the *ceteris paribus* ordering could be further refined without going numerical (i.e. unlike UCP-nets). It is clear therefore that the constraints added to refine this order should, in this case, be consistent with *ceteris paribus*. Finally, π-pref nets are sometimes able to represent preference orderings when CP-nets fail to do it, as shown in the example below [1].

Example 8. *Let us consider two binary variables A and B standing respectively for "vacations" and "good weather". Suppose that we have the following preference ordering:* $ab \succ \neg a\neg b \succ a\neg b \succ \neg ab$. *We observe that this complete order cannot be represented by a CP-net. In fact, given two variables we can define two possible structures: either A depends on B or conversely. But, none of them are capable to capture this total ordering in the CP-net setting. Indeed, this total order exhibits a violation of the Ceteris Paribus principle. However, such preferences can be represented by a joint possibility distribution such that:* $\pi(ab) > \pi(\neg a\neg b) > \pi(a\neg b) > \pi(\neg ab)$. *Thus, we have* $\top : a \succ \neg a$, $a : b \succ \neg b$ *and* $\neg a : \neg b \succ b$. *It corresponds to a network with two nodes and their corresponding conditional possibility distributions are:* $\pi(a) = 1$, $\pi(\neg a) = \alpha$, $\pi(b|a) = 1$, $\pi(b|\neg a) = \gamma$, $\pi(\neg b|a) = \beta$ *and* $\pi(\neg b|\neg a) = 1$. *This yields* $\pi(ab) = 1 > \pi(\neg a\neg b) = \alpha > \pi(a\neg b) = \beta > \pi(\neg ab) = \alpha\gamma$ *taking* $\alpha > \beta$ *and* $\beta = \gamma$.

5 Related Works

Despite the existence of various graphical models for preferences [3], only few works have been concerned in comparing their expressive power. We can, in particular, mention two interesting results. The first concerns OCF-nets, which are preference networks where possibility distributions are replaced by ranking (ordinal conditional) functions (OCF) valued in the set of integers and the chain rule is additive. These functions may be transformed into possibility distributions [12]. [13] proved that OCF-nets can refine CP-net orderings. Precisely, OCF-nets will always lead to total orderings that are compatible with CP-nets. To do so they use a set of particular constraints to be imposed on their integer weights, which basically correspond to our constraints (3), albeit between numerical values. In contrast, the use of symbolic weights in our approach preserves the partiality of the ordering, and, if the CP-net order does refine the inclusion ordering, the CP-net configuration graph can be exactly recovered. Moreover the use of symbolic weights does not commit us to the choice of particular numerical values.

There were several attempts to represent CP-net orderings using a possibilistic logic base (a logical counterpart of π-pref nets), where the product is replaced by the minimum. See [10] for a bibliography and a discussion. It was observed that an exact logical representation of CP-nets was not possible when variables in the net have several children variables even though good approximations could be built. This is because additional constraints in this framework compare individual symbolic weights, not product thereof.

Some extensions of CP-nets can be considered as akin to π-pref nets. TCP-nets [9] also add priority constraints between variable nodes, that we can render in π-pref nets by inequalities between symbolic weights pertaining to different CP-tables. Utility-enhanced CP-nets (UCP-nets) [6] add additive utility functions to CP-nets in order to encode total orderings consistent with the ceteris paribus assumption. To do so, linear constraints are added on utility values that are somewhat similar to constraints (4). They express that for any variable, given an instantiation of its parents, the utility gain in choosing the good value rather than the bad one in this context, should be more important than the maximum value of the sum of the possible utility loss for its children over all possible instantiations of the other related variables. Up to a log transformation this is like comparing products.

π-pref nets can also be compared with so-called CP-theories [16]. The latter interpret conditional preference statements assuming they hold *irrespectively* of the values of other variables. It means that any configuration ω such that $\omega_A = a^+$ and $\omega_{\mathcal{P}(A)} = p(A)$ is preferred to any configuration ω such that $\omega_A = a^-$ and $\omega_{\mathcal{P}(A)} = p(A)$. In terms of possibility functions, it reads $\Delta(p(A) \wedge a^+) > \Pi(p(A) \wedge a^-)$, where $\Delta(\varphi) = \min_{\omega \models \varphi} \pi(\omega)$. In [16] are studied hybrid nets where some variables are handled ceteris paribus, while the preference holds irrespectively of other variables. In π-pref nets preference statements are interpreted by $\pi(a^+|p(A)) > \pi(a^-|p(A))$ which is provably equivalent to $\Pi(p(A) \wedge a^+) > \Pi(p(A) \wedge a^-)$, i.e. comparing best configurations. It is clear that if $\omega \succ_{CP} \omega'$ holds, then $\omega \succ \omega'$ holds in a CP-theory, where conditional preference holds irrespectively of other variables, because the CP-theory generates more preference constraints between configurations, including the ones induced by the *ceteris paribus* assumption. Constraints induced by CP theories can thus be captured in π-pref nets by adding more constraints between products of symbolic weights.

6 Conclusion

In this paper, we have explored the expressive power of π-pref nets. First, we have proved that the CP-net configuration orderings cannot contradict those of the π-pref nets and we found suitable additional constraints to refine π-pref net orderings in order to encompass ceteris paribus constraints of CP-nets. CP-nets would then be exactly captured by π-pref nets with constraints if their configuration graph did refine the inclusion-based ordering. This indicates that CP-nets potentially represent a subclass of π-pref nets with constraints. One may further refine CP-net preferences by adding more constraints between symbolic weights appearing in π-pref nets. For instance, one may introduce priorities between two parents nodes or between two child nodes.

Regarding query processing, finding an optimal configuration is straightforward, for both CP-nets and π-pref nets. In fact, it consists in traversing the network from root to leaves and choose the best value for each variable depending on its parents configuration. The complexity is linear with the size of the network. As to comparing two configurations, the dominance query for CP-nets consists in finding a chain of worsening flips from one configuration to the other. It is NP-complete to PSPACE-complete depending on the graph structure [8]. For π-pref nets, without constraints this query comes down to a Pareto-comparison of vectors symbolic weights. If there are constraints, the approach requires a reordering of coefficients and the complexity is at most equal to $O(n!)$ [2]. Dominance and optimization queries on instantiated π-pref nets respecting constraints will have the same complexities as for UCP-nets.

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