# On the coefficients of power sums of arithmetic progressions

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**Abstract**

We investigate the coefficients of the polynomial

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n.$$

We prove that these can be given in terms of Stirling numbers of the first kind and $r$-Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.

*Key words:* arithmetic progressions, power sums, Stirling numbers, $r$-Whitney numbers, Bernoulli polynomials
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## 1 Introduction

Let $n$ be a positive integer, and let

$$S_n(\ell) = 1^n + 2^n + \cdots + (\ell-1)^n$$

be the power sum of the first $\ell - 1$ positive integers. It is well known that $S_n(\ell)$ is strongly related to the Bernoulli polynomials $B_n(x)$ in the following

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way

$$S_n(\ell) = \frac{1}{n+1}(B_{n+1}(\ell) - B_{n+1}).$$

where the polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and $B_n = B_n(0)$ is the $n$th Bernoulli number.

It is possible to find the explicit coefficients of $\ell$ in $S_n(\ell)$ [9]:

$$S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left( \sum_{k=0}^{n} S_2(n,k)S_1(k+1,i)\frac{1}{k+1} \right), \tag{1}$$

where $S_1(n,k)$ and $S_2(n,k)$ are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bazsó et al. [1] considered the more general power sum

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n,$$

where $m \neq 0, r$ are coprime integers. Obviously, $S_{1,0}^n(\ell) = S_n(\ell)$. They, among other things, proved that $S_{m,r}^n(\ell)$ is a polynomial of $\ell$ with the explicit expression

$$S_{m,r}^n(\ell) = \frac{m^n}{n+1}\left( B_{n+1}\left(\ell + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right)\right). \tag{2}$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.

For some related diophantine results on $S_{m,r}^n(\ell)$ see [3,10,15,16,2] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial $S_{m,r}^n(\ell)$, thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called $r$-Whitney numbers introduced by the second author [13].

The $r$-Whitney numbers $W_{m,r}(n,k)$ of the second kind are generalizations of the usual Stirling numbers of the second kind with the exponential generating function

$$\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{z^n}{n!} = \frac{e^{rz}}{k!}\left(\frac{e^{mz}-1}{m}\right)^k.$$

For algebraic, combinatoric and analytic properties of these numbers see [5,14] and [6,7], respectively.

First, we prove the following.

**Theorem 1** *For all parameters $\ell > 1, n, m > 0, r \geq 0$ we have*

$$S_{m,r}^{n}(\ell) = \sum_{i=0}^{n+1} \ell^{i} \left( \sum_{k=0}^{n} \frac{m^{k} W_{m,r}(n,k)}{k+1} S_{1}(k+1,i) \right).$$

*Proof.* The formula which connects the power sums and the $r$-Whitney numbers is the next one from [13]:

$$(mx+r)^{n} = \sum_{k=0}^{n} m^{k} W_{m,r}(n,k) x^{\underline{k}}.$$

Here $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$ is the falling factorial. We can see that it is enough to sum from $x = 0, 1, \ldots, \ell - 1$ to get back $S_{m,r}^{n}(\ell)$. Hence

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} m^{k} W_{m,r}(n,k) \sum_{x=0}^{\ell-1} x^{\underline{k}}.$$

The inner sum can be determined easily (see [9]):

$$\sum_{x=0}^{\ell-1} x^{\underline{k}} = \frac{\ell^{\underline{k+1}}}{k+1} + \delta_{k,0}.$$

The Kronecker delta will never appear, because if $k = 0$ then the $r$-Whitney number is zero (unless the trivial case $n = 0$, which we excluded). Therefore, as an intermediate formula, we now have that

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} m^{k} W_{m,r}(n,k) \frac{\ell^{\underline{k+1}}}{k+1}.$$

The falling factorial $\ell^{\underline{k+1}}$ is a polynomial of $\ell$ with Stirling number coefficients:

$$\ell^{\underline{k+1}} = \sum_{i=0}^{k+1} S_{1}(k+1,i) \ell^{i}.$$

Substituting this to the formula above, we obtain:

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} \frac{m^{k} W_{m,r}(n,k)}{k+1} \sum_{i=0}^{k+1} S_{1}(k+1,i) \ell^{i}.$$

Since $S_1(k+1,i)$ is zero if $i > k+1$, we can run the inner summation up to $n+1$ (this is taken when $k = n$) to make the inner sum independent of $k$.

Altogether, we have that

$$S_{m,r}^n(\ell) = \sum_{i=0}^{n+1} \ell^i \sum_{k=0}^{n} \frac{m^k W_{m,r}(n,k)}{k+1} S_1(k+1,i).$$

This is exactly the formula that we wanted to prove. □

Now we give some elementary consequences of the theorem. The proofs are trivial.

**Remark.** The next properties of the polynomial $S_{m,r}^n(\ell)$ hold true for all parameters $\ell > 1, n > 0, r, m \geq 0$:

(i) The constant term of $S_{m,r}^n(\ell)$ is 0,
(ii) The leading coefficient of $S_{m,r}^n(\ell)$ is $m^n/(n+1)$,
(iii) $S_{m,r}^n(\ell)$ is a polynomial of $\ell$ of degree $n+1$ unless $m = 0$; in this latter case the degree is $n$.

The above statements also follow from (2).

## 2 The integer property of the coefficients in $S_{m,r}^n(\ell)$

The coefficients of the polynomial $S_{m,r}^n(\ell)$ are not integer in the overwhelming majority of the cases:

$$S_{1,0}^1(\ell) = \frac{\ell(\ell-1)}{2},$$
$$S_{2,5}^2(\ell) = \frac{1}{3}\ell(47 + 24\ell + 4\ell^2),$$

etc.

However, we revealed that in special cases the polynomial $S_{m,r}^n(\ell)$ has integer coefficients. Several parameters are in the next table.

| $m$ | $r$ | $n$ |
|---|---|---|
| 2 | 1 | 3 |
| 2 | 3 | 3 |
| 2 | 5 | 3 |
| 4 | 3 | 3 |
| 4 | 5 | 3 |

For example,

$$S_{2,1}^3(\ell) = \ell^2(2\ell^2 - 1),$$

or

$$S_{2,3}^3(\ell) = \ell(2+\ell)(2\ell^2 + 4\ell + 3).$$

From the formula of Theorem 1 it can be seen that if

$$(k+1) \mid m^k W_{m,r}(n,k) \quad (k = 1, 2, \ldots, n),$$

then we get integer coefficients.

To find another condition which is necessary and sufficient for the integrity of the coefficients in $S_{m,r}^n(\ell)$, we recall the following well known properties of Bernoulli polynomials and Bernoulli numbers.

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k(y) x^{n-k}; \tag{3}$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}; \tag{4}$$

$$B_3 = B_5 = B_7 = \ldots = 0. \tag{5}$$

By the *denominator* of a rational number $q$ we mean the smallest positive integer $d$ such that $dq$ is an integer. We recall also the von Staudt theorem

$$\Lambda_{2n} = \prod_{\substack{(p-1)\mid 2n \\ p \text{ prime}}} p, \tag{6}$$

where $\Lambda_n$ is the denominator of $B_n$. In particular, $\Lambda_n$ is a square-free integer, divisible by 6. For the proofs of (3)-(5) see e.g. the work of Brillhart [4].

Let $2 \le j \le n$ be an even number and put

$$f(n,j) := \operatorname{lcm}\left( \frac{\Lambda_j}{\gcd\left(\Lambda_j, \binom{n+1}{j}\binom{j}{j}\right)}, \frac{\Lambda_j}{\gcd\left(\Lambda_j, \binom{n+1}{j+1}\binom{j+1}{j}\right)}, \ldots, \frac{\Lambda_j}{\gcd\left(\Lambda_j, \binom{n+1}{n}\binom{n}{j}\right)} \right). \tag{7}$$

Further, we define

$$F(n) := \begin{cases} \operatorname{lcm}\left(\operatorname{rad}(n+1), f(n,2), f(n,4), \ldots, f(n,n)\right) & \text{if } n \text{ is even,} \\ \operatorname{lcm}\left(\operatorname{rad}(n+1), f(n,2), f(n,4), \ldots, f(n,n-1)\right) & \text{if } n \text{ is odd,} \end{cases} \tag{8}$$

where

$$\operatorname{rad}(n) = \prod_{\substack{p\mid n \\ p \text{ prime}}} p.$$

**Theorem 2** *The polynomial $S_{m,r}^n(\ell)$ has integer coefficients if and only if $F(n) \mid m$.*

*Proof.* By relations (2), (3) and (4) we can rewrite $S_{m,r}^n(\ell)$ as follows:

$$S_{m,r}^n(\ell) = \frac{m^n}{n+1}\left(B_{n+1}\left(\ell+\frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right)\right) = \tag{9}$$

$$= \frac{m^n}{n+1}\left(\left(\sum_{k=0}^{n+1}\binom{n+1}{k}B_k\left(\frac{r}{m}\right)\ell^{n+1-k}\right) - B_{n+1}\left(\frac{r}{m}\right)\right) = \tag{10}$$

$$= \frac{m^n}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k\left(\frac{r}{m}\right)\ell^{n+1-k} = \tag{11}$$

$$= \frac{m^n}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\left(\sum_{j=0}^{k}\binom{k}{j}B_j\cdot\left(\frac{r}{m}\right)^{k-j}\right)\ell^{n+1-k} \tag{12}$$

We denote the common denominator of the coefficients of $S_{m,r}^n(\ell)$ by $Q$. One can see from (9) that the polynomial has integral coefficients if and only if $m$ is divisible by $Q$. Thus we have to determine $Q$.

By (12) we observe that neither $m$ nor $r$ occurs in $Q$. Moreover, the only algebraic expressions which may affect $Q$ in (12) are on one hand $n+1$ and on the other hand, the denominators of the Bernoulli numbers involved, which are $2, \Lambda_j (2 \le j \le n$ even) by (5) and the von Staudt theorem.

It can easily be seen that $n+1 \mid m^n$ if $\mathrm{rad}(n+1) \mid m$. Indeed, supposing the contrary, i.e., that $\mathrm{rad}(n+1) \mid m$ and $n+1 \nmid m^n$, it implies that there is a prime factor $p$ of $n+1$ such that $p^{n+1}$ divides $n+1$. Hence $2^{n+1} \le p^{n+1} \le n+1$, which is a contradiction.

Let $2 \le j \le n$ be an even index. It follows from (12) that the contribution of $\Lambda_j$ to the common denominator $Q$ is precisely $f(n,j)$ defined in (7). In other words, if $f(n,j) \mid m$, then every term of (12) containing the factor $B_j$ has integer coefficients.

In conclusion, we obtained that $Q$ is the least common multiple of $\mathrm{rad}(n+1)$ and $f(n,j)$ for all even $j \in [2,n]$, which number we denoted in (8) by $F(n)$. The theorem is proved. □

**Remark.** An easy consequence of our Theorem 2 is that $S_n(\ell) = S_{1,0}^n(\ell) \notin \mathbb{Z}[x]$ for any positive integer $n$.

Some small values of $F(n)$ are listed in the following table. These are results of an easy computation in MAPLE.

| $n$ | $F(n)$ | $n$ | $F(n)$ | $n$ | $F(n)$ | $n$ | $F(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 42 | 11 | 6 | 16 | 510 |
| 2 | 6 | 7 | 6 | 12 | 2730 | 17 | 30 |
| 3 | 2 | 8 | 30 | 13 | 210 | 18 | 3990 |
| 4 | 30 | 9 | 10 | 14 | 30 | 19 | 210 |
| 5 | 6 | 10 | 66 | 15 | 6 | 20 | 2310 |

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