A Comparative Study of Directional Connections in Popular U.S. and Chinese High School Mathematics Textbook Problems

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ABSTRACT

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Mathematical connection has received increasing attention and become one major goal in mathematics education. Two types of connections are distinguished: (a) between-concept connection, which cuts across two concepts; and (b) within-concept connection, which links two representations of one concept. For example, from the theoretical probability to experimental probability is a between-concept connection; generate a graph of a circle from its equation is a within-concept connection. Based on the directionality, unidirectional and bidirectional connections are discerned. Bidirectional connection portrays a pair of a typical and a reverse connection. The benefits of connections, especially bidirectional connections, are widely endorsed. However, researchers indicated that students and even teachers usually make unidirectional connections, and underlying reasons may be the curriculum and cognitive aspects. Previous studies have reported differences in learning opportunities for bidirectional connections in U.S. and Chinese textbook problems, but few have explored the high school level.

This study addressed this issue by comparing the directionality of mathematical connections and textbook-problem features in popular U.S. (the UCSMP series) and Chinese (the PEP-A series) high school mathematics textbook problems. The results indicated that the between-concept condition and unidirectional connections dominated textbook problems.

Mathematical topic, contextual feature, and visual feature were most likely to contribute to different conditions of connections. Overall, problems dealing with quadratic relations from Chinese textbooks presented a vigorous network of more unique and total between-concept connections with balanced typical and reverse directions than the U.S. counterparts. Problems from U.S. textbooks showed a denser network of (a) within-concept connections in two topics and (b) between-concept connections in probability and combinatorics than the Chinese counterparts, but still exhibited an emphasis on specific concepts, representations, and directionality. The study reached a generalized statement that the new-to-prior knowledge direction was largely overlooked in textbook problems. The results have implications for adopting graph theory and Social Network Analysis to visualize and evaluate mathematical connections and informing mathematics teachers and textbook authors to pay attention to the new-to-prior knowledge connection.

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S. L.

DEDICATION

To my beloved parents, who have always encouraged me and believed in me to pursue my dreams and complete my dissertation. Thank you for giving me the life I love today.

Chapter I

INTRODUCTION

Need for Study

Fostering connections among mathematical ideas has become one major goal of mathematics education (National Council of Teachers of Mathematics [NCTM], 2000). Researchers distinguish between two types of mathematical connections: between-concept, which cuts across two concepts; and within-concept, which links two representations of the same concept (Selinski, Rasmussen, Wawro, & Zandieh, 2014). The benefits of connections are widely endorsed, such as cultivating deep understanding, extending mathematical topics, supporting error detection, and piquing interest in math (Barmby, Harries, Higgins, & Suggate, 2009; Boaler & Humphreys, 2005; Bossé, 2003; Karp, 2002). However, students and teachers still struggle to make connections (Olson, 2016; Prodromou, 2012). As such, many studies investigated characteristics of connections to support connection-making moves.

Directionality is one of these characteristics. Researchers use this term to describe the particular direction in which a mathematical connection is developed from one mathematical entity to another within the context (Janvier, 1987; Lesser, 2001; Marshall, Superfine, & Canty, 2010; Woods, 1975). For example, generating a graph from an algebraic equation involves the reverse directionality of producing an algebraic equation from a graph (Goldin & Shteingold, 2001; Leinhardt, Zaslavsky, & Stein, 1990); connection from the theoretical-to-experimental probability holds the reverse directionality of connection from the experimental-to-theoretical probability (Prodromou, 2012). Generally, *bidirectional connections* are used to portray a pair of connections: (i) a typical connection and (ii) a reverse connection (Ding & Li, 2010; Jin & Wong, 2015; Leinhardt et al., 1990; Lesser, 2001; Prodromou, 2012).

Bidirectional connections have several benefits, such as assisting conceptual understanding, having productive knowledge backward-transfer, and aiding in problem solving (Brenner et al., 1997; Heid, 1988; Hohensee, 2016; Knuth, 2000b; Wilson, 1994). However, both students and teachers often limit connection-making moves in one direction. For example, many students fail to move from Algebra to Arithmetic (Lee & Wheeler, 1989); even many preservice teachers struggle with the experimental-to-theoretical probability connection (Prodromou, 2012). In addition, logical analysis and empirical work suggest that a connection in a particular direction, such as a graph-to-equation direction, is usually more difficult than the reverse (Confrey & Smith, 1995; Stein & Leinhardt, 1989). These studies suggest that two aspects—the curriculum emphasis and cognitive obstacles—may prohibit bidirectional connections.

Among various curriculum materials, mathematics textbooks played and continue playing a central role in classrooms for mathematics teaching and learning (Stein, Remillard, & Smith, 2007). The majority of teachers regard textbooks as an authority and the primary teaching tool, and most students work on problems in textbooks on a daily basis (Grouws, Smith, & Sztajn, 2004). Accordingly, textbooks may be a useful resource to analyze the directionality of mathematical connections. Given the growing interest in problem-solving issues, textbook problems receive much attention. Moreover, international comparisons of textbook problems are receiving increased attention. These studies can provide insights into students' achievement differences in mathematics and credible outcomes of missed content or pedagogical factors in textbook problems, and then, in turn, they can elicit educational improvements in textbook problem design (Cao, 2018; Kubow & Fossum, 2007; Zhu & Fan, 2006).

Prior cross-national studies have reported considerable differences in students' mathematical performance and standards-based U.S. and Chinese elementary or lower secondary school mathematics textbook problems in terms of connections and problem features. In almost all existing global comparative studies, Chinese students outperformed American students across grade levels and mathematical topics (Cai & Nie, 2007). Regarding mathematical connections in textbook problems, researchers reported that some standards-based U.S. elementary and middle school mathematics textbooks lacked learning opportunities for some reverse connections, such as additive inverses, multiplicative inverses, the graphical-to-symbolic connections (e.g., Chang, Cromley, & Tran, 2016; Ding, 2016). In contrast, the Chinese counterparts adopted bidirectional connections in many topics, such as addition-subtraction (Sun, 2011b), multiplication-division (Xin, Liu, & Zheng, 2011), and the bidirectional use of the distributive property (Ding & Li, 2010). But the analysis was conducted mostly on elementary and middle school-level topics. Regarding problem features, prior textbook comparisons suggested that many standards-based U.S. elementary and middle school mathematics textbooks contained more single-step, real-life, visual problems with more exercises (e.g., Zhu & Fan, 2006). In contrast, the Chinese counterparts had more multi-step, purely mathematical, non-visual problems with more worked-out examples. However, few studies have analyzed high school textbook problems (J. Wang & Lu, 2018), even though they have a wider range of informative data.

In summary, a comparison of directional connections in popular U.S. and Chinese high school mathematics textbook problems may yield new information about the balance between typical and reverse within-concept and between-concept connections in textbook problems. It may expose relationships between the directionality of connections and problem features, and therefore help us to reflect beyond the context of a specific system and explore possible ways to embed bidirectional connections. This can, in turn, enhance the teaching and learning of mathematics in the United States and China. Therefore, there is a need to explore mathematical connections in popular U.S. and Chinese high school mathematics textbook problems.

Purpose for Study

In my study, I examined mathematical connections in popular U.S. and Chinese high school mathematics textbook problems. There were three purposes for my research: (i) to compare features of problems with or without connections and examine associations between mathematical connections and problem features as a way to show cross-national similarities and differences in embedding connections; (ii) to compare and visualize the network of within-concept and between-concept connections in textbook problems as a way to illuminate cross-national similarities and differences in the directionality of two types of connections; and (iii) to provide exemplary practices and suggestions for designing textbook problems supporting bidirectional connections. To achieve these purposes, my study was guided by the following research questions:

1. What are the similarities and differences in the feature of problems with or without mathematical connections from popular U.S. and Chinese high school mathematics textbooks?
2. What are the similarities and differences in the directionality of mathematical connections embedded in problems from popular U.S. and Chinese high school mathematics textbooks?
3. Which structural differences in popular U.S. and Chinese high school mathematics textbook problems may promote or hinder bidirectional connections?

Procedure for Study

Sample Textbooks and Data Collection

To keep comparisons neutral, I selected textbooks stressing connections with a similar textbook organization, in which each chapter has several sections with worked-out examples and exercises. In the United States, standards-based curricula view learning as developing understanding by constructing concept connections (Stein et al., 2007). I chose *University of Chicago School Mathematics Project (UCSMP) Textbooks*, that represent one of the largest projects reflecting curriculum reform (Fan & Kaeley, 2000). In China, *People's Education Press* (*PEP) Textbooks* are the most widely used curricula having multiple connections (Fan & Zhu, 2007). I specifically used *PEP General High School Curriculum Standard Experimental Textbook Mathematics, A Version*, which is the most widely circulated version (Cao, 2018).

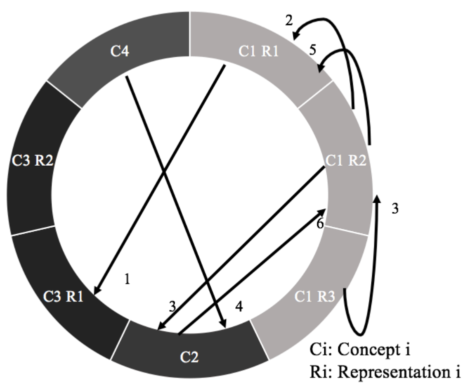
I first collected the students' edition and the teachers' edition for these two textbook series. The teachers' edition includes detailed solutions to problems, which are data essential for identifying connections. Next, I chose two topics: (a) quadratic relations (Algebra and Geometry strands); and (b) probability and combinatorics (Probability, Statistics, and Discrete Mathematics strands)—all difficult core topics from different strands (e.g., Bulone, 2017; Leinhardt et al., 1990). Finally, I compiled all worked-out examples, exercises, and their solutions in each section in their original sequence together as a single set, since every problem applies to and serves the same mathematical topic.

Data Coding

Phase 1 involved dividing collected sets into items and coding relevant features. Textbook problems have one or two levels of sequence numbers. The first-level numbering was by 1, 2, …; the second-level numbering was by (1), (2), ... in PEP-A and a, b, … in UCSMP. For problems having the second-level numbering, I first divided them into basic items. For problems only having the first-level sequence numbering, I kept them as basic items. Then, I assigned an item number to basic items one by one as the new sequence number. Then, I coded each basic item for its (i) topic: quadratic relations, probability and combinatorics; (ii) presentational feature: worked-out example, exercise; (iii) contextual feature: purely mathematical, real-life; (iv) mathematical feature: single-step, multi-step; and (v) visual feature: visual, non-visual.

Phase 2 was designed to identify connections. I first collected the vocabulary checklist in the chapter review and the textbooks' glossary for the whole textbook to compile the Concepts Table. Next, I made the Representations Table: words, tables, graphs, symbolic expressions, and concrete/pictorial representations (Marshall et al., 2010). Because some concepts have various symbolic representations (e.g., quadratic functions have the standard, intercept, and vertex form), I expanded symbolic expressions to S1, S2, and so on. Then, I compiled all possible connections (from a single concept in a specific representation to another concept in a specific representation or to the same concept in a different representation) in a complete Connection Table and coded solutions of each item step by step in terms of corresponding connections found in the Connection Table as well as their categories: no-connection, between-concept connections only, within-concept connections only, and the mixed condition of both between-concept and within-concept connections. All identified connections were compiled in a table.

Phase 3 was drafted to recognize bidirectional connections in Phase 2 data and produce related digraphs and adjacency matrices. I used the NodeXL, a network analysis software package, to filter out bidirectional connections by the reciprocated function. All bidirectional connections were compiled in a table. Digraph theory, which has a long history as an analytical tool in applied mathematics, has been used successfully to examine the quality (e.g., strength, connectivity, centrality) and the structure of connections with a vivid graphical representation (Selinski et al., 2014; Strom, Kemeny, Lehrer, & Forman, 2001). I thus constructed digraphs (see Figure 1) with: (a) vertices: concepts with representations (e.g., C1R2 stands for Concept 1 in Representation 2); (b) edges with number x: connections in problem item x (e.g., an edge from C4 to C2 with number 4 shows a connection from Concept 4 to Concept 2 in problem item 4); and (c) arrows: the directionality. Finally, I produced the corresponding adjacency matrix with one row and one column for each vertex (see Figure 1). An entry of k in row X and column Y indicates there are k connections from X to Y (Chartrand & Lesniak, 2005), e.g., the entry "2" in Row 2 and Column 1 shows that there are two connections of Concept 1 from Representation 2 to Representation 1 (one in problem item 2 and one in problem item 5 shown in the digraph).



	C1R1	C1R2	C1R3	C2	C3R1	C3R2	C4
C1R1					1		
C1R2	2			1			
C1R3		1					
C2		1					
C3R1							
C3R2							
C4				1			

Figure 1. A sample digraph and its corresponding adjacency matrix

Subsequently, I invited two math teachers (one in China and one in the United States), who have more than 5 years of teaching experience, to recode and add missed connections to compile the final coding. Finally, I invited four graduate students in math education, who are proficient in English and Chinese, to agree or disagree with part of the final coding and calculated the overall percentage of agreement. The percentage of agreement and the overall percentage of agreement all passed 80%. The final coding reached the reliability requirement.

Data Analysis

To address the first research question, I first used the word frequency cloud in NVivo, a qualitative data analysis software package, to generalize the difference in the usage of concepts and representations in textbook content. Then, I used tables and charts to compare the frequency of textbook problems in terms of four conditions of mathematical connections across topics, presentational, contextual, mathematical, and visual feature. Frequency of between-concept and within-concept connections was explored across topics and textbooks. Loglinear analysis was conducted to explore associations among textbook series, connections, and problem features.

To address the second research question, I first used a table and chart to compare the frequency of unidirectional and bidirectional connections across textbooks. Then, I compared bidirectional within-concept connections and between-concept pairs across textbooks to check the integration of bidirectional connections. Next, I examined digraphs for each subtopic and topic. Digraphs were classified into dense, moderate, sparse, the sparsest, and aggregated digraphs. In moving from a digraph to its adjacency matrix, quantitative characteristics can be attained (Strom et al., 2001). I checked whether on-diagonal and off-diagonal block submatrices had symmetrical entries and similar weight, and non-zero entries in the diagonal for self-loops. I compared: (a) in- vs. out-connections: the number of unique connections connecting to vs. emanating from a vertex; (b) in- vs. out-degree: the number of connections leading to vs. out of a vertex; and (c) other indices related to the directionality (Smith et al., 2010; Strom et al., 2001). I used the above analysis to generalize similarities and differences in the directionality issue.

To address the third research question, structural differences in textbook problems, such as the placement of subtopics, unique practices in each textbook series, were reviewed to explore their potential influence on bidirectional connections.

Chapter II

LITERATURE REVIEW

Overview

Contemporary literature in mathematics education and psychology suggests that mathematical connections have received increasing attention despite being rarely defined in the literature (Payton, 2017). Therefore, many researchers have conceptualized and explored characteristics of mathematical connections. Focusing on one characteristic—directionality, prior studies have reported both the benefits of bidirectional connections and students' and teachers' difficulty in bidirectional connections. They indicated that two aspects—the curriculum and cognitive aspects—may hinder bidirectional connections. To discuss the literature on this topic, I have divided this chapter into five sections: (a) overview; (b) defining and conceptualizing mathematical connections; (c) mathematical connections in mathematics textbooks; (d) mathematical connections in cognitive psychology; and (e) summary.

Defining and Conceptualizing Mathematical Connections

What Is a Mathematical Connection?

Mathematical connection, which is widely used and referred to in the literature, seems to be somewhat ambiguous and rarely defined in the literature (Payton, 2017). From a historical perspective, Hau (1993) analyzed definitions and synonyms for "connect" and "mathematical" in dictionaries and thesauri, and defined *mathematical connection* as "mathematical concepts or procedures or activities may be coupled, or tied, or linked, or attached, or conjoined to one another or to other concepts or procedures or activities" and "mathematical concepts, or procedures, or activities may be related, or correlated, or bracketed to, or identified with, or equated to other concepts or procedures or activities" (p. 50). Currently, researchers usually use links or relationships between two entities to describe mathematical connections in practice. For example, Businskas (2008) defined the term as "a true relationship between two mathematical ideas, A and B" (p. 18). Singletary (2012) defined it as "a relationship between a mathematical entity and another mathematical or nonmathematical entity" (p. 10), where a mathematical entity is "any mathematical object from any area of curricular mathematics" (Zbiek & Conner, 2006, p. 92). I adopted this definition because it provides a basis for interpreting characteristics of mathematical connections, which have been used in other research as well (Payton, 2017).

Mathematical connections were emphasized in reforms in mathematics education in the United States. Two reports, the 1923 Report of the National Committee on Mathematical Requirements of the Mathematical Association of America and the 1940 Report of the Commission on the Secondary School Curriculum of the Progressive Education Association, revealed the initial desire to emphasize mathematical connections to encourage integrated and connected curricula (Coxford, 1995). Hereafter, the "new math movement" of 1957-1970 emphasized the interrelationships of mathematical ideas and the structure of mathematics (Begle, 1970). After that, the National Council of Teachers of Mathematics (NCTM, 1980) responded to the "back to basics" movement in the 1970s and recommended "a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them" (p. 2).

Later, mathematical connections constituted an essential component of reforms and standard documents in mathematics education, as illustrated in consecutive NCTM Standards (1989, 1991, 2000). For example, NCTM (1989) stressed "modeling connections between problem situations that may arise in the real world or in disciplines other than mathematics and their mathematical representation(s)"; and "mathematical connections between two equivalent representations and between corresponding processes in each" (p. 146). Later, NCTM (2000) stated that mathematics is "a web of closely connected ideas," in which ideas are linked by connections. Specifically, NCTM addressed connections as one process standard as follows:

> **Connections.** When students connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Through instruction that emphasizes the interrelatedness of mathematical ideas, students not only learn mathematics, they also learn about the utility of mathematics. Mathematics is not a collection of separate strands or standards, even though it is often partitioned and presented in this manner. Rather, mathematics is an integrated field of study. (p. 64)

Moreover, the Mathematical Association of America (MAA) proposed six common standards for teacher preparation, in which Standard 2 addressed connections as follows:

> **Standard 2: Connecting Mathematical Ideas.**
> The mathematical preparation of teachers must provide experiences in which they:
> - develop an understanding of the interrelationships within mathematics and an appreciation of its unity;
> - explore the connections that exist between mathematics and other disciplines;
> - apply mathematics learned in one context to the solution of problems in other contexts. (Leitzel, 1991, p. 3)

In 2010, the *Common Core State Standards for Mathematics* (CCSS-M) was released (CCSSI, 2010), which was intended as the national standards in the United States. Particularly, it stressed connections between different problem-solving approaches, connections between a given problem situation and its abstraction, and connections from the Standards for Mathematical Practice to the Standards for Mathematical Content.

All of the above reform movements and standards documents in the United States share the common belief that mathematical connection is an important, valuable, and essential aspect of teaching, learning, and understanding mathematics (Singletary, 2012). Consequently, many studies have concentrated on conceptualizing mathematical connections. Based on the nature of mathematics, mathematical connections can be identified *a priori* and as part of a connected discipline, which exists independently of learners.

Mathematical Connections: A Feature of Mathematics

The "common theme" view. Coxford (1995) shared the "common theme" view of mathematical connections, which conceived mathematical connections as broad ideas or processes connecting multiple topics. He identified three categories: (a) unifying themes, (b) mathematical processes, and (c) mathematical connectors.

Unifying theme is defined as a theme "that may be used to pay attention to the connected nature of mathematics" (Coxford, 1995, p. 4), such as change, data, and shape. The following example explains how the theme "change" connects Algebra, Geometry, Discrete Mathematics, and Calculus:

> For example, how is a constant rate of change related to lines and linear equations? What changes occur in the graph of a function when a coefficient in the equation of the function is changed?... How does the perimeter or area of a plane shape change when it is transformed using isometries, size transformations, shears, or some unspecified linear transformation?... Each of these questions suggests opportunities to connect mathematical topics by relating them through the theme of change. (pp. 4-5)

Researchers have identified other possible themes. For example, Crowley (1995) suggested transformation (using multiple representations) as a unifying theme and listed examples showing how it connects Plane Geometry, Matrices, Compositions, Conic Sections, and Trigonometry. Also, Bossé (2003) identified conjunction (a logical connective '*and*') and disjunction (a logical connective '*or*') as unifying themes connecting the fields of Logic, Set Theory, Algebra, Number Theory, and Probability.

Mathematical process embodies (a) representation, (b) application, (c) problem solving, and (d) reasoning, which exists as activities that "continue in all the mathematical work done by students from kindergarten through independent learning as an adult" (Coxford, 1995, p. 7). The following is one example of "representation":

> For example, upper elementary school students should develop facility in moving back and forth among the concrete and the pictorial models, the oral name, and the symbolic representation of any fraction or decimal. These connections are vital if students are to make sense out of later operations on numbers. (p. 7)

Coxford (1995) advocated that the above four mathematical processes, which form a continuous web of emphasis, should occur regularly in mathematical instruction. Under this circumstance, the mathematics itself is seen as interrelated and connected.

Mathematical connector is defined as a mathematical idea "that arises in relation to the study of a wide spectrum of topics" (Coxford, 1995, p. 10), such as function, matrix, algorithm, graph, variable, and ratio. The following shows how "graphs" work as the mathematical connector in the curriculum:

> Later in the curriculum, graphs are used to represent solutions to equations or inequalities; to represent functions and relations; to represent problem situations; to display data visually so that trends and tendencies can be observed; to represent patterns found in all strands; and, in discrete mathematics, to serve as an object of study and to model a variety of situations. (pp. 10-11)

In sum, the "common theme" view stresses the nature of mathematics. However, looking into this view alone may leave connections at a general level (Businskas, 2008).

Concept-to-concept links. Businskas (2008) employed concept-to-concept links to describe particular relationships between two fine-grain-sized concepts, which are conceived as mathematical connections. For example, Zazkis (2000) demonstrated some cases of concept-to-concept links as follows:

> The mathematical connection among a factor, a divisor, and a multiple is expressed in the equivalence of the following three statements, for any two natural numbers A and B:
>
> - B is a factor of A;
> - B is a divisor of A;
> - A is a multiple of B. (p. 212)

Other researchers have illustrated abundant concept-to-concept links, such as fraction-division (Weinberg, 2001), addition-subtraction (Cai & Moyer, 2008), and multiplication-division (Xin et al., 2011). These studies indicated that concept-to-concept links might be a productive way to conceptualize mathematical connections in practice.

Representational links. Two types of representational links—translations between representations and transformations within a representation—are conceptualized as mathematical connections. Lesh, Post, and Behr (1987) defined *translations* as "between-system mappings," i.e., moving from one representation system to another. For example, Janvier (1987) presented potential translations among four representations of variables in Cartesian graphs in a 4*4 table (see Figure 2) and considered these to be mathematical connections.

TRANSLATION PROCESSES

From \ To	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Figure 2. Translation processes (Janvier, 1987, p. 28)

Subsequent researchers have also supported translations between representations. Goldin and Shteingold (2001) argued that translating from a graph of a circle of radius one centered at the origin to its an algebraic equation $x^2+y^2 = 1$ is a mathematical connection. Businskas (2008) showed examples of connections in terms of mappings between two representations as follows:

In the equation, $y = mx + b$:
- Characteristics of the graph of the line are equivalent to parts of the equation;
- Slope is equivalent to m;
- Y-intercept is equivalent to b. (p. 12)

Moreover, researchers have used translations between two representations to recognize and assess learners' connection-making moves (e.g., Zazkis & Liljedahl, 2004).

In addition to translations, researchers have also discussed transformations as mathematical connections. Lesh et al. (1987) defined *transformations* as "within-system operations," i.e., moving within one representation system. Some mathematical ideas have alternative representations in one representation system. For example, the symbolic representations of linear equations include a standard form, a slope-intercept form, and a point-slope form. Burkett (1998) presented an example of a mathematical connection in a problem requiring the transformation of the linear equation:

> Determine the y-intercept of the linear equation $2x + 3y = 6$ without graphing the line…. First, the student algebraically changes the given equation, $2x + 3y = 6$, into slope-intercept form, $y = -\frac{2}{3}x + 2$. This change of the given equation into the slope-intercept form is the transformation. (p. 11)

In reality, translations and transformations tend to be interdependent, conjointly conceptualizing mathematical connections in terms of representations (Lesh et al., 1987).

Proposed Framework for Conceiving Mathematical Connections

Based on the nature of mathematics, the "common theme" view, concept-to-concept links, and representational links are used to conceptualize mathematical connections. The "common theme" view considers connections at a super general level. In comparison, sorting out connections by concepts and representations details connections in the usage of specific concepts and particular representations in mathematics. Integrating the concept and the representation perspective seems to be a productive way to conceptualize mathematical connections. Therefore, based on the combined perspective of concepts and representations, the proposed framework consists of two types: within-concept connections and between-concept connections.

Within-concept connection describes the mathematical connection involving two representations of the same concept (see Figure 3) (Selinski et al., 2014), which includes representations in different representation systems and alternative representations within one representation system. For example, generating a graph of a circle from an algebraic equation is a within-concept connection (Goldin & Shteingold, 2001). This category echoes the translation and the transformation process.

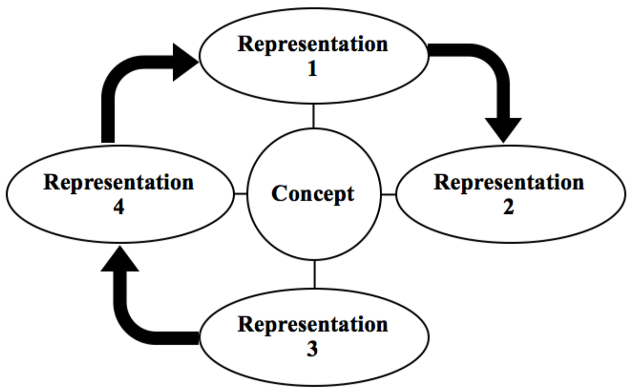


Figure 3. Within-concept connections

Between-concept connection describes the mathematical connection cutting across different concepts in mathematics (see Figure 4) (Selinski et al., 2014). For example, moving from theoretical probability to experimental probability is a between-concept connection (Prodromou, 2012). This category coins concept-to-concept links.

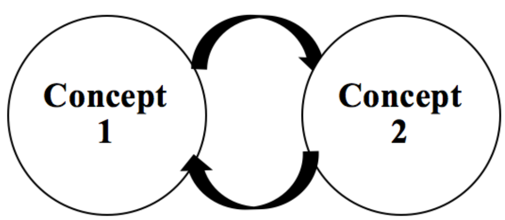


Figure 4. Between-concept connections

For the overall structure (see Figure 5), white arrows characterize within-concept connections and black arrows represent between-concept connections. For example, the white arrow from concept 1 in representation 1 to representation 2 represents a within-concept connection of concept 1 from representation 1 to representation 2 (translation). The white arrow from concept 2 in representation 1 to itself represents a within-concept connection of concept 2 within representation 1 (transformation). Between-concept connections cover both connections between two concepts in two different representations (e.g., from concept 2 in representation 3 to concept 3 in representation 1) and connections between two concepts in one representation (e.g., from concept 1 in representation 2 to concept 2 in representation 2).

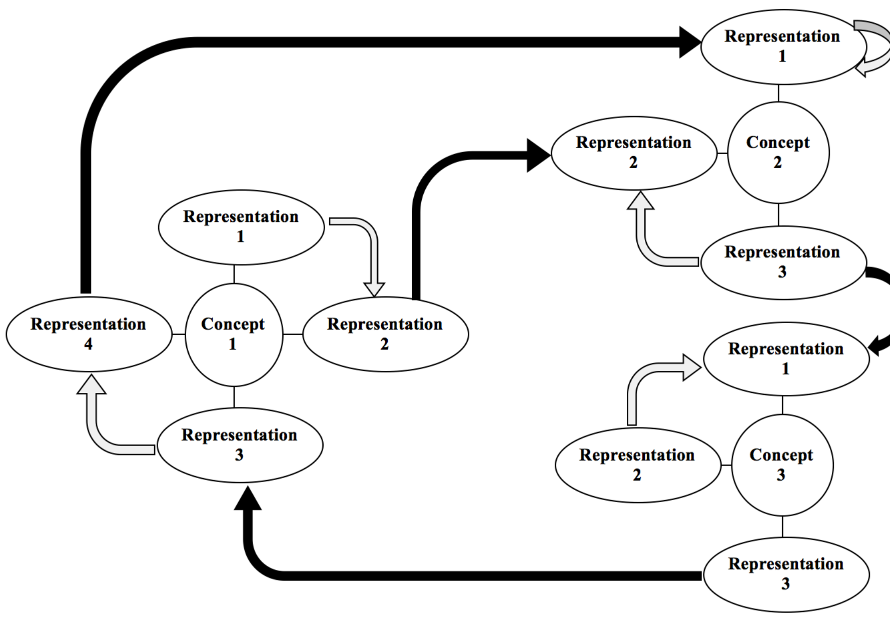


Figure 5. A framework of between-concept and within-concept connections

Selinski et al. (2014) used within-concept and between-concept connections to analyze mathematical connections in Linear Algebra successfully and posited that this model could be used to analyze connections in other mathematical content.

A New Perspective: Directionality

In mathematics education, the directionality of connections is receiving growing attention due to the benefits and learners' difficulty in bidirectional connections. The term *directionality* is generally used to describe the particular direction in which a mathematical connection is from one mathematical entity to another within the context (Janvier, 1987; Lesser, 2001; Marshall et al., 2010; Woods, 1975). It has been discussed in the literature in terms of within-concept connections and between-concept connections. For example, Goldin and Shteingold (2001) stated that connections between two representations of the same concept are bidirectional in nature, thus supporting a previous study that the connection of generating a graph from an algebraic equation involved the reverse directionality of the connection of producing an algebraic equation from a graph (Leinhardt et al., 1990). For between-concept connections, Prodromou (2012) indicated that moving in the theoretical-to-experimental probability direction holds the reverse directionality of moving in the experimental-to-theoretical probability direction. Researchers identified two types of directionality: unidirectional and bidirectional.

Unidirectional connections: Typical vs. reverse direction. In the introduction of the foundations for semantic networks, Woods (1975) used the term *inverse link* to denote the reverse of a given connection as follows:

> …by storing a sentence such as "John hit Mary" as a link named HIT from the node for John to the node for Mary, as in the structure…and perhaps placing an **inverse link** under Mary "Mary hit* John." (p. 53)

The above example characterizes a connection from vertex A to vertex B and a reverse one from vertex B to vertex A. Then, researchers employ *typical*[1] and *reverse connections* to depict

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[1] Some researchers used "regular." The word "regular" depicts connections conforming to or governed by some rules or standards, which implies that the reverse may be out of the ordinary or incorrect. Comparatively, the word "typical" portrays connections that occur often and usually not a surprise, which implies that the reverse may rarely appear. This study adopts "typical" instead of "regular" to describe connections that occur often and avoid the conception that the reverse is incorrect.

unidirectional connections in mathematics (Cai & Moyer, 2008; Davydov, 1990; Hohensee, 2014; Leinhardt et al., 1990; Lesser, 2001; Usiskin, 2018). Some logical analysis and empirical work have suggested that a connection in a particular direction, such as a graph-to-equation direction, is usually more difficult than the reverse (Confrey & Smith, 1995; Stein & Leinhardt, 1989). It demonstrates that the difficulty levels of grasping a connection in two directions may also be different, thereby further strengthening the ground for examining the directionality.

Bidirectional connections. This term is used to portray a pair of typical and reverse connections (Ding & Li, 2010; Ellis, 2007; Jin & Wong, 2015; Leinhardt et al., 1990; Lesser, 2001; Prodromou, 2012). Several studies have reported the benefits of bidirectional connections.

First, Heid (1988) reported that U.S. students using the new curriculum with bidirectional connections demonstrated a better conceptual understanding of covered concepts than the control group using traditional curriculum. Researchers also suggested that bidirectional connections make mathematics meaningful and build a more coherent understanding of mathematical concepts (Marshall et al., 2010). Also, the web of bidirectional connections allows learners to extend mathematical concepts (Confrey & Smith, 1995), regenerate forgotten results, make remembering correct results more likely, and play a major role in error detection (Schoenfeld, Smith, & Arcavi, 1993), thereby enhancing their conceptual understanding. Second, Hohensee (2014) demonstrated that U.S. students' reasoning about their previously-learned concepts (linear functions) was productively influenced by newly-learned concepts (quadratic functions) in significant aspects. Connections from newly constructed to previously learned concepts in mathematics do bring significant productive backward reasoning and meaningful learning. Moreover, bidirectional connections not only help students form an understanding of the relative costs and benefits of two representations (Dufour-Janvier, Bednarz, & Belanger, 1987) but also provide students the flexibility to work with a wide range of problems with the appropriate representation (Piez & Voxman, 1997). In sum, the benefits of bidirectional connections are widely endorsed. However, prior studies have shown that both students and teachers often limited their connection-making moves in one direction (e.g., Knuth, 2000b; Prodromou, 2012).

Several studies reported students' struggle in making bidirectional connections. For example, bidirectional Arithmetic-Algebra connections were problematic for many 7th graders (Herscovics & Linchevski, 1994). The Arithmetic-to-Algebra transition was difficult for many junior high school students in the United States (Brenner et al., 1997). Many 10th graders in Canada might fail in the reverse—Algebra-to-Arithmetic—connection as well (Lee & Wheeler, 1989). In Early Algebra, Li, Ding, Capraro, and Capraro (2008) reported that many 6th graders in the United States had misconceptions in moving between two sides of the equal sign, whereas the Chinese counterparts exhibited their understanding of these connections. Similarly, Blanton et al. (2015) reported that before the intervention many U.S. 3rd graders showed an operational understanding of the equal sign. It was consistent with a previous study conducted by Stephens et al. (2013) indicating that many U.S. Grade 3-5 students had an operational view of the equal sign and exhibited difficulties in recognizing connections between underlying structures of equations. In contrast, Yang, Huo, and Yan (2014) reported that many Grade 3-5 students in China demonstrated a relational view of the equal sign. Later, Ding, Li, Hassler, and Barnett (2019) reported that 94% of Chinese 4th graders in the study applied the reverse direction of the distributive property, whereas only 6% of the U.S. counterparts could make such connections.

One of the biggest stumbling blocks for Algebra students was translating among a variety of representations: algebraic expression, equation, graph, word problem, and verbal description (Seeley & Schielack, 2007). For example, many Beginning Algebra students in the United States failed in graphical-to-algebraic and tabular-to-algebraic directions of functions (McCoy, 1994). In quadratic functions, many high school students in Israel preferred the equation-to-graph connection than the reverse connection (Zaslavsky, 1997). In linear functions, researchers revealed not only U.S. high school students' over-reliance on algebraic methods in a simpler-graphical-favored situation but also the superficial mastery of bidirectional algebraic-graphical connections, especially the graph-to-equation direction (Knuth, 2000a, 2000b). These findings were consistent with previous studies conducted with other age groups, such as Stein and Leinhardt (1989) with 10/11-year-olds and Markovits, Eylon, and Bruckheimer (1986) with 14/15-year-olds. In functions, Stylianou (2011) showed that many U.S. middle school students had limited usage of connections that representations worked as a monitoring tool to move between subsequent goals and current problem-solving plans. Some secondary school students in Cyprus also presented a gap in moving among the tabular, graphical, symbolic, and other representations of functions (Elia, Panaoura, Eracleous, & Gagatsis, 2007). Later, Adu-Gyamfi and Bossé (2014) found that some U.S. high school students were able to connect from domain to co-domain but made limited reverse connections from co-domain to domain.

Still, many U.S. students leave high school without an understanding of bidirectional connections among the numeric, symbolic, and graphical representations of functions (Blume & Heckman, 1997). In a National Assessment of Educational Progress (NAEP) study, 18% of 17-year-olds in the United States made correct equation-to-graph connections, whereas only 5% generated graph-to-equation connections (Leinhardt et al., 1990). Many college students in the United States still had difficulties in the graphical-to-symbolic connection of a logarithmic function (Confrey, Millman, & Piliero, 1993). Trigueros and Martínez-Planell (2010) reported undergraduate students' limited connections between various representations of two-variable functions in Puerto Rico. Some first-year university students in Belgium made most errors in connecting two representations of decreasing functions, and the most difficult within-concept connection was between a formula and a graph and vice versa (De Bock, Van Dooren, & Verschaffel, 2015). In a large-scale survey of 34,412 Grade 8 students in China, He and Qi (2017) reported that students preferred the symbolic the most, and the pictorial, the linguistic, and the structural representation in descending order.

For concepts in Linear Algebra, many university students in Canada struggled in moving within and across their abstract, algebraic, and geometric representations (Hillel, 2000). Several second-year university students in New Zealand missed bidirectional connections involving basis in Linear Algebra (Stewart & Thomas, 2008). Many U.S. undergraduate students also struggled with connections among the symbolic and geometric representation of three interpretations of the matrix equation (Larson & Zandieh, 2013) and connections from the augmented matrix to the linear system (Zandieh & Andrews-Larson, 2015). In Discrete Mathematics, Eizenberg and Zaslavsky (2004) found that some undergraduate students in Israel failed to solve combinatorics problems correctly and then were not able to have efficient verification strategies to detect the error or correct their solutions by making connections. In a recent investigation of U.S. postsecondary students' understanding of combinatorics problems, Bulone (2017) found that many students failed in using connections to previous problems and struggled with connections between problems of the same type with altered contexts.

From the above review, students' struggles with bidirectional connections in the Arithmetic-Algebra transition, Early Algebra (the equal sign, equation, and basic properties of operation), Functions (connections between various representations), Linear Algebra (basis, linear system), and Discrete Mathematics (combinatorics problems) were reported.

Furthermore, preservice teachers showed difficulties in making or identifying bidirectional connections as well. For example, in a case study of one preservice secondary mathematics teacher in the United States, Wilson (1994) reported that this teacher had difficulties in making connections between functions and the real world, as well as connections between functions and other areas of mathematics before the intervention. Later, Lesser (2001) reported that some preservice secondary school teachers in the United States relied on the tabular and numerical representations of Simpson's Paradox and struggled with connecting other representations. Some sophomore preservice teachers in Cyprus over-relied on the algebraic approach and struggled with bidirectional connections among different representations in functions (Mousoulides & Gagatsis, 2004). Zazkis and Liljedahl (2004) reported that many preservice elementary school teachers in Canada struggled with identifying connections between prime and composite numbers and connections between the factored form representation and the numerical representation of numbers. Later, Eli, Mohr-Schroeder, and Lee (2011) found that many prospective middle-grade teachers in the United States made far fewer derivational mathematical connections: from one concept to build upon or explain another concept. Also, some preservice primary school teachers in Australia failed to build experimental-to-theoretical probability connections (Prodromou, 2012). Olson (2016) surveyed some preservice secondary mathematics teachers in the United States and reported their inability to self-identify connections between the CCSS-M content and college-level mathematics coursework.

Researchers have indicated that there could be two aspects influencing students' and teachers' bidirectional connection-making moves: curriculum and cognitive.

From the curriculum aspect, limited learning opportunities for bidirectional connections in curriculum materials may significantly contribute to learners' difficulties in making bidirectional connections. For example, Lee and Wheeler (1989) showed that the emphasized curricular track in an Arithmetic-to-Algebra direction brought obstacles to bridging the Arithmetic and Algebra worlds bidirectionally. Moreover, most mathematical problems in curriculum materials were within the symbolic representation, and routine translation tasks required connections from the algebraic to graphical representation. Emphasized representations and translation tasks of the algebraic-to-graphical direction offered students limited opportunities to build reverse graphical-algebraic connections (Knuth, 2000b). Additionally, the distributive property in the reverse direction rarely appeared in U.S. textbooks (Ding & Li, 2010). It was consistent with U.S. students' difficulties in using the reverse direction of the distributive property (Ding et al., 2019) as limited learning opportunities were provided in textbooks.

From the cognitive aspect, Goldin and Shteingold (2001) described situations that children manipulated signed numbers meaningfully in one representation but not in another, thus indicating cognitive obstacles to moving from one familiar representation to another difficult representation. Also, a logical analysis showed that the graph-to-equation task covered difficult pattern detection while the equation-to-graph task involved a relatively straightforward series of steps (Leinhardt et al., 1990). These cognitive obstacles reflected empirical work that a connection in a particular direction, such as the graphical-to-algebraic and the Algebra-to-Arithmetic, was usually more difficult than the reverse direction (Confrey & Smith, 1995; Knuth, 2000b; Stein & Leinhardt, 1989).

In summary, even though bidirectional connections bring several benefits, they may still be challenging for students and even teachers. The following sections review these two perspectives one by one to examine the status quo of the use of bidirectional connections and explore possible ways to support bidirectional connections.

Mathematical Connections in Mathematics Textbooks

Selection of Curriculum Materials

Mathematics textbooks, historically the main curriculum material for mathematics teaching and learning, still play a central role in classrooms today (Stein et al., 2007). The Third International Mathematics and Science Study (TIMSS) survey reported that the majority of mathematics teachers adopted textbooks as the main teaching tool (Beaton, 1996). Two widely used models validate the unique status of mathematics textbooks, which may be a proper and effective curriculum material to explore the directionality of mathematical connections.

First, Stein et al. (2007) illustrated a framework of temporal phases of curriculum (see Figure 6)—*written curriculum*, *intended curriculum*, and *enacted curriculum*—in which textbooks belonged to the beginning phase, the *written curriculum*. Under this model, mathematics textbooks are of fundamental importance as they influence what and how topics are covered and presented in classrooms (Alajmi, 2012), which reflect the directionality of mathematical connections. This model was echoed with the statement that "what is actually taught in classrooms is strongly influenced by the available textbooks" (Kilpatrick, Swafford, & Findell, 2001, p. 36).

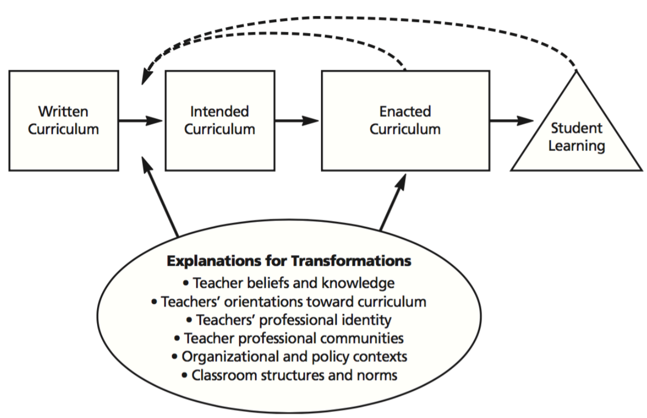


Figure 6. Temporal phases of curriculum use (Stein et al., 2007, p. 322)

Second, in a tripartite model (see Figure 7), Valverde, Bianchi, Wolfe, Schmidt, and Houang (2002) considered textbooks as part of the *potentially implemented curriculum*, which is a significant bridge connecting the *intended curriculum* (intentions, aims, and goals) that is given at the level of a system or state and the *implemented curriculum* (strategies, practices, and activities) that is presented at the level of a classroom.

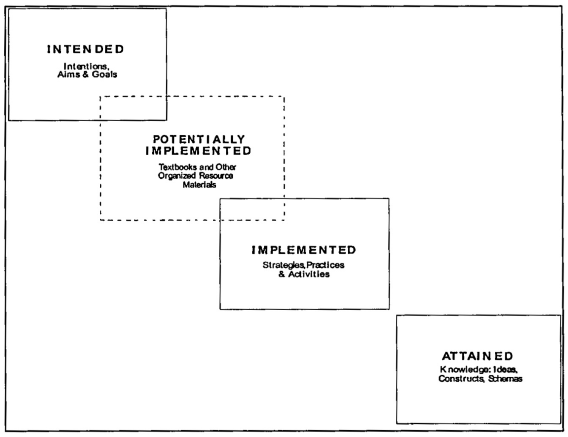


Figure 7. Textbooks and the tripartite model (Valverde et al., 2002, p. 13)

Relatively, textbooks provide not only a clearer picture of what is to be taught and learned in classrooms than the *intended curriculum* but also a more accessible way of documenting the long-time development of teaching and learning in a large population than the *implemented curriculum* (Li, Chen, & An, 2009). This model also sustains mathematics textbooks as an effective resource to explore the directionality of mathematical connections.

Textbook-Problem Analysis

Growing attention to textbook-problem analysis. Previous mathematics textbook analysis was mainly in two aspects: more in content analysis and less in problem analysis (Stein et al., 2007). Recently, textbook-problem analysis has received growing attention due to the emphasis on problem solving and the benefits of problem analysis. Problems have played a central role in school mathematics since antiquity (Stanic & Kilpatrick, 1989), whereas the first major call for problem solving occurred in the late 1970s. Later, more researchers started to examine problem solving, which became a main theme at the ICME-5 in 1984 (Fan & Zhu, 2007). The TIMSS 1999 video study technical report indicated that solving problems constituted 80% of the lessons (Hiebert et al., 2003). Most students work on textbook problems for both in-class exercises and homework on a daily basis (Grouws et al., 2004). Moreover, the types, forms, and sequence of problems offered in a textbook form a picture of how the textbook authors envisioned the lesson: what problems they wanted to teach students to solve and how they wished to attain the goal (Karp, 2015). Furthermore, textbook-problem analysis produces meaningful information about learning opportunities available to students and performance expectations for students, which may reflect the current usage and potential supports to bidirectional connections, guide the curriculum development, and eventually improve the teaching and learning of mathematics (Ding, 2016; Li, 2000; Li, Chen, & An, 2009).

To examine the directionality of mathematical connections addressed in textbook problems, we need to clarify what a problem is. Fan and Zhu (2007) defined a *problem* as "a situation that requires a solution and/or decision, no matter whether the solution is readily available or not to the solvers" (p. 64), which is more operational in textbook-problem analysis.

Burgeoning textbook-problem comparisons. Cross-national textbook comparisons have played an important role in the mathematics education field for at least the past 30 years (Star & Rittle-Johnson, 2016). In a survey of textbook research in mathematics education, Fan, Zhu, and Miao (2013) reported that the largest body of work (63%) was found on textbook analysis and comparison, in which analysis of textbooks from different countries are more frequently conducted with a focus on identifying their similarities and differences. Some researchers suggested that U.S. textbooks covered more topics in each grade, which implied less depth of any one topic and less emphasis on a connected map of mathematics (Ginsburg, Leinwand, Anstrom, & Pollock, 2005). The TIMSS textbook analysis labeled U.S. curricula as "a mile wide and an inch deep" and showed that U.S. content standards lacked coherence and focus compared with those in mathematically high-achieving countries (Schmidt, Wang, & McKnight, 2005). Similarities and differences in the integration of mathematical connections in textbook problems from the United States and a mathematically high-achieving country may provide valuable insights in supporting bidirectional connections and guiding textbook design.

Prior international comparisons on students' mathematical achievements (e.g., the 2007 and 2011 TIMSS, and the 2015 and 2018 Program for International Student Assessment [PISA]) showed that Chinese students outperformed their counterparts in the West, including U.S. students (Son & Hu, 2016; Wong, 2008). Even though it is difficult to link students' mathematical performance directly to textbooks, studies exploring possible contributing factors to this cross-cultural difference indicated that the textbook was potentially one of the key factors (Li, 2000). Considering consistent good mathematical performance of Chinese students in international comparisons, many researchers have conducted U.S. and Chinese textbook-problem comparisons as a way to show similarities and differences in mathematical expectations (Li, 2000), educational practices (Li, Chen, & An, 2009), learning opportunities (Ding, 2016), educational policy (J. Wang & Lu, 2018), and so on. Focusing on the integration of bidirectional connections, prior textbook-problem comparisons indicated substantial differences in U.S. and Chinese textbooks, which were illustrated as follows.

Textbook-problem comparisons in terms of mathematical connections. Cai and his colleagues compared problems from U.S. and Chinese textbooks on the averaging algorithm (the typical and reverse application) (Cai et al., 2002); algebra concepts and representations (Cai et al., 2005); and the addition-subtraction pair and the multiplication-division pair (Cai & Moyer, 2008). They found that some standards-based elementary school mathematics textbooks in the United States not only rarely included the reverse use of the averaging algorithm but also de-emphasized algebraic symbols, which might inhibit bidirectional connections within the symbolic representation and prohibit the smooth Arithmetic-to-Algebra transition. It was consistent with a prior study conducted by Flanders (1994) that U.S. textbooks had an emphasis on Arithmetic (84% items) and less on Algebra or Geometry. In contrast, Cai and his colleagues suggested that the Chinese counterparts adopted the bidirectional use of the averaging algorithm with flexibility, integrated reverse operations with equation solving (the addition-subtraction pair and the multiplication-division pair), presented multiple worked-out examples with solutions in algebraic and arithmetic approaches, and promoted generalization of concrete representations. It corroborated another study that many problems from Chinese elementary school mathematics textbooks embedded bidirectional addition-subtraction connection (Zhou & Peverly, 2005).

Later, Ding and her colleagues compared U.S. and Chinese elementary school textbook problems on the equal sign (Li, Ding, Capraro, & Capraro, 2008); the distributive property (the typical, reverse, and dual direction) (Ding & Li, 2010); the associative property in multiplication (Ding, Li, Capraro, & Capraro, 2012); and inverse relations (additive inverses and multiplicative inverses) (Ding, 2016). They indicated that many Chinese textbooks presented these topics bidirectionally in deliberately constructed problems stressing the underlying structural relations and provided multiple solutions in arithmetic and algebraic approaches. In contrast, the U.S. counterparts covered few solutions in the algebraic approach and failed to present these topics in the reverse direction of use (e.g., the distributive property in a reverse direction). Researchers also revealed that some U.S. elementary school mathematics textbooks lacked explicit inverse relations between multiplication and division (Xin et al., 2011) and exhibited the emphasis on the left side operation with less than 5% of two-sided instances (Rittle-Johnson, 2013).

Routine translation tasks in U.S. high school mathematics textbooks required connections of functions mostly in the equation-to-graph direction (Knuth, 2000b). In linear equations, Huntley and Terrell (2014) found that only one in five popular U.S. secondary school textbook series included many tasks of connecting linear equations symbolically with linear functions. Later, Chang, Cromley, and Tran (2016) examined coordination tasks of multiple representations (symbolic, graphical, tabular, and text) in a widely used U.S. reformed Calculus textbook. They found that tasks from the symbolic to graphical representation accounted for 32.8%. In contrast, the reverse tasks accounted for only 6.4%. Unbalanced learning opportunities for bidirectional symbolic-graphical connections were demonstrated. Recently, Ma and Cao (2018) reported that the Geometry content accounted for 25.96% of one standards-based U.S. middle school textbooks and 40.18% for the Chinese counterparts, which indicated the de-emphasis of Algebra-Geometry connections in the U.S. series.

In the multiplication principle, Lockwood, Reed, and Caughman (2017) reported that 46 of 64 university-level Combinatorics, Discrete, and Finite Mathematics textbooks in the United States lacked the bridge statement connecting counting processes and set of outcomes. Tran and Tarr (2018) found that many association tasks of bivariate data in traditional and standards-based U.S. high school textbooks eliminated students' need to decide the proper data representation, which might restrict bidirectional connections between two data representations.

From the above, varied learning opportunities for connections were observed in textbook-problem comparisons, and the analysis was conducted mostly on elementary and middle school levels. Comparisons of connections in high school or university-level topics were underexplored.

Textbook-problem comparisons in terms of problem features. Prior studies suggested that some features of textbook problems may influence learning opportunities for mathematical connections. Exploring the association between connection and problem features may produce valuable insights to support connection-making moves. The following textbook-problem features are reviewed based on their potential influence on the presence or absence of connections.

The first feature is *presentational feature*. In general, there are two categories: *worked-out example* (i.e., worked examples; a complete solution provided), which is designed for teachers' instruction; and *exercise* (no solution provided), which is presented for students' practice (Li, 1999). Worked-out examples play a critical role in scaffolding student understanding, set a model to which students can refer, promote initial skill acquisition and later transfer of learning, and facilitate acquisitions of problem schema (Chi & VanLehn, 2012) (discussed later in worked-out example effects). Exercises, i.e., *to-be-solved problems*, embody the expectation for developing students' mathematics competencies (Li, 1999). On one hand, the difference in the ratio of worked-out examples to exercises included in textbooks indicated the difference in curricular emphasis between problem-solving processes and results (Mayer, Sims, & Tajika, 1995). On the other hand, studies showed that properly designed example-exercise pairs could be more effective than either exercises or examples only (Pashler et al., 2007).

Previous textbook-problem comparisons indicated that U.S. and Chinese textbooks usually exhibited different trends in presentational feature. In addition and subtraction, Mayer et al. (1995) indicated that four traditional 7th-grade U.S. textbooks devoted 45% of page space to exercises while the Japanese counterparts used only 19% for exercises. For 8th-grade textbooks, Li (1999) found that the ratio of exercises to worked-out examples is about 9.1 for Chinese textbooks while five U.S. textbooks exhibited a higher ratio, from 19.3 to 39.3. Then, Li, Chen, and An (2009) reported that 6th-grade Chinese textbooks used real-life worked-out examples with a clear verbal and pictorial explanation to show fraction division-multiplication connections, whereas the U.S. counterparts provided abundant exercises and few worked-out examples and one solution method without further explanation. In quadratic equations, Hong and Choi (2014) found that one standards-based U.S. secondary school textbook series did not present worked-out examples with complete solutions. Later, Ding (2016) found that two widely used U.S. elementary school textbooks had a much smaller portion of worked-out examples than the Chinese series for additive inverses (U.S.: 9.0% and 5.7%; Chinese: 24.1%). However, for multiplicative inverses, one U.S. series contained more worked-out examples than the Chinese series (U.S.: 12.0% and 6.8%; Chinese: 9.5%). It suggested that mathematical topics might influence presentational feature. In trigonometric functions, Fu and Zhang (2018) reported that one U.S. high school textbook series exhibited a lower portion of worked-out examples (13.0%) than the Chinese counterparts (42.5%). The above review suggested that more emphasis on worked-out examples might exist in Chinese textbooks than in U.S. textbooks.

The second feature is *contextual feature.* The first type—*purely mathematical*—is the problem formulated with numbers, symbols, geometric figures, and other purely mathematical representations verbally or only with purely mathematical representations (Li, 1999). The second type—*real-life*—is the problem involving a real-life situation (Hong & Choi, 2018). Purely mathematical problems tend to stress abstract mathematics, whereas real-life problems tend to emphasize the real-world application of mathematics. When solving real-life problems, combining multiple representations is usually required and abundant mathematical connections are generated in the problem-solving process (discussed later in multiple representation section).

Contextual feature is widely used in textbook-problem comparisons. For example, Li (2000) found that most problems dealing with addition and subtraction of integers in five 7th-grade U.S. textbooks and four Chinese textbooks are purely mathematical (87% for U.S. and 90% for China). Similarly, Zhu and Fan (2006) reported that the majority of problems in both Chinese and U.S. lower secondary school textbooks were not situated in real-world situations.

The recent reform call for real-life problems brought changes to contextual feature of textbook problems in the United States and China, as some researchers reported a decrease in the percentage of purely mathematical problems. For example, in functions, Son and Hu (2016) reported that the Chinese middle school textbooks used more purely mathematical problems (51.5%) than one standards-based U.S. textbook series (6.7%). Similarly, in statistics content of junior high schools, J. Wang (2017) reported that one standards-based U.S. textbook series had more real-life problems than the Chinese counterparts. In linear functions, Hong and Choi (2018) also found that one standards-based U.S. secondary school textbook series included more real-life worked-out examples (62.5%). For problems dealing with rational numbers in 7th-grade standards-based U.S. and Chinese textbooks, X. Wang and Zhang (2018) reported the Chinese series had more purely mathematical worked-out examples (88.10%) than the U.S. counterparts (77.97%). However, most worked-out examples in both series were purely mathematical. A similar situation was reported for trigonometric functions that both U.S. and Chinese high school textbooks exhibited a high portion of purely mathematical problems, especially for the Chinese series (92.1% for China; 85.3% for U.S.) (Fu & Zhang, 2018). The above textbook-problem comparisons indicated that many standards-based U.S. textbooks might have more problems set in real-life contexts than the Chinese counterparts. Different ratios of real-life to purely mathematical problems might occur in varied topics, which suggested that the fulfillment of reform call for real-life problems might be different in various mathematical topics.

The third feature *mathematical feature*—single-step and multi-step—is a long-standing indicator used by researchers. The *single-step* problem is defined as a problem that can be solved by one direct step or operation; and the rest conditions, *multi-step* problems (Zhu & Fan, 2006). This feature displays the number of steps required to solve the problem, indicating whether textbook problems are complex or simple. On one hand, a single-step problem is solved by one direct operation, which may include one or no connection. Multi-step problems may embed more than one connection. On the other hand, multi-step problems provide more space to increase the variability of worked-out examples (discussed later in worked-out example effects) and decrease over-repetition of simple exercises, which may yield a rich network of connections.

Striking differences in mathematical feature between U.S. and Chinese textbook problems were reported. For example, Stigler et al. (1986) reported that only 7% of the problems in addition and subtraction across four U.S. elementary school textbooks were multi-step. Then, Zhu and Fan (2006) found that one U.S. standards-based lower secondary school textbook series had more single-step problems than the Chinese counterparts. Another study conducted by Son and Senk (2010) also indicated that the majority of problems from one standards-based U.S. elementary school textbook are single-step. Similarly, in multiplication of fractions, Kar, Güler, Şen, and Özdemir (2018) found that the majority of problems from two widely used U.S. elementary school textbooks were single-step (53.7% and 80.1%, respectively).

The fourth feature is *visual feature*, which indicates the usage of visual information in textbook problems. The *visual* problem is defined as the problem includes visual information like figures, pictures, graphs, charts, tables, diagrams, and so on; and *non-visual*: the rest conditions

(Zhu & Fan, 2006). Visual information, compared to verbal information, usually shows its efficiency, which may influence the presence or absence of connections (discussed later in efficiency of external representations section).

Researchers reported some inconsistent results of visual feature of problems from U.S. and Chinese elementary and secondary school mathematics textbooks. For example, Zhu (2003) reported that one standards-based U.S. lower secondary school textbook series contained problems with more visual information in pictures, figures, or tables, than the Chinese counterparts. Similarly, in quadratic equations, Hong and Choi (2014) found that problems from a standards-based U.S. secondary school textbook series required multiple representations involving visual information (e.g., graphs, tables, etc.). In rational numbers, X. Wang and Zhang (2018) reported the Chinese 7th-grade textbooks had more worked-out examples with only verbal information than the U.S. counterparts. In linear functions, Hong and Choi (2018) found that one standards-based U.S. secondary school textbook adopted visual information (graphs or tables) in 23.0% of worked-out examples and 34.9% of exercises. However, for problem-posing tasks, Cai and Jiang (2017) reported that less than 25% of the problem-posing tasks in two standards-based U.S. elementary school textbooks included pictures, figures, and tables (7.70% and 23.75%, respectively), which was lower than that for the 2010s Chinese series (46.21%).

Last but not least, *cognitive demands* were widely used by researchers with a focus on connections. The Task Analysis Guide developed by Stein (2000) is widely used, which consists of two levels: (a) high-level cognitive demand (doing mathematics and procedures with connections), and (b) low-level cognitive demand (memorization and procedures without connections). Moreover, Schmidt, Raizen, Britton, Bianchi, and Wolfe (1997) identified five types of cognitive demands: knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating. Generally, problems with higher-level cognitive demands tend to include mathematical connections. Analysis of cognitive demands explicitly suggested the presence or absence of connections but lacked concepts, representations, and the directionality of connections. Further review of this feature is not included here.

In sum, the above comparisons of problem features indicated that standards-based U.S. elementary and middle school mathematics textbooks might have more single-step, real-life, visual exercises with few worked-out examples, whereas the Chinese counterparts were likely to embed more multi-step, purely mathematical, non-visual worked-out examples with few exercises. Different mathematical topics might contribute to problem features. It was noticeable that previous textbook-problem comparisons in terms of mathematical connections and problem features were mainly concentrated on the elementary or middle school level. Few analyses on the high school level were conducted. Associations between mathematical connections and textbook-problem features were also underexplored.

Focus: High school mathematics textbook-problem comparisons. J. Wang and Lu (2018) indicated that previous textbook problem comparisons were more on the elementary school level than on the high school level. Even though few studies have focused on the high school level, it is still of great importance for the teaching and learning of mathematics.

First of all, there are potentially different trends between high school and elementary or middle school mathematics textbooks. Hong and Choi (2014) found that some textbook features reflected in elementary school mathematics textbooks were not reflected in secondary school mathematics textbooks. Similarly, problem features reflected in the elementary or middle school level may not be reflected in the high school level. Second, problems at the high school level are more complex, diversified, and challenging, compared to problems at the lower grade levels.

This indicates that connections addressed in problems from high school mathematics textbooks may be more diverse, intricate, and substantial. Also, mathematical topics may influence problem features and conditions of connections. Problems from high school mathematics textbooks may provide comprehensive data to analyze the directionality of mathematical connections addressed in textbooks and, in turn, lead to new insights into cross-cultural differences in U.S. and Chinese mathematics textbooks. Last but not least, comparisons of high school textbook problems may produce meaningful insights for textbook authors and publishers to make changes in the textbooks, and thus help students make a smooth transition from high school mathematics to college-level mathematics since high school mathematics textbooks tend to have a strong influence on this issue (Raman, 2004). The following section reviews the status quo of high school mathematics textbooks and related critical curriculum reforms and standards in the United States and China.

High school mathematics textbooks in the United States. Historically, the United States has exhibited much variation in textbooks and has no national textbooks. Even available textbooks in the United States differ significantly, the most consistent differences are found between conventional-based and standards-based textbooks due to the *Curriculum and Evaluation Standards* released by NCTM (1989). Results from the Second International Mathematics Study (SIMS) and the Fourth National Assessment of Educational Progress (NAEP), academic studies in mathematics education (e.g., Fey & Good, 1985; Usiskin, 1985), and the burgeoning field of computer science in the 1980s all called for a revolution in the U.S. high school mathematics curriculum, which led to the release of the NCTM *Standards* in 1989. These *Standards* articulated five goals for students: “learning to value mathematics; becoming confident in one’s own ability; becoming a mathematical problem solver; learning to communicate mathematically; learning to reason mathematically" (p. 5). These goals called for attention to real-world problems, calculator and computer usage, and bidirectional within-concept connections in functions.

To develop standards-based curriculum materials aligned with the NCTM (1989) *Standards*, the National Science Foundation (NSF) initiated systemic reform and provided extensive support in the late 1980s and early 1990s (Senk & Thompson, 2003). Standards-based curriculum materials challenge the status quo by embodying a different approach: focusing on the students' active creation of important ideas and concepts. They usually begin with an immersive group-work task involving active exploration of new concepts in real-world situations; they also have a heavy balance towards the development of concepts and problem solving set in realistic contexts, and use a *modular* approach (Stein et al., 2007). On the contrary, conventional-based textbooks present content directly and expect teachers to explicitly teach students skills, concepts, applications, problem solving, and procedures. They tend to rely on direct applications, have a heavy balance towards procedures, and organize units and chapters according to topics.

Later in 2010, the CCSS-M released eight Standards for Mathematical Practice and Standards for Mathematical Content for High School, which specified the mathematics that all students should study to be college and career ready (CCSSI, 2010). In particular, it stressed connections from Algebra to Functions and Modeling; from Functions to Expressions, Equations, Modeling, and Coordinates; from Geometry to Equations; from Statistics and Probability to Functions and Modeling, as well as related reverse connections. These bidirectional connections may appear in textbooks developed aligned to the CCSS-M.

High school mathematics textbooks in China. Compared with the diversity of textbooks in the United States, until 2000, China owned a centralized education system and adopted a national unified mathematics textbook (Li, Zhang, & Ma, 2009). After the founding of the People's Republic of China in 1949, the Ministry of Education (MOE) published eight waves of curriculum standards, each following new mathematics textbooks. To avoid the drilling of "Two Basics" which may prohibit creativity and critical thinking, the eighth curricular reform to push forward the implementation of *Quality-Oriented Education* started in 1999 (Cui & Zhu, 2014), leading to dramatic changes in textbooks. The MOE began to draft the new curriculum and then formally published the *High School Mathematics Curriculum Standards (Trial version)* in 2003, which called for three transformations: (a) from centralization to decentralization; (b) from scientific discipline-centered to society construction-centered curriculum; and (c) from transmission-centered to inquiry-centered teaching (Zhong, 2006). The curriculum standard calls for developing students' abilities to pose, analyze, and solve problems from both mathematics and real life. Correspondingly, textbook development is open to all publishers to decentralize the curriculum—similar to the U.S. situation (Hirsch, 2007). Some complicated, insignificant, and outdated content was deleted to provide flexibility for students' self-directed learning and real-world contexts. Textbooks tend to be fundamental, diversified, and optional, thereby endorsing conceptual understanding, basic skills, and problem solving (Li, Zhang, & Ma, 2009).

Since 2000, standards-based textbooks have been created aligning with the *MOE 2003 Standards*. By 2012, they were implemented across China, marking the completion of the experimental stage (L. Wang, Liu, Du, & Liu, 2017). Six series of high school mathematics textbooks have been officially approved (Li, Zhang, & Ma, 2009). Standards-based textbooks include more open-ended and real-life problems and fewer complex computations and reasoning than old ones (Bao, 2004). This allows for more flexibility and adoption of a new three-course structure: (a) compulsory, taken by all students who want to graduate; (b) elective, only for students who take the China's National College Entrance Examination; and (c) optional (J. Wang & Lu, 2018). Current textbooks reflect content changes in the inclusion of Calculus content and a special focus on Statistics and Probability. Exercises are divided into Group A and Group B in terms of difficulty, with Group A being fundamental and Group B applying "Two Basics," to improve mathematical abilities and meet the needs of high-achievement students.

In sum, mathematics textbooks in two countries have gone through critical curriculum reforms. Similar standards-based textbooks appeared, both with an emphasis on mathematical connections and real-life problem solving. Therefore, comparing connections in high school textbook problems in the United States and China may reflect the fulfillment of the call for mathematical connections and problem-solving requirements articulated in curriculum reforms, explore exemplary ways to embed bidirectional connections in textbook problems, and provide insights into curriculum development.

The following section reviews the second perspective: the cognitive aspect, focusing on its potential influence on the presence and absence of between-concept connections and within-concept connections, as well as the directionality of connections.

Mathematical Connections in Cognitive Psychology

External Representations

Definition. Representations (i.e., representational forms/formats/modes) in different domains (e.g., Formal Mathematics, Cognition, and Epistemology) are defined differently. Hiebert and Carpenter (1992) used external and internal representation to show the structure of knowledge in mathematics. External representations take the form of spoken language, written symbols, pictures, or physical objects, which restricts the external embodiments of students' internal conceptualization (Lesh et al., 1987); internal representations are unobservable mental representations that show how ideas are represented inside the head. As connections between internal representations are assumed to be influenced and stimulated by building connections between corresponding external representations (Hiebert & Carpenter, 1992), this section focuses on external representations and their potential influence on mathematical connections.

Researchers have suggested that external representations play a critical role in learning and understanding mathematics, especially in constructing mathematical connections (Ainsworth & Th Loizou, 2003). Two types of connections between external representations can be constructed: (a) between different representations of the same mathematical idea; and (b) between related ideas within the same representation. The first type supports within-concept connections, which is often based on relationships of similarity and difference. The second type supports between-concept connections, which are generally promoted by noticing patterns or regularities in the system. These two types of external connections play a role in learning mathematics with understanding (Hiebert & Carpenter, 1992), which backs the proposed framework of within-concept and between-concept connections.

Many researchers have discussed different types of external representations. Lesh et al. (1987) identified five types and bidirectional connections among them (see Figure 8). They are: (a) real scripts (around real-world events); (b) manipulative models (like arithmetic blocks, fraction bars, etc.); (c) static pictures; (d) spoken language; and (e) written symbols. Other researchers have also indicated different categories of external representations: concretes or manipulatives, pictorials or diagrams, tables, graphs, symbols, numerals, written descriptions, and verbal descriptions (Goldin & Shteingold, 2001; Marshall et al., 2010). In all, mathematical ideas can be presented in a variety of representations. These representations are important in their own right, and connections among them are important as well (Lesh et al., 1987).

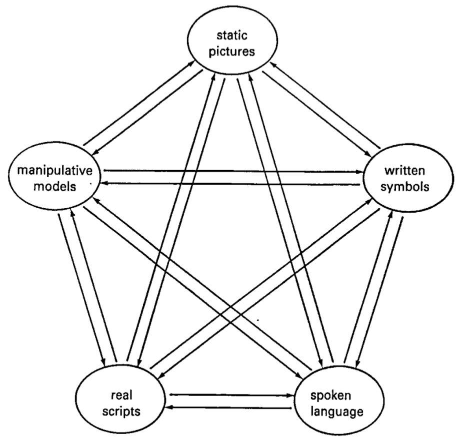


Figure 8. External representations (Lesh et al., 1987, p. 34)

Efficiency of external representations. *Informational efficiency* and *computational efficiency* are used to evaluate the value of different representations (Larkin & Simon, 1987). The first term refers to whether all of the information in one representation is inferable from the other and vice versa, whereas the latter term refers to the ease and rapidity with which inferences can be drawn from a representation. If two representations are informationally equivalent, one representation can be more effective and superior than another due to its high computational efficiency. In the problem of a thief guessing a four-digit PIN-code (5526), different but informationally equivalent ways of representing the guessing process—(a) diagram, (b) text, and (c) arithmetic—are shown in Figure 9 (Kolloffel, Eysink, de Jong, & Wilhelm, 2009).

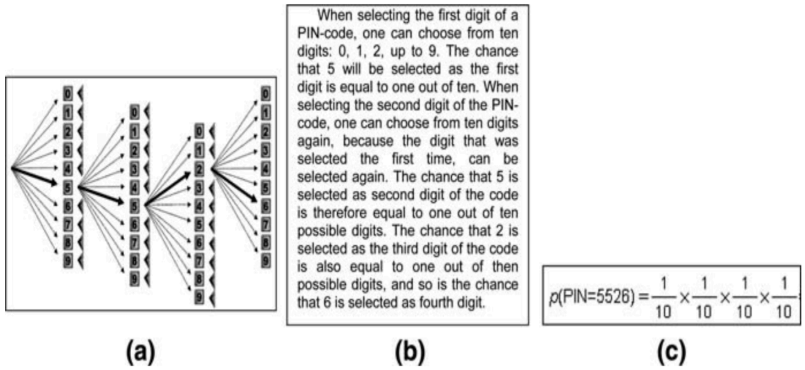


Figure 9. Three representations for the PIN-code problem (Kolloffel et al., 2009, p. 505)

In the above example, visual representation (i.e., diagram) shows its efficiency because inferences can be drawn more quickly and easily, compared with verbal representation (i.e., text). In addition, computational efficiency also explains Knuth's (2000a) finding that both students and teachers showed their over-reliance on algebraic representations.

Furthermore, abundant studies compared learning effects of two representations: verbal representations (i.e., text) and visual representations (i.e., graph, diagram), and generally indicated that visual representation yields superior performance compared with verbal representation (Marcus, Cooper, & Sweller, 1996) due to the following reasons.

First, memory capacity differs in verbal and visual information. Good memory is essential for the learning and understanding of mathematics and the construction of mathematical connections. Regarding memory for verbal information, people generally remember just its meaning instead of its exact wording. In terms of memory for visual information, people attend to and remember best those aspects that they consider meaningful (Anderson, 2005). Prior studies have suggested that memory for visual information often seems much better than memory for verbal information. In an experiment of a picture-memory task and a sentence/word-memory task, Shepard (1967) demonstrated that memory for visual information was virtually perfect compared with memory for verbal material. Many subsequent studies have also shown the high capacity for remembering pictures, i.e., visual information (Anderson, 2005). However, people do not always show good memory for visual information since memory depends critically on an individual's ability to interpret that material meaningfully. Therefore, making sense of visual representations may be helpful for promoting connection-making moves.

Second, visual representation is holistic in nature, whereas verbal encoding cannot be grasped "at one glance" (Sfard, 1991). Various aspects of a mathematical construct in the visual representation can be extracted by random simultaneous access. In contrast, this process should be processed sequentially in the verbal representation, which brings a burden to working memory and more cognitive loads for learners (discussed in Cognitive Load Theory later).

Moreover, visualization makes abstract ideas more tangible and encourages learners to treat them as if they are real entities. This promotes students' learning of abstract mathematical ideas, representations, and their connections. What is more, the effectiveness of the visual representation depends on the visual-indicator, the complexity of instructional materials, consistency and coherence of format use, and other factors (Jeung, Chandler, & Sweller, 1997; Murata, 2008). For example, Xing, Cai, and Shan (2014) reported that elementary school students in China showed better problem-solving performance in problems with informational pictures than that with decorative pictures and the verbal representation. More subsequent studies are needed to check the effective use of visual information in mathematics textbooks and its potential influence on learning opportunities for mathematical connections.

Multiple representation. The term *multiple representation* describes the result of combining two or more external representations together (Van Someren, Reimann, & Boshuizen, 1998), which is assumed to have additional benefits in learning and understanding mathematics and constructing mathematical connections since different representations can complement each other, and one representation may constrain the interpretation of the other (Ainsworth, 2006). Several studies were conducted to compare the effects of single and multiple representations. For example, Tindall-Ford, Chandler, and Sweller (1997) reported that the dual-mode of auditory text and visual diagrams could result in superior learning to a single representation (e.g., visual-only). Then, Kolloffel et al. (2009) compared the effects of five conditions—Diagram, Arithmetic, Text, Text+Arithmetic, and Diagram+Arithmetic—in combinatorics problems, and found that Text+Arithmetic representation was the most beneficial for learning. However, multiple representations may contain redundant information and increase the cognitive load on a learner's cognitive system (illustrated later in Cognitive Load Theory).

Solving real-life problems, which are strongly emphasized in mathematics, mostly require combining multiple representations. For example, in the two pizza problems:

> Show a 6th grader one-fourth of a real pizza, and then ask, "If I eat this much pizza, and then one-third of another pizza, how much will I have eaten altogether?"
>
> Show a 6th grader one-third of a real pizza, and then ask, "If I already ate one-fourth of a pizza, and now eat this much, how much will I have eaten altogether?" (Lesh et al., 1987, p. 37)

These two problems include a real object (pizza) and a spoken word (to represent past or future situations), which are "pizza-word" problems. Combining two modes into a homogeneous mode is one difficulty students have in solving real-life problems. Moreover, solution paths also weave back and forth among different representations, which have abundant bidirectional connections.

In sum, external representations may influence connections, affect the problem-solving process, and power the interaction between mathematical connections and problem solving. Appropriate usage of visual information and real-life contexts may promote connections.

Cognitive Load Theory (CLT)

Definition. CLT is concerned with "the development of instructional methods that efficiently use people's limited cognitive processing capacity to stimulate their ability to apply acquired knowledge and skills to new situations" (Paas, Tuovinen, Tabbers, & Van Gerven, 2003, p. 63) due to limited working memory capacity. CLT adopts interactions between information structures and knowledge of human cognition to determine instructional methods to guarantee that available cognitive resources can be fully devoted to learning. The term *cognitive load* is defined as a construct demonstrating the load that performing a particular task imposes on the learner's cognitive system (Paas & Van Merriënboer, 1994). Three types of cognitive load are distinguished: intrinsic, extraneous, and germane.

Intrinsic load arises from element interactivity within a task, which represents the nature of instructional materials and cannot be directly influenced by instructional designers (Paas et al., 2003). The intrinsic load will be high if interactions between elements must be learned simultaneously. In contrast, the intrinsic load will be low if elements can be learned successively and do not interact (Sweller, 1994). For example,

> For a percentage change problem such as, "A discount of 10% was given for a digital camera with a marked price of $350. Find the price paid after the discount.", the learner needs to identify relevant information in different units (10%, $350), specify key words such as 'price paid,' and construct a relation between values and variable (price paid) in an equation: price paid = $350–$350*10 %. Although there are only three elements (price paid, $350, 10%), the interaction between these elements must be considered simultaneously to allow understanding to occur. (Ngu, Yeung, & Tobias, 2014, pp. 687-688)

This example exhibits a high intrinsic cognitive load for most novice students learning percentage change problems. However, an expert can process many elements as a single unit (e.g., $350*10%), which reduces the intrinsic load associated with the problem.

Extraneous load is the extra load resulting from inappropriate instructional design (Paas et al., 2003). Different organizations and presentations of instructional material differ in extraneous loads. For example, the instruction combining diagrams and texts may reduce extraneous loads compared with informationally equivalent materials with separated sources (Sweller, Chandler, Tierney, & Cooper, 1990). Also, hardly legible text, irrelevant side notes in a textbook, inapplicable information in a problem, and unnecessary sound effects in a presentation may generate extraneous loads (McCarron, 2011). Furthermore, prior studies have suggested that only in the case of high intrinsic loads, extraneous loads seemed to be critical and designing instructional material to reduce extraneous loads was shown to be highly effective (Sweller, Van Merrienboer, & Paas, 1998). When the intrinsic load was low, the instructional design was of little consequence (Sweller & Chandler, 1994).

Germane load is relevant to processes contributing to schema acquisition controlled by instructional designers, such as organizing the material and relating it to prior knowledge (Paas et al., 2003). The process of asking students to solve a variety of problems generates germane load. However, the diversified worked-out examples can generate "healthy" germane load as they facilitate the acquisition of identical structure essence across different contexts (Paas & Van Merriënboer, 1994). Three types of loads are additive. Making sure that the sum of loads related to the instructional design is within working memory limits is essential (Paas et al., 2003).

Worked-out example effect and example-problem pairs. Earlier CLT research focused on using appropriate worked-out examples to reduce the extraneous load. A large number of experiments and a small number of classroom studies have demonstrated the learning efficiency and learning outcomes of *example-problem pairs*, in which students study worked-out examples first and then solve similar or isomorphic problems, instead of simply solving problems on their own (Pashler et al., 2007). For example, Sweller and Cooper (1985) conducted an early test of worked-out example effect by comparing differences between the conventional group (only exercises) and the worked-out example group (worked-out examples and exercises that had been carefully studied) in Algebra. They found that the worked-out example group spent less time studying the problem and completing the test with significantly fewer errors, which indicated that worked-out examples focused the learners' attention on problem states and useful solution paths, thereby reducing the extraneous cognitive load caused by weak-method problem solving and documenting the advantage of example-problem pairs over problems. A classroom study conducted by Zhu and Simon (1987) showed the feasibility and effectiveness of a 3-year curriculum of teaching factorization from worked-out examples than from conventional instruction in a middle school in China. Worked-out examples play a critical role in scaffolding students' understanding, set a model that students can refer to and emulate, promote initial skill acquisition and later transfer, and facilitate acquisitions of problem schema (Chi & VanLehn, 2012). Trafton and Reiser (1993) tested the order of example-problem pairs: Interleaved Example (a source problem, e.g.,1a, 2a, as an example immediately followed by solving the related target problem, e.g., 1b, 2b), Interleaved Solve, Blocked Example (all source problems as worked-out examples followed by solving all target problems), and Blocked Solve (see Figure 10) and found that students learned significantly more from interleaved examples and problems.

Interleaved Example	Interleaved Solve	Blocked Example	Blocked Solve
Example 1a Solve 1b	Solve 1a Solve 1b	Example 1a Example 2a	Solve 1a Solve 2a
Example 2a Solve 2b	Solve 2a Solve 2b	Example 3a Example 4a	Solve 3a Solve 4a
⋮	⋮	⋮	⋮
Example 4a Solve 4b	Solve 4a Solve 4b	Solve 1b Solve 2b	Solve 1b Solve 2b
Example 5a Solve 5b	Solve 5a Solve 5b	Solve 3b Solve 4b	Solve 3b Solve 4b
⋮	⋮	⋮	⋮

Figure 10. Example-problem pairs in four conditions (Trafton & Reiser, 1993, p. 8)

In another lab experiment, Paas and Van Merriënboer (1994) showed that students who studied worked-out examples gained most from high-variability examples, invested less time and mental effort, and attained better transfer performance than students who solved problems first and then studied worked-out examples. It demonstrated not only the advantage of example-to-problem sequence over problem-to-example sequence but also the benefits of high-variability in worked-out examples. As discussed before, high variability generates germane load. However, increasing variability in worked-out examples can generate "healthy" germane load as they facilitate the acquisition of identical structure essence across different contexts. Ngu et al. (2014) used multiple example-problem pairs similar to the high variability of worked-out examples that introduce "healthy" germane load to facilitate problem-solving and the mastery of key concepts. Compared to single-step simple problems, multi-step problems are more likely to increase the variability of both worked-out examples and exercises.

Moreover, researchers found that worked-out examples became redundant and exercises proved superior when learners gained more expertise in the problem-solving domain (Kalyuga, Chandler, Tuovinen, & Sweller, 2001), which suggested that the relative effectiveness of either worked-out examples or exercises depends heavily on learners' expertise in problem solving.

Therefore, the ratio of worked-out examples to exercises should be adapted to the development of learners' expertise.

In practice, most mathematics textbooks contain worked-out examples followed by exercise problems. Researchers indicated different usage of example-problem pairs in textbooks from different countries, which was discussed in presentational feature before. The practice guide published by the Institute of Education Sciences (IES) indicated that the U.S. curricular materials did not offer teachers with many interleaved example-problem pairs (Pashler et al., 2007). What is more, explanation in worked-out examples is important since obscure examples without an explanation paired with exercises may lead students to incorrect conclusions or problem-solving paths. For instance,

> A classic example from mathematics involves showing children an example like 3*2+5=6+5=11 and then asking them to solve 4+6*2=? Many students will give 20 as the answer, mistakenly adding 4 and 6 and then multiplying that by 2. (Anderson, 2005, p. 188)

In this case, an explanation, like the fact of multiplication first instead of performing the first operation in the expression, is needed for students to solve the followed problem. Worked-out examples lacking necessary explanation may fail to reduce the extraneous load and hinder acquisitions of problem schema and mathematical connections.

Connectivism and Social Network Analysis (SNA)

New technology, especially the emergence of the internet, has reorganized how we live and communicate over the last 20 years, opening up opportunities for new forms of communication and knowledge formation (Goldie, 2016). Looking into principles of Connectivism (a theory of learning for a digital age), three of the eight principles of connectivism address mathematical connections directly as follows:

Principles of Connectivism:

- Learning is a process of connecting specialized nodes or information sources.
- Nurturing and maintaining connections is needed to facilitate continual learning.
- Ability to see connections between fields, ideas, and concepts is a core skill. (Siemens, 2004, p. 4)

According to these principles, mathematical connections can be characterized as edges between certain vertices (can be concepts in mathematics), and the network of mathematical connections can be an interconnected world with weak or strong edges. The likelihood that a vertex will be connected depends on how well it is currently connected (Siemens, 2004). Alternations within the network, such as gaining or losing one connection, have ripple effects on the whole. Connectivism supports characterizing directional connections by weak or strong directed edges between vertices in a digraph. Moreover, it indicates that SNA can be used to visualize and analyze characteristics of concepts, representations, connections, and the network of mathematical connections. Several indices in SNA, which were used to evaluate vertices (concepts and representations), edges (mathematics connections), and the whole network (the integrated network of mathematical connections), are introduced as follows.

From the perspective of the whole network, size and density are largely used. The *size* of a network is indexed by the number of vertices in a network, and the *density* of a binary network is defined as the sum of the edges divided by the number of possible edges (Hanneman & Riddle, 2011). The size is important because of the limited resources and capacities that each vertex has for building and maintaining edges with other vertices. Consider a network containing k vertices; there are k*(k-1) possible unique directional edges in a binary network. The density, the proportion of all possible connections that are actually present, offers insights into the speed at which information diffuses among vertices (Hanneman & Riddle, 2011). For multigraphs or graphs with self-loops, the density can be higher than one. Under this circumstance, another two indices—*unique edges* and *total edges*—are used to quantize the whole network of connections. The first index is the number of connections where multiple connections between two vertices A and B are counted only once; the second index is the number of connections where multiple connections between two vertices A and B are all counted (Smith et al., 2010). The edge with the larger weight (more multiple connections) seems to be stronger and likely to be more durable than the reverse edge between the same two vertices with a smaller weight. It shows the relative emphasis between the connection from vertex A to vertex B and the connection from vertex B to vertex A. Jin and Wong (2015) analyzed the number of incoming and outgoing connections between pairs of concepts, which implied the relative strength of typical and reverse connections. Researchers have assessed the growth of expertise by more vertices (size) and connections among them (unique edges and total edges), which was consistent with the previous finding that the structure of an expert's knowledge is flexible and robust (Knuth, 2000b).

Furthermore, the *reciprocated connection* is introduced by NodeXL to explore the frequent usage of bidirectional connections. If there is an edge from vertex A to vertex B and another edge from vertex B to vertex A, then the connections between A and B are reciprocated, i.e., bidirectional. The reciprocated edge ratio (the percentage of edges that have a reciprocal relationship)—indicates the degree of the integration of bidirectional connections. Besides, self-loops (an edge that starts and ends in the same vertex) also exhibit the extent of the usage of bidirectional connections.

Focusing on vertices, connectivity and centrality are widely used in quantifying the influence of the directionality issue. *Connectivity* is defined as the number of unique connections to the given particular vertex (Strom et al., 2001, p. 752). Two indices—in-connections and out-connections—are generally employed to quantify the connectivity of the graph. In-connections refers to the number of unique connections connecting to a particular vertex; out-connections refers to the number of unique connections emanating from a specific vertex. Strom et al. (2001) used in-connections and out-connections to present a schema-based view of the most central features of mathematical argumentation. Another widely used approach to perceiving the structural resources of a particular vertex's advantage and disadvantage relative to vertices in their neighbors is *centrality* (Hanneman & Riddle, 2011). This quantifies the importance or influence or degree of participation of a specific vertex in a network. Different indices are used to portray centrality, such as in-degree, out-degree, closeness, and betweenness centrality.

In-degree centrality, which is interpreted as a form of popularity, measures the number of edges directed to a vertex, i.e., summing the number of connections leading to a particular vertex; out-degree centrality quantifies the number of edges but self-reported, i.e., summing the number of connections leading out of a particular vertex (Costenbader & Valente, 2003). In an exploration of 8th-grade students' understanding of algebraic concepts, Jin and Wong (2015) employed in-degree and out-degree to assess connections associated with individual concepts. They found that the concept *equation* had the highest out-degree and the concept *unknown* had the highest in-degree centrality. As the influence of connections may gradually dissipate and cease to have a noticeable effect on vertices with distance over three, the section leaves other centrality indices involving path issues, e.g., closeness centrality, betweenness centrality. Similar to the reciprocated edges ratio, the reciprocated vertex pair ratio (the percentage of vertices that have a reciprocal relationship) is also employed as a way to compare the integration of bidirectional connections.

Adjacency matrices were also used to explore the network of mathematical connections. For example, Selinski et al. (2014) identified three types of matrices to capture the diversity of the overall network of connections that students constructed: (a) a dense adjacency matrix (many between-concept and within-concept connections); (b) a sparse adjacency matrix (mainly between-concept and limited within-concept connections); and (c) a hub adjacency matrix (typically within-concept connections). They also suggested that possible chains of connections can be examined by the matrix A to the 3rd power, A^3.

In sum, connectivism supports using networks of vertices and directed edges to represent directional within-concept and between-concept connections. Some approaches from Social Network Analysis (SNA) may be used to evaluate mathematical connections.

Summary

Mathematical connections receive great attention in mathematics education and cognitive psychology. Based on the nature of mathematics, mathematical connections are conceived as the "common theme," concept-to-concept links, and representational links. The "common theme" view leaves connections at a super general level. From the combined perspective of concept-to-concept links and representational links, a framework of between-concept and within-concept connections is illustrated. In terms of characterizing mathematical connections, directionality is receiving growing attention. Two types are identified: unidirectional and bidirectional (a pair of typical and reverse connections). The importance and benefits of bidirectional connections are widely endorsed. However, students and teachers usually build unidirectional connections. Two aspects—curricular emphasis and cognitive obstacles—may prohibit bidirectional connections.

From the curriculum aspect, mathematical textbooks are likely to be a productive artefact to examine mathematical connections and their directionality. Due to the benefits of problem analysis and the recent growing attention to problem-solving issues and international comparisons, abundant studies have focused on textbook-problem comparisons. Prior comparisons of the U.S. and Chinese textbook problems have suggested many differences in (a) mathematical connections, and (b) textbook-problem features. This may influence learning opportunities for connections. But few comparisons have been conducted at the high school level. Furthermore, both countries have released new standards-based high school mathematics textbooks, with an emphasis on both mathematical connections and real-life contexts, that have not yet been analyzed in terms of learning opportunities for mathematical connections.

From the cognitive aspect, external representations and Cognitive Load Theory suggest that well-designed interleaved examples-problem pairs with different features (purely mathematical or real-life contexts, visual or verbal, multi-step or single-step) may promote connections. Connectivism supports mathematical connections as directed edges between two vertices (concepts and representations), which opens up the possibility of using Social Network Analysis to analyze and assess mathematical connections.

Therefore, comparing mathematical connections in high school mathematics textbook problems may yield insights into sustaining bidirectional connections, reflecting beyond the context of a specific system, providing insights into curriculum reforms and development, and eventually improving the teaching and learning of mathematics in the United States and China.

Chapter III

METHODOLOGY

Overview

This chapter describes the methods and framework used to collect, code, and analyze mathematical connections in the textbook problems. It begins with the selection and acquisition of textbook problems, then moves to the data coding schema and the final coding, and finally presents analysis tools used to answer research questions.

Data Collection

Sample High School Mathematics Textbooks

Selection criteria. To keep comparisons neutral, the first criterion in selecting textbooks was a similar textbook-problem organization, e.g., each chapter has a chapter review section and several sections with worked-out examples and exercises. The second criterion was popularity. Examining mathematics textbooks that students are most commonly exposed to and teachers are most frequently referred to contributes to a relatively meaningful image of connections. The third criterion was to highlight mathematical connections. The result can be different in selecting different textbooks. This study was not present to be general, i.e., examining the status quo of connections in all U.S. and Chinese textbooks. Instead, this study intentionally chose textbooks stressing connections to explore exemplary practices to promote bidirectional connections.

High school mathematics textbooks in China. In the 1950s, the Ministry of Education (MOE) founded *People's Education Press* (PEP) to study, compile, and publish national textbooks and curriculum standards. Until 1988, PEP served as the only official developer of textbooks and curriculum standards in China (Li, 2004). Then, after 2000, the Chinese Education

Administration approved more publishing presses to publish textbooks and furthered their efforts in loosening central control (Li, Zhang, & Ma, 2009). Overall, an estimate of 90 billion elementary and secondary school students are using the PEP series (calculated from the estimated market share and the total number of students from the MOE [2018] website). Five textbook series (around 13 versions) are currently in use, which organize problems under 'Examples->In-class Exercise-> Examples->In-class Exercise->…->After-class Exercises' with a chapter review and a self-test. Among these versions, the most widely used and circulated version is still from PEP, *General High School Curriculum Standard Experimental Textbook Mathematics, A Ver.* (Cao, 2018), named PEP-A. It emphasizes real-life contexts, mathematical reasoning, underlying mathematical thinking and application, and connections between different content knowledge, which may be productive materials for probing the directionality issue.

The PEP-A series has 20 high school mathematics textbooks. Five of them are compulsory textbooks designed for mandatory mathematical content. Compulsory 1 and 2 are for 10th grade, Compulsory 3 and 4 are for 11th grade, and Compulsory 5 is for 12th grade. Another five are elective textbooks prepared for high school students who take the China National College Entrance Examination. Elective 1-1 and 1-2 are for 10th grade; Elective 2-1 and 2-2 are for 11th grade; and Elective 2-3 are for 12th grade. They are required by most of the provinces in China and are treated essentially as compulsory textbooks. To be specific, Elective 1-1 and Elective 1-2 are generally for students majoring in liberal arts, and the rest are for students majoring in science-related areas. So two pairs—(1) Elective 1-1 and 2-1, and (2) Elective 1-2 and 2-2—share similar topics with increased mathematical demands. The study selected Elective 2-1 and 2-2 for its broader scope of content. The remaining 10 are optional textbooks developed mainly for some students' interest in particular mathematical topics (e.g., Number Theory, History of Mathematics, Spherical Geometry). In terms of the compulsory essence, this study covered only Compulsory and Elective textbooks.

High school mathematics textbooks in the United States. Given the diversity and consistent differences between conventional-based and standards-based curriculum in the United States, this study selected standards-based textbooks for their emphasis on connections and problem solving. There are only seven standards-based high school mathematics textbooks on the list of the most well-known U.S. curriculum materials (Stein et al., 2007). Among these, the *University of Chicago School Mathematics Project Grade 6-12* (named UCSMP) was one of the largest and most progressive projects on the curriculum in the United States (Fan & Kaeley, 2000). The UCSMP materials are CCSS-M aligned, which are used by an estimated 4.5 million elementary and secondary school students in the United States (UCSMP, n.d.). They stressed representations, a real-world orientation, and mathematical connections (Usiskin, 2018), which may offer rich opportunities to probe the directionality. Also, it employs a similar textbook-problem organization of 'Examples->Exercises,' which has been used in numerous cross-cultural mathematics textbook comparisons (e.g., Cai & Jiang, 2017; Ding, 2016). These studies enhanced the feasibility and value of using UCSMP as a window to reveal the cross-cultural difference in the directionality of connections. The current UCSMP series (3rd Edition) covers high school-level content mainly by *Advanced Algebra* (Grade 9-12), *Functions, Statistics, and Trigonometry* (Grade 10-12), and *Pre-calculus and Discrete Mathematics* (Grade 11-12).

Supplementary materials: Teachers' edition. Textbooks usually have two editions: students' edition and teachers' edition (i.e., teachers' guide, teachers' guidebook, teachers' manual) (McNeil, 1991). Focusing on textbook problems, both UCSMP and PEP-A teachers' editions include (i) additional worked-out examples to accommodate students' needs, and (ii) detailed step-by-step solutions to exercises for which no solutions are provided in the students' edition (Li, 2004). These books offer valuable auxiliary data for identifying mathematical connections in textbook problems. Moreover, studies have revealed the benefits of teachers' editions to resolve possible discrepancies in coding data, e.g., in the bidirectional use of the distributive property (Ding & Li, 2010). Therefore, this study adopted the teachers' edition as supplementary materials. In summary, Table 1 shows the background of sample textbooks.

Table 1. *Textbooks Included in the Study*

Country	Textbook			Publisher & Year	Author	Simplified Name
	Grade	Title	Supplementary			
China	10	Compulsory 1; Compulsory 2	Teaching Guidebook	People's Education Press, 2007	Liu, Shaoxue; Qian, Peiling; Zhang, Jianyue; et al.	PEP-A-C1; PEP-A-C2
	11	Compulsory 3; Compulsory 4; Elective 2-1; Elective 2-2				PEP-A-C3; PEP-A-C4; PEP-A-E2.1; PEP-A-E2.2
	12	Compulsory 5; Elective 2-3				PEP-A-C5; PEP-A-E2.3
U.S.	9~12	Advanced Algebra (Volume 1 & 2)	Teachers' Edition	McGraw-Hill, 2010	James Flanders; Zalman Usiskin; et al.	UCSMP-AA
	10~12	Functions, Statistics, and Trigonometry (Volume 1 & 2)		UChicago Solutions, 2016	John W. McConnell; Susan A. Brown; et al.	UCSMP-FST
	11~12	Precalculus and Discrete Mathematics (Volume 1 & 2)		McGraw-Hill, 2010	Anthony L. Peressini; Peter D. DeCraene; et al.	UCSMP-PDM

Sample Topics and Problems

Sample topics. It is necessary to select representative topics to keep comparisons neutral and meaningful as the coverage and sequence of content vary in the United States and China.

Quadratic relations. Functions, a particular type of relation in which no two ordered pairs (x, y) have the same first component x, are often considered as one of the most important topics in mathematics education (Burkett, 1998), with its recognized organizing power from middle school mathematics through more advanced topics in high school and college (Leinhardt et al., 1990). Quadratic functions, as the first non-linear polynomial function, is a traditional core topic that is essential to building the transition from linearity to non-linearity and laying the foundation for Pre-calculus and Calculus in the United States (Nielsen, 2015; Parent, 2015). In China, quadratic functions are embedded in middle school mathematics, laying the foundation and transition for the learning of quadratic relations in high schools. Quadratic relations address a broader area of non-linear polynomial equations and involve multiple connections in Algebra and Geometry strands, which may offer ample opportunities to investigate the directionality.

Lastly, researchers reported students' limited understandings of quadratics. For example, 10th and 11th graders in Israel showed their over-reliance on the equation-to-graph direction of quadratic functions (Zaslavsky, 1997). Nielsen (2015) found that 65% of 20 U.S. high school students were able to make the connection from the quadratic equation to the graph. Even U.S. undergraduate students showed not much flexibility in moving between two representations of quadratic functions (Metcalf, 2007). Second-year university students in the United States still showed strong preference for the standard form, rather than the vertex form or the factor form of quadratic functions in the tasks of transforming quadratic functions (Vaiyavutjamai, Ellerton, & Clements, 2005). Few studies have focused on the more general topic, i.e., quadratic relations (a relation between two variables that follows: $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, where A, B, C, D, E, and F are real numbers and at least one of A, B, C is not zero). The literature review also suggested that bidirectional connections between multiple representations were difficult for students and teachers, and routine tasks in textbooks were mostly in the equation-to-graph direction (Knuth, 2000b). Therefore, examining quadratic relations may offer a comparison of the current usage of mathematical connections, present possible exemplary ways to embed bidirectional connections in textbook problems, and in the end, help learners conquer learning difficulties in quadratic relations.

Probability and combinatorics. Besides Algebra and Geometry strands, probability and combinatorics comprise a rich structure of mathematical principles that underlie Probability, Statistics, and Discrete Mathematics strands. They not only foster deep mathematical thinking and meaningful representations but also offer a problem-heavy field with a variety of solution approaches and representations with rich bidirectional connections, which lead to difficulties for students to identify problem structures and mathematical connections (Lockwood, 2011; Sriraman & English, 2004). For example, Bulone (2017) found that many U.S. postsecondary students struggled with connections between problems of the same type with altered contexts. Even novice undergraduate students enrolled in Calculus in the United States may recognize the first type but not the second type of combinatorics problems (Lockwood, Wasserman, & McGuffey, 2018). Additionally, researchers reported some Australian preservice teachers' failures to move from experimental probability to theoretical probability (Prodromou, 2012). The effects of single and multiple representation on learning probability and combinatorics were also examined (Kolloffel et al., 2009). Moreover, probability and combinatorics, compared with quadratic relations, usually involve more problems in real-life contexts and reveal different mathematical structures. Furthermore, the eighth curriculum reform in China placed a special focus on statistics and probability, which did not get close attention for a long time, whereas quadratic relations are long-standing emphasized content (Li, Zhang, and Ma, 2019). Therefore, examining probability and combinatorics may offer a comparison of the curricular emphasis of different mathematical topics and explore exemplary usage of representations to reveal practical implications for the teaching and learning of probability and combinatorics.

Sample problems. The UCSMP and PEP-A series contain several chapters involving selected topics which are under the organization of 'Section 1->Section 2->…->Chapter Review.' For each section, the PEP-A series includes several leading words, like "Example" or "In-class Exercises" or "After-class Exercises" to indicate worked-out examples and exercises; the UCSMP series uses leading words like "Example" or "Questions" with sub-heading words—"Covering the Ideas," "Applying the Mathematics," "Review," and "Exploration"—to denote worked-out examples and exercises. Also, additional worked-out examples provided in the teachers' edition are included as they are valuable supplementary materials. Problems in the projects, explorations, reading, the chapter review, and the self-test in each chapter are not included in the analysis as their random sequence and frequency of use are not comparable.

Regarding selected topics, the UCSMP and PEP-A series cover seven subtopics: four for quadratic relations (circle, ellipse, hyperbola, and parabola) and three for probability and combinatorics (probability, counting problems, and binomial theorem). Therefore, I compiled all of the above worked-out examples, exercises, and their solutions (some in the teachers' edition) in related chapters in their original sequence as a separate single set. Table 2 summarizes the corresponding problem sets and related chapters included in the study.

Table 2. *Corresponding Problem Sets Included in the Study*

Pair	Subtopic	Related Chapter	
1	Circle	U.S.	UCSMP-AA-Chapter 12
		China	PEP-A-C2-Chapter 4; PEP-A-E2.1-Chapter 2
2	Ellipse	U.S.	UCSMP-AA-Chapter 12
		China	PEP-A-E2.1-Chapter 2
3	Hyperbola	U.S.	UCSMP-AA-Chapter 12
		China	PEP-A-E2.1-Chapter 2
4	Parabola	U.S.	UCSMP-AA-Chapter 12
		China	PEP-A-E2.1-Chapter 2
5	Probability	U.S.	UCSMP-FST-Chapter 6
		China	PEP-A-C3-Chapter 3
6	Counting Problems	U.S.	UCSMP-PDM-Chapter 12
		China	PEP-A-E2.3-Chapter 1
7	Binomial Theorem	U.S.	UCSMP-PDM-Chapter 12
		China	PEP-A-E2.3-Chapter 1

Data Coding

Designing a Schema for Coding

Phase 1 started with dividing collected data into separate instances and then coding relevant features. Some problems in both series have two levels of numbering and share the first-level numbering by 1, 2, …. They differ in the second-level numbering as PEP-A uses (1), (2), … and UCSMP adopts a, b, … (see Figure 11). For problems with the first-level numbering only, I divided data into basic items by the first-level numbering. For problems having two levels of numbering, I divided data into basic items by the second-level numbering. Finally, I assigned an item number one by one. In Figure 11, sample problems of the PEP-A series are divided into 5 items: item 1 (for 1), item 2 (for 2(1)), item 3 (for 2(2)), item 4 (for 3(1)), and item 5 (for 3(2)); sample problems of the UCSMP series are divided into 8 items: item 1 (for 1), item 2 (for 2a), item 3 (for 2b), item 4 (for 3a), item 5 (for 3b), item 6 (for 3c), item 7 (for 4), and item 8 (for 5).

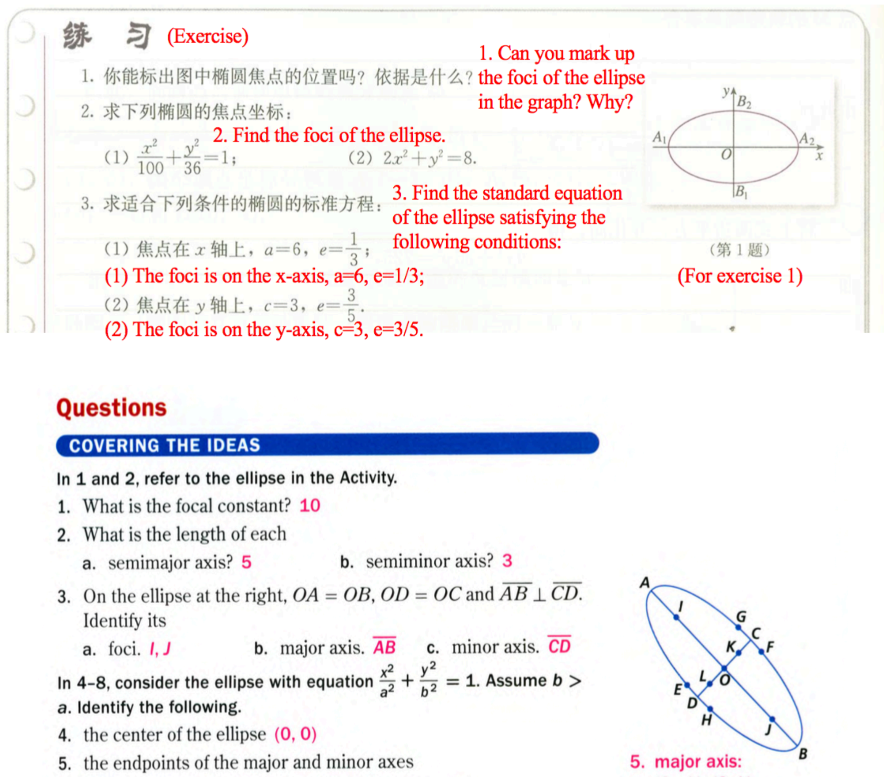


Figure 11. Textbook problem samples (above: PEP-A; below: UCSMP)

Next, I coded each item for its mathematical topic: Quadratic Relations (QR); and Probability and Combinatorics (PC). Then, I coded each item for its presentational feature: Worked-out Example (WE): problems with a complete solution which are usually designed for teachers' instruction and students' reference; and Exercise (EX): problems with no solution in students' edition (except for possible answers provided at the end of textbooks), which provide students with practice opportunities (Li, 1999). Then, I coded each item for its contextual feature: Purely Mathematical (PM): problems formulated with numbers, symbols, geometric figures, and other purely mathematical representations verbally or only with purely mathematical representations (Li, 1999); and Real-life (RL): real-life contexts. In Figure 11, all sample problems are purely mathematical exercises in quadratic relations, coded as QR, EX, and PM, respectively. Next, I coded each item for its mathematical feature: Single-step (S): problems that can be solved by one direct step or operation; and Multi-step (M): rest conditions (Zhu & Fan, 2006). Finally, I coded each item for its visual feature: Non-visual (N); and Visual (V) problems with visual information like pictures, graphs, charts, tables, diagrams, and so on (Zhu & Fan, 2006). In Figure 11, the sample problem 3a of the UCSMP series is a single-step problem including visual information, coded as S and V; the sample problem 3(1) of the PEP-A series is a non-visual problem that cannot be solved by one direct step, coded as M and N.

Phase 2 was designed to build the Connection Table and code each instance in Phase 1 for corresponding connections. I first collected the vocabulary checklist in the chapter summary and relevant items in the glossary in the PEP-A and UCSMP series to compile the Concepts Table (see Appendix A). Then, I built the Representations Table: written description, numerals, symbolic expressions (S1: with numerical coefficients; S2: with letter coefficients), tables, graphs, diagrams, charts, pictures, and concrete/manipulative representations (Marshall et al., 2010). Next, I compiled all possible connections (an ordered pair of two concepts with its representation from the Concepts Table and Representations Table) in the Connection Table and finally coded each item in terms of relevant connections in the table. All identified connections were compiled in a table. Each item may have zero or one or multiple connections. I coded each item for the presence or absence of connections into: the no-connection condition (0), the between-concept condition (1), the within-concept condition (2), and the mixed condition of both between-concept and within-concept connections (3). For instance, the connection in sample problem 2(1) of the PEP-A series is ellipse-to-foci, the between-concept condition, coded as 1.

Phase 3 was designed to recognize bidirectional connections and transfer connections into digraphs and adjacency matrices. For connections in Phase 2, I used NodeXL to filter out bidirectional connections by the reciprocated function (designed to recognize reciprocated edges) and sorted each connection for its directionality: Unidirectional (Uni-) and Bidirectional (Bi-). All bidirectional connections were compiled in a table. For each subtopic, a digraph with vertices and directed edges with problem item numbers showing directional connections was produced. Vertices denote mathematical concepts with or without representations, which are placed in a circle or grid. Edges with arrows and number x display connections in problem item x, in which arrows illustrate the directionality. I also constructed digraphs for each topic without problem item number x. Finally, I generated a corresponding adjacency matrix (a square matrix with one row and one column for each vertex). In sum, Table 3 summarizes the coding framework.

Table 3. *Textbook-Problem Analysis Coding Framework*

Dimension	Feature	Category and Coding
Problems	Mathematical topic	Quadratic Relations (QR)
		Probability and Combinatorics (PC)
	Presentational feature	Worked-out Example (WE)
		Exercise (EX)
	Contextual feature	Purely Mathematical (PM)
		Real-life (RL)
	Mathematical feature	Single-step (S)
		Multi-step (M)
	Visual feature	Non-visual (N)
		Visual (V)
Mathematical Connections	Conditions of connections	No-connection condition (0)
		Between-concept condition (1)
		Within-concept condition (2)
		Mixed condition (3)
	Directionality	Unidirectional (Uni-)
		Bidirectional (Bi-)

Prior Coding and Adjustment

To make sure the coding framework was appropriate for the UCSMP and PEP-A series, prior coding (one section for each topic) was performed to check the feasibility of the model.

Different representations of concepts were identified in the prior coding. During this process, it was found that for some concepts the symbolic representation was more complex than assumed originally. Problems in the section on permutations contained concepts in the symbolic representation of original expression, or polynomial expansion, or factorial expansion. Therefore, the previous categorization—symbolic expression with or without numerical coefficients—might be insufficient to portray within-concept connections. Additionally, less than 5% of the total were within-concept connections in the prior coding. Also, identified within-concept connections were concentrated, which covered less than 10 vertices. Under this circumstance, digraphs of within-concept connections for each topic seemed to be more meaningful compared to fewer edges in smaller digraphs for each subtopic. Therefore, I made several revisions. To reflect different external representations, I updated the Representations Table (see Appendix A) based on the nature of concepts (illustrated in the Concepts Table) in textbooks. Instead of producing digraphs for seven subtopics and digraphs for two topics, I constructed digraphs of (a) between-concept connections for seven subtopics (edges with problem item number x), and (b) between-concept connections and within-concept connections, respectively, for two topics.

Final Coding and Reliability

To reduce my own bias and keep the final coding as neutral as possible, I recruited two experienced high school mathematics teachers (one in China and one in the United States), who had more than 5 years of teaching experience. I first explained to them the coding rubrics in detail, then recorded their agreement or disagreement with my coding for each item in their language, and finally detected missed mathematical connections. We discussed and resolved disputes and then generated the final coding. I invited four doctoral students (a pair of students for problems of one subtopic) majoring in mathematics education, who were both proficient in English and Chinese. I first trained them on the coding rubrics and then asked them to agree or disagree with part of the final coding for the PEP-A (29 items in the subtopic probability; 45 items in the subtopic parabola) and UCSMP series (48 items in the subtopic probability; 29 items in the subtopic parabola), respectively. Both the percentage of agreement (the number of ratings in agreement over the total number of ratings) for each feature and the overall percentage of agreement (an index of inter-coder reliability, calculated by the number of ratings in agreement by both raters over the total number of ratings) for each feature were calculated to check the coding reliability. The percentage of agreement for each coder and the overall percentage of agreement for coder pairs surpassed 80% across coding features and textbooks. The final coding reached the reliability requirement (see Table 4).

Table 4. *Textbook-Problem Coding Reliability*

Textbook	Feature	% of AG-L	% of AG-X	Overall (%)	% of AG-S	% of AG-C	Overall (%)
PEP-A (China)	Presentational	100.0	100.0	100.0	100.0	100.0	100.0
	Contextual	100.0	100.0	100.0	100.0	100.0	100.0
	Mathematical	100.0	100.0	100.0	100.0	100.0	100.0
	Visual	100.0	100.0	100.0	100.0	100.0	100.0
	Connections	96.6	100.0	96.6	84.4	97.8	82.2
UCSMP (U.S.)	Presentational	100.0	100.0	100.0	100.0	100.0	100.0
	Contextual	100.0	100.0	100.0	100.0	100.0	100.0
	Mathematical	100.0	100.0	100.0	100.0	100.0	100.0
	Visual	100.0	100.0	100.0	100.0	100.0	100.0
	Connections	100.0	97.9	97.9	100.0	96.6	96.6

Notes: % of AG stands for the percentage of agreement; Overall (%) stands for the overall percentage of agreement.

Data Analysis

For Research Question 1

1. What are the similarities and differences in the feature of problems with or without mathematical connections from popular U.S. and Chinese high school mathematics textbooks?

I used NVivo to generate a word frequency cloud (a visualization in which the words are different sizes according to their frequency of use in a given text) for (a) quadratic relations, and (b) probability and combinatorics, in related UCSMP and PEP-A textbooks contents. Non-mathematical concepts or representations were listed as stop words (removed from the analysis). The top 20 items were compared to see the general difference in the use of concepts and representations in two textbook series.

Based on four conditions of mathematical connections (no-connection, between-concept, within-concept, and mixed), I generalized a frequency table and distribution chart of problems across topics and textbooks, which showed general information of textbook problems covered in the study and potential associations between mathematical topics and connections. To check potential cognitive supports of worked-out example effects, I produced a frequency table of problems in terms of connections across textbooks and presentational feature, as well as a chart of ratios of worked-out examples to exercises for problems with connections across topics. Considering real-world contexts, I produced a frequency table of problems in terms of connections across textbooks and contextual feature, as well as a chart of ratios of real-life context to purely mathematical for problems with connections across topics. To explore the potential influence of complex problems to connections, I produced a frequency table of problems in terms of connections across textbooks and mathematical feature, as well as a chart of ratios of multi-step to single-step for problems with connections across topics. To analyze the potential support of visual information to connections, I produced a frequency table of problems in terms of connections across textbooks and visual feature, as well as a chart of ratios of visual to non-visual information for problems with connections across topics. To examine the frequent usage of within-concept and between-concept connections, I produced a frequency table of two types of connections across textbooks and topics. To explore potential associations between mathematical connections and problem features across textbooks, I used SPSS to conduct loglinear analysis among textbook series, conditions of mathematical connections, and five problem features: mathematical topic, presentational feature, contextual feature, mathematical feature, and visual feature. Similarities and differences are compared across countries.

For Research Question 2

2. What are the similarities and differences in the directionality of mathematical connections embedded in problems from popular U.S. and Chinese high school mathematics textbooks?

First, I produced a frequency table of unidirectional and bidirectional connections and a chart of ratios of bidirectional to unidirectional connections in terms of types of connections and topics across textbooks. Then, I explored trends of frequently used bidirectional within-concept and between-concept connections, as well as frequently used concepts and representations involved in bidirectional connections, across topics and textbook series. It exhibited similarities and differences in the integration of bidirectional connections.

In a pair of digraphs depicting corresponding sets, I scrutinized: (a) size, (b) unique edges and total edges, (c) density of arrows, and (d) flow of edges to show similarities and differences in the network of connections. Based on the size and edges, dense, moderate, sparse, the sparsest, and aggregated digraphs were identified. The digraph analysis indicated the diversity, weight, sequence, and relative emphasis between typical and reverse connections across textbooks.

More quantitative characteristics can be attained when moving from a digraph to its adjacency matrix (Strom et al., 2001). For each matrix, an entry of non-negative k in row X and column Y indicates there are k connections from X to Y (Chartrand & Lesniak, 2005). I first checked whether on-diagonal and off-diagonal block submatrices had symmetrical entries, which suggested the curriculum emphasis of connections in a particular direction as being strong or weak. Then, I analyzed the adjacency matrix by various indices (see Table 5). In-degree and out-degree centrality, together with in-connection and out-connection connectivity, indicated the curriculum emphasis leading to or leading out of a specific vertex (concepts and representations). The further analysis of reciprocated vertex pair ratio, self-loops, bidirectional pairs, and reciprocated edge ratio suggested the extent of the usage of bidirectional connections and the curriculum emphasis on unidirectional connections. Finally, I generalized the similarities and differences in the directionality of mathematical connections in problems from high school mathematics textbooks in the United States and China.

Table 5. *Digraph and Adjacency Matrix Analysis Dimensions*

Dimension	Index	Feature
The Whole Network	Size	The number of vertices in a network
	Unique Edges	The number of connections where multiple connections from vertex A to vertex B are counted only once
	Total Edges	The number of connections where multiple connections from vertex A to vertex B are all counted
Vertex	In-degree	The number of connections leading to a specific vertex
	Out-degree	The number of connections leading out of a specific vertex
	In-connection	The number of unique connections leading to a specific vertex
	Out-connection	The number of unique connections leading out of a specific vertex
Edge	Reciprocated-Vertex-Pair Ratio	The percentage of vertices that have a reciprocal relationship of total vertices (When an edge from vertex A to vertex B is joined by another edge from B to A, then their connection is reciprocated)
	Reciprocated-Edge Ratio	The percentage of edges that have a reciprocal relationship of total edges
	Bidirectional Pairs	The number of pairs of bidirectional connections (Except self-loops)
	Self-loops	The number of edges that starts and ends in the same vertex

For Research Question 3

3. Which structural differences in popular U.S. and Chinese high school mathematics textbook problems may promote or hinder bidirectional connections?

I reviewed structural differences existed in textbook problems from the UCSMP and PEP-A series, such as the placement of subtopics, unique practices in each textbook series. Correspondences between differences in textbook problem structure and differences in the directionality of mathematical connections were explored to unpack potential factors promoting or hindering bidirectional connections.

Chapter IV

RESULTS

Overview

This chapter presents the results of the analysis of the data collected on selected high school mathematics textbooks in the United States (the UCSMP series) and China (the PEP-A series) for the following research questions:

1. What are the similarities and differences in the feature of problems with or without mathematical connections from popular U.S. and Chinese high school mathematics textbooks?
2. What are the similarities and differences in the directionality of mathematical connections embedded in problems from popular U.S. and Chinese high school mathematics textbooks?
3. Which structural differences in popular U.S. and Chinese high school mathematics textbook problems may promote or hinder bidirectional connections?

This chapter addresses the above three research questions by the following analysis. Word frequency clouds were developed to show a vivid image of emphasized mathematical concepts and representations in textbook content. Frequency of textbook problems across five textbook problem features (mathematical topic, presentational feature, contextual feature, mathematical feature, and visual feature) in terms of four conditions of connections, frequency of between-concept and within-concept connections across topics, and loglinear analysis among textbook series, mathematical connections, and five textbook problem features were examined to answer the first research question. For the second research question, frequency of unidirectional and bidirectional connections was compared across textbooks and topics. Detailed comparisons of bidirectional within-concept and between-concept connections were conducted. Digraph and adjacency matrix analysis were used to investigate the overall directionality of mathematical connections in textbook problems. For the third research question, structural differences in U.S. and Chinese high school textbook problems were analyzed to explore potential factors influencing the directionality of mathematical connections.

Research Question 1

Word Frequency Clouds

To visually show the coverage and emphasis of concepts and representations addressed in the overall content of textbooks of selected chapters, I conducted a word frequency cloud query in NVivo across topics and textbooks (see Figure 12).

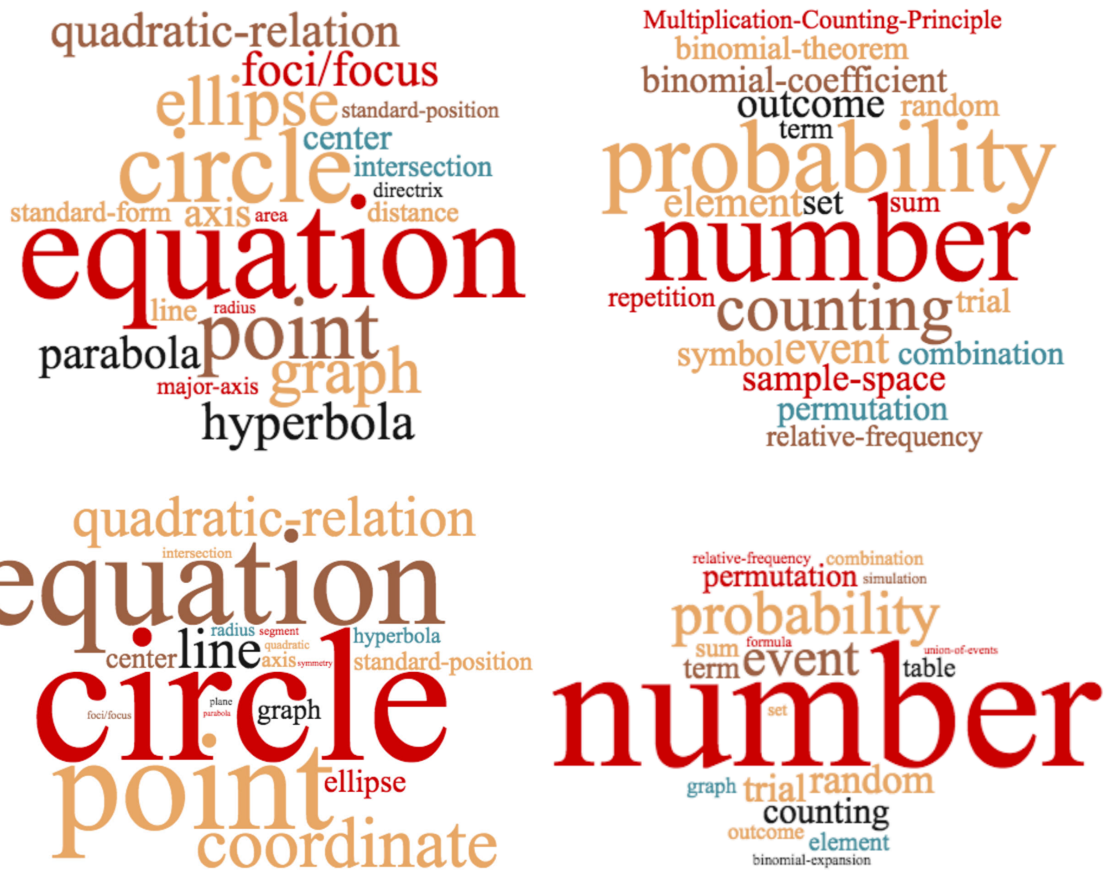


Figure 12. Word frequency clouds (above: UCSMP; below: PEP-A)

From Figure 12, the dominating words in the UCSMP series for quadratic relations are "Equation; Circle; Point; Ellipse; Graph; Hyperbola; Foci/Focus; Quadratic Relation," and for probability and combinatorics the words are "Number; Probability; Counting; Event; Element; Outcome; Sample Space; Binomial Coefficient." For the PEP-A series, the dominating words are "Circle; Point; Equation; Coordinate; Line; Quadratic Relation; Ellipse; Center," and "Number; Probability; Event; Random; Trial; Counting; Permutation; Term" respectively. Based on differences in the size of listed words, the UCSMP series seems to have a more balanced distribution of mathematical concepts and representations in two topics than the PEP-A series.

Top 20 Items

To compare the curricular emphasis illustrated across textbooks, the frequency and weighted percentage (the frequency of words relative to the frequency of total words) of the top 20 concepts and representations are presented in Table 6 and Table 7.

Table 6. *Top 20 Items in Quadratic Relations*

Textbook	Word	Fre-quency	Weighted Percentage	Textbook	Word	Fre-quency	Weighted Percentage
UCSMP (U.S.)	equation	598	2.50%	PEP-A (China)	circle	1127	4.70%
	circle	430	1.79%		point	810	3.38%
	point	356	1.49%		equation	808	3.37%
	ellipse	286	1.19%		coordinate	425	1.77%
	graph	272	1.13%		line	337	1.40%
	hyperbola	216	0.90%		QR	337	1.40%
	foci/focus	201	0.84%		ellipse	188	0.78%
	QR	199	0.83%		center	173	0.72%
	parabola	190	0.79%		graph	168	0.70%
	axis	168	0.70%		STD position	153	0.64%
	center	147	0.61%		hyperbola	130	0.54%
	line	126	0.53%		axis	128	0.53%
	intersection	121	0.50%		radius	107	0.45%
	STD form	115	0.48%		intersection	89	0.37%
	distance	113	0.47%		segment	70	0.29%
	focal constant	107	0.45%		foci/focus	66	0.28%
	major axis	97	0.40%		plane	62	0.26%
	vertices/vertex	96	0.39%		symmetry	52	0.22%
	STD position	92	0.38%		parabola	47	0.20%
	directrix	84	0.35%		tangent	45	0.19%

Notes: QR stands for quadratic relation; STD stands for standard.

In Table 6, both the PEP-A series and UCSMP series stress circle, point, and equation (top 3) with less attention on graph. The PEP-A series emphasizes more quadratic relations in the standard position (centered at (0, 0) with its foci/focus on an axis) (top 10), whereas the UCSMP series stresses more the standard form of a quadratic relation ($Ax^2+Bxy+Cy^2+Dx+Ey+F=0$) than the standard position (0.48%>0.38%). The UCSMP series emphasizes circle, ellipse, hyperbola, and parabola in descending order, while the PEP-A series focuses more on circle, ellipse, and hyperbola, with less attention to parabola. Both series illustrate a clear focus on some attributes of quadratic relations (center, foci/focus). Additionally, the PEP-A series stresses some concepts related to linear functions (coordinate, line, axis, segment, tangent).

Table 7. *Top 20 Items in Probability and Combinatorics*

Textbook	Word	Fre-quency	Weighted Percentage	Textbook	Word	Fre-quency	Weighted Percentage
UCSMP (U.S.)	number	410	2.74%	PEP-A (China)	number	1488	5.66%
	probability	336	2.25%		probability	505	1.92%
	counting	221	1.48%		event	447	1.70%
	event	156	1.04%		random	338	1.29%
	element	143	0.96%		trial	325	1.24%
	outcome	140	0.94%		counting	300	1.14%
	sample space	123	0.82%		permutation	265	1.01%
	BI coefficient	122	0.82%		term	256	0.97%
	set	120	0.80%		table	225	0.86%
	symbol	118	0.79%		sum	216	0.82%
	permutation	115	0.77%		element	211	0.80%
	trial	111	0.74%		outcome	184	0.70%
	combination	108	0.72%		graph	182	0.69%
	BI theorem	106	0.71%		combination	161	0.61%
	random	105	0.70%		set	144	0.55%
	sum	99	0.66%		RF	134	0.51%
	term	96	0.64%		simulation	125	0.48%
	RF	96	0.64%		BI expansion	123	0.47%
	repetition	91	0.61%		formula	119	0.45%
	MCP	85	0.57%		union of events	97	0.37%

Notes: BI stands for binomial; RF stands for relative frequency; MCP stands for multiplication counting principle.

In Table 7, both series highlight the numerical representation and probability (top 2). The UCSMP series pays extra attention to the symbolic representation (top 10), whereas the PEP-A series highlights the tabular and graphical representation (top 15). The UCSMP series stresses probability, counting, permutation, combination, and binomial theorem in descending order (all>0.70%), whereas the PEP-A series highlights probability, counting, permutation (all>1.00%), with less coverage on combination and binomial theorem (both<0.70%). Both series stress event, element, outcome, set, and trail.

Mathematical Topic

This study classified each problem for the presence or absence of connections into: the no-connection condition (0), the between-concept condition (1), the within-concept condition (2), and the mixed condition (3). In terms of these conditions, the frequency of textbook problems across textbook series and topics are shown in Table 8.

Table 8. *Frequency of Problems with Different Connections Across Textbooks and Mathematical Topic*

Textbook			Topic		Total
			PC (% of subtotal)	QR (% of subtotal)	
PEP-A (China)	Mathematical Connection	0	19 (8.1%)	0 (0.0%)	19 (4.3%)
		1	130 (55.3%)	188 (90.4%)	318 (71.8%)
		2	51 (21.7%)	4 (1.9%)	55 (12.4%)
		3	35 (14.9%)	16 (7.7%)	51 (11.5%)
	Subtotal		235 (53.0%)	208 (47.0%)	443 (100%)
UCSMP (U.S.)	Mathematical Connection	0	11 (3.1%)	15 (4.6%)	26 (3.8%)
		1	223 (63.2%)	242 (73.6%)	465 (68.2%)
		2	93 (26.3%)	63 (19.1%)	156 (22.9%)
		3	26 (7.4%)	9 (2.7%)	35 (5.1%)
	Subtotal		353 (51.8%)	329 (48.2%)	682 (100%)

Notes: PC stands for probability and combinatorics; QR stands for quadratic relations; Code 0 stands for problems without connection; Code 1 stands for problems with between-concept connections only; Code 2 stands for problems with within-concept connections only; Code 3 stands for problems with between-concept and within-concept connections.

As shown in Table 8, this study covered 1,125 problems in total, with a similar number across topics. Overall, the UCSMP series includes more problems in two topics than the PEP-A series (353>235 and 329>208, respectively). The between-concept connection is the most frequently used condition regardless of textbook series and topics, especially in problems dealing with quadratic relations in the PEP-A series (90.4% of subtotal, which is the highest percentage).

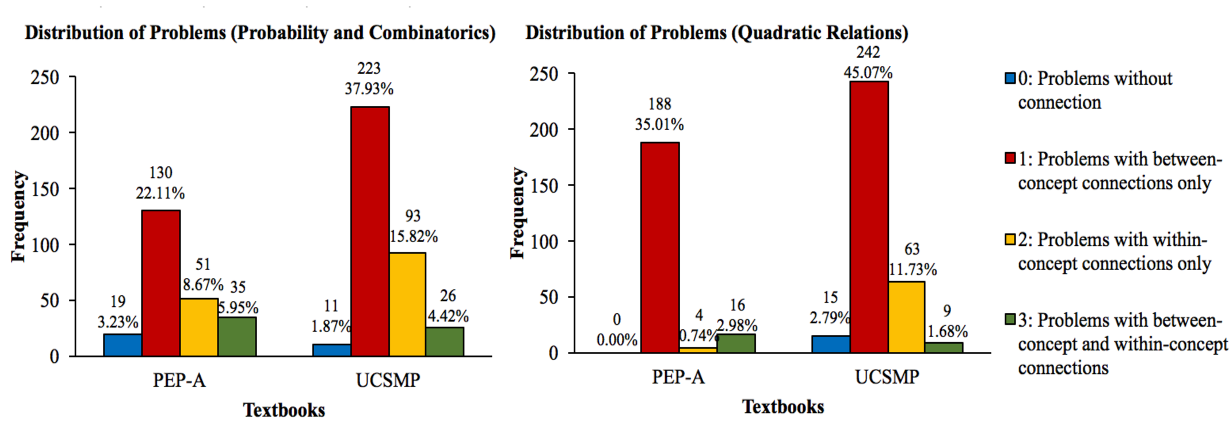


Figure 13. Distribution of conditions of mathematical connections across textbooks and topics

Figure 13 displays the distribution of four conditions across textbooks and topics. Both series emphasize the between-concept condition (top 1) in two topics, especially in quadratic relations. Moreover, the distribution of four conditions in probability and combinatorics is more balanced than in quadratic relations. For probability and combinatorics, both series show the between-concept, the within-concept, the mixed, and the no-connection condition in descending order. For quadratic relations, the PEP-A series illustrates slightly more the mixed than the within-concept condition (2.98%>0.74%). In contrast, the UCSMP series highlights more the within-concept than the mixed condition (11.73%>1.68%). This indicates that two series differed in the usage of the mixed and the within-concept condition in quadratic relations.

Presentational Feature

This study used presentational feature—Worked-out Example (WE) and Exercise (EX)—to explore worked-out example effects. Table 9 presents the frequency of problems in terms of connections across textbooks and presentational feature.

Table 9. *Frequency of Problems with Different Connections Across Textbooks and Presentational Feature*

Textbook			Presentational Feature		Ratio (WE: EX)
			WE (% of subtotal)	EX (% of subtotal)	
PEP-A (China)	Mathematical Connection	0	4 (4.5%)	15 (4.2%)	0.27
		1	70 (78.7%)	248 (70.1%)	0.28
		2	9 (10.1%)	46 (13.0%)	0.20
		3	6 (6.7%)	45 (12.7%)	0.13
	Subtotal		89 (20.1%)	354 (79.9%)	0.25
UCSMP (U.S.)	Mathematical Connection	0	0 (0.0%)	26 (4.8%)	0.00
		1	102 (73.9%)	363 (66.7%)	0.28
		2	24 (17.4%)	132 (24.3%)	0.18
		3	12 (8.7%)	23 (4.2%)	0.52
	Subtotal		138 (20.2%)	544 (79.8%)	0.25

Notes: WE stands for worked-out examples; EX stands for exercises; Code 0 stands for problems without connection; Code 1 stands for problems with between-concept connections only; Code 2 stands for problems with within-concept connections only; Code 3 stands for problems with between-concept and within-concept connections.

In Table 9, both series share a similar ratio of worked-out examples to exercises, overall 0.25. The UCSMP series shows a higher ratio for the mixed condition than the PEP-A series (0.52>0.13). That is, for one worked-out example of the mixed condition, there are approximately two exercises in the UCSMP series and around eight exercises in the PEP-A series. The mixed-condition problems tend to be more challenging for students than the rest conditions. This suggests that the UCSMP series may provide less training exercises of the mixed condition than the PEP-A series. This study then compared ratios of worked-out examples to exercises for problems with connections across textbooks and topics (see Figure 14).

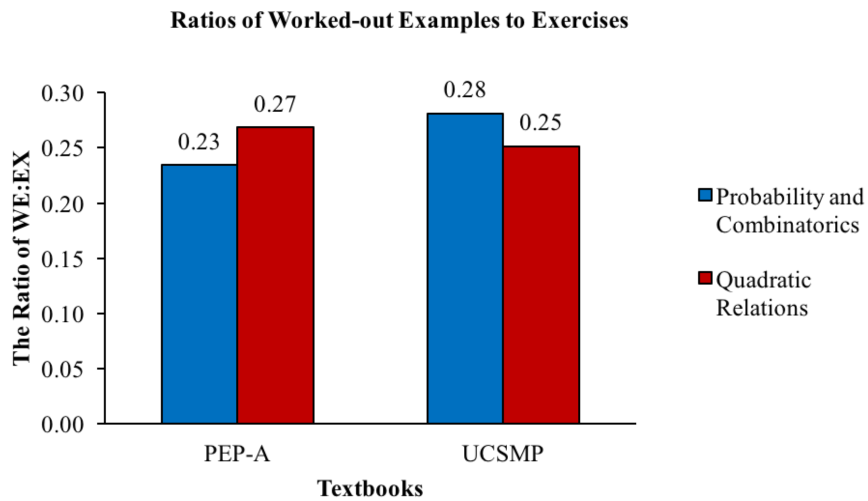


Figure 14. Ratios of worked-out examples to exercises for problems with connections across textbooks and topics

From Figure 14, the PEP-A series demonstrates a slightly higher ratio of worked-out examples to exercises in quadratic relations (0.27>0.25) and a slightly lower ratio in probability and combinatorics (0.23<0.28) than the UCSMP series. This suggests that presentational feature of problems with mathematical connections are similar across topics in both series.

Contextual Feature

This research adopted contextual feature—Purely Mathematical (PM) and Real-life (RL)—to evaluate the call for real-life problems. Table 10 shows the frequency of problems in terms of connections across textbooks and contextual feature.

Table 10. *Frequency of Problems with Different Connections Across Textbooks and Contextual Feature*

Textbook			Contextual Feature		Ratio (RL:PM)
			RL (% of subtotal)	PM (% of subtotal)	
PEP-A (China)	Mathematical Connection	0	0 (0.0%)	19 (6.9%)	0.00
		1	114 (67.5%)	204 (74.5%)	0.56
		2	23 (13.6%)	32 (11.7%)	0.72
		3	32 (18.9%)	19 (6.9%)	1.68
	Subtotal		169 (38.1%)	274 (61.9%)	0.62
UCSMP (U.S.)	Mathematical Connection	0	12 (3.9%)	14 (3.7%)	0.86
		1	223 (73.4%)	242 (64.0%)	0.92
		2	48 (15.8%)	108 (28.6%)	0.44
		3	21 (6.9%)	14 (3.7%)	1.50
	Subtotal		304 (44.6%)	378 (55.4%)	0.80

Notes: RL stands for real-life; PM stands for purely mathematical; Code 0 stands for problems without connection; Code 1 stands for problems with between-concept connections only; Code 2 stands for problems with within-concept connections only; Code 3 stands for problems with between-concept and within-concept connections.

From Table 10, the UCSMP series shares a higher ratio of real-life context to purely mathematical problems than the PEP-A series (0.80>0.62). Both series have a higher ratio of real-life context to purely mathematical for problems of the mixed condition (1.68 and 1.50) than the rest conditions (all<1.00). That is, the mixed condition is often illustrated in more real-life than purely mathematical problems. Also, the no-connection condition happens only in purely mathematical problems for the PEP-A series. Overall, purely mathematical problems still account for a larger part than real-life problems (61.9%>38.1% for PEP-A; 55.4%>44.6% for UCSMP). This study then compared ratios of real-life context to purely mathematical for problems with mathematical connections across textbooks and topics (see Figure 15).

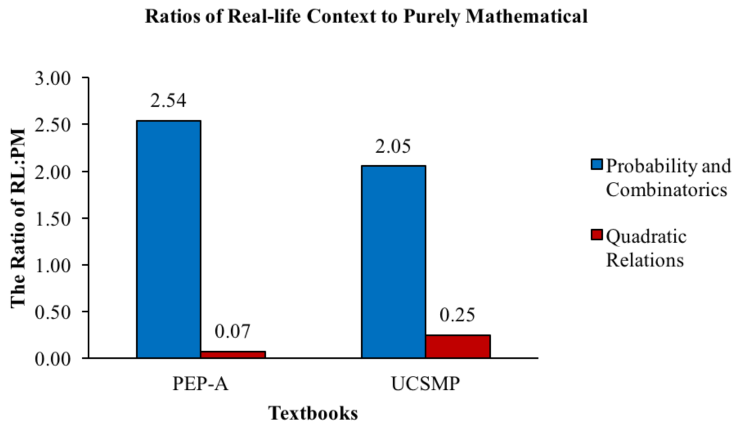


Figure 15. Ratios of real-life context to purely mathematical for problems with connections across textbooks and topics

In Figure 15, striking differences of contextual feature are demonstrated across topics. The UCSMP series has a higher ratio of real-life context to purely mathematical problems in quadratic relations (0.25>0.07) and a lower ratio in probability and combinatorics (2.05<2.54) than the PEP-A series. Both series demonstrate an extremely lower ratio of real-life context to purely mathematical problems in quadratic relations than probability and combinatorics, especially in the PEP-A series. The analysis suggests that the reform call for real-life problems differs in topics, with a better fulfillment in probability and combinatorics. Overemphasizing purely mathematical problems at the cost of real-life context problems may influence connections, particularly the mixed condition.

Mathematical Feature

This research used Single-step (S) and Multi-step (M) to evaluate the potential influence of complex problems to connections. Table 11 displays the frequency of problems in terms of mathematical connections across textbooks and mathematical feature.

Table 11. *Frequency of Problems with Different Connections Across Textbooks and Mathematical Feature*

Textbook			Mathematical Feature		Ratio (M: S)
			M (% of subtotal)	S (% of subtotal)	
PEP-A (China)	Mathematical Connection	0	0 (0.0%)	19 (27.5%)	0.00
		1	292 (78.1%)	26 (37.7%)	11.23
		2	31 (8.3%)	24 (34.8%)	1.29
		3	51 (13.6%)	0 (0.0%)	Undefined
	Subtotal		374 (84.4%)	69 (15.6%)	5.42
UCSMP (U.S.)	Mathematical Connection	0	5 (1.1%)	21 (9.9%)	0.24
		1	344 (73.3%)	121 (56.8%)	2.84
		2	85 (18.1%)	71 (33.3%)	1.20
		3	35 (7.5%)	0 (0.0%)	Undefined
	Subtotal		469 (68.8%)	213 (31.2%)	2.20

Notes: M stands for multi-step; S stands for single-step; Code 0 stands for problems without connection; Code 1 stands for problems with between-concept connections only; Code 2 stands for problems with within-concept connections only; Code 3 stands for problems with between-concept and within-concept connections.

As shown in Table 11, the PEP-A series exhibits a higher ratio of multi-step to single-step problems than the UCSMP series (5.42>2.20). Both series address the mixed condition only in multi-step problems. This suggests that multi-step problems can be used to promote the mixed condition. Additionally, the PEP-A series shows a significantly higher ratio of multi-step to single-step problems for problems with between-concept connections only, than the UCSMP series (11.23>2.84). Moreover, the ratio of multi-step to single-step problems for the no-connection condition approaches 0, especially in the PEP-A series (0.00 for PEP-A and 0.24 for UCSMP). This suggests that multi-step problems tend to address more between-concept connections, and single-step problems tend to have no connection. Then, this study analyzed ratios of multi-step to single-step for problems with connections (see Figure 16).

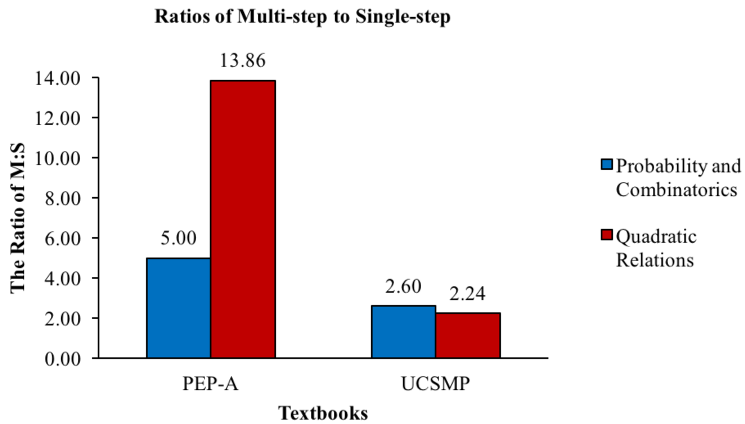


Figure 16. Ratios of multi-step to single-step for problems with connections across textbooks and topics

From Figure 16, the PEP-A series exhibits a higher ratio of multi-step to single-step problems with connections across topics (5.00 and 13.86), than the UCSMP series, which shares a similar ratio across topics (2.60 and 2.24). The extremely high ratio of multi-step to single-step problems in quadratic relations of the PEP-A series is observed (13.86). This suggests that the PEP-A series may have more connections than the UCSMP series in two topics.

Visual Feature

This study used Visual (V) and Non-visual (N) to examine the influence of visual information (e.g., pictures, graphs, charts, tables) to different conditions of connections. Table 12 shows the frequency of problems in terms of connections across textbooks and visual feature.

Table 12. *Frequency of Problems with Different Connections Across Textbooks and Visual Feature*

<table>
<tr><th colspan="3" rowspan="2">Textbook</th><th colspan="2">Visual Feature</th><th rowspan="2">Ratio (Visual: Non-visual)</th></tr>
<tr><th>Visual (% of subtotal)</th><th>Non-visual (% of subtotal)</th></tr>
<tr><td rowspan="5">PEP-A (China)</td><td rowspan="4">Mathematical Connection</td><td>0</td><td>1 (0.8%)</td><td>18 (5.5%)</td><td>0.06</td></tr>
<tr><td>1</td><td>103 (87.3%)</td><td>215 (66.2%)</td><td>0.48</td></tr>
<tr><td>2</td><td>2 (1.7%)</td><td>53 (16.3%)</td><td>0.04</td></tr>
<tr><td>3</td><td>12 (10.2%)</td><td>39 (12.0%)</td><td>0.31</td></tr>
<tr><td colspan="2">Subtotal</td><td>118 (26.6%)</td><td>325 (73.4%)</td><td>0.36</td></tr>
<tr><td rowspan="5">UCSMP (U.S.)</td><td rowspan="4">Mathematical Connection</td><td>0</td><td>4 (2.0%)</td><td>22 (4.6%)</td><td>0.18</td></tr>
<tr><td>1</td><td>136 (66.3%)</td><td>329 (69.0%)</td><td>0.41</td></tr>
<tr><td>2</td><td>54 (26.3%)</td><td>102 (21.4%)</td><td>0.53</td></tr>
<tr><td>3</td><td>11 (5.4%)</td><td>24 (5.0%)</td><td>0.46</td></tr>
<tr><td colspan="2">Subtotal</td><td>205 (30.1%)</td><td>477 (69.9%)</td><td>0.43</td></tr>
</table>

Notes: Code 0 stands for problems without connection; Code 1 stands for problems with between-concept connections only; Code 2 stands for problems with within-concept connections only; Code 3 stands for problems with between-concept and within-concept connections.

As can be seen from Table 12, the UCSMP series shows a higher ratio of visual to non-visual problems than the PEP-A series (0.43>0.36), especially for problems of the within-concept condition (0.53>0.04). This implies that the UCSMP series may have more visual problems with within-concept connections than the PEP-A series. Overall, the majority of problems are non-visual (73.4% for PEP-A and 69.9% for UCSMP). Then, this study compared ratios of visual to non-visual information for problems with connections (see Figure 17).

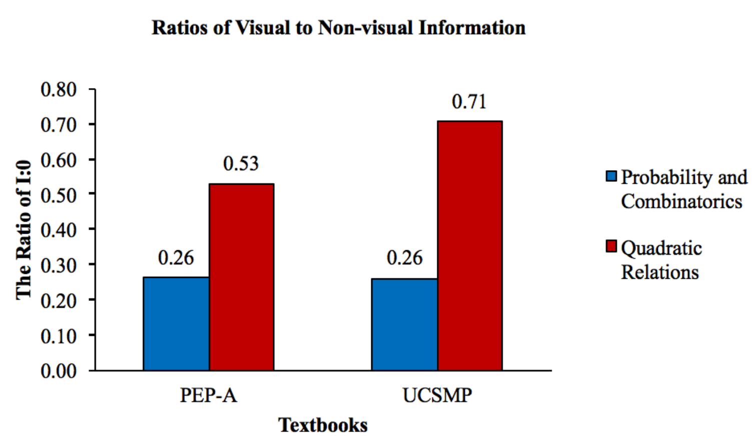


Figure 17. Ratios of visual to non-visual information for problems with connections across textbooks and topics

In Figure 17, the UCSMP series displays a higher ratio of visual to non-visual information for problems with connections in quadratic relations (0.71>0.53) than the PEP-A series. Both series show a lower ratio in probability and combinatorics than quadratic relations (0.26<0.53<0.71). The higher ratio in quadratic relations is consistent with the analysis of word frequency at the beginning that the graph is in the top 10 list of concepts and representations in quadratic relations other than probability and combinatorics. More concepts in quadratic relations tend to be demonstrated with graphs than concepts in probability and combinatorics. This suggests that the UCSMP series may embed more within-concept connections involving the graphical representation in quadratic relations than the PEP-A series.

Between-concept and Within-concept Connections

To compare the usage of between-concept and within-concept connections, this study analyzed the frequency of between-concept and within-concept connections across topics and textbooks (see Table 13).

Table 13. *Frequency of Between-concept and Within-concept Connections Across Textbooks and Topics*

Textbook			Topic		MC/PB
			PC (% of subtotal)	QR (% of subtotal)	
PEP-A (China)	Type	Between-concept	293 (76.7%)	605 (96.6%)	2.03
		Within-concept	89 (23.3%)	21 (3.4%)	0.25
	MC/PB		1.63	3.01	2.28
UCSMP (U.S.)	Type	Between-concept	426 (77.6%)	423 (84.3%)	1.24
		Within-concept	123 (22.4%)	79 (15.7%)	0.30
	MC/PB		1.56	1.53	1.54

Notes: PC stands for probability and combinatorics; QR stands for quadratic relations; MC/PB stands for the number of mathematical connections per problem on average.

In Table 13, this study identified 2,059 mathematical connections, much more between-concept than within-concept connections in two topics and two series. Overall, compared to the

UCSMP series, the PEP-A series includes more connections per problem overall (2.28>1.54), and in two topics, especially in quadratic relations (3.01>1.53). The PEP-A series presents more between-concept connections per problem than the UCSMP series (2.03>1.24), whereas the UCSMP series shows slightly more within-concept connections per problem than the PEP-A series (0.30>0.25). The striking difference between two series in the number of connections per problem in quadratic relations (3.01>1.53) is consistent with the noticeable difference in the ratio of multi-step to single-step problems with connections (13.86>2.24). This suggests that a multi-step problem on average include more connections than a single-step problem. Then, this study compared the number of between-concept and within-concept connections per problem.

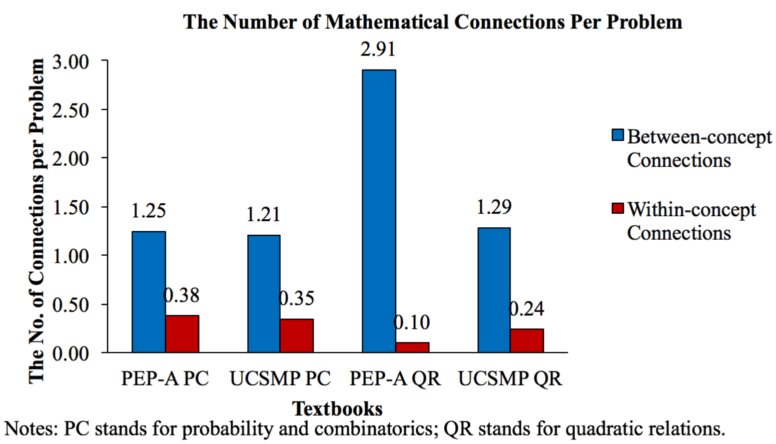


Figure 18. The number of mathematical connections per problem across textbooks and topics

In Figure 18, the number of between-concept connections per problem in two topics (>1.00) is much higher than that of within-concept connections (<0.50), especially for quadratic relations of the PEP-A series (2.91>0.10). That is, for one problem in quadratic relations of the PEP-A series, there is on average about three between-concept connections compared to 0.10 within-concept connections. This indicates the PEP-A series may lack within-concept connections in quadratic relations. In two topics, the UCSMP series exhibits a similar number of between-concept connections per problem (1.21 and 1.29) and within-concept connections per problem (0.35 and 0.24). In contrast, the PEP-A series shows a larger number of between-concept per problem and a smaller number of within-concept connections per problem in quadratic relations than in probability and combinatorics.

Loglinear Analysis

The above analysis suggests that the PEP-A series and UCSMP series are similar in presentational feature and differ in topic, contextual, mathematical, and visual feature. Problem features seem to be dependent, e.g., striking differences in contextual feature between topics are observed. Problems of four conditions of mathematical connections in two series exhibit similarities and differences in problem features.

This study aimed to find a model depicting associations between mathematical connections and problem features in two textbook series without substantial loss of predictive power of all five problem features. Also, this study tried to determine which model components, i.e., one-way or higher-order interactions among problem features, textbook series, and connections, were necessary to retain or contribute more to best account for the data. Therefore, this study conducted hierarchical loglinear analysis, including seven categorical variables: (i) mathematical connection (no-connection, between-concept only, within-concept only, or the mixed); (ii) textbook series (PEP-A or UCSMP); (iii) topic (probability and combinatorics or quadratic relations); (iv) presentational (exercise or worked-out example); (v) contextual (purely mathematical or real-life); (vi) mathematical (single-step or multi-step); and (vii) visual (non-visual or visual).

The assumptions of the loglinear analysis with more than two variables were that there are no more than 20% of cells with expected frequencies less than 5, all cells must have expected frequencies greater than 1, and all variables are independent (Field, 2013). Analysis of cross-tabulation results indicated that 76.56 % of cells had expected frequencies less than 5. This could be explained by the fact that problem features are not independent of each other. For example, both series have no problems with a combination of problem features of particular categories, e.g., single-step purely mathematical visual worked-out examples in two topics. Also, a combination of problem features that hindering connections also leads to many empty cells. For example, there are no multi-step problems in the PEP-A series without connections. However, since the data set is large (N=1125) and all seven variables are essential in understanding relationship between problem features and connections, the decrease in statistical power due to violations of assumptions might not result in substantial loss of predictive power of the model.

The initial loglinear analysis was performed with all seven variables (see Table 14). The analysis suggests that removing four-way and higher-order effects will not significantly affect the fit of the model ($p=1$); removing three-way and lower interactions has a significant detrimental effect on the model ($p<0.001$). The final model retains three-way and lower interactions. The likelihood ratio of this model is $\chi^2(173)=84.570$, $p=1$. This indicates that some of the three-way interactions are significant, $\chi^2(31)=335.642$, $p<0.001$.

Table 14. *K-way and Higher-order Effects Results*

K		df	Likelihood Ratio	
			Chi-Square	Sig.
K-way and Higher-order Effects[a]	1	255	3481.191	<0.001
	2	246	1354.274	<0.001
	3	213	444.060	<0.001
	4	148	60.599	1.000
	5	73	1.441	1.000
	6	22	0.000	1.000
	7	3	0.000	1.000
K-way Effects[b]	1	9	2126.917	<0.001
	2	33	910.214	<0.001
	3	65	383.461	<0.001
	4	75	59.157	0.910
	5	51	1.441	1.000
	6	19	0.000	1.000
	7	3	0.000	1.000

Notes:
a. Tests that k-way and higher-order effects are zero.
b. Tests that k-way effects are zero.

Based on the analysis of partial associations, statistically significant three-way and lower interactions in the model were identified (see Table 15). Since the study focused on exploring the similarities and differences between the PEP-A series and UCSMP series in embedding connections and problem features, further analysis was completed for the three-way interactions involving the Textbook Series variable (highlighted in the color purple).

Table 15. *Partial Associations of Statistically Significant Three-way or Lower Interactions*

Effect	Partial χ^2	Sig.	Effect	Partial χ^2	Sig.	Effect	Partial χ^2	Sig.
MC	1113.398	<0.001	T*C	438.920	<0.001	MC*Text*T	27.881	<0.001
P	428.153	<0.001	M*MC	161.189	<0.001	MC*Text*C	23.566	<0.001
M	292.682	<0.001	T*V	105.081	<0.001	MC*Text*V	19.924	<0.001
V	210.606	<0.001	T*MC	88.847	<0.001	Text*T*C	19.857	<0.001
Text	51.163	<0.001	C*V	71.447	<0.001	T*V*MC	17.414	0.001
C	28.602	<0.001	C*MC	59.123	<0.001	T*C*M	15.464	<0.001
			M*V	30.507	<0.001	T*M*MC	13.817	0.003
			Text*M	29.782	<0.001	C*M*MC	13.013	0.005
			Text*MC	25.229	<0.001	Text*P*V	10.543	0.001
			Text*C	13.464	<0.001	Text*T*M	8.868	0.003
			P*V	9.722	0.002	Text*M*V	8.414	0.004
			V*MC	9.118	0.028	P*M*V	5.776	0.016
			P*M	7.432	0.006			
			Text*T	5.813	0.016			

Notes: MC stands for Mathematical Connection; P stands for Presentational; M stands for Mathematical; V stands for Visual; Text stands for Textbook Series; C stands for Contextual; T stands for Topic.

The following analysis examines the highlighted interactions one by one in the order of descending partial χ^2, which reflects the contribution of effects to the model. In order to do that, separate loglinear analysis was performed on these interactions involving *Textbook Series*.

*1. Mathematical Connection * Textbook Series * Topic.* The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)=0$, p=1. This indicates that Mathematical Connection * Textbook Series * Topic is statistically significant, $\chi^2(3)=50.194$. To break down the three-way effect, Chi-square tests on connection and topic were performed separately for two series. The PEP-A series shows a strong statistically significant association between connection and topic, $\chi^2(3)=75.455$ and Cramer's V=0.413, whereas the UCSMP series shows a weak statistically significant association, $\chi^2(3)=14.592$ and Cramer's V=0.146. For the PEP-A series, the odds of problems of the within-concept condition are 14.14 times higher when problems deal with probability and combinatorics than quadratic relations; the odds of problems of the between-concept condition are 7.59 times higher when problems deal with quadratic relations than probability and combinatorics. Overall, the odds of problems with connections are 37.57 times higher when problems deal with quadratic relations than probability and combinatorics in the PEP-A series, but only 0.67 in the UCSMP series. Therefore, this seems to reveal a fundamental difference between the two series: the PEP-A series is more likely to embed connections in problems dealing with quadratic relations than probability and combinatorics, whereas the UCSMP series is more likely to embed connections in problems dealing with probability and combinatorics than quadratic relations.

There is a weak statistically significant association between textbook series and connection, $\chi^2(3)=30.617$ and Cramer's V=0.165. The odds of problems of the mixed condition are 2.41 times higher when problems are from the PEP-A series than the UCSMP series. The odds of problems of the within-concept condition are 2.09 times higher when problems are from the UCSMP series than the PEP-A series. Therefore, this seems to reveal a fundamental difference between the two series: the PEP-A series is more likely to have problems of the mixed condition than the UCSMP series, whereas the UCSMP series is more likely to have problems of the within-concept condition than the PEP-A series.

*2. Mathematical Connection * Textbook Series * Contextual.* The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)=0$, p=1. This indicates that Mathematical Connection * Textbook Series * Contextual is statistically significant, $\chi^2(3)=21.480$. To break down the three-way effect, Chi-square tests on connection and contextual feature were performed separately for two series. The PEP-A series presents a moderate statistically significant association between connection and contextual feature, $\chi^2(3)=25.822$ and Cramer's V=0.241, whereas the UCSMP series shows a weak statistically significant association, $\chi^2(3)=17.585$ and Cramer's V=0.161. The odds of problems of the mixed condition are 3.13 times higher when problems are real-life than purely mathematical in the PEP-A series, but also 1.93 in the UCSMP series. Overall, the odds of problems with connections are 25.87 times higher when problems are real-life than purely mathematical in the PEP-A series, but only 0.94 in the UCSMP series. Therefore, this seems to reveal a fundamental difference between the two series: the PEP-A series is more likely to embed connections in real-life than purely mathematical problems, whereas the UCSMP series is slightly more likely to include connections in purely mathematical than real-life problems. Also, there is a weak statistically significant association between textbook series and contextual feature, $\chi^2(1)$=4.551 and Cramer's V=0.064.

*3. Mathematical Connection * Textbook Series * Visual.* The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)$=0, p=1. This shows that Mathematical Connection * Textbook Series * Visual is statistically significant, $\chi^2(3)$=26.839. To break down the three-way effect, Chi-square tests on connection and visual feature were performed separately for two series. There is a moderate statistically significant association between connection and visual feature only in the PEP-A series, $\chi^2(3)$=24.969 and Cramer's V=0.237. The odds of problems of the within-concept condition are 11.30 times higher when problems are non-visual than visual, which is consistent with the previous low ratio of visual to non-visual problems of the within-concept condition (0.04) in the PEP-A series. Overall, the odds of problems with connections are 6.86 times higher when problems are visual than non-visual in the PEP-A series, but also 2.43 in the UCSMP series. This seems to reveal that both series are more likely to embed connections in visual than non-visual problems.

*4. Textbook Series * Topic * Contextual*. The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)$=0, p=1. This shows that Textbook Series * Topic * Contextual is statistically significant, $\chi^2(1)=12.116$. To break down the three-way effect, Chi-square tests on topic and contextual feature were performed separately for two series. There is a strong statistically significant association between topic and contextual feature in the PEP-A series, $\chi^2(1)=164.033$ and Cramer's V=0.609; and in the UCSMP series, $\chi^2(1)=154.610$ and Cramer's V=0.476. The odds of real-life problems are 26.85 times higher when problems dealing with probability and combinatorics than quadratic relations in the PEP-A series, but also 8.25 in the UCSMP series. This implies that for both series, problems dealing with quadratic relations are more likely to be purely mathematical than real-life, whereas problems dealing with probability and combinatorics are more likely to be real-life than purely mathematical, which is consistent with the previous analysis of contextual feature.

*5. Textbook Series * Presentational *Visual*. The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)=0$, p=1. This shows that Textbook Series * Presentational * Visual is statistically significant, $\chi^2(1)=13.564$. To break down the three-way effect, Chi-square tests on presentational and visual feature were conducted separately for two series. There is a moderate statistically significant association between presentational and visual feature only in the PEP-A series, $\chi^2(1)=32.625$ and Cramer's V=0.271. The odds of worked-out examples are 3.94 times higher when problems are visual than non-visual. This implies that worked-out examples in the PEP-A series are more likely to be visual than non-visual, whereas exercises are more likely to be non-visual than visual.

*6. Textbook Series * Topic * Mathematical*. The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)=0$, p=1. This shows that Textbook Series * Topic * Mathematical is statistically significant, $\chi^2(1)=23.095$. To break down the three-way effect, Chi-square tests on topic and mathematical feature were performed separately for two series. There is a moderate statistically significant association between topic and mathematical feature only in the PEP-A series, $\chi^2(1)=23.327$ and Cramer's V=0.229. The odds of multi-step problems are 4.23 times higher when problems deal with quadratic relations than probability and combinatorics. This implies that problems dealing with quadratic relations are more likely multi-step than single-step, whereas problems dealing with probability and combinatorics are more likely single-step than multi-step. Additionally, there is a weak statistically significant association between textbooks and mathematical feature, $\chi^2(1)=35.045$ and Cramer's V=0.176. The odds of multi-step problems are 2.46 times higher when problems are from the PEP-A series than the UCSMP series. This seems to reveal a fundamental difference that the PEP-A series is more likely to present multi-step problems than the UCSMP series.

*7. Textbook Series * Mathematical *Visual*. The three-way loglinear analysis produced a final model retaining all effects. The likelihood ratio of this model is $\chi^2(0)=0$, p=1. This shows that Textbook Series * Mathematical * Visual is statistically significant, $\chi^2(1)=14.039$. To break down the three-way effect, Chi-square tests on mathematical and visual feature were performed separately for two series. There is a moderate statistically significant association between topic and mathematical feature in the PEP-A series, $\chi^2(1)=26.533$ and Cramer's V=0.245; and a weak association in the UCSMP series, $\chi^2(1)=17.216$ and Cramer's V=0.159. The odds of multi-step problems are 30.96 times higher when problems are visual than non-visual in the PEP-A series, but also 2.26 in the UCSMP series. This implies that multi-step problems are more likely to be visual than non-visual, while single-step problems are more likely to be non-visual than visual.

Summary

The word frequency analysis shows the most frequently used words in textbook content. In quadratic relations, both series stress circle, point, and more on the equation than graphical representation. The UCSMP series highlights ellipse, hyperbola, and parabola, whereas the PEP-A series stresses coordinate, line, ellipse, and hyperbola. In probability and combinatorics, both series stress the numerical representation, probability, and more on permutation than combination. The UCSMP series strengthens the symbolic representation and binomial theorem, whereas the PEP-A series highlights the tabular and graphical representation.

Focusing on textbook problems, the between-concept condition dominates problems across textbooks and topics, especially in quadratic relations and the PEP-A series. Four conditions are more evenly distributed in probability and combinatorics than quadratic relations. Both series exhibit a similar ratio of worked-out examples to exercises. For problems of the mixed condition, the UCSMP series provides quite fewer exercises than the PEP-A series. The UCSMP series exhibits a higher ratio of real-life to purely mathematical contexts than the PEP-A series. However, purely mathematical problems still account for a larger part in both series, especially in quadratic relations of the PEP-A series. Problems of the mixed condition are more likely to be set in real-life contexts. The PEP-A series exhibits a higher ratio of multi-step to single-step problems across topics than the UCSMP series, especially in quadratic relations. Complex multi-step problems tend to contribute to more between-concept connections in the PEP-A series, especially in quadratic relations. The UCSMP series shows a higher ratio of visual to non-visual problems than the PEP-A series, especially for problems of the within-concept condition. Overall, the PEP-A series exhibits more connections per problem in total and across topics, especially for between-concept connections in quadratic relations, than the UCSMP series, which addresses more within-concept connections per problem in total and in quadratic relations. Between-concept connections in quadratic relations and within-concept connections in probability and combinatorics are richer than the other topic.

Mathematical topic, contextual feature, and visual feature are most likely to contribute to the presence of four conditions of mathematical connections. The PEP-A series is more likely to embed connections in real-life than purely mathematical problems and problems dealing with quadratic relations than probability and combinatorics, whereas the UCSMP series is more likely to embed connections in purely mathematical than real-life problems and problems dealing with probability and combinatorics than quadratic relations. Both series are more likely to embed connections in visual than non-visual problems, have more problems of the between-concept condition in quadratic relations than probability and combinatorics, and include more problems of the within-concept condition in probability and combinatorics than quadratic relations. The PEP-A series is more likely to have problems of the mixed condition than the UCSMP series, while the UCSMP series is more likely to have problems of the within-concept condition than the PEP-A series. Statistically significant associations among problem features are observed. Problems dealing with quadratic relations are more likely to be purely mathematical than real-life, whereas problems dealing with probability and combinatorics are more likely to be real-life than purely mathematical. Multi-step problems are more likely to be visual than non-visual, while single-step problems are more likely to be non-visual than visual. For the PEP-A series, problems dealing with quadratic relations are more likely to be multi-step than single-step, whereas problems dealing with probability and combinatorics are more likely to be single-step than multi-step. Worked-out examples are more likely to be visual than non-visual, while exercises are more likely to be non-visual than visual.

Research Question 2

Unidirectional and Bidirectional Connections

This study classified the directionality into two types—Unidirectional (Uni-) and Bidirectional (Bi-)—and analyzed their frequency for problems with different types of connections and topics across textbooks (see Table 16).

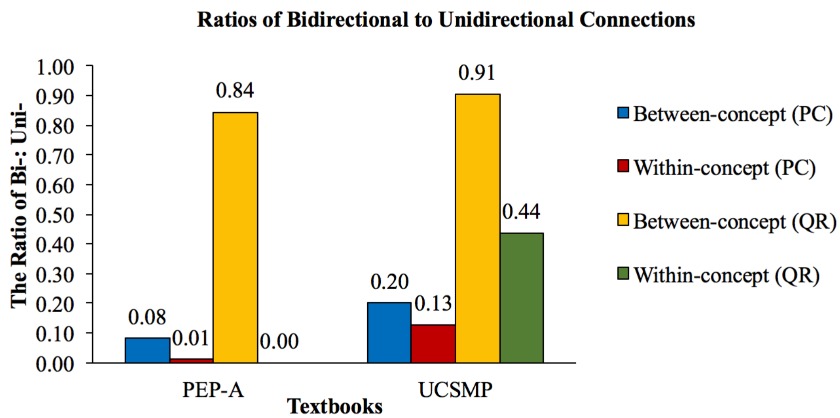
Table 16. *Frequency of Unidirectional and Bidirectional Connections Across Textbooks in Terms of Types of Connections and Mathematical Topics*

Textbook			Type		Topic		Total
			BCC	WCC	PC	QR	
PEP-A (China)	Directi-onality	Bi	299 (33.3%)	1 (0.9%)	23 (6.0%)	277 (44.2%)	300 (29.8%)
		Uni	599 (66.7%)	109 (99.1%)	359 (94.0%)	349 (55.8%)	708 (70.2%)
	Ratio (Bi: Uni)		0.50	0.01	0.06	0.79	0.42
UCSMP (U.S.)	Directi-onality	Bi	272 (32.0%)	38 (18.8%)	85 (15.5%)	225 (44.8%)	310 (29.5%)
		Uni	577 (68.0%)	164 (81.2%)	464 (84.5%)	277 (55.2%)	741 (70.5%)
	Ratio (Bi: Uni)		0.47	0.23	0.18	0.81	0.42

Notes:
a. BCC stands for between-concept connections; WCC stands for within-concept connections; PC stands for probability and combinatorics; QR stands for quadratic relations; Bi stands for bidirectional; Uni stands for unidirectional.
b. The percentage of subtotal appears in parentheses after frequency.

In Table 16, both series exhibit a similar ratio of bidirectional to unidirectional connections, overall 0.42. The majority of connections (more than 70%) are unidirectional. Both series show a similar ratio of bidirectional to unidirectional between-concept connections (0.50 for PEP-A and 0.47 for UCSMP). Additionally, a higher ratio of bidirectional to unidirectional connections is observed in quadratic relations than probability and combinatorics (0.79>0.06 for PEP-A and 0.81>0.18 for UCSMP). It is noted that the ratio of bidirectional to unidirectional within-concept connections is extremely low in the PEP-A series (0.01), which indicates that the PEP-A series may lack learning opportunities for bidirectional within-concept connections.

Then, this study compared ratios of bidirectional to unidirectional connections combining types and topics across textbooks (see Figure 19).



Notes: PC stands for probability and combinatorics; QR stands for quadratic relations.

Figure 19. Ratios of bidirectional to unidirectional connections across textbooks

In Figure 19, the UCSMP series exhibits a higher ratio of bidirectional to unidirectional for two types of connections in two topics than the PEP-A series, especially for within-concept connections in quadratic relations (0.44>0.00). Both series show a higher ratio of bidirectional to unidirectional between-concept connections in quadratic relations than the rest of the three groups. This suggests that problems dealing with quadratic relations show a richer network of bidirectional between-concept connections than problems dealing with probability and combinatorics.

Next, the analysis moved to an in-depth comparison of bidirectional connections.

Integration of Bidirectional Connections

The study compared bidirectional within-concept connections first and then bidirectional between-concept pairs. Table 17 summarizes bidirectional within-concept connections across topics and textbooks. Self-loops are highlighted in the color purple.

Table 17. *Bidirectional Within-concept Connections*

No.	Textbook	Topic	Source Vertex	Target Vertex	Weight
1	UCSMP	PC	Permutation (n, r); S	Permutation (n, r); S2	3
2	UCSMP	PC	Permutation (n, r); S2	Permutation (n, r); S	1
3	UCSMP	PC	Combination (n, r); S	Combination (n, r); W1	1
4	UCSMP	PC	Combination (n, r); W1	Combination (n, r); S	1
5	UCSMP	PC	$(x + y)^n$; S	$(x + y)^n$; S3	6
6	UCSMP	PC	$(x + y)^n$; S3	$(x + y)^n$; S	1
7	UCSMP	PC	n factorial; S	n factorial; S	1
1	UCSMP	QR	Exterior of a circle; G	Exterior of a circle; S1	1
2	UCSMP	QR	Exterior of a circle; S1	Exterior of a circle; G	1
3	UCSMP	QR	Interior of a circle; W	Interior of a circle; S1	1
4	UCSMP	QR	Interior of a circle; S1	Interior of a circle; W	1
5	UCSMP	QR	Ellipse; W	Ellipse; G	1
6	UCSMP	QR	Ellipse; G	Ellipse; W	1
7	UCSMP	QR	Ellipse; W	Ellipse; S	1
8	UCSMP	QR	Ellipse; S	Ellipse; W	2
9	UCSMP	QR	Circle; S1	Circle; W	1
10	UCSMP	QR	Circle; W	Circle; S1	2
11	UCSMP	QR	Ellipse; G	Ellipse; S1	3
12	UCSMP	QR	Ellipse; S1	Ellipse; G	6
13	UCSMP	QR	Ellipse; S	Ellipse; S	1
14	UCSMP	QR	Parabola; W	Parabola; W	1
15	UCSMP	QR	Quadratic relation; S	Quadratic relation; S	1
1	PEP-A	PC	n factorial; S	n factorial; S	1

Notes: PC stands for probability and combinatorics; QR stands for quadratic relations; S stands for the symbolic representation; W stands for the written description; G stands for the graphical representation.

Both the PEP-A and UCSMP series share one self-loop of n factorial from and to the symbolic representation. Except for this self-loop, all the rest of bidirectional within-concept connections are in the UCSMP series. Overall, the UCSMP series highlights connections of ellipse from the standard symbolic to graphical representation than the reverse (6 typical vs. 3 reverse). The symbolic representation is dominating all bidirectional within-concept connections (20 in 23 unique edges, 36 in 39 total edges). Ellipse, circle, and the nth power of the binomial (x + y) are core concepts for bidirectional within-concept connections in the UCSMP series.

Table 18 and Table 19 list bidirectional between-concept connections across textbooks in two topics, respectively. Same pairs across textbooks are highlighted in the same color.

Table 18. *Bidirectional Between-concept Connections (Probability and Combinatorics)*

No.	Textbook	Source Vertex	Target Vertex	Weight	Textbook	Source Vertex	Target Vertex	Weight
1	PEP-A	Probability	Frequency	1	UCSMP	Probability	Relative frequency	1
2	PEP-A	Frequency	Probability	1	UCSMP	Relative frequency	Probability	13
3	PEP-A	Probability	Event	1	UCSMP	Probability	Mutually exclusive events	1
4	PEP-A	Event	Probability	5	UCSMP	Mutually exclusive events	Probability	4
5	PEP-A	Probability	Combination (n, r)	1	UCSMP	Probability	Union of events	1
6	PEP-A	Combination (n, r)	Probability	7	UCSMP	Union of events	Probability	11
7	PEP-A	>1 combination	MCP	5	UCSMP	Probability	Overlapping events	1
8	PEP-A	MCP	>1 combination	1	UCSMP	Overlapping events	Probability	6
9					UCSMP	Probability	Trial	7
10					UCSMP	Trial	Probability	1
11					UCSMP	Event	No. of outcomes (event)	2
12					UCSMP	No. of outcomes (event)	Event	1
13					UCSMP	Independent events	MCP	3
14					UCSMP	MCP	Independent events	1
15					UCSMP	Combination	Pascal's Triangle	1
16					UCSMP	Pascal's Triangle	Combination	1
17					UCSMP	(x+y)^n	Binomial coefficient	12
18					UCSMP	Binomial coefficient	(x+y)^n	2
19					UCSMP	(x+y)^n	Sum (binomial coefficients)	1
20					UCSMP	Sum (binomial coefficients)	(x+y)^n	1

Note: MCP stands for Multiplication Counting Principle.

In Table 18, bidirectional between-concept pairs in probability and combinatorics are all different across textbooks. Overall, the UCSMP series includes more distinct bidirectional between-concept pairs than the PEP-A series (10>4). Both series emphasize bidirectional pairs ending in probability or multiplication counting principle. For example, the PEP-A series stresses connections from event to probability (5 typical vs. 1 reverse); from combination (n, r) to probability (7 typical vs. 1 reverse); and from more than one combination to multiplication counting principle (5 typical vs. 1 reverse). The UCSMP series highlights connections from relative frequency to probability (13 typical vs. 1 reverse); from union of events to probability (11 typical vs. 1 reverse); from overlapping events to probability (6 typical vs. 1 reverse); and from independent events to multiplication counting principle (3 typical vs. 1 reverse). Additionally, the UCSMP series exhibits the emphasis on connections from the nth power of the binomial $(x + y)$ to binomial coefficient (12 typical vs. 2 reverse).

Table 19. *Bidirectional Between-concept Connections (Quadratic Relations)*

No.	Textbook	Source Vertex	Target Vertex	Weight	Textbook	Source Vertex	Target Vertex	Weight
1	PEP-A	Circle	Center	18	UCSMP	Circle	Center	5
2	PEP-A	Center	Circle	22	UCSMP	Center	Circle	13
3	PEP-A	Circle	Radius	16	UCSMP	Circle	Radius	6
4	PEP-A	Radius	Circle	15	UCSMP	Radius	Circle	13
5	PEP-A	Circle	Point (circle)	11	UCSMP	Circle	Point (circle)	6
6	PEP-A	Point (circle)	Circle	10	UCSMP	Point (circle)	Circle	1
7	PEP-A	Circle	Line	1	UCSMP	Circle	Area	1
8	PEP-A	Line	Circle	1	UCSMP	Area	Circle	2
9	PEP-A	Line	Point (line)	1	UCSMP	Circle	a	1
10	PEP-A	Point (line)	Line	8	UCSMP	a	Circle	1
11	PEP-A	Line	Center	4	UCSMP	Circle	b	1
12	PEP-A	Center	Line	2	UCSMP	b	Circle	1
13	PEP-A	Center	Midpoint	1	UCSMP	Circle	Ellipse	6
14	PEP-A	Midpoint	Center	1	UCSMP	Ellipse	Circle	3
15	PEP-A	Center	Radius	1	UCSMP	Point	Point (circle)	1
16	PEP-A	Radius	Center	1	UCSMP	Point (circle)	Point	1
17	PEP-A	Ellipse	Point (ellipse)	1	UCSMP	Semicircle	Point (semicircle)	8
18	PEP-A	Point (ellipse)	Ellipse	5	UCSMP	Point (semicircle)	Semicircle	1
19	PEP-A	Ellipse	Foci	8	UCSMP	Ellipse	Foci	6
20	PEP-A	Foci	Ellipse	3	UCSMP	Foci	Ellipse	2
21	PEP-A	Ellipse	2a	3	UCSMP	Ellipse	x-intercepts	9
22	PEP-A	2a	Ellipse	4	UCSMP	x-intercepts	Ellipse	1
23	PEP-A	Ellipse	2b	3	UCSMP	Ellipse	y-intercepts	8
24	PEP-A	2b	Ellipse	1	UCSMP	y-intercepts	Ellipse	1
25	PEP-A	Ellipse	2c	1	UCSMP	Ellipse	Focal constant	3
26	PEP-A	2c	Ellipse	4	UCSMP	Focal constant	Ellipse	3
27	PEP-A	Ellipse	a	3	UCSMP	Ellipse	Scale change	2
28	PEP-A	a	Ellipse	6	UCSMP	Scale change	Ellipse	4
29	PEP-A	Ellipse	e	3	UCSMP	Ellipse	Area	4
30	PEP-A	e	Ellipse	5	UCSMP	Area	Ellipse	1
31	PEP-A	Ellipse	PF1	1	UCSMP	Ellipse; Hyperbola	Four intersections	3
32	PEP-A	PF1	Ellipse	3	UCSMP	Four intersections	Ellipse; Hyperbola	1
33	PEP-A	Ellipse	PF2	1	UCSMP	Hyperbola	Focal constant	5
34	PEP-A	PF2	Ellipse	1	UCSMP	Focal constant	Hyperbola	9
35	PEP-A	PF2	2a	3	UCSMP	Hyperbola	Point (hyperbola)	1
36	PEP-A	2a	PF2	1	UCSMP	Point (hyperbola)	Hyperbola	1
37	PEP-A	Hyperbola	Vertices	4	UCSMP	Hyperbola	Vertices	6
38	PEP-A	Vertices	Hyperbola	1	UCSMP	Vertices	Hyperbola	1
39	PEP-A	Hyperbola	Foci	8	UCSMP	Hyperbola	Foci	7
40	PEP-A	Foci	Hyperbola	6	UCSMP	Foci	Hyperbola	9
41	PEP-A	Hyperbola	a	2	UCSMP	Parabola	Directrix	8
42	PEP-A	a	Hyperbola	2	UCSMP	Directrix	Parabola	6
43	PEP-A	Hyperbola	b	2	UCSMP	Parabola	Focus	10
44	PEP-A	b	Hyperbola	1	UCSMP	Focus	Parabola	6
45	PEP-A	Hyperbola	e	7	UCSMP	Parabola	Point (parabola)	3
46	PEP-A	e	Hyperbola	4	UCSMP	Point (parabola)	Parabola	1
47	PEP-A	Hyperbola	2a	4	UCSMP	Point (parabola)	Focus	3
48	PEP-A	2a	Hyperbola	5	UCSMP	Focus	Point (parabola)	1
49	PEP-A	Hyperbola	2b	4	UCSMP	Point (parabola)	Directrix	2
50	PEP-A	2b	Hyperbola	2	UCSMP	Directrix	Point (parabola)	1
51	PEP-A	Parabola	Directrix	10	UCSMP	Directrix	Vertex	1
52	PEP-A	Directrix	Parabola	4	UCSMP	Vertex	Directrix	1
53	PEP-A	Parabola	Focus	13				
54	PEP-A	Focus	Parabola	4				
55	PEP-A	Parabola	Point (parabola)	6				
56	PEP-A	Point (parabola)	Parabola	6				
57	PEP-A	Parabola	Axis of symm.	1				
58	PEP-A	Axis of symm.	Parabola	4				
59	PEP-A	QR	Point (QR)	1				
60	PEP-A	Point (QR)	QR	3				

Notes: QR stands for quadratic relation; Axis of symm. stands for axis of symmetry.

In Table 19, both series share nine bidirectional between-concept pairs in quadratic relations. Overall, the PEP-A series employs more distinct pairs (30>26) with more balanced typical and reverse directions than the UCSMP series. For example, the PEP-A series covers connections from circle to center (18 typical and 22 reverse); from circle to radius (16 typical and 15 reverse); from circle to a point on the circle (11 typical and 10 reverse); from ellipse to 2a (3 typical and 4 reverse); and from parabola to a point on the parabola (6 typical and 6 reverse) in a balanced way. In comparison, the UCSMP series highlights more on connections from center to circle (13 typical vs. 5 reverse); from radius to circle (13 typical vs. 6 reverse); from circle to a point on the circle (6 typical vs. 1 reverse); from semicircle to a point on the semicircle (8 typical vs. 1 reverse); from ellipse to x-intercept (9 typical vs. 1 reverse); from ellipse to y-intercept (8 typical vs. 1 reverse); and from parabola to a point on the parabola (3 typical vs. 1 reverse), than the reverse direction.

Additionally, both series include more connections from ellipse to foci (8 typical vs. 3 reverse for PEP-A; 6 typical vs. 2 reverse for UCSMP); from hyperbola to vertices (4 typical vs. 1 reverse for PEP-A; 6 typical vs. 1 reverse for UCSMP); and from parabola to focus (13 typical vs. 4 reverse for PEP-A; 10 typical vs. 6 reverse for UCSMP), than reverse connections. What is more, both series embed connections between hyperbola and foci in a balanced way (8 typical and 6 reverse for PEP-A; 7 typical and 9 reverse for UCSMP). It is surprising that both series stress connections from ellipse or parabola to foci/focus, but integrate hyperbola-to-foci and foci-to-hyperbola connections in a balanced way.

Next, the study moved to the analysis of the overall network of mathematical connections by digraphs and adjacency matrices.

Digraph Analysis for Subtopics

This study then compared digraphs for between-concept connections in seven subtopics. Based on the number of vertices and edges (unique edges and total edges), dense, moderate, sparse, the sparsest, and aggregated digraphs are demonstrated. The following results follow the decreasing order of density, from dense to moderate, sparse, and the sparsest one. The special case of the aggregated digraph is analyzed at the end.

Dense digraphs. Digraphs for the subtopic circle of both series are cases of dense digraphs, which includes more than 35 vertices, 60 unique edges, and 120 total edges. Figure 20 and Figure 21 illustrate the digraphs for between-concept connections in the subtopic circle for the PEP-A series and UCSMP series, respectively.

In Figure 20 and Figure 21, the PEP-A series presents a denser digraph with a similar number of vertices (48 and 49) but more unique edges (96>78) and total edges (240>175) than the UCSMP series. In looking at arrows to a specific vertex, connections ending in circle, center, and radius are emphasized in both series. Additionally, the PEP-A series stresses connections ending in line. Following the edges label (the item number indicating the sequence of textbook problems), both series start with connections between circle and its attributes (radius, center, etc.). Then, the PEP-A series addresses connections between circle and line or point (chord, perpendicular bisector, midpoint, a point on the circle, etc.), and finally between circle and ellipse or hyperbola. In contrast, the UCSMP series stresses connections between special circles (semicircle, interior circle, exterior circle, inner circle, outer circle) and their attributes, and finally between circle and line or unit circle or parabola or ellipse.

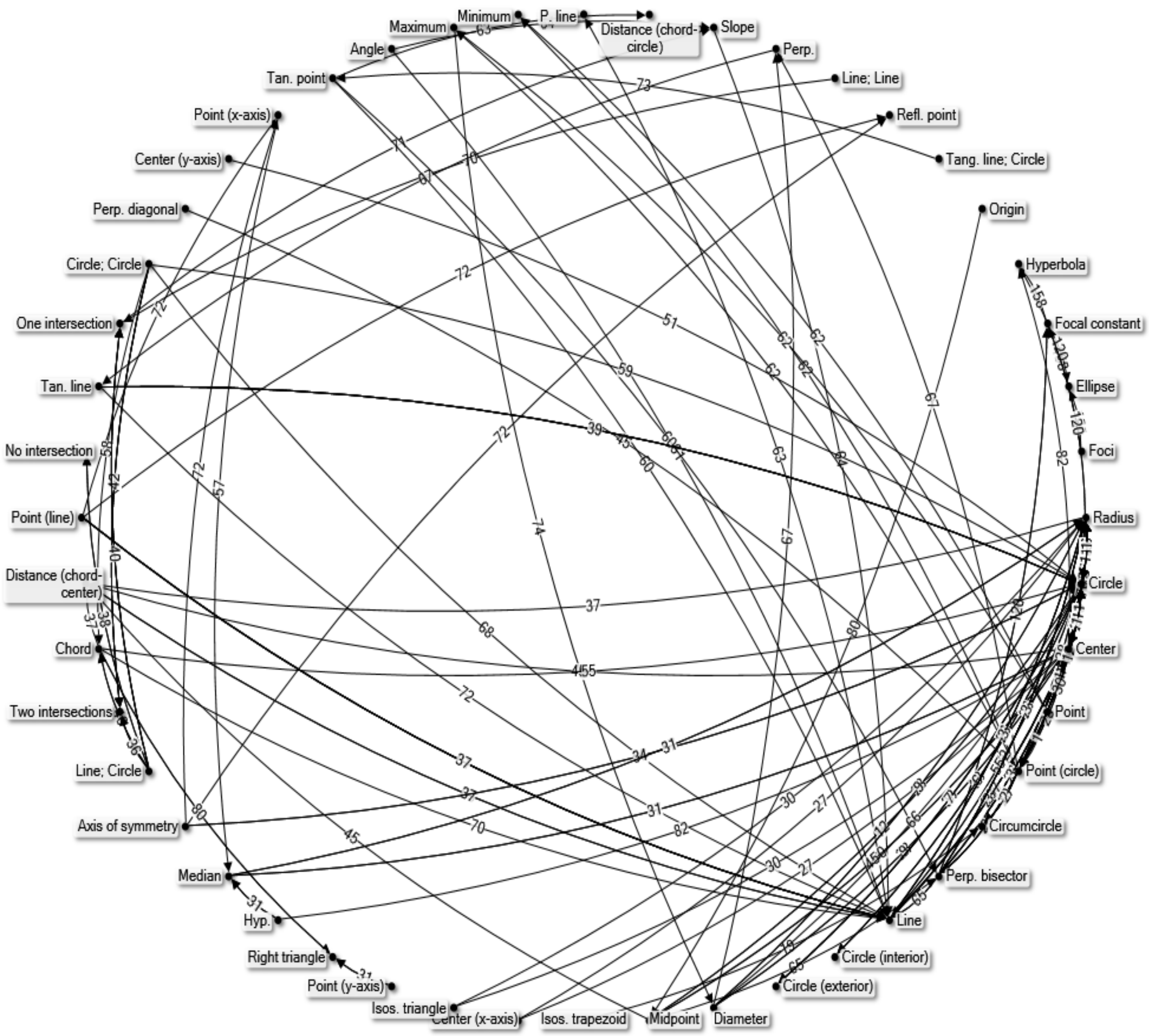


Figure 20. Digraphs for the subtopic circle (PEP-A)

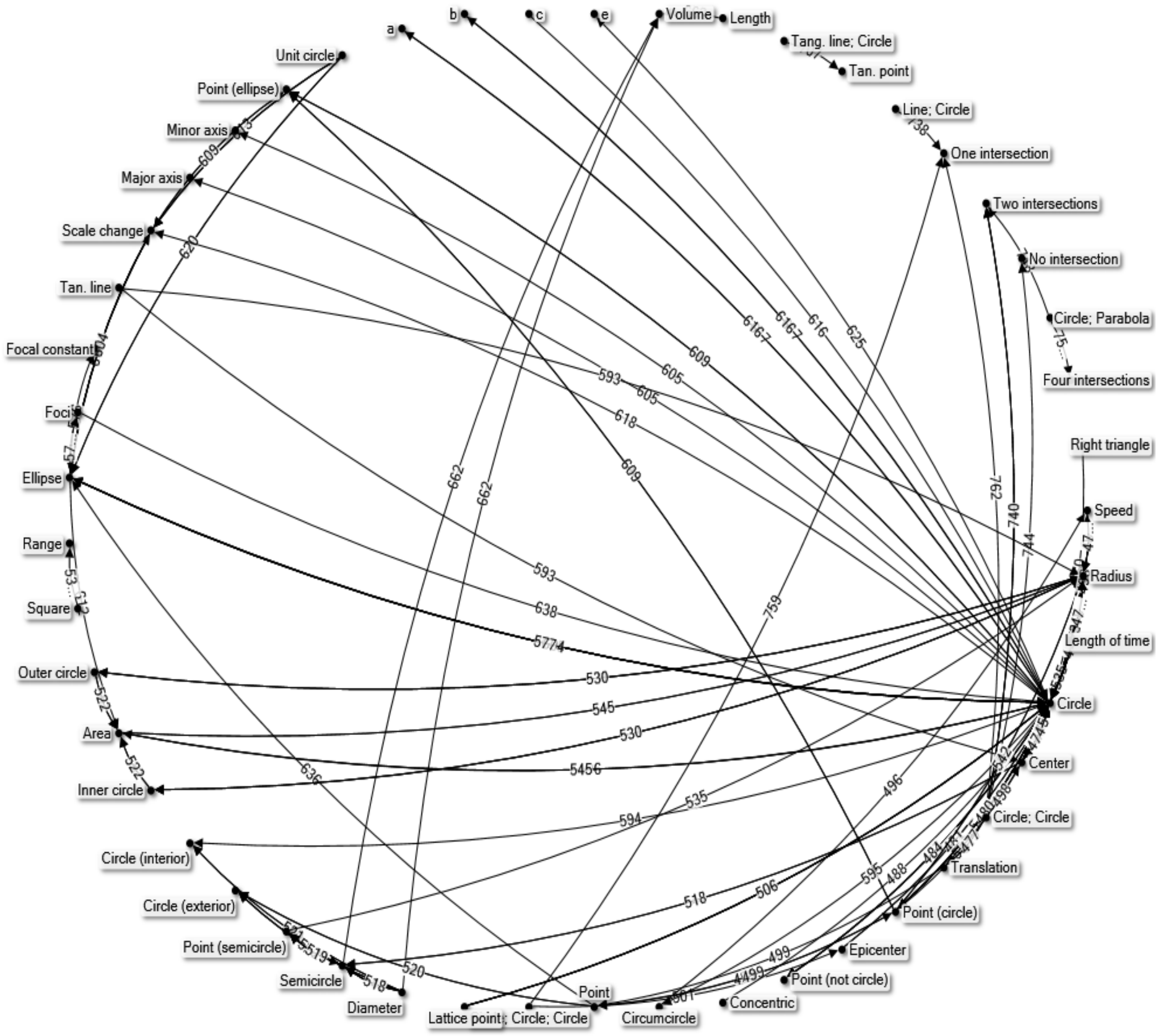


Figure 21. Digraphs for the subtopic circle (UCSMP)

Digraphs for the subtopic ellipse (see Appendix B for full-size digraphs) for both series also belong to dense digraphs. Overall, the PEP-A series shows a denser network with more vertices (46>38), unique edges (71>66), and total edges (135>128) than the UCSMP series. For the density of arrows, the PEP-A series emphasizes bidirectional pairs involving ellipse, whereas the UCSMP series highlights connections starting from ellipse. This is consistent with the previous analysis that the UCSMP series integrates bidirectional pairs with an emphasis on connections starting from ellipse, e.g., from ellipse to x-intercept (9 typical vs. 1 reverse); from ellipse to y-intercept (8 typical vs. 1 reverse); and from ellipse to foci (6 typical vs. 2 reverse). Following the label of edges, the PEP-A series starts with connections between ellipse and its attributes, then between ellipse and line or point, and finally between ellipse and circle or hyperbola. The UCSMP series starts with connections from ellipse to its attributes, then between ellipse and circle, and finally between ellipse and line or hyperbola. This is consistent with the previous analysis that line is in the list of top 5 mathematical concepts and representations in the content of quadratic relations of the PEP-A series, but not in the UCSMP series.

Moderate and sparse digraphs. The density of moderate digraphs is between dense and sparse digraphs, in which sparse digraphs address less than 30 vertices, 40 unique edges, and 100 total edges. Figure 22 presents the digraphs for between-concept connections in the subtopic parabola for the PEP-A (a moderate digraph) and UCSMP series (a sparse digraph).

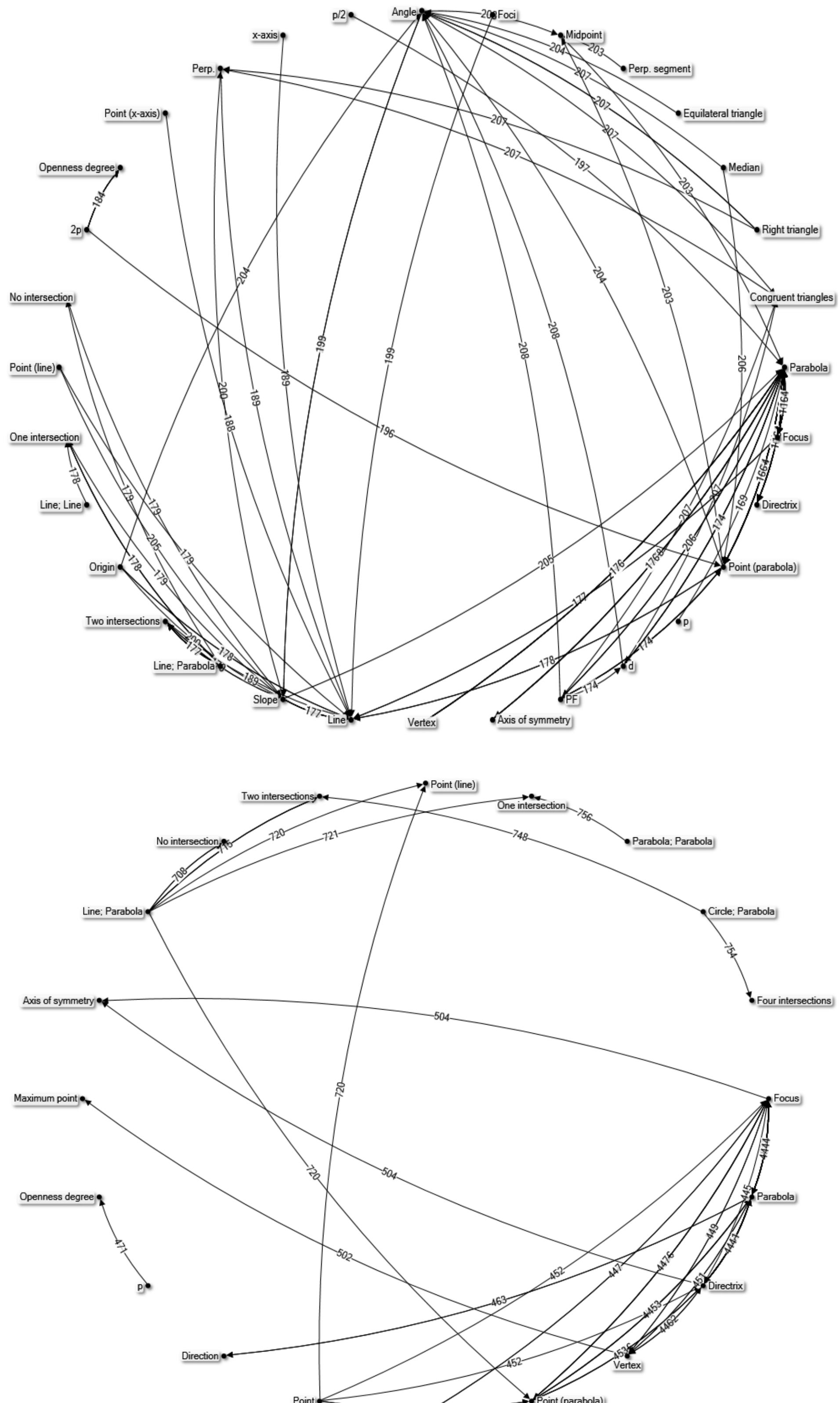


Figure 22. Digraphs for the subtopic parabola (above: PEP-A; below: UCSMP)

In Figure 22, for the subtopic parabola, the PEP-A series covers more vertices (32>20), distinct edges (56>33), and total edges (128>72) than the UCSMP series. Overall, the PEP-A series shows a moderate network, and the UCSMP series shows a sparse network due to fewer vertices and edges. For the density of arrows, both series embed connections involving parabola and a point on the parabola. Additionally, the PEP-A series stresses connections ending in line, angle, and slope, whereas the UCSMP series emphasizes connections involving directrix or vertex and connections ending in focus. Following the edges label, the PEP-A series starts with connections between parabola and its attributes, then between parabola and line or slope or intersection, and finally between parabola and triangle-related concepts. In comparison, the UCSMP series starts with connections between parabola and its attributes, then between attributes of parabola, and finally between parabola and line or circle.

Digraphs for the subtopic hyperbola (see Appendix B for full-size digraphs) for the PEP-A series is a moderate digraph, whereas for the UCSMP series is a sparse digraph. The PEP-A series includes more vertices (42>24), distinct edges (51>30), and total edges (110>82) than the UCSMP series. For the density of arrows, both series address abundant connections involving hyperbola, especially the PEP-A series. Following the edges label, both series begin with connections between hyperbola and its attributes, then between hyperbola and line or intersection point, and finally the PEP-A series has connections between hyperbola and circle or ellipse whereas the UCSMP series has connections between hyperbola and ellipse.

Sparse and the sparsest digraphs. There is an extreme sparse digraph which covers less than 10 vertices, 10 unique edges, and 40 total edges. Figure 23 shows the digraphs for between-concept connections in the subtopic binomial theorem for the PEP-A (the sparsest digraph) and UCSMP series (a sparse digraph).

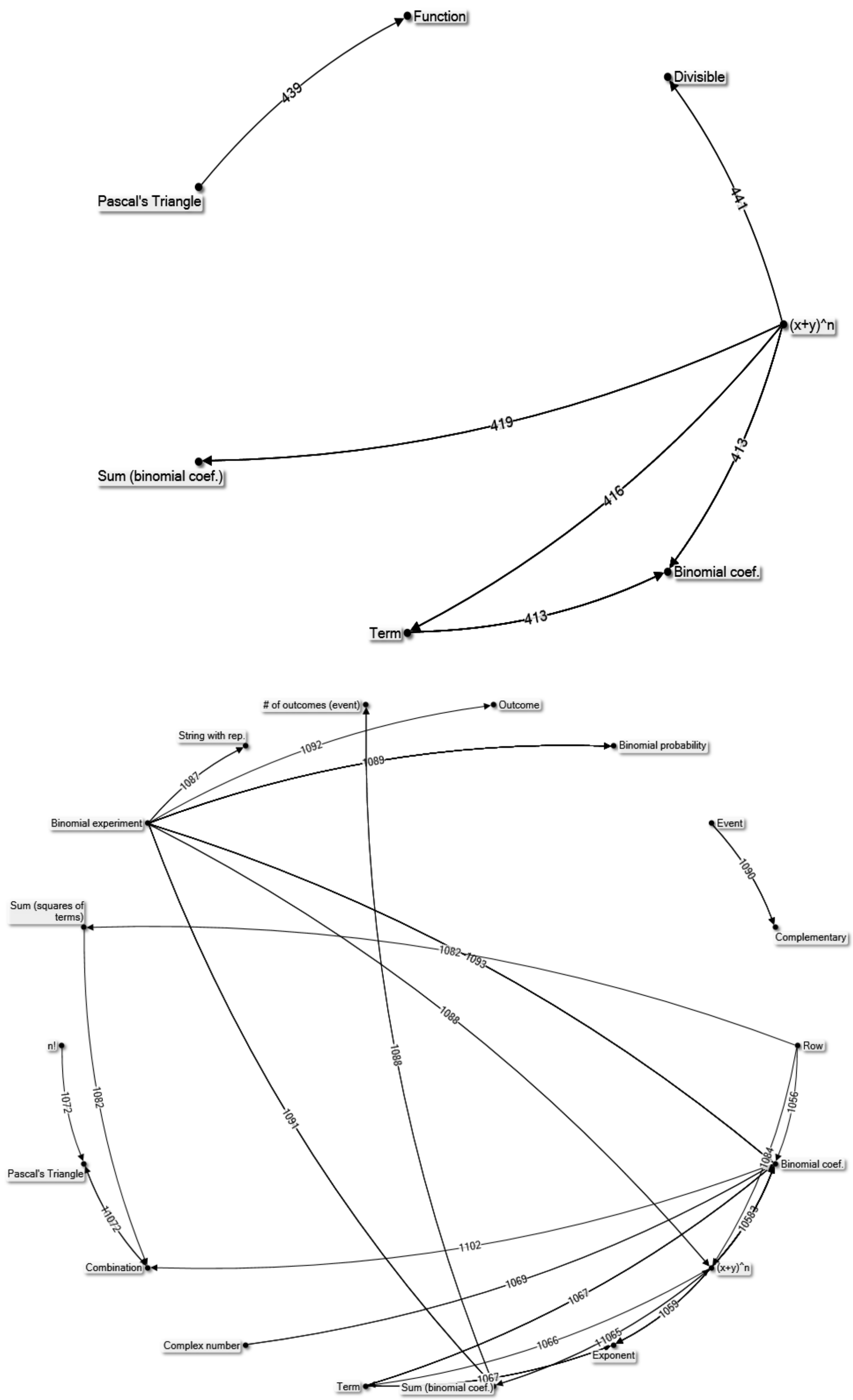


Figure 23. Digraphs for the subtopic binomial theorem (above: PEP-A; below: UCSMP)

In Figure 23, for the subtopic binomial theorem, the digraph for the PEP-A series is the sparsest of all digraphs. The sparse digraph for the UCSMP series shows more vertices (18>7), distinct edges (25>6), and total edges (77>30) than the PEP-A series. By the density of arrows, the PEP-A series stresses connections starting from the nth power of the binomial (x + y). The UCSMP series emphasizes connections ending in binomial coefficient. For the label of edges, the PEP-A series starts with connections from the nth power of the binomial (x + y) to binomial coefficient or term, then connections from term to binomial coefficient, and finally connections from the nth power of the binomial (x + y) to the sum of binomial coefficients. The UCSMP series starts with connections from the nth power of the binomial (x + y) to binomial coefficient and connections from term to exponent, then between Pascal's triangles and combinations, and finally connections from binomial experiment to binomial coefficient or binomial probability.

Digraphs for the subtopic counting problems (see Appendix B for full-size digraphs) are sparse digraphs. The UCSMP series includes more vertices (28>21) and distinct edges (28>21) but fewer total edges (71<77) than the PEP-A series. This indicates that connections in the PEP-A series are more likely with heavy weights. Considering the density of arrows, both series stress connections ending in multiplication counting principle. The PEP-A series shows extra attention to connections ending in probability. Following the edges label, the PEP-A series starts with connections from counting problems to probability, then between event and two basic counting principles, and finally from counting problems to multiplication counting principle. In comparison, the UCSMP series starts with connections between set theory-related concepts and counting problems, then between event and multiplication counting principle, and finally between counting problems and multiplication counting principle.

Special: Aggregated digraphs. Digraphs for the subtopic probability are cases for aggregated digraphs, which include a moderate number of vertices and unique edges but a huge number of total edges. That is, the digraph contains few distinct edges but edges with large weights. The average weight of a unique edge is 4.0 or above. Figure 24 and Figure 25 present digraphs for between-concept connections in the subtopic probability for the PEP-A series and UCSMP series, respectively.

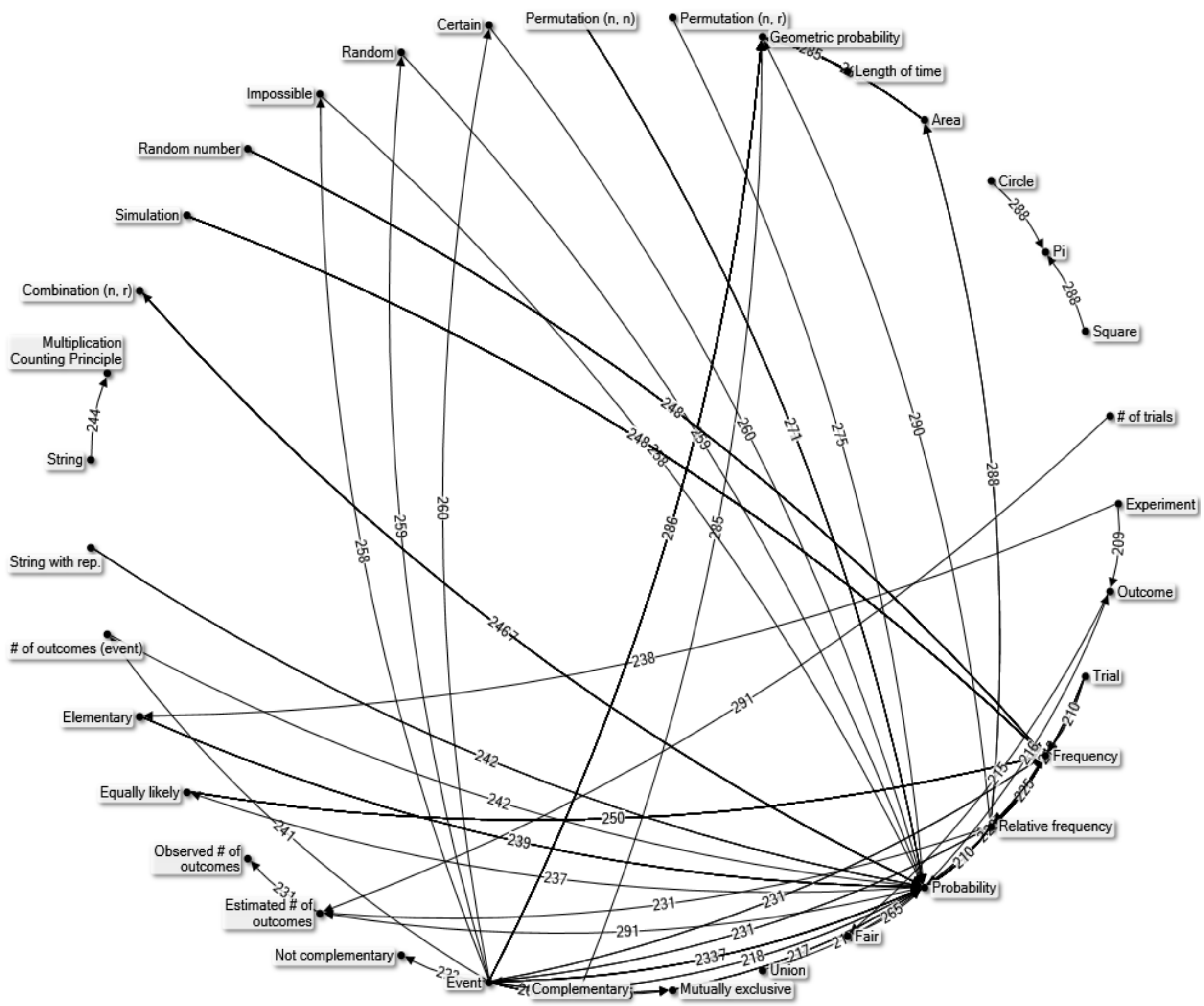


Figure 24. Digraphs for the subtopic probability (PEP-A)

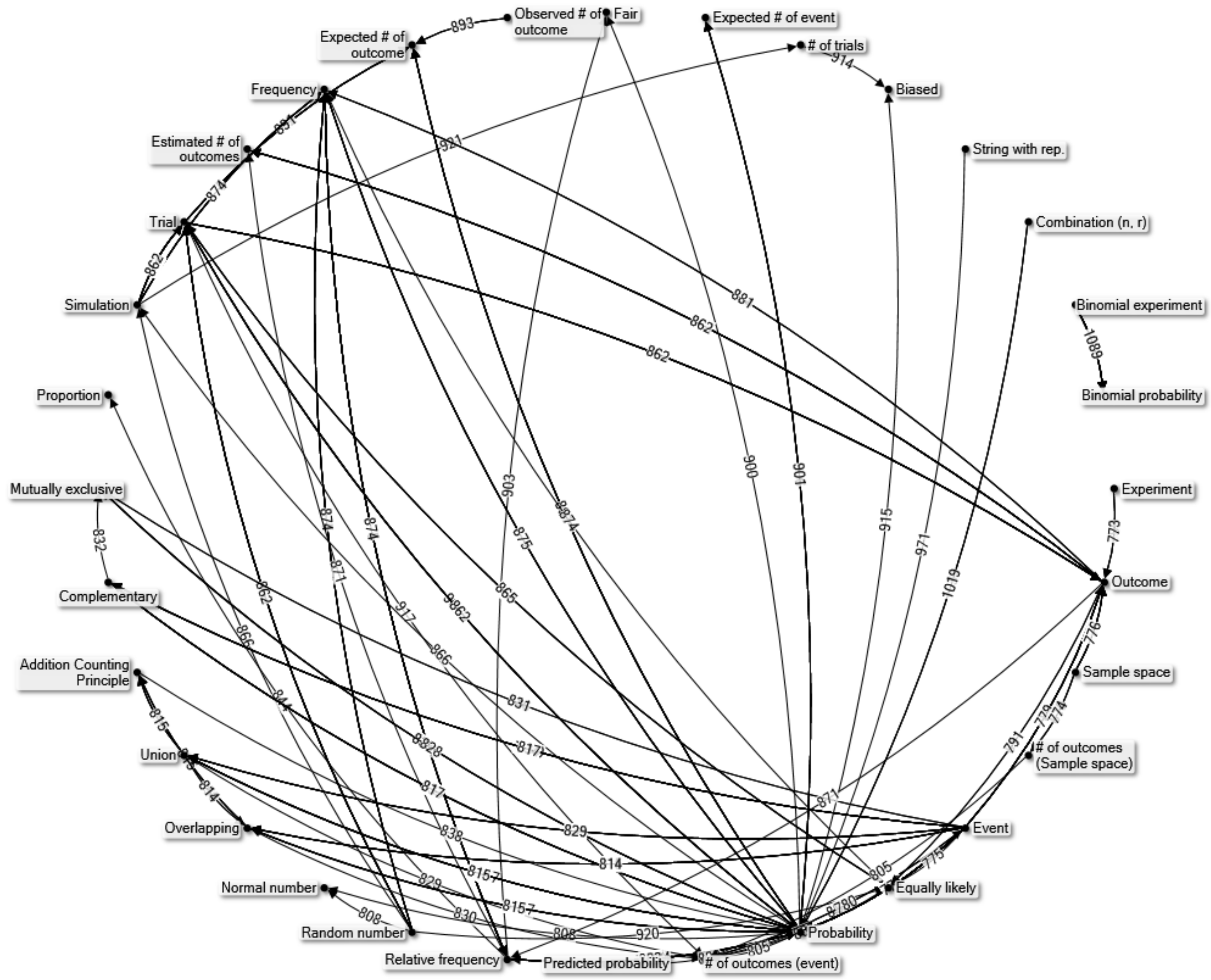


Figure 25. Digraphs for the subtopic probability (UCSMP)

In Figure 24 and Figure 25, for the subtopic probability, the UCSMP series includes a similar number of concepts (32 and 35) with more unique edges (67>53) and total edges (298>209) than the PEP-A series. Overall, the UCSMP series presents a more aggregated digraph of edges with heavy weights than the PEP-A series. The average weight of a unique edge is 4.4 for the UCSMP series and 4.0 for the PEP-A series. For the density of arrows, both series highlight connections involving probability, connections starting from event, and connections ending in frequency. The PEP-A series also stresses connections involving relative frequency and connections ending in geometric models of probability, whereas the UCSMP series highlights connections ending in outcome in the event. Following the edges label, the PEP-A series starts with connections between probability and trial or frequency or relative frequency, then between probability and event or counting problems, and finally between concepts in geometric models of probability. Connections between concepts in geometric models of probability involve the usage of tables and graphs. This is consistent with the previous word frequency analysis that the tabular and graphical representations are stressed in the content of probability and combinatorics of the PEP-A series, but not in the UCSMP series. In contrast, the UCSMP series starts with connections between probability and event or outcome, then between probability and simulation-related concepts or event, and finally between probability and counting problems or binomial experiments.

Digraph Analysis for Topics

This study then analyzed digraphs for two topics across textbooks. Figure 26 presents the thumbnail of digraphs for within-concept connections in quadratic relations for two series (see Appendix B for full-size digraphs).

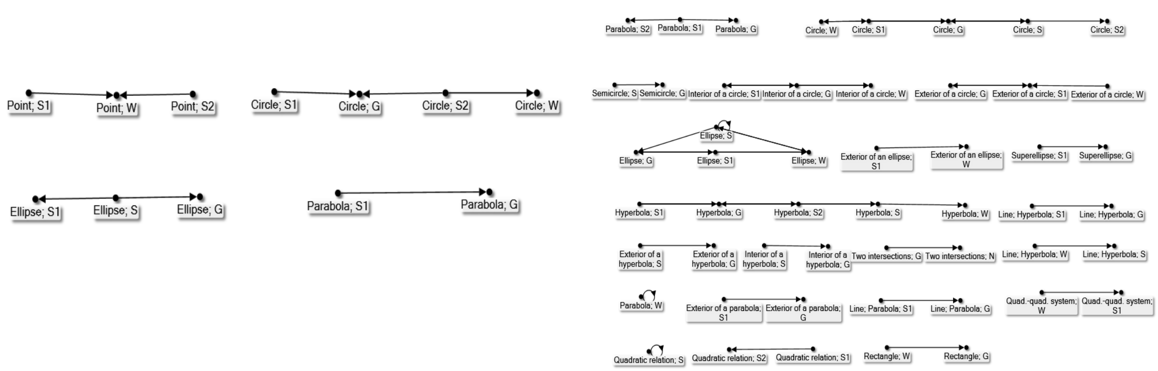


Figure 26. Digraphs for quadratic relations, within-concept connections (left: PEP-A; right: UCSMP)

From Figure 26, for problems dealing with quadratic relations, the UCSMP series shows a denser digraph of within-concept connections with more vertices (51>12), unique edges (44>8), and total edges (79>21) than the PEP-A series. The UCSMP series has unidirectional and bidirectional connections for four subtopics: (a) circle, semicircle, interior or exterior of a circle; (b) ellipse, exterior of an ellipse, and superellipse; (c) hyperbola, interior or exterior of a hyperbola, and line-hyperbola systems; and (d) parabola, exterior of a parabola, and line-parabola systems, as well as self-loops. It is noticeable that the symbolic representation is largely involved and the direction from the symbolic to graphical representation is stressed in the whole network. However, the PEP-A series addresses only unidirectional within-concept connections of point, circle, ellipse, and parabola, mostly in the direction from the symbolic to graphical representation. For between-concept connections (see Appendix B for full-size digraphs), the PEP-A series presents a denser digraph for between-concept connections with more vertices (101>94), unique edges (244>187), and total edges (605>423) than the UCSMP series. Both series place ellipse and circle as two central concepts, and address hyperbola and parabola in descending order. It is consistent with previous digraph analysis for subtopics ellipse and circle are dense digraphs in both series, and digraphs for subtopics hyperbola and parabola are moderate (PEP-A) and sparse (UCSMP). The PEP-A series exhibits extra attention to line.

Problems dealing with probability and combinatorics exhibit different trends of between-concept and within-concept connections (see Appendix B for full-size digraphs). Overall, the digraph of between-concept connections for probability and combinatorics is sparser than that for quadratic relations, which is consistent with previous digraph analysis for subtopics. The

UCSMP series presents a denser digraph of both between-concept and within-concept connections with more vertices (63>50; 46>27), unique edges (115>69; 36>18), and total edges (426>293; 123>89) than the PEP-A series. Probability is the central topic for embedding between-concept connections in both series. The PEP-A series shows extra attention to multiplication counting principle, whereas the UCSMP series highlights trial and binomial coefficient. For within-concept connections, both series address one self-loop of the symbolic representation of n factorial, which is the only bidirectional within-concept connection in the PEP-A series. The UCSMP series addresses unidirectional and bidirectional connections for four types of counting problems: stings with repetition, unordered symbols with repetition, permutations, and combinations, mostly in the direction from the written description to the numerical or the symbolic representation or between two different written descriptions. Additionally, the UCSMP series covers within-concept connections for concepts in probability and binomial theorem. In contrast, the PEP-A series addresses only unidirectional connections for three types of counting problems (except unordered symbols with repetition).

Adjacency Matrix Analysis for Topics

This study then used corresponding adjacency matrix analysis to further explore the directionality of connections from a quantitative perspective. Figure 27 shows the adjacency matrix of within-concept connections in quadratic relations for the PEP-A series and UCSMP series, respectively (see Appendix C for full-size adjacency matrices).

	Point; S1	Point; W	Point; S2	Circle; G	Circle; S1	Circle; S2	Circle; W	Ellipse ; G	Ellipse ; S	Ellipse ; S1	Parab ola; G	Parab ola; S1	Out-degree	Out-Connect ion
Point; S1		1											1	1
Point; W													0	0
Point; S2		1											1	1
Circle; G													0	0
Circle; S1				4									4	1
Circle; S2				4			2						6	2
Circle; W													0	0
Ellipse ; G													0	0
Ellipse ; S								2		1			3	2
Ellipse ; S1													0	0
Parab ola; G													0	0
Parab ola; S1											6		6	1
In-degree	0	2	0	8	0	0	2	2	0	1	6	0	21	
In-connecti on	0	2	0	2	0	0	1	1	0	1	1	0		8

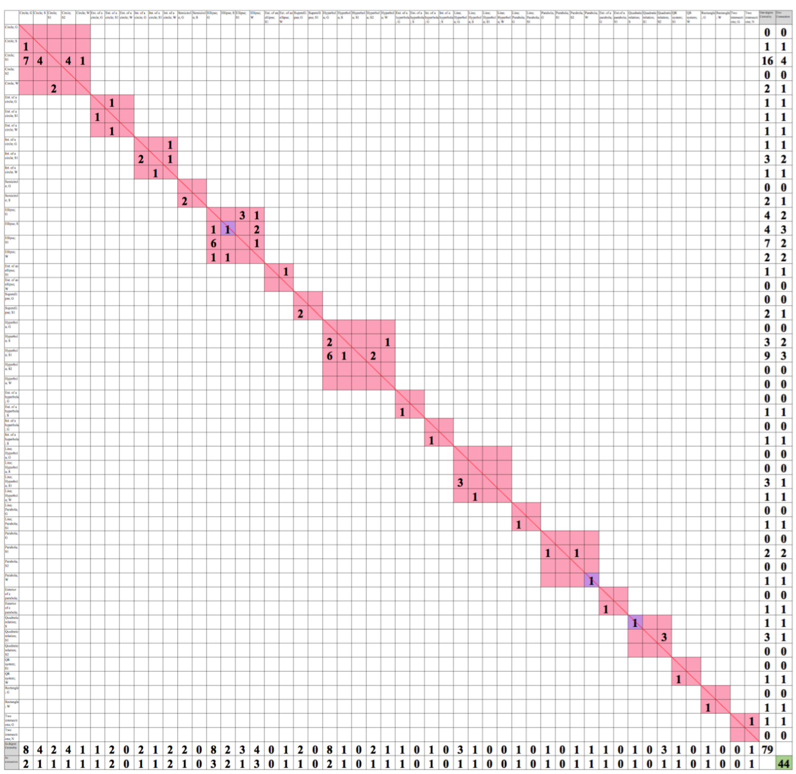


Figure 27. Adjacency matrices for quadratic relations, within-concept connections (above: PEP-A; below: UCSMP)

In Figure 27, for quadratic relations, the UCSMP series exhibits a larger 51*51 matrix than the PEP-A series (12*12). Looking at the on-diagonal block submatrices (highlighted in the pink color), the UCSMP series includes a more even distribution of entries in the submatrices than the PEP-A series, which covers large entries mostly in the left-down part of the submatrices. This suggests that within-concept connections are largely unidirectional in the PEP-A series, and more bidirectional pairs appear in the UCSMP series. For the diagonal, the PEP-A series has no non-zero entry, and the UCSMP series has three non-zero entries (highlighted in the purple color): (a) ellipse within the symbolic representation, (b) quadratic relation within the symbolic representation, and (c) parabola within the written description.

Looking at each vertex for the in-degree or out-degree centrality, in the PEP-A series, circle and parabola in the graphical representation (8; 6) are the top vertices, which have the most connections leading to. Circle in the standard symbolic form for a quadratic equation and parabola in the standard symbolic form (6; 6) are the top vertices, which have the most connections leading out of. This indicates that most within-concept connections in the PEP-A series may be circle or parabola from the symbolic to graphical representation. In the UCSMP series, circle, ellipse, and hyperbola in the graphical representation (8; 8; 8) are the top vertices, which have the most within-concept connections leading to. Circle, hyperbola, and ellipse in the standard symbolic representation (16; 9; 7) are the top vertices, which have the most within-concept connections leading out of. Similarly, several within-concept connections in the UCSMP series may be circle, ellipse, and hyperbola from the symbolic to graphical representation. For in-connection and out-connection, most vertices in both series share values of 0 to 3, except for the out-connection of circle in the standard symbolic representation (4) in the UCSMP series.

Then, the adjacency matrix of within-concept connections in probability and combinatorics for the PEP-A and UCSMP series (see Appendix C for full-size adjacency matrices) are compared. The UCSMP series is a larger 46*46 matrix than the PEP-A series (27*27). Looking at the on-diagonal block submatrices (highlighted in the yellow color), the UCSMP series shows a more even distribution of entries than the PEP-A series, which has large entries mostly in the left-down part of the submatrices. This suggests that the UCSMP series embeds more bidirectional connections than the PEP-A series. For the diagonal, both series share one entry (highlighted in the purple color): one self-loop of n factorial within the symbolic representation.

Looking at the in-degree and out-degree of each vertex, in the PEP-A series, combination (n, r), permutation (n, r), permutation (n, n) in the numerical representation, and the nth power of the binomial (x + y) in the binomial expansion symbolic representation (35; 11; 10; 10) are the top vertices, which have the most connections leading to. Combination (n, r), permutation (n, r), permutation (n, n) in the written description (real-world context), and the nth power of the binomial (x + y) in the original symbolic expression (36; 12; 10; 10) are the top vertices, which have the most connections leading out of. This indicates that several within-concept connections in the PEP-A series are combinations or permutations from the written description (real-world context) to the numerical representation, and connections of the nth power of the binomial (x + y) within the symbolic representation. In the UCSMP series, the top four vertices for the in-degree centrality are combination (n, r), permutation (n, r), string with repetition, and permutation in the numerical representation (21; 16; 12; 12). The top four vertices for the out-degree centrality are the written description (real-world) of string with repetition, combination (n, r), permutation, and permutation (n, n) (16; 14; 14; 12). Similarly, the UCSMP series emphasizes unidirectional connections of different counting problems from the written description (real-world context) to the numerical representation. For in-connection and out-connection, each vertex has a value of 0 to 3, except for the out-connection of combination (n, r) in the symbolic representation (4) in the UCSMP series. Table 20 lists the indices calculated from adjacency matrices for within-concept connections.

Table 20. *Adjacency Matrices Indices for Within-concept Connections*

Textbook & Topic	Reciprocated Vertex Pair Ratio	Self-loop	Bidirectional Pair	Reciprocated Edge Ratio
PEP-A-QR	0	0	0	0
UCSMP-QR	0.1714	3	6	0.2927
PEP-A-PC	0	1	0	0
UCSMP-PC	0.0938	1	3	0.1714

Notes: QR stands for quadratic relations; PC stands for probability and combinatorics.

As shown in Table 20, except for one self-loop in probability and combinatorics, there are no bidirectional within-concept connections in the PEP-A series. The UCSMP series exhibits a low number of bidirectional within-concept connections, especially in probability and combinatorics. Ratios for reciprocated vertex pair and edge are still low. This indicates that most within-concept connections in probability and combinatorics are still unidirectional.

Then, the analysis moves to between-concept connections. For quadratic relations, the corresponding adjacency matrix for the PEP-A series is 101*101, which is larger than the 94*94 matrix for the UCSMP series (see Appendix C for full-size adjacency matrices). Looking at the non-zero entries (highlighted in the light blue color) above and below the diagonal, the PEP-A series has more symmetrical entries than the UCSMP series. Particular rows and columns have more non-zero entries than other rows or columns. This reveals that particular concepts are dominating the network of connections in quadratic relations.

For in-degree and out-degree centrality of vertices, in the PEP-A series, circle (64:53), ellipse (51:38), hyperbola (46:36), line (42:16), center (39:28), radius (35:19), and parabola (28:37) are the top vertices, which have more than 30 connections leading to or leading out of. In the UCSMP series, circle (46:38), ellipse (21:61), hyperbola (25:32), radius (22:19), and parabola (13:25) are the top vertices, which have more than 20 connections leading to or leading out of. It reveals that circle is a central concept involved in abundant connections for both series. Connections leading to line are stronger in the PEP-A series and connections leading out of ellipse are stronger in the UCSMP series, than the reverse direction.

For in-connection and out-connection connectivity, the PEP-A series stresses ellipse (21:15), line (20:8), hyperbola (19:9), circle (12:6), and center (12:6) as the top vertices, which have more than 10 distinct connections leading to or leading out of. Even though circle is involved in more connections than ellipse, line, and hyperbola, the diversity of connections involving circle is limited. This indicates that connections involving circle may have large weights. What is more, ellipse, hyperbola, and line frequently appear as the starting or ending vertex of distinct between-concept connections in the PEP-A series. In particular, there are more distinct connections leading to line than leading out of line. In comparison, the UCSMP series stresses circle (14:15) and ellipse (9:21), which have more than 10 distinct between-connections leading to or leading out of. This reveals that many distinct connections in the UCSMP series may involve circle or start from ellipse.

For probability and combinatorics, the matrix for the PEP-A series is 50*50, which is smaller than the 63*63 matrix for the UCSMP series (see Appendix C for full-size adjacency matrices). These matrices are smaller than that for quadratic relations, which is consistent with previous digraph analysis that digraphs for subtopics in quadratic relations are denser than digraphs for probability and combinatorics. Comparing the non-zero entries (highlighted in the grey color) above-diagonal and below-diagonal, the UCSMP series includes more entries in the symmetrical position than the PEP-A series. It suggests that the UCSMP series embeds more bidirectional connections in probability and combinatorics than the PEP-A series. Also, both series have many large entries. It indicates that the weights of particular connections are large.

For in-degree and out-degree centrality, probability (65:7), frequency (48:26), event (1:42), geometric models of probability (35:0), multiplication counting principle (34:1), and relative frequency (31:30) are the top vertices in the PEP-A series, which have more than 30 between-concept connections leading to or leading out of. In the UCSMP series, probability (77:38), event (1:71), trial (34:21), and outcome (32:17) are the top vertices, which have more than 30 between-concept connections leading to or leading out of. This indicates that both series emphasize between-concept connections starting from event or ending in probability. The PEP-A series includes several connections ending in geometric models of probability and multiplication counting principle, as well as many bidirectional connections between frequency and relative frequency. The UCSMP series contains many connections involving trial and outcome. For in-connection and out-connection connectivity, both series stress probability (15:7 for PEP-A; 10:14 for UCSMP), event (1:11 for PEP-A; 1:8 for UCSMP), and multiplication counting principle (7:1 for PEP-A; 7:3 for UCSMP) as the top vertices, which have the most distinct between-concept connections leading to or leading out of. This indicates that probability is the core concept as the starting or ending vertex of many distinct connections. Additionally, both series show many distinct connections starting from event and ending in multiplication counting principle. Compared to connectivity values of vertices in quadratic relations, concepts in probability and combinatorics have smaller in-connection and out-connection values, which suggests the limited diversity of unique connections in probability and combinatorics. It is consistent with the previous analysis that the digraph for probability is aggregated. Table 21 summarizes the indices calculated from adjacency matrices for between-concept connections.

Table 21. *Adjacency Matrices Indices for Between-concept Connections*

Textbook & Topic	Reciprocated Vertex Pair Ratio	Bidirectional Pairs	Reciprocated Edge Ratio
PEP-A (Quadratic Relations)	0.1402	30	0.2459
UCSMP (Quadratic Relations)	0.1615	26	0.2781
PEP-A (Probability and Combinatorics)	0.0615	4	0.1159
UCSMP (Probability and Combinatorics)	0.0952	10	0.1739

As shown in Table 21, the PEP-A series embeds more bidirectional between-concept pairs than the UCSMP series (30>26) in quadratic relations, whereas the UCSMP series shows a higher reciprocated vertex pair ratio (0.0952>0.0615; 0.1615>0.1402) and reciprocated edge ratio (0.1739>0.1159; 0.2781>0.2459) than the PEP-A series in two topics. Both series exhibit richer bidirectional between-concept connections in problems dealing with quadratic relations than problems dealing with probability and combinatorics.

Summary

Similarities and differences in the directionality of mathematical connections appeared in problems from popular U.S. and Chinese high school mathematics textbooks. Still, unidirectional connections account for a large part in both series. The UCSMP series shows a higher ratio of bidirectional to unidirectional connections than the PEP-A series across topics, especially for within-concept connections. More bidirectional connections are identified in quadratic relations and between-concept connections. For bidirectional pairs, the UCSMP series integrates more unique bidirectional within-concept connections in two topics than the PEP-A series, but shows more typical than reverse connections. Connections involving specific concepts, such as ellipse, circle, and the nth power of the binomial (x + y), as well as more connections starting from the symbolic representation than the reverse direction are observed. For bidirectional between-concept pairs, both series stress connections ending in probability or multiplication counting principle. In quadratic relations, the PEP-A series embeds more distinct bidirectional between-concept pairs with more balanced typical and reverse directions than the UCSMP series. In probability and combinatorics, the UCSMP series integrates more distinct bidirectional between-concept pairs than the PEP-A series, but still in an unbalanced way.

Consider the network of between-concept connections for each subtopic, both series stress more on circle, ellipse (dense digraph), and probability (aggregated digraph) than the rest subtopics. Overall, the PEP-A series shows a denser digraph of between-concept connections for four subtopics of quadratic relations than the UCSMP series. In contrast, the UCSMP series presents denser digraphs of between-concept connections for three subtopics of probability and combinatorics than the PEP-A series. The density of arrows indicates that the PEP-A series highlights connections ending in line, angle, slope, whereas the UCSMP series stresses connections starting from ellipse and ending in focus, the number of outcomes in the event, and binomial coefficient. The flow of connections suggests that the PEP-A series shows extra attention to connections between subtopics of quadratic relations and point or line, whereas the UCSMP series stresses more on connections between subtopics of quadratic relations. The PEP-A series highlights connections between concepts in geometric models of probability, whereas the UCSMP series pays attention to connections between probability and binomial theorem or binomial experiment.

The network of between-concept and within-concept connections for each topic follows the trends exhibited in digraphs for seven subtopics. From the digraph analysis and the adjacency matrix analysis, the UCSMP series demonstrates a better condition in more distinct and total unidirectional and bidirectional within-concept connections in two topics than the PEP-A series (limited and almost unidirectional except for one self-loop). Both series stress within-concept connections of the symbolic-to-graphical representation of quadratic relations and the written description-to-numerical representation of counting problems, and share one same self-loop. The PEP-A series shows a denser network of between-concept connections in quadratic relations with balanced typical and reverse directions than the UCSMP series. In probability and combinatorics, the UCSMP series presents more distinct and total between-concept connections than the PEP-A series, but in unbalanced typical and reverse directions. Both series highlight connections involving circle, probability, and connections starting from event. Additionally, the PEP-A series stresses connections leading to line or geometric models of probability, whereas the UCSMP series stresses connections involving trial and outcome, and leading out of ellipse.

Research Question 3

The Placement of Mathematical Subtopics

Similarities and differences in the placement of subtopics may influence the directionality of mathematical connections.

For probability and combinatorics, both series follow the order of probability, counting problems (first permutation and then combination), and binomial theorem. Previous analysis indicated that both series stress probability (aggregated digraph) the most, then counting problems (sparse digraph), and the binomial theorem (sparsest digraph) the least. More connections in permutation than combination are also identified. This consistency supports the conjecture that the emphasized direction is consistent with the sequence of subtopics in textbooks. Emphasized connections are mostly involved with concepts that appeared early in the sequence. Consider the underlying reasons, textbooks tend to extend prior knowledge to new concepts when presenting a new concept. Prior concepts are enhanced several times when introducing new knowledge. Concurrently, textbook problems follow a similar trend. It explains why the direction of strong connections (edges with large weight) is consistent with the direction that the curricular proceed.

By contrast, in quadratic relations, two series have different placement of subtopics. The PEP-A series addresses circle in one chapter for Grade 10 and places ellipse, hyperbola, and parabola in another chapter for Grade 11. Due to the similar number of pages of one chapter, circle is the most emphasized subtopic in the PEP-A series. On the contrary, the UCSMP series places all subtopics in the order of parabola, circle, ellipse, and hyperbola in one chapter. Four subtopics are more equally emphasized in the UCSMP series. For both series, linear functions and quadratic relations are two topics placed far away in both textbooks. By the connection analysis, the PEP-A series exhibits more attention to connections involving the subtopic circle with large weights. The UCSMP series emphasizes more on connections between quadratic relations-related subtopics, whereas the PEP-A series stresses more on connections between quadratic relations-related subtopics and linear functions-related subtopics. The separation of subtopics in two chapters or textbooks or grade levels may weaken bidirectional connections between these subtopics. For example, the UCSMP series shows limited connections between linear function-related concepts and quadratic relations-related concepts. In comparison, the PEP-A series still pays attention to the transition from linear functions to quadratic relations, as well as connections from quadratic relations to linear functions, even though they are placed far away in the curriculum. Son and Hu (2016) indicated that Chinese secondary school mathematics textbooks tended to give coherence in the organization of concepts across the curriculum. The culture of emphasizing the coherence of curriculum contributes to more connections between concepts that are placed far away. This indicates that the coherence and connectivity of the whole curriculum can be strengthened by providing more bidirectional connections between the prior and new knowledge that connected in nature.

What is more, both series proceed in the algebraic-to-graphical direction for quadratic relations. At the same time, both series include few within-concept connections from the graphical to symbolic representation. This is consistent with previous studies that emphasized the curricular track, such as in Arithmetic-to-Algebra (Flanders, 1994; Lee & Wheeler, 1989) and in algebraic-to-graphical (Knuth, 2000b), which might lead to an emphasis on connections in a particular direction and thus hinder bidirectional connections.

In sum, all the above supports the conjecture that the placement of subtopics may contribute to the relative strength of typical and reverse connections. The mathematical concepts and representations that appeared early in the curriculum seem to be stressed more than ones appearing later, which contributes to unbalanced typical and reverse connections. This suggests that connections from new concepts and representations to old ones could be strengthened to support bidirectional connections. Even though some subtopics may be placed far apart in the whole curriculum, intentional bidirectional connections between these subtopics are viable to enhance the connectivity of mathematics textbooks and the whole curriculum.

Unique Practices in the PEP-A Series

The PEP-A series owns two specific practices that may influence learning opportunities for bidirectional connections. The first practice is interleaved example-exercise pairs. In an examination of approaches and practices in developing mathematics textbooks in China, Li, Zhang, and Ma (2019) suggested that the major design principle in selecting and arranging textbook problems is to match exercises with given worked-out examples. For each subtopic, the PEP-A series places all textbook problems in the layout: Interleaved Set 1 (worked-out examples-to-in-class exercises), Interleaved Set 2, …, After-class exercises (set A and set B). For example, for the subtopic ellipse (see Figure 28), the PEP-A series addresses example 1, 2, and 3, then a set of in-class exercises, next example 4, 5, and 6, then a set of in-class exercises, and finally the after-class exercise set A and set B.

Figure 28. Layout of textbook problems for the subtopic ellipse (PEP-A)

In contrast, the UCSMP series owns the layout of example-to-exercise. For example (see Figure 29), the UCSMP series follows the order of example 1, 2, and 3, and finally, a set of exercises.

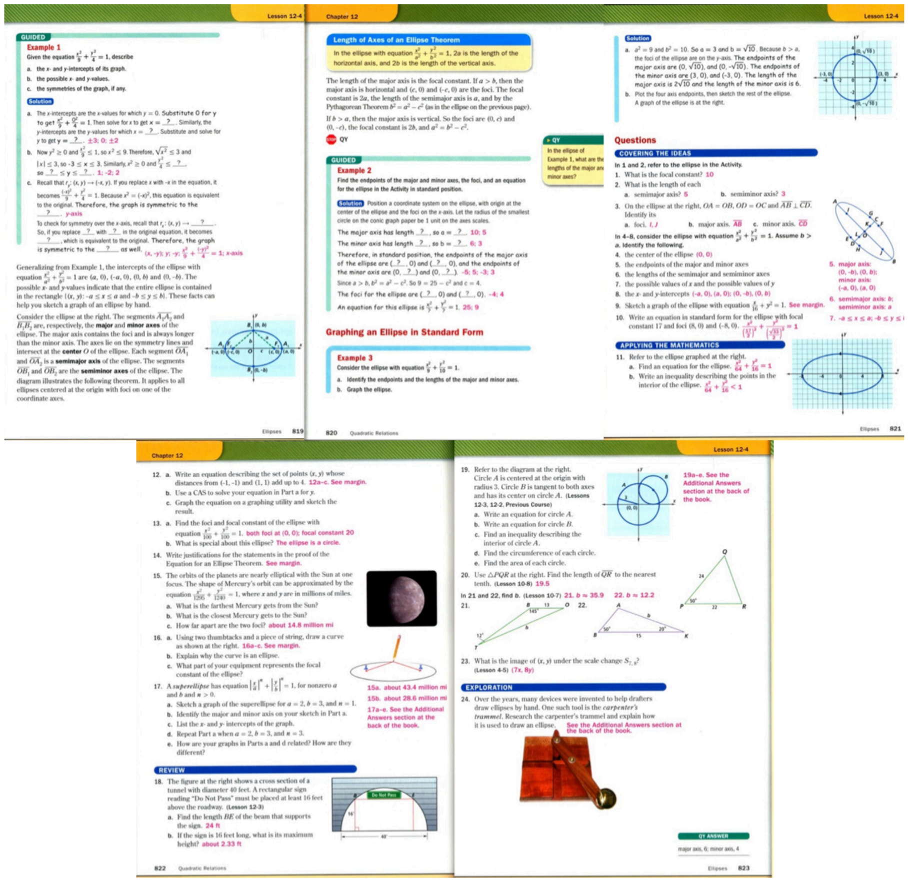


Figure 29. Layout of textbook problems for the subtopic ellipse (UCSMP)

As discussed before, interleaved examples and exercises may help students attain better transfer performance. Even though ratios of worked-out examples to exercises are similar across topics and textbooks, as indicated before, the interleaved example-exercise pairs in the PEP-A series may provide students with more cognitive support in attaining connections than the UCSMP series.

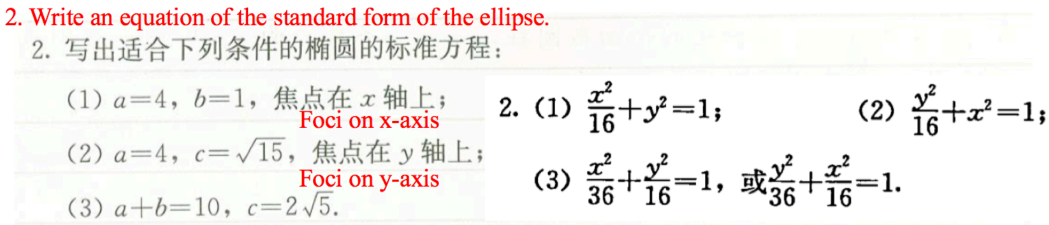
The second practice is an indigenous practice, called *Bianshi* problems (where *Bian* stands for "changing" and *shi* means "form" in Chinese, can be translated as "variation" in English) (Sun, 2011a). *Bianshi* problems are defined as a group of mathematical isomorphic problems by changing the conditions, conclusions, or deduction process of the example problem, which facilitates connections by adding proper variations, discerning and comparing the invariant essence, and prompting representational transitions step by step (Sun, 2011b; Sun, Wong, & Lam, 2005). For example, Table 22 illustrates an example of *Bianshi* problems.

Table 22. *An Example of* Bianshi *Problems in Factorization of Polynomial* (Sun et al., 2005)

Level 1: Given: $x^2+5x+6=(x+m)(x+n)$; $m+n=5$, $m*n=6$.	
Bianshi: Find the possible value of a and b that the polynomial can be factored.	
Level 2	x^2+ax+6
Level 3	x^2+5x+b
Level 4	x^2+ax+b
Level 5	x^3+ax+b
Level 6	x^n+ax+b

Prior studies have suggested that *Bianshi* problems might provide cognitive supports for connections. *Bianshi* emphasizes “general relationship” rather than “one-thing-at-the-time” design (Sun, 2011b). The “one-thing-at-the-time” design may miss the chance of discerning critical aspects between two or more topics. While in “general relationship” design, connections are created in comparing the invariant feature since comparisons seem to be the pre-condition to perceive the structure, dependencies, and relations (Sun, 2011a, 2011b). *Bianshi* problems not only draw learners into a “space of relations,” but also may work as an exemplar to increase variability in worked-out examples, exercises, and example-exercise pairs.

Researchers have indicated that Chinese elementary school mathematics textbooks utilize *Bianshi* problems to support bidirectional connections in numerous topics, such as addition and subtraction (Sun, 2011b), the distributive property (Ding & Li, 2010), and multiplication and division (Xin et al., 2011). This study suggested that the Chinese high school mathematics textbook series, the PEP-A series, also adopts *Bianshi* problems to support bidirectional between-concept connections in quadratic relations. For example, Figure 30 shows a *Bianshi* problem sample in which step-by-step variations promote bidirectional connections.


2. Write an equation of the standard form of the ellipse.

2. 写出适合下列条件的椭圆的标准方程：

(1) $a=4$，$b=1$，焦点在 x 轴上；
Foci on x-axis

(2) $a=4$，$c=\sqrt{15}$，焦点在 y 轴上；
Foci on y-axis

(3) $a+b=10$，$c=2\sqrt{5}$.

2. (1) $\frac{x^2}{16}+y^2=1$； (2) $\frac{y^2}{16}+x^2=1$；

(3) $\frac{x^2}{36}+\frac{y^2}{16}=1$，或$\frac{y^2}{36}+\frac{x^2}{16}=1$.

Figure 30. Bianshi problem sample and solutions (PEP-A-E2.1, p. 42)

As can be seen from Figure 30, exercise 2(1) involves connections from a, b, and foci on x-axis to ellipse; exercise 2(2) covers one more connection from a and c to b, then from a, b, and foci on y-axis to ellipse; and exercise 2(3) is a problem with multiple solutions, which is more complex than the previous two problems. In the last exercise, the first step is to get a and b calculated from (a+b) and c. The second step is to consider whether the foci of ellipse is on the x-axis and y-axis and then apply the appropriate formula. Exercise 2(3) relies on the knowledge students gain from the first two exercises 2(1) and 2(2). The sample *Bianshi* problem contains flexible bidirectional connections among ellipse, a, b, c, foci on x-axis, and y-axis (see Table 23).

Table 23. *Flexible Uses of Connections in* Bianshi *Problem Sample*

Exercise	Given Information (Source)	Need to Know (Target)
2 (1)	a, b, Foci on x-axis	Standard equation of the ellipse
2 (2)	a, c, Foci on y-axis	b, Standard equation of the ellipse
2 (3)	a+b, c	a, b, Foci on x-axis, Foci on y-axis, Standard equation of the ellipse

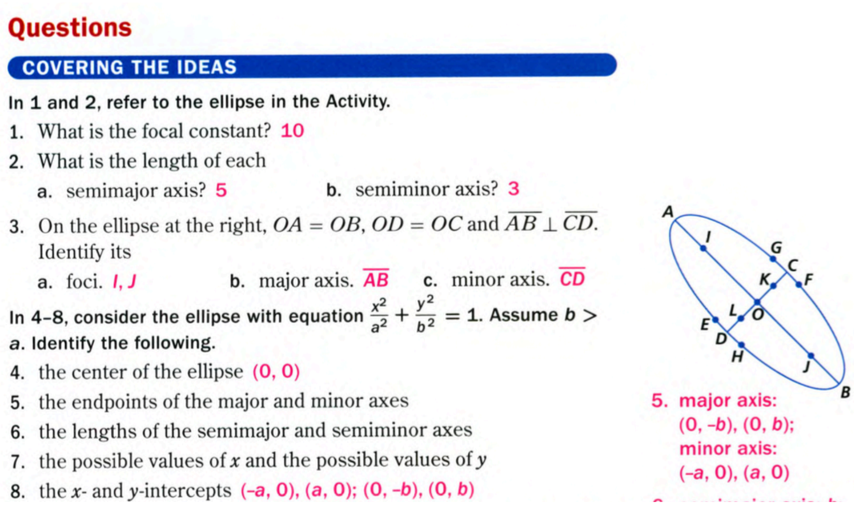
The *Bianshi* practice brings step-by-step variation, which makes problems more complex with more concepts. At the same time, the step-by-step variation provides proper cognitive supports to solve multi-step problems. Students gradually build diverse bidirectional connections in the problem-solving process. Overall, 82.26% of *Bianshi* problems dealing with quadratic relations include bidirectional between-concept connections. It supports the conjecture that *Bianshi* problems may promote diverse bidirectional between-concept connections.

Unique Practices in the UCSMP Series

The UCSMP series owns two particular practices that may influence learning opportunities for bidirectional connections.

The first practice is four different sections in the exercises: (a) covering the ideas (40%~ 50% of total), (b) applying the mathematics (20%~30% of total), (c) review (15%~25% of total), and (d) exploration (less than 5% of total) (see Figure 29). The first part, covering the ideas, usually contains more single-step problems with repetition. The second part, applying the mathematics, emphasizes real-life context problems. The third part, review, includes questions related either to previous content in the same chapter or to earlier chapters. Only problems related to quadratic relations and probability and combinatorics are included in this study.

Several previous studies reported that many U.S. elementary school and middle school mathematics textbooks had a great deal of repetition (e.g., Alajmi, 2012; Pickle, 2012). This study identified consistent results that the UCSMP series exhibits many single-step problems without much variation, especially in the first section of exercises. For example, Figure 31 presents sample exercises for the subtopic ellipse in the UCSMP series.



Questions

COVERING THE IDEAS

In 1 and 2, refer to the ellipse in the Activity.

1. What is the focal constant? 10
2. What is the length of each
   a. semimajor axis? 5 b. semiminor axis? 3
3. On the ellipse at the right, $OA = OB$, $OD = OC$ and $\overline{AB} \perp \overline{CD}$. Identify its
   a. foci. I, J b. major axis. $\overline{AB}$ c. minor axis. $\overline{CD}$

In 4–8, consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Assume $b > a$. Identify the following.

4. the center of the ellipse (0, 0)
5. the endpoints of the major and minor axes
6. the lengths of the semimajor and semiminor axes
7. the possible values of x and the possible values of y
8. the x- and y-intercepts $(-a, 0)$, $(a, 0)$; $(0, -b)$, $(0, b)$

Figure 31. Sample exercises in the subtopic ellipse (from UCSMP-AA, p. 821)

All these single-step exercises in Figure 31 include only unidirectional connections from ellipse to its attribute (e.g., focal constant, foci, x-intercept, y-intercept). By the connection analysis, this study reported that the UCSMP series stressed connections starting from ellipse, such as from ellipse to x-intercept (9 typical vs. 1 reverse); from ellipse to y-intercept (8 typical vs. 1 reverse); and from ellipse to foci (6 typical vs. 2 reverse). Also, the in-degree and out-degree analysis showed that the UCSMP series had 21 connections leading to ellipse and 61 connections leading out of ellipse. This supports the conjecture that repetition of simple single-step problems may shift the balance of typical and reverse connections.

Additionally, the second part stresses the real-world application of mathematics. For the problem feature analysis, the UCSMP series exhibits a higher ratio of real-life to purely mathematical problems than the PEP-A series. This suggests that a separate section for real-life applications may be helpful to fulfill the reform call for real-life context problems. Furthermore, the third part includes a set of problems involving prior knowledge only, which may increase the weights of connections involving prior concepts and representations. If the review section addresses prior knowledge, new knowledge, as well as connections from the new to prior concepts and representations, a network of connections in balanced typical and reverse directions may be presented to learners.

Consider the underlying reasons for the high degree of repetition in the UCSMP series, the intentions of textbook authors, as well as the cultural differences between the United States and China, may explain them. In an informal talk with Professor Zalman Usiskin, the former overall director of UCSMP, he indicated that their team intended to include such repetition to provide students with more opportunities to practice because students in the United States have fewer opportunities to practice, compared to students in China. High school students in China generally have more opportunities for repetition by multiple rounds of exam-based reviews in school and classes in afterschool learning centers, like Kumon and Mathnasium. Even so, embedding too much repetition may shift the balance of typical and reverse connections in textbook problems.

The second practice is the explicit objective: representations. The UCSMP series has been known for the SPUR (skills, properties, uses, and representations) approach to understanding. The representation dimension focuses on the ability to use concrete materials and models, or graphs and other pictorial representations. From Figure 32, bidirectional connections between the graphical and symbolic representation of quadratic relations are explicitly stressed. By the connection analysis, the UCSMP series demonstrates more unidirectional and bidirectional within-concept connections in quadratic relations than the PEP-A series. This supports the conjecture that the explicit objective of representations in textbooks may promote the diversity of within-concept connections. The former overall director of UCSMP indicated that they wanted the readers to make connections between representations (Usiskin, 2018).

Representations		
I	Graph quadratic relations when given equations for them in standard form, and vice versa.	12-2, 12-4, 12-6, 12-7
J	Graph interiors and exteriors of ellipses when given inequalities for them, and vice versa.	12-3
K	Interpret representations of quadratic-linear and quadratic-quadratic systems.	12-8, 12-9
L	Draw a graph or interpret drawings or graphs of conic sections based on their definitions.	12-1, 12-2, 12-4, 12-6

Figure 32. Summary of objectives of representations (from UCSMP-AA-T, p. 796B)

Summary

The placement of mathematical subtopics may contribute to the relative strength of typical or reverse connections. The PEP-A series tends to adopt the setting of interleaved example-problem pairs and *Bianshi* problems to enhance learning opportunities for bidirectional connections. Excessive repetition in the UCSMP series may shift the balance of typical and reverse connections. Moreover, the explicit objective of incorporating representations in the UCSMP series may increase the diversity of bidirectional within-concept connections.

Chapter V

CONCLUSIONS

Summary

Mathematical connection has been a major goal in mathematics education. Researchers have categorized two types: between-concept connections (cutting across two concepts) and within-concept connections (linking two representations of one concept). Based on directionality, unidirectional and bidirectional connections (a pair of a typical connection and a reverse connection) are recognized. The benefits of bidirectional connections are widely endorsed. However, prior studies have reported learners' struggles in making bidirectional connections, and indicated that some curriculum materials and cognitive obstacles of reverse connections might hinder bidirectional connection-making moves.

From the curriculum aspect, prior cross-national studies reported that some U.S. elementary and middle school mathematics textbooks lack learning opportunities for particular reverse connections, such as multiplication-division connections, while the Chinese counterparts include such reverse connections. Also, researchers have previously addressed differences in textbook-problem features. Standards-based U.S. elementary and middle school textbooks tend to have a higher ratio of exercises to worked-out examples and highlight real-life, single-step, visual problems, whereas the Chinese counterparts tend to show a higher ratio of worked-out examples to exercises and emphasize purely mathematical, multi-step, non-visual problems. However, few studies have explored high school textbook problems. From the cognitive aspect, reverse connections may bring certain cognitive obstacles. External representations and Cognitive Load Theory have suggested that well-designed interleaved example-exercise pairs with particular problem features might promote mathematical connections. However, few studies have provided evidence on the potential cognitive supports in various problem features.

Therefore, this study compared the directionality of mathematical connections in problems from popular U.S. and Chinese high school mathematics textbooks. It aimed to illuminate cross-national similarities and differences in embedding mathematical connections in textbook problems, as well as examine the relationship between mathematical connections and textbook-problem features, and thus provide suggestions for developing textbook problems with learning opportunities for balanced typical and reverse connections.

This study selected popular high school mathematics textbooks stressing connections with similar textbook problem structures, PEP-A for China and UCSMP for the United States. It focused on (a) quadratic relations and (b) probability and combinatorics, which were identified as challenging core topics in which students had difficulties in making bidirectional connections. The results indicated that mathematical topic, contextual feature, and visual feature most likely influence the four conditions of connections. The PEP-A series presented a vigorous network of more unique and total between-concept connections, as well as more bidirectional pairs with balanced typical and reverse directions than the UCSMP series in problems dealing with quadratic relations. The UCSMP series showed a denser network of between-concept connections in probability and combinatorics, as well as within-concept connections in two topics than the PEP-A series, but in unbalanced typical and reverse directions. The placement of subtopics, interleaved example-exercise pairs, *Bianshi* problems, repetition, and the explicit objective of representation may influence the directionality of mathematical connections.

Conclusions

For Research Question 1

The word frequency analysis indicated that both series highlighted circle, probability, more on permutation and less on combination, the numerical representation in probability and combinatorics, more on the equation representation and less on the graphical representation in quadratic relations. In quadratic relations, the PEP-A series exhibited extra attention to line and less attention to parabola. In probability and combinatorics, the PEP-A series highlighted the tabular representation and the graphical representation, whereas the UCSMP series showed a focus on binomial theorem and the symbolic representation.

This study examined more than 1,000 textbook problems, with a larger number of problems in the UCSMP series than in the PEP-A series, and slightly more in probability and combinatorics than in quadratic relations. The between-concept condition dominated problems across textbooks and topics, especially in quadratic relations. The loglinear analysis showed that the Mathematical Connection * Textbook Series * Topic interaction was statistically significant. A strong statistically significant association between connection and topic in the PEP-A series and a weak association in the UCSMP series were observed. Problems of the between-concept condition were more likely in quadratic relations than probability and combinatorics. Problems of the within-concept condition were more likely in probability and combinatorics than quadratic relations. Overall, the PEP-A series was more likely to embed connections in problems dealing with quadratic relations than the other topic, whereas the UCSMP series was more likely to have connections in problems dealing with probability and combinatorics than the other topic.

Both the PEP-A and UCSMP series showed a similar presentational feature. Previous studies have shown that compared to the Chinese series, two widely used U.S. elementary school textbooks had a much smaller portion of worked-out examples for additive inverses (U.S.: 9.0% and 5.7%; Chinese: 24.1%) while more worked-out examples for multiplicative inverses (U.S.: 12.0% and 6.8%; Chinese: 9.5%) (Ding, 2016). For problems dealing with trigonometric functions, one U.S. high school textbook series exhibited a lower portion of worked-out examples (13.0%) than the Chinese counterparts (42.5%) (Fu & Zhang, 2018). For problems dealing with quadratic relations and probability and combinatorics, this study indicated that both series showed a similar portion of worked-out examples, around 20%. This suggests that presentational feature may differ in mathematical topics. Furthermore, the UCSMP series provided fewer exercises on average for problems of the mixed condition than the PEP-A series. Particularly, the PEP-A series exhibited a moderate statistically significant association between presentational and visual features. Worked-out examples were more likely to be visual than non-visual in the PEP-A series.

Considering the real-life problem orientation, the UCSMP series had a higher ratio of real-life to purely mathematical problems than the PEP-A series. This was consistent with previous studies showing that U.S. secondary school textbooks tended to have more real-life problems than the Chinese counterparts (e.g., J. Wang, 2017; X. Wang & Zhang, 2018). However, the majority of problems in both series were still purely mathematical. Furthermore, the fulfillment of the reform call for real-life problems differed in topics. Both series exhibited a strong statistically significant association between topic and contextual feature. Problems dealing with quadratic relations were more likely to be purely mathematical than real-life, whereas problems dealing with probability and combinatorics were more likely to be real-life than purely mathematical. What is more, the Mathematical Connection * Textbook Series * Contextual interaction was statistically significant. A moderate statistically significant association between connection and contextual feature was observed in the PEP-A series and a weak statistically significant association in the UCSMP series. Overall, the PEP-A series was more likely to embed connections in real-life than purely mathematical problems, whereas the UCSMP series was slightly more likely to address connections in purely mathematical than real-life problems.

For mathematical feature, the PEP-A series showed an extremely higher ratio of multi-step to single-step problems than the UCSMP series, especially in quadratic relations. This finding was consistent with a previous study showing that UCSMP textbooks (Grades 7 and 8) had a larger portion of single-step problems than PEP textbooks (62.9% for UCSMP and 52.1% for PEP) (Zhu & Fan, 2006). In a study on two widely used U.S. elementary school textbooks, Kar et al. (2018) showed that the majority of problems were single-step (53.7% and 80.1%, respectively). In contrast, this study indicated that the majority of high school textbook problems in both series were multi-step. This suggests that mathematical feature may differ in grade levels. Additionally, the ratio of the between-concept condition was much higher than the rest three conditions. Furthermore, the PEP-A series exhibited a moderate statistically significant association between mathematical feature and (a) topic and (b) visual feature. Multi-step problems in the PEP-A series were more likely to be visual than non-visual problems, dealing with quadratic relations than probability and combinatorics.

For visual feature, the UCSMP series exhibited a higher ratio of visual to non-visual problems than the PEP-A series. This was consistent with a previous study conducted by Hong and Choi (2018) that the UCSMP textbooks used visual information (graphs, tables, etc.) in 23.0% of worked-out examples and 34.9% of exercises, as well as another previous comparison on U.S. and Chinese middle school mathematics textbook problems (Zhu, 2003). The ratio of visual to non-visual problems of the within-concept condition was particularly low in the PEP-A series and high in the UCSMP series. Problems dealing with quadratic relations included more visual information than problems dealing with probability and combinatorics in both series. What is more, the Mathematical Connection * Textbook Series * Visual interaction was statistically significant. A moderate statistically significant association between connection and visual feature was observed in the PEP-A series. The PEP-A series was more likely to embed within-concept connections in non-visual than visual problems. Overall, problems with connections in both series were more likely to be visual than non-visual.

This study identified more than 2,000 mathematical connections, mostly between-concept connections, especially in quadratic relations. The PEP-A series showed more connections per problem in total and across topics than the UCSMP series, especially for between-concept connections in quadratic relations. The UCSMP series had more within-concept connections per problem in total and in quadratic relations than the PEP-A series. Overall, between-concept connections in quadratic relations and within-concept connections in probability and combinatorics were richer than the other topic in both series. There was a weak statistically significant association between connection and textbook series. In terms of four conditions of connections, the PEP-A series was more likely to have problems of the mixed condition than the UCSMP series, whereas the UCSMP series was more likely to have problems of the within-concept condition than the PEP-A series.

For Research Question 2

Different trends in the directionality of mathematical connections in popular U.S. and Chinese high school mathematics textbook problems were observed. Overall, more than 70% of connections were unidirectional. The UCSMP series showed a higher ratio of bidirectional to unidirectional for two types of connections in two topics than the PEP-A series. The ratio was higher for between-concept than within-concept connections, as well as in quadratic relations than probability and combinatorics. This indicates that most bidirectional connections may appear as between-concept connections or in quadratic relations.

For the integration of bidirectional connections, almost all bidirectional within-concept connections were in the UCSMP series, except for one self-loop in the PEP-A series. The symbolic representation was dominating, especially in the direction starting from the symbolic representation. Additionally, bidirectional within-concept connections concentrated on certain concepts, such as circle, ellipse, and the nth power of the binomial (x + y). Different trends were observed for bidirectional between-concept connections. In probability and combinatorics, the UCSMP series embedded more bidirectional pairs than the PEP-A series, but in unbalanced typical and reverse directions. Both series stressed connections ending in probability or multiplication counting principle. In quadratic relations, the PEP-A series employed more distinct bidirectional between-concept connections in a balanced way (the number of typical connections was similar to the number of reverse connections) than the UCSMP series. For example, in the UCSMP series, circle-to-center, circle-to-radius, x intercept-to-ellipse, foci-to-ellipse, and focus-to-parabola were weaker than the reverse direction.

Based on digraphs for between-concept connections for seven subtopics, both series presented dense digraphs for the subtopics circle and ellipse, and aggregated digraphs for the subtopic probability. All the rest of the digraphs for subtopics were sparse in the UCSMP series. The PEP-A series showed moderate digraphs for the subtopics hyperbola and parabola, a sparse digraph for the subtopic counting problems, and the sparsest digraph for the subtopic binomial theorem. Overall, the PEP-A series presented denser digraphs with more vertices and rich connections in quadratic relations-related subtopics than the UCSMP series, whereas the

UCSMP series showed denser digraphs in probability and combinatorics-related subtopics than the PEP-A series. The curriculum emphasis on circle, ellipse, and probability was evidenced again as the digraphs for these subtopics were denser than others. For the density of arrows, the PEP-A series had some aggregated connections ending in line, angle, and slope. The UCSMP series showed aggregated connections starting from ellipse and aggregated connections ending in focus, the number of outcomes in the event, and binomial coefficient. Following the flow of connections in quadratic relations, both series started with connections between subtopics and their attributes and ended with connections among these subtopics. The PEP-A series stressed connections between subtopics and line or point, whereas the UCSMP series emphasized connections among special circles, ellipse-circle, hyperbola-line or point, and among attributes of parabola. It implied that the UCSMP series stressed connections between quadratic relations-related concepts, whereas the PEP-A series highlighted connections between quadratic relations-related and linear function-related concepts. For probability and combinatorics, the PEP-A series stressed connections between concepts in geometric models of probability, whereas the UCSMP series stressed connections between probability and binomial theorem or binomial experiment.

Considering the digraph and adjacency matrix for each topic, the UCSMP series showed a stronger network of within-concept connections with more concepts and representations, as well as more unique and total unidirectional and bidirectional connections in two topics than the PEP-A series. Compared to the UCSMP series, the PEP-A series addressed fewer within-concept connections that were almost all unidirectional except for one self-loop. Both series stressed some unidirectional connections with heavy weights, such as counting problems from the written description of a real-world context to numerical representation, and quadratic relations from the symbolic to graphical representation. For between-concept connections, the PEP-A series presented a more balanced and stronger network of between-concept connections in quadratic relations with more concepts, connections (unique and total), and balanced bidirectional pairs (typical and reverse) than the UCSMP series. Circle was the central concept for both series to embed total between-concept connections in and out of. But connections involving circle had large weights due to their limited diversity in the PEP-A series. Ellipse and hyperbola were the central concepts to embed distinct connections for both series. The PEP-A series showed extra attention to connections leading to line, whereas the UCSMP series showed extra attention to connections involving special circles and connections leading to ellipse. More bidirectional pairs in quadratic relations appeared in the PEP-A series than the UCSMP series. The ratio of bidirectional to unidirectional connections, the reciprocated vertex pair, and the reciprocated edge in the PEP-A series were slightly lower than that in the UCSMP series. For probability and combinatorics, the UCSMP series showed a stronger network of between-concept connections with more concepts, as well as more unique and total unidirectional and bidirectional connections than the PEP-A series, but still in unbalanced typical and reverse directions. Probability was the core concept as the starting or ending vertex of many distinct and total connections. Both series also stressed distinct and total between-concept connections leading out of event. The PEP-A series paid extra attention to connections leading in geometric models of probability and bidirectional connections between frequency and relative frequency. The UCSMP series stressed connections involving outcome and trial. Overall, concepts in probability and combinatorics have smaller connectivity values than that in quadratic relations. This not only indicated the limited diversity of connections in probability and combinatorics, but also coincided with the digraph analysis for seven subtopics in which the subtopic probability showed aggregated digraphs.

Overall, the UCSMP series presented a more extensive network of within-concept connections as well as between-concept connections in problems dealing with probability and combinatorics than the PEP-A series. However, the UCSMP series usually shifted the balance of typical and reverse connections. In contrast, the PEP-A series presented a stronger and balanced network of between-concept connections in problems dealing with quadratic relations.

For Research Question 3

Different placement of subtopics might be an underlying reason for the relative strength of typical and reverse connections. Concepts and representations that were taught early in the curriculum sequence tended to be overemphasized in the network of connections than those that were introduced later. For probability and combinatorics, both series followed the order of probability, counting problems, and binomial theorem. By the connection analysis, both series stressed probability (aggregated digraph) the most, then counting problems (sparse digraph), and the binomial theorem (sparse and the sparsest digraph) the least. The consistency supported the conjecture that the emphasized direction was consistent with the sequence of subtopics in textbooks. For quadratic relations, the PEP-A series addressed circle in one chapter and placed ellipse, hyperbola, and parabola in another chapter, whereas the UCSMP series placed all subtopics in one chapter in the order of parabola, circle, ellipse, and hyperbola. Both series placed linear functions and quadratic relations far apart. By the connection analysis, the PEP-A series stressed connections involving the subtopic circle with larger weights and highlighted connections between quadratic relations-related subtopics and linear functions-related subtopics, whereas the UCSMP series stressed more connections between quadratic relations-related subtopics. This suggested that the separation of subtopics may weaken bidirectional connections between these subtopics. Furthermore, intentional connections between concepts that were placed far away may strengthen bidirectional connections and the connectivity of curriculum. Additionally, both series proceeded in the algebraic-to-graphical direction and embedded few within-concept connections of quadratic relations from the graphical to symbolic representation in textbook problems, which was consistent with many previous studies (e.g., Knuth, 2000b).

Although both series had similar presentational features, the PEP-A series adopted carefully designed interleaved example-problem pairs, whereas the UCSMP series followed the layout of worked-out example-to-exercise. The PEP-A series might provide more cognitive support in making connections. Previous studies have suggested that Chinese elementary school textbooks utilize *Bianshi* problems to support bidirectional connections in numerous topics (e.g., Ding & Li, 2010; Sun, 2011b). This study found that the majority of *Bianshi* problems in the PEP-A series promoted bidirectional between-concept connections in quadratic relations. It was consistent with the connection analysis that the PEP-A series presented a dense network of between-concept connections in quadratic relations with balanced typical and reverse connections than the UCSMP series. The UCSMP series presented a degree of repetition of simple single-step problems, which might shift the balance between typical and reverse connections. The UCSMP series also used the explicit objective of representations to promote bidirectional within-concept connections. It was consistent with the connection analysis that the UCSMP series presented a denser network of within-concept connections than the PEP-A series.

Theoretical Contributions

Mathematical Connections and Directionality

Previous studies of mathematical connections suggested that some standards-based U.S. elementary and middle school mathematics textbooks lacked learning opportunities for some reverse connections compared to their Chinese counterparts (e.g., Cai & Moyer, 2008; Ding, 2016). It was consistent with part of my result that the standards-based U.S. high school mathematics textbook problems in this study exhibited unbalanced learning opportunities for typical and reverse between-concept connections in quadratic relations, whereas the Chinese counterparts showed balanced typical and reverse connections. However, for problems dealing with probability and combinatorics, the standards-based U.S. high school mathematics textbook problems exhibited more learning opportunities for unidirectional and bidirectional between-concept connections than the Chinese counterparts, but still in unbalanced typical and reverse directions. Furthermore, differences between the overall network of between-concept connections in two topics of the Chinese series may be explained by the differences in time and attention given to two topics. Statistics and probability were not required content until the late 1990s, and their practical applications were largely ignored before the eighth curriculum reform in China (Li, Zhang, and Ma, 2019). In contrast, quadratic relations are long-standing emphasized content. Problems dealing with quadratic relations have gone through several revisions and improvements in past curriculum reforms. During this process, intentional bidirectional connections between concepts belonging to quadratic relations or topics that were placed far away in the curriculum sequence but connected in nature were adopted.

Additionally, previous studies on U.S. high school mathematics textbooks conducted by Knuth (2000b) and a U.S. university-level Calculus textbook conducted by Chang, Cromley, and Tran (2016) indicated that most tasks in textbooks were in the symbolic-to-graphical direction. It was consistent with this study that current popular U.S. and Chinese high school textbook problems in this study still lacked learning opportunities for connections moving from the graphical to symbolic representation of quadratic relations. Over the past 30 years, researchers have reported students' difficulties in graphical-to-algebraic/symbolic connections or over-reliance on the algebraic/symbolic methods in graphical-favored situations worldwide, such as in the United States (e.g., Blume &Heckman, 1997; Confrey, Millman, & Piliero, 1993; Knuth, 2000b; Larson & Zandieh, 2013; Leinhardt et al., 1990; McCoy, 1994; Trigueros & Martínez-Planell, 2010); China (He & Qi, 2017); Israel (Zaslavsky, 1997); Canada (Hillel, 2000); Cyprus (Elia et al., 2007); and Belgium (De Bock, Van Dooren, & Verschaffel, 2015). The consistency between students' difficulties and the lack of learning opportunities offered in textbook problems for graphical-to-symbolic connections supported the conjecture that limited learning opportunities in textbook problems might contribute to students' difficulties in making bidirectional connections. This leads to some practical implications that a sparse network of connections, i.e., the lack of learning opportunities for particular connections in textbook problems, may hinder learners' connection-making moves and thus influence their learning progress in mathematics. Also, this study supplemented previous studies that standards-based U.S. high school mathematics textbook problems embedded more unidirectional and bidirectional within-concept connections than the Chinese counterparts, especially in quadratic relations, but still in an unbalanced way.

Furthermore, previous studies on the directionality have usually focused on one particular bidirectional pair (e.g., Cai & Moyer, 2008; Prodromou, 2012). This study supplemented previous studies by examining more than 50 bidirectional pairs and the network of connections and reaching a more generalized conjecture about the directionality. The stressed directionality was consistent with the prior-to-new knowledge direction. Connections from new to prior knowledge were largely overlooked, except for between-concept connections in the quadratic relations of Chinese textbooks. Repetition of single-step problems may shift the balance of typical and reverse connections.

Textbook-Problem Features

Previous studies of textbook-problem features have suggested that standards-based U.S. elementary and middle school mathematics textbooks usually have more real-life, single-step, visual problems than the Chinese counterparts (e.g., J. Wang, 2017; Zhu, 2003; Zhu & Fan, 2006). My study supported this finding by showing that the PEP-A series had more purely mathematical, multi-step, non-visual problems than the UCSMP series. Additionally, my study suggested that the fulfillment of the reform call for real-life problems differed in topics. Both series showed a strong statistically significant association between topic and contextual feature. Problems dealing with quadratic relations were largely set in purely mathematical than real-life contexts, whereas problems dealing with probability and combinatorics were more likely to be real-life than purely mathematical. What is more, there was a statistically significant association between textbook series and mathematical feature. Multi-step problems were more largely employed in the PEP-A series than the UCSMP series.

However, for presentational feature, previous studies have indicated that many standards-based U.S. mathematics textbooks included more exercises than worked-out examples, compared to their Chinese counterparts (e.g., Ding, 2016; Fu & Zhang, 2018; Li, Chen, & An, 2009). My study reported that the PEP-A series and the UCSMP series had a similar ratio of worked-out examples to exercises, which differed from the results of previous studies. On one hand, this could indicate that the presentational feature reflected in textbook problems of the elementary or lower secondary school level might not exist at the high school level. It also supports the finding by Hong and Choi (2014) that some of the characteristics of elementary school mathematics textbooks were not reflected in the analysis of secondary school mathematics textbooks. On the other hand, this could suggest that problems dealing with different topics showed varied presentational feature. Ding's (2016) study supported the finding that U.S. textbooks have a smaller portion of worked-out examples in additive inverses, but a larger portion of worked-out examples in multiplicative inverses than the Chinese counterparts. What is more, my study supplemented previous studies by showing a statistically significant association between presentational and visual feature in the PEP-A series. Worked-out examples were more likely to be visual than non-visual, whereas exercises were more likely to be non-visual than visual.

Previous studies have indicated that curriculum and cognitive aspects may influence connection-making moves (e.g., Goldin & Shteingold, 2001; Knuth, 2000b). My study also supplemented previous studies by demonstrating that mathematical topic, contextual feature, and visual feature had a statistically significant association with mathematical connections and textbook series. Problems of the between-concept condition were more likely to deal with quadratic relations than probability and combinatorics. This was consistent with the finding that the network of between-concept connections in quadratic relations was denser than that in probability and combinatorics. This suggests that the strength of the network of connections of a specific topic might be related to the nature of mathematics itself. This leads to some practical implications that the richness of the network of connections for different topics is associated with its own nature, which may have an upper limit.

New Methodology

The new methodology proposed in my study not only broadens the scope of mathematical connection analysis, but also opens up the possibility of adopting new and efficient analytical tools to visualize, evaluate, and generalize features of connections. It supplements previous studies by combining concept, representation, connection, and, importantly, the whole network of connections and the directionality to visualize and assess connections. My study suggests that the new methodology is a theoretical contribution to the current analysis of mathematical connections, which has several practical implications.

Regarding the scope of the connection analysis, previous analysis has usually focused on a specific connection or concept or representation, e.g., graph-equation (Knuth, 2000b), fraction-division (Weinberg, 2001), addition-subtraction (Cai & Moyer, 2008), multiplication-division (Xin et al., 2011), theoretical-experimental probability (Prodromou, 2012), Simpson's Paradox (Lesser, 2001), the averaging algorithm (Cai, Lo, & Watanabe, 2002), the distributive property (Ding & Li, 2010), and two-variable functions (Trigueros & Martínez-Planell, 2010). However, merely analyzing concepts or representations may lose the other critical aspect and miss the structural characteristics of the network. Missing one connection/concept/representation would influence the rest of the connected components in the whole network. Therefore, my study not only examined a particular concept or representation or connection, but also evaluated the structure of the network and directionality. For example, the digraph and adjacency matrix analysis examined the network of connections in three dimensions: (a) the network (the digraph of varied density), (b) concepts and representations (vertices), and (c) connections and its directionality (directed edges). It broadened the scope of the current analysis of connections.

Regarding the analytical tools used in previous studies, in general, researchers have counted the number or percentage of particular connections/concepts/representations or listed exemplary examples (e.g., Cai et al., 2005; Ding, 2016). In my study, several indices from Social Network Analysis (SNA), e.g., size, unique/total edges, in/out-degree, in/out-connection, reciprocated vertex/edge pair ratio, and self-loops, were used successfully to characterize the directionality and the network of connections. Directionality is a vital feature of connections. The strength of directionality, i.e., the relative strength of typical and reverse connections, demonstrated a new perspective to evaluate the quality of connections. The proposed digraph and adjacency matrix analysis in this study was a successful attempt at visualizing and evaluating the structure of within-concept and between-concept connections from both qualitative and quantitative aspects. It produced meaningful insights into the flexibility, strength, connectivity, and extensiveness of the network to support bidirectional connections. My study not only validated that SNA could be used successfully to explore the quality of connections, but also to open up the possibility of adopting other useful tools from SNA to visualize and examine the quality of mathematical connections.

Additionally, the new methodology had several practical implications. For the new learning theory for a digital age, Downes (2007) stated that *connectivism* is the thesis that "knowledge is distributed across a network of connections, and therefore that learning consists of the ability to construct and traverse those networks" (n.p.), which implied a relationship between the network of mathematical connections and the learning of mathematics. To be specific, diverse networks of connections can influence the learning of mathematics, and in reverse, the learning of mathematics can be assessed by the connection network that learners construct and traverse. Researchers have assessed learners' understanding of mathematics by mathematical connections they construct. For example, Selinski et al. (2014) used the adjacency matrix to analyze connections that students made within and between concepts in Linear Algebra, and showed the usefulness of comparing differences in the structure of connections that students made as a way to examining their understanding. Jin and Wong (2015) investigated the number of incoming and outgoing connections (a) within individual concepts, (b) between pairs of concepts, and (c) all constructed by the whole class to evaluate the conceptual understanding of a class of 8th graders. They captured a gap in students' understanding of equations and functions.

My study indicated that the assessment of students' learning progress could be more comprehensive by adding the analysis of the network of connections they generated and the directionality. From this point of view, the new analysis may yield valuable insights into (a) learners' difficulties in understanding concepts, representations, and connections; (b) gaps between the relative strength of typical and reverse connections; (c) a vivid demonstration of the knowledge network; and (d) the structural hole and flexibility of making within-concept and between-concept connections across the whole network. This leads to some practical conclusions that adopting the proposed framework to analyze the network of connections learners make can produce valuable information to improve the teaching and learning of mathematics.

Recommendations and Limitations

Recommendations

The following are recommended for mathematics teachers and textbook authors to provide balanced learning opportunities for typical and reverse connections.

For the direction in which the curricula proceed and the reverse direction, it seems viable for teachers and textbook authors to consider new-to-prior knowledge and prior-to-new knowledge connections at the same time, especially the new-to-prior knowledge connection. When two connected subtopics are split into two chapters or textbooks, teachers and textbook authors are urged to pay special attention to bidirectional connections between them. Connections articulated in the *Common Core State Standards for Mathematics* and learning trajectory maps are helpful in identifying subtopics or concepts being connected in nature.

For external representations, current popular U.S. and Chinese high school mathematics textbook problems in this study showed the emphasis on the symbolic and numerical representation. Particular emphasis is recommended for having tasks leading out of the graphical representation and leading to the symbolic representation in classroom instruction and textbooks. Clearly, it seems viable for teachers and textbook authors to consider increasing the diversity of representations used in instruction and textbook problems to provide various learning opportunities for rich and balanced typical and reverse connections.

Digital interactive software can be used to embed multiple representations in textbook problems to support within-concept connections, such as interactive exercises and electronic textbooks. For example, several e-exercise platforms were developed and used in many countries for all levels (Gueudet, 2006). On one hand, the technology can be utilized to enhance the diversity of representations, e.g., the tabular representation (Gueudet, 2008), the graphical representation (Gueudet, Pepin, Restrepo, Sabra, & Trouche, 2018), and so on. Dynamic representations and animated help in the feedback section can integrate various representations, which embed rich learning opportunities for traversing the network of distinct representations and thus support the grasp of within-concept connections. On the other hand, these platforms can adjust the exercises to meet the needs of different students and embed variations of the same exercise. Similar exercises or the same exercise with different values, together with animated help that embeds multiple representations, can be arranged if students give an incorrect response and have difficulties in making connections. For high-achieving students, drill exercises can be avoided for the same type of problems if they produce a consistently correct answer. More multi-step problems can be assigned to them to help students construct stronger and richer connections. In addition, e-textbooks afford dynamic representations which can display change of representations over time, transformations of graphical figures, and the like (Usiskin, 2018). More distinct within-concept connections may be available in e-textbooks.

For problem features supporting or hindering mathematical connections, teachers and textbook authors are encouraged to be cautious with the over-repetition of single-step exercises and the over-reliance on purely mathematical contexts with non-visual information. This may lead to connections in a particular direction with heavy weights and limited within-concept connections. Unbalanced opportunities for typical and reverse connections may hinder students from grasping bidirectional connections. Mathematics teachers are recommended to select textbook problems with balanced learning opportunities for typical and reverse connections.

Limitations

First, the word frequency analysis approach overemphasized some single-word concepts and representations in textbook content. For example, the single-word concept "event" is over-counted as it also appears in other multi-word concepts, such as overlapping events, complementary events, independent events, and mutually exclusive events. Therefore, the word frequency of the top 20 terms was not accurate due to some overemphasized single-word terms.

Second, the classification of external representations also had limitations. The categories for the symbolic representation are different for two topics. For example, for probability and combinatorics, the symbolic representation (listed in textbooks) is divided into: original, polynomial, factorial, and binomial expansion; for quadratic relations, the symbolic representation is split into: standard form for a circle/ellipse/hyperbola/parabola, standard form for a quadratic relation, and other forms. It is hard to reach a common conclusion on whether these categories are enough or whether the classification is necessary.

Third, the coding of mathematical connections was not entirely objective. Due to the limitation of time, the pre-coded connection list was provided to experienced mathematics teachers as the basis to recode and compile the final coding. The pre-coded list created bias.

Even though graduate students majoring in mathematics education conducted an inter-rater reliability check, some disagreements still existed. More coders from diversified fields, e.g., mathematicians and textbook authors, may improve the reliability and validity of the coding since the final coding significantly influences the results.

Finally, the violation of assumptions for the loglinear analysis was a limitation. Problem features were not independent of each other. As the data set was large and variables were all essential, this study accepted the decrease in statistical power due to violations as it might not result in substantial loss of predictive power of the model.

Future Research

First, this study indicated that mathematical topic, contextual feature, and visual feature supported four conditions of mathematical connections. Future research can further determine the appropriate ratio of (a) real-life context to purely mathematical and (b) visual to non-visual problems, such that the real-life context and visual support in textbook problems can benefit the development of bidirectional connections. Also, cognitive load analysis of problems with connections in the weak direction, as well as an analysis of efficiency of underrepresented representation involved in within-concept connections, can be conducted to (a) understand students' difficulties in grasping particular connections and (b) find possible ways to enhance students' learning of the weaker direction. To improve the limited usage of meaningful visual information, new technology, such as dynamic geometry software (e.g., GeoGebra, Geometry Sketchpad, SketchUp), graphing technology (e.g., graphing calculators, Desmos), simulation software (e.g., Fathom, Flash), statistics software (e.g., Excel-Spreadsheet, Fathom), VR/AR-based apps, interactive whiteboard, and online learning systems, can be used to embed more graphical support to promote connection-making moves. Future studies may be done on the development of e-exercises embedding more visual information.

Second, the analytical framework proposed to evaluate mathematical connections can be valuable in many other studies. Previous studies on mathematical connections may miss the fundamental structure of the network of connections. Therefore, it is recommended that examining concepts, representations, connections, and the whole network altogether may produce a complete, meaningful, and comprehensive analysis. For example, it can be used to assess mathematical connections in (a) other mathematical topics, (b) textbooks or curriculum standards in other countries, and (c) e-textbooks. Different trends of the directionality exist in two topics and two textbook series in this study. Using the framework to examine connections in other topics may provide a full image of the directionality issue. Exploring connections in textbooks or curriculum standards of other countries may unpack associations among the nature of mathematics, connections, and textbooks or curriculum standards in different cultural and social contexts. The development of technology has opened up opportunities for the progress of e-textbooks, and new theoretical frameworks are needed to analyze e-textbooks (Gueudet et al., 2018). Currently, researchers use connectivity at the macro and micro levels to examine e-textbooks. Instead of merely counting connections, future studies can adopt more ideas from SNA to analyze the overall connectivity in e-textbooks. The digital affordance makes the connection analysis quicker and easier.

Finally, the methodology used in my study can be extended to examine the network of mathematical connections that learners construct as a way to assess their conceptual understanding. Future studies can use this innovative method to visualize the network of mathematical connections and evaluate the progress of learners' conceptual understanding.

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Appendix A

Tables

1. The Concepts and Related Representations Table

The following tables present the concepts in (a) probability and combinatorics and (b) quadratic relations.

The Concepts and Related Representations Table (Probability and Combinatorics)														
#	Concepts	Representations												Synonyms
1	Experiment	W	W1	W2	W3	N	S	S1	S2	S3	D	G	P	
2	Outcome													
3	Sample space													
4	Event													
5	Number of outcomes in the sample space													N(S)
6	Number of outcomes in the event													N(E)
7	Probability													
8	Predicted probability													
9	Fair													Unbiased
10	Biased													
11	Randomly													At random
12	Empty set													Null set
13	Set													
14	Subset													
15	Equally likely													
16	Frequency													
17	Relative frequency													
18	Union of events													Union of sets
19	Mutually exclusive events													Disjoint sets
20	Overlapping events													Intersection of sets
21	Complementary events													Complement of E; Not E
22	Addition Counting Principle													
23	Independent events													
24	Dependent events													
25	Certain event													
26	Impossible event													

27	Random event													
28	Elementary event													
29	String													Ordered symbols
30	Length of a string													
31	String with repetition													String with replacement
32	More than one string with repetition													More than one string with replacement
33	Permutation													String without replacement; String without repetition; Arrangement
34	More than one permutation													More than one string without repetition
35	Multiplication Counting Principle													
36	Permutations of n elements taken r at a time													P (n, r)
37	Permutations of n elements taken n at a time													P (n, n)
38	n factorial													n!
39	Products of consecutive integers													
40	Derangement													
41	Simulation													
42	Random number													Random digit
43	Trial													
44	The number of trials													
45	Monte Carlo method													
46	Estimated outcome/probability/ number of trials/number of outcomes													
47	Expected count of an outcome													
48	Expected count of an event													
49	Observed count of an outcome													
50	Law of Large Numbers													
51	Possibility tree													

52	Branch point													Node
53	Leaves													
54	Principle of Mathematical Induction													
55	Combination													Unordered symbols without repetition; Unordered symbols without replacement
56	Combination of n elements taken r at a time													C (n, r)
57	Combination of n elements taken n at a time													C (n, n)
58	Unordered symbols with repetition													Unordered symbols with replacement
59	Pascal's Triangle													
60	Row													Row of Pascal's Triangle
61	Binomial coefficients													
62	The nth power of the binomial x + y													$(x + y)^n$
63	Exponent													
64	Term													The nth element/term
65	Binomial experiment													
66	Binomial probability													
67	The sum of binomial coefficients													Sum (binomial coefficients)
68	The sum of the squares of terms													
69	The number of 0s													
70	Probability theory													
71	Classical models of probability													
72	Geometric models of probability													
73	Circle													
74	Square													
75	π													
76	Area													

77	Length of time													
78	Complex number													
79	Normal number													

The Concepts and Related Representations Table (Quadratic Relations)											
#	Concepts	Representations									Synonyms
1	Point	W	N	S	S1	S2	G	T	D	P	P
2	Midpoint										
3	Trisection point										
4	Quarter point										
5	Point on the line										
6	Point on the perpendicular bisector										
7	Point on the circle										
8	Point on the semicircle										
9	Point on the parabola										
10	Point on the ellipse										
11	Point on the hyperbola										
12	Point on the x-axis										
13	Point on the quadratic relation										
14	Point outside the quadratic relation										
15	Origin										O
16	Lattice point										
17	Reflection point										
18	Reflection										
19	Angle										
20	Line										
21	Slope										
22	Intercept										
23	Median										
24	Perpendicular										
25	Perpendicular bisector										
26	Perpendicular diagonal										
27	Perpendicular segment										
28	Right triangle										
29	Hypotenuse										
30	Equilateral triangle										

31	Isosceles triangle										
32	Isosceles trapezoid										
33	Chord										
34	Distance from the chord to the center										
35	Distance from the chord to the point on the circle										
36	Tangent point										
37	Tangent line										
38	Tangent circle										
39	Parabola										
40	Focus										F
41	Vertex										
42	Directrix										
43	Direction										Open up or open down
44	Distance between the focus and the directrix										p
45	Distance between the focus and the vertex										p/2
46	Coefficients of x in the standard equation for a parabola										2p
47	Distance between a point on the parabola and the directrix										d
48	Distance between a point on the parabola and the focus										PF
49	Openness degree										
50	Axis of symmetry										
51	Circle										
52	Radius										
53	Diameter										
54	Center										
55	Concentric										
56	Epicenter										
57	Circumstance										
58	Area										
59	Speed										
60	Length of time										
61	Interior of a circle										
62	Exterior of a circle										
63	Semicircle										Half-circle

64	Circumcircle										
65	Inner circle										
66	Outer circle										
67	Ellipse										
68	x-intercept										The x-coordinate of the point at which a graph crosses the x-axis
69	y-intercept										The y-coordinate of the point at which a graph crosses the y-axis
70	Range										
71	Foci										(c, 0) & (-c, 0); or (0, c) & (0, -c); F1 & F2
72	Foci on x-axis										
73	Foci on y-axis										
74	Distance between two foci										F1F2; 2c
75	Distance between a point on the ellipse and the left focus										PF1
76	Distance between a point on the ellipse and the right focus										PF2
77	Major axis										A1A2
78	Minor axis										B1B2
79	2a										Length of the major axis; a>b
80	2b										Length of the minor axis; a>b
81	Focal constant										PF1+PF2; The constant sum of the distances from any point P on an ellipse to the foci
82	Semimajor axis										

83	Semiminor axis										
84	a										Length of the semimajor axis; a>b
85	b										Length of the semiminor axis; a>b
86	Center										
87	Interior of an ellipse										
88	Exterior of an ellipse										
89	Superellipse										
90	Shape of ellipse										
91	Shape of superellipse										
92	Scale change										Sa,b; A horizontal scale change of magnitude a and a vertical change of magnitude b maps (x, y) onto (ax, by)
93	Translation										Ta,b; A transformation for all x and y that maps (x, y) onto (x+a, y+b)
94	Eccentricity										e; c/a
95	Hyperbola										
96	Focal constant										2a; The absolute value of the difference of the distances from a point on a hyperbola to the two foci of the hyperbola
97	Foci										(c, 0) & (-c, 0); or (0, c) & (0, -c); F1 & F2

98	Vertices										A1 & A2; (-a, 0) & (a, 0)
99	Vertices on x-axis										
100	Vertices on y-axis										
101	Distance between a point on the hyperbola and the left focus										PF1
102	Distance between a point on the hyperbola and the right focus										PF2
103	Real axis										2a
104	Imaginary axis										2b
105	Semi-real axis										a
106	Semi-imaginary axis										b
107	Length										
108	Volume										
109	Perimeter										
110	Asymptote										
111	Perpendicular asymptotes										
112	Rectangular hyperbola										Equilateral hyperbola
113	Exterior of a hyperbola										
114	Interior of a hyperbola										
115	Quadratic relation										
116	Coefficients of a quadratic relation										
117	Shape of a quadratic relation										
118	Line; Line										
119	Line; Parabola										
120	Tangent line; Circle										
121	Quadratic-linear system										
122	Line; Hyperbola										
123	Line; Ellipse										
124	Line; Circle										
125	Quadratic-quadratic system										
126	Circle; Circle										
127	Circle; Circle; Circle										
128	Ellipse; Hyperbola										
129	Hyperbola; Hyperbola										
130	Circle; Parabola										
131	Parabola; Parabola										

132	Absolute-value function										
133	Function										
134	Non-function										
135	No intersection										
136	One intersection										
137	Two intersections										
138	Four intersections										
139	Exterior of a parabola										
140	Center on x-axis										
141	Center on y-axis										
142	Maximum										
143	Minimum										
144	Maximum point										
145	Minimum point										
146	Square										
147	Exponent										
148	Code										
149	Unit circle										
150	Transformation										

2. The Representations Table

The following tables show the representations in (a) probability and combinatorics and (b) quadratic relations.

The Representations Table (Probability and Combinatorics)	
Representations	Coding and Explanation
Written Description	W: Situations except W1, W2, W3
	W1: In real-world context without mathematical feature
	W2: With mathematical feature
	W3: List of outcomes
Numerals	N
Symbolic Expressions	S: Original symbolic expression
	S1: Polynomial expansion
	S2: Factorial expansion
	S3: Binomial expansion
Diagrams	D
Pictures	P
Tables	T
Graphs	G

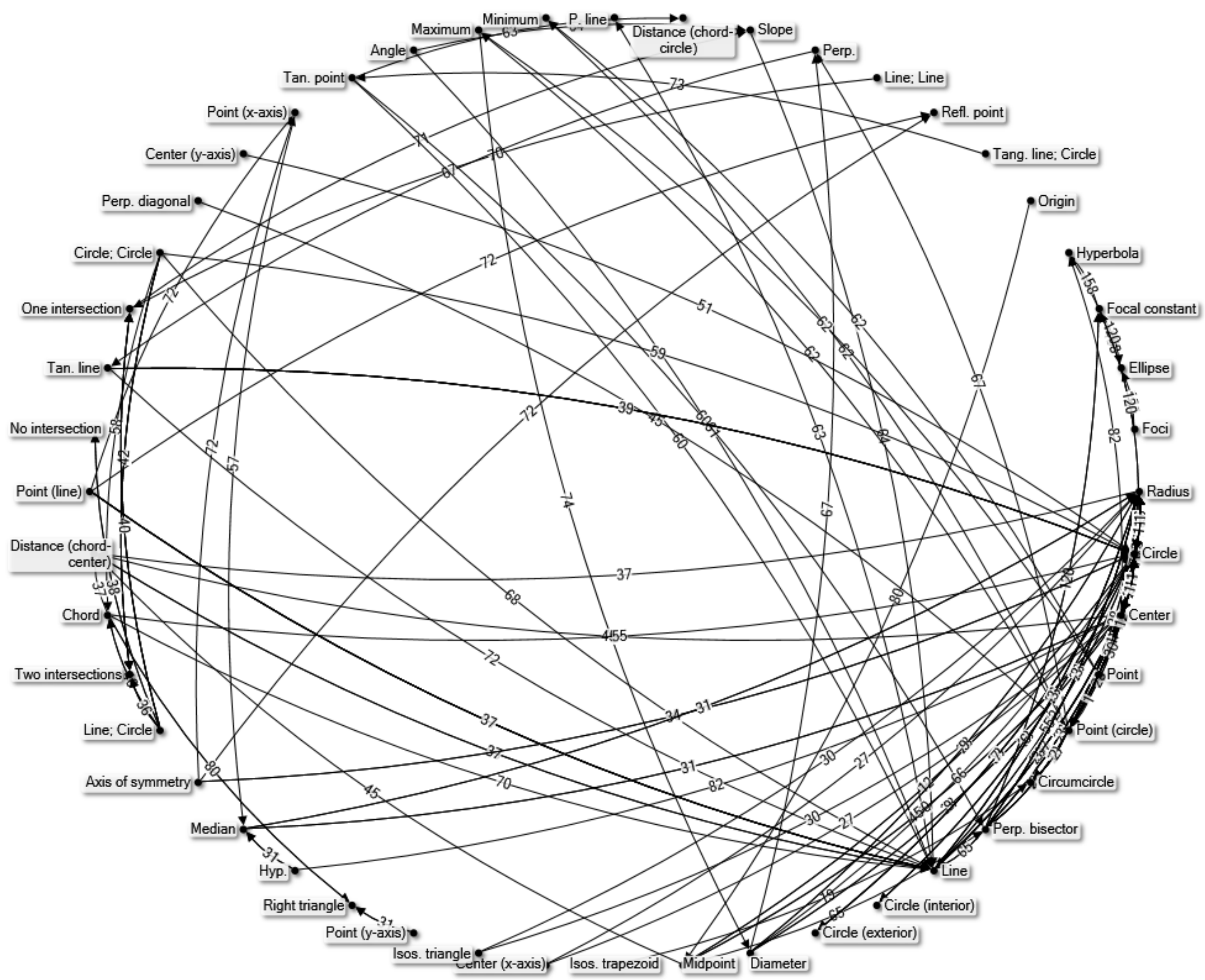
The Representations Table (Quadratic Relations)	
Representations	Coding and Explanation
Written Description	W
Numerals	N
Symbolic Expressions	S: Other forms except S1 and S2
	S1: Standard form for a circle/ellipse/hyperbola/parabola
	S2: Standard form for a quadratic equation: $Ax^2+Bxy+Cx^2+Dx+Ey+F=0$
Diagrams	D
Pictures	P
Tables	T
Graphs	G

Appendix B

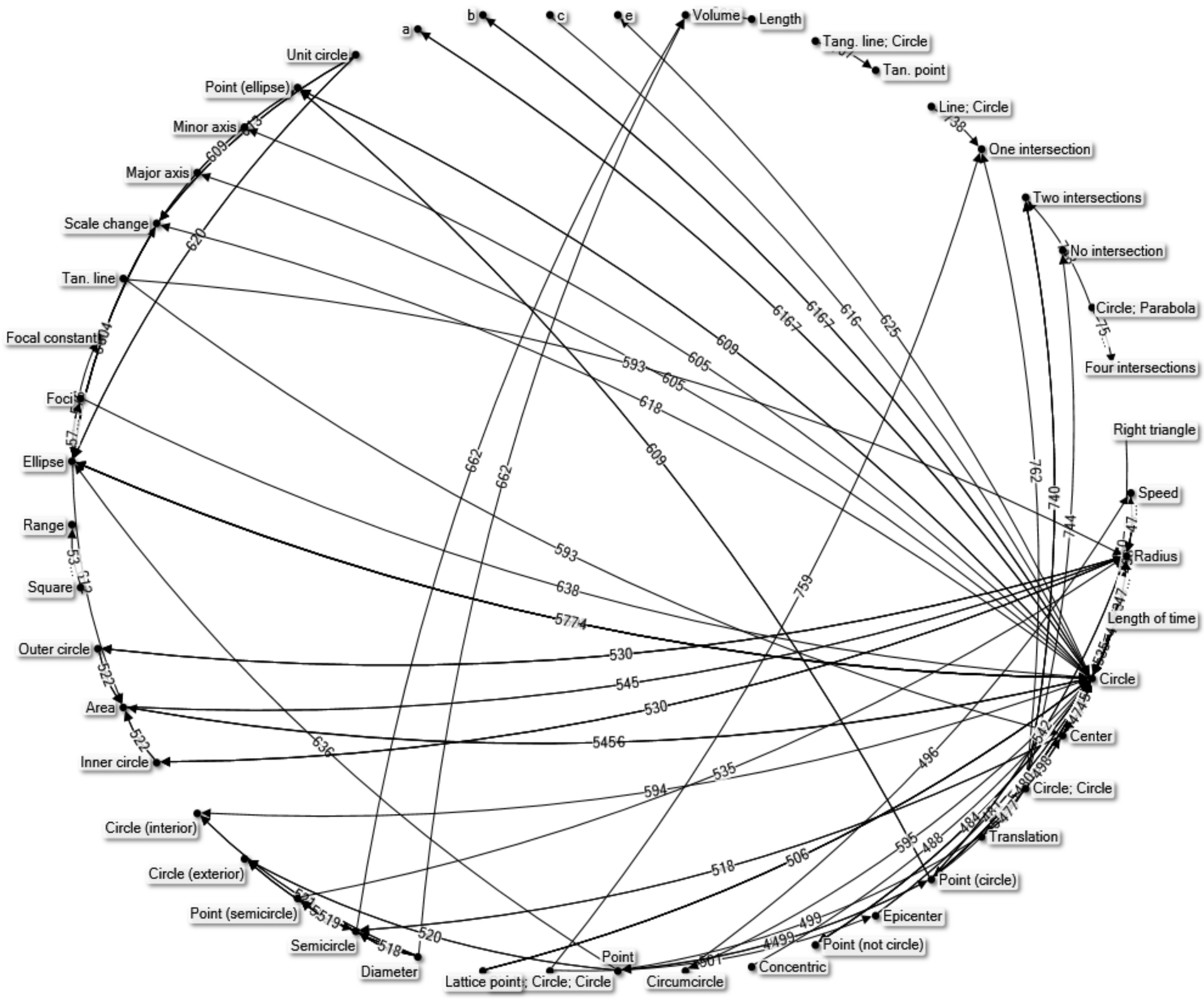
Digraphs

1. Digraphs for Subtopics

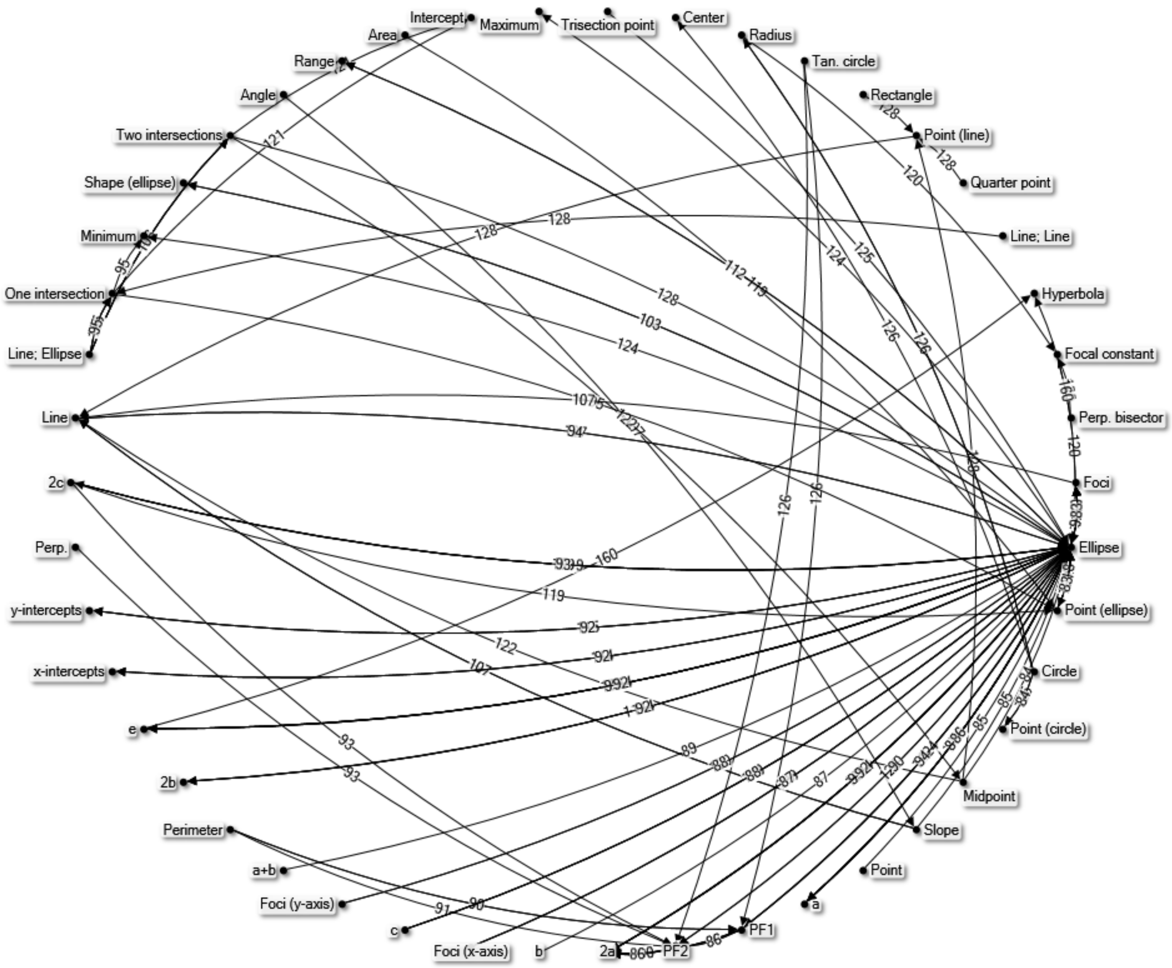
The following digraph is for the subtopic circle for the PEP-A series.



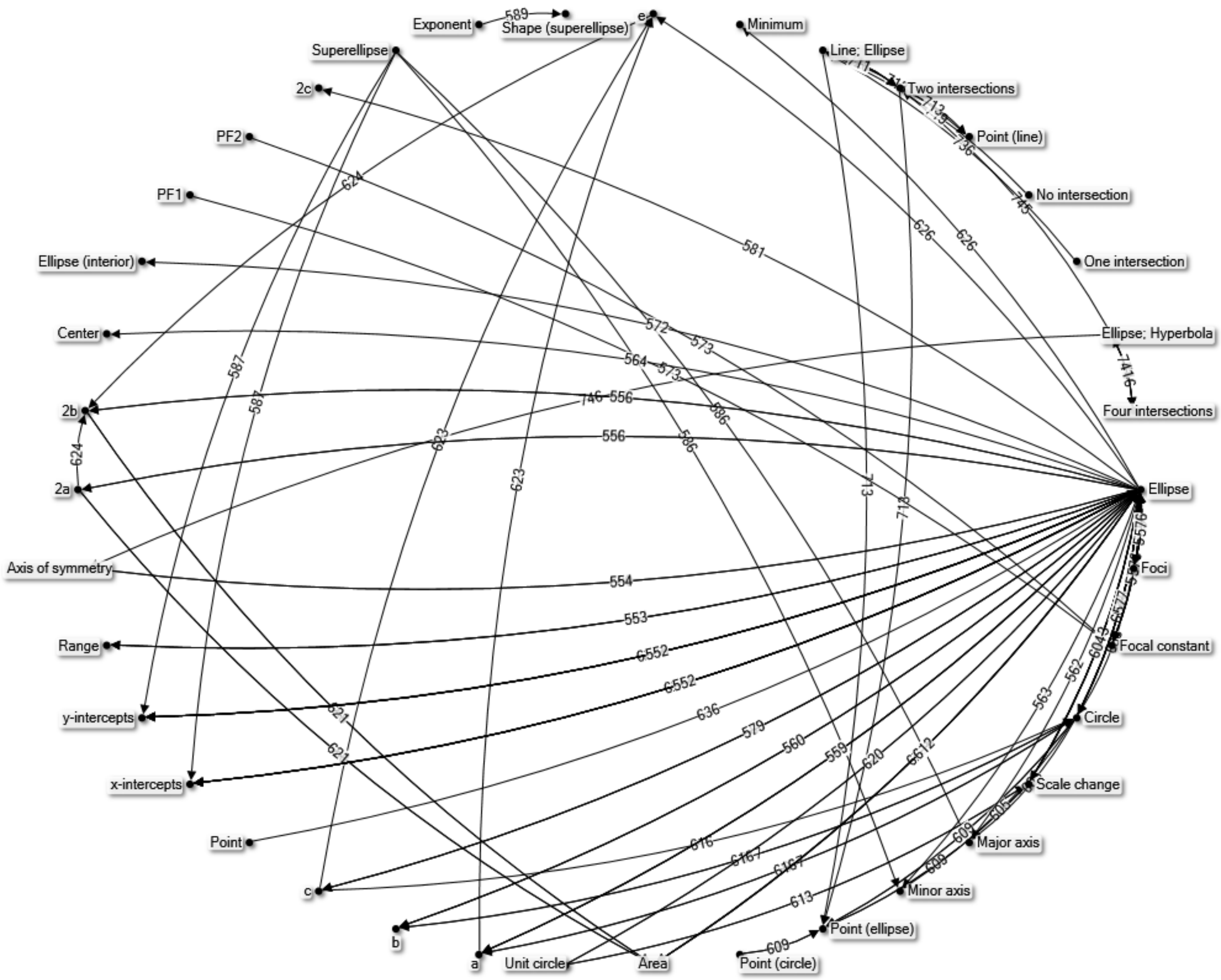
The following digraph is for the subtopic circle for the UCSMP series.



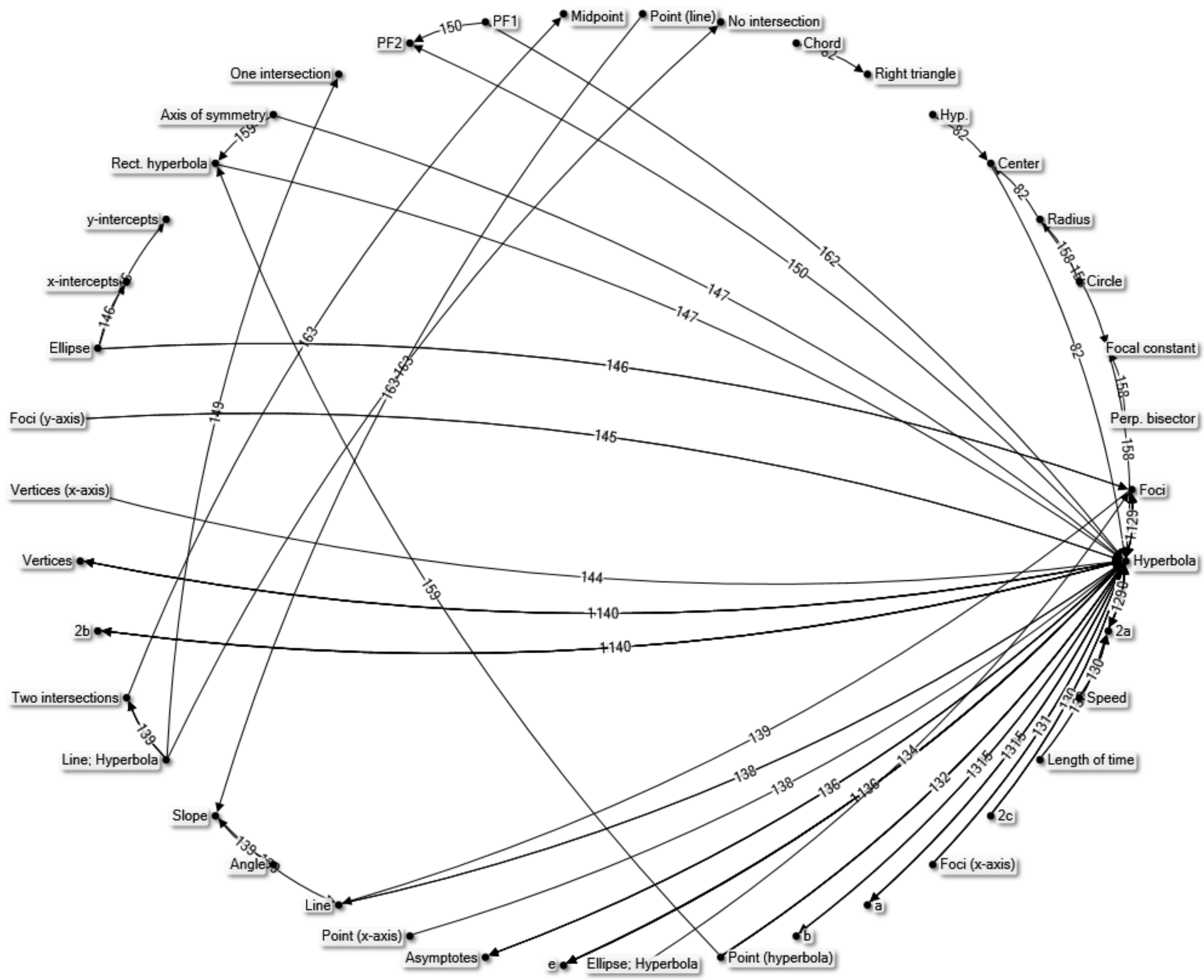
The following digraph is for the subtopic ellipse for the PEP-A series.



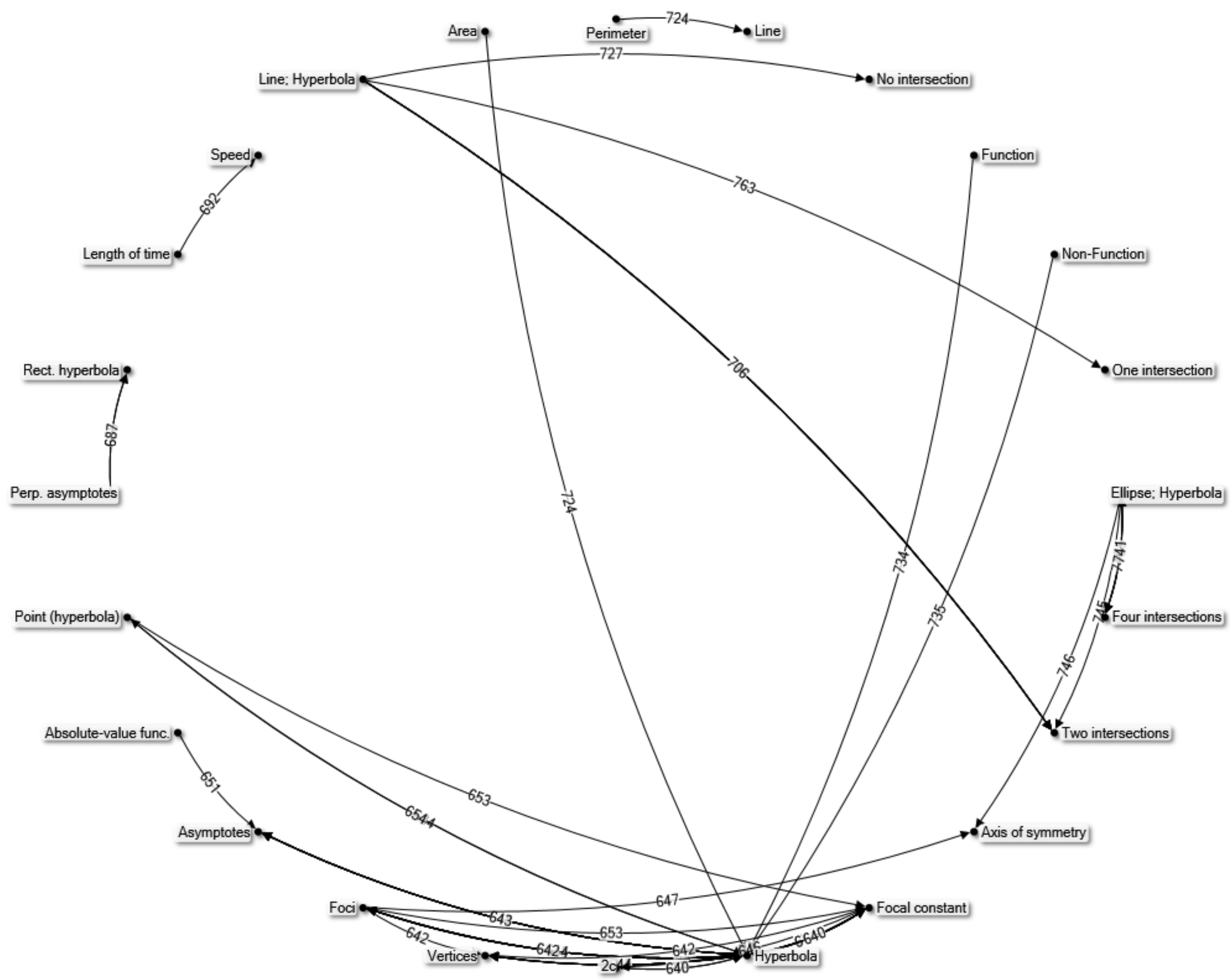
The following digraph is for the subtopic ellipse for the UCSMP series.



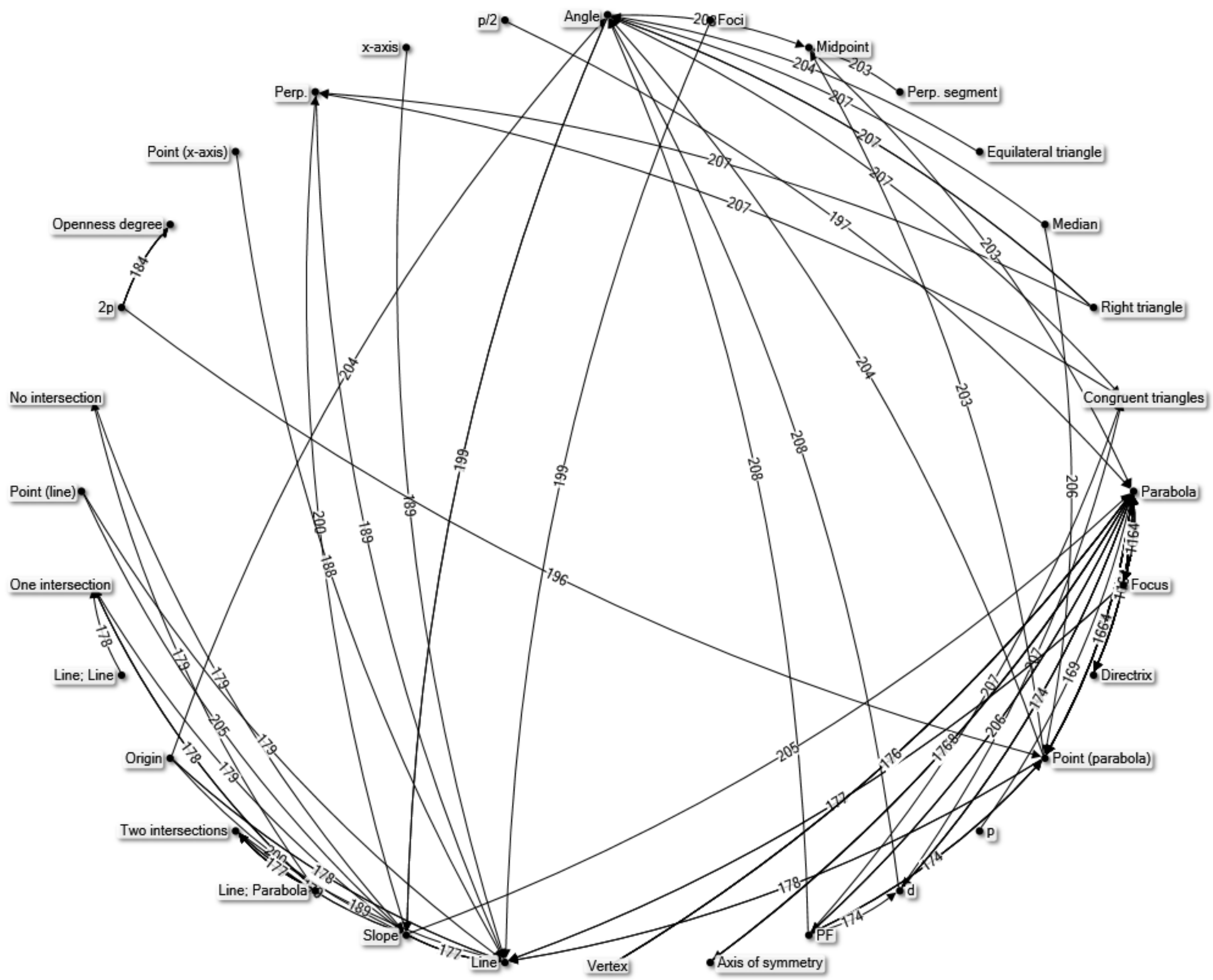
The following digraph is for the subtopic hyperbola for the PEP-A series.



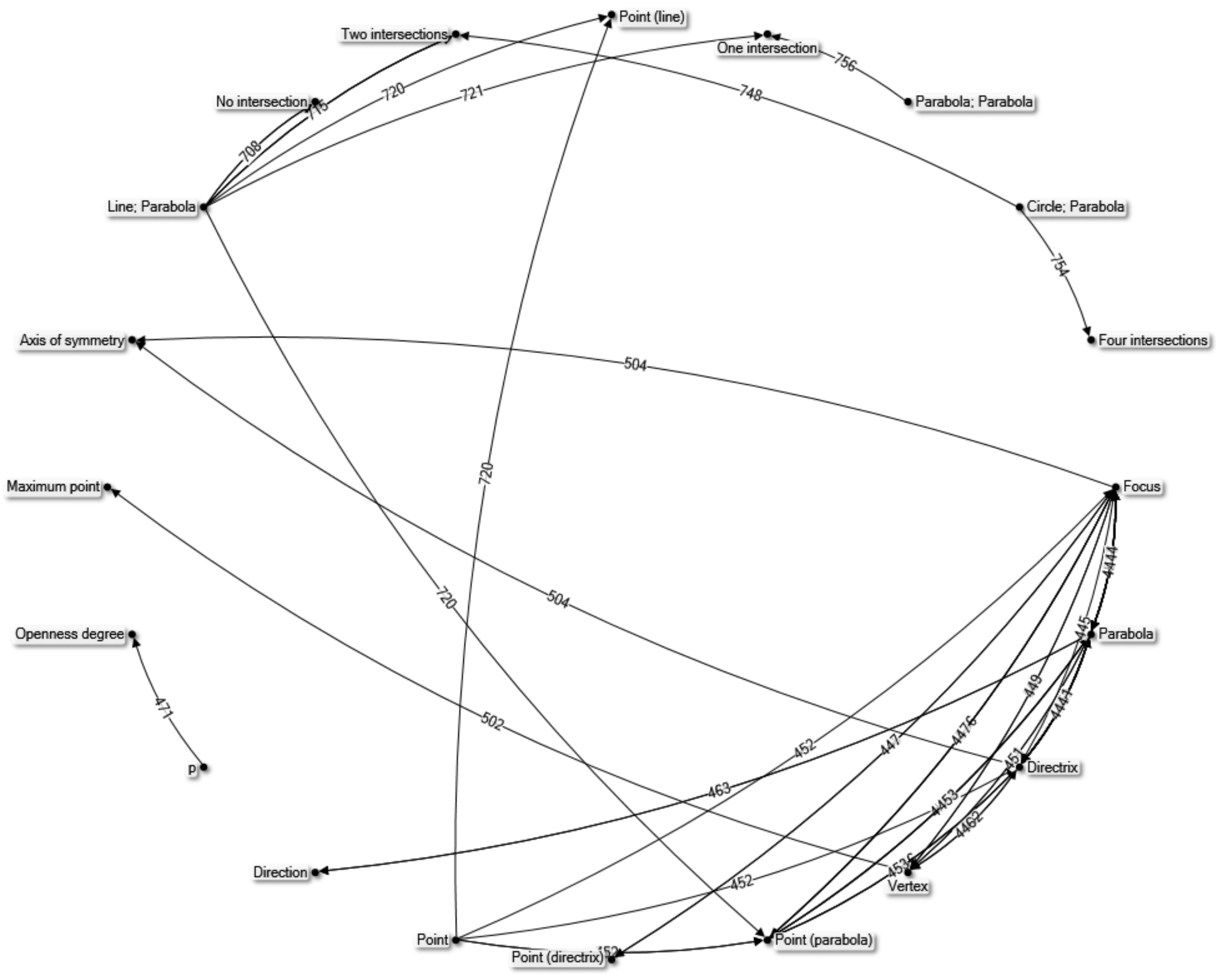
The following digraph is for the subtopic hyperbola for the UCSMP series.



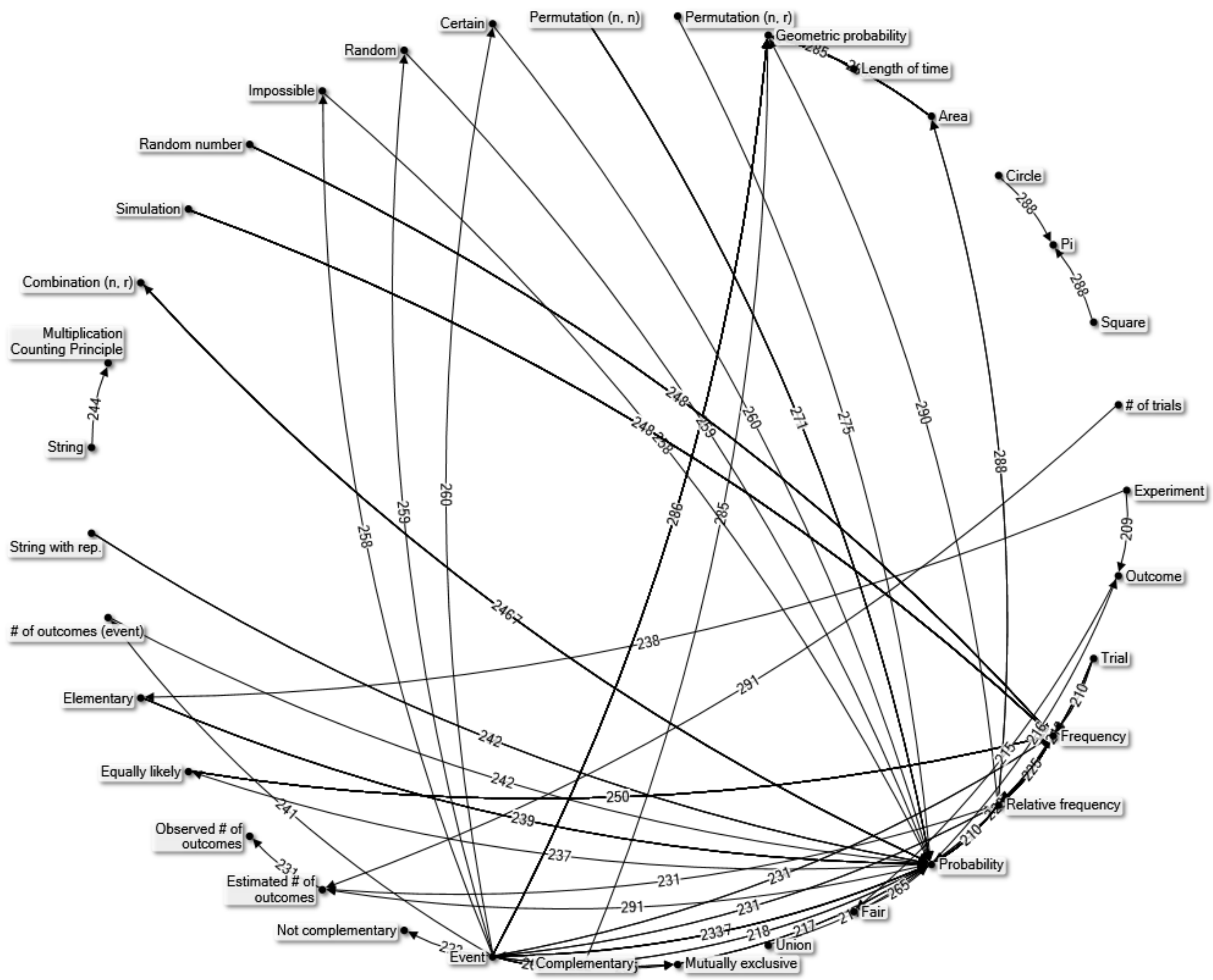
The following digraph is for the subtopic parabola for the PEP-A series.



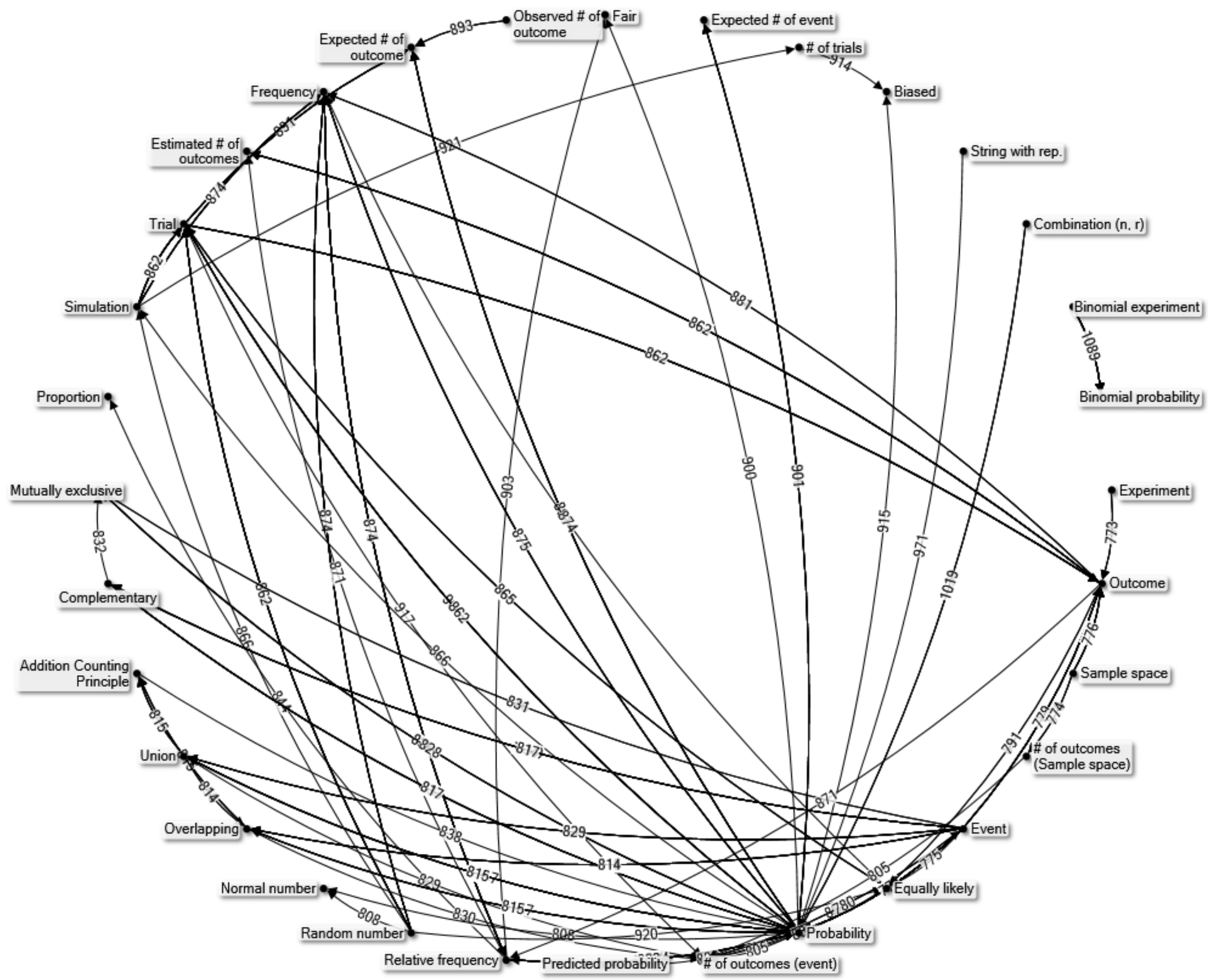
The following digraph is for the subtopic parabola for the UCSMP series.



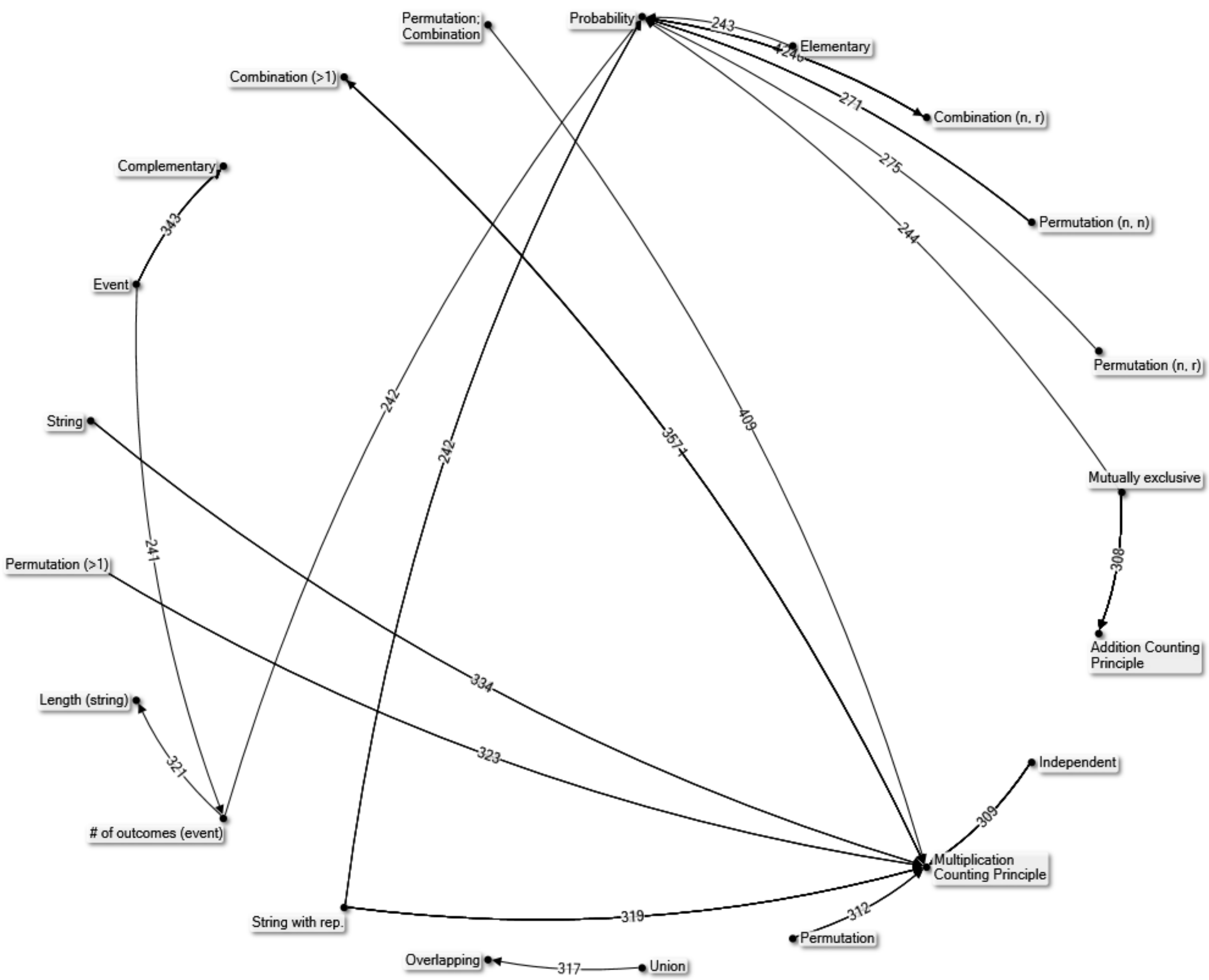
The following digraph is for the subtopic probability for the PEP-A series.



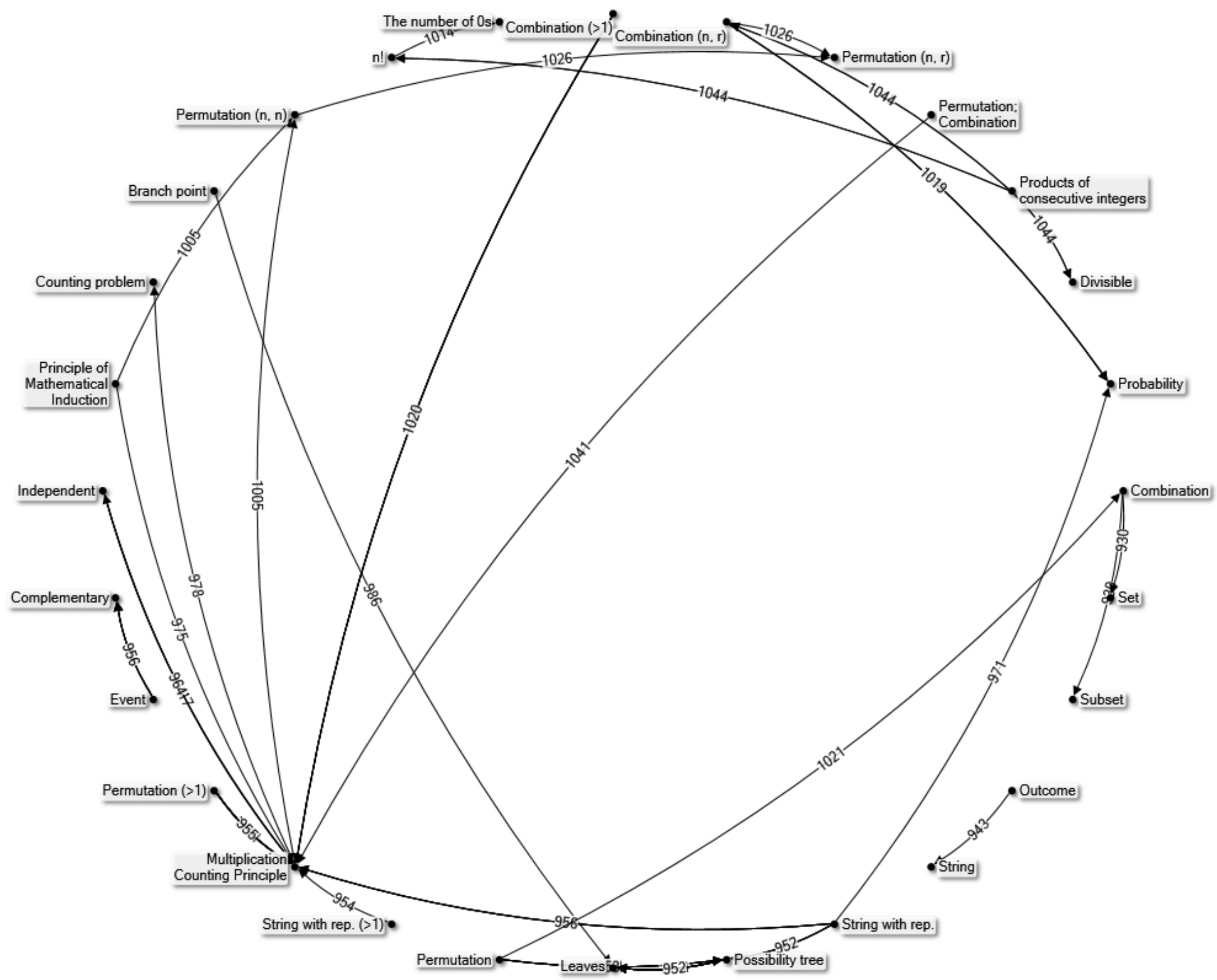
The following digraph is for the subtopic probability for the UCSMP series.



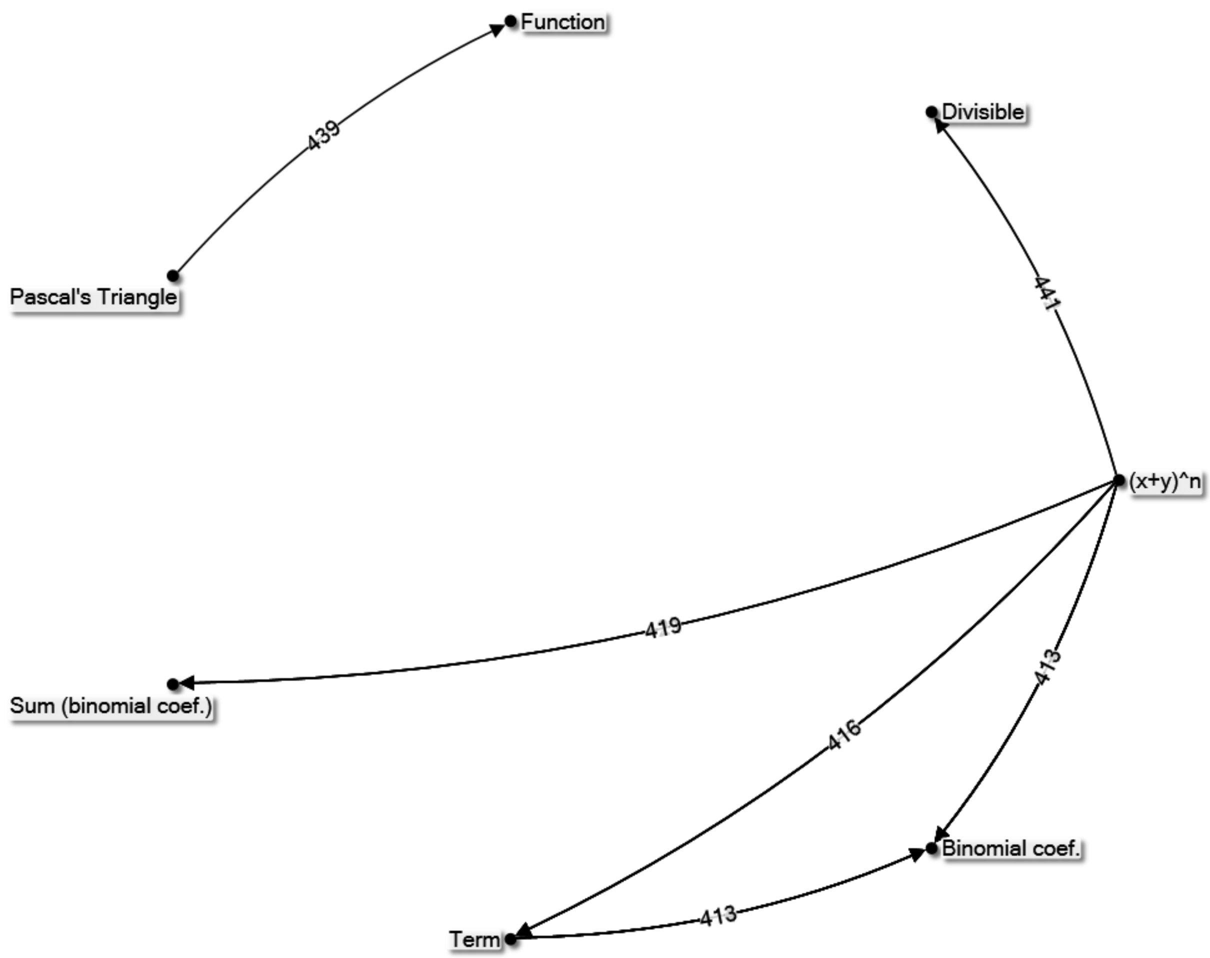
The following digraph is for the subtopic counting problems for the PEP-A series.



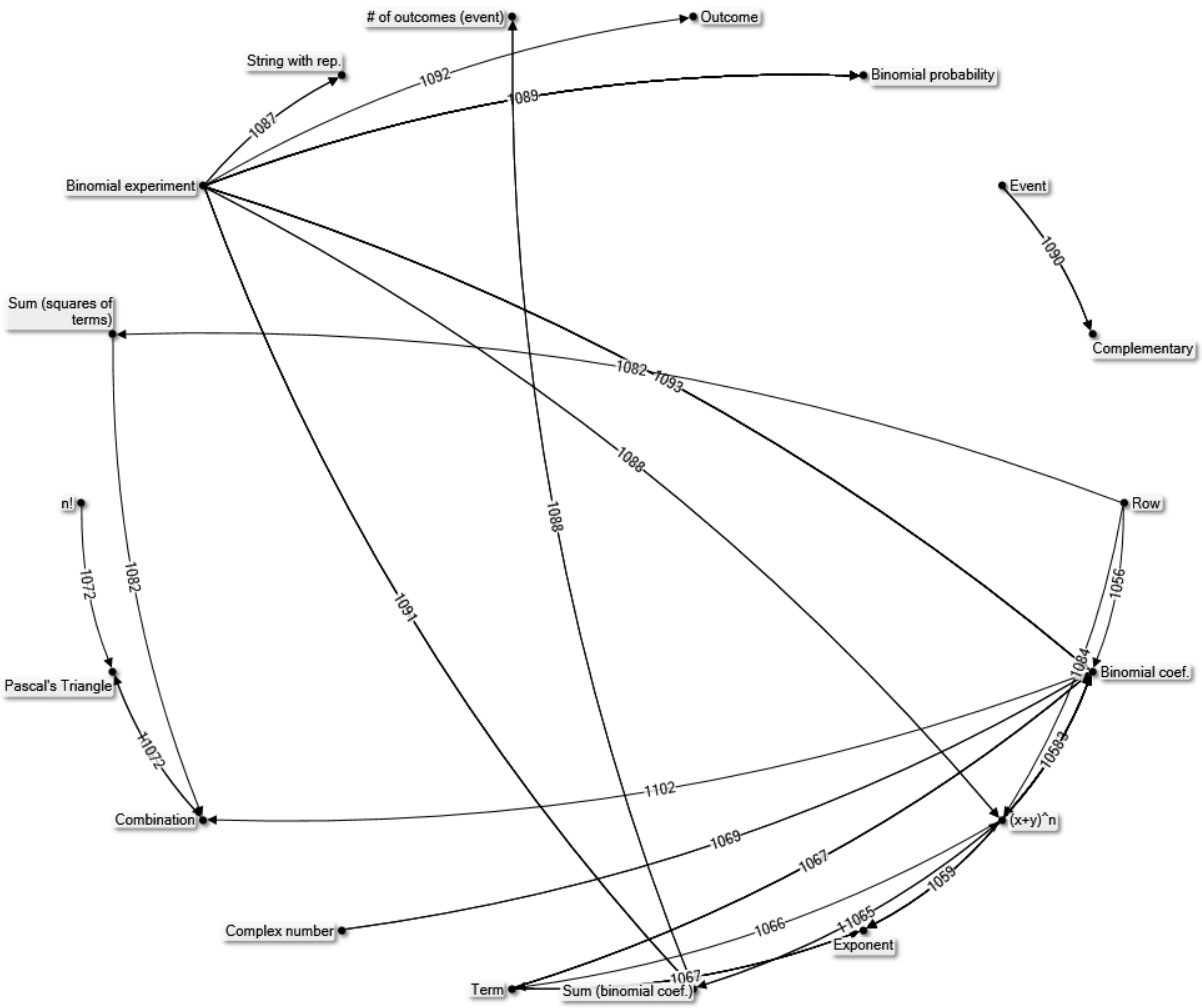
The following digraph is for the subtopic counting problems for the UCSMP series.



The following digraph is for the subtopic binomial theorem for the PEP-A series.

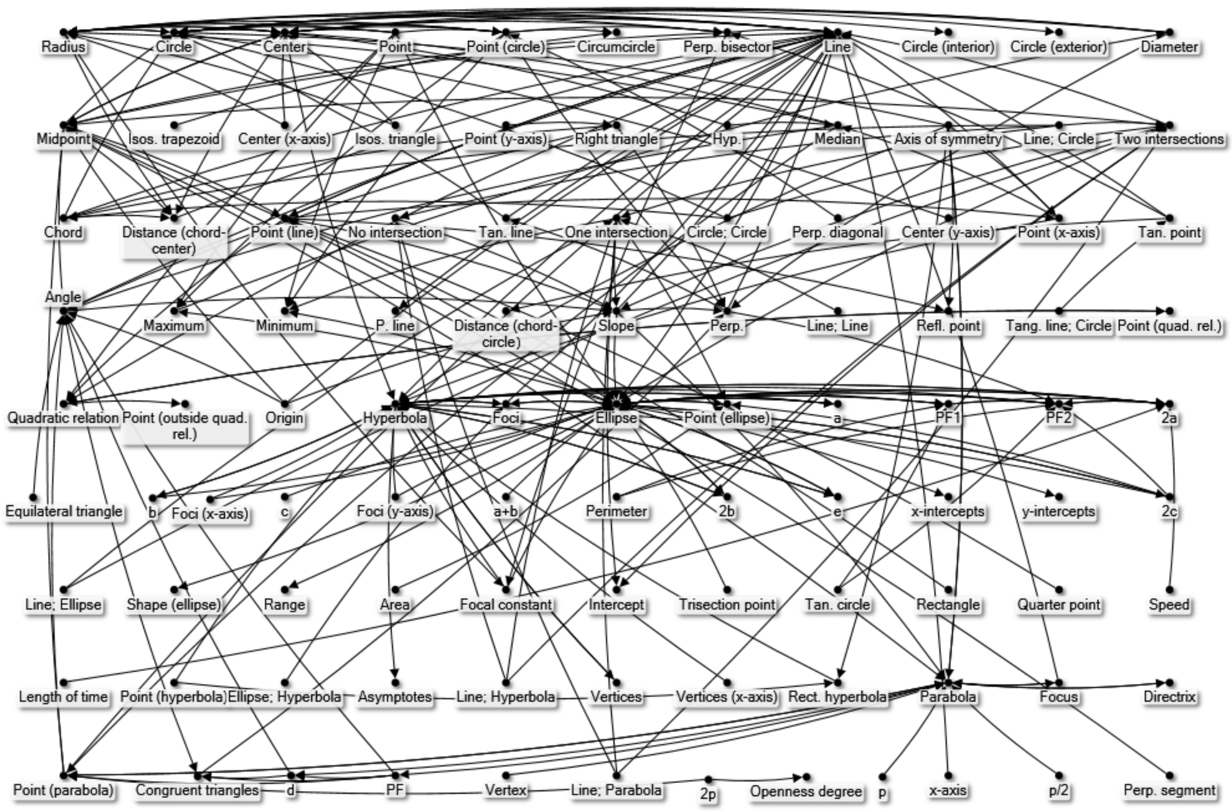


The following digraph is for the subtopic binomial theorem for the UCSMP series.

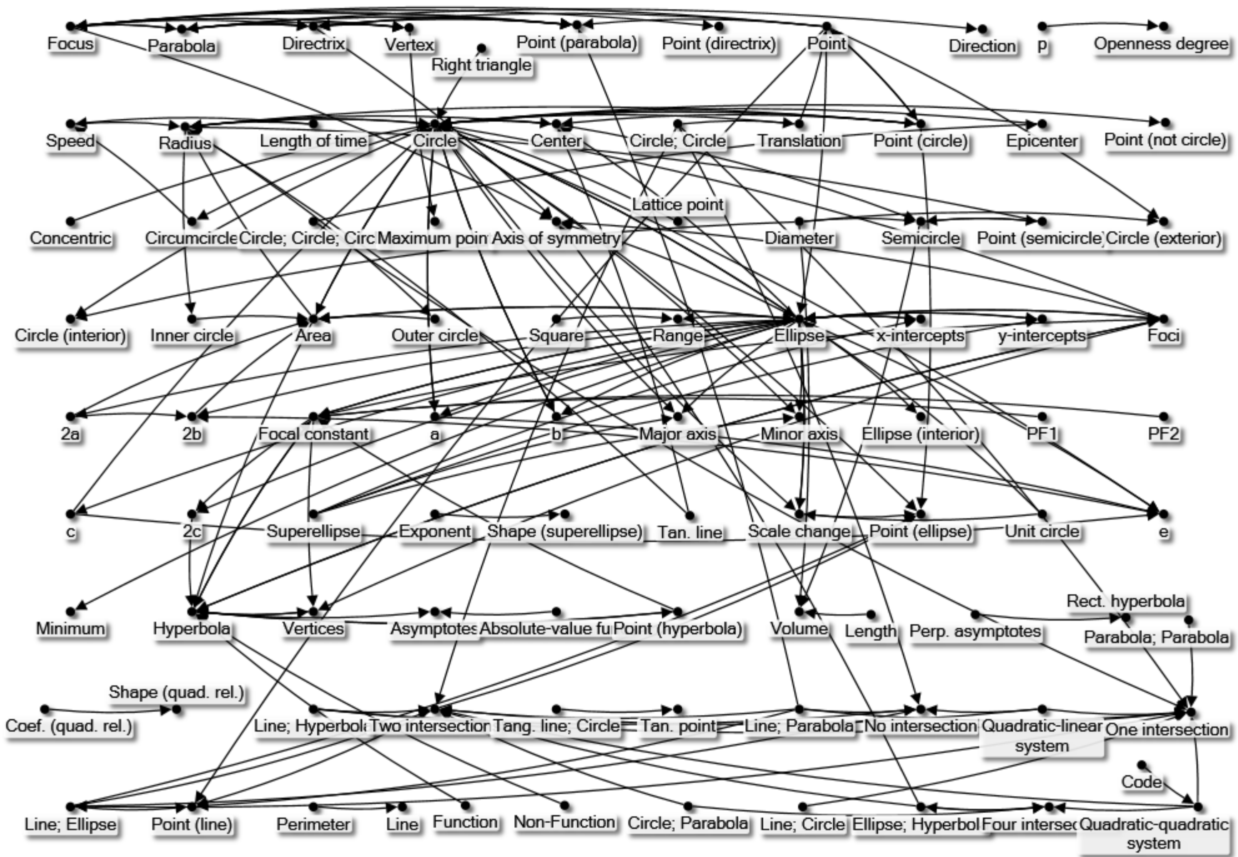


2. Digraphs for Topics

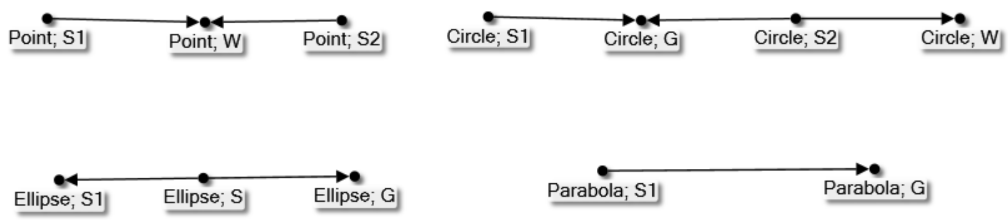
The following digraph is for quadratic relations (between-concept connections) for the PEP-A series.



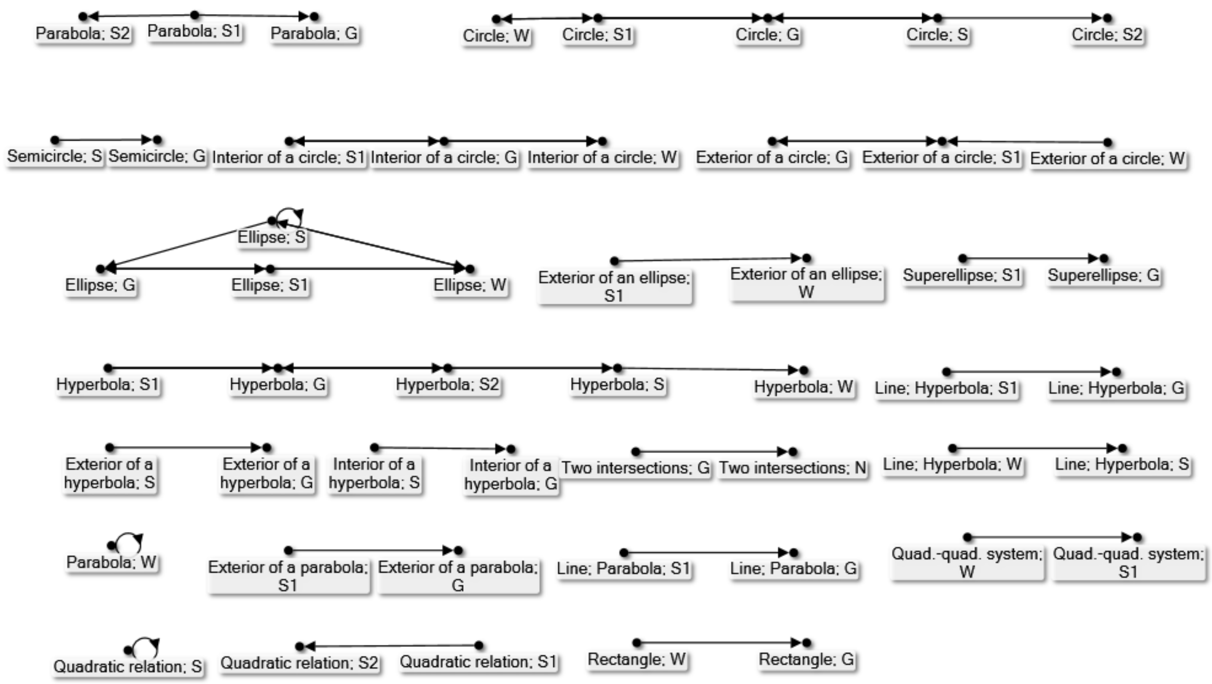
The following digraph is for quadratic relations (between-concept connections) for the UCSMP series.



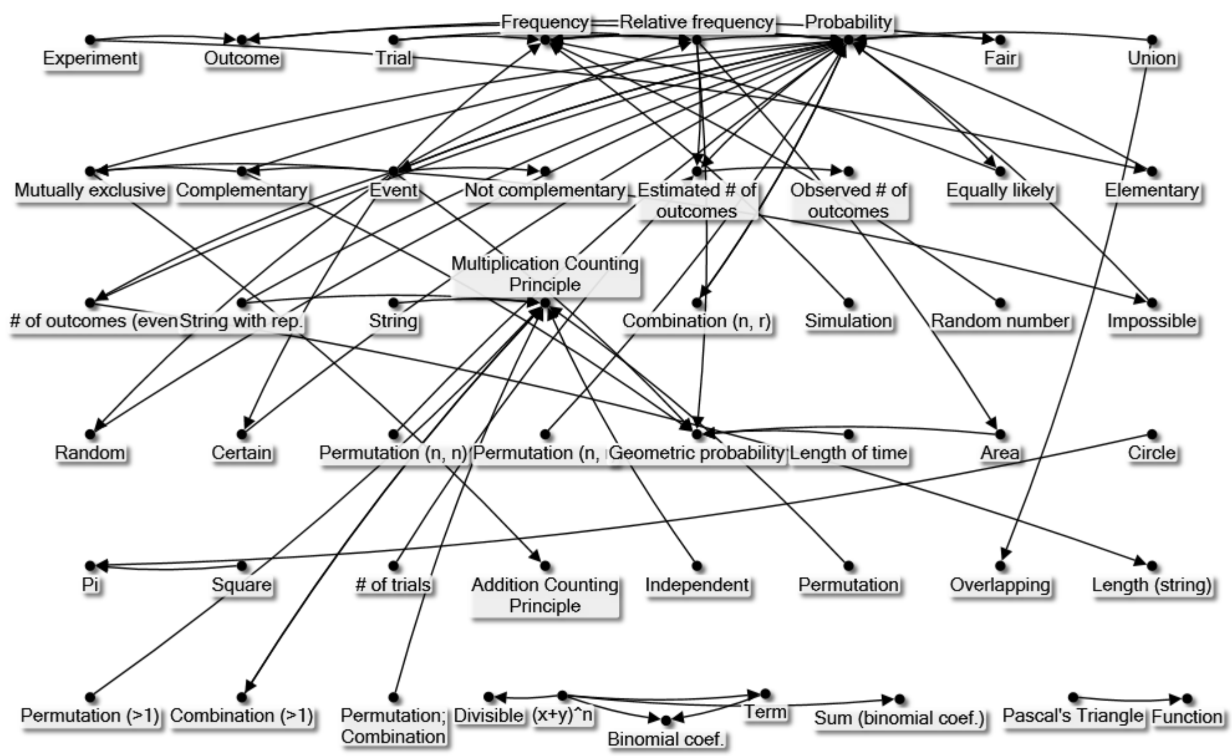
The following digraph is for quadratic relations (within-concept connections) for the PEP-A series.



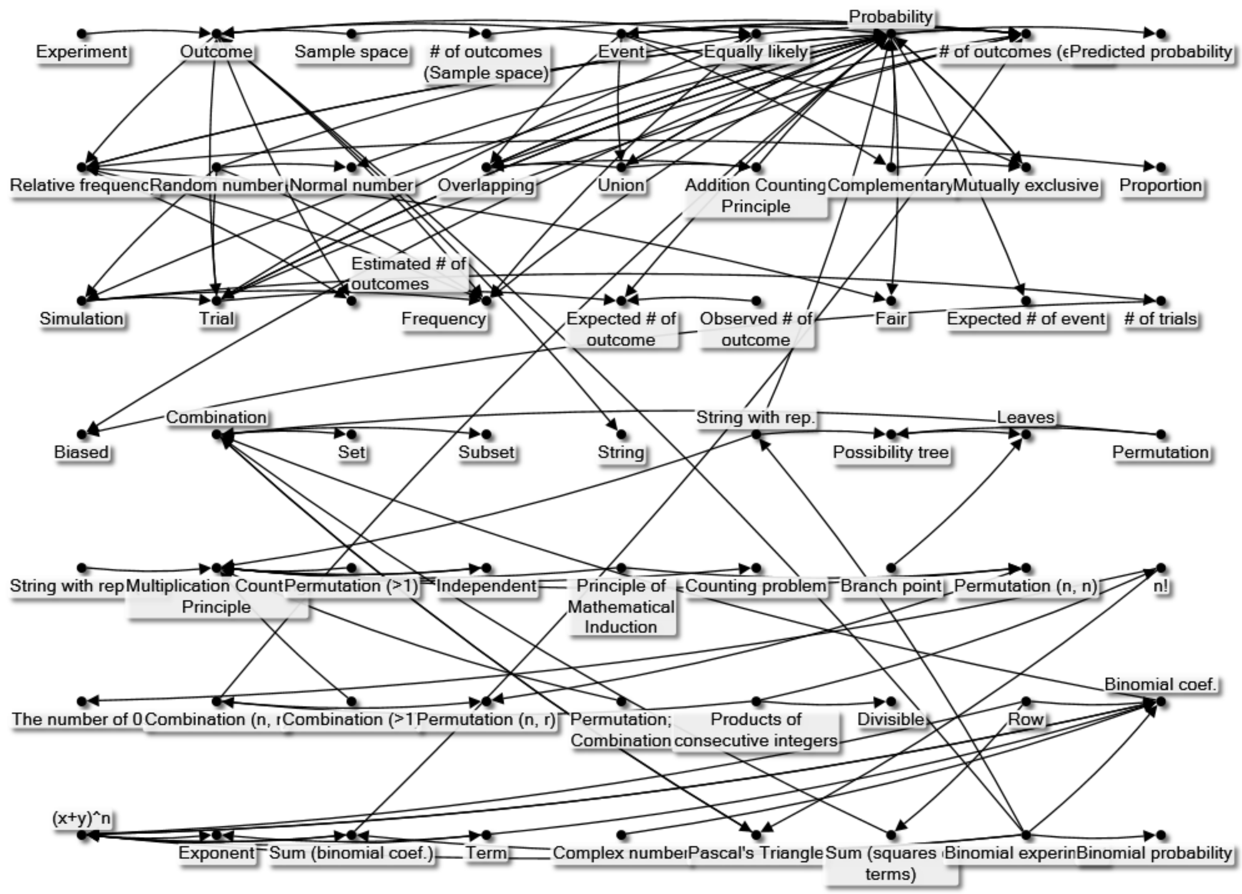
The following digraph is for quadratic relations (within-concept connections) for the UCSMP series.



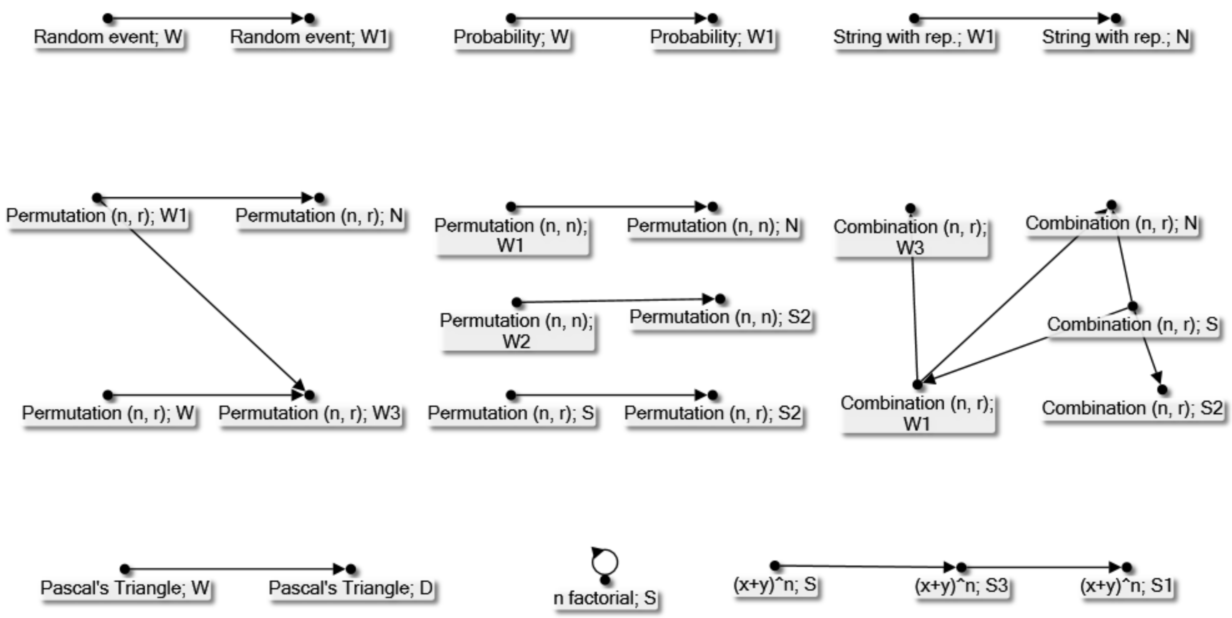
The following digraph is for probability and combinatorics (between-concept connections) for the PEP-A series.



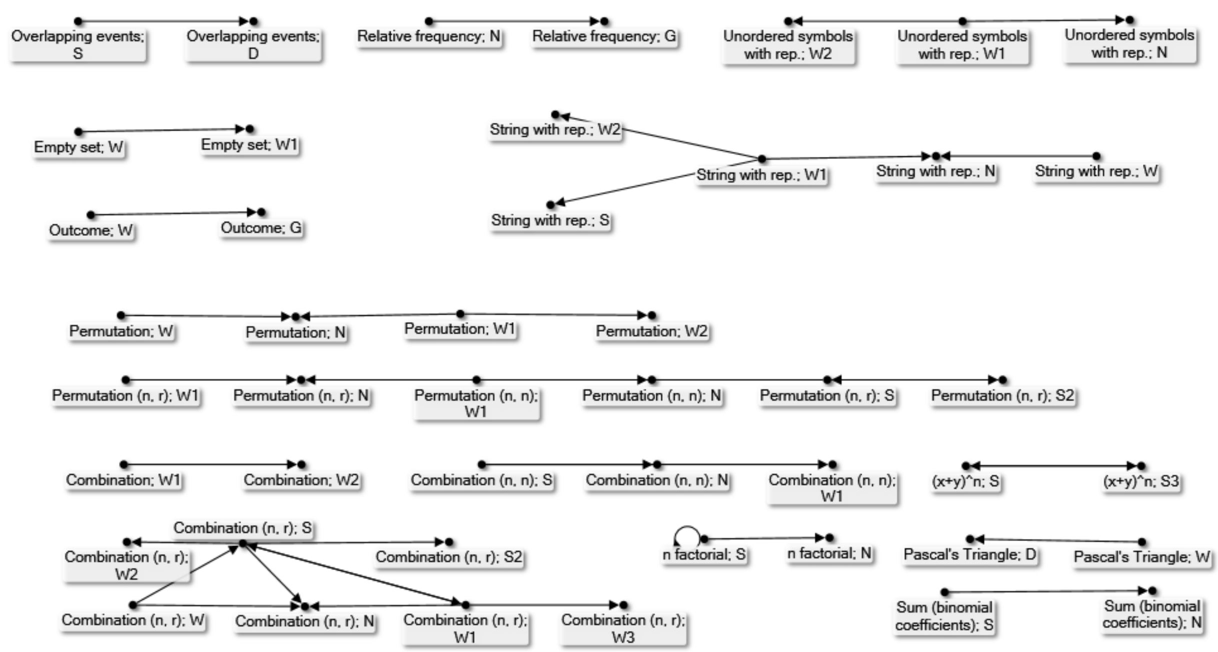
The following digraph is for probability and combinatorics (between-concept connections) for the UCSMP series.



The following digraph is for probability and combinatorics (within-concept connections) for the PEP-A series.



The following digraph is for probability and combinatorics (within-concept connections) for the UCSMP series.



Appendix C

Adjacency Matrices

1. Adjacency Matrix for Within-concept Connections

The following digraph is for probability and combinatorics (within-concept connections) for the PEP-A series.

	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13	X14	X15	X16	X17	X18	X19	X20	X21	X22	X23	X24	X25	X26	X27	Out-degree	Out-Connection
X1		1																										1	1
X2																												0	0
X3				2																								2	1
X4																												0	0
X5																												0	0
X6					6																							6	1
X7																												0	0
X8																												0	0
X9							10																					10	1
X10								1																				1	1
X11																												0	0
X12													3															3	1
X13																												0	0
X14																2												2	1
X15											11					1												12	2
X16																												0	0
X17																												0	0
X18																	1		1	1								3	3
X19																												0	0
X20																	34				2							36	2
X21																												0	0
X22																						1						1	1
X23																												0	0
X24																							1					1	1
X25																											10	10	1
X26																												0	0
X27																										1		1	1
In-degree	0	1	0	2	6	0	10	1	0	0	11	0	3	0	0	3	35	0	1	1	2	1	1	0	0	1	10	89	
In-connection	0	1	0	1	1	0	1	1	0	0	1	0	1	0	0	2	2	0	1	1	1	1	1	0	0	1	1		18

Notes: X1: Probability; W. X2: Probability; W1. X3: Random event; W. X4: Random event; W1. X5: String with repetition; N. X6: String with repetition; W1. X7: Permutation (n, n); N. X8: Permutation (n, n); S2. X9: Permutation (n, n); W1. X10: Permutation (n, n); W2. X11: Permutation (n, r); N. X12: Permutation (n, r); S. X13: Permutation (n, r); S2. X14: Permutation (n, r); W. X15: Permutation (n, r); W1. X16: Permutation (n, r); W3. X17: Combination (n, r); N. X18: Combination (n, r); S. X19: Combination (n, r); S2. X20: Combination (n, r); W1. X21: Combination (n, r); W3. X22: n factorial; S. X23: Pascal's Triangle; D. X24: Pascal's Triangle; W. X25: $(x + y)^n$; S. X26: $(x + y)^n$; S1. X27: $(x + y)^n$; S3.

The following digraph is for probability and combinatorics (within-concept connections) for the UCSMP series.

	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8	Y9	Y10	Y11	Y12	Y13	Y14	Y15	Y16	Y17	Y18	Y19	Y20	Y21	Y22	Y23	Y24	Y25	Y26	Y27	Y28	Y29	Y30	Y31	Y32	Y33	Y34	Y35	Y36	Y37	Y38	Y39	Y40	Y41	Y42	Y43	Y44	Y45	Y46	Out-degree	Out-Connectio
Y1																																															0	0
Y2	1																																														1	1
Y3																																															0	0
Y4			1																																												1	1
Y5						1																																									1	1
Y6																																															0	0
Y7																																															0	0
Y8							1																																								1	1
Y9																																															0	0
Y10									2		4																																				6	2
Y11																																															0	0
Y12																																															0	0
Y13																																															0	0
Y14												2																																			2	1
Y15												10	1			5																															16	3
Y16																																															0	0
Y17																																															0	0
Y18																	1																														1	1
Y19																	4			10																											14	2
Y20																																															0	0
Y21																																															0	0
Y22																					12																										12	1
Y23																																															0	0
Y24																							5		3																						8	2
Y25																								1																							1	1
Y26																							11																								11	1
Y27																												2																			2	1
Y28																																															0	0
Y29																																															0	0
Y30																													5		1																6	2
Y31																																															0	0
Y32																																															0	0
Y33																																7		2		1	1										11	4
Y34																																															0	0
Y35																																2	1														3	2
Y36																																12	1					1									14	3
Y37																																															0	0
Y38																																															0	0
Y39																																															0	0
Y40																																							2	1							3	2
Y41																																															0	0
Y42																																									1						1	1
Y43																																															0	0
Y44																																											1				1	1
Y45																																														6	6	1
Y46																																													1		1	1
In-degree	1	0	1	0	0	1	1	0	2	0	4	12	1	0	0	5	5	0	0	10	12	0	16	1	3	0	0	2	5	0	1	21	2	2	0	1	1	1	2	1	1	0	1	0	1	6	123	
In-connection	1	0	1	0	0	1	1	0	1	0	1	2	1	0	0	1	2	0	0	1	1	0	2	1	1	0	0	1	1	0	1	3	2	1	0	1	1	1	1	1	1	0	1	0	1	1		36

Notes: Y1: Outcome; G. Y2: Outcome; W. Y3: Overlapping events; D. Y4: Overlapping events; S. Y5: Empty set; W. Y6: Empty set; W1. Y7: Relative frequency; G. Y8: Relative frequency; N. Y9: Unordered symbols w/ rep.; N. Y10: Unordered symbols w/ rep.; W1. Y11: Unordered symbols w/ rep.; W2. Y12: String with rep.; N. Y13: String with rep.; S. Y14: String with rep.; W. Y15: String with rep.; W1. Y16: String with rep.; W2. Y17: Permutation; N. Y18: Permutation; W. Y19: Permutation; W1. Y20: Permutation; W2. Y21: Permutation (n, n); N. Y22: Permutation (n, n); W1. Y23: Permutation (n, r); N. Y24: Permutation (n, r); S. Y25: Permutation (n, r); S2. Y26: Permutation (n, r); W1. Y27: Combination; W1. Y28: Combination; W2. Y29: Combination (n, n); N. Y30: Combination (n, n); S. Y31: Combination (n, n); W1. Y32: Combination (n, r); N. Y33: Combination (n, r); S. Y34: Combination (n, r); S2. Y35: Combination (n, r); W. Y36: Combination (n, r); W1. Y37: Combination (n, r); W2. Y38: Combination (n, r); W3. Y39: n factorial; N. Y40: n factorial; S. Y41: Pascal's Triangle; D. Y42: Pascal's Triangle; W. Y43: Sum (binomial coefficients); N. Y44: Sum (binomial coefficients); S. Y45: $(x + y)^n$; S. Y46: $(x + y)^n$; S3.

The following digraph is for quadratic relations (within-concept connections) for the PEP-A series.

	Point; S1	Point; W	Point; S2	Circle; G	Circle; S1	Circle; S2	Circle; W	Ellipse; G	Ellipse; S	Ellipse; S1	Parabola; G	Parabola; S1	Out-degree	Out-Connection
Point; S1		1											1	1
Point; W													0	0
Point; S2		1											1	1
Circle; G													0	0
Circle; S1				4									4	1
Circle; S2				4			2						6	2
Circle; W													0	0
Ellipse; G													0	0
Ellipse; S								2		1			3	2
Ellipse; S1													0	0
Parabola; G													0	0
Parabola; S1											6		6	1
In-degree	0	2	0	8	0	0	2	2	0	1	6	0	21	
In-connection	0	2	0	2	0	0	1	1	0	1	1	0		8

The following digraph is for quadratic relations (within-concept connections) for the UCSMP series.

	Z1	Z2	Z3	Z4	Z5	Z6	Z7	Z8	Z9	Z10	Z11	Z12	Z13	Z14	Z15	Z16	Z17	Z18	Z19	Z20	Z21	Z22	Z23	Z24	Z25	Z26	Z27	Z28	Z29	Z30	Z31	Z32	Z33	Z34	Z35	Z36	Z37	Z38	Z39	Z40	Z41	Z42	Z43	Z44	Z45	Z46	Z47	Z48	Z49	Z50	Z51	Out-degree	Out-Connectio
Z1																																																				0	0
Z2	1																																																			1	1
Z3	7	4		4	1																																															16	4
Z4																																																				0	0
Z5			2																																																	2	1
Z6							1																																													1	1
Z7						1																																														1	1
Z8							1																																													1	1
Z9											1																																									1	1
Z10									2		1																																									3	2
Z11										1																																										1	1
Z12																																																				0	0
Z13												2																																								2	1
Z14																3	1																																			4	2
Z15														1	1		2																																			4	3
Z16														6			1																																			7	2
Z17														1	1																																					2	2
Z18																			1																																	1	1
Z19																																																				0	0
Z20																																																				0	0
Z21																				2																																2	1
Z22																																																				0	0
Z23																						2				1																										3	2
Z24																						6	1		2																											9	3
Z25																																																				0	0
Z26																																																				0	0
Z27																																																				0	0
Z28																											1																									1	1
Z29																																																				0	0
Z30																													1																							1	1
Z31																																																				0	0
Z32																																																				0	0
Z33																															3																					3	1
Z34																																1																				1	1
Z35																																																				0	0
Z36																																			1																	1	1
Z37																																																				0	0
Z38																																					1		1													2	2
Z39																																																				0	0
Z40																																								1												1	1
Z41																																																				0	0
Z42																																									1											1	1
Z43																																											1									1	1
Z44																																													3							3	1
Z45																																																				0	0
Z46																																																				0	0
Z47																																														1						1	1
Z48																																																				0	0
Z49																																																1				1	1
Z50																																																			1	1	1
Z51																																																				0	0
In-degree	8	4	2	4	1	1	2	0	2	1	2	2	0	8	2	3	4	0	1	2	0	8	1	0	2	1	1	0	1	0	3	1	0	0	1	0	1	0	1	1	1	0	1	0	3	1	0	1	0	0	1	79	
In-connection	2	1	1	1	1	1	2	0	1	1	2	1	0	3	2	1	3	0	1	1	0	2	1	0	1	1	1	0	1	0	1	1	0	0	1	0	1	0	1	1	1	0	1	0	1	1	0	1	0	0	1		44

Notes: Z1: Circle; G. Z2: Circle; S. Z3: Circle; S1. Z4: Circle; S2. Z5: Circle; W. Z6: Ext. of a circle; G. Z7: Ext. of a circle; S1. Z8: Ext. of a circle; W. Z9: Int. of a circle; G. Z10: Int. of a circle; S1. Z11: Int. of a circle; W. Z12: Semicircle; G. Z13: Semicircle; S. Z14: Ellipse; G. Z15: Ellipse; S. Z16: Ellipse; S1. Z17: Ellipse; W. Z18: Ext. of an ellipse; S1. Z19: Ext. of an ellipse; W. Z20: Superellipse; G. Z21: Superellipse; S1. Z22: Hyperbola; G. Z23: Hyperbola; S. Z24: Hyperbola; S1. Z25: Hyperbola; S2. Z26: Hyperbola; W. Z27: Ext. of a hyperbola; G. Z28: Ext. of a hyperbola; S. Z29: Int. of a hyperbola; G. Z30: Int. of a hyperbola; S. Z31: Line; Hyperbola; G. Z32: Line; Hyperbola; S. Z33: Line; Hyperbola; S1. Z34: Line; Hyperbola; W. Z35: Line; Parabola; G. Z36: Line; Parabola; S1. Z37: Parabola; G. Z38: Parabola; S1. Z39: Parabola; S2. Z40: Parabola; W. Z41: Ext. of a parabola; G. Z42: Ext.of a parabola; S1. Z43: Quadratic relation; S. Z44: Quadratic relation; S1. Z45: Quadratic relation; S2. Z46: Quadratic-quadratic system; S1. Z47: Quadratic-quadratic system; W. Z48: Rectangle; G. Z49: Rectangle; W. Z50: Two intersections; G. Z51: Two intersections; N.

2. Adjacency Matrix for Between-concept Connections

The following digraph is for probability and combinatorics (between-concept connections) for the PEP-A series.

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20	A21	A22	A23	A24	A25	A26	A27	A28	A29	A30	A31	A32	A33	A34	A35	A36	A37	A38	A39	A40	A41	A42	A43	A44	A45	A46	A47	A48	A49	A50	Out-degree	Out-Connection
A1	0	0	0	9	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	4	0	0	21	4
A2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	13	1
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
A6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	0	7	1
A8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	4	3
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	0	7	1
A11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	1
A12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A13	0	0	0	0	1	0	0	10	0	0	0	0	0	0	0	2	0	16	1	0	0	0	0	0	0	2	1	1	0	0	0	0	0	0	0	0	0	0	5	1	0	2	0	0	0	0	0	0	0	0	42	11
A14	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
A15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	25	0	0	0	0	0	0	0	0	26	2
A17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
A20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	1
A21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	1
A23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	1
A24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	1
A25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A26	0	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	16	2
A27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	2	2
A29	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	1
A35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	5	1
A36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
A37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A39	0	0	0	0	0	0	1	0	0	0	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	7
A40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
A41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	1
A42	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	26	0	0	0	0	0	0	0	0	0	0	0	30	4
A43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17	1
A44	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
A45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	1
A46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	11	2
A47	0	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	1
A48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A49	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	6	2
A50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	2	2
In-degree	0	13	2	17	1	0	1	10	2	1	1	3	1	0	2	48	1	35	1	0	1	0	1	0	34	3	1	1	0	1	2	1	0	0	0	0	0	2	65	1	0	31	0	0	0	0	6	4	0	0	293	
In-Connection	0	1	1	2	1	0	1	1	1	1	1	3	1	0	2	6	1	5	1	0	1	0	1	0	7	2	1	1	0	1	2	1	0	0	0	0	0	2	15	1	0	3	0	0	0	0	1	1	0	0		69

Notes: A1: $(x + y)^n$; A2: Addition Counting Principle; A3: Area; A4: Binomial coefficient; A5: Certain event; A6: Circle; A7: Combination of n elements taken r at a time; A8: Complementary events; A9: Divisible; A10: Elementary event; A11: Equally likely; A12: Estimated number of outcomes; A13: Event; A14: Experiment; A15: Fair; A16: Frequency; A17: Function; A18: Geometric models of probability; A19: Impossible event; A20: Independent events; A21: Length of a string; A22: Length of time; A23: More than one combination; A24: More than one permutation; A25: Multiplication Counting Principle; A26: Mutually exclusive events; A27: Not complementary event; A28: Number of outcomes in the event; A29: Number of trials; A30: Observed number of outcomes; A31: Outcome; A32: Overlapping events; A33: Pascal's Triangle; A34: Permutation; A35: Permutation (n, n); A36: Permutation (n, r); A37: Permutation; Combination; A38: π; A39: Probability; A40: Random event; A41: Random number; A42: Relative frequency; A43: Simulation; A44: Square; A45: String; A46: String with repetition; A47: Term; A48: Sum (binomial coefficients); A49: Trial; A50: Union of events.

The following digraph is for probability and combinatorics (between-concept connections) for the UCSMP series.

	B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	B21	B22	B23	B24	B25	B26	B27	B28	B29	B30	B31	B32	B33	B34	B35	B36	B37	B38	B39	B40	B41	B42	B43	B44	B45	B46	B47	B48	B49	B50	B51	B52	B53	B54	B55	B56	B57	B58	B59	B60	B61	B62	B63	Out-degree	Out-Connection
B1	0	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	20	4
B2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
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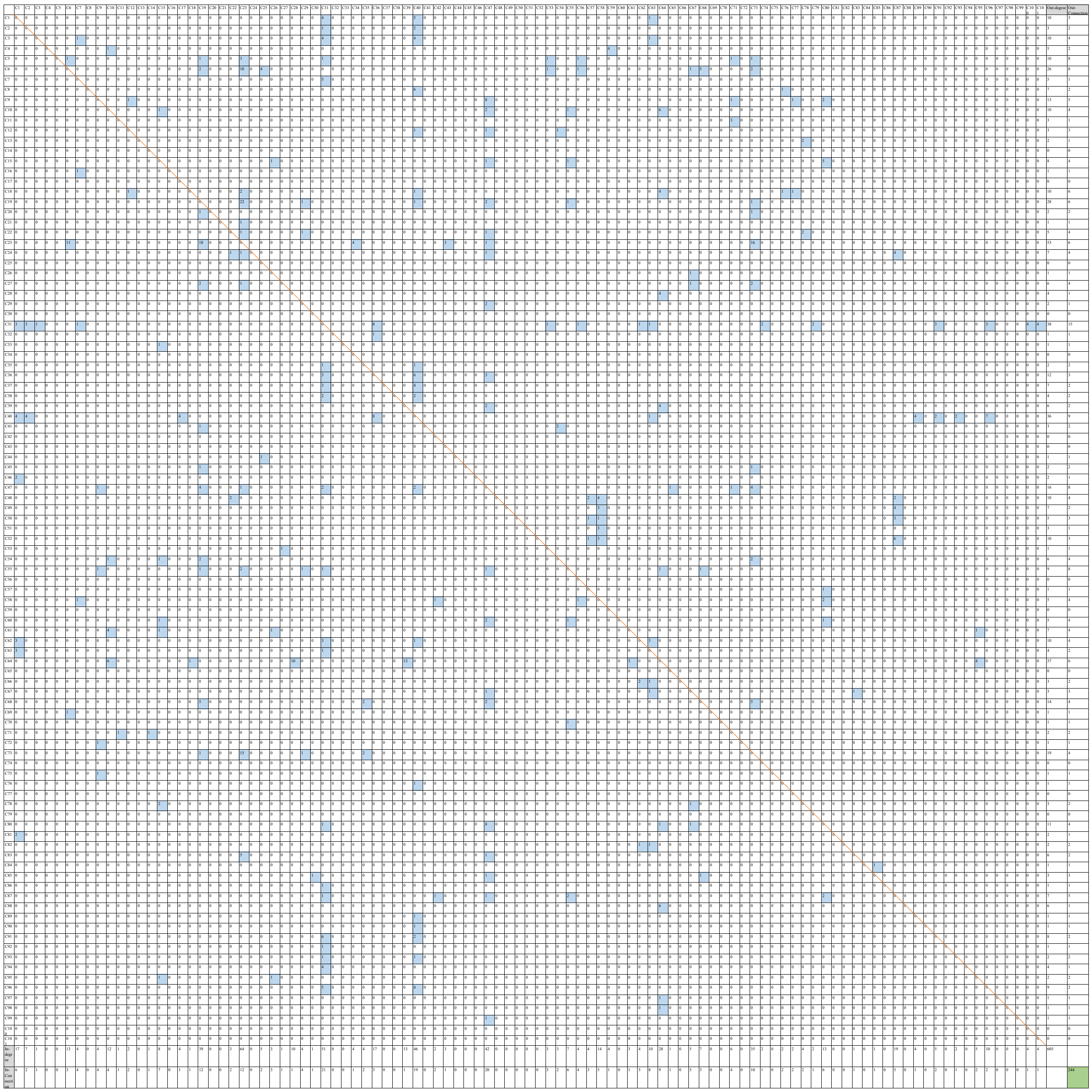
B60	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
B61	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	12	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	21	4
B62	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0	0	0	0	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	18	4
B63	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2	2
In-degree	6	8	2	27	0	13	0	4	2	15	0	1	2	4	11	1	6	19	0	11	2	24	1	11	0	0	0	24	4	2	3	10	1	0	32	10	2	0	2	2	0	10	1	0	77	0	1	0	17	0	0	1	2	1	2	1	1	1	5	1	34	7	2	426	
In-Connection	4	2	2	5	0	1	0	4	1	1	0	1	1	3	2	1	1	3	0	2	2	5	1	2	0	0	0	7	3	2	1	8	1	0	5	3	2	0	2	2	0	2	1	0	10	0	1	0	3	0	0	1	2	1	1	1	1	1	2	1	4	2	1		115

Notes: B1: $(x + y)^n$; B2: Addition Counting Principle; B3: Biased; B4: Binomial coefficient; B5: Binomial experiment; B6: Binomial probability; B7: Branch point; B8: Combination; B9: Combination (n, r); B10: Complementary events; B11: Complex number; B12: Counting problem; B13: Divisible; B14: Equally likely; B15: Estimated number of outcomes; B16: Event; B17: Expected count of an event; B18: Expected count of an outcome; B19: Experiment; B20: Exponent; B21: Fair; B22: Frequency; B23: Independent events; B24: Leaves; B25: More than one combination; B26: More than one permutation; B27: More than one string with repetition; B28: Multiplication Counting Principle; B29: Mutually exclusive events; B30: Normal number; B31: Number of outcomes in the sample space; B32: Number of outcomes in the event; B33: Number of trials; B34: Observed count of an outcome; B35: Outcome; B36: Overlapping events; B37: Pascal's Triangle; B38: Permutation; B39: Permutation (n, n); B40: Permutation (n, r); B41: Permutation; Combination; B42: Possibility tree; B43: Predicted probability; B44: Principle of Mathematical Induction; B45: Probability; B46: Products of consecutive integers; B47: Proportion; B48: Random number; B49: Relative frequency; B50: Row; B51: Sample space; B52: Set; B53: Simulation; B54: String; B55: String with repetition; B56: Subset; B57: Term; B58: The number of 0s; B59: Sum (binomial coefficients); B60: The sum of the squares of terms; B61: Trial; B62: Union of events; B63: n factorial.

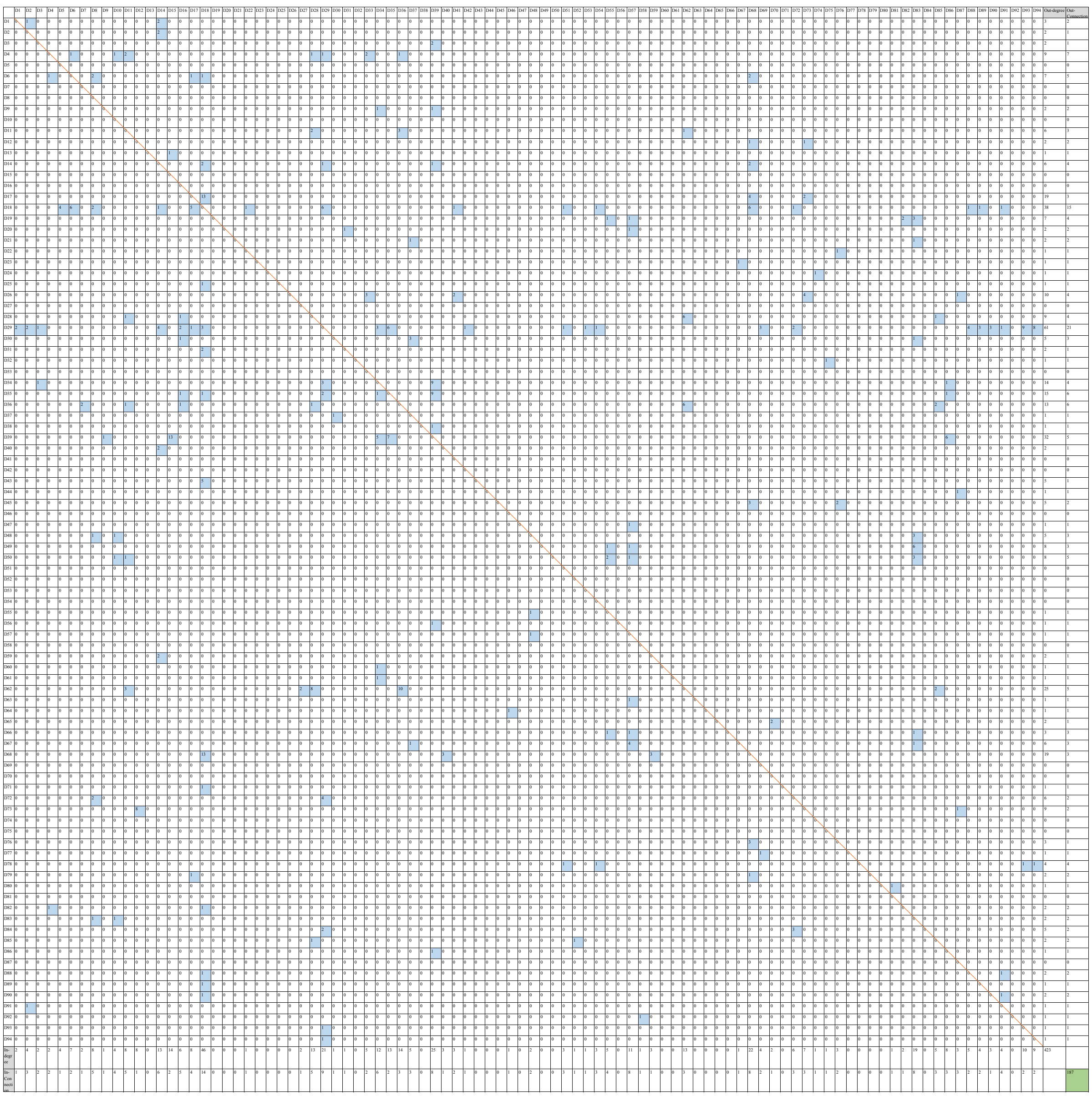
The following digraph is for quadratic relations (between-concept connections) for the PEP-A series.



Notes: C1: 2a; C2: 2b; C3: 2c; C4: 2p; C5: Point; C6: Point on the circle; C7: Point on the ellipse; C8: Point on the hyperbola; C9: Point on the line; C10: Point on the parabola; C11: Point on the quadratic relation; C12: Point on the x-axis; C13: Point on the y-axis; C14: Point outside the quadratic relation; C15: Angle; C16: Area; C17: Asymptote; C18: Axis of symmetry; C19: Center; C20: Center on x-axis; C21: Center on y-axis; C22: Chord; C23: Circle; C24: Circle; Circle; C25: Circumcircle; C26: Congruent triangles; C27: Diameter; C28: Directrix; C29: Distance from the chord to the center; C30: Distance from the chord to the point on the circle; C31: Ellipse; C32: Ellipse; Hyperbola; C33: Equilateral triangle; C34: Exterior of a circle; C35: Focal constant; C36: Foci; C37: Foci on x-axis; C38: Foci on y-axis; C39: Focus; C40: Hyperbola; C41: Hypotenuse; C42: Intercept; C43: Interior of a circle; C44: Isosceles trapezoid; C45: Isosceles triangle; C46: Length of time; C47: Line; C48: Line; Circle; C49: Line; Ellipse; C50: Line;

Hyperbola; C51: Line; Line; C52: Line; Parabola; C53: Maximum; C54: Median; C55: Midpoint; C56: Minimum; C57: No intersection; C58: One intersection; C59: Openness degree; C60: Origin; C61: PF; C62: PF1; C63: PF2; C64: Parabola; C65: Parallel line; C66: Perimeter; C67: Perpendicular; C68: Perpendicular bisector; C69: Perpendicular diagonal; C70: Perpendicular segment; C71: Quadratic relation; C72: Quarter point; C73: Radius; C74: Range; C75: Rectangle; C76: Rectangular hyperbola; C77: Reflection point; C78: Right triangle; C79: Shape of ellipse; C80: Slope; C81: Speed; C82: Tangent circle; C83: Tangent line; C84: Tangent line; Circle; C85: Tangent point; C86: Trisection point; C87: Two intersections; C88: Vertex; C89: Vertices; C90: Vertices on x-axis; C91: a; C92: a+b; C93: b; C94: c; C95: d; C96: e; C97: p; C98: p/2; C99: x-axis; C100: x-intercept; C101: y-intercept.

The following digraph is for quadratic relations (between-concept connections) for the UCSMP series.



Notes: D1: 2a; D2: 2b; D3: 2c; D4: Point; D5: Point not on the circle; D6: Point on the circle; D7: Point on the directrix; D8: Point on the ellipse; D9: Point on the hyperbola; D10: Point on the line; D11: Point on the parabola; D12: Point on the semicircle; D13: Absolute-value function; D14: Area; D15: Asymptote; D16: Axis of symmetry; D17: Center; D18: Circle; D19: Circle; Circle; D20: Circle; Circle; Circle; D21: Circle; Parabola; D22: Circumcircle; D23: Code; D24: Coefficients of quadratic relations; D25: Concentric; D26: Diameter; D27: Direction; D28: Directrix; D29: Ellipse; D30: Ellipse; Hyperbola; D31: Epicenter; D32: Exponent; D33: Exterior of a circle; D34: Focal constant; D35: Foci; D36: Focus; D37: Four intersections; D38: Function; D39: Hyperbola; D40: Inner circle; D41: Interior of a circle; D42: Interior of an ellipse; D43: Lattice point; D44: Length; D45: Length of time; D46: Line; D47: Line; Circle; D48: Line; Ellipse; D49: Line; Hyperbola; D50: Line; Parabola; D51: Major axis; D52: Maximum point; D53: Minimum; D54: Minor axis; D55: No intersection; D56: Non-function; D57: One intersection; D58: Openness Degree; D59: Outer circle; D60: PF1; D61: PF2; D62: Parabola; D63: Parabola; Parabola; D64: Perimeter; D65: Perpendicular asymptotes; D66: Quadratic-linear system; D67: Quadratic-quadratic system; D68: Radius; D69: Range; D70: Rectangular hyperbola; D71: Right triangle; D72: Scale change; D73: Semicircle; D74: Shape of quadratic relations; D75: Shape of superellipse; D76: Speed; D77: Square; D78: Superellipse; D79: Tangent line; D80: Tangent line; Circle; D81: Tangent point; D82: Translation; D83: Two intersections; D84: Unit circle; D85: Vertex; D86: Vertices; D87: Volume; D88: a; D89: b; D90: c; D91: e; D92: p; D93: x-intercept; D94: y-intercept.