Bilinear Control of a Binary Distillation Column

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by

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"... scientific research, no matter how 'pure' and useless it may seem has an annoying habit of paying for itself many times, in the long run, ..."

R. A. Heinlein.

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Summary

Bilinear systems are an attractive alternative to the traditional linearisation approach for many chemical plant items. Techniques for the identification and control of discrete time bilinear systems were examined and developed.

The performance of four reccursive identification methods was compared for a discrete bilinear system with white noise contamination of the output. Reccursive least squares methods gave the best performance for a number of criteria. A reccursive maximum likelihood gave similar performance to standard reccursive least squares despite having double the computational requirements.

A design method for a discrete time, globally asymptotically stabilising, optimal controller with a quadratic performance function was developed based on the solution to the algebraic matrix Riccati equation. The controller design was successfully and safely applied to both simulated and pilot scale, constant volume, heated tank systems and a simulated binary distillation column.

Application of the discrete-time, bilinear controller to the heated tank system gave good control over the full operating range. Conventional linear and PID controllers, while accurate near the tuning point, were unable to cope when away from this region. The linear controller gave large steady state offset, while the PID controller suffered from stability problems. A method of deadtime compensation, based on a discrete time bilinear model of the system, reduced deadtime induced overshoot after set point changes or disturbances, however, steady state offset resulted, due to the amplification of errors in the model.

The discrete bilinear controller gave good, safe, control of a simulated binary distillation column. A reduction in steady state offset was observed when compared to a linear optimal regulator with similar weighting matrices. The weakly bilinear nature of the distillation simulation did not threaten the stability of either the linear regulator or a PID controller with static decoupling. Versions of both the linear regulator and bilinear controller with added integral action gave almost identical performance. The presence of integral action dominated the system response.

Significant improvements in control and safety may be achieved for strongly bilinear systems such as the constant volume heated tank. For systems which display weak bilinearity, such as the distillation simulation, the bilinear controller may improve the steady state performance, eliminating the need for controllers with integral action in some applications.

Contribution

A number of innovations and advances in the identification and control of chemical plant items which display bilinear behaviour have been made. These include :

1) A review of the theory of bilinear systems as relevant to chemical process control. Particular emphasis is given to discrete time systems as part of the growing trend towards digital rather than analogue instrumentation.

2) A comparison of four recursive estimation methods for the identification of a discrete time model of a bilinear chemical process, including trends in the characteristic parameters for increasing "white" measurement noise. The characteristic parameters being the process time constants and gains at selected operating points.

3) The development of a method for the design of discrete time optimal stabilising controllers for bilinear systems using a quadratic performance index.

4) The development of methods for including feedforward and integral elements in the bilinear controller design process.

5) Application of method for deadtime compensation based on the use of a bilinear process model to predict system states in the near future. The control calculations are made using these predictions rather than the measured values. An analysis of the limitations of this approach is also included.

6) Application of bilinear controller designs to simulated and pilot scale constant volume tank systems and a comparison of performance against traditional controller designs.

7) The use of contour plots to represent the steady state behaviour of the binary distillation system, and the steady state errors in the identified models.

8) Modification of the Cohen-Coon PID controller tuning equations to garanttee the stability of discrete time PID controllers.

9) Application of the bilinear controller design method to the control of a simulated binary distillation column. As a basis for comparison PID and linear optimal regulators were also implemented.

CHAPTER 1

Introduction

Most chemical plant items behave in a complex, non-linear manner. This is particularly true of multi-stage separation processes such as distillation columns, which are based on non-linear equilibrium relationships in addition to considerations of fluid dynamics. Despite this, the systems used for the control of unit operations are usually based on linear system theory.

The use of linear system theory has the advantage of a well understood theoretical basis with a range of analytical tools available to the control system designer. As important, the theory and controller designs which result from the application of linear theory are relatively simple and the control may be realised through the use of analogue equipment.

The complex, usually non-linear, equations which fully describe plant behaviour, however accurate, may not be readily used to design controllers. In many cases the complexity of the models alone prevents such application, without consideration of the non-linear effects involved. However, the behaviour of non-linear systems and the design of controls for such systems is not well understood. Many of the analytical tools used for linear systems are not applicable when linearity is lost and the effect of disturbances or control action can only be predicted through the use of digital computer simulation methods which are too expensive for most applications.

Bilinear Systems

A particular class of non-linear system which may provide a useful first step away from the linear tradition is the group of bilinear systems. These systems are linear in both the states and inputs when considered separately but not when considered jointly. The form of multiplicative interaction which gives bilinear systems their name occurs naturally in a variety of processes.

Much of the initial impetus for research into bilinear systems was due to their natural occurrence in open loop nuclear reactor dynamics (Mohler and Shen 1970). Bilinear systems have since been found to occur naturally in a wide range of processes. Bilinear population models (Mohler and Frick 1979) have been applied to a variety of systems including human demography, biological cells and the manufacture and distribution of products.

In engineering applications, in addition to the previously mentioned nuclear reactor models, bilinear systems provide important approximations in vehicle braking and certain aircraft dynamics (Mohler 1973). In the process industries bilinear systems arise naturally in many items of constant volume plant. España and Landau (1978) and España (1977) develop a bilinear equation set to describe the dynamic behaviour of the continuous multistage distillation process and Janssen (1986) investigated the identification of discrete time bilinear models for a binary distillation column.

In addition to these naturally occurring examples, the use of bilinear systems has been advocated by Svoronos, Stephanopoulos and Aris (1980) as an alternative to linear systems for modeling the behaviour of general non-linear processes. This application is described as bilinearisation.

Identification of Bilinear Systems

A variety of methods have been proposed for the identification of dynamic models of bilinear systems. Many of these are based on methods developed for use with linear systems.

Among the more traditional approaches a significant amount of work has been done on the use of recursive identification techniques for discrete time bilinear models. A recursive least squares estimation via UD factorisation was used by Janssen (1986) to identify bilinear models for a binary distillation column, an on-line application of the same method is used by Fletcher (1987) for a constant volume tank system. On-line implementation of least squares algorithms has been used as the basis for adaptive deadbeat control systems by Goodwin, McInnis and Long (1981), Ohkawa and Yonezawa (1983), Dochain and Bastin (1984) and Cho and Marcus (1987).

In addition to the basic least squares algorithm, a variety of recursive methods which claim to eliminate or reduce parameter biasing in noisy systems have been investigated. Methods suggested include extended least squares methods. Two approaches have been advocated, models linear in the error were used by Fnaiech and Ljung (1987) and models which include multiplicative terms between the errors and the inputs by Gabr (1986). A batchwise instrumental variable method was used by Ahmed (1986) and a recursive formulation is described by Fnaiech and Ljung (1987). A recursive method based on a Newton-Raphson iterative approach to the maximum likelihood parameter estimates was applied to bilinear systems by Gabr (1986).

Other approaches to the identification of bilinear systems include the use of Walsh functions (Rao, Frick and Mohler 1978), Laguerre polynomials (Ranganathan, Jha and Rajamani 1986) and Legendre polynomials (Hwang and Chen 1986).

Control of Bilinear Systems

In recent years, attention has shifted to the problem of the control of bilinear systems. Initial methods called for linearisation at some selected operating point and the use of the wealth of accumulated knowledge on linear systems control. Although this approach produces acceptable results close to the set-point, the stability and quality of control cannot be guarantied away from this point.

Stabilising Control

A number of control methods for bilinear systems have been proposed based on stabilisation approaches. Closed loop asymptotic stability was obtained by Ionescu and Monopoli (1975) through the use of feedback control laws quadratic in the state. Other researchers have concentrated on the local asymptotic stabilisation with a sufficiently large region of attraction in the state-space (Derese and Noldus 1980).

Part of the difficulty in devising control schemes for bilinear systems lies in the nature of the resulting closed loop system equations. For linear systems subject to a linear feedback control, the resulting closed loop system is linear and has only one equilibrium point. In the bilinear case, the application of linear feedback results in a closed loop equation which is quadratic in the state, giving a number of possible equilibrium points. The characterisation of these equilibrium sets has been explored by Benallou, Mellichamp and Seborg (1983).

Optimal Control

The optimal regulator problem for linear systems has a solution via the algebraic matrix Riccati equation. For bilinear systems the presence of the bilinearity matrices prevents such a solution. Derese and Noldus (1980) presented a controller design method for bilinear systems based on the solution of the Riccati equation to produce a linear regulator. The magnitude of the weighting matrices was determined based on the desired controller response and stability region.

Benallou, Mellichamp and Seborg (1988) have presented a controller design method which globally asymptotically stabilized a continuous bilinear system and minimised a general quadratic performance index.

Adaptive Control

A number of researchers have investigated the use of adaptive control methods based on bilinear systems. The general approach has been through the use of a recursive identification procedure coupled to a minimum variance or deadbeat controller. Goodwin, McInnis and Long (1980) applied these methods to the control of waste water treatment and pH neutralization systems in simulation studies, other works include Ohkawa and Yonezawa (1983) and Dochain and Bastin

(1984). A weighted minimum variance controller was proposed by Cho and Marcus (1987) as displaying boundedness in the closed loop control variables, the lack of which causes problems in traditional minimum variance control.

This Work

In this work, a discrete-time version of an optimal stabilising controller for bilinear systems is developed resulting in a practical design procedure. The controller is applied to the control of both simulated and pilot-scale, constant volume, heated tank systems, and a simulated binary distillation column.

In chapter 2 the structure and properties which make bilinear systems attractive for modeling chemical plant items are reviewed. Much of this is necessary background to the work in later chapters. Chapter 3 deals with practical methods for the identification of discrete time bilinear systems and includes a comparison of four such methods for the identification of a single input, single output bilinear system. Chapter 4 examines some of the methods available for the control of bilinear systems and develops a design procedure for a discrete time bilinear optimal controller. Feed-forward compensation and integral action are incorporated into controller designs. These methods are applied to both simulated and pilot scale tank systems in chapter 5. Chapters 6 and 7 deal with the identification and control of a simulated binary distillation process.

At the end of each chapter a list of the references and nomenclature used in the chapter is given. Included in the back of this thesis is an 800K floppy disk in Apple Macintosh format which contains an executable copy of the batch identification program developed in this work, with associated documentation and data samples from the simulations and pilot plant studies in chapters 5 and 6.

Portions of this work have been previously published at CHEMECA'90 (Fletcher and Allen 1990). A copy of this paper is included in Appendix 1.

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CHAPTER 2

Bilinear System Theory

Overview

This chapter presents an introduction to the structure and properties of the class of bilinear systems and to how they are suited to the modeling of chemical plant items. Structural forms important to the identification and control of discrete time bilinear systems are examined as a basis for work in later chapters.

Linear Systems

Traditionally, chemical engineers have used linearised models to describe the dynamic behaviour of plant items. Linear systems have the advantages of a well developed theoretical base and a relative lack of complexity.

A linear system may be described by the continuous time state space formulation in Equation 2.1.

$$\dot{x} = A\,x + \sum_{i=1}^{m} u_i\, b_i \qquad (2.1)$$

where x = the system state vector in deviations from a known steady state
A = the state coefficient matrix
u_i = the ith input in deviation variable form
b_i = the coefficient vector for the ith input
m = the number of inputs.

The rate of change of the states is a linear sum of the effect of the current state of the system and the effect of the current inputs.

A linear system is not a true representation of the behaviour of most chemical plant items. For such non-linear plant, the conventional approach has been to select some desired operating point, and to linearise the behaviour of the plant about this point. The result of this approach is a process model which is only accurate over a portion of the possible operating range, the width of this region is dependent upon the degree of non-linearity of the process.

Bilinear Systems

A more general class of systems may be obtained by the addition of a number of terms which represent multiplicative interactions between the states and the inputs, these are termed bilinear. The general continuous time bilinear state space representation is:

$$\dot{x} = A\,x + \sum_{i=1}^{m} u_i\, b_i + \sum_{i=1}^{m} u_i\, C_i\, x \qquad (2.2)$$

where C_i = coefficient matrix for interaction between input i and the states.

The last set of terms describes a form of interaction common in chemical plant items of constant volume.

Constant Volume Heated Tank

Figure 2.1 shows the flow diagram of a simple heated tank system. The level of the tank is maintained through the use of a weir governing the outlet. The cold water flowrate into the tank is the control variable, and the temperature of the outlet stream is the state.

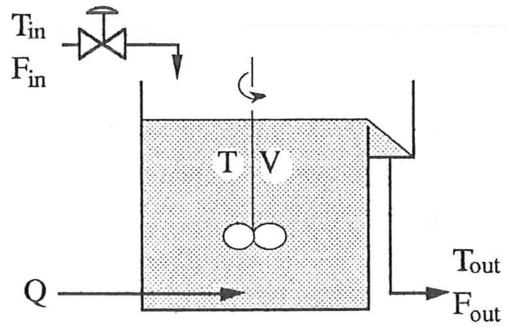

Figure 2.1 Heated Tank Flow Diagram

The tank is assumed to be well mixed, the outlet temperature being equal to the temperature in the tank. It is also assumed that there are no heat losses from the system except in the outlet water. The specific heat and density of the water remain constant over the entire range of the plant.

Heat and mass balances over this system yield the equations 2.3 and 2.4.

$$\frac{dV}{dt} = F_{in} - F_{out} = 0 \;\therefore\; F = F_{in} = F_{out} \qquad (2.3)$$

$$V\,C_p\,\rho\,\frac{dT}{dt} = C_p\,\rho\,F\,(T_{in} - T) + Q \qquad (2.4)$$

where C_p = the specific heat of the liquid

ρ = the liquid density

The first term on the right side of equation 2.4 contains the state and the input multiplied together, causing the system to be non-linear. This particular type of interaction is described as bilinear, and the system above may be exactly modelled using the bilinear state-space form of equation 2.2. This form of interaction should occur whenever the flowrate through a piece of constant volume plant is used as a control variable, or acts as a measured disturbance.

Comparing equations 2.1 and 2.2, a linear system is merely a bilinear system without the interaction terms. It follows that the set of linear systems is a subset of the set of bilinear systems.

A larger group of systems exists, in addition to those systems which are naturally bilinear, which are inherently bilinear or show bilinear tendencies to varying degrees. The modeling and control of these systems can be improved through the use of bilinear rather than linear models, an operation termed bilinearisation (Svoronos et. al. 1980). An example of such a system is the operation of a distillation column which has been shown by España (1977) to display bilinear tendencies. Further work on the identification of bilinear models for a distillation column was carried out by Janssen (1986).

Discrete Bilinear State-Space Representation

With the development of digital computing hardware over the last two decades, the control of chemical plants has shifted from simple analog instrumentation toward distributed digital control systems. These systems not only perform the basic low-level control of the individual plant items, but may also perform higher level functions and provide accurate and up to the minute analysis of the operation and efficiency of the entire site.

With digital computer control in mind, it is necessary to have a discrete time equivalent of the bilinear system described by equation 2.2. This may be achieved by applying the central difference approximations :

$$x = \frac{x(k+1) + x(k)}{2} \qquad (2.5)$$

$$\dot{x} = \frac{x(k+1) - x(k)}{h} \qquad (2.6)$$

to the continuous system (2.2) giving :

$$\frac{x(k+1) - x(k)}{h} = A\,\frac{x(k+1) + x(k)}{2} + \sum_{i=1}^{m} u_i\, b_i + \sum_{i=1}^{m} u_i\, C_i\, x \qquad (2.7)$$

The x in the last term will be substituted at a later stage. Rearranging the above equation gives:

$$x(k+1) = x(k) + \frac{Ah}{2}\,[x(k+1) + x(k)] + h \sum_{i=1}^{m} u_i\,[b_i + C_i\, x\,] \qquad (2.8)$$

$$\left[I - \frac{Ah}{2}\right] x(k+1) = \left[I + \frac{Ah}{2}\right] x(k) + h \sum_{i=1}^{m} u_i \, [b_i + C_i \, x\,] \tag{2.9}$$

$$x(k+1) = \left[I - \frac{Ah}{2}\right]^{-1} \left[I + \frac{Ah}{2}\right] x(k) + h\left[I - \frac{Ah}{2}\right]^{-1} \sum_{i=1}^{m} u_i \, [b_i + C_i \, x\,] \tag{2.10}$$

The structure of equation 2.10 is similar to that of the continuous system and by combining portions of the expression the discrete state space model may be established

$$x(k+1) = \alpha \, x(k) + \sum_{i=1}^{m} u_i \left[\beta_i + \gamma_i \, x\right] \tag{2.11}$$

where

$$\alpha = \left[I - \frac{Ah}{2}\right]^{-1} \left[I + \frac{Ah}{2}\right] \tag{2.12}$$

$$\beta_i = h\left[I - \frac{Ah}{2}\right]^{-1} b_i \tag{2.13}$$

$$\gamma_i = h\left[I - \frac{Ah}{2}\right]^{-1} C_i \tag{2.14}$$

or by grouping the input terms :

$$\delta_i(x) = \beta_i + \gamma_i \, x = h\left[I - \frac{Ah}{2}\right]^{-1} [\, b_i + C_i \, x] = h\left[I - \frac{Ah}{2}\right]^{-1} d_i \, (x) \tag{2.15}$$

These relations enable the parameters of a discrete model to be obtained from those of a continuous model. It is also possible to obtain an approximate continuous model from the parameters of a discrete system.

$$A = \frac{2}{h} \left[\alpha - I\right] \left[\alpha + I\right]^{-1} \tag{2.16}$$

$$b_i = \frac{1}{h} \left[I - \left[\alpha - I\right] \left[\alpha + I\right]^{-1}\right] \beta_i \tag{2.17}$$

Stability

A sufficient condition for a continuous time system to be open loop stable is the existence of a symetric positive definite matrix S that satisfies the Lyapunov equation(Elbert 1984).

$$S\,A + A^T S = - Q \tag{2.18}$$

Where Q is a symetric positive definite matrix.

Using equation 2.16 it is possible to substitute for A and after some algebra arrive at equation 2.19.

$$\frac{2}{h} S \left[\alpha - I\right]\left[\alpha + I\right]^{-1} + \frac{2}{h} \left[\alpha^T + I\right]^{-1} \left[\alpha^T - I\right] S = - Q$$

$$\frac{2}{h} \left\{\left[\alpha^T + I\right] S \left[\alpha - I\right] + \left[\alpha^T - I\right] S \left[\alpha + I\right]\right\} = - \left[\alpha^T + I\right] Q \left[\alpha + I\right]$$

$$\alpha^T S \, \alpha - S = - \frac{h}{4} \left[\alpha^T + I\right] Q \left[\alpha + I\right] \tag{2.19}$$

The above equation is equivalent to the discrete time Lyapunov equation 2.20 (Elbert 1984) in that provided Q is positive definite, the right hand side of the expression will be negative definite.

$$\alpha^T S \alpha - S = - Q^* \qquad (2.20)$$

For a continuous time system sampled at intervals h, equation 2.19 will yield the same solution matrix S as the continuous Lyapunov equation for the system.

Identification

Application of identification techniques to discrete bilinear systems is covered in chapter 3, including details of the algorithms used. This section shows how a discrete bilinear system may be rewritten for identification purposes, and discusses how a model may account for deadtime and also store information about the steady states of the process.

Deadtime

The presence of deadtime in a system may be represented in two ways.

The first case is the normal physical reality where there exists some delay between the actual process and the point at which the outputs become measurable (ie. the system boundary). This concept is illustrated in figure 2.2.

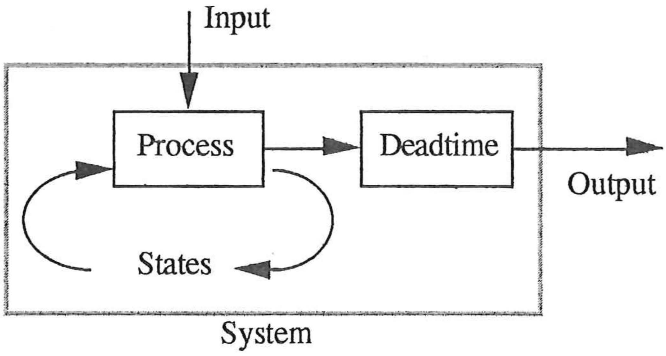

Figure 2.2 Deadtime as a delay on Outputs.

The alternative is to consider a delay between the time an input enters the system and the point at which it begins to affect the process. This is illustrated in figure 2.3. Although this is not always the case it provides a useful basis for adapting discrete time identification procedures to cope with the presence of deadtime. The delay on inputs approach also enables different deadtimes to be used for each input, giving greater flexibility.

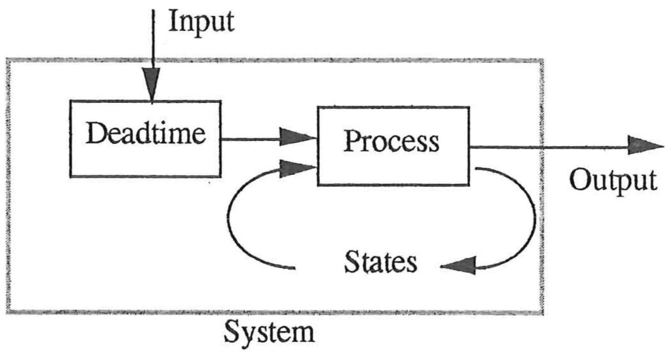

Figure 2.3 Deadtime as a delay on Inputs.

A first order discrete linear difference equation

$$y(k+1) = a\,y(k) + b\,u(k) \qquad (2.21)$$

may be modified to include deadtime simply by replacing $u(k)$ with $u(k-l)$, where l is the deadtime measured in sampling intervals:

$$y(k+1) = a\,y(k) + b\,u(k-l) \qquad (2.22)$$

Difference Equations

A difference equation form for the modelis required to facilitate the identification of a system. Papers by Goodwin and Sin (1984) and Beghelli and Guidorzi (1976) present methods for converting the state-space representation into a difference equation form.

$$y(k+1) = \sum_{i=1}^{n} \beta_i\, y(k+1-i) + \beta_0 \qquad (2.23)$$

Where $y(k)$ is the value of the measured variable at a time k.

β_i are non-linear functions of $u(k), u(k-1),\ldots u(k+1-n)$

β_0 is a linear function of $u(k),\ldots u(k+1-n)$

This full difference equation contains a large number of terms, many of which do not significantly improve the accuracy of the model whilst slowing convergence of the identification method.

A more manageable form may be obtained by taking the β_i s as linear functions of $u(k), u(k-1),\ldots u(k+1-n)$. This method was used by Janssen (1986) and defines the reduced bilinear form:

$$\begin{aligned} y(k) = {} & a_1 y(k-1) + \ldots a_n y(k-n) + b_1 u(k-1) + \ldots b_n u(k-n) \\ & + c_{11} y(k-1)u(k-1) + \ldots c_{n1} y(k-1)u(k-n) \\ & + \ldots\ldots \\ & + c_{1n} y(k-n)u(k-1) + \ldots c_{nn} y(k-n)u(k-n) \end{aligned} \qquad (2.24)$$

Although many terms have been omitted, this form is still maintains the multiplicative non-linearity which provides the improvement over a linear approximation. However, the number of parameters involved is still proportional to n^2 compared with n for a linear system. It has been suggested by previous workers (Janssen 1986, Rao and Gabr 1984) that acceptable accuracy may be obtained using a diagonal bilinear model. In such a model only those terms on the diagonal of the matrix of c terms are considered, the other elements of this matrix are assumed to be zero, leading to a model

$$y(k) = a_1 y(k-1) + \ldots a_n y(k-n) + b_1 u(k-1) + \ldots b_n u(k-n)$$
$$+ c_1 y(k-1)u(k-1) + c_2 y(k-2)u(k-2) + \ldots c_n y(k-n)u(k-n) \tag{2.25}$$

Another approach is to consider the case where the states of the system are themselves measurable (ie. the measurement equation is $y(k) = I\,x(k)$). Often it is the case that the measured variables are the current states of the system or the states of the system at some point in the near past where deadtime is involved. In this case the structure of the difference equation may be directly derived from the state-space expression. A general second order, SISO discrete bilinear state space relation including deadtime:

$$\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \end{bmatrix} + u(k-l) \left\{ \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \end{bmatrix} \right\} \tag{2.26}$$

may be rewritten to form a difference equation :

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_1 u(k-l) + c_1 u(k-l) y(k) + c_2 u(k-l) y(k-1) \tag{2.27}$$

Having converted the equation to difference form by one of the above methods, it may be rewritten as the dot product of two vectors, equation 2.28. Hence for a reduced bilinear model :

$$y(k) = \theta^T(k) \,.\, \phi(k) \tag{2.28}$$
$$\phi^T(k) = \left[\, y(k-1), y(k-2), \ldots y(k-n), y(k-1)u(k-l-1), y(k-2)u(k-l-1), \ldots \right.$$
$$\left. \ldots y(k-n)u(k-l-n), u(k-l-1), \ldots u(k-l-n) \,\right] \tag{2.29}$$
$$\theta^T(k) = \left[\, a_1, a_2, \ldots a_n, c_{11}, c_{12}, \ldots c_{nn}, b_1, \ldots b_n \,\right] \tag{2.30}$$

The measurement vector ϕ is a non-linear function of the outputs (y) and the inputs (u). However the parameter vector θ is linear in the model parameters. It is therefore possible to identify the parameters for the system using the techniques developed for linear systems.

Measured Variables

The models so far have been given in terms of deviation variables about some steady state. In order to convert the measured values of the states and inputs into deviation variable form it is neccessary to have accurate *a priori* knowledge of at least one steady state of the system. This has the effect of tying the the model to this steady state even if in error. To overcome this differculty the model may be modified to use the measured values directly. The deviation variables are defined

$$y(k-i) = Y(k-i) - Y_S \tag{2.31}$$
$$u(k-j) = U(k-j) - U_S \tag{2.32}$$

It follows that a bilinear term becomes

$$u(k-j)\, y(k-i) = U(k-j)\, Y(k-i) - U_S\, Y(k-i) - U(k-j)\, Y_S + U_S Y_S \tag{2.33}$$

Substituting these expressions into equation 2.24 leads to :

$$
\begin{aligned}
Y(k) = \hat{a}_1 Y(k\text{-}1) + \ldots \hat{a}_n Y(k\text{-}n) + \hat{b}_1 U(k\text{-}1) + \ldots \hat{b}_n U(k\text{-}n) \\
+ c_{11} Y(k\text{-}1)U(k\text{-}1) + \ldots c_{n1} Y(k\text{-}1)U(k\text{-}n) \\
+ \ldots\ldots \\
+ c_{1n} Y(k\text{-}n)U(k\text{-}1) + \ldots c_{nn} Y(k\text{-}n)U(k\text{-}n) + DC
\end{aligned}
\quad (2.34)
$$

Where :

$$\hat{a}_i = a_i - U_S \sum_{j=1}^{n} c_{ji} \quad (2.35)$$

$$\hat{b}_j = b_j - Y_S \sum_{i=1}^{n} c_{ji} \quad (2.36)$$

$$DC = U_S Y_S \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ji} + Y_S\left(1 - \sum_{i=1}^{n} a_i\right) - U_S \sum_{j=1}^{n} b_j \quad (2.37)$$

The *DC* term contains information about the steady states of the system. Analogous results are obtained for the other difference equation representations (equations 2.25 and 2.27).

The measurement and parameter vectors for the system are now

$$
\begin{aligned}
\phi^T(k) = [\, Y(k\text{-}1), Y(k\text{-}2), \ldots Y(k\text{-}n), Y(k\text{-}1)U(k\text{-}l\text{-}1), Y(k\text{-}2)U(k\text{-}l\text{-}1), \ldots \\
\ldots Y(k\text{-}n)U(k\text{-}l\text{-}n), U(k\text{-}l\text{-}1), \ldots U(k\text{-}l\text{-}n), 1 \,]
\end{aligned}
\quad (2.38)
$$

$$\theta^T(k) = [\, \hat{a}_1, \hat{a}_2, \ldots \hat{a}_n, c_{11}, c_{12}, \ldots c_{nn}, \hat{b}_1, \ldots \hat{b}_n, DC \,] \quad (2.39)$$

The application of identification techniques to bilinear systems is examined in more detail in chapter 3, including details of practical methods.

Conversion from Difference Equation to State Space Form

A discrete difference model of a process can be converted to state space form by selecting $y(k+1\text{-}i)$,$i = 1..n$ as the states at a time $k\,\Delta t$:

$$x(k) = \begin{bmatrix} y(k) \\ y(k\text{-}1) \\ \ldots \\ y(k+1\text{-}n) \end{bmatrix} \quad (2.40)$$

The states at a time $(k+1)\,\Delta t$ are found by :

$$\therefore \begin{bmatrix} y(k+1) \\ y(k) \\ \cdots \\ y(k+2-n) \end{bmatrix} = \begin{bmatrix} a_1 a_2 \ldots a_{n-1} a_n \\ 1\ 0 \ldots\ 0\ \ 0 \\ \ldots\ldots\ldots\ \ldots\ \ldots \\ 0\ 0 \ldots\ 1\ \ 0 \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \cdots \\ y(k+1-n) \end{bmatrix} + \sum_{i=1}^{m} u_i \begin{bmatrix} d_i(x(k)) \\ 0 \\ \cdots \\ 0 \end{bmatrix} \qquad (2.41)$$

Where :

$$d_i(x(k)) = b_i + \begin{bmatrix} c_{i1}\ c_{i2} \ldots c_{in} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \cdots \\ y(k+1-n) \end{bmatrix} \qquad (2.42)$$

This form of model may now be used as the basis for some form of control design procedure.

Multivariable Systems

In the case of Multiple Input Multiple Output systems the concepts presented above still hold.

To identify MIMO systems it is usual to break the system down into a number of multiple input single output (MISO) sub-systems which can be easily identified in the manner described above. Once this is completed the overall state-space relation may be found by grouping all the resulting equations.

A two input, two output, second order system :

$$\begin{aligned} y_1(k+1) &= a_1 y_1(k) + a_2 y_1(k-1) + b_1 u_1(k-l) + c_{11} u_1(k-l) y_1(k) + c_{12} u_1(k-l) y_1(k) \\ &\quad + b_2 u_2(k-l) + c_{21} u_2(k-l) y_1(k) + c_{22} u_1(k-l) y_1(k) \\ y_2(k+1) &= a_3 y_2(k) + a_4 y_2(k-1) + b_3 u_1(k-l) + c_{31} u_1(k-l) y_2(k) + c_{32} u_1(k-l) y_2(k) \\ &\quad + b_4 u_2(k-l) + c_{41} u_2(k-l) y_2(k) + c_{42} u_1(k-l) y_2(k) \end{aligned}$$

Becomes :

$$\begin{aligned} \begin{bmatrix} y_1(k+1) \\ y_1(k) \\ y_2(k+1) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_1(k-1) \\ y_2(k) \\ y_2(k-1) \end{bmatrix} + u_1 \left\{ \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{31} & c_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_1(k-1) \\ y_2(k) \\ y_2(k-1) \end{bmatrix} \right\} \\ &\quad + u_2 \left\{ \begin{bmatrix} b_2 \\ 0 \\ b_4 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{41} & c_{42} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_1(k-1) \\ y_2(k) \\ y_2(k-1) \end{bmatrix} \right\} \end{aligned} \qquad (2.43)$$

Determination of Steady States

One of the important advantages of a bilinear model over a linear one is the increase in the range over which a bilinear model remains valid for many real systems. It is important to be able to determine the correct input values to correspond to a desired output.

For a single input, single output model, the input and output variables ($u(k-j)$ & $y(k-i)$) can be replaced with their steady state values (u_s & y_s). Rearranging to make u_s the subject of the resulting equation yields an expression relating the steady states. For a reduced bilinear model :

$$u_s = \frac{\left[1 - \sum_{i=1}^{n} a_i\right] y_s - DC}{\sum_{j=1}^{n} \left[b_j + y_s \sum_{i=1}^{n} c_{ji}\right]} \qquad (2.44)$$

For a multiple input, multiple output system the problem is more complex as there are a number of equations which must be solved simultaneously. Beginning with a state-space model :

$$x(k+1) = \alpha\, x(k) + \sum_{i=1}^{m} u_i\, \delta_i(x) + DC \qquad (2.45)$$

Substituting, rearranging and combining the u_i into a single vector gives :

$$\left[I - \alpha\right] x_s - DC = \Delta(x_s)\, u_s \qquad (2.46)$$

The above equation cannot usually be solved directly as $\Delta(x_s)$ will not normally be square, may contain one or more rows of zeros, and thus may not be readily inverted. If those rows of the equation which contain only zeros in $\Delta(x_s)$ are removed, what remains should be a well conditioned set of simultaneous equations.

If there are still more rows remaining than inputs the system is uncontrollable as written. If less rows remain then an excess of control variables exists and the value of one must be assigned before the others may be calculated. When the number of equations equals the number of unknown inputs the equations may be solved using the standard methods.

eg. For the multivariable system in equation 2.47. (note : as this model is already in deviation variables about a known steady state the vector ***DC*** contains only zeros and has been omitted.)

$$\begin{bmatrix} y_1^s \\ y_1^s \\ y_2^s \\ y_2^s \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1^s \\ y_1^s \\ y_2^s \\ y_2^s \end{bmatrix} + u_1^s \left\{ \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{31} & c_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^s \\ y_1^s \\ y_2^s \\ y_2^s \end{bmatrix} \right\}$$

$$+ u_2^s \left\{ \begin{bmatrix} b_2 \\ 0 \\ b_4 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{41} & c_{42} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^s \\ y_1^s \\ y_2^s \\ y_2^s \end{bmatrix} \right\} \tag{2.47}$$

Reducing :

$$\begin{bmatrix} y_1^s \\ y_2^s \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & a_3 + a_4 \end{bmatrix} \begin{bmatrix} y_1^s \\ y_2^s \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \begin{bmatrix} u_1^s \\ u_2^s \end{bmatrix} \tag{2.48}$$

Where
$$\begin{aligned} d_1 &= b_1 + y_1^s\,(c_{11} + c_{12}) \\ d_2 &= b_2 + y_1^s\,(c_{21} + c_{22}) \\ d_3 &= b_3 + y_2^s\,(c_{31} + c_{32}) \\ d_4 &= b_4 + y_2^s\,(c_{41} + c_{42}) \end{aligned}$$

Giving a steady state solution :

$$\begin{bmatrix} u_1^s \\ u_2^s \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}^{-1} \begin{bmatrix} 1 - a_1 - a_2 & 0 \\ 0 & 1 - a_3 - a_4 \end{bmatrix} \begin{bmatrix} y_1^s \\ y_2^s \end{bmatrix} \tag{2.49}$$

Process Gains & Time Constants

A discrete bilinear model, although accurate, does not lend itself to an appreciation of the actual plant behaviour. System parameters which aid in understanding the behaviour of a piece of plant include the time constants, which illustrate the relative speed of the process, and the gains with respect to the inputs, which enable prediction of the response to a known change in an input.

To express a bilinear model in terms of gains and time constants it is necessary to linearise the model about some operating point. A deviation variable model is obtained at this point. This model is then linearised by dropping out all the bilinear terms. A linear difference equation will remain. ie.

$$y(k+1) = \sum_{i=1}^{n} a_i\, y(k+1-i) + \sum_{i=1}^{n} b_i\, u(k+1-i) \tag{2.50}$$

A transfer function expression using the z operator is then obtained by rearranging.

$$\frac{y(z)}{u(z)} = \frac{\sum_{i=1}^{n} b_i z^{-i}}{1 - \sum_{i=1}^{n} a_i z^{-i}} \tag{2.51}$$

The gain of the process is found by setting $z = 1$ and evaluating the resulting fraction. To evaluate the time constants it is necessary to consider the denominator of an nth order model to be the product of the denominators of n first order processes, where the 1st order denominator is given by equation 2.52 and the nth order case by 2.53.

$$D(z) = 1 - z^{-1} e^{\frac{-\tau}{T}} \tag{2.52}$$

$$D(z) = \prod_{i=1}^{n} \left(1 - z^{-1} e^{\frac{-\tau}{T_i}} \right) \tag{2.53}$$

Multiplying the denominator by z^n will yield a polynomial with roots defined by equation 2.54 and the time constants may be estimated using 2.55.

$$z_i = e^{\frac{-\tau}{T_i}} \tag{2.54}$$

$$T_i = \frac{-\tau}{\log_e z_i} \tag{2.55}$$

Nomenclature

Continuous State Space

x	State Vector
x_i	i th element of State Vector
u	Input Vector
u_i	i th element of Input Vector
n	Number of States
m	Number of Inputs
A	State Coefficient Matrix
B	Input Coefficient Matrix
b_i	i th Column of B, Coefficient vector for u_i
C_i	Bilinear Coefficients for Input i
t	Time
Q, S	Symetric Positive Definite Matrices

Discrete State Space

$x(k)$	State Vector sampled at $t = k * h$
k	Discrete Time variable
h	Sampling Interval
l	Discrete Deadtime in sampling intervals
α	State Coefficient Matrix
β_i	Coefficient Vector for i th input

γ_i	Bilinear Coefficient Matrix for i th input
$\delta_i(x(k))$	Combined Coefficient Vector for Input i. $\delta_i(x(k)) = \beta_i + \gamma_i x(k)$
$\Delta(x(k))$	Combined Input Coefficient Matrix, columns are $\delta_i(x(k)), i = 1,m$

Discrete Diference Equations

$y(k)$	Deviation Variable Output at time $k*h$
$u(k)$	Deviation Variable Input at $k*h$
a_i	Coefficient of $y(k+1-i)$
b_j	Coefficient of $u(k+1-j-l)$
c_{ij}	Coefficient of $y(k+1-i).u(k+1-j-l)$
$\theta(k)$	Parameter Vector
$\phi(k)$	Measurement Vector
$Y(k)$	Measured Output
$U(k)$	Measured Input
Y_S, U_S	Steady State Values
$\hat{a}_i$, $\hat{b}_j$	Coefficients in Measured Variable Model
DC	Constant Term

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CHAPTER 3

Identification of Bilinear Systems

Overview

A number of standard identification procedures were examined, the objective being their application to the identification of discrete time models of bilinear systems.

Four methods were tested for the identification of a known third order discrete time bilinear model with varying levels of measurement noise. The relative performance was gauged by use of three properties of the identified models, the process gain, the principle time constant and the variance of the identified model response against that of the noise free original system.

Theory

The identification of a process model involves finding the values of the parameters which minimise some function of the errors between the model predictions and the measured values for a set of data. The most commonly used cost function is the sum of the square of the prediction error at each data point. Methods using such a cost function are termed Least Squares (LS) methods. There are two ways in which least squares methods may be used to estimate model parameters, batchwise or recursively.

In batch estimation the data collection and parameter estimation operations are performed separately. The data is first collected and stored, then a multivariable search procedure is used to find the parameter values which minimise the cost function. A commonly used search procedure is the Newton-Raphson method. Batchwise methods often require the storage and manipulation of large amounts of data.

Alternatively, successive estimates to the values may be obtained as the data becomes available by using recursion techniques developed from statistical theory. After each data point is measured, it is used to generate a new estimate of the model parameters. The storage requirements are significantly lower than for batch or off-line methods although the overall computational requirements may be greater. Recursive estimation methods may also be operated in an off-line fashion, this approach has been used throughout this work.

Recursive Estimation

Recursive Least Squares (RLS) is the simplest formulation which may be used to estimate parameters on-line.

Given a set of data:

$$\left\{ \phi(k),\, y(k) \right\},\ k = 1..N \tag{3.1}$$

it is possible to find a set of parameters θ that minimise the square of the error through :

$$e(k) = y(k) - \phi^T(k).\theta(k-1) \tag{3.2}$$

$$R(k) = R(k-1) + \phi(k)\,\phi^T(k) \tag{3.3}$$

$$\theta(k) = \theta(k-1) + R^{-1}(k)\,\phi(k)\,e(k) \tag{3.4}$$

where $e(k)$ = the error at step k

$\phi(k)$ = the measured values of the independent variables at step k

$\theta(k)$ = the estimate of the parameters at step k

$R(k)$ = the information matrix at step k

Replacement of $R^{-1}(k)$ with $P(k)$ leads to a form which is computationally more efficient. The covarience matrix $P(k)$ can be updated by applying the matrix-inversion lemma (Friedmann 1954) to equation (3.3). The recursive equation set now becomes :

$$e(k) = y(k) - \phi^T(k).\theta(k-1) \tag{3.5}$$

$$P(k) = P(k-1) - \frac{P(k-1)\,\phi(k)\,\phi^T(k)\,P^T(k-1)}{1 + \phi^T(k)\,P(k-1)\,\phi(k)} \tag{3.6}$$

$$\theta(k) = \theta(k-1) + P(k)\,\phi(k)\,e(k) \tag{3.7}$$

For the purposes of identifying dynamic systems, equation 3.6 is further modified by the addition of a forgetting factor $\lambda(k)$. This parameter allows the system to 'forget', or reduce the importance of events that occurred in the distant past, placing more importance on recent events. A forgetting factor is useful in adaptive control applications or where the parameters of a system may change with time.

$$P(k) = \frac{1}{\lambda(k)}\left[P(k-1) - \frac{P(k-1)\,\phi(k)\,\phi^T(k)\,P^T(k-1)}{\lambda(k) + \phi^T(k)\,P(k-1)\,\phi(k)}\right] \tag{3.8}$$

Under some circumstances the recursive least squares algorithm converges poorly or not at all. It has been shown by Bierman (1977) to become unstable if the error covarience matrix P loses positive definiteness.

UD Factorisation Algorithm

An efficient and stable method for solving the recursive equation set is provided by the UD factorisation algorithm of Bierman (1977). Cholesky decomposition is used to factorise the error covarience matrix P into the form $U D U^T$.

At each measurement point the following procedure is calculated :

1. $f = U^T(k\text{-}1)\,\phi(k)$
2. $g = D(k\text{-}1)\,f$
3. $\alpha_1 = \lambda + g_1 f_1$
4. $d_1(k) = d_1(k\text{-}1) / \alpha_1$
5. $l_2^T = [g_1, 0, \ldots 0]$
6. Repeat steps 7 to 10 for $j = 2$ to m (m is the order of D)
7. $\alpha_j = \alpha_{j\text{-}1} + g_j f_j$
8. $d_j(k) = d_j(k\text{-}1)\,\alpha_{j\text{-}1} / \alpha_j \lambda$
9. $u_j(k) = u_j(k\text{-}1) + \mu_j l_j$ where $\mu_j = -f_j / \alpha_{j\text{-}1}$
10. $l_{j+1} = l_j + g_j u_j(k\text{-}1)$

The parameter gain and the new values of the parameters may be found

$$L(k) = \frac{l_{m+1}}{\alpha_m} \qquad (3.9)$$

$$\theta(k) = \theta(k\text{-}1) + L(k)\,e(k) \qquad (3.10)$$

In practise it is not necessary to calculate $L(k)$ directly, but to use the following expression :

$$\theta(k) = \theta(k\text{-}1) + l_{m+1}\frac{e(k)}{\alpha_m} \qquad (3.11)$$

To start the algorithm the elements of U and the initial parameter estimates θ should be set to *0*. The initial values for the elements on the diagonal of D should be assigned a large value. The forgetting factor λ should be in the range *0.9 - 1.0.*

Recursive Least Squares

From a discrete bilinear system described by a difference equation 3.12 it is possible to obtain a vector dot-product equation 3.13.

$$y(k) = \sum_{i=1}^{n} a_i y(k\text{-}i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji} u(k\text{-}j\text{-}l) y(k\text{-}i) + \sum_{i=1}^{n} b_i u(k\text{-}i\text{-}l) + DC \qquad (3.12)$$

$$y(k) = \theta^T . \phi(k) \qquad (3.13)$$

$$\phi^T(k) = \big[\, y(k\text{-}1), y(k\text{-}2), \ldots y(k\text{-}n), y(k\text{-}1)u(k\text{-}l\text{-}1), y(k\text{-}2)u(k\text{-}l\text{-}1), \ldots$$
$$\ldots y(k\text{-}n)u(k\text{-}l\text{-}n), u(k\text{-}l\text{-}1), \ldots u(k\text{-}l\text{-}n), 1 \,\big] \qquad (3.14)$$

$$\theta^T = \big[\, a_1, a_2, \ldots a_n, c_{11}, c_{12}, \ldots c_{nn}, b_1, \ldots b_n, DC \,\big] \qquad (3.15)$$

The vector $\phi(k)$ contains the measured values of the inputs and outputs at time *k* and θ contains the model parameters. The parameter vector θ is linear in the parameters although the overall system is not linear. The linearity of θ enables the application of recursive least squares estimation techniques to the bilinear system.

The recursive least squares estimator will always converge to a set of parameters using the UD factorisation algorithm. However, if there is any noise present in the process, these parameters will suffer from biasing. The identification procedure determines the parameters which best model the "noisy" response of the process to the input sequence. The accidental correlation of the noise with the input sequence results in the parameters being different from those of the "true" process. The extent of this bias will vary depending upon the amount of noise present and to what extent the noise is correlated with the process response.

There are three approaches to reducing this bias :

1.Modify the process and / or the sensors to reduce the noise.

2.Use some form of filter to attempt to remove or reduce the noise.

3.Use an identification procedure which is less susceptible to noise-induced biasing.

Combinations of the above methods may be applied to a process. In most cases with existing plant, it is impossible or expensive to make changes to the physical equipment of the process so the first option has not been considered further.

There are many methods of filtering currently available, ranging from simple analog low-pass devices to very complex, software-based, digital methods. The important consideration when filtering process response data is to select a filter which will remove the noise without removing important information about the process.

A number of alternatives to RLS have been reported in the literature, many of these have been adapted for the identification of discrete time bilinear systems (Fnaiech and Ljung 1987, Gabr 1986). Three such methods are examined in this work.

Recursive Extended Least Squares

In RLS estimation the model structure is assumed to be deterministic with no random components affecting the system behaviour.

An alternative is to consider the system as a combination of a deterministic process and a random process. In this manner a linear system may be written:

$$y(k) = \sum_{i=1}^{n} a_i\, y(k-i) + \sum_{i=1}^{n} b_i\, u(k-i-l) + DC + \sum_{i=1}^{n} \varepsilon_i\, e(k-i) \quad (3.16)$$

Where $e(k)$ = the random error at a time k found by $e(k) = y(k) - y(k)_{predicted}$

The additional terms in this expression may be separated and attached on the end of the parameter and measurement vectors and these 'extended' vectors used in the recursive identification procedure. This is termed Recursive Extended Least Squares (RELS) (Isermann 1981).

For bilinear systems there are two possible methods of extending the model structure. The simpler method is to use additional terms identical to those for a linear system of the same order (Fnaiech and Ljung 1987). For a reduced bilinear difference equation this gives :

$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji} u(k-j-l) y(k-i) + \sum_{i=1}^{n} b_i u(k-i-l) + DC + \sum_{i=1}^{n} \varepsilon_i e(k-i) \quad (3.17)$$

This method is referred to as RELS throughout this work.

The second method is the use of a more complete formulation, including bilinear terms between the errors and the inputs (Gabr 1986). The reduced bilinear difference equation becomes :

$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji} u(k-j-l) y(k-i) + \sum_{i=1}^{n} b_i u(k-i-l) + DC$$

$$+ \sum_{i=1}^{n} \varepsilon_i e(k-i) + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ji} u(k-j-l) e(k-i) \quad (3.18)$$

To prevent confusion this has been referred to as REELS.

Recursive Maximum Likelihood

The fourth method used was the Recursive Maximum Likelihood method of Gabr (1986). A derivation of this method is not given here. Unlike the other three methods RML is based on the Newton-Raphson method. The recursive equation set for this method is given below.

$$V(k) = -\phi(k) - \sum_{i=1}^{n} \varepsilon_i V(k-i) - \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ji} u(k-j-l) V(k-i) \quad (3.19)$$

$$P(k) = \frac{1}{\lambda(k)}\left[P(k-1) - \frac{P(k-1)\, V(k)\, V^T(k)\, P^T(k-1)}{\lambda(k) + V^T(k)\, P(k-1)\, V(k)} \right] \quad (3.20)$$

$$\theta(k) = \theta(k-1) - P(k)\, V(k)\, e(k) \quad (3.21)$$

This method uses the same structure as REELS for the measurement and parameter vectors, and may be solved using the UD factorisation algorithm by substituting $V(k)$ for each occurrence of $\phi(k)$ and calculating $V(k)$ recursively using equation 3.19.

Input Sequence and Sampling Interval

A suitable input sequence and sampling interval is required, to accurately identify a model of a process.

The sampling rate should be fast enough to provide good modeling of the fastest time constant of interest in the system, but not so fast that it correlates high frequency noise. A general rule for selecting sampling intervals is given by Isermann (1981).

$$\frac{1}{15} T_{95} < T_s < \frac{1}{4} T_{95} \tag{3.22}$$

Where T_{95} = the time taken for the process to reach *95%* of its final value after a step change in one of the inputs.

An input sequence should have the following properties:

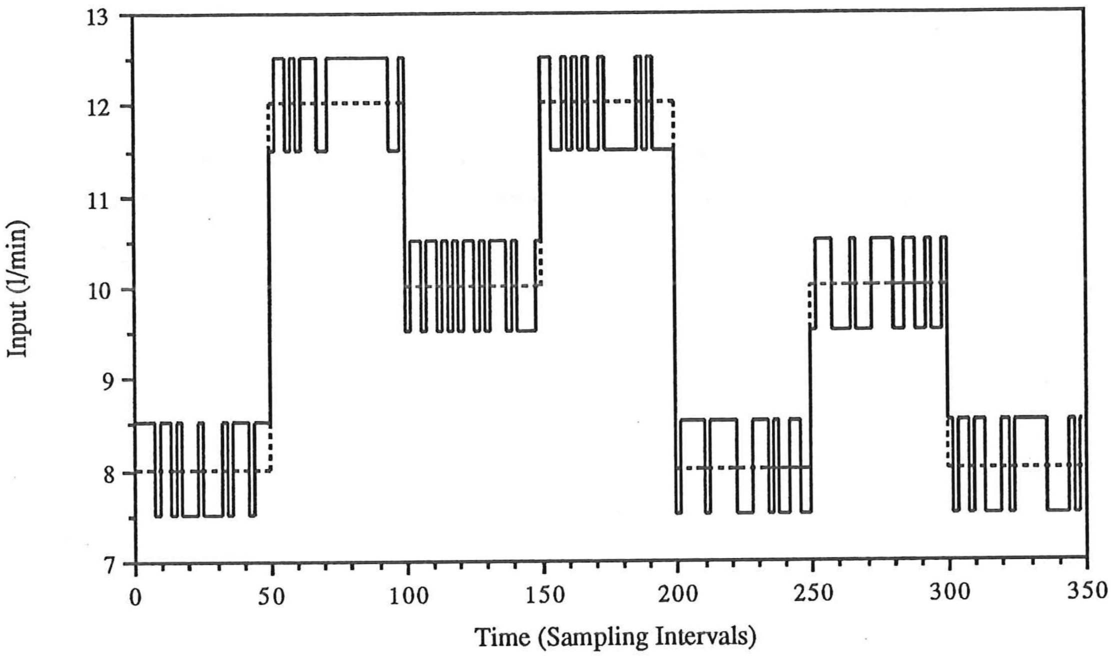
1. The sequence must be sufficiently stimulating to excite all the non-linearities in the process. A common method for achieving this is to use a pseudo-random binary sequence as this contains a wide range of frequencies. The use of such a sequence has been mathematically shown to satisfy the persistent excitation criterion for both linear (Isermann 1980) and bilinear systems (Janssen 1986).

2. The sequence should drive the plant over its entire operating range. A plant model should be identified over the desired range of operation. This is essential for bilinear models where the gain is subject to variation over the operating range. To achieve this, a series of operating points should be selected and these in conjunction with a pseudo random binary sequence should be used to form the input sequence.

3. The identification run should be long enough to provide a good base for estimating the parameters of the model. As a guide, Gustavsson (1975) suggests the length be at least ten times the major time constant of the system. Longer runs may be required if the system is subject to excessive amounts of noise.

A suitable input sequence for identifying bilinear systems is a series of step changes with a superimposed pseudo-random binary sequence (PRBS). An example of such a sequence is shown in graph 3.1. Three input values were used as a basis with a PRBS of amplitude *1.0* superimposed.

Graph 3.1 Step sequence with superimposed PRBS



Trials

A series of tests were carried out to assess the effectiveness of these identification methods for a bilinear system.

A third order diagonal bilinear model of a steam heated tank, identified by an on-line application of RLS (Fletcher 1987), was used as the basis for method trials.

$$Y(k) = 0.8602\ Y(k-1) + 0.2206\ Y(k-2) - 0.0914\ Y(k-3)$$
$$- 0.01008\ U(k-2)\ Y(k-1) - 0.00313\ U(k-3)\ Y(k-2) - 0.00078\ U(k-4)\ Y(k-3)$$
$$+ 0.1124\ U(k-2) + 0.0200\ U(k-3) + 0.0354\ U(k-4) + 4.9379 \qquad (3.23)$$

The process gain of the system was calculated at steady states corresponding to selected values of the input U.The principle time constant was also estimated at $U = 10\ l/min$ and is shown in table 3.1 as a number of sampling intervals.

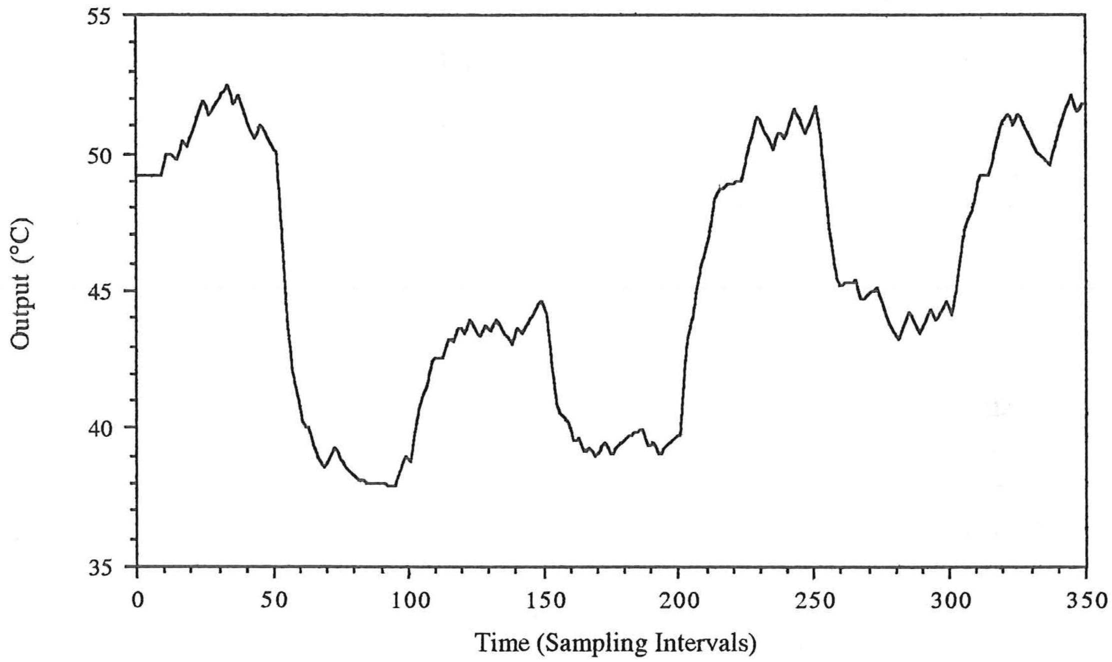
Table 3.1 Model Gain and Time Constant at Selected Steady States

U l/min	Y °C	K_p	T_1 ($\times T_s$)
8	51.3	-4.482	-
10	44.0	-2.9707	5.916
12	38.9	-2.1124	-

The known discrete time bilinear model was subjected to the input sequence shown in graph 3.1. The response of the model is shown in graph 3.2. This response then became the noise-free process.

To gauge the effect of noise on the performance of the identification methods a normally distributed random noise signal of varying amplitude was superimposed on the above response. The parameters of the resultant noisy system were then identified using the four methods described earlier. Three criteria were then used to evaluate the performance of the methods.

Graph 3.2 Model Response to Input Sequence 3.1



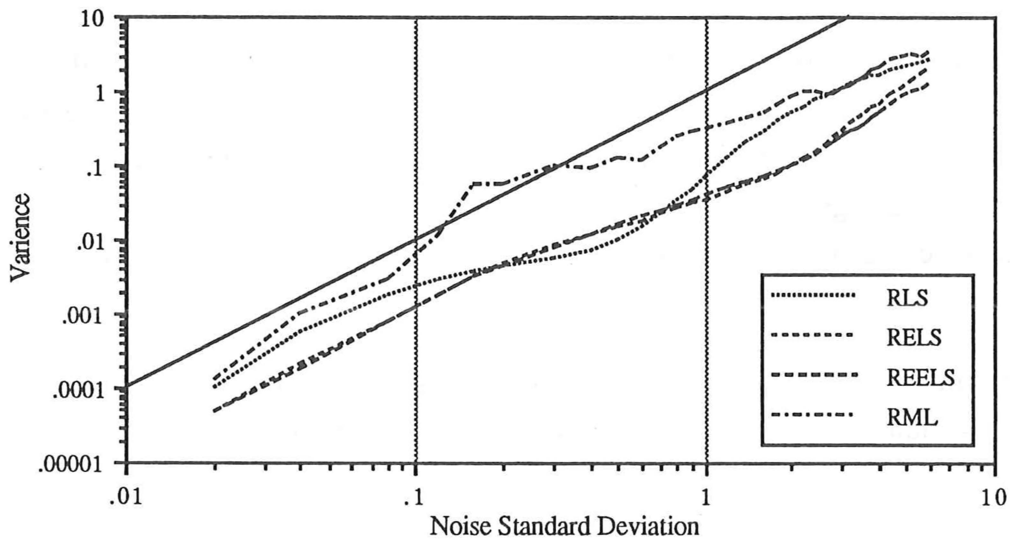
Variance

The most natural measure of the success of an identification procedure is to see how well the model predicts the behaviour of the system. This involves calculation of the variance between model predictions and plant output.

The identified models were subjected to the same input sequence used in graph 3.1. The variance between the response of the model and the original noise-free system was calculated and the results are shown on graph 3.3. The solid line indicates a variance equivalent to the noise standard deviation (σ) at which the model was identified.

All methods apart from RML produced models with variances less than that of the noise signal over the range examined. The performance of the two extended least squares methods (RELS & REELS) was similar and apart from a small region between $.2 \le \sigma \le .7$ was better than the other two methods. The performance of recursive maximum likelihood was poor, giving variances

Graph 3.3 Variance between Model Output and Original System

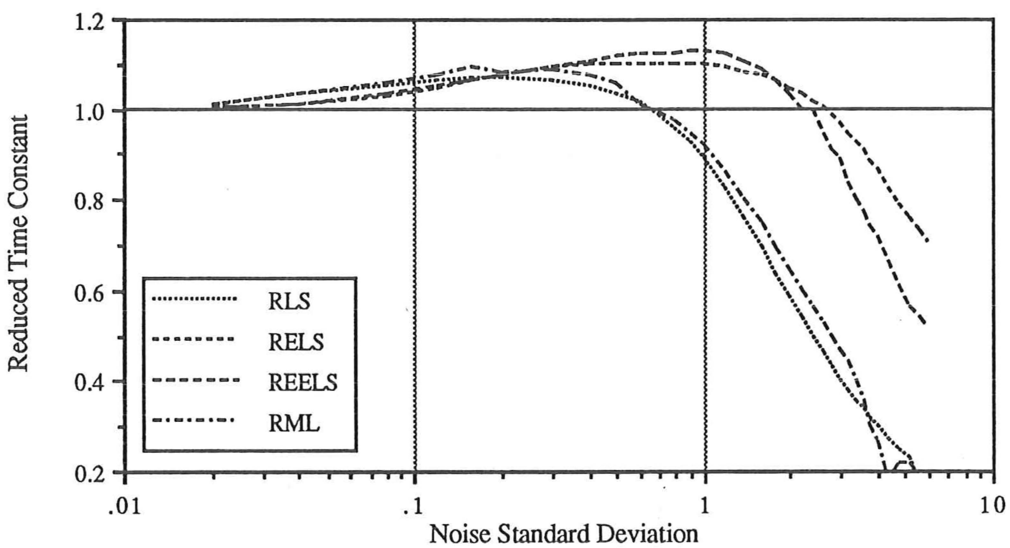


greater than that of the noise signal in one region and at least $\sqrt{10}$ greater than RELS over the entire range.

Time Constants

A second measure of the suitability of an identification method may be obtained by comparing the principle time constant of the identified model with that of the known process. The estimation of these time constants is described in chapter 2.

Graph 3.4 Reduced Time Constant vs Noise for 4 Identification Methods



The time constants of the identified models were estimated at the steady state corresponding to an input value of *10 l/min*. The largest of these was the principle time constant for the model.

This value was divided by the time constant of the original system and the results plotted on graph 3.4. The solid line represents the true time constant.

The results for all four model types follow the same general trends. For low noise the estimator slightly overestimated the value of the time constant. As the noise increased further the time constant estimate decreased rapidly, suggesting that the faster noise signal was dominating the process. For RLS at high noise a levelling off was observed which was due to the time constant estimate approaching the size of one sampling interval. Given a sufficiently large noise signal all four methods should exhibit this behaviour.

The best results were obtained using the extended least squares methods (RELS & REELS), with the maximum likelihood method (RML) on a par with normal recursive least squares (RLS).

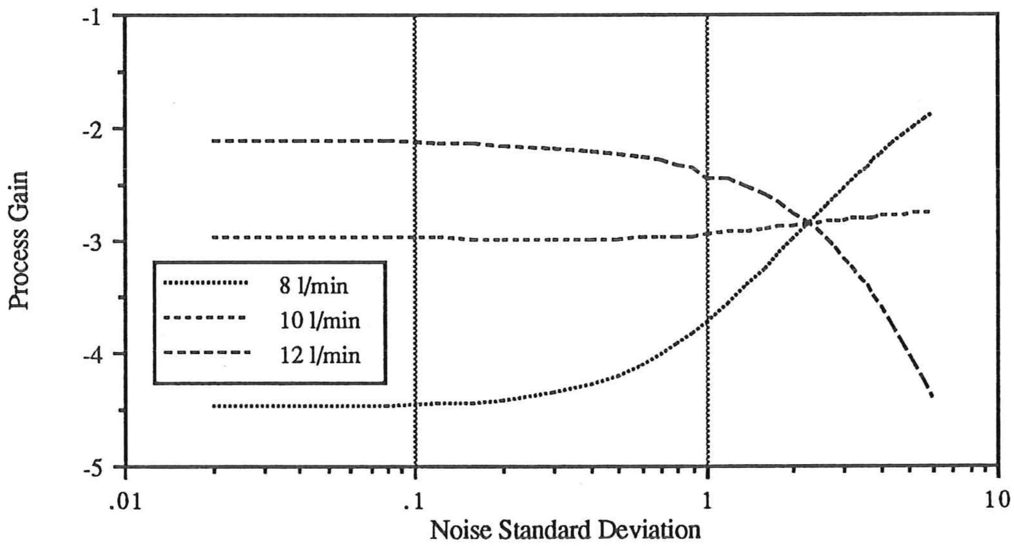
Process Gains

The third criterion for judging the performance of a method for identifying bilinear systems was the ability of the model to determine the gains of the process at various steady states. A major advantage of bilinear models over linear models is the ability of the bilinear model to account for changes in the process gain over the full operating range.

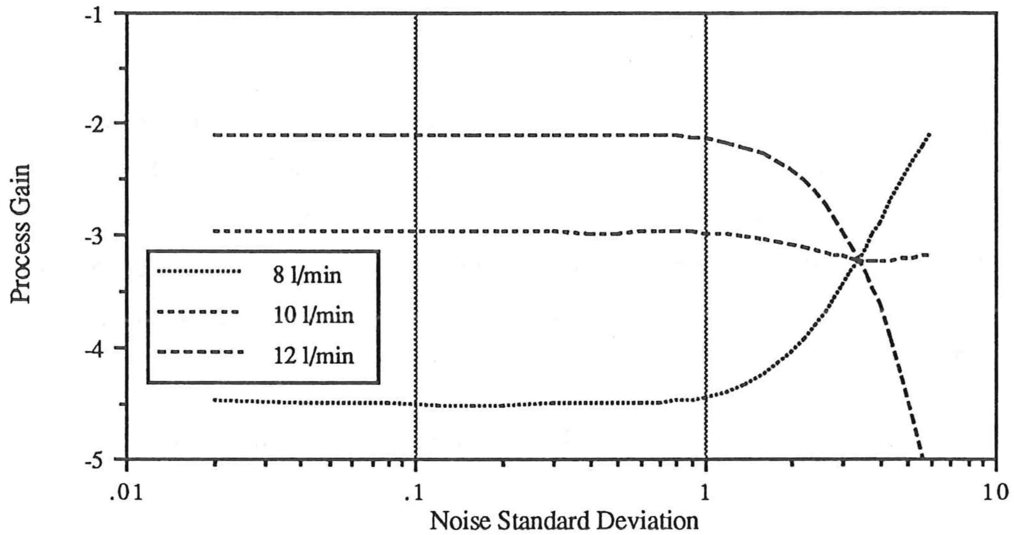
The gain was calculated at three steady states for each identified model, corresponding to input values of *8*, *10* and *12 l/min*. These results were then plotted for each method. (Graphs 3.5 - 3.8)

As for the time constant biasing, all four methods produced graphs with the same general form. At low noise levels the estimated gains correspond to the gains of the original process. As the

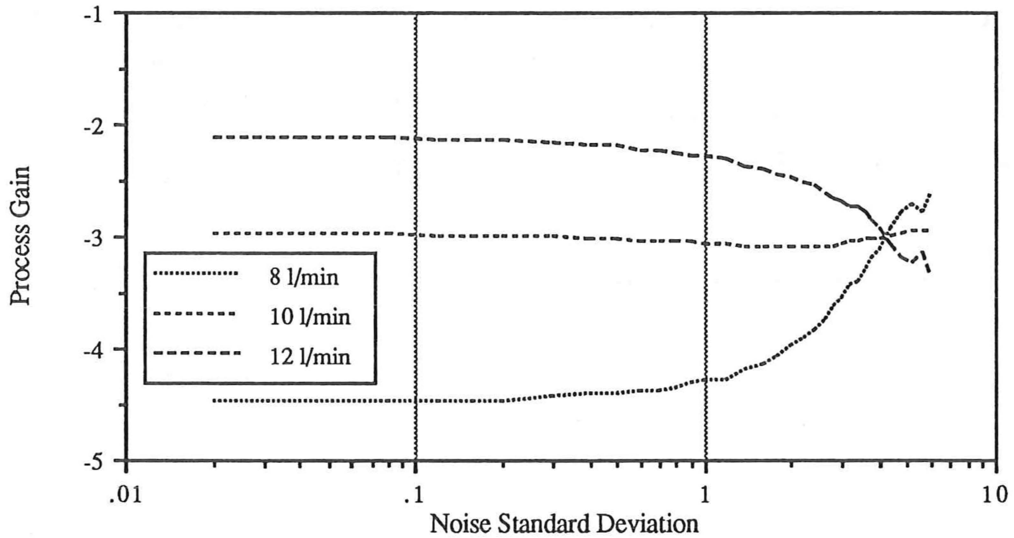
Graph 3.5 Gain Variation for RLS



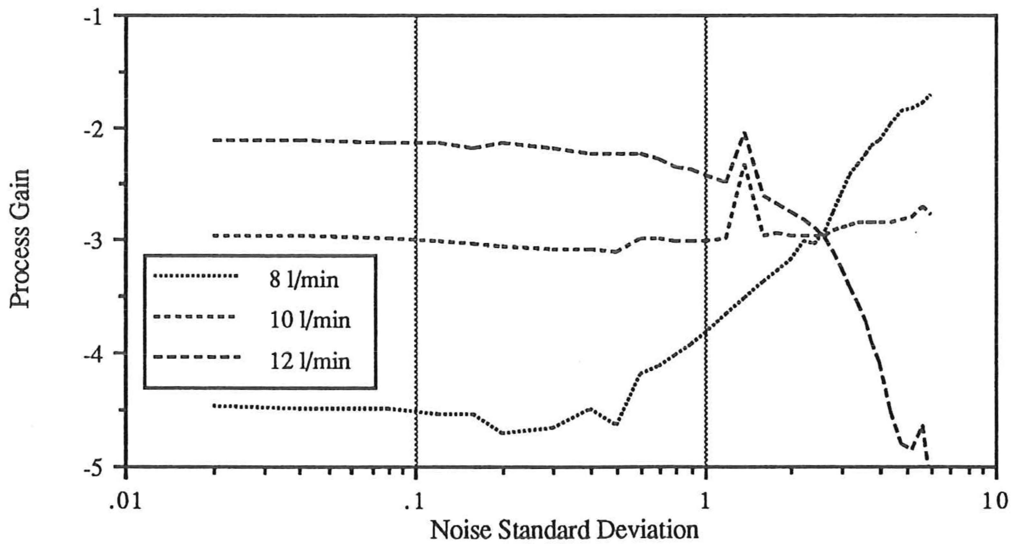
Graph 3.6 Gain Variation for RELS



Graph 3.7 Gain Variation for REELS



Graph 3.8 Gain Variation for RML



amount of noise present increases the separation between the gains begins to disappear until at some value of the noise standard deviation the gains at the operating points are the same. This point indicates where a bilinear model ceases to give any advantage over a linear process model.

As was found with both the variance and time constant comparisons the best results were obtained using the Recursive Extended Least Squares methods (RELS & REELS). There was almost no change in the gain estimates for RELS until the noise standard deviation reached *1°C* and the point at which the gains were equal occurred at a noise of $\sigma = 3.5°C$. Convergence of the gain estimates for REELS began earlier but the gains did not become equal until the noise reached $\sigma = 4.2°C$.

The performance of the recursive maximum likelihood method was similar to ordinary recursive least squares.

CPU Usage

The identification procedures were carried out using a FORTRAN program on a VAX 11/730 minicomputer. Table 3.2 gives the CPU requirements for each method, including the initialisation section of the program.

Table 3.2 CPU usage by method

Method	Number of Terms	CPU usage
RLS	*10*	*31 s*
RELS	*13*	*41 s*
REELS	*16*	*51 s*
RML	*16*	*73 s*

For RLS, RELS and REELS the CPU usage was almost exactly proportional to the number of terms in the model. RML used extra CPU time because of the additional recursive calculation required to obtain $V(k)$.

Computational requirements of this order are becoming trivial for many applications as many modern desk-top computers have more calculating power than the VAX system used for these trials. The exception to this is in the design of adaptive control systems in which the parameter estimation is carried out on-line, as part of the control calculation.

Conclusion

It is possible to identify bilinear process models using many of the methods designed for linear systems. Although bilinear models are not linear, they may be separated to form a vector dot product in which the parameter vector is linear.

A comparison of four recursive identification methods was carried out using a known discrete bilinear model of a steam heated constant volume tank as the process. Important properties of the identified models were compared to the original system. Under low noise conditions all four identification methods gave similar results. Recursive extended least squares gave the best performance for the trial system over a wide range of superimposed white noise strengths with a noise rejection of approximately 10^{-1} on the variance.

As a result of these trials, implementations of RLS, RELS and REELS were used in the batch identification program developed for the Apple Macintosh and used to identify the models in chapter 6.

Nomenclature

$y(k)$	Output value at time k
$u(k)$	Input value at time k
k	Discrete Time variable
n	Model order
m	Number of terms in model
l	Discrete Deadtime
N	Number of Data Points
$e(k)$	Prediction error at time k
$\phi(k)$	Measurement Vector at time k
$\theta(k)$	Parameter Vector Estimate at time k
$R(k)$	Information Matrix
$P(k)$	Error Covarience Matrix
$\lambda(k)$	Forgetting factor at time k
a_i	State Coefficient
b_j	Input Coefficient
c_{ij}	Bilinear Coefficient
DC	Constant Term
ε_i	Error Coefficient (RELS, REELS & RML)
γ_{ij}	Bilinear Error Coefficient (REELS & RML)
$V(k)$	First order derivatives of $e(k)$ with respect to θ (RML)

UD Factorisation Algorithm

$U(k)$	Upper triangular matrix
$D(k)$	Diagonal Matrix
f, g	Vectors

l_i, $i=1,m+1$	Gain Vectors
$L(k)$	Parameter Gain Matrix

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CHAPTER 4

Control of Bilinear Systems

Overview

In this chapter the methods currently available for the control of bilinear systems are reviewed and a controller design method for discrete time bilinear systems is developed.

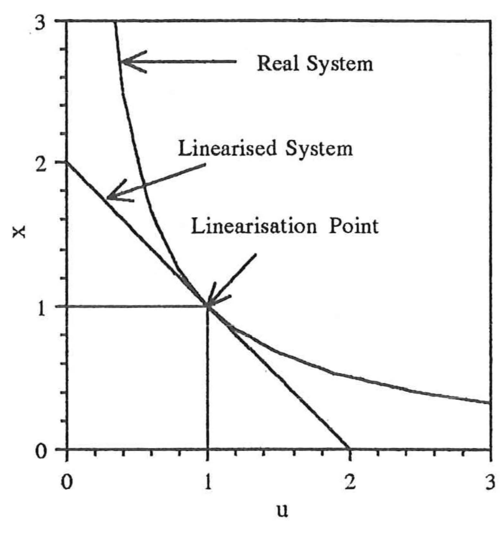
The results of Benallou et. al. are presented. Discrete approximation for the controller is derived. Control systems with feed forward and integral elements are examined.

Traditional Methods

The traditional approach to the control of bilinear systems or any other form of non-linear system is to linearise the behaviour of the plant about some operating point. The linearisation approach is represented graphically in graph 4.1.

The resulting linear model is used to design a controller for the plant. Although accurate near the linearisation point a linear approximation may become inaccurate with relatively small changes in the state of the plant.

Graph 4.1 Process Linearisation for a Constant Volume Tank System

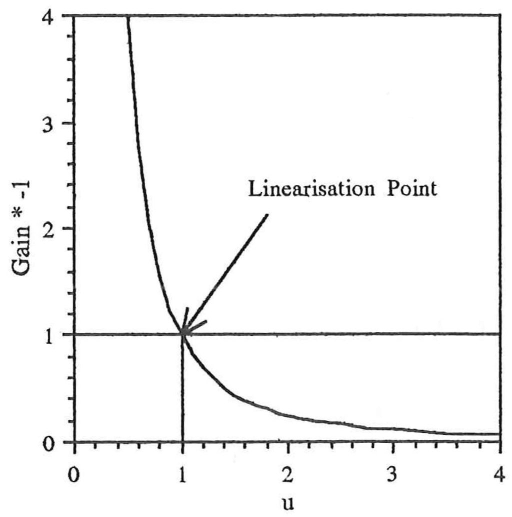


The curve in graph 4.1 represents the behaviour of a constant volume heated tank and its linearised model at a selected operating point.The gain of the process is inversely proportional to the square of the flowrate through the plant.

A PID controller tuned using the ultimate method will become unstable if the process gain reaches *1.7* times the gain at the linearisation point, this being the gain margin used in the Ziegler-Nichols tuning technique (Stephanopoulos 1984). For the constant volume heated tank system, this corresponds to a *23.3%* reduction in the inlet water flowrate, severely limiting the operating range of the plant. Similarly as the flowrate of water through the plant increases, the gain of the system decreases, stabilising the system but decreasing the effectiveness of the controller.

There exists a narrow region where the performance of a standard controller is acceptable. The width of this region depends upon the degree of non-linearity of the process.

Graph 4.2 Comparison of Linearised Model & Plant Gains for a Tank System



Gain Scheduling

The simplest way to extend the viable region of a controller is to use the technique of parameter scheduling. As suggested by the name, this method involves the use of a schedule, table or formula to alter the parameters of the controller based on the current set point and / or the current states of the plant. Often only the gain of the controller is changed leading to the term gain scheduling.

Parameter scheduled controllers may be divided into two types. Set point based controllers where the parameters are adjusted whenever the set point of the plant is changed and state based controllers where the controller gain is expressed as a function of the current states of the plant.

Set Point Based Scheduling

A set point based schedule can be developed in two ways.

1) By tuning the controller at a number of set points and using these settings to create a reference table. When the set point is altered the new parameter values are extracted from the table.

2) By using a non-linear model of the plant to obtain an expression for the process gain as a function of the operating point. A gain modification function can be obtained from this expression by applying some suitable constraint such as maintaining a constant open loop gain. For the heated tank system described above :

$$K_p \propto \frac{1}{F^2} \tag{4.1}$$

$$K_c \propto \frac{1}{K_p} \propto F^2 \tag{4.2}$$

where F is the steady state flowrate corresponding to the desired set point.
K_p is the process gain
K_c is the controller gain

The non-linear plant model used may be derived using knowledge of the behaviour of the plant or by using plant behaviour to identify the parameters of a general non-linear model eg. bilinear.

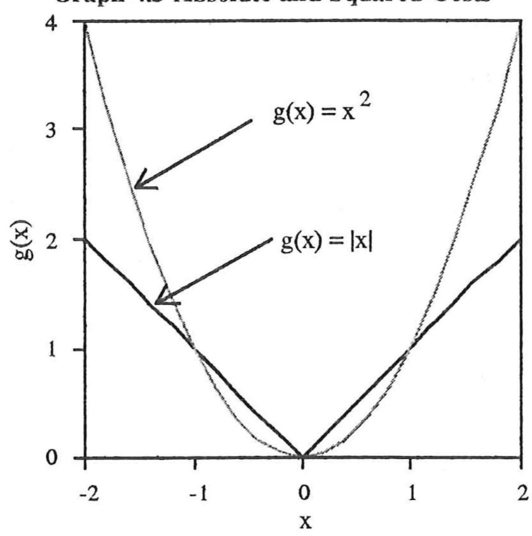
State Based Scheduling

State based scheduling is more complex, and requires a non-linear dynamic model of the plant. The non-linear model is used to construct a function which determines the appropriate controller parameters at any point in time. The structure of the modifications is determined from the results of linear control theory or relevant non-linear theory.

Optimal Control

For a physical process it is possible to formulate an expression which associates costs with deviations from the steady state values of the states and controlled inputs. The resulting equation is referred to as a cost function. The cost function must be designed with a global minimum at the set point. One approach would be to take the absolute values of the errors and use these in the cost function. This method introduces a discontinuity in the first derivative of the cost function as shown in graph 4.3. A better method is to use the squares of the errors as these will always be positive and there is no discontinuity.

Graph 4.3 Absolute and Squared Costs



The usual cost function for a continuous system is

$$J = x^T(t_f)\, W\, x(t_f) + \frac{1}{2}\int_{t=0}^{t_f}[x^TQx + u^TRu]dt \qquad (4.4)$$

Where *W* is the weighting matrix for the final states
Q represents the costs associated with the states during the run,
R gives the cost weightings for the controller action.

In many control applications it is possible to consider the final time t_f to be a long time in the future. The control objective is now to maintain the system as close as possible to the desired states by the use of a reasonable amount of control and without regard for the terminal state, this is expressed in equation 4.5.

$$J = \frac{1}{2}\int_{t=0}^{t_f} L(x,u,t)\, dt = \frac{1}{2}\int_{t=0}^{t_f}[x^TQx + u^TRu]dt \qquad (4.5)$$

The Hamiltonian of the system is defined as:

$$H(x,\lambda,u,t) = L(x,u,t) + \lambda^T(t).f(x,u,t) \qquad (4.6)$$

Where $\dot{x}$ $= f(x,u,t)$ describes the system behaviour,
$L(x,u,t)$ $= x^TQx + u^TRu$
and $\lambda(t)$ is a vector of Lagrange multipliers.

A sufficient condition for optimality is that the minimum value of the Hamiltonian function is equal to zero (Athans & Falb 1966). The control scheme which satisfies this is found from the stationary points of the Hamiltonian.

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial L}{\partial u} + \lambda^T.\frac{\partial f}{\partial u} \tag{4.7}$$

For a time invarient linear system with quadratic performance index as above the optimal control is given by:

$$u(t) = -R^{-1}\, B^T\, S\, x(t) \tag{4.8}$$

where the matrix S is the symmetric positive definite solution to the algebraic matrix Riccati equation.

$$S\,A + A^T S + Q - S\,B\,R^{-1}\,B^T S = 0 \tag{4.9}$$

An optimal state variable feedback controller (4.8), will be globally asymptotically stabilisingfor strictly linear systems. The controlled system will return to a stable steady state regardless of its starting point, as changes in the control variable have a constant effect over the entire operating range of the plant. The gain of a non-linear system may vary to such an extent over the operating region that the controlled plant will not return to the desired state from some initial conditions. In designing controllers for non-linear systems it is therefore necessary to ensure that the control system is stable for the entire operating range of the plant.

Lyapunov stability theory states that if, for a homogeneous system

$$\dot{x} = f(x,t) \tag{4.10}$$

there exists a scalar function $V(x(t),t)$ such that :

I. The partial derivatives with respect to x and t are continuous,
II. $V(x(t),t) > 0$ for all $x \neq 0$ and for all t,
III. $dV/dt = \dot{V} < 0$ for all $x \neq 0$ and for all t,
IV. $V(x(t),t) \to \infty$ as $x^T(t)\, x(t) \to \infty$.

then the system has asymptotic, global stability (Elbert 1984).

Continuous Time Bilinear Optimal Controller

For the case of continuous time bilinear systems a globally asymptotically stabilizing optimal controller design was established by Benallou et. al. (1988). What follows is a summary of their results, aside from differences in nomenclature.

For a continuous bilinear system

$$\dot{x} = A\,x + \sum_{i=1}^{m} u_i\,[b_i + C_i\,x] \qquad (4.11)$$

Let $d_i(x) = b_i + C_i\,x$ and $D(x) = \left[d_1(x)\,\middle|\,d_2(x)\,\middle|\ldots\middle|\,d_m(x)\right]$ and the process may be written:

$$\dot{x} = A\,x + D(x)\,u \qquad (4.12)$$

If the control objective is to minimise the cost function

$$J = \frac{1}{2}\int_0^{\infty}\left\{x^TQx + \sum_{i=1}^{m}\frac{1}{r_i}\,[x^TSd_i(x)]^2 + u^TRu\right\}dt \qquad (4.13)$$

Where Q is the symmetric, positive definite state weighting matrix,
R is the diagonal matrix of control weightings and
S is the symmetric positive definite solution to the continuous time Lyapunov equation $A^TS + SA = -Q$.

then the optimal control policy is given by :

$$u_i^* = -\frac{1}{r_i}\,x^TSd_i(x) \qquad (4.14)$$

which may also be written:

$$u^* = -\,R^{-1}D^T(x)\,S\,x \qquad (4.15)$$

Implementation of this control results in the closed loop system :

$$\begin{aligned}\dot{x} &= A\,x - D(x)\,R^{-1}D^T(x)\,S\,x \\ &= [A - D(x)\,R^{-1}D^T(x)\,S]x \end{aligned} \qquad (4.16)$$

Proof of Stability

A suitable Lyapunov candidate which meets conditions I, II and IV is:

$$V = \frac{1}{2}\,x^TS\,x \qquad (4.17)$$

differentiating with respect to time gives:

$$\dot{V} = \frac{1}{2}\,x^TS\,\dot{x} + \frac{1}{2}\,\dot{x}^TS\,x \qquad (4.18)$$

substituting for $\dot{x}$:

$$\dot{V} = \frac{1}{2}\,x^TSAx + \frac{1}{2}\,x^TA^TS\,x - x^TS\,D(x)\,R^{-1}D^T(x)\,S\,x \qquad (4.19)$$

as $A^TS + SA = -Q$:

$$\dot{V} = -\frac{1}{2}\,x^TQ\,x - x^TS\,D(x)\,R^{-1}D^T(x)\,S\,x \qquad (4.20)$$

Both Q and R are positive definite so the above expression will be negative for all values of x other than zero, thus fulfilling criterion III. The controlled system has global, asymptotic stability.

Proof of Optimality

Using the Hamiltonian:

$$H(x,u) = L(x,u) + V_x(x)\,[Ax + D(x)\,u] \tag{4.21}$$

$$= \tfrac{1}{2}\{x^TQx + x^TSD(x)\,R^{-1}D^T(x)\,Sx + u^TRu\} + x^TS\,[Ax + D(x)\,u] \tag{4.22}$$

$$\frac{\partial H}{\partial u_i} = 0 = r_i\,u_i + x^T S\,d_i(x) \tag{4.23}$$

$$u_i^* = -\frac{1}{r_i}x^T S\,d_i(x) \quad \text{or} \quad u^* = -R^{-1}D^T(x)\,S\,x \tag{4.24}$$

Substituting this back into the Hamiltonian function gives :

$$H(x,u) = \tfrac{1}{2}\{x^TQx + x^TSD(x)\,R^{-1}D^T(x)\,Sx + x^TSD(x)\,R^{-1}D^T(x)\,Sx\}$$

$$+ V_x(x)\,[Ax - D(x)\,R^{-1}D^T(x)\,S\,x]$$

$$= x^T\left\{\tfrac{1}{2}[Q + SA + A^TS] + SD(x)R^{-1}D^T(x)\,S - SD(x)R^{-1}D^T(x)\,S\right\}x \tag{4.25}$$

which is zero for all values of x, so the control is optimal for the performance function (4.13).

Effect of Set Point Changes

The work of Benallou et al stopped short of considering the effect of changes in set point on the behaviour of the controlled system. A major advantage of using bilinear models of processes is the ability to use the model over a greater range of the operating region than is possible with linear models. Any controller design method should have provision for dealing with changes in the operating or set point.

Set point changes alter the parameters of the state matrix A,

$$A_{new} = A_{old} + \sum_{i=1}^{m}\tilde{u}_i\,C_i \tag{4.26}$$

where $\tilde{u}_i$ is the value of the *ith* input, required to achieve a steady state at the new set point, as a deviation from the current steady state input.

To ensure the system remains stable after such a change, the Lyapunov solution must be recalculated. If this is done while holding Q constant, then although the system must necessarily remain stable the nature of the control and process response will change. The reasons for this become clear when the structure of the performance index is examined.

The performance index for the continuous controller is given in (4.27). This may be reorganised to combine the state weighting terms giving an expression of the form used in standard optimal control theory (4.28).

$$J = \int_0^{\infty}\left\{x^TQx + \sum_{i=1}^{m}\frac{1}{r_i}\left[x^TSd_i(x)\right]^2 + u^TRu\right\}dt \tag{4.27}$$

$$j(t) = x^TQx + x^TSD(x)R^{-1}D^T(x)Sx + u^TRu$$

$$j(t) = x^T\left[Q + SD(x)R^{-1}D^T(x)S\right]x + u^TRu \qquad (4.28)$$

The value of the overall state weighting matrix is dependent upon the current operating region of the system as both S & $D(x)$ are dependent upon the controller set point. If we desire the response of the system to be consistent for the entire range of operation then whenever the set point is changed the value of Q must be re-evaluated, so that $Q + SD(x)R^{-1}D^T(x)S$ remains constant at the various set points. Let this be equal to a symmetric positive definite matrix P equation.

$$Q + SD(x)R^{-1}D^T(x)S = P \qquad (4.29)$$

By substituting for Q from the Lyapunov equation and defining $K(x) = D(x)R^{-1}D^T(x)$ the expression becomes:

$$A^TS + S\,A + P - S\,K(x)\,S = 0 \qquad (4.30)$$

which has the same form as the Algebraic Matrix Riccati Equation. Thus in order to maintain the same overall weighting matrices at each set point, it is necessary to solve the Algebraic Riccati equation whenever the set point is altered, using $K(x)$ calculated at the set point.

In the case of a linear system, $K(x)$ becomes $B\,R^{-1}\,B^T$, giving the standard continuous time optimal regulator equation.

Discrete Time

To obtain a discrete time controller the central difference equations 4.31 and 4.32 were applied to the continuous equations resulting in the following substitutions developed in chapter 2.

$$x = \frac{x(k+1) + x(k)}{2} \qquad (4.31)$$

$$\dot{x} = \frac{x(k+1) - x(k)}{h} \qquad (4.32)$$

$$A = \frac{2}{h}\left[\alpha - I\right]\left[\alpha + I\right]^{-1} \qquad (4.33)$$

$$d_i = \frac{1}{h}\left[I - \left[\alpha - I\right]\left[\alpha + I\right]^{-1}\right]\delta_i \qquad (4.34)$$

When these are applied to equation 4.14 the control equation 4.35 results, or 4.37 in matrix form.

$$u_i^* = -\frac{1}{r_i}x^TS\frac{1}{h}\left[I - \left[\alpha - I\right]\left[\alpha + I\right]^{-1}\right]\delta_i(x) \qquad (4.35)$$

$$\Delta = \left[\delta_1 \,|\, \delta_2 \,|\, \dots \,|\, \delta_m\right] \qquad (4.36)$$

$$u^* = -\frac{1}{h}R^{-1}\Delta^T(x)\left[I - \left[\alpha - I\right]\left[\alpha + I\right]^{-1}\right]^T S\,x \qquad (4.37)$$

The value of x is found by substituting this expression back into the discrete time equations for the system, to obtain an expression for $x(k+1)$.

$$x(k+1) = \alpha\,x(k) + \frac{1}{h}\Delta(x)\,R^{-1}\Delta^T(x)\left[I - \left[\alpha - I\right]\left[\alpha + I\right]^{-1}\right]^T S\,x \qquad (4.38)$$

$$x(k+1) = \alpha\,x(k) - \frac{1}{8h}\left[\Delta(x(k+1)) + \Delta(x(k))\right]R^{-1}\left[\Delta^T(x(k+1)) + \Delta^T(x(k))\right]$$
$$\left[I - \left[\alpha - I\right]\left[\alpha + I\right]^{-1}\right]^T S\left[x(k+1) + x(k)\right] \qquad (4.39)$$

Defining a function $G(x(k+1),x(k)) =$

$$\frac{1}{8h}\left[\Delta(x(k+1)) + \Delta(x(k))\right] R^{-1}\left[\Delta^T(x(k+1)) + \Delta^T(x(k))\right]\left[I - [\alpha - I][\alpha + I]^{-1}\right]^T S \qquad (4.40)$$

Substituting this expression into equation 4.39 yields the pseudo-linear equation 4.41. Together these two equations make up an iterative pair which may be used to find $x(k+1)$ and therefore x.

$$x(k+1) = \left[I + G(x(k+1),x(k))\right]^{-1}\left[\alpha - G(x(k+1),x(k))\right] x(k) \qquad (4.41)$$

In a linear system, the function G is independent of the state and therefore constant over the entire operating range of the plant. For bilinear systems, however, the value of G can only be found by iterating the equations above. An approximation may be obtained by using one of a number of simplifying assumptions, to avoid the computation associated with solving exactly. In order of complexity, or computational load.

1) As the sampling time of the system becomes very small in relation to the time constant the state of the system will not change much between sampling points and $x(k)$ may be used in place of x in the control equation.

2) Use the linear system case and assume G is independent of the state, giving

$$x(k+1) = \left[I + G(0,0)\right]^{-1}\left[\alpha - G(0,0)\right] x(k) \qquad (4.42)$$

3) Take the first estimate of $x(k+1)$ from the iterative sequence and use this to estimate x.

$$x(k+1) = \left[I + G(x(k),x(k))\right]^{-1}\left[\alpha - G(x(k),x(k))\right] x(k) \qquad (4.43)$$

4) The last case is the complete iterative method using the equation pair to obtain $x(k+1)$ to the desired accuracy.

A comparison of these four methods is given in table 4.1 for a sampled continuous time bilinear system with the following characteristics.

System	$\dot{x} = -0.25\,x - 0.025\,x\,u - 0.5\,u$
Control	$u = 2\ (0.5 + 0.025\,x)\,x$
Current State	$x(k) = 10$

The data is presented as the estimated value of x divided by the initial value $x(k)$. The 'true' values are given by method 4.

Graph 4.4 Comparison of $\frac{x}{x(k)}$ Values for a 1st Order Bilinear System

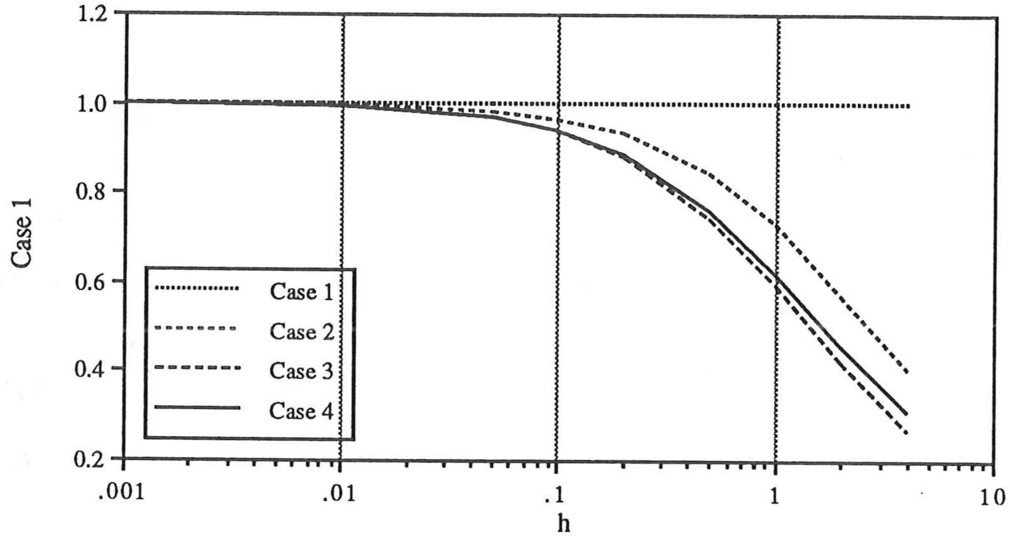


Table 4.1 Comparison of $\frac{x}{x(k)}$ Values for a 1st Order Bilinear System

using different approximations for x(k+1)

Sampling Interval, h	*Method 1*	*Method 2 Linear*	*Method 3 1 Iteration*	*Method 4 3 Iterations*
.001	*1.0*	*0.9996*	*0.9993*	*0.9993*
.01	*1.0*	*0.9963*	*0.9932*	*0.9932*
.05	*1.0*	*0.9816*	*0.9668*	*0.9671*
.1	*1.0*	*0.9639*	*0.9357*	*0.9367*
.2	*1.0*	*0.9302*	*0.8791*	*0.8825*
.5	*1.0*	*0.8421*	*0.7442*	*0.7568*
1.0	*1.0*	*0.7273*	*0.5926*	*0.618*
2.0	*1.0*	*0.5714*	*0.4211*	*0.4585*
4.0	*1.0*	*0.4*	*0.2667*	*0.3067*

There is little difference between the four methods at very small sampling times ($h<0.05$ *min*). The state of the system does not change significantly over such a small interval and under these circumstances acceptable control may be obtained using $x = x(k)$. As the sampling period increases, the differences between the assumptions increases. The two bilinear methods produce very similar results for sampling intervals up to one minute, suggesting the use of iteration to produce a more accurate result is not justified. The linear assumption falls in the mid-range and may provide a useful compromise as it is only re-evaluated when the set point changes thus reducing computation at each sampling point.

The selection of an appropriate approximation depends upon the size of the sampling interval and the degree of non-linearity present in the process.

Discrete Bilinear Controller Design Method

An efficient design procedure results:

1) An input weighting matrix R and an approximate overall state weighting matrix P should be selected for the desired response, P satisfying:

$$P = Q + S\,D\,R^{-1}D^{T}S \qquad (4.44)$$

2) The equivalent continuous model parameters A and D may be calculated at the set point, x^{ss}.

$$\frac{Ah}{2} = [\alpha - I][\alpha + I]^{-1} \qquad (4.45)$$

$$D = \frac{1}{h}\left[I - \frac{Ah}{2}\right]\Delta(x^{ss}) \qquad (4.46)$$

3) The algebraic Riccati equation may be solved to obtain S.

$$A^{T}S + S\,A + P - S\,D\,R^{-1}\,D^{T}\,S = 0 \qquad (4.47)$$

4) The constant factor of the controller, L, can be obtained from S.

$$L = -\frac{1}{h}S\left[I - \frac{Ah}{2}\right] \qquad (4.48)$$

5) The control variable may now be calculated using Equation 4.49 where x is determined using a suitable approximation to Equation 4.38.

$$u_i^*(k) = \frac{1}{r_i}x^{T}L\,\delta_i(x) \qquad (4.49)$$

A discrete optimal controller for bilinear systems can be obtained with little more effort than for a linear system.

Limitations

Many chemical plant items are subject to environmental effects or disturbances of long duration. In addition, the behaviour of some plant items may alter over time, for example, fouling of heat exchangers. Under such circumstances a state variable feedback controller, such as the bilinear optimal controller, will suffer from offset and will not bring the plant back to the desired set point. To overcome this:

1) Incorporate as many of the variables which affect the plant into the process model. A rigourous treatment of the dynamic equations of the system may be used to determine how each influences the plant and a suitable parametric model structure devised. The resulting model may be used to develop a feed-forward control strategy for the plant. The feedforward design is limited to those variables which may be changed or that vary sufficiently on their own to provide sufficient information for good parameter identification.

2) Eliminate or reduce as many of the disturbances as possible, either by physical modifications to the plant or its surrounds, or by improving the control on upstream units. In many cases, this approach is not possible or cannot be justified for economic reasons.

3) Implement adaptive control. An identification procedure can be included in the control system to continually update the parameters of the model. Adaptive control has usually been implemented using minimum variance or deadbeat control strategy (Goodwin, McInnis & Long (1981), Ohkawa & Yonezawa (1983)). However, these designs have the drawback of unboundedness in the controlled inputs and may result in unrealisable control action. The adaptive implementation of a more complex control strategy has many difficulties, not the least of which being the amount of computation which may be required.

4) Include integral action in the controller design to correct for small amounts of drift.

Of these four methods the simplest to implement is the use of control based on the integral of the state. Feed forward and integral methods have been examined in further detail, as they apply to the bilinear controller design.

Feedforward Control

An expression which relates the measured disturbances to the system states is required for the design of a feedforward control or compensation system.

A dynamic system subject to measured disturbances m, may be modeled:

$$\dot{x} = Ax + D(x)\,u + G\,m \qquad (4.50)$$

Two methods exist for developing a feedforward strategy based on such a model, by cancellation or by an augmented optimal control method.

Cancellation Approach

The objective for the cancellation method, is to find a control setting that removes the effect of the disturbance from the system. Let the total control response be a linear sum of the response to the states and the control required to offset the disturbances. The system equation becomes

$$\dot{x} = Ax + D(x)\,u_x + D(x)\,u_m + G\,m \qquad (4.51)$$

To cancel the disturbances the last two terms must add to zero. The disturbance portion of the controller action is

$$u_m = -\,D(x)^{-1}\,G\,m \qquad (4.52)$$

In order to obtain an inverse of $D(x)$ it is necessary to use only those rows of D & G which contain non-zero elements. In the case of a linear system the value of $D(x)^{-1}\,G$ is independent of x, and need only be calculated once. For a bilinear system it is necessary to perform an $m*m$ matrix inversion at each control point.

The overall controller response may be written:

$$u = - D(x)^{-1}\, G\, m \;-\; R^{-1}\, D^T(x)\, S\, x \qquad (4.53)$$

For many systems the solution of equation 4.52 may not be found and it is not possible to obtain feedforward cancellation of the disturbance effects.

Augmented Optimal Control Method

If the state vector is augmented to include the disturbances, the modified system becomes:

$$\dot{y} = \left[\begin{array}{c|c} A & G \\ \hline 0 & 0 \end{array}\right] y + \left[\begin{array}{c} D(x) \\ \hline 0 \end{array}\right] u \text{ with } y = \left[\begin{array}{c} x \\ \hline m \end{array}\right] \qquad (4.54)$$

where the first derivatives of the disturbances are assumed to be random functions with a mean of zero. The vector m can be referred to as the vector of pseudo-states. The pseudo-states of the system may not be controlled and should not be represented in the cost function so the state weighting matrix for the modified system is that of the simple system with zeros added to bring it to the correct order.

$$P_N = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 0 \end{array}\right] \qquad (4.55)$$

The augmented state matrix A_N and the augmented state weighting matrix P_N are both singular. A number of the terms in the Riccati equation solution, S_N, will be undefined. However, the leading $n*n$ sub matrix of S_N will be the same as S obtained from the original system:

$$S_N = \left[\begin{array}{c|c} S & S_2 \\ \hline S_2^T & \# \end{array}\right] \qquad (4.56)$$

The control policy is

$$\begin{aligned} u &= - R^{-1} D_N^T S_N y \\ &= - R^{-1} \left[D(x) \mid 0\right] \left[\begin{array}{c|c} S & S_2 \\ \hline S_2^T & \# \end{array}\right] \left[\begin{array}{c} x \\ \hline m \end{array}\right] \\ &= - R^{-1} D^T S\, x - R^{-1} D^T S_2\, m \end{aligned} \qquad (4.57)$$

This method will not eliminate the disturbances but will mitigate their effects through the use of a reasonable amount of control as defined by the cost function. Unlike the cancellation method, this approach will always give a result.

The following example is given to illustrate the differences between the two methods:

$$\frac{d\theta_o}{dt} = .25\left[\theta_i - \theta_o\right] - .5f + .025f\left[\theta_i - \theta_o\right] \qquad (4.58)$$

where θ_o is the state, θ_i is the disturbance and f is the input. The state and input weights for the optimal control calculation are both one. The optimal control for the system, ignoring the disturbance, is:

$$f = 1.236\left[.5 + .025\,\theta_o\right]\theta_o \qquad (4.59)$$

Using the cancellation method to control the disturbance effect gives:

$$f = 1.236\left[.5 + .025\,\theta_o\right]\theta_o + \frac{.25\,\theta_i}{.5 + .025\left[\theta_o - \theta_i\right]} \qquad (4.60)$$

and the augmented optimal method gives:

$$f = \left[\; .5 + .025\left\{\theta_o - \theta_i\right\} \quad 0 \;\right]\begin{bmatrix} 1.236 & .5528 \\ .5528 & 2.577e{+}10 \end{bmatrix}\begin{bmatrix} \theta_o \\ \theta_i \end{bmatrix} \qquad (4.61)$$

$$= \left[.5 + .025\left\{\theta_o - \theta_i\right\}\right]\left[1.236\,\theta_o + .5528\,\theta_i\right]$$

In the case where $\theta_i = 1$ and $\theta_o = 0$ the two methods give $f = .526$ and $f = .263$ respectively. When feed into the system equation (4.58) the cancellation method completely removes the effect of the disturbance. The augmented state method halves the first derrivative, reducing but not removing the disturbance effect.

Integral Action and the Bilinear Controller

For a general bilinear system :

$$\dot{x} = A\,x + D(x)\,u \qquad (4.62)$$

extra virtual states may be added to the system which when evaluated represent the integrals with respect to time of actual states. To integrate a state x_i define a virtual state x_{n+j} such that

$$\dot{x}_{n+j} = x_i$$

if the augmented system equation is used to design a state feedback controller the resulting control policy will include terms based on this additional state. Thus it is possible to include integral action into bilinear controller design method.

For the second order, single input bilinear system :

$$\dot{x} = A\,x + D(x)\,u \text{ where } x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad (4.63)$$

an extra virtual state may be included which represents the integral of x. The system becomes :

$$\dot{x} = \begin{bmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 1\;0 & 0 \end{bmatrix} x + \begin{bmatrix} D(x) \\ \hline 0 \end{bmatrix} u \text{ where } x = \begin{bmatrix} x \\ \dot{x} \\ x_3 \end{bmatrix} \qquad (4.64)$$

It is not possible to obtain S from the Lyapunov equation for the augmented system as the augmented matrix A is singular. However, it is possible to calculate Q from a known S. A value for S may be obtained by solving the Riccati equation for some selected P. The resulting controller expression is:

$$u = -\frac{1}{r}\,[D^T(x) \mid 0]\; S \begin{bmatrix} x \\ \dot{x} \\ x_3 \end{bmatrix} \tag{4.65}$$

If the system has the standard second order structure:

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \text{ and } D(x) = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{4.66}$$

then as $x_3 = \int x\, dt$ the control becomes :

$$u = -\frac{1}{r}\,\{b_1 + c_1 x + c_2 \dot{x}\}\left\{s_{21} x + s_{22} \dot{x} + s_{23} \int x\, dt\right\} \tag{4.67}$$

which describes a PID controller with state based gain scheduling defined by equation 4.68.

$$\frac{Gain_x}{Gain_0} = \frac{\{b_1 + c_1 x + c_2 \dot{x}\}}{b_1} \tag{4.68}$$

where $Gain_x$ = the controller gain at a state x
$Gain_0$ = the controller gain at the tuning point.

Extending this further, the performance of a PID controller for a bilinear system might be improved by implementing a state-based schedule defined in the same manner.

Nomenclature

K_P	Process Gain
F	Flowrate into Heated Tank
K_C	Controller Gain

Optimal Control

x	State Vector
u	Input Vector
$x(t_f)$	Final value of State
W	Symmetric Positive Definite Final State Weighting Matrix
Q	Symmetric Positive Definite State Weighting Matrix
R	Positive Definite Diagonal Input Weighting Matrix
r_i	Diagonal Element of R
J	Performance Criterion
$H(x,\lambda,u,t)$	Hamiltonian Function
$\lambda(t)$	Lagrange multipliers
S	Symmetric Positive Definite Matrix

$V(x(t),t)$	Positive Valued Scalar Function
A	State Coefficient Matrix
B	Input Coefficient Matrix
b_i	i th Column of B, Coefficient vector for u_i
C_i	Bilinear Coefficients for Input i
$d_i(x)$	Combined Coefficient Vector for Input i. $d_i(x) = b_i + g_i\, x(k)$
$D(x)$	Combined Input Coefficient Matrix, columns are $d_i(x)$, $i = 1,m$
P	Symmetric Positive Definite Overall State Weighting Matrix
$K(x)$	Positive Definite Matrix Function

Discrete Time

$x(k)$	State Vector sampled at $t = k * h$
k	Discrete Time variable
h	Sampling Interval
α	State Coefficient Matrix
β_i	Coefficient Vector for i th input
γ_i	Bilinear Coefficient Matrix for i th input
$\delta_i(x(k))$	Combined Coefficient Vector for Input i. $\delta_i(x(k)) = \beta_i + \gamma_i\, x(k)$
$\Delta(x(k))$	Combined Input Coefficient Matrix, columns are $\delta_i(x(k))$, $i = 1,m$
$G(x(k+1),x(k))$	Matrix Function
L	Constant portion of Control Equation.

Feedforward Control & Compensation

G	Disturbance Coefficient Matrix
m	Vector of Measured Disturbances
y	Augmented State Vector

Integral Action

$Gain_x$	Controller gain at state x
$Gain_0$	Controller gain at the tuning point.

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CHAPTER 5

Bilinear Control of a Heated Tank

Overview

The control methods described in the previous chapter were applied to a constant volume steam heated tank system. A sequence of set point and heat input disturbances was used to compare the performance of the controller designs.

The results of digital computer simulations are presented along with data collected from pilot plant trials.

Tank System

As mentioned in previous chapters the constant volume heated tank system is one of the simplest physical processes which displays bilinear behaviour. The tank system is thus ideal for trials of control methods for bilinear systems, before attempting to implement them on more complex systems.

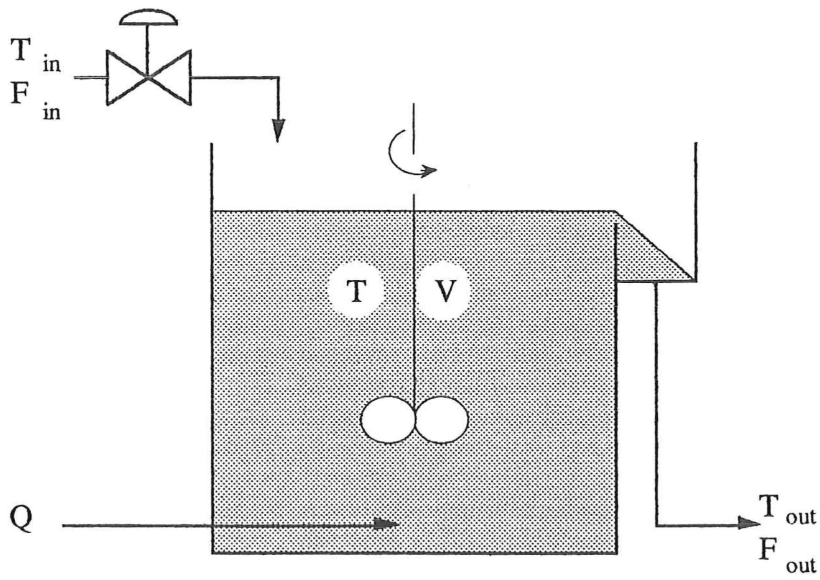
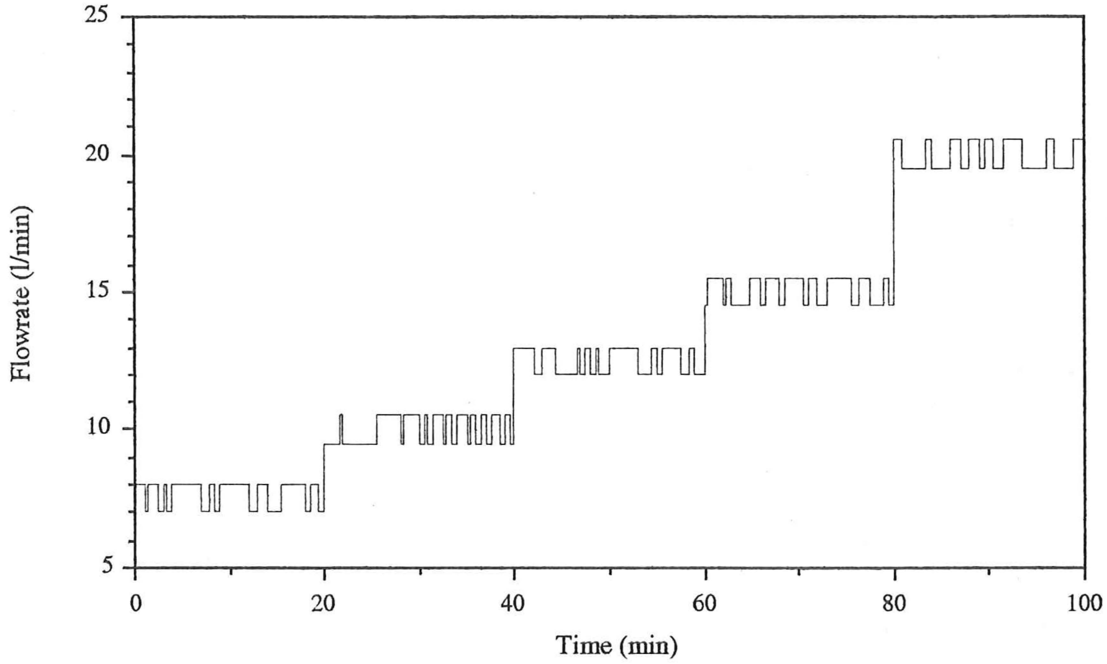
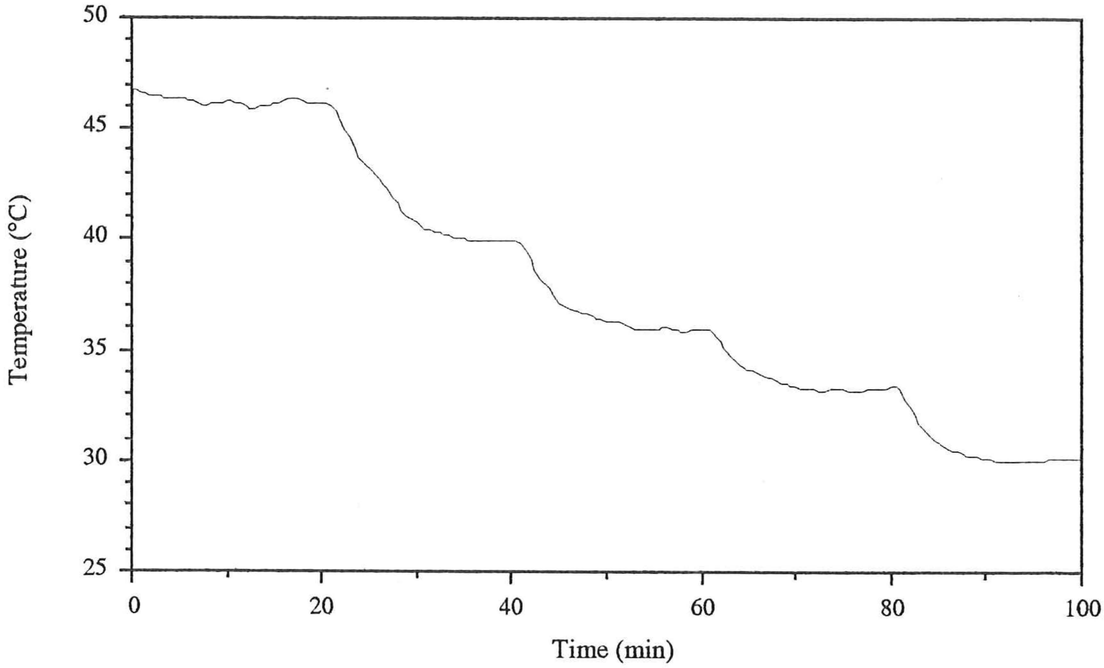


Figure 5.1 Diagram of Tank System

Graph 5.1 Input Sequence for Simulation Identification



Graph 5.2 Response of Simulation to 5.1



The control objective was to maintain the temperature of the outlet water at a desired value, by manipulating the flowrate of cold water into the tank. Selecting the control variable in this manner enables the full use of all the available heat to provide water of the desired temperature but results in a process which is inherently bilinear.

Digital Computer Simulation

A digital computer simulation of the tank system was used for initial trials. The differential equations governing the tank were integrated using a modified Euler method with an integration step size of *0.01* minutes. The process response was sampled at *0.5* minute intervals.

Identification

The simulation was run using a series of flowrate step changes with a superimposed pseudo random binary sequence to drive the system. The input sequence and resulting process output are shown in graphs 5.1 and 5.2. Using this data, discrete time models were obtained by least squares identification.

The bilinear model used for the control simulations is given in equation 5.1 in terms of deviations from a steady state output of *40°C* corresponding to a cold water flowrate of *10 l/min*.

$$x(k+1) = \begin{bmatrix} 1.480 & -0.523 \\ 1 & 0 \end{bmatrix} x(k) + u \left\{ \begin{bmatrix} -.08756 \\ 0 \end{bmatrix} + \begin{bmatrix} -.00246 & -.00199 \\ 0 & 0 \end{bmatrix} x(k) \right\} \qquad (5.1)$$

Where the state vector is defined, $x(k) = \begin{bmatrix} T_{out}(k) \\ T_{out}(k-1) \end{bmatrix}$

A linear model was obtained, for the design of linear controllers, by ignoring the bilinear terms.

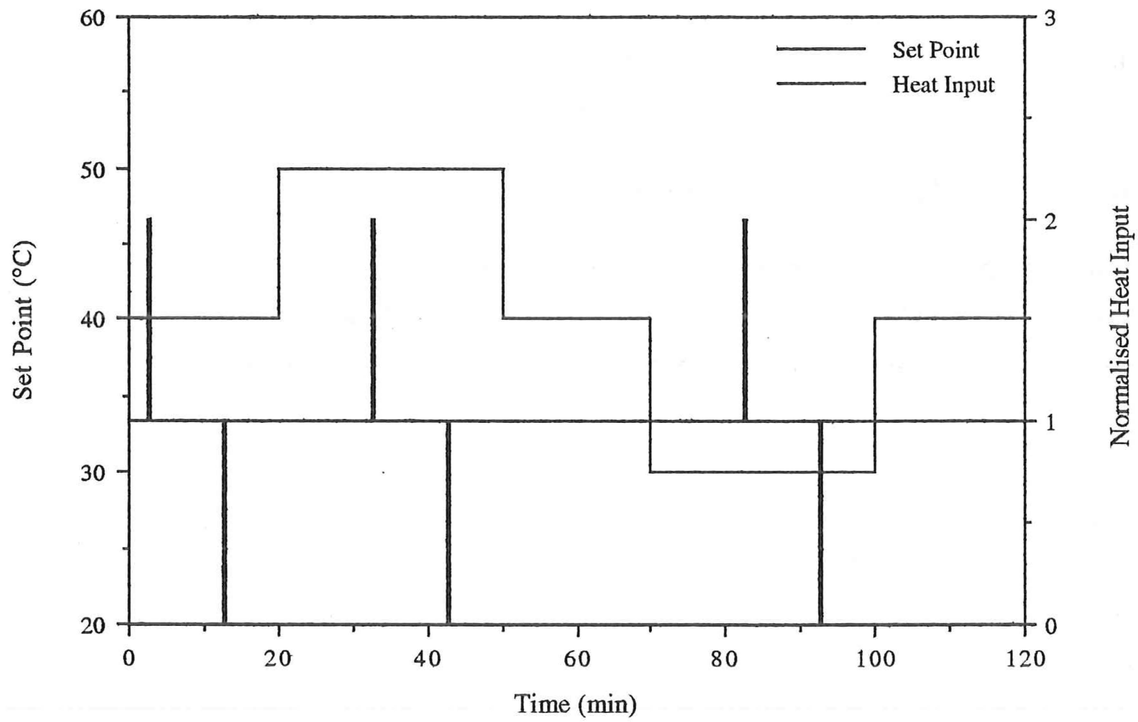
Control

The simulation was modified to enable a variety of controllers to be implemented by linking with different subroutines. An interrupt system was designed to provide a series of set point and heat flow disturbances to test the various control methods. The source code for the simulation may be found on the appendix disk. The operating sequence for the control trials is given below.

Operating Sequence

The operating sequence for the control trials was designed to test plant response over a wide range of operation. The plant was subjected to a combination of set point changes and heat input disturbances.

Graph 5.3 Operating Sequence for Simulation Control Trials



Three outlet temperature set points were used, at *30°C*, *40°C* and *50°C*. At each of these points the system was subjected to both an increase and a decrease in the heat input by *100%* for *0.4* minutes each. At least 10 minutes was allowed after each disturbance or set point change to enable the system to regain the set point. The overall length of the trial runs was *120* minutes.

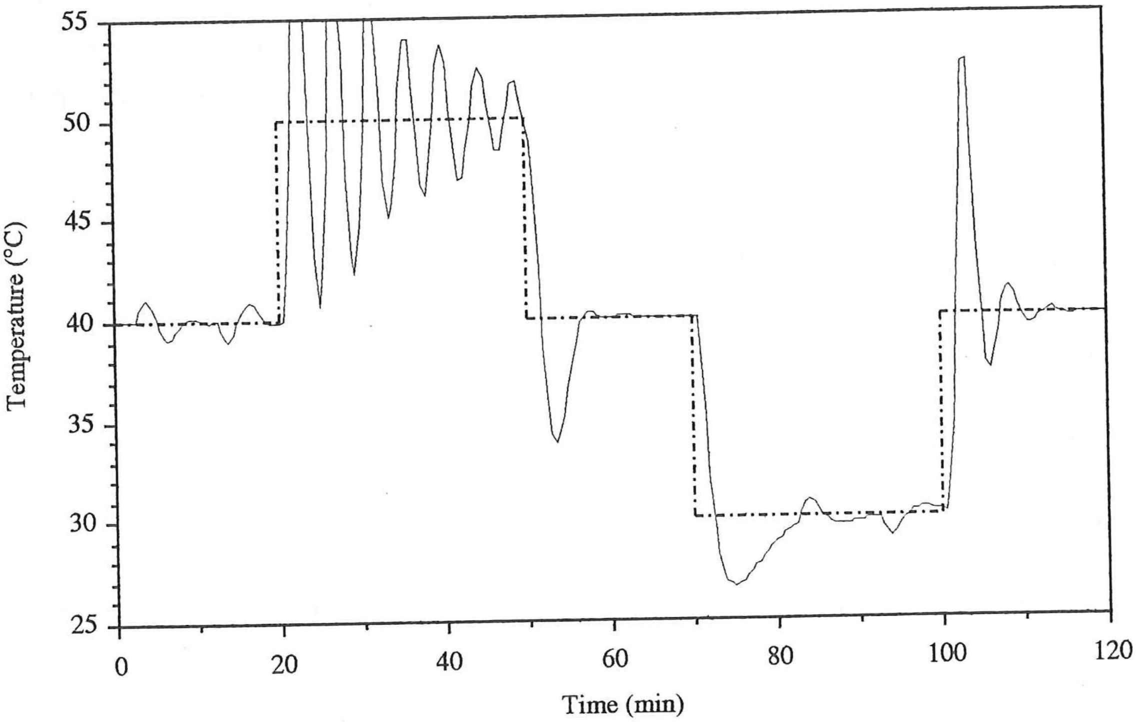
PID Control

Proportional only control was used for a disturbance at the initial set point to obtain constant amplitude oscillations from the plant. A PID controller was tuned using the ultimate method (Stephanopoulos, 1984) from this data. Graph 5.4 shows the plant response with this control.

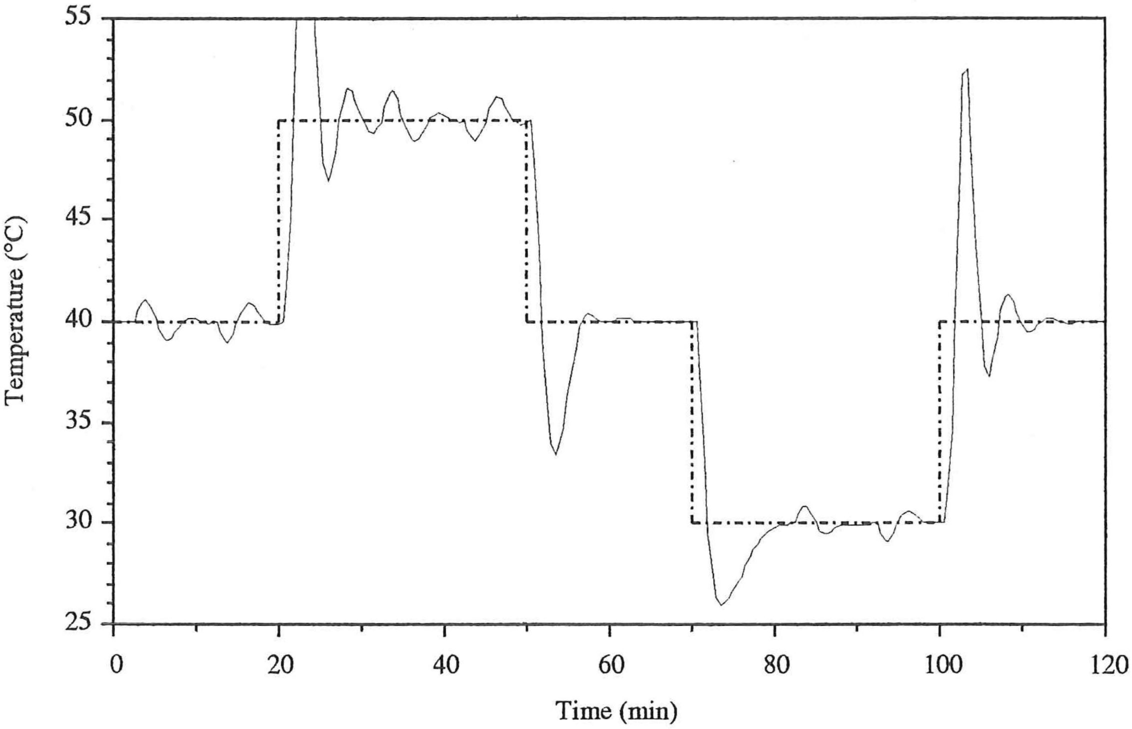
$$K_U = -4 \text{ °C } l/min, \; P_U = 10.1\dot{6} \; min$$
$$K_C = -2.4 \text{ °C } l/min, \; T_I = 5.08\dot{3} \; min, \; T_D = 1.271 \; min \qquad (5.2)$$

The controller gave adequate performance in the close vicinity of the tuning point. At the higher temperature set point the system became unstable. At the third set point the system became sluggish.

5.4 Simulation Response with PID Control



Graph 5.5 Simulation Response with Gain Scheduled PID Control



Gain Scheduled PID Control

A gain scheduled PID controller was obtained by modifying the gain of the above PID controller to maintain a constant open loop gain at different set points. The gain modifying relation was obtained from the bilinear differential equations of the system. The response is shown in graph 5.5.

The introduction of gain scheduling resulted in a significant improvement in the stability of the controlled system. Oscillation at the *50°C* set point was reduced, however a steady state was not achieved before the heat disturbance at *42.5* minutes occurred. The system response time at the low temperature set point was reduced.

The integral portion of the control caused large overshoot after set point changes.

Linear Optimal Regulator

A Linear Optimal Regulator (Elbert, 1984) was obtained for the process by solving the Riccati Equation, using a linearised form of equation 5.1 . The weighting matrices used were

$$P = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad R = [1] \qquad (5.3)$$

The system response is shown by graph 5.6.

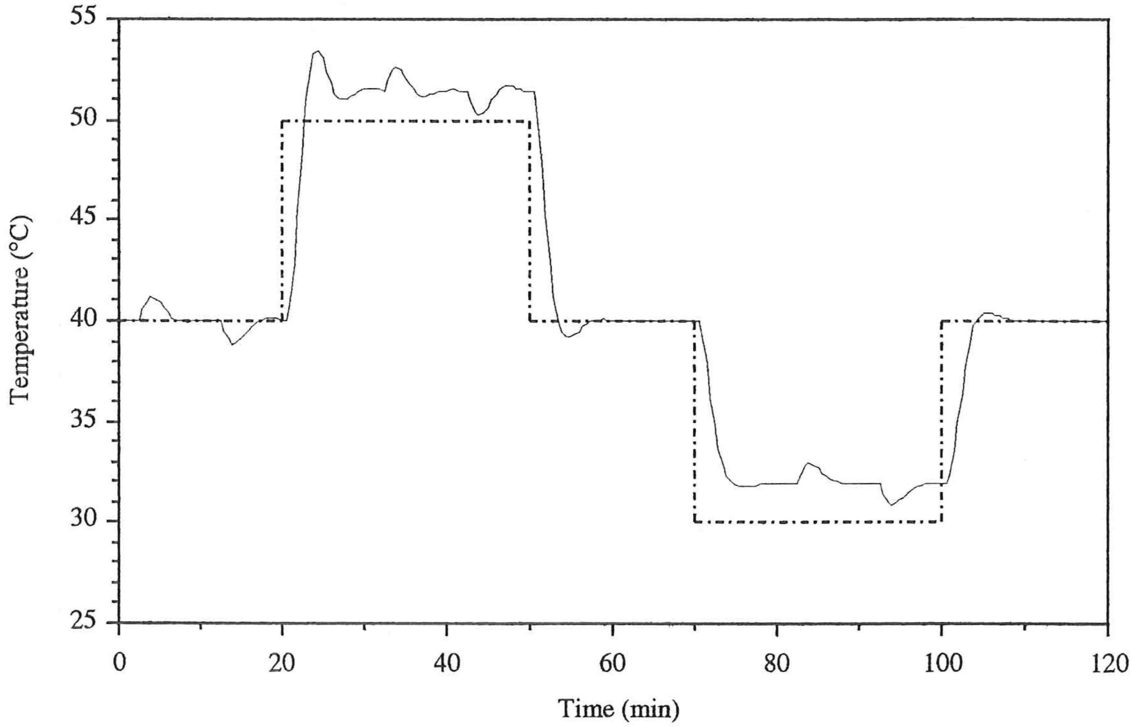
Good control was observed at the linearisation point. Slight overshoot, attributable to deadtime, occurred when returning to this point from the remote set points. At the remote set points the system remained stable with no long term oscillatory effects. However, an offset of approximately *2°C* was observed due to the inability of the linear model to accurately predict the steady states of the system.

Bilinear Controller

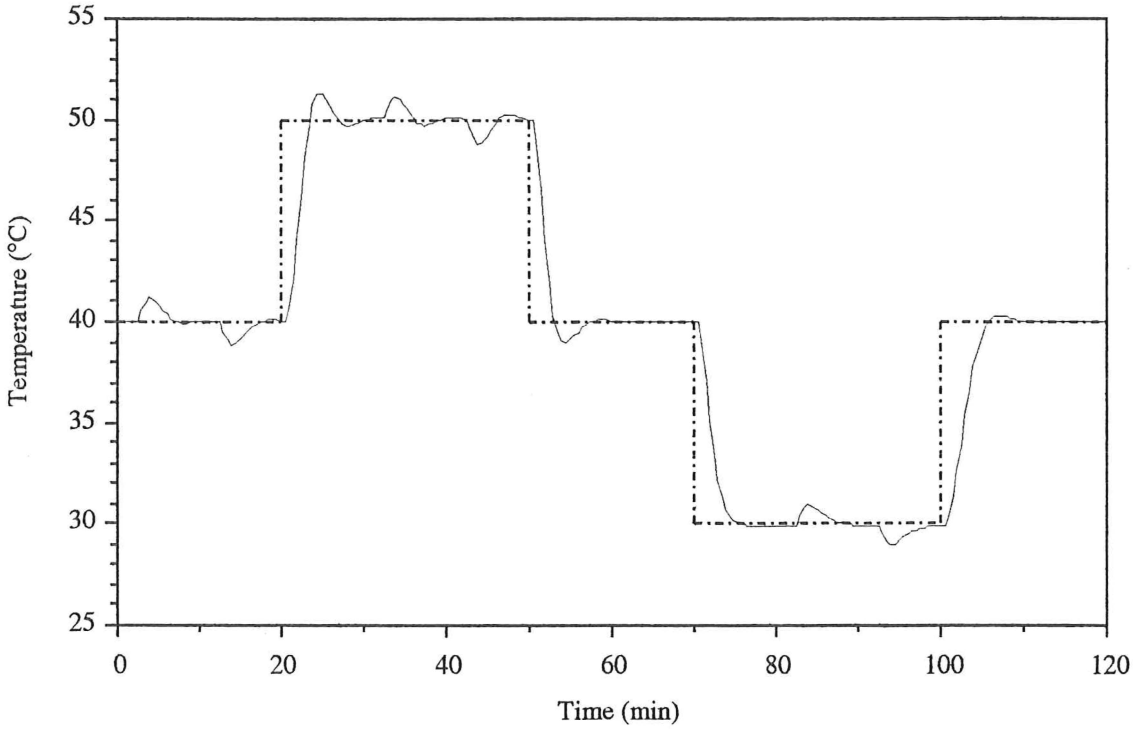
The bilinear controller design was applied to the simulation using the weighting matrices in equation 5.3. The response of the system with this control is shown in graph 5.7.

The bilinear controller gave precise response at all three set points. The nature of the response was similar over the entire range. However, some overshoot was observed when recovering from set point changes which is attributable to the effect of uncompensated deadtime.

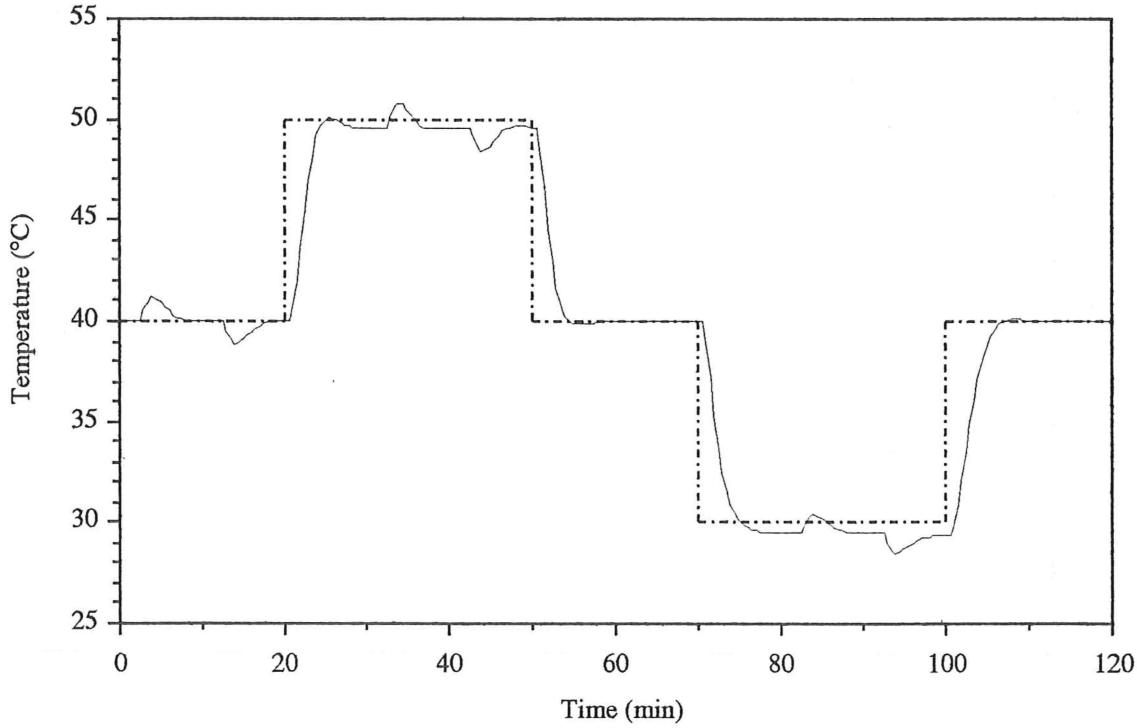
5.6 Simulation Response with Linear Optimal Control



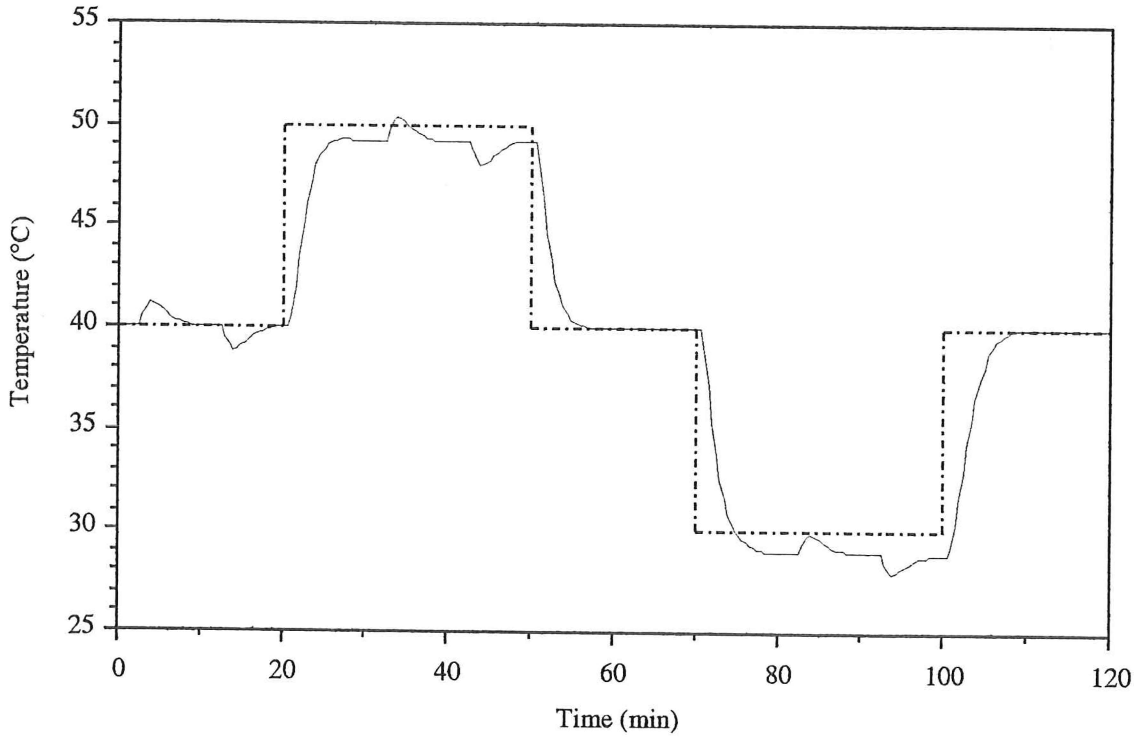
Graph 5.7 Simulation Response with Bilinear Optimal Control



Graph 5.8 Simulation Response with Bilinear Optimal Control and Deadtime Compensation of 1 Sampling Interval



Graph 5.9 Simulation Response with Bilinear Optimal Control and Deadtime Compensation of 2 Sampling Intervals



Deadtime Compensation

A form of deadtime compensation was tested in an effort to reduce the overshoot caused by deadtime present in the system. The bilinear model of the process was used to predict the state of the plant at points 1 or 2 sampling intervals into the future. The predicted state was then used to calculate the control to be applied.

Graphs 5.8 and 5.9 show the results of trials with deadtime prediction of *1* and *2* sampling intervals respectively. The overshoot after set point changes was significantly reduced with 1 sampling interval and eliminated with 2 sampling interval prediction. In general the use of prediction resulted in a more cautious controller. However, the effect of errors in the process model was amplified as predictions further into the future were used, resulting in increased offset from the set point.

Pilot Plant Trials

The pilot plant tank system as instrumented for these trials is shown in figure 5.2. Cascade control was used on the cold water flowrate to remove non-linearities associated with the control valve and to eliminate disturbances due to supply pressure changes.

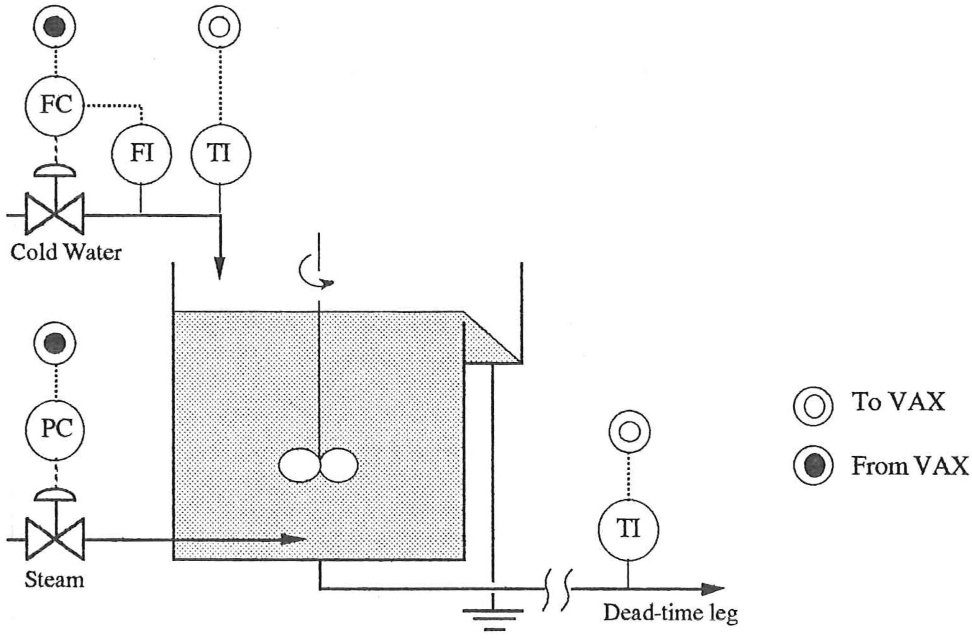
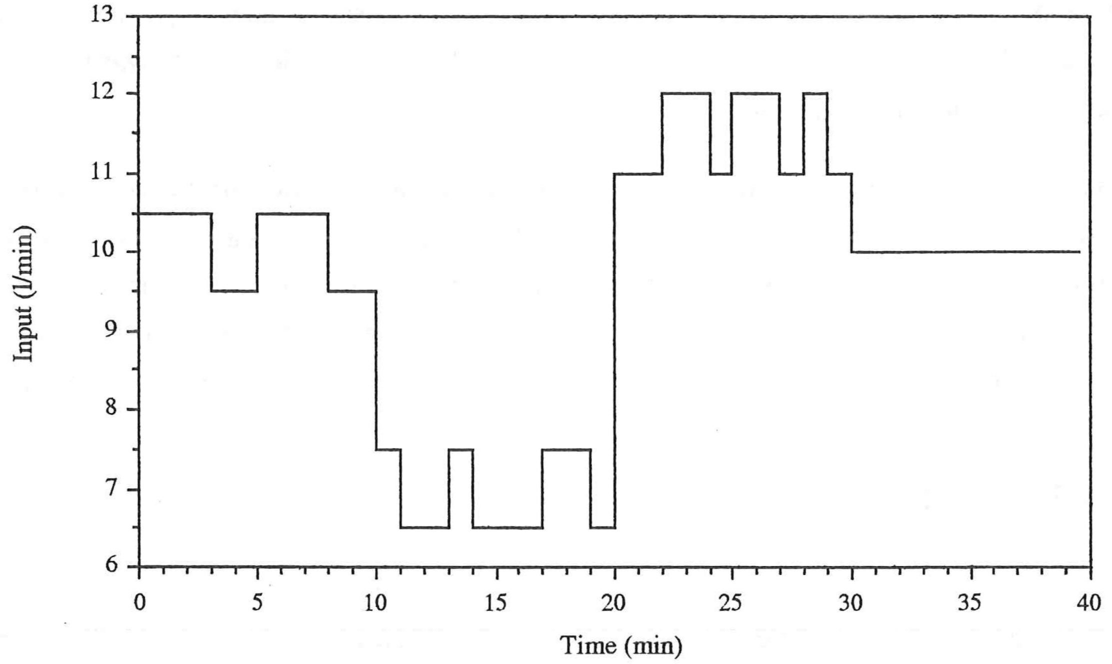


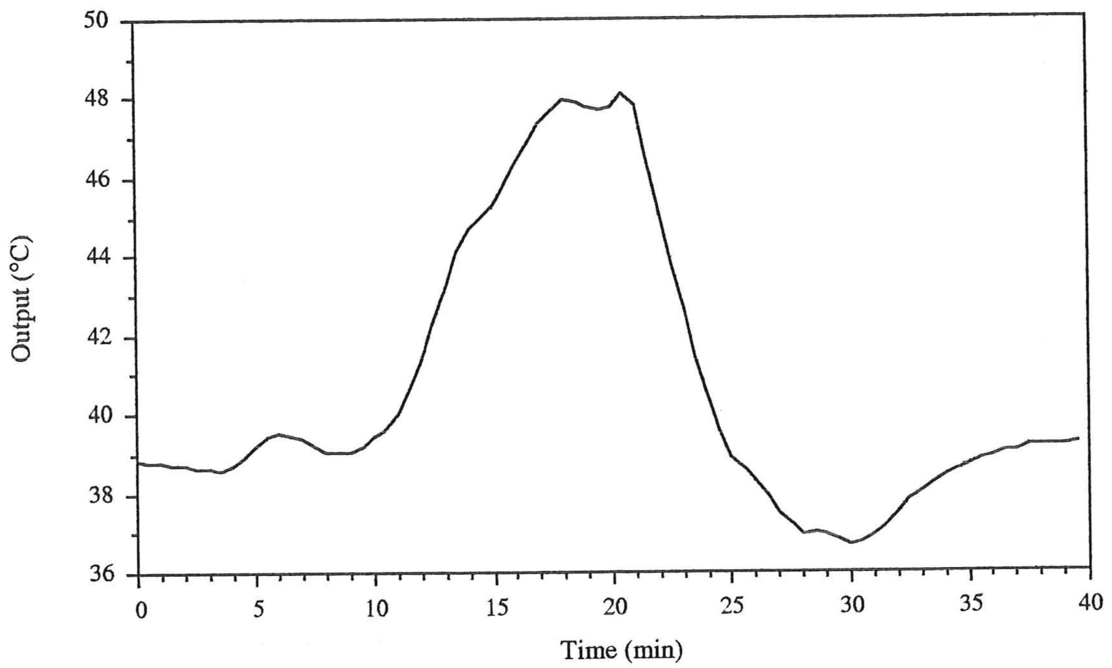
Figure 5.2 Experimental Tank System

The temperature of the inlet cold water was measured and the results passed to the computer program. The inlet temperature data was used to ensure no additional disturbances were entering the system and disrupting the basis for comparison of the various controller types.

Graph 5.10 Input Sequence for Pilot Plant Identification



Graph 5.11 Response of Pilot Plant to 5.10



The steam valve was positioned remotely by the operating program to provide identical disturbances for all control trials. The flowrate of hot water through the deadtime leg was held constant to provide a uniform delay on all output measurements.

Sampling was performed every 30 seconds.

Identification

A sequence of flowrate step changes and a superimposed pseudo random binary sequence was used to drive the plant, graph 5.10, the response of the system is shown in graph 5.11. A bilinear model of the system was identified from this data resulting in equation 5.4.

$$x(k+1) = \begin{bmatrix} 1.083 & -.1981 \\ 1 & 0 \end{bmatrix} x(k) + u \left\{ \begin{bmatrix} -.2691 \\ 0 \end{bmatrix} + \begin{bmatrix} -.00634 & -.00441 \\ 0 & 0 \end{bmatrix} x(k) \right\} \qquad (5.4)$$

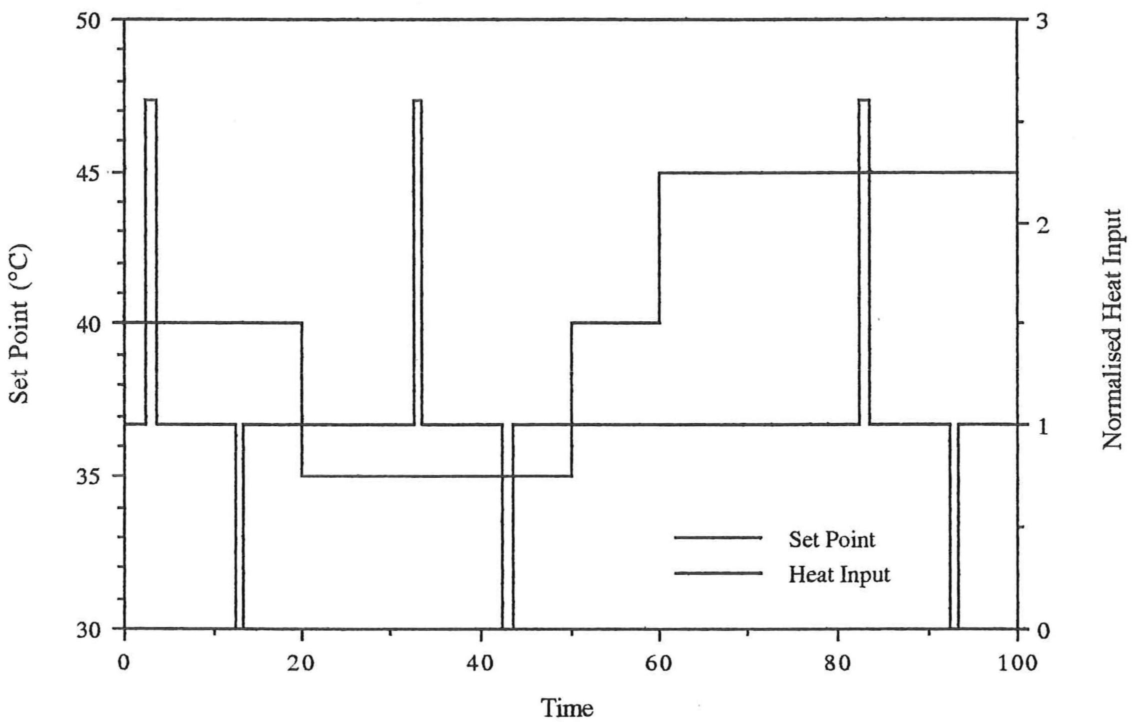
Control

An operating sequence similar to the one used for the simulation trials was devised and used to test the behaviour of the various controller types. The operating sequence is shown in graph 5.12.

Operating Sequence

The operating sequence for the pilot plant tank was similar to that used by the simulation.

Graph 5.12 Operating Sequence for Pilot Plant



However some changes were required due to physical constraints not present in the previous case. The set points used were *35°C, 40°C* and *45°C.* It was intended to use the same set points as the simulation but the PID controller proved to be dangerously unstable at a set point of *50°C.* The size and duration of the heat disturbances was also altered. The size of the increase was *160%* of the normal value and the decrease *100%*. The duration of both disturbances was one minute.

Manual control was used at the start of each run to bring the system to a steady state near the first set point. Hence the initial offset present in some of the response plots.

PID Control

A standard PID controller was designed using the ultimate method at an initial set point of 40°C. The controlled plant was then subjected to the operating sequence described above. The results are shown in graph 5.13.

$$K_U = -2.4 \text{ °C l/min},\ P_U = 3.8 \text{ min}$$

$$\therefore K_C = -1.44 \text{ °C l/min},\ T_I = 1.9 \text{ min},\ T_D = 0.45 \text{ min} \qquad (5.5)$$

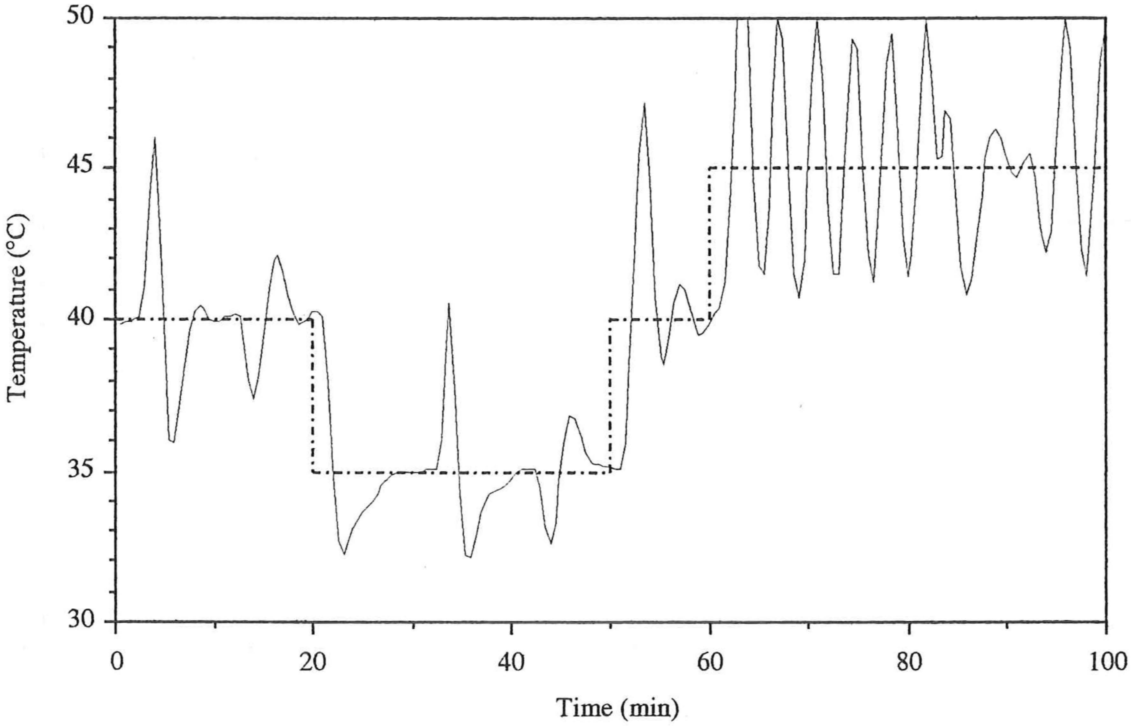
The system response gave similar behaviour to the digital simulation. Acceptable control was obtained at the tuning point but performance away from this set point was poor, particularly at the *45°C* set point.

Gain Scheduled PID Control

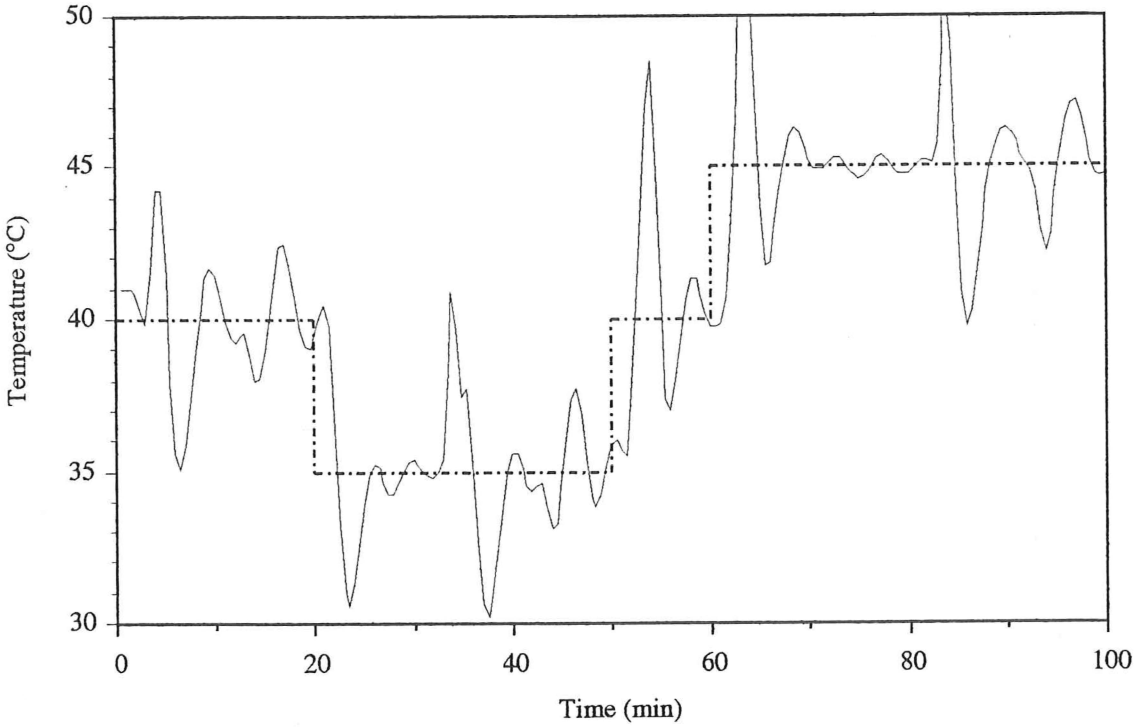
The gain scheduled PID controller was obtained using the bilinear process model (5.4) to modify the gain of the above PID controller when the set point was altered. Graph 5.14 shows the response of the system with this controller.

Many of the stability problems of the PID controller were reduced by the introduction of gain scheduling. However the system was still prone to large overshoot after set point changes or large disturbances. The controller also required a considerable length of time to damp out oscillations after each disturbance.

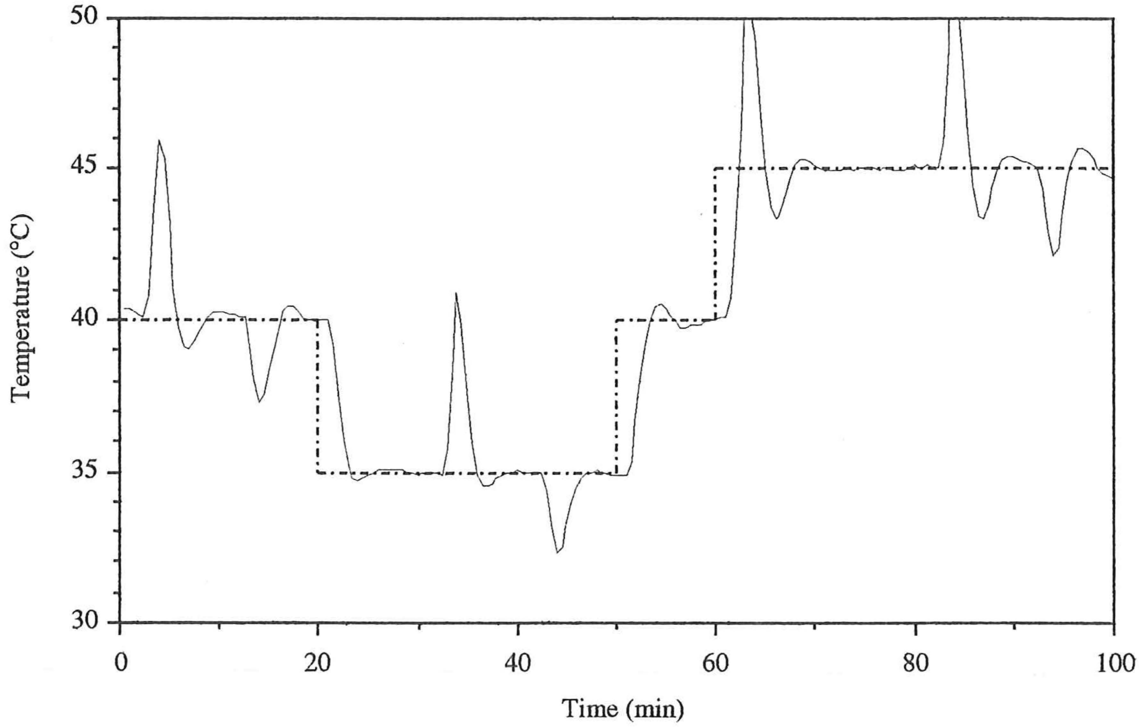
Graph 5.13 Tank Response with PID Control



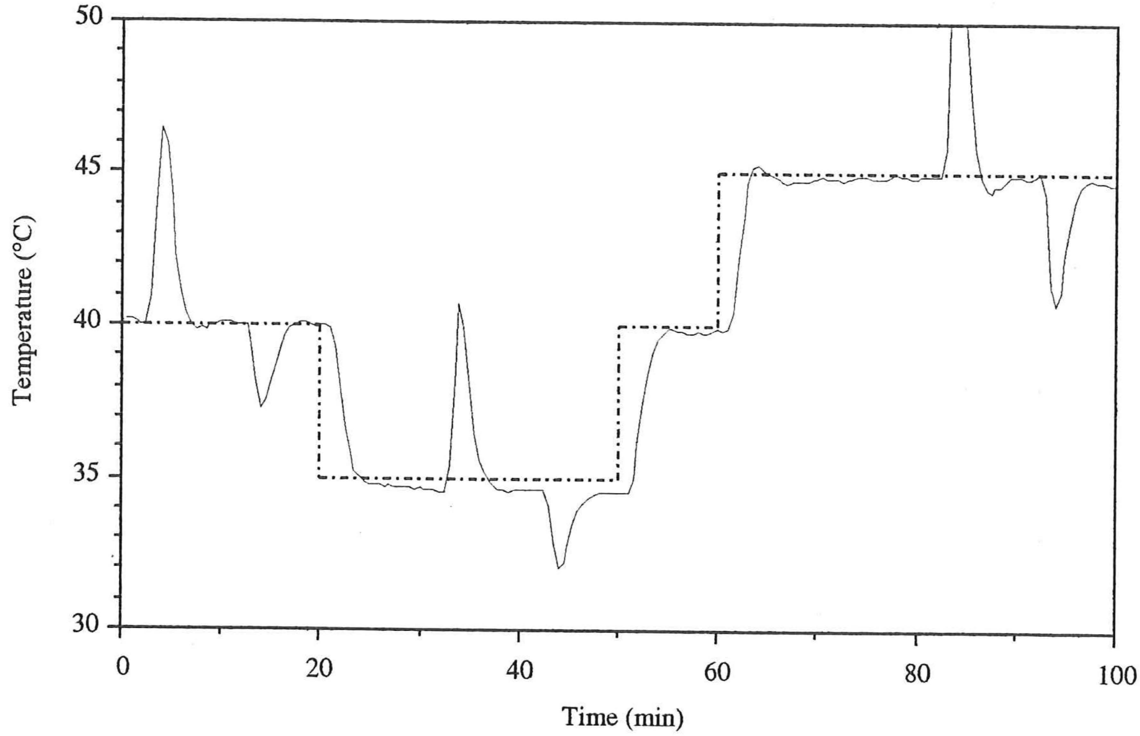
Graph 5.14 Tank Response with Gain Scheduled PID Control



Graph 5.15 Tank Response with Bilinear Optimal Control



Graph 5.16 Tank Response with Bilinear Optimal Control and Deadtime Compensation of 1 Sampling Interval



Bilinear Controller

The bilinear control design method was applied to the system using the process model (5.4), and the weighting matrices

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = [1] \tag{5.6}$$

The response of the system is shown in graph 5.15. The bilinear controller provided precise stable control at all set points. However, as with the simulation, the deadtime present in the process resulted in overshoot when the set point was changed or when recovering from large disturbances.

Deadtime Compensation

The deadtime compensation method, described earlier, was applied to the pilot scale tank. The controlled response is shown in graphs 5.16 & 5.17. The behaviour of the system was similar to the simulation trials. Increasing the prediction time reduced the amount of overshoot that occurred after set point changes and lead to a more cautious controller. However, extending the forecast time resulted in an increase in steady state offset. A mathematical treatment explains this phenomena.

If a first order discrete time linear system

$$\tilde{x}(k+1) = a\,\tilde{x}(k) + b\,u(k-l) + dc \tag{5.7}$$

is modelled giving

$$x(k+1) = a\,x(k) + b\,u(k-l) + dc + \delta \tag{5.8}$$

where δ represents the error in the identified model.

This model is used to predict the state of the system a distance l into the future, giving

$$x(k+l) = a\,x(k+l-1) + b\,u(k-1) + [dc + \delta] \tag{5.9}$$

The prediction error at this point may be defined

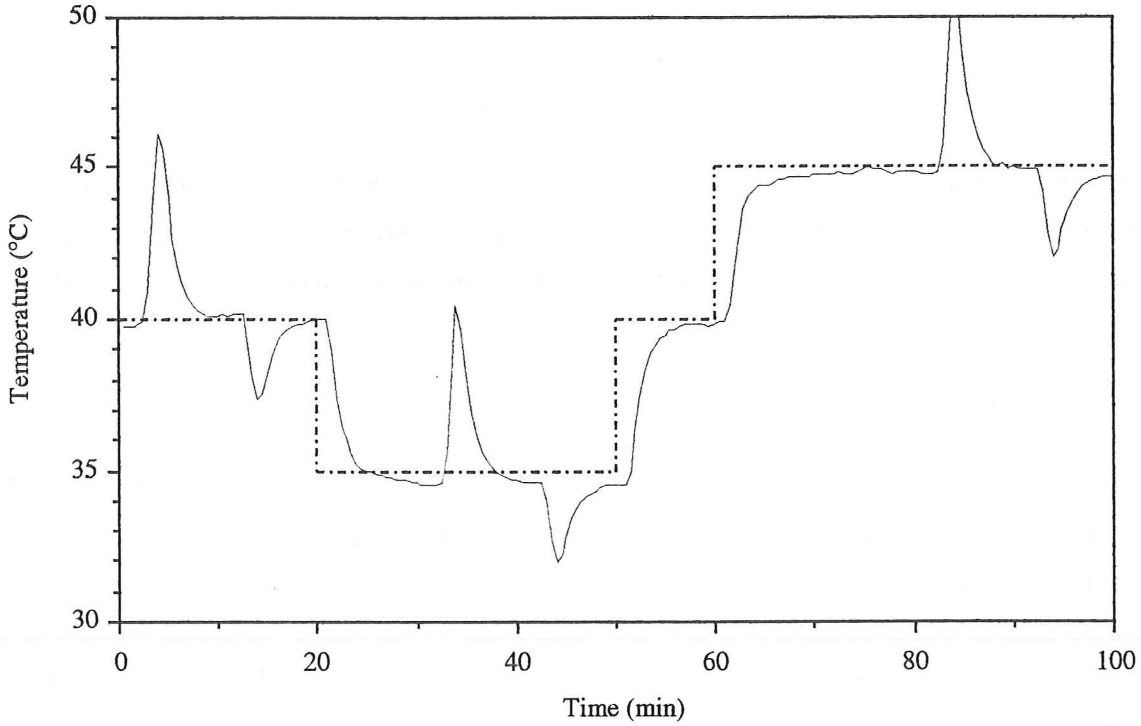
$$\begin{aligned} e(k+l) &= x(k+l) - \tilde{x}(k+l) \\ &= a\,x(k+l-1) + b\,u(k-1) + dc + \delta - a\,\tilde{x}(k) - b\,u(k-l) - dc \\ &= a\,e(k+l-1) + \delta \end{aligned}$$

$$e(k+l) = \sum_{i=1}^{l} a^{i-1}\,\delta \tag{5.10}$$

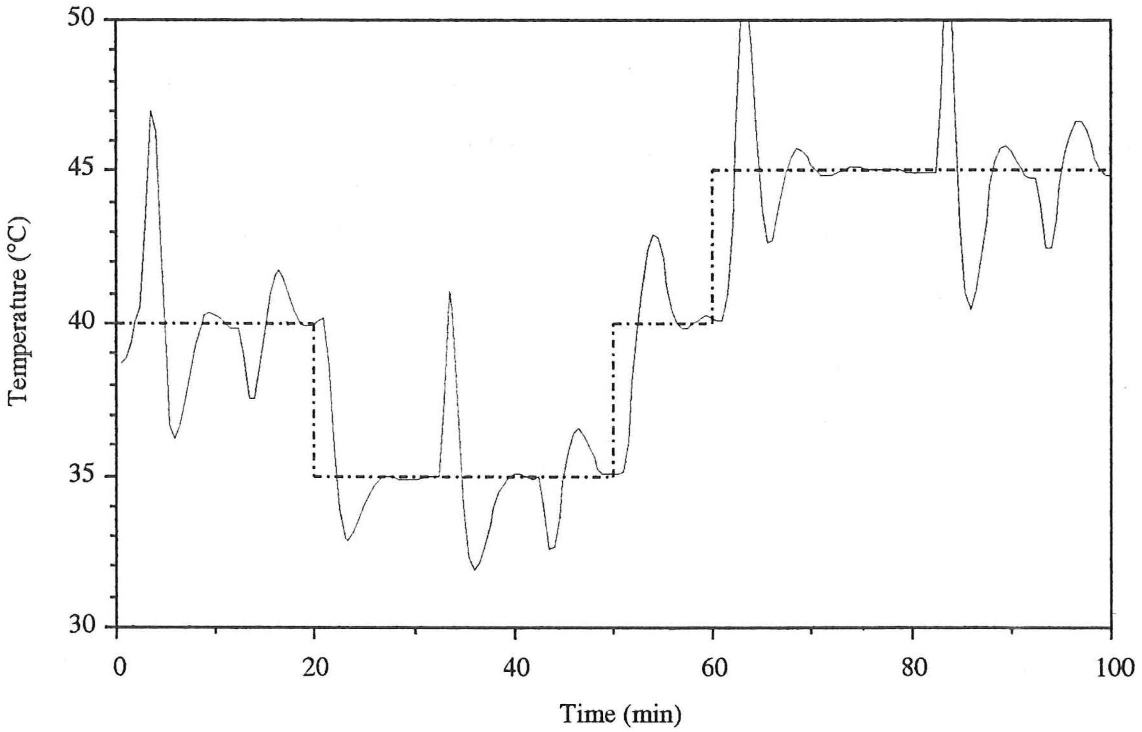
If a state variable feedback controller is designed using this prediction then the final steady state of the controller is given by the simultaneous equation set

$$\tilde{x}_{SS} = a\,\tilde{x}_{SS} + b\,u_{SS} + dc \tag{5.11}$$

Graph 5.17 Tank Response with Bilinear Optimal Control and Deadtime Compensation of 2 Sampling Intervals



Graph 5.18 Tank Response with Bilinear Optimal PID Control



$$u_{SS} = K\,x(+l)$$
$$x(+i) = a\,x(+i\text{-}1) + b\,u_{SS} + dc + \delta \quad \text{for } i = 1..l$$

Solving by successive substitution gives the final steady state as

$$x_{SS} = \frac{Kb}{1 - a - Kb} \sum_{i=1}^{l} a^{i-1}\,\delta + \frac{dc}{1 - a - Kb} \tag{5.12}$$

Table 5.1 gives the final steady states for a system with the parameters $a = .9$, $b = .1$, $dc = 0$ and $\delta = 1$ subject to a state variable feedback controller with gain $K = -1$.

Table 5.1 Predictive Controller Offset

l	0	1	2	3	5
x_{SS}	0	0.5	0.95	1.355	2.048

Similar results may be obtained for bilinear systems and higher order systems, although the equation complexity increases rapidly.

The controller offset due to the modelling error increases as the model is used to predict the system state further into the future.

Bilinear Controller with Integral Action

A discrete time bilinear controller incorporating integral action was designed using the augmented state method described in chapter 4. The modified system model was

$$x(k+1) = \begin{bmatrix} 1.083 & -.1981 & 0 \\ 1 & 0 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} x(k) + u \left\{ \begin{bmatrix} -.2691 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -.00634 & -.00441 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(k) \right\} \tag{5.13}$$

with the state vector is defined as $x(k) = \begin{bmatrix} T_{out}(k) \\ T_{out}(k\text{-}1) \\ \int_0^k T_{out}dt \end{bmatrix}$

and using the weighting matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } R = [1] \tag{5.14}$$

The response of the controlled system is given in graph 5.18. At all three set points the quality of control was better than the gain scheduled PID controller with less overshoot and faster damping of oscillations. The performance of this controller was not as precise as the normal bilinear controller as the presence of integral action tended to increase overshoot and oscillation effects when recovering from disturbances.

Integral action enables a control system to adjust for changes in the process. In addition, integral action has been used to cover up the non-linearity present in most plant items. The disadvantages of using integral action are the overshoot and oscillation effects that occur when the system recovers from set point changes or disturbances. In situations where such offset is unacceptable the use of integral based control should be replaced by the use of suitable non-linear controller designs, possibly including feed forward and/or adaptive mechanisms.

Conclusions

In both simulation and pilot plant trials traditional PID controllers performed poorly, being unable to maintain good control in the face of changes in the process gain. A significant improvement was obtained by using set point based gain scheduling to maintain a constant open loop gain based on a bilinear model of the process.

The bilinear controller provided precise control for the entire range of plant operation. Stability was maintained even in the presence of small amounts of uncompensated dead-time.

Bilinear models were successfully used to provide deadtime compensation in an attempt to eliminate overshoot. A more cautious controller resulted, but the effect of modeling errors was amplified, causing process offset in the controlled system.

Nomenclature

General

$x(k)$	State vector at time k.
u	Input

PID Controller

K_U	Controller Gain for constant amplitude oscillation
P_U	Period of constant amplitude oscillation
K_C	PID Controller Gain
T_I	Integral Time
T_D	Derivative Time

Optimal Control

P	Overall State Weighting Matrix
R	Input Weighting Matrix

Deadtime Compensation Analysis

$\tilde{x}(k)$	Actual System State at time k

l	discrete deadtime in sampling intervals
a, b & dc	System Parameters
δ	Error in *dc* term in prediction model
$e(k+i)$	Prediction Error at time $k+i$
$\tilde{x}_{SS}$, x_{SS}, u_{SS}	Steady State values of actual state, predicted state and inputs
K	Controller Gain

References

Elbert,T.F., *Estimation and Control of Systems*, Van Nostrand Reinhold, (1984)

Stephanopoulos,G., *Chemical Process Control : An Introduction to Theory and Practice*, Prentise-Hall, (1984)

CHAPTER 6

Bilinear Modeling of a Binary Distillation Column

Overview

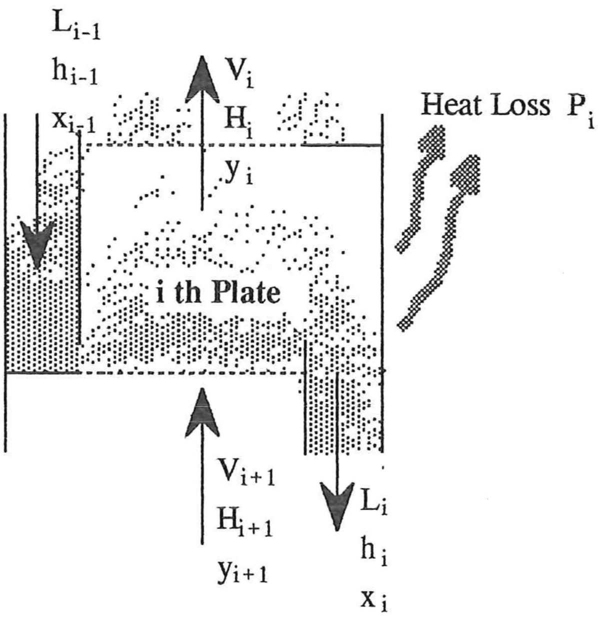
A rigourous treatment of the system dynamics is examined to devise suitable structures for the parametric modeling of distillation processes. The selection of control variables and determination of operating region is also examined.

Linear and bilinear models of a simulated binary distillation column were identified and their performance compared. The identified models were used in chapter 7 as the basis for controller designs for the simulated column.

Distillation Dynamics

Distillation and other separation processes are among the most complex unit operations regularly used in chemical plants. The dynamics of these processes is compounded by the non-linear nature of most equilibrium relationships and the number of operations occurring in each stage. A rigourous model of the distillation process is unsuitable for control purposes, even with modern computer hardware, due to this complexity. Simpler control models which characterise the response of the system are required.

Figure 6.1 General Plate



In the development of parametric models for chemical plant units it is important to examine the basic equations governing the systems. A general plate in a binary distillation column is shown in figure 6.1. If the vapour and liquid on the plate are assumed to be perfectly mixed, the vapour leaving is at equilibrium with the liquid on the plate, and of the holdup of liquid on the plate is time invarient then the equations governing this plate can be written as follows:

Mass Balance: $$\frac{dc_i}{dt} = L_{i-1} + V_{i+1} - V_i - L_i \qquad (6.1)$$

Component Balance: $$\frac{d}{dt}(c_i x_i) = L_{i-1} x_{i-1} + V_{i+1} y_{i+1} - V_i y_i - L_i x_i \qquad (6.2)$$

Energy Balance: $$\frac{d}{dt}(c_i h_i) = L_{i-1} h_{i-1} + V_{i+1} H_{i+1} - V_i H_i - L_i h_i - P_i \qquad (6.3)$$

Constant Volumetric Holdup: $$\dot{v}_i = 0 \qquad (6.4)$$

Enthalpy Relations: $$h_i = h(x_i) \quad H_i = H(x_i) \qquad (6.5)$$

Molar Holdup: $$c_i = v_i\, g(x_i) \qquad (6.6)$$

Equilibrium Relation: $$y_i = f(x_i) \qquad (6.7)$$

where
- L_i Liquid molar flow from plate *i*
- V_i Vapour molar flow from plate *i*
- h_i Liquid enthalpy on plate *i*
- H_i Vapour enthalpy on plate *i*
- x_i Liquid concentration on plate *i*
- y_i Vapour concentration on plate *i*
- P_i Heat Loss from plate *i*
- c_i Molar holdup on plate *i*
- v_i Volumetric holdup on plate *i*
- $g(x_i)$ Liquid Density Function

By using the multiplication rule for differentiation and equation 6.1, the component and enthalpy balances may be rewritten to obtain expressions for the derivatives of x_i and h_i respectively.

$$c_i \frac{dx_i}{dt} = L_{i-1}(x_{i-1} - x_i) + V_{i+1}(y_{i+1} - x_i) - V_i(y_i - x_i) \qquad (6.8)$$

$$c_i \frac{dh_i}{dt} = L_{i-1}(h_{i-1} - h_i) + V_{i+1}(H_{i+1} - h_i) - V_i(H_i - h_i) - P_i \qquad (6.9)$$

The enthalpy relations (6.5) give the enthalpy as an explicit function of the concentration. The differential of the enthalpy with respect to time can therefore be written:

$$\frac{dh_i}{dt} = \frac{dh_i}{dx_i}\frac{dx_i}{dt} = h_x'(x_i)\,\dot{x}_i \qquad (6.10)$$

Using this relationship and equation 6.9 it is possible to convert the dynamic enthalpy balance into an algebraic equation.

$$-L_{i-1}(h_x'(x_i)d_{i-1,i} - \delta_{i-1,i}) - V_{i+1}(h_x'(x_i)D_{i+1,i} - \Delta_{i+1,i}) + V_i(h_x'(x_i)D_{i,i} - \Delta_{i,i}) = P_i \qquad (6.11)$$

where the following symbols have been introduced:

$$d_{i-1,i} = x_{i-1} - x_i \quad \delta_{i-1,i} = h_{i-1} - h_i$$
$$D_{i+1,i} = y_{i+1} - x_i \quad \Delta_{i+1,i} = H_{i+1} - h_i$$
$$D_{i,i} = y_i - x_i \quad \Delta_{i,i} = H_i - h_i$$

In the same way the constant volume holdup equation may be used to reduce the dynamic molar holdup equation to an algebraic relationship, equations 6.12 and 6.13.

$$\frac{dc_i}{dt} = v_i \frac{dg(x_i)}{dx_i}\frac{dx_i}{dt} = v_i\, g'(x_i)\,\dot{x}_i \qquad (6.12)$$

$$L_i + L_{i-1}\left(\frac{g'(x_i)}{g(x_i)} d_{i-1,i} - \right) + V_{i+1}\left(\frac{g'(x_i)}{g(x_i)} D_{i+1,i} - 1\right) - V_i\left(\frac{g'(x_i)}{g(x_i)} D_{i,i} - 1\right) = 0 \qquad (6.13)$$

The dynamic behaviour of the plate has been reduced to one differential equation (6.8) and two algebraic equations (6.11 & 6.13), all of which are non-linear.

A distillation column contains a number of specialised plates in addition to the standard plate in the above development. These include the reboiler, feed plate and the condenser-top plate group. The development of the equations for these special plates is not given here. The full derivations are given by España (1977) it suffices to say that similar equation structures result.

The dynamic behaviour of a binary distillation column, with *B* plates, may be described by a set of *2B+3* algebraic and *B+1* differential equations, all non-linear and interacting. The differential equation set may be expressed in matrix form as

$$\dot{x} = \begin{bmatrix} -\frac{V_1}{c_C} & \frac{f_1 V_1}{c_C} & 0 & 0 & 0 \\ \frac{L_0}{c_1} & -\frac{L_0+V_2+V_1(f_1-1)}{c_1} & \frac{V_2 f_2}{c_1} & 0 & 0 \\ 0 & \frac{L_1}{c_2} & -\frac{L_1+V_3+V_2(f_2-1)}{c_2} & \frac{V_3 f_3}{c_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{L_{jF-1}}{c_{jF}} & -\frac{L_{jF-1}+L_F+V_{jF+1}+V_{jF}(f_{jF}-1)}{c_{jF}} & \frac{V_{jF+1}f_{jF+1}}{c_{jF}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{L_{B-1}}{c_B} & -\frac{L_{B-1}+V_B(f_B-1)}{c_B} \end{bmatrix} x$$

$+$

$$\begin{bmatrix} \frac{V_1 y_1^0}{c_C} \\ \frac{V_2 y_2^0 - V_1 y_1^0}{c_1} \\ \frac{V_3 y_3^0 - V_2 y_2^0}{c_2} \\ \vdots \\ \frac{V_{jF+1} y_{jF+1}^0 - V_{jF} y_{jF}^0 + L_F x_F}{c_{jF}} \\ \vdots \\ \frac{-V_B y_B^0}{c_B} \end{bmatrix} \quad (6.14)$$

where the equilibrium has been linearised on each plate, to give

$$y_i = y_i^0 + f_i x_i$$

The state coefficient matrix is tridiagonal. The coefficients of both matrices are determined by the current inputs and the solutions of the algebraic equations obtained from the plate energy and mass balances. This system is bilinear in the feed and reflux flowrates (L_F & L_O), and boilup rate (V_B), and linear in the product of feed concentration and flowrate ($L_F x_F$).

It is reasonable to consider a discrete time, parametric model with a similar structure to the above matrix differential equation.

Simulation Structure

A simulation of a *9"* distillation column with *8* sieve plates, developed by Janssen (1986) was used in this work. The simulation used LSODE to integrate the differential and algebraic equation set for the column, including the fluid mechanical relations for the tray holdups. Feed for the simulation was a mixture of methanol and water.

The simulation was operated as a batch job on a VAX minicomputer. The feed, reflux and steam condensate flowrates and the feed concentration were read from a data file. Output was in the form of a data file with the following columns.

Time	*Feed Flow*	*Reflux Flow*	*Steam Flow*	*Feed Conc.*	*Tops Conc.*	*Bottoms Conc.*	*Tops Flow*	*Bottoms Flow*
(min)	*(l / min)*			*(mole fraction MeOH)*			*(mol / min)*	

FORTRAN source for a modified version of the simulation which incorporates control is included on the Appendix disk.

Identification Program

The Macintosh identification program was developed to enable the identification of mixed linear and bilinear models. The program was written in Turbo Pascal and makes full use of Macintosh toolbox calls to provide an easy to use interface.

Flexibility of operation was obtained by the use of a number of specialised file types.

1) Data files, which contain the process input and output data in columns.

2) Configuration files, which define the inputs and outputs of the system in terms of the columns of the data file. These files also define the structure of the process model to be identified.

3) Model files, which contain the identified model parameters, in addition to a copy of the Configuration file used for the identification.

All three file types can be exported as TEXT files for use by other applications, such as word processors. The configuration report in figure 6.2 was generated in this manner. Data input in tab delimited TEXT form was implemented.

A choice of identification algorithm was given based on the results from chapter 3, with three identification methods available to the user (RLS, RELS and REELS).

A copy of the application along with more detailed information in the form of a HyperCard 2.0 stack, is included in the disk appendix.

Figure 6.2 Sample Model Configuration

```
Configuration Report

Time : Column 1
State : Column 6
Inputs : Columns 2 3 4 5
Constant term included

Length of Model Vector 25

Detailed Configuration

State
  Order : 3
  Bilinear Links
    Diagonal with Input 1
    Diagonal with Input 2
    Diagonal with Input 3

Inputs
  No : 1   Order : 3   DeadTime : 0
  No : 2   Order : 3   DeadTime : 0
  No : 3   Order : 3   DeadTime : 0
  No : 4   Order : 3   DeadTime : 0
```

Selection of Control Variables

In the control of binary distillation processes there are four basic control objectives and six possible manipulated variables. These are listed in table 6.1.

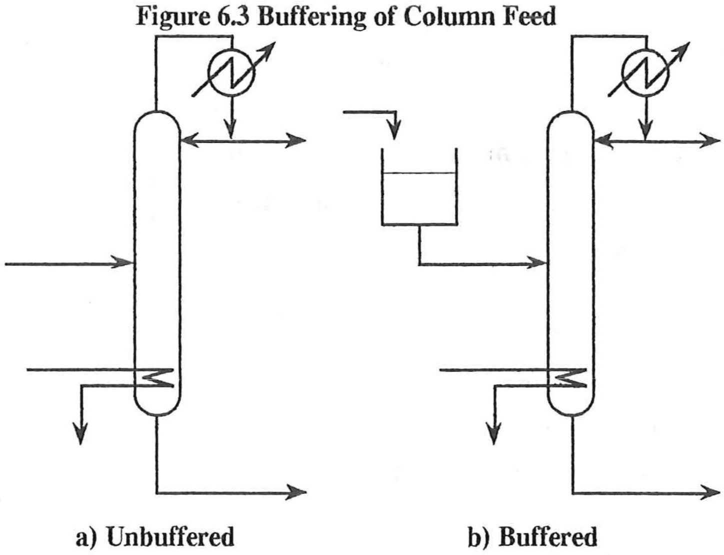
The two product flowrates were used to control the reflux accumulator and reboiler liquid levels, as this gave two relatively quick control loops which were ignored when considering the composition control. The preceding dynamic analysis of column behaviour incorporated this strategy.

Table 6.1 Inputs & Control Objectives for a Binary Distillation Column

Inputs	*Control Objectives*
Feed Flowrate	*Distillate Conc.*
Feed Conc.	*Bottoms Conc.*
Tops Flow	*Reflux Accumulator Level*
Reflux Flow	*Reboiler Level*
Bottoms Flow	
Heat Input	

In many cases the configuration of the plant and the position of the unit operation within the plant determines which of the remaining variables are available for control use. Generally the feed for the process is the output stream of another unit such as a reactor and may not be easily manipulated as part of the distillation control, although it may be possible to buffer some of the concentration and flow disturbances by using a feed tank, figure 6.3.

Figure 6.3 Buffering of Column Feed



If the feed flow and/or concentration is subject to disturbances and these are measured, then they may be incorporated in the parametric model structure for later use in devising a feed-forward strategy.

The product concentrations are bilinearly dependent upon the reflux flow, heat input and feed flowrate and linearly dependent upon the feed rate of the key component. The structure of parametric models for the plant should reflect this. A possible model configuration for the tops concentration of the simulation, which incorporates this structure is shown in figure 6.2.

If the feed variables are not measured or do not vary sufficiently for successful identification then two inputs remain with which to achieve two control objectives. Having reduced the control problem to two interacting loops a suitable parametric model structure using these variables can be devised. The states are the tops and bottoms product concentrations, the inputs are the reflux flowrate and the heat input to the reboiler. A model with fewer parameters results.

Input Sequence Design

The design of an input sequence for a multivariable system requires careful consideration of the modeling objectives. A good fit for both plant steady state and dynamic behaviour is required over the range of possible operating conditions.

As a basis for the input sequence design, the combination of preset values and superimposed pseudo random binary sequence discussed in chapter 3 was used. Without careful selection of the base points this method alone cannot guarantee a useful result from whatever identification method is used.

To determine the required input values to obtain a desired set point, a control system based on a parametric model must solve the matrix equation:

$$u = \Delta_R(x_S)^{-1}\left[\left(I - \alpha_R\right) x_S - DC_R\right] \qquad (6.15)$$

The chosen operating sequence must produce a wide range of output values for a robust solution, particularly if the output signal is subject to significant amounts of noise.

The predicted shape of the steady state response surface was another consideration. The steady states of a two-input linear system may be represented by an inclined plane in three dimensional space (equation 6.16). The position and inclination of such a plane can be defined with knowledge of but three points on its surface. A bilinear system, on the other hand, will give a curved surface which requires that a greater number of points be found, even though the general form is known. The general bilinear steady state equation (6.17) contains five unknowns, therefore at least five points should be used to define the surface. This may be extended to cover systems with more than two inputs. For the n input case the linear system has $n+1$ unknowns and the bilinear system has $2n+1$.

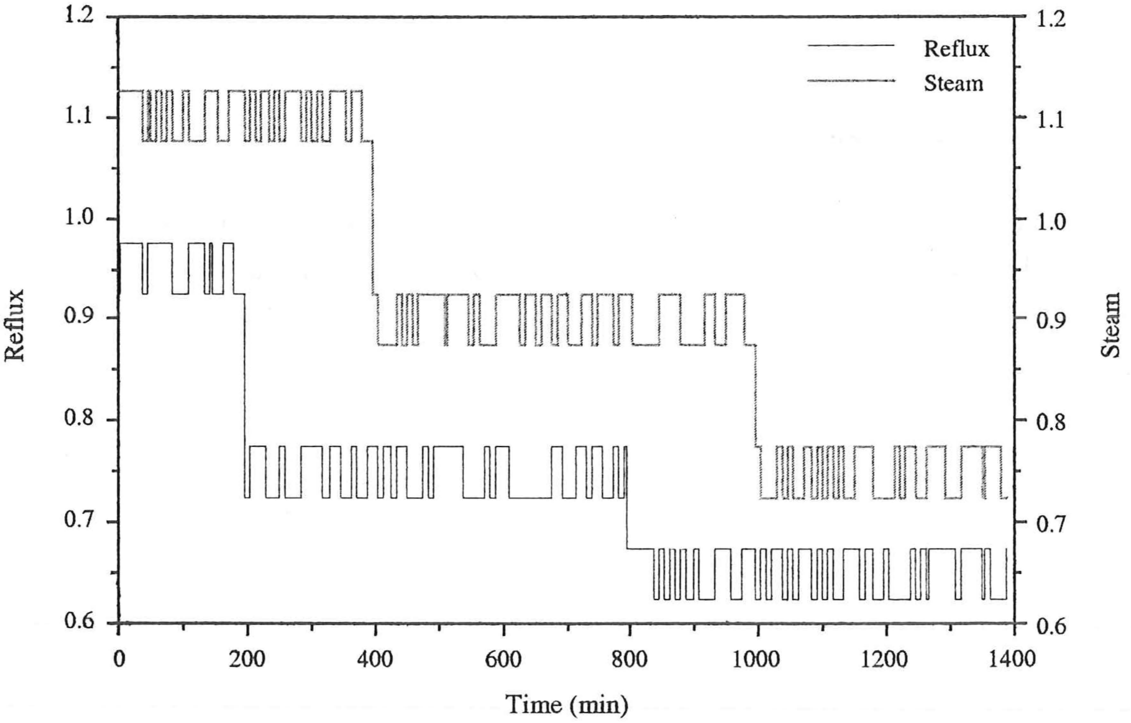
$$z = ax + by + c \qquad (6.16)$$

$$z = \frac{ax + by + c}{1 - dx - ey} \qquad (6.17)$$

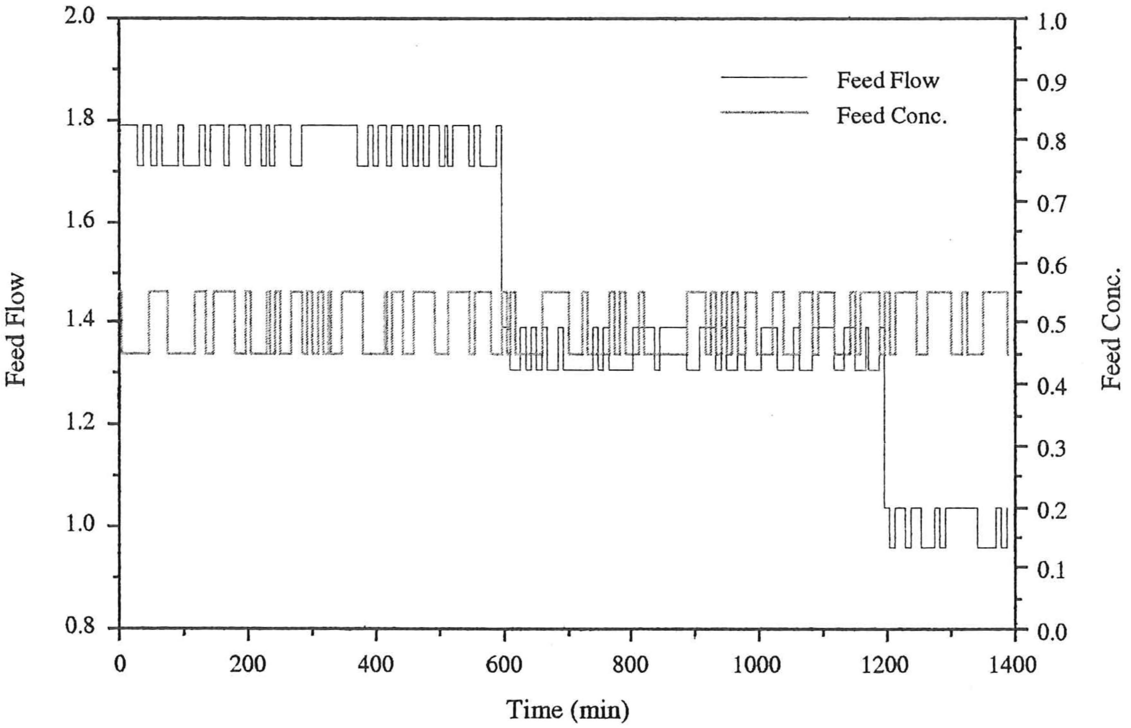
The location of these points plays an important part in the performance of the identification procedure. The operating points for the identification of a linear system should not lie along a line, but be arranged to form a triangle. A different approach is desired for bilinear systems, to obtain a good representation of the curved portion of the surface. The input points should be arranged about a central point with lines to the outer points having a wide angular separation.

The operating points for the identification of existing plant should be selected with regard to plant operating data to ensure a safe and realistic range of operation.

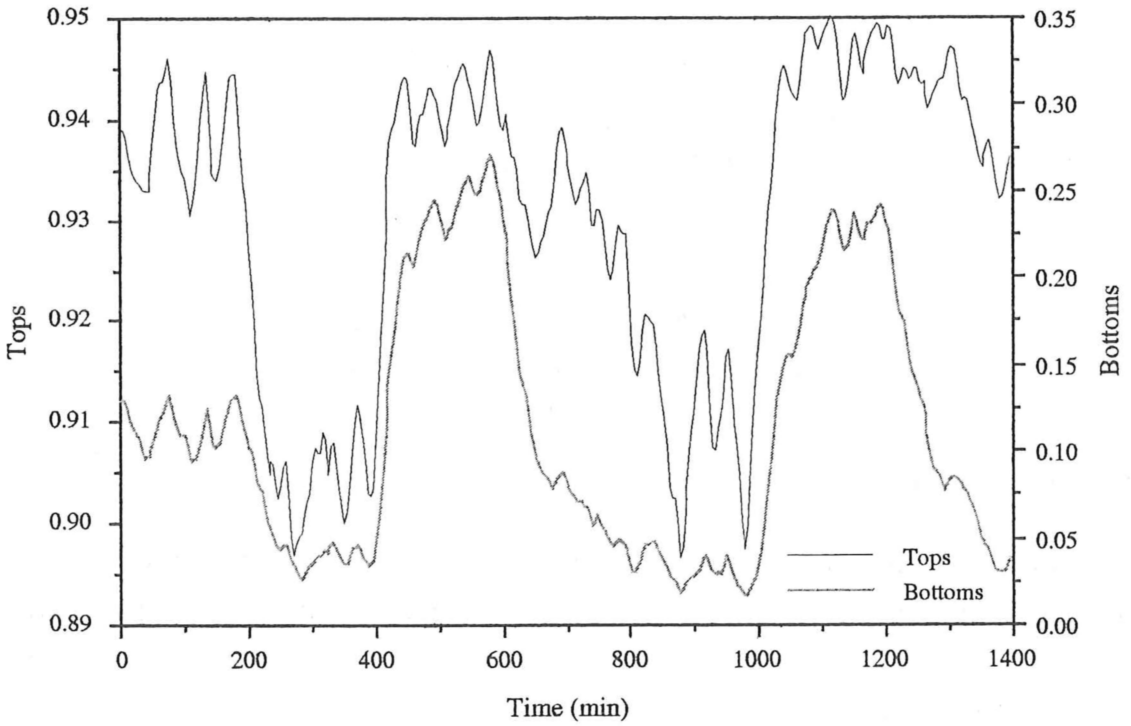
Graph 6.1(a) Reflux and Steam Flowrates



Graph 6.1(b) Feed Flowrate and Concentration



Graph 6.2 Simulation Response to Inputs in Graph 6.1



Four Input Models

Janssen (1987) used a series of seven points obtained from downward step sequences in the feed, reflux and steam flowrates with superimposed PRBS to identify a three input model for the binary distillation system. This sequence of points was used as the basis for the identification of a four input model of the simulated distillation column[1] using the model structure given in figure 6.2. In addition to these points and their superimposed PRBS, the feed concentration was forced using a PRBS of amplitude *0.1* centred about $x_F = 0.5$. The input sequence and system response are shown in graphs 6.1 and 6.2.

Both linear and diagonal bilinear models were identified from this data[2].

Two variances were calculated for each model, the "full run" variance and the "one step prediction" variance. The full run variance is obtained by using the original input sequence to drive the model and then comparing the response of the model to that of the original system. The one step prediction variance is obtained by using the model to predict the state of the system at the next sampling point based on the measured states and inputs of the system, and comparing this with the measured value. The variances are shown in table 6.2.

---

[1]Details and Source code for the Binary Distillation Simulation are included on the enclosed disk.

[2]Data included as "DC with 4 Inputs" on the enclosed disk, model structure files "L Bots 4In.C", "L Tops 4In.C", "B Bots 4In.C" and "B Tops 4In.C".

The use of bilinear models effected reductions of *85%* and *53%* in the full run variances for the bottoms and tops, respectively.

Table 6.2 Model Variances for 4 Input Models ($*10^{-7}$)

Model Type		*Bottoms*	*Tops*
Linear	*Full Run*	*861.6*	*91.0*
	1 Step	*11.0*	*4.0*
Bilinear	*Full Run*	*129.8*	*42.7*
	1 Step	*9.7*	*3.2*
Improvement	*Full Run*	*85 %*	*53 %*
	1 Step	*12 %*	*20 %*

Two Input Models

When the feed flowrate and concentration are not manipulated and do not vary sufficiently of their own accord to enable modeling, the distillation column model becomes a two input - two output system. The states or outputs are the tops and bottoms concentrations, the inputs are the reflux flowrate and heat input to the reboiler.

A contour plot of the simulation steady states was obtained by running the simulation at a range of reflux and steam flowrates. The feed flowrate and concentration were held constant at *1.35l/min* and *0.5* respectively, the desired operating region for the plant. The steady state data was plotted using a contouring package on a VAX 11/730 minicomputer giving graph 6.3. This approach is equivalent to the use of historical data from existing plant.

The distance between the contour lines increases towards the bottom right of the graph, indicating non-linear behaviour. The top left corner represents a region of rapid change in the composition profile in the column as the bottoms concentration approaches zero. The bottom plates of the column provide very little separation while operating in this region.

The operating points for the simulation identification were selected to fall within the region not affected by the equilibrium non-linearity, yet still provide a wide range of output values. The selected points are given in table 6.3. The input sequence was generated by superimposing a PRBS of suitable amplitude on these points. The final input sequences are shown in graph 6.4.

The input sequence in graph 6.4 was used to drive the distillation simulation giving the response shown in graph 6.5[1]. Linear and bilinear models were then identified for the tops and bottoms concentrations from this data using recursive least squares. The resulting discrete time models are shown in figures 6.4 and 6.5.

[1]Data included as "DC with 2 Inputs" on the enclosed disk.

Graph 6.3 Contour plot of Steady States for the Distillation Column Simulation

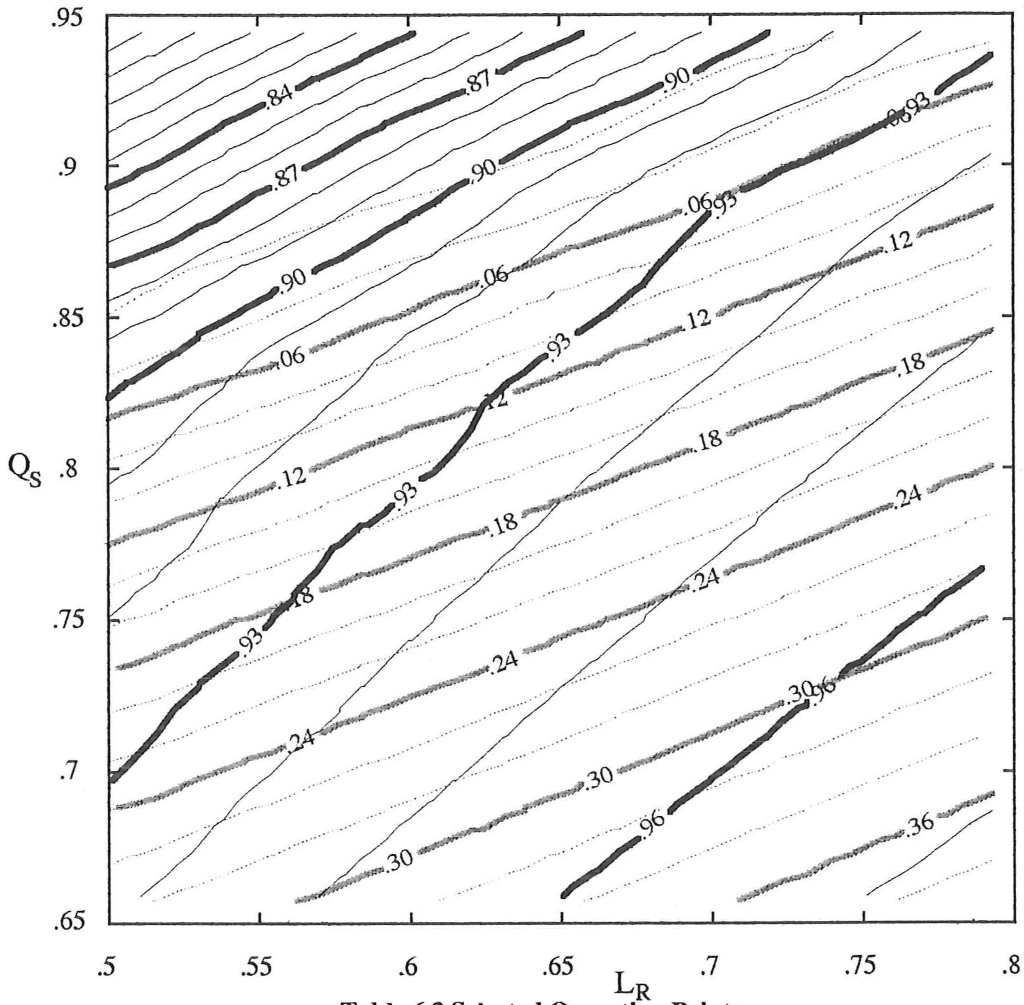


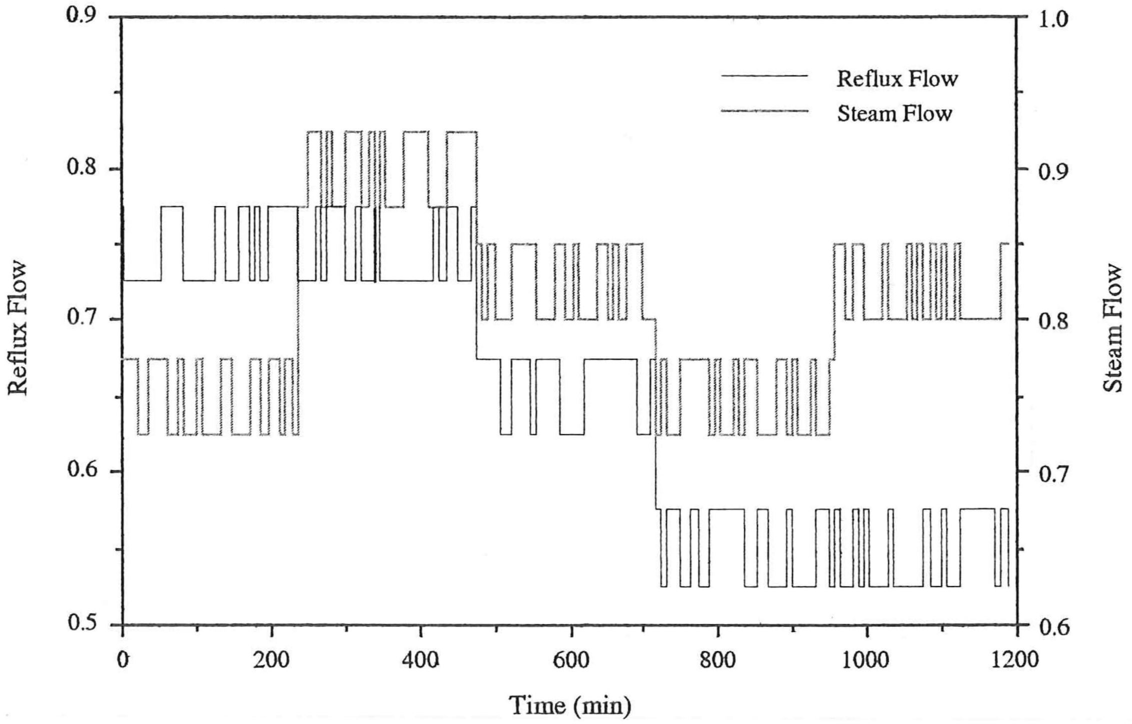
Table 6.3 Selected Operating Points

No	L_R	Q_S
1	*.75*	*.75*
2	*.75*	*.9*
3	*.65*	*.8*
4	*.55*	*.75*
5	*.55*	*.825*

Contour plots of the error between the model and the simulation were prepared to assess the steady state performance of the models. Graphs 6.6 and 6.7 show the sum of the absolute errors in the tops and bottoms concentrations for the two models.

Both models gave poor steady state performance towards the top left of the graph. The operating sequence did not drive the simulation in this region due to the equilibrium non-linearity. The bilinear model remained accurate over a larger portion of the operating region, particularly towards the bottom right corner.

Graph 6.4 Input Sequence for Simulation Identification



Graph 6.5 Simulation Response to Inputs in Graph 6.4

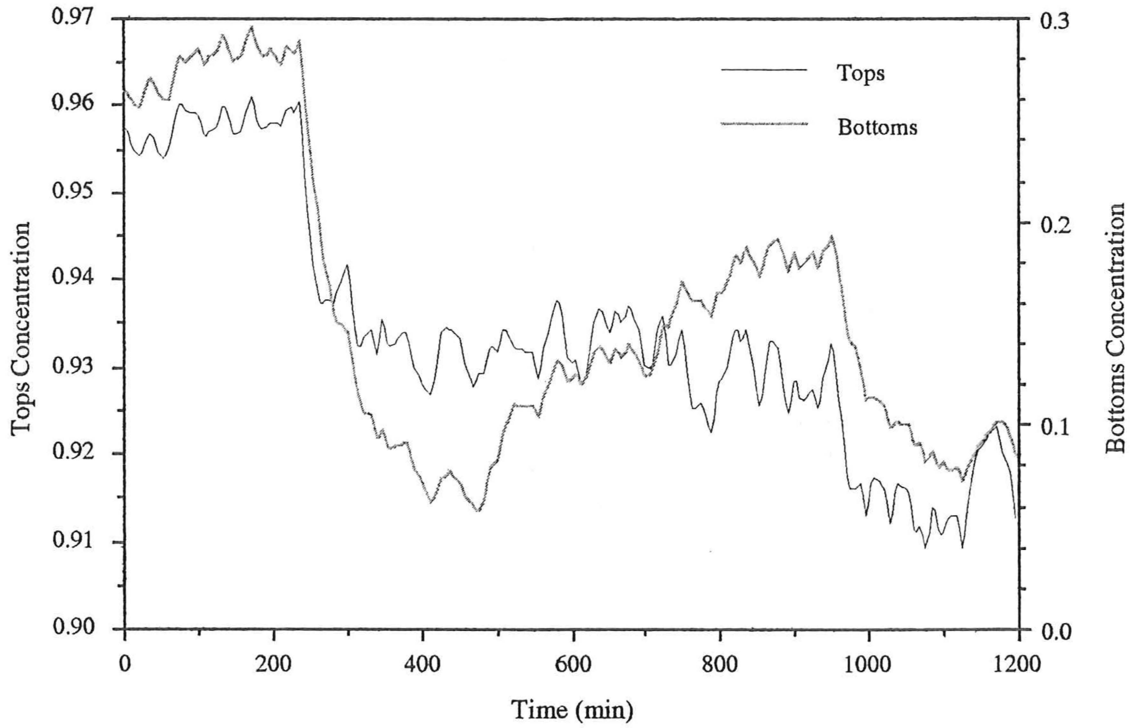


Figure 6.4 Linear models of Bottoms and Tops Composition

Discrete Time Model Report	Discrete Time Model Report
1 State in Column 7	1 State in Column 6
2 Inputs in Columns 3,4	2 Inputs in Columns 3,4
State	State
Order = 3	Order = 3
5.4674e-1 3.3881e-1 7.8814e-3	5.8174e-1 1.9314e-1 -3.5408e-2
Input 1	Input 1
Order 2 Deadtime 0	Order 2 Deadtime 0
3.5689e-2 1.8784e-2	2.1485e-2 1.8769e-2
Input 2	Input 2
Order 2 Deadtime 0	Order 2 Deadtime 0
-1.0271e-1 -4.5704e-2	-2.2699e-2 -1.9988e-2
Constant Term = 1.0071e-1	Constant Term = 2.5161e-1

Graph 6.6 Steady State Errors for Linear Model

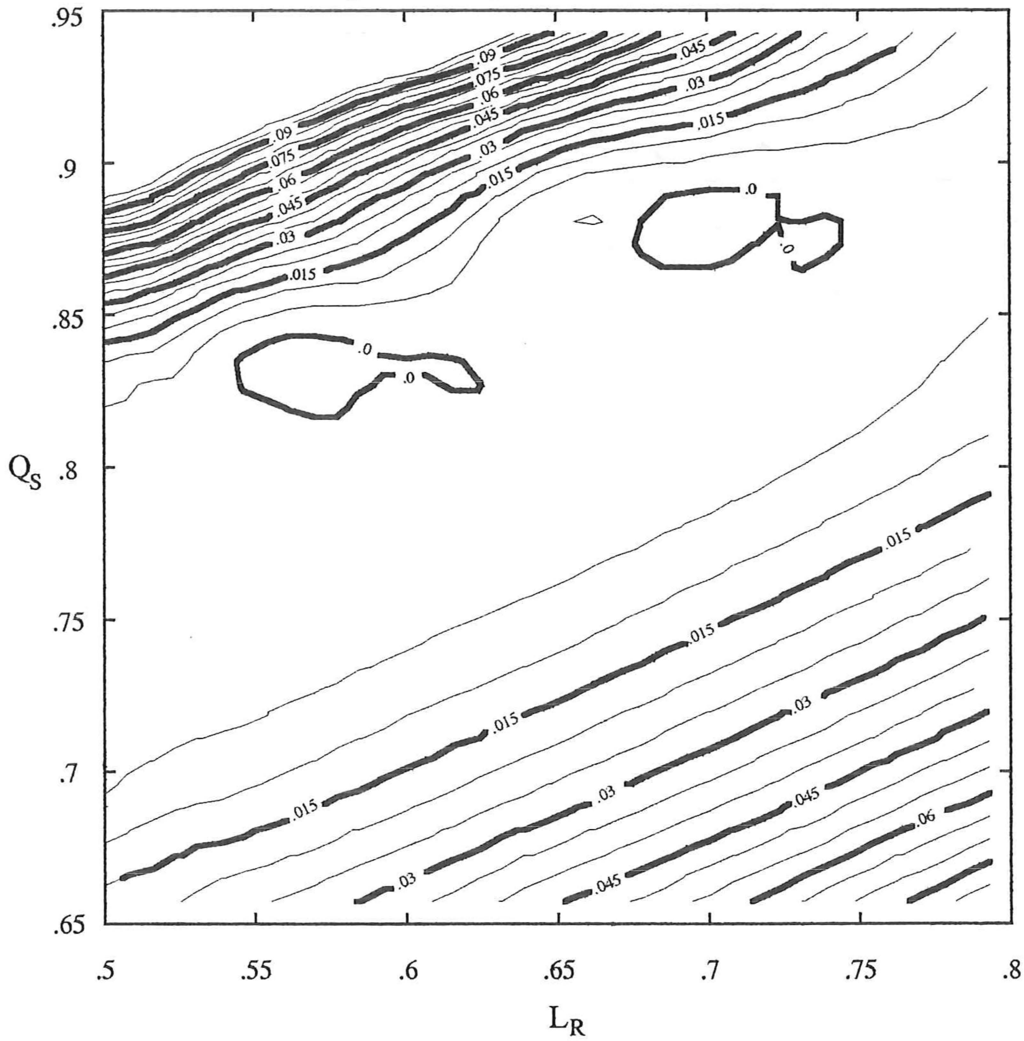
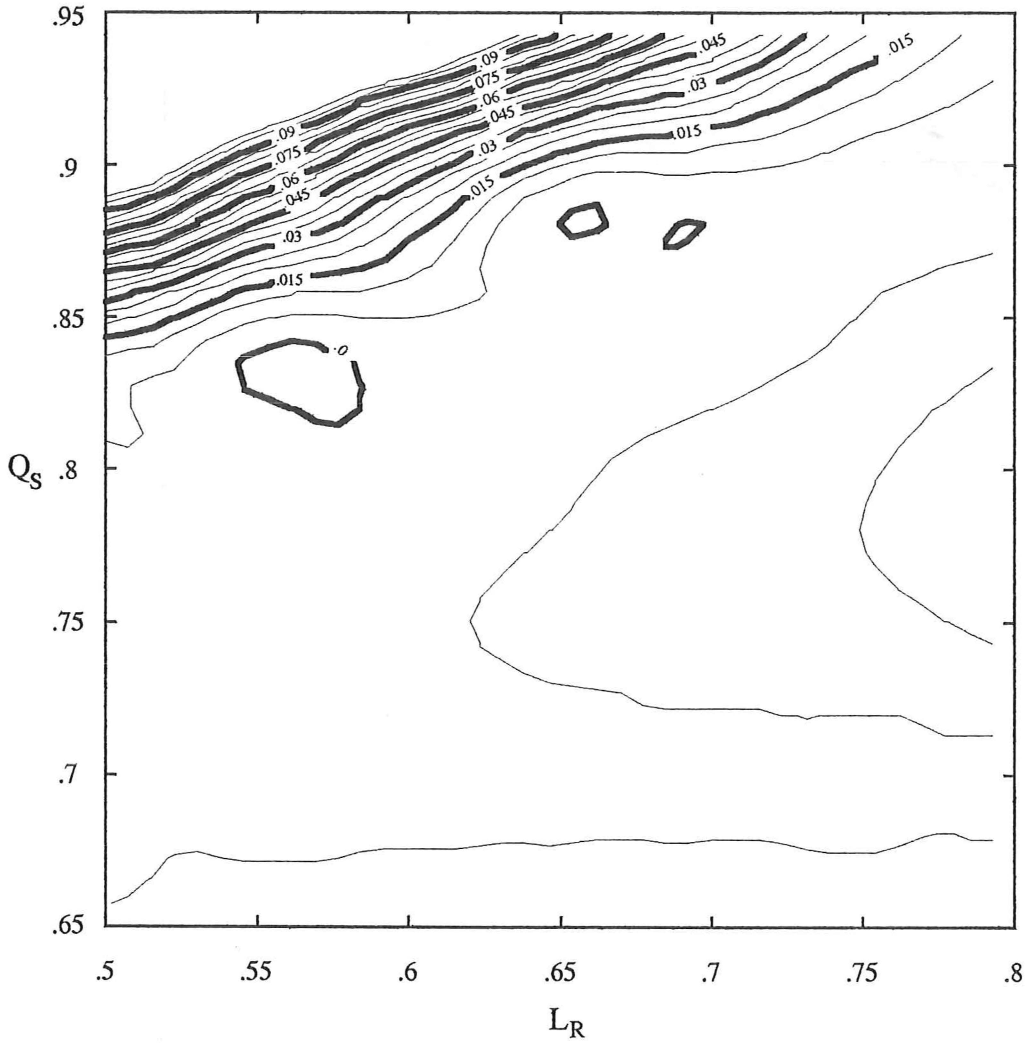


Figure 6.5 Diagonal Bilinear Models of Bottoms and Tops Composition

```
Discrete Time Model Report

1 State in Column 7
2 Inputs in Columns 3,4

State
  Order = 3
  5.5252e-1  3.7188e-1 -1.9987e-2
  Diagonal Bilinear Link, Input : 1
  7.3105e-3 -1.2058e-2 -1.7353e-2
  Diagonal Bilinear Link, Input : 2
 -2.3055e-2  1.4525e-2  1.6040e-2

Input 1
  Order 2  Deadtime 0
  3.4652e-2  2.2696e-2

Input 2
  Order 2  Deadtime 0
 -9.8919e-2 -4.8993e-2

Constant Term =  9.8097e-2
```

```
Discrete Time Model Report

1 State in Column 6
2 Inputs in Columns 3,4

State
  Order = 3
  4.7226e-1  1.9342e-1  1.4215e-2
  Diagonal Bilinear Link, Input : 1
 -4.7592e-2 -1.1586e-1  1.8130e-3
  Diagonal Bilinear Link, Input : 2
  1.4161e-1  4.7584e-2 -4.1844e-3

Input 1
  Order 2  Deadtime 0
  6.6666e-2  1.2801e-1

Input 2
  Order 2  Deadtime 0
 -1.5622e-1 -6.4348e-2

Constant Term =  3.0932e-1
```

Graph 6.7 Steady State Errors for Bilinear Model



The response of both models to the original input sequence was obtained and compared to the output data to calculate the variances in table 6.4. The use of bilinear models effected useful reductions in both the tops and bottoms variances particularly for the "full run".

Table 6.4 Model Variances for 2 Input Models ($*10^{-8}$**)**

Model Type		*Bottoms*	*Tops*
Linear	*Full Run*	*283.2*	*137.7*
	1 Step	*11.8*	*16.8*
Bilinear	*Full Run*	*137.8*	*59.8*
	1 Step	*8.9*	*9.5*
Improvement	*Full Run*	*51 %*	*57 %*
	1 Step	*25 %*	*44 %*

A measure of the degree of non-linearity or bilinearity present in a plant or model may be obtained by comparing the values of the process gains at a number of points in the operating range. The gains of the models at the operating points used in the identification are given in table 6.5.

Table 6.5 Comparison of Model Gains

Model	*Point*	*Tops*		*Bottoms*	
		Reflux	*Steam*	*Reflux*	*Steam*
Bilinear	*1*	*.132*	*-.143*	*.48*	*-1.368*
	2	*.159*	*-.174*	*.528*	*-1.398*
	3	*.156*	*-.171*	*.517*	*-1.411*
	4	*.165*	*-.18*	*.522*	*-1.435*
	5	*.183*	*-.2*	*.548*	*-1.451*
Linear	*1 - 5*	*.154*	*-.164*	*.511*	*-1.393*

The behaviour of the distillation column simulation shows weak bilinearity over the selected region with the tops gains varying by *38 %* and *42 %* between points *1* and *5*, the bottoms gains show rather less variation, *14 %* and *6 %*. Gain variations of this order are not sufficient to threaten the stability of a PID controller tuned in the middle of the operating region.

Conclusions

The dynamic equations which govern the behaviour of the distillation process show bilinear interactions between the states and inputs.

A significant improvement in model performance was obtained by the use of bilinear rather than linear models for both four and two input systems.

For the two input case, contour plots provided a useful representation of information about the steady states of the plant, enabling the selection of suitable operating points for identification trials. Contour plots also provide a useful tool for comparing the errors in model steady state estimates and determining valid regions for such models.

A comparison of process gains for identified linear and bilinear models of the distillation system showed weak bilinearity over the operating range used. The stability of a PID controller tuned in the middle of the operating region would not be threatened by the changes in process gain that occur towards the region boundary.

Nomenclature

Distillation Dynamic Analysis

L_i	Liquid flow from plate *i*
V_i	Vapour flow from plate *i*
h_i	Liquid enthalpy on plate *i*
H_i	Vapour enthalpy on plate *i*
x_i	Liquid concentration on plate *i*
y_i	Vapour concentration on plate *i*
P_i	Heat Loss from plate *i*
c_i	Molar holdup on plate *i*
v_i	Volumetric holdup on plate *i*
$g(x_i)$	Liquid Density Function
$d_{i-1,i}$, $D_{i+1,i}$	Concentration difference functions
$\delta_{i-1,i}$, $\Delta_{i+1,i}$	Enthalpy difference functions
jF	Feed Plate Number
B	Reboiler
L_F	Feed Flowrate
L_O	Reflux Flowrate
V_B	Boilup Rate or Vapour flow from Reboiler
x_F	Feed Concentration
Q_S	Steam Condensate flow from reboiler

Discrete State Space

u	Input Vector
Δ_R	Reduced Input Coefficient Matrix

x_S	Steady State
α_R	Reduced State Coefficient Matrix
DC_R	Reduced Constant Matrix

Three Dimensional Surface Equations

x, y, z	Variables
a, b, c, d, e	Constants

References

España,M.D., *Modelisation Bilineaire de Colonnes a Distiller*, Docteur Ingenieur Thesis, Institut National Polytechnique de Grenoble, France, (1977)

Janssen,P.W.M., *Bilinear Identification of a Binary Distillation Column*, Ph.D. Thesis, University of Canterbury, New Zealand, (1986)

CHAPTER 7

Bilinear Control of a Binary Distillation Column

Overview

The control methods developed in Chapter 4 were applied to the binary distillation process. Digital Computer Simulation was used to evaluate the performance of a range of controller design methods.

Digital Computer Simulation

The digital computer simulation described in the previous chapter was modified to provide simulations for a variety of controller configurations[1]. A sequence of operating points and feed concentration disturbances was used to test the performance of the controllers.

Operating Sequence

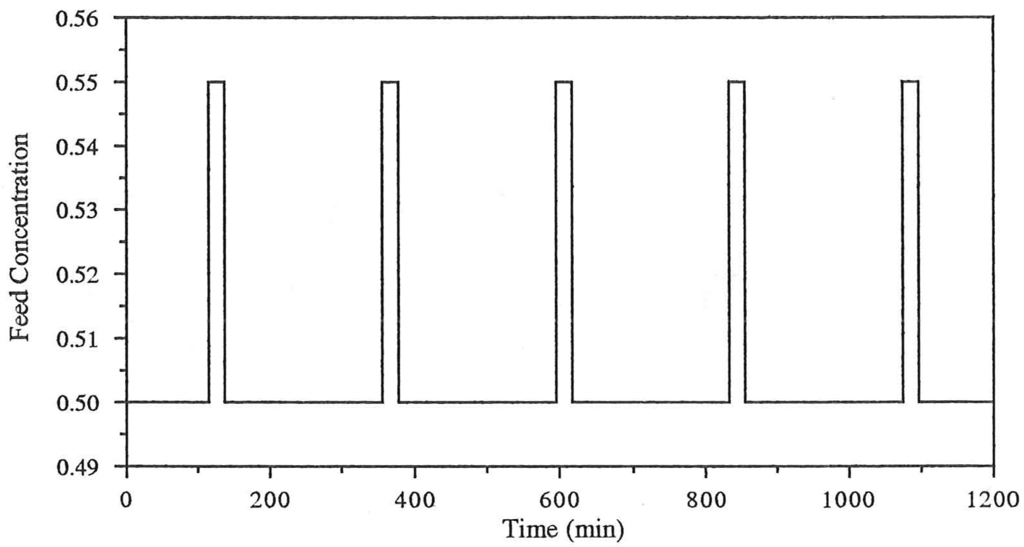
Using the steady state information from graph 6.3 a series of set points was selected, table 7.1. *240* minutes were spent at each set point. Feed concentration increases of *0.05* mole fraction and duration *20* minutes occurred midway between set point changes, these are shown in graph 7.1. The set points were selected to cover a wide range of operation and to include changes in the tops and bottoms set points both individually and jointly. The set points are represented by dashed lines on the process response graphs.

Table 7.1 Set Points for Control Simulation Operation

Set Point	*Bottoms*	*Tops*
1	*0.06*	*0.92*
2	*0.14*	*0.92*
3	*0.14*	*0.94*
4	*0.26*	*0.94*
5	*0.3*	*0.96*

[1]FORTRAN source code for the simulation is contained on the disk appendix.

Graph 7.1 Feed Concentration Disturbances for Controller Trials



The following control methods were examined.

PID Control with Static Decoupling

For multivariable systems, such as distillation columns, in which there is strong interaction between the states and inputs it is necessary to decouple the control loops before tuning the individual controllers. Both the decoupling and controller tuning may be accomplished through the use of the process reactions to step changes in each of the inputs.

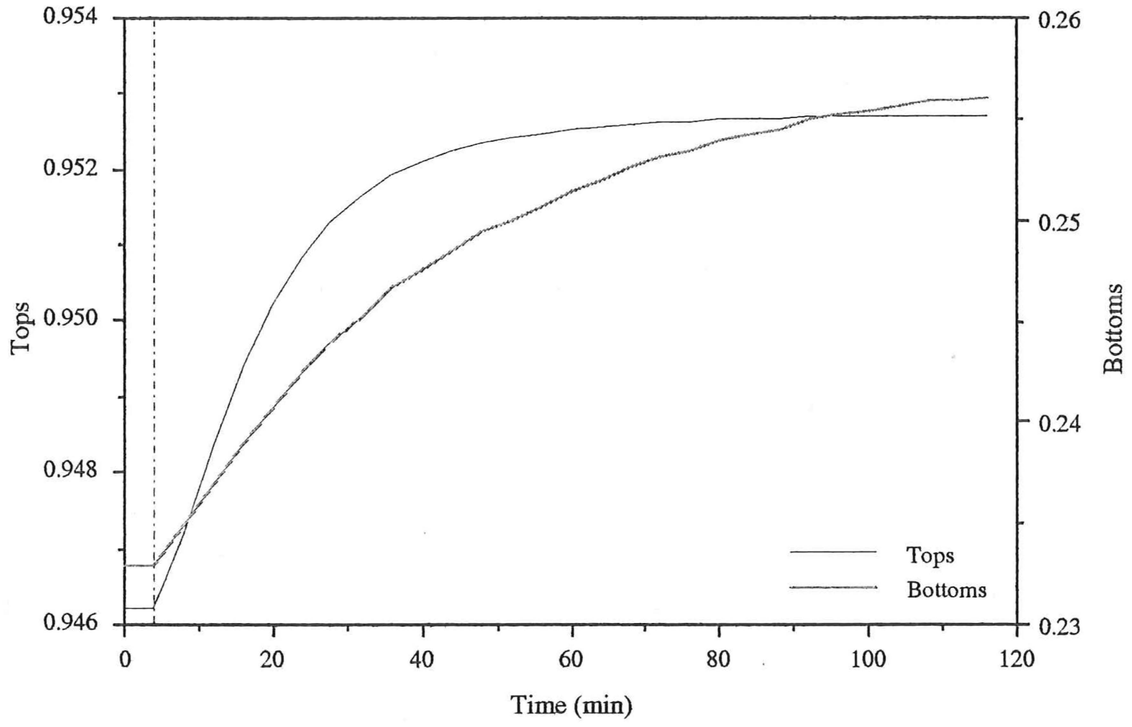
The process reaction curves of the system for reflux and steam input step changes were obtained, these are shown in graphs 7.2(a) and 7.2(b). From these graphs a gain matrix G for the system was generated:

$$G = \begin{bmatrix} 0.128 & -0.172 \\ 0.49 & -1.346 \end{bmatrix} \text{ where } \begin{bmatrix} x_D \\ x_B \end{bmatrix} = G \begin{bmatrix} L_R \\ Q_S \end{bmatrix} \qquad (7.1)$$

The inverse of the gain matrix provides a steady state or static decoupler for the process[1]. The block diagram of the decoupled system is shown in figure 7.1. The control problem was reduced to controlling the two decoupled systems $x_D = f(u_D)$ and $x_B = f(u_B)$. By using the inverse of the gain matrix as the decoupler, the process gains of the decoupled system were both forced to unity and the controllers designed, based solely on the time characteristics of the process dynamics.

[1]A mathematical proof of this is given in Appendix II.

Graph 7.2(a) Process Reaction Curve for step in Reflux Flow



Graph 7.2(b) Process Reaction Curve for step in Steam Flow

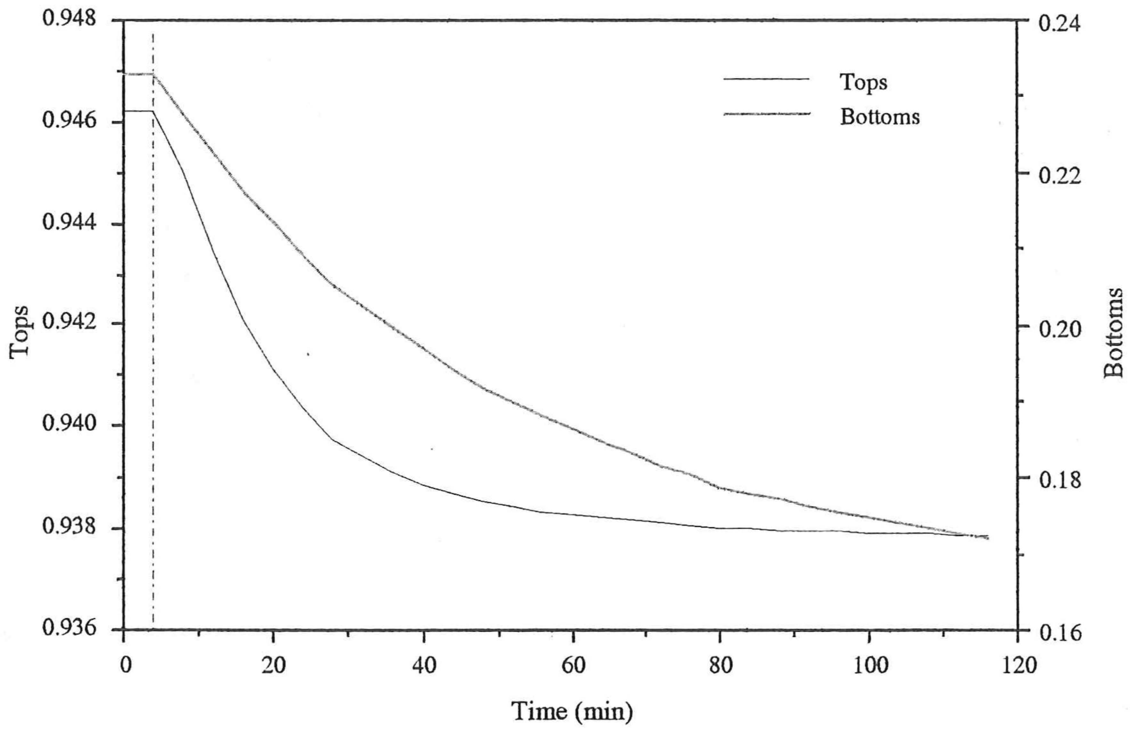
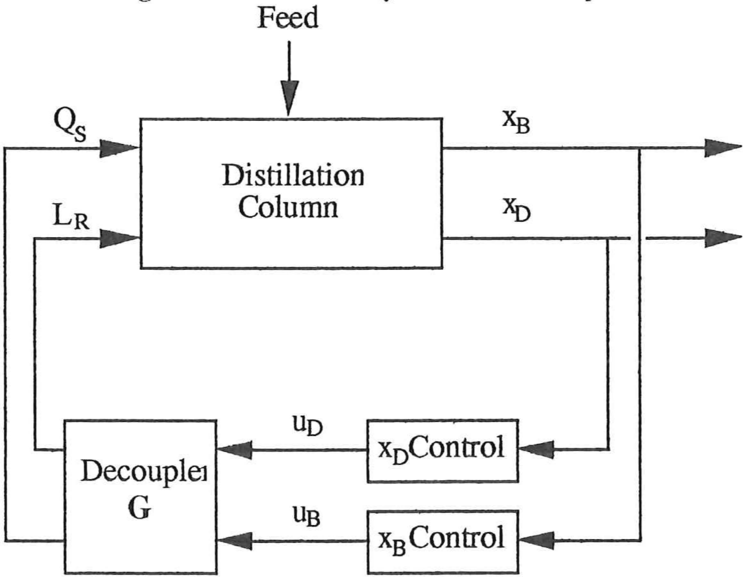


Figure 7.1 Distillation System with Decoupler



The equivalent first order plus deadtime parameters for the decoupled system were calculated from graphs 7.2 (a & b), and are given in table 7.2. The sampling period for the simulation trials was four minutes.

Table 7.2 Equivalent First Order + Deadtime Systems

	Bottoms	*Tops*
Gain	*1*	*1*
Time Constant	*34*	*14*
Deadtime	*3*	*3*

The parameters of the discrete PID controller were found using a modified Cohen - Coon relationship to ensure stability despite the relatively large sampling period, ie. T_S is larger than the deadtime of the first order approximation.

The modification is achieved by multiplying the gain calculated using the Cohen - Coon formulae by an exponential relating the sampling interval to the deadtime of the first order approximation for the system. This form was arrived at empirically and tested for a variety of first order systems and sampling times using a digital computer simulation. The resulting formulae are shown below :

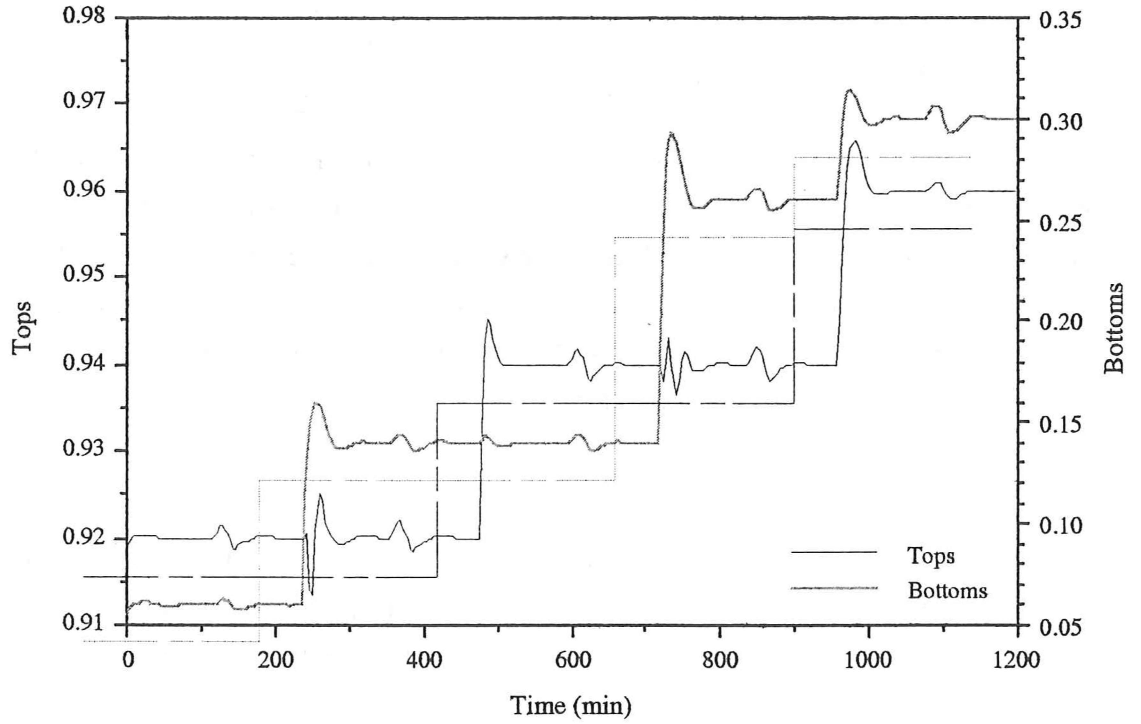
$$K_C = \frac{1}{K_P}\frac{\tau}{L}\left(\frac{4}{3} + \frac{L}{4\tau}\right)e^{-T_S/L} \qquad (7.2)$$

$$T_I = L\frac{32 + 6\,L/\tau}{13 + 8\,L/\tau} \qquad (7.3)$$

$$T_D = L\frac{4}{11 + 2\,L/\tau} \qquad (7.4)$$

The response of the controlled system was evaluated using the parameters in table 7.3. The State and Input responses are shown in graphs 7.3 and 7.4.

Graph 7.3 State Response with PID Control & Static Decoupler



Graph 7.4 Input Response with PID Control & Static Decoupler

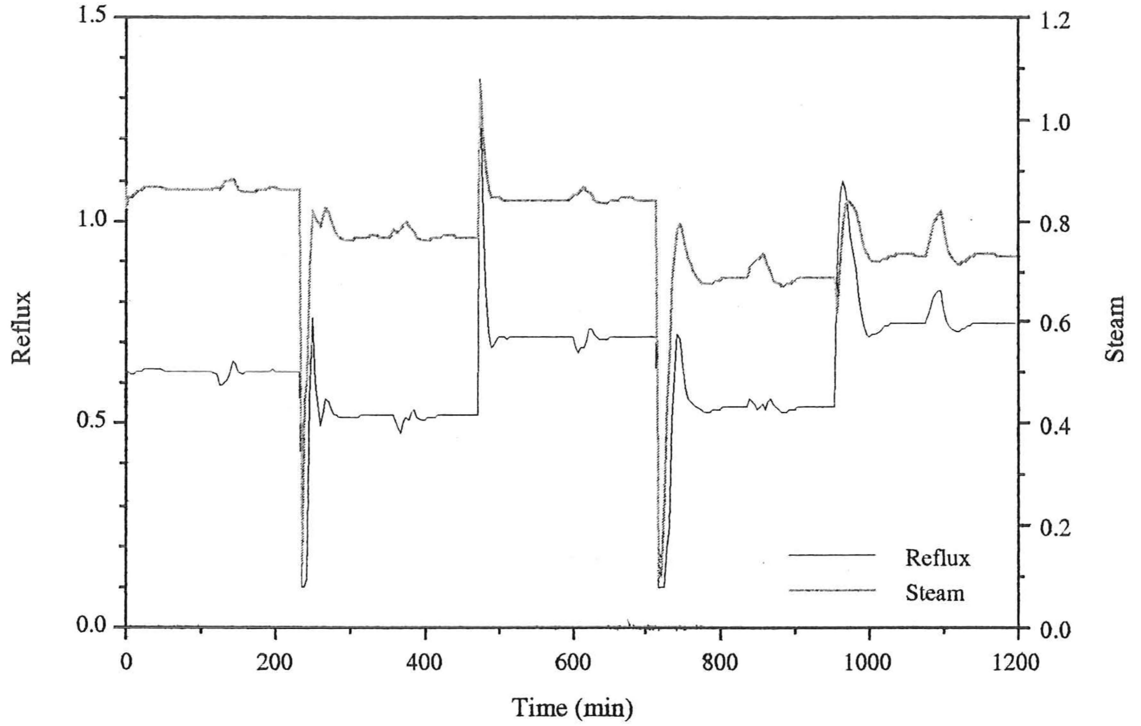


Table 7.3 PID Controller Parameters

	Bottoms	*Tops*
K_C	-4.049	-1.706
T_I	7.12 min	6.786 min
T_D	1.074 min	1.05 min

The modified Cohen - Coon method provided a reasonably well tuned, stable PID controller. It was however necessary to set minimum values for the reflux and steam flows to ensure that plates of the column did not become dry. During the second set point change the reflux flowrate exceeded the total rate of liquid formation in the condenser for eight minutes which would cause problems in a real plant. The integral action of the controller caused some overshoot when recovering from both set point and load disturbances.

Linear Optimal Control

A linear optimal regulator was designed using the linear models of tops and bottoms composition from Figure 6.4. The compound model structure is shown in equation 7.5. The weighting matrices were selected to be inversely proportional to the desired operating ranges for the tops and bottoms concentrations.

$$x(k+1) = \alpha\, x(k) + \sum_{i=1}^{m} u_i(k).\delta_i(x(k)) \text{ where } x(k) = \begin{bmatrix} x_D(k) \\ x_D(k-1) \\ x_D(k-2) \\ x_B(k) \\ x_B(k-1) \\ x_B(k-2) \end{bmatrix} \qquad (7.5)$$

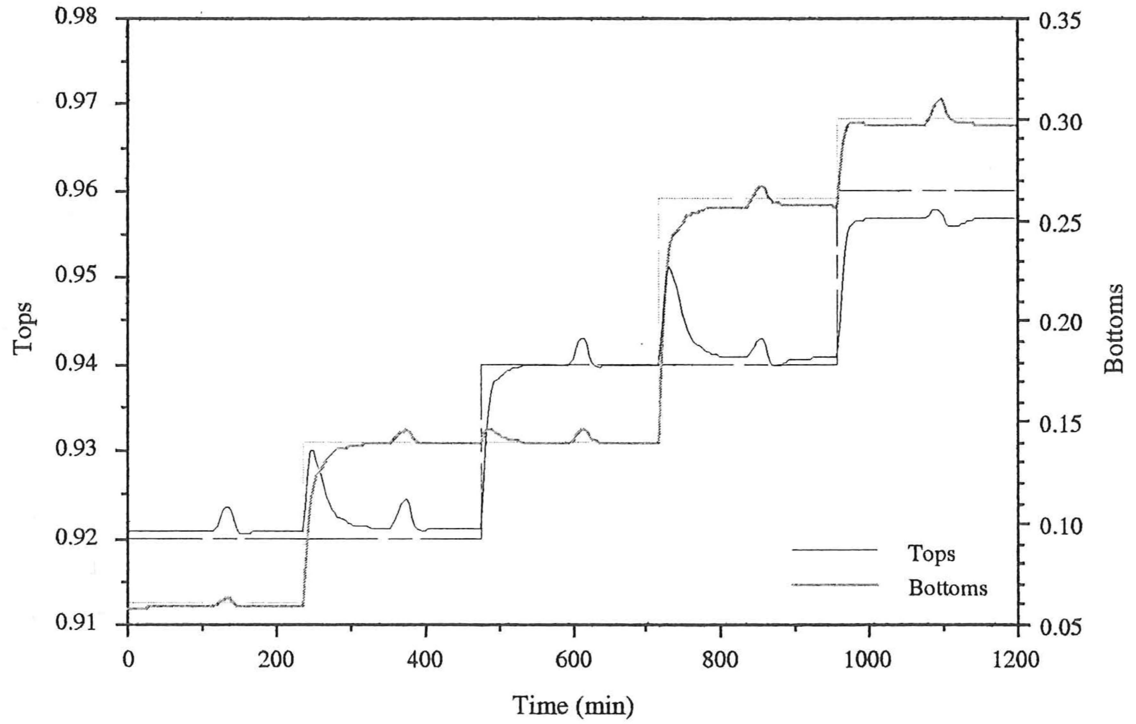
$$Q = \begin{bmatrix} 20.\boldsymbol{I}_3 & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & 5.\boldsymbol{I}_3 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (7.6)$$

Where $\boldsymbol{I}_3$ is the $3*3$ identity matrix and $\boldsymbol{0}_3$ is the $3*3$ matrix of zeros.

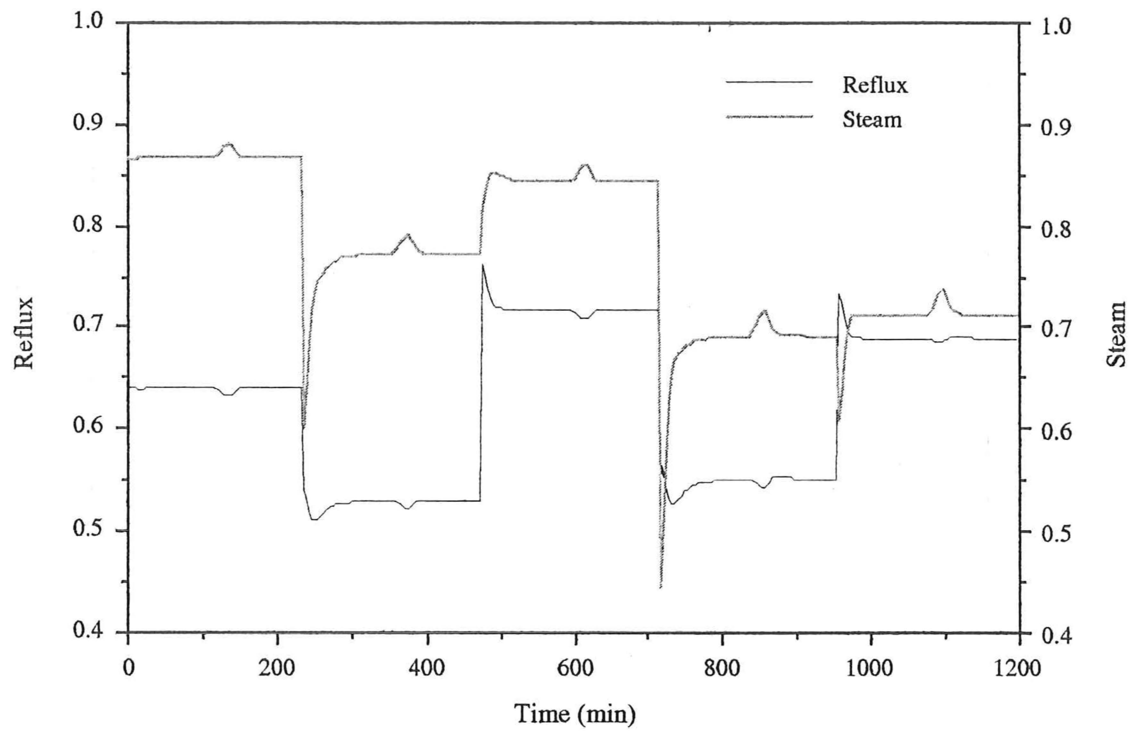
The response of the system is shown in graphs 7.5 and 7.6.

The linear regulator displayed good stability over the entire operating range. However, as expected based on the results of the tank simulation, small amounts of offset were observed at most set points due to model inaccuracy. The control action used during set point changes was within an acceptable range for the safe operation of a real plant. Significant disturbances were observed in the tops concentration during bottoms set point changes these are a direct result of the optimal regulator design procedure. The controller was designed to minimise a quadratic performance index, rather than provide decoupling of control loops.

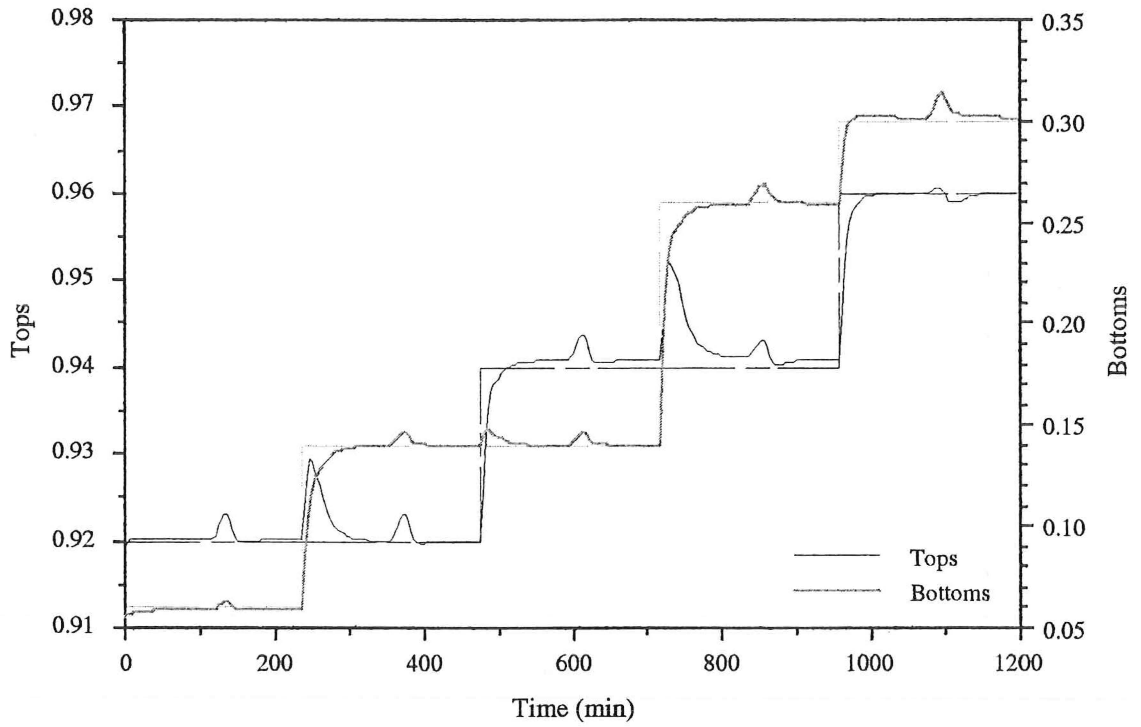
Graph 7.5 State Response with Linear Control



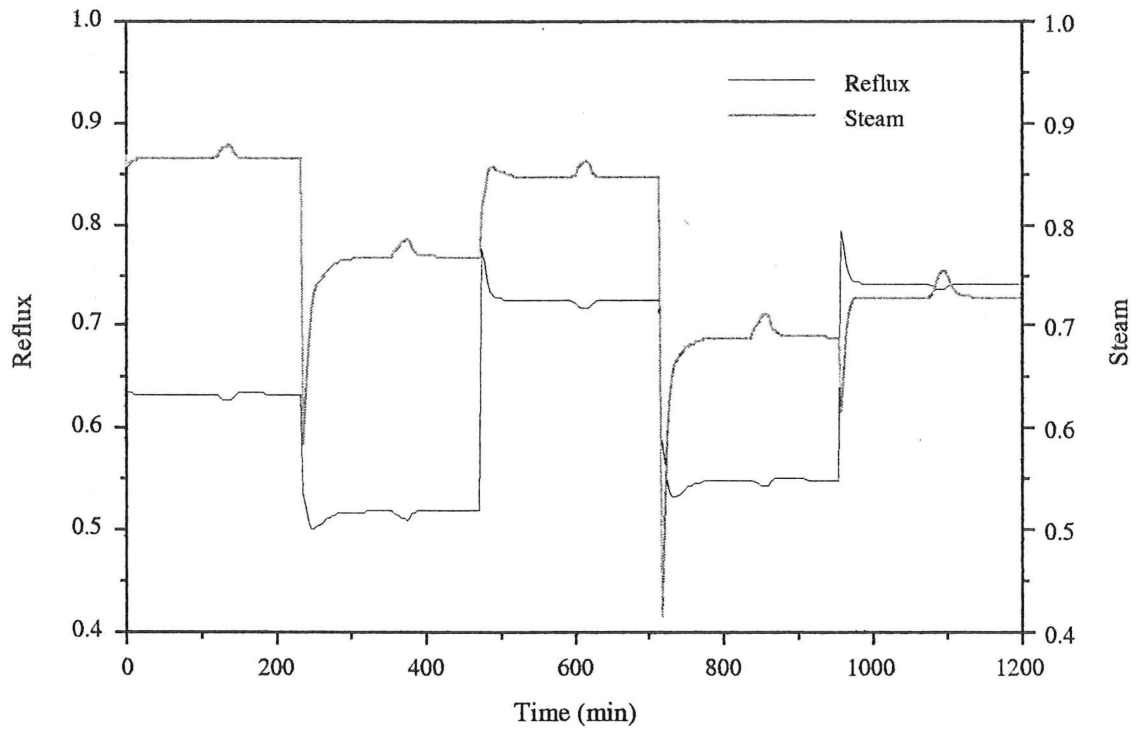
Graph 7.6 Input Response with Linear Control



Graph 7.7 State Response with Bilinear Control



Graph 7.8 Input Response with Bilinear Control



Bilinear Optimal Controller

A discrete bilinear controller was implemented using the composition models from 6.5 and the weighting matrices

$$P = \begin{bmatrix} 20.I_3 & 0_3 \\ 0_3 & 5.I_3 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.7}$$

The system response is given in graphs 7.7 and 7.8.

The performance of the bilinear controller was superior to the linear regulator, with smaller steady state offsets being observed. However, the response of the system to disturbances was similar for both controllers, due to the weakly bilinear nature of the control model. Minimal overshoot occurred and the control values were within a realisable range.

Linear and Bilinear Controllers with Integral Action

Integral action was added to the previous two controllers by the augmented state method described in chapter 4. The augmented state vector and weighting matrices used were

$$x(k) = \begin{bmatrix} x_D(k) \\ x_D(k-1) \\ x_D(k-2) \\ S_D(k) \\ x_B(k) \\ x_B(k-1) \\ x_B(k-2) \\ S_B(k) \end{bmatrix} \quad P = \begin{bmatrix} \begin{matrix} 20.I_3 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\;0\;0 & 4 \end{matrix} & 0_4 \\ 0_4 & \begin{matrix} 5.I_3 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\;0\;0 & 0.5 \end{matrix} \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.8}$$

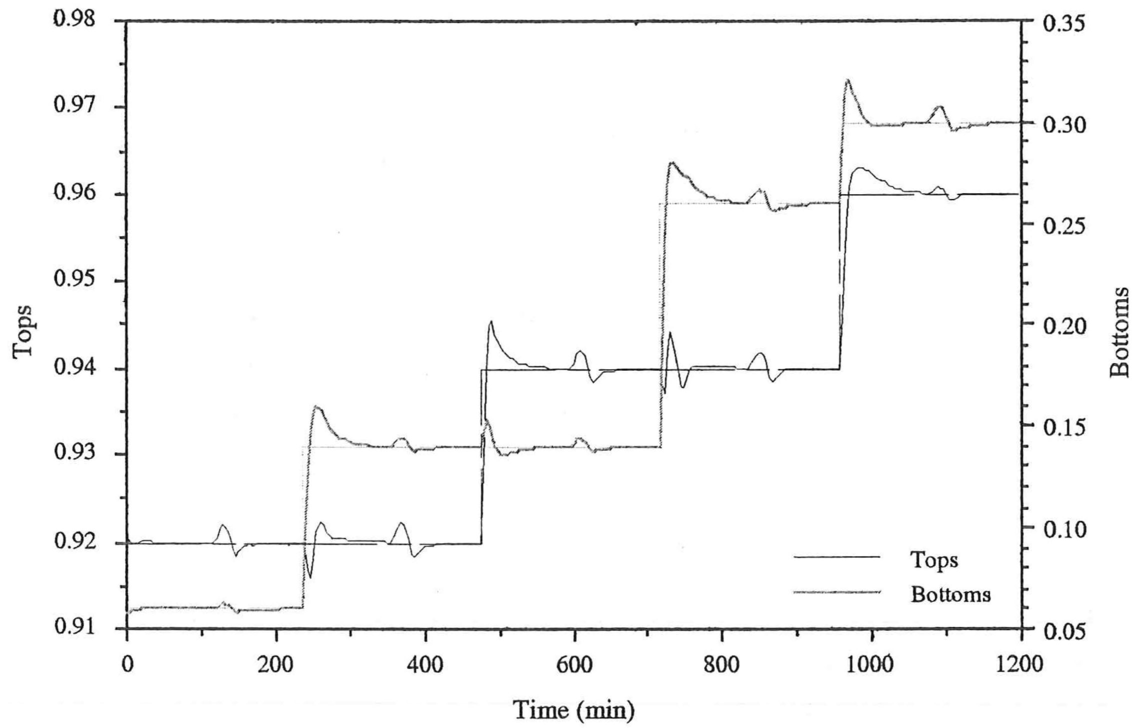
where the integral terms were calculated using the rectangular method:

$$S_D(k) = S_D(k-1) + T_S * x_D(k), \quad S_B(k) = S_B(k-1) + T_S * x_B(k) \tag{7.9}$$

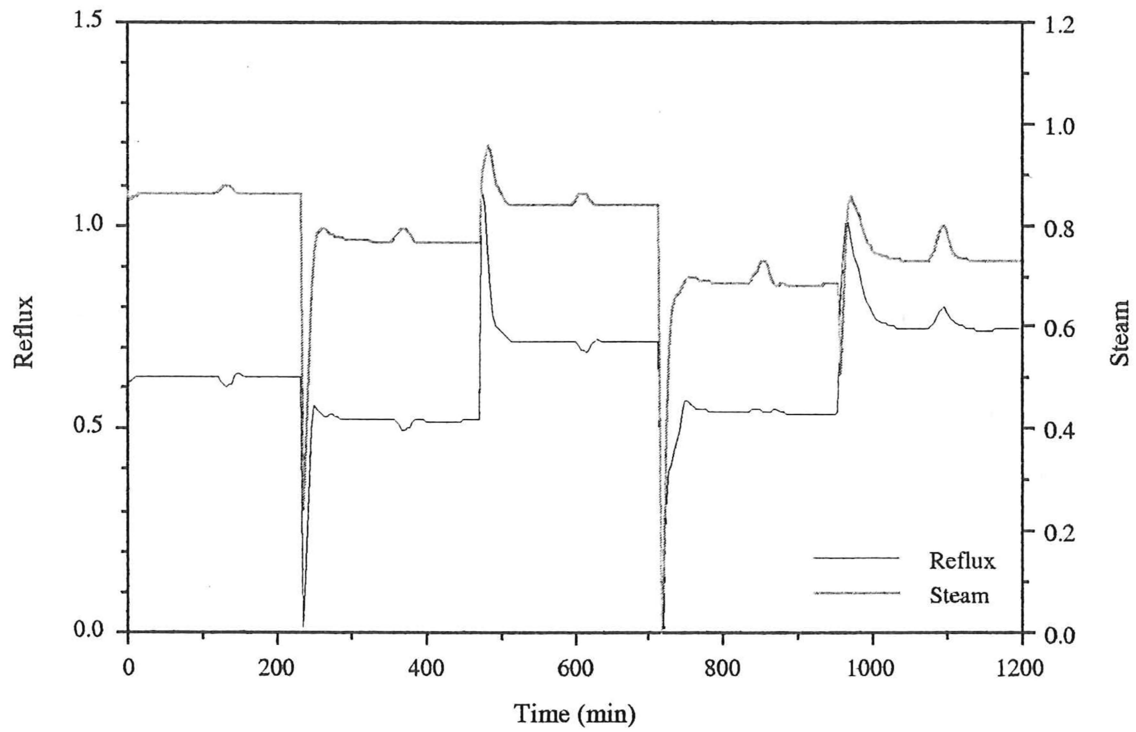
The system responses are shown in graphs 7.9 to 7.12.

The performance of both controllers was similar with the integral action negating any advantage gained through the use of a bilinear model. However the performance was not as good as the standard bilinear controller because of overshoots and unattainable input values occurring during set point changes.

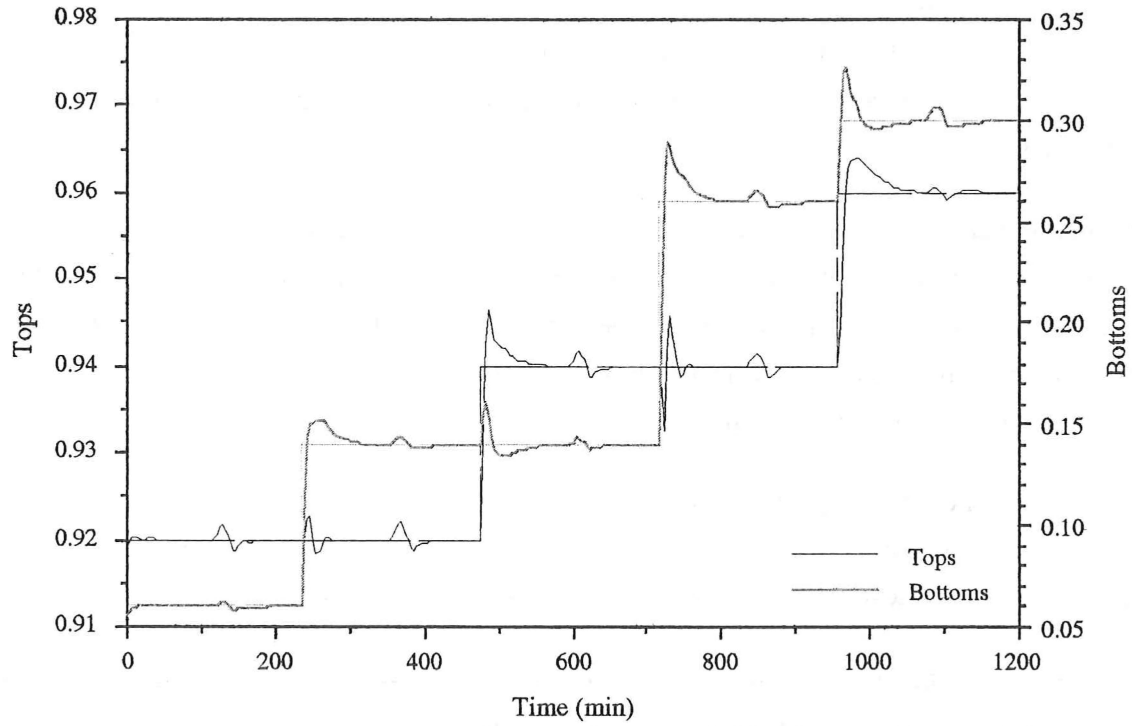
Graph 7.9 State Response with Linear Control + Integral Action



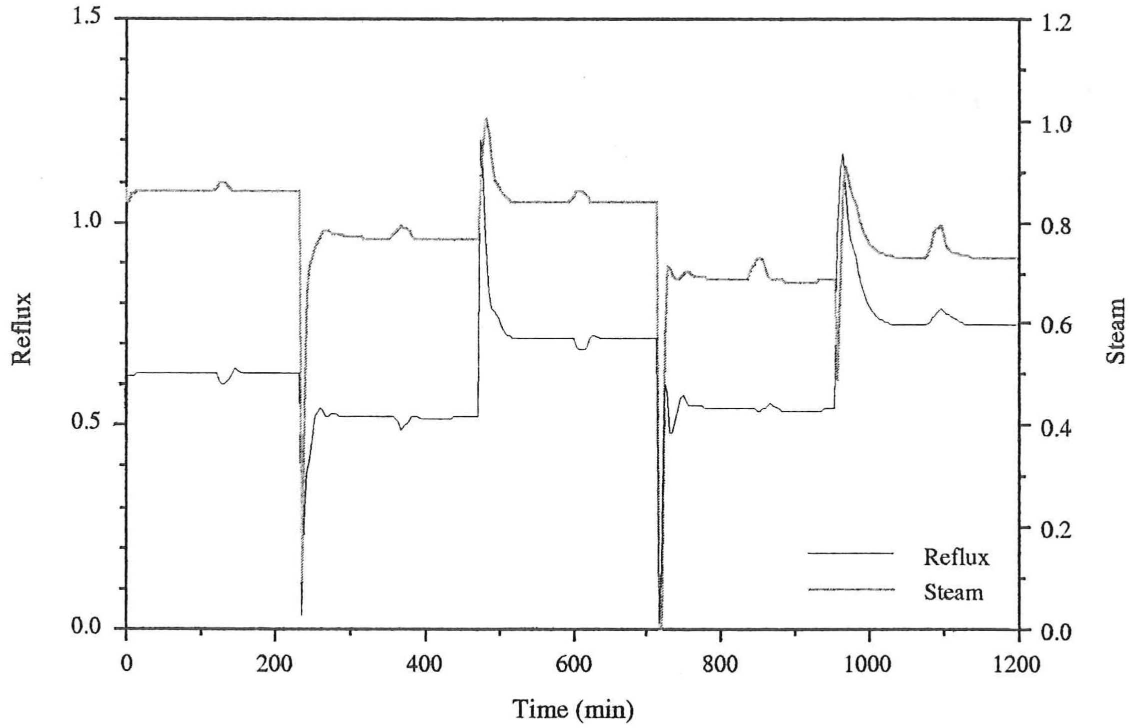
Graph 7.10 Input Response with Linear Control + Integral Action



Graph 7.11 State Response with Bilinear Control + Integral Action



Graph 7.12 Input Response with Bilinear Control + Integral Action



Conclusions

The discrete time bilinear controller design method in chapter 4 was successfully applied to a simulated binary distillation column. For a sequence of set point changes and feed concentration disturbances the bilinear controller gave good control through the use of a realisable range of input values.

Although the distillation system showed only weak bilinearity over the selected operating range the bilinear controller displayed a significant improvement over a linear regulator by reducing process offset through better modeling of the system steady states. In this case a suitable design compromise might be reached by using a bilinear model to determine the steady state inputs and implementing the control using a linear regulator.

As a consequence of the performance function and weighting matrices used in the design procedure, the control of tops and bottoms concentrations was not independent and significant disturbances in each were observed when the other set point was altered.

Versions of both the linear regulator and bilinear controller were devised incorporating integral action. The performance of both controllers was almost identical, and was very similar to that of a discrete time PID controller with static decoupling, designed using a modified Cohen-Coon method. In all three cases the presence of integral action resulted in sizable overshoot when recovering from set point changes. Both the standard linear and bilinear designs were free of this problem. The use of integral action also resulted in physically impossible or unsafe control values.

Nomenclature

x_D	Distillate Concentration
x_B	Bottoms Concentration
Q_S	Steam Condensate Flowrate
L_R	Reflux Flowrate

PID Control and Static Decoupler

G	Matrix of System Gains
u_D	Distillate Control Value
u_B	Bottoms Control Value
K_C	Controller Gain
K_P	Process Gain
τ	Time Constant of equivalent first order plus deadtime system

L	Deadtime of equivalent first order plus deadtime system
T_S	Sampling Period
T_I	Integral Time
T_D	Derivative Time

Optimal Control

$x(k)$	State at time k
α	State Coefficient Matrix
u_i	Value of i th Input
$\delta_i(x(k))$	Combined Coefficient Vector for Input i. $\delta_i(x(k)) = \beta_i + \gamma_i x(k)$
Q	Symmetric Positive Definite State Weighting Matrix
R	Diagonal Positive Definite Input Weighting Matrix
P	Symmetric Positive Definite Overall State Weighting Matrix
I_n	$n*n$ Identity matrix
0_n	$n*n$ Zero Matrix
$S_D(k)$	Discrete Integral of Distillate Concentration Deviations
$S_B(k)$	Discrete Integral of Bottoms Concentration Deviations

CHAPTER 8

Conclusions

Objectives

The structural properties of bilinear systems make them a particularly attractive choice for the modelling of many chemical plant dynamics. Bilinear systems occur naturally in a number of applications such as the constant volume tank and binary distillation systems used in this work.

The object of this work was the development of a discrete time controller design method for bilinear systems and application of such a method to both heated tank and simulated binary distillation systems. In addition a number of recursive identification procedures for bilinear systems were trialled and an Apple MacIntosh program developed, in Turbo Pascal, for batchwise identification of sampled linear, bilinear and mixed linear/bilinear systems.

Identification

A comparison of four recursive identification methods for bilinear systems yielded significant differences in performance. In a low noise environment, all four methods gave similar results for a number of performance criteria. When white noise was added to the system a considerable variation resulted. Recursive extended least squares methods gave the best overall performance. A recursive maximum likelihood implementation was disappointing, giving results similar to standard recursive least squares despite requiring double the computation. The maximum likelihood method assumes a coloured noise signal and has little effect on systems with white noise contamination.

Heated Tank Control

Standard PID controllers performed satisfactorily at the original tuning point but performance was poor away from this region with the stability of the system being endangered under some conditions. Using a bilinear model of the system to develop a set point based gain schedule went a long way in improving the stability and control of the system.

A linear state feedback controller gave good stability but suffered from offset at set points other than the tuning point. The discrete time bilinear controller yielded a stable system with good steady state accuracy at all set points despite the presence of uncompensated deadtime.

A form of deadtime compensation was successfully implemented by using a discrete time bilinear model to predict future states of the system, and using these predictions to calculate the control response. While it reduced deadtime induced overshoot, this method resulted in steady state offset by amplification of errors in the prediction model.

Distillation Simulation

Significant improvements in fit were achieved by using bilinear rather than linear models for a binary distillation system with two or four inputs. The four input case is of limited application as most industrial columns have little freedom or control over the feed concentration and flowrate. However, if either variable is subject to significant and frequent disturbances then it may be incorporated into a parametric plant model with a view to developing a feed forward control strategy.

The use of bilinear models for the two input case gave a reduction in "full run" prediction variance of the order of *50%* when compared with linear models. The steady state estimates generated by the model showed a corresponding improvement. A comparison of model gains over the operating range revealed changes of approximately *40%* for the tops concentration relative to the reflux and steam condensate flowrates and approximately *10%* for the bottoms concentration suggesting that the distillation system was only weakly bilinear over the selected range. This was also reflected in the "one step" prediction variance where the reduction was in the region of *20%* for the bottoms.

Due to the weakly bilinear nature of the distillation simulation the performance of the bilinear controller was only slightly better than a linear regulator with the same weighting matrices. The performance improvement was achieved through better estimates of the inputs required to obtain a desired steady state.

A disadvantage of state variable feedback controllers is that offset occurs when the plant is subject to prolonged disturbances. Integral action may be included in the controller design to overcome this, by an augmented state approach. The incorporation of integral action into both linear and bilinear controller designs resulted in almost identical responses. The presence of integral action also lead to sizeable offshoot when recovering from disturbances and set point changes as well as requiring physically impossible control settings during set point changes.

Overview

The bilinear controller design method may be successfully and safely applied to chemical plant items. Significant improvements in control and safety may be achieved for strongly bilinear systems such as the constant volume tank in chapter 5. On weakly bilinear systems such as the distillation column simulation where stability is not a problem, the steady state behaviour may be improved enabling the use of a controller design without integral action in some applications.

Appendix I

Published Papers

The paper on the following pages was presented at CHEMECA'90.

Fletcher, A.J. and Allen, R.M., *Bilinear Control for Chemical Plant*, Proceedings CHEMECA'90, Auckland, New Zealand, (1990)

This paper was reprinted with a slightly extended summary in *Automation and Control*, the journal of the Institute of Measurement and Control (N.Z.).

18TH AUSTRALASIAN CHEMICAL ENGINEERING CONFERENCE
CHEMECA '90
27th - 30th AUGUST 90
AUCKLAND • NEW ZEALAND

BILINEAR CONTROL FOR CHEMICAL PLANT

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Summary

Bilinear systems provide a convenient compromise between the inaccuracy of linearised models operating away from their set point and the computational load and complexity associated with many non-linear models. This class of system is particularly suited to the multiplicative non-linearities often found in constant volume chemical plant such as mixers and distillation columns. Their structure enables the use of conventional linear identification techniques.

A design procedure for a discrete, optimal, globally asymptotically stabilizing controller has been developed using the solution of the Lyapunov equation. Trials of the method both on a digital computer simulation and on pilot scale plant showed precise, reliable control for a wide range of process set points and disturbances. Conventional linear and PID controllers, while accurate near the tuning point, were not able to cope with large disturbances, or set point changes.

Introduction

Common practice is to consider the behaviour of a piece of chemical plant as a linear sum of the effects of the current states and the effects of the current inputs. This assumption has been applied even when it is known the plant is not linear, because the mathematics of these linear models is well understood, but the assumption is only valid for the process near the linearisation point.

With the rapid development of microprocessor technology, more complicated control calculations can now be performed at an acceptable speed and cost, allowing us to consider and work with more accurate descriptions of unit operations.

Bilinear Systems

Bilinear systems provide a logical first step away from the linear tradition. A general continuous bilinear model is:

$$\dot{x} = A x + B u + \sum_{i=1}^{m} u_i C_i x$$

The first two terms on the right hand side are the usual linear system and the remaining summation describes the non-linearity. These extra terms account for a form of interaction common in chemical plant items such as constant volume tanks or reactors and provides a good approximation for many others, for example distillation columns (España, 1977, Janssen, 1986).

In keeping with digital computer control, a discrete version of the above equation may be obtained:

$$\dot{x}\,(k+1) = \alpha\, x(k) + \beta\, \mathbf{u}(k) + \sum_{i=1}^{m} u_i(k)\, \gamma_i\, x(k) \tag{2}$$

This model may be separated into two vectors, one containing the parameters and the other the variables, exactly as for linear systems. The variable, or measurement, vector is a non-linear function of $x(k)$ and $\mathbf{u}(k)$. The parameter vector, however, remains linear. The wealth of knowledge available for the identification of linear discrete time systems may be directly applied (Gabr, 1986, Ahmed, 1986).

Controller Design

Controllers to operate with bilinear models may be designed in two ways. The simple approach is to use the bilinear model to modify the parameters of a conventional controller, usually by maintaining a constant open loop gain. This form of retuning is termed *gain scheduling* (Stephanopoulos, 1984).

The second method is to develop a controller which makes full use of the process knowledge contained in the model to give a design which will optimise some performance function. Benallou et. al. (1988) used this approach to derive an optimal controller

$$u_i^* = -\frac{1}{r_i}\, \mathbf{x}^T \mathbf{S}\, \mathbf{d}_i(\mathbf{x}) \tag{3}$$

for a continuous bilinear system

$$\dot{x} = A\, x + \sum_{i=1}^{m} u_i\, \mathbf{d}_i(x) \tag{4}$$

subject to a performance function.

$$j(\mathbf{x},\mathbf{u}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{m} \frac{1}{r_i}\left[\mathbf{x}^T\mathbf{S}\mathbf{d}_i(\mathbf{x})\right]^2 + \mathbf{u}^T\mathbf{R}\mathbf{u} \tag{5}$$

The matrix S is the solution to the continuous time Lyapunov equation.

$$S\,A + A^T S = -\,Q \tag{6}$$

An equivalent discrete time controller, Equations 7 to 9, may be derived through application of finite differences to the above equations. Application of central difference approximations result in the controller equations being implicit, requiring that their solutions be found iteratively. However, a number of approximations to the solution of Equation 8 may be used to reduce the computation.

$$u_i^*(k) = -\frac{1}{r_i h}\, \mathbf{x}^T\mathbf{S}\left\{ I - [\alpha - I][\alpha + I]^{-1} \right\}[\beta_i + \gamma_i\, \mathbf{x}] \tag{7}$$

$$\mathbf{x} = \frac{x(k+1) + x(k)}{2} \tag{8}$$

$$\alpha^T S\, \alpha - S = -\frac{h}{4}\left[\alpha^T + I\right] Q \left[\alpha + I\right] \tag{9}$$

This controller can be shown to provide optimal globally asymptotically stabilizing control.

Set point changes alter the state matrix (A or α) and the Lyapunov equation must accordingly recalculated to ensure stability. If this is done holding Q constant, then the system must necessar remain stable, but the nature of the response will change. Consequently, the algebraic matrix Ricc equation must be solved to maintain the desired balance between state and input weightings.

An efficient design procedure results;

1) An input weighting matrix R and an approximate overall state weighting matrix P shou be selected for the desired response, P satisfying:

$$\mathbf{P} = \mathbf{Q} + \mathbf{S}\,\mathbf{D}\,\mathbf{R}^{-1}\mathbf{D}^{T}\mathbf{S} \qquad ($$

2) The equivalent continuous model parameters A and D may be calculated at the set poin

$$\frac{Ah}{2} = [\alpha - I][\alpha + I]^{-1} \qquad ($$

$$D = \frac{1}{h}\left[I - \frac{Ah}{2}\right]\Delta(x(k)) \qquad ($$

3) The algebraic Riccati equation may be solved to obtain S.

$$\mathbf{A}^{T}\mathbf{S} + \mathbf{S}\,\mathbf{A} + \mathbf{P} - \mathbf{S}\,\mathbf{D}\,\mathbf{R}^{-1}\,\mathbf{D}^{T}\,\mathbf{S} = \mathbf{0} \qquad ($$

4) The constant factor of the controller, L, can be obtained from S.

$$L = -\frac{1}{h}S\left[I - \frac{Ah}{2}\right] \qquad ($$

5) The control variable may now be calculated using Equation 15 where x is determi using a suitable approximation to Equation 8.

$$u_i^*(k) = \frac{1}{r_i}x^{T}\,L\,\delta_i(x) \qquad ($$

The structure of this controller is similar to that obtained for discrete linear optimal regulators, exc the last term $\delta_i(x)$ is not constant, to reflect system bilinearity.

Simulation

A digital computer simulation of a heated stirred tank was used to evaluate the controller performar Figure 1. Mass and energy balances on the tank led to the continuous bilinear model, Equation These equations were integrated using a modified Euler method with a step size of 0.01 minutes. process exit temperature was controlled using the inlet water flowrate.

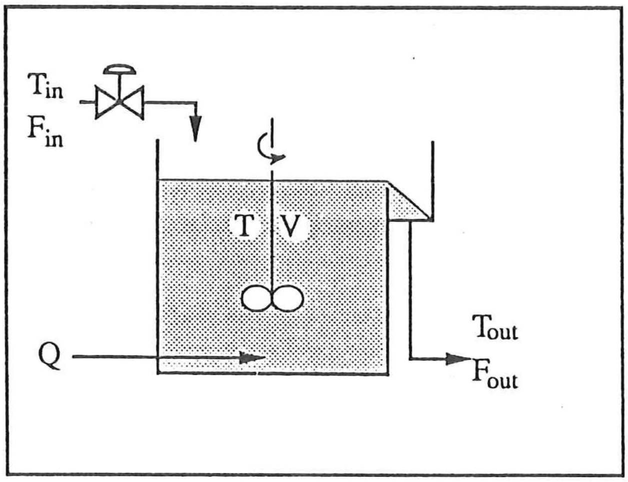


Figure 1 Heated Tank Flow Diagram

$$\frac{dV}{dt} = F_{in} - F_{out} = 0 \qquad ($$

$$C_p\,\rho\,V\frac{dT}{dt} = C_p\,\rho\,F\,(T_{in} - T) + Q \qquad ($$

$$\frac{dT_{out}}{dt} = T - T_{out} \qquad ($$

With plant parameters

$$V = 40\ l,\ Q = 840W$$

operating at

$$T^{ss} = 40°C \text{ and } F^{ss} = 10\ l/min;$$

$$\dot{x} = \begin{bmatrix} -.25 & 0 \\ 1 & -1 \end{bmatrix}x + \begin{bmatrix} -.5 \\ 0 \end{bmatrix}u + u\begin{bmatrix} -.025 & 0 \\ 0 & 0 \end{bmatrix}x$$

Sample points were 0.5 minutes apart, about 15% of the process time constant. The simulation also contained a deadtime of 1 sampling interval to accurately reflect real plant conditions.

Results

A Recursive Least Squares method was applied to input / output data from the simulation to obtain a discrete bilinear model. This model was then used as the basis for the controller design procedure.

$$x(k+1) = \begin{bmatrix} 1.462 & -0.507 \\ 1 & 0 \end{bmatrix} x(k) + u \left\{ \begin{bmatrix} -.09039 \\ 0 \end{bmatrix} + \begin{bmatrix} -.00249 & -.00220 \\ 0 & 0 \end{bmatrix} x(k) \right\} \qquad (20)$$

For comparison three conventional controllers, a PID, a gain scheduled PID and a linear optimal regulator, were also designed and tuned temperature and flow set points of 40°C and 10 l/min.

All four controllers were subjected to a sequence of set point changes and heat input disturbances. Three different set points were chosen; the tuning point and 10°C either side. At each set point the process was subject to two pulse disturbances to the heat input, increasing and decreasing the heat by 840W for 0.4 minutes. Graphs 1 to 4 show the response of the controllers, the dotted line indicating the set point and the solid line the process response.

The standard, constant parameter PID controller was tuned using the ultimate method (Stephanopoulos, 1984) and gave adequate performance in the close vicinity of the tuning point, but proved unsuitable with even modest (10°C) set point changes, resulting in unstable behaviour in one case and very slow response in another (Graph 1).

A gain scheduled variant of the same controller, designed to maintain a constant open loop gain showed noticeable improvement. However, Graph 2 shows that merely altering the gain is not sufficient to adjust for the non-linearities present in the system.

A linearised model of the system was used to design a linear multivariable optimal regulator (Elbert, 1984). This gave precise control near the tuning point but was subject to large offset when operating at other set points (Graph 3).

The Bilinear controller gave precise response at all three set points. However, some overshoot was observed when recovering from set point changes, attributable to the effect of deadtime (Graph 4).

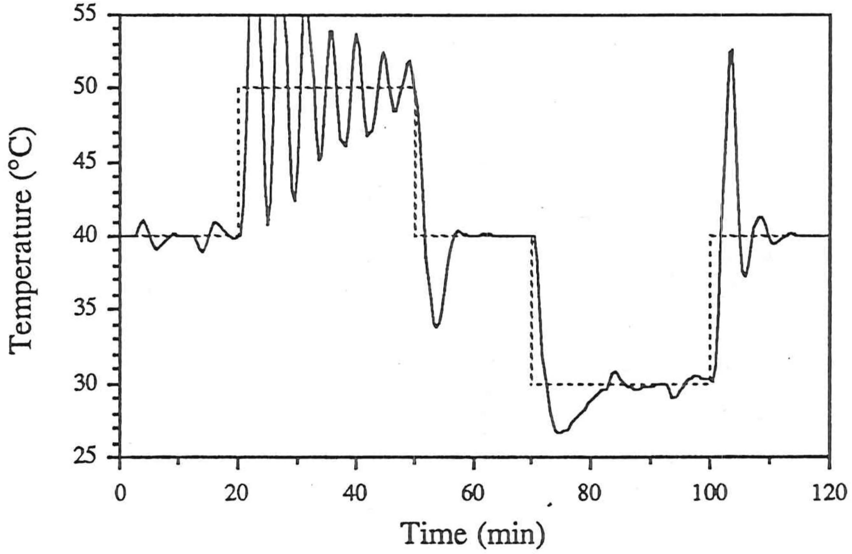
Deadtime compensation was added to the bilinear controller to correct for deadtime and remove this overshoot. The discrete bilinear model was used to predict the state of the system at one and two sampling intervals into the future and the control calculation based on these predictions. The results are shown in Graphs 5 and 6.

Deadtime compensation was effective in correcting for deadtime offset resulted but if the predictive model was imperfect. Offset was caused at some set points due to increased sensitivity to modeling errors. However, use of the predictor made the controller more cautious, reducing or eliminating overshoot during recovery from disturbances. Over-estimating the deadtime when using a predictor led to larger offset and slower return from disturbances.

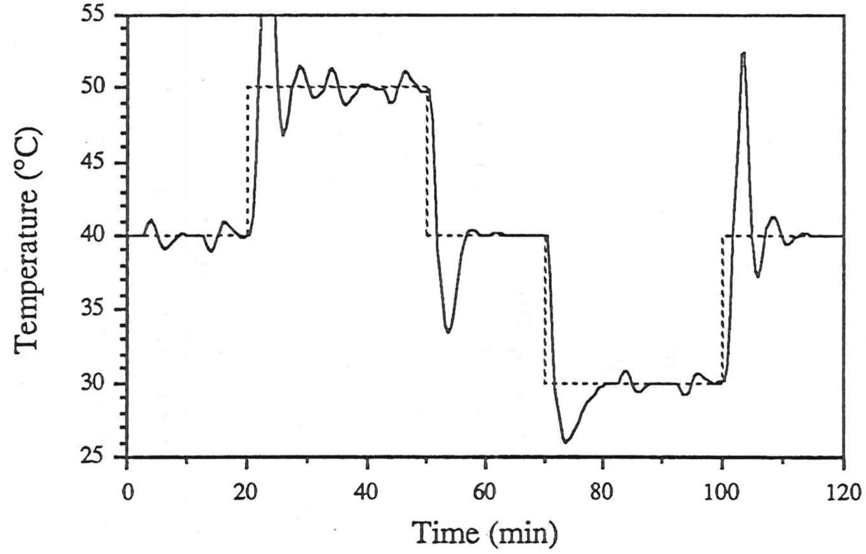
Pilot Scale Tank

Further controller trials were performed on the pilot scale steam heated tank shown in Figure 2. Cold water flow into the plant was regulated, using an analog PI controller in cascade configuration, to remove the effect of valve non-linearities. The steam valve was used to provide heat input disturbances.

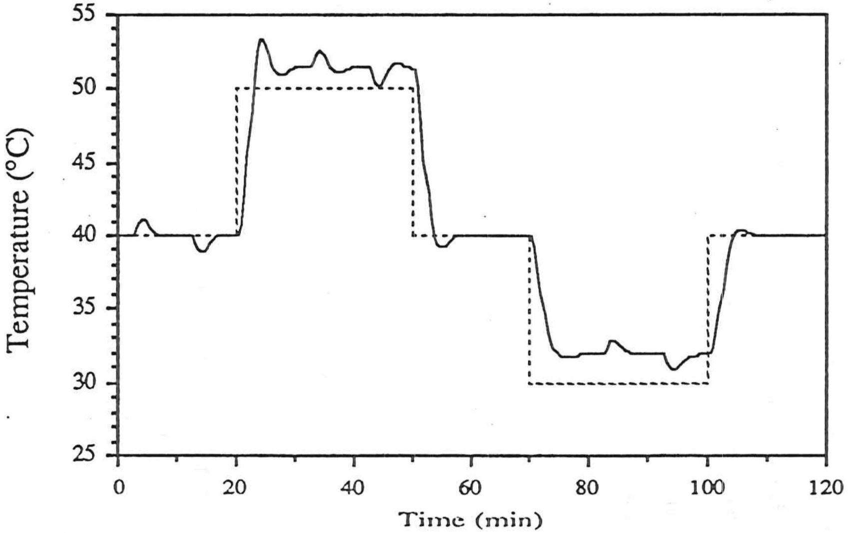
Graph 1 Simulation Response with PID Control



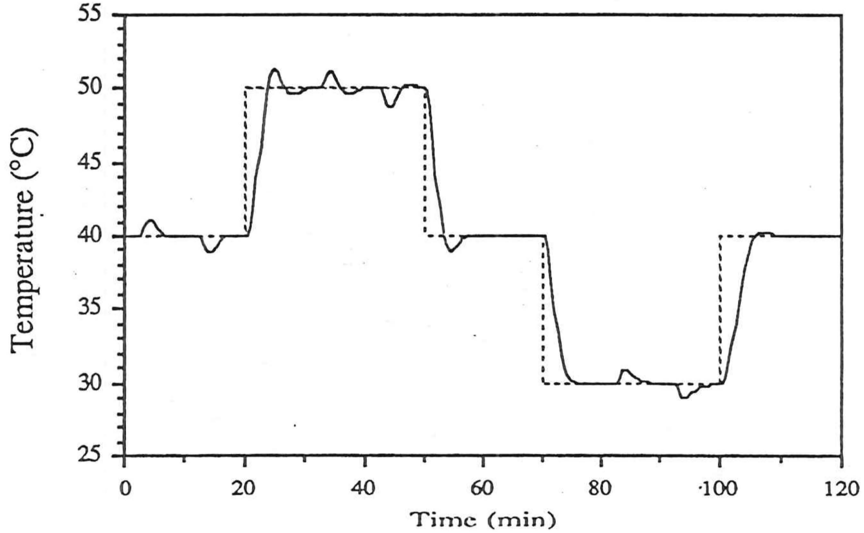
Graph 2 Simulation Response with Gain Scheduled PID Control



Graph 3 Simulation Response with Linear Optimal Control



Graph 4 Simulation Response with Bilinear Optimal Control



Sample points were 0.5 minutes apart and the flowrate through the deadtime leg corresponded to a deadtime of 0.5 to 0.8 minutes.

The operating sequence was similar to that used for the simulation, with three set points and heat disturbances at each. The duration of the heat disturbances was increased to 1 minute (2 sample intervals) and the magnitude of the disturbances altered to +160% and -100% of the normal steam flow.

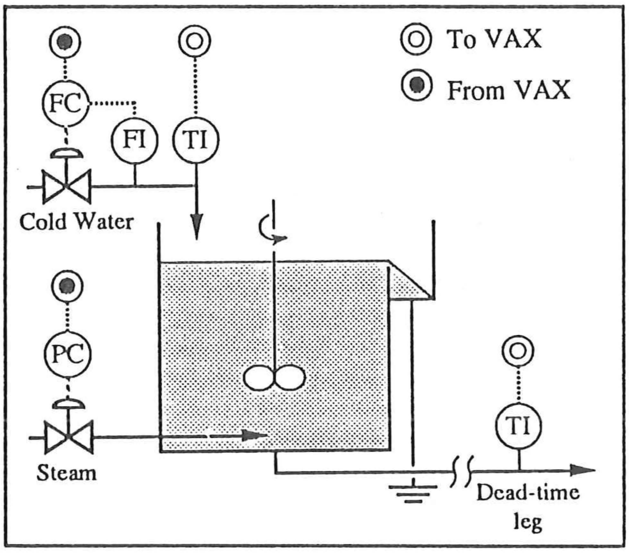


Figure 2 Pilot Tank PID

Results

A gain scheduled PID controller and a bilinear optimal controller were applied to this system. The results are shown in Graphs 7 and 8.

The gain scheduled PID controller was tuned at 40°C and a flowrate of 9.8 litres/min. Large overshoot occurred after set point changes or after large disturbances, especially at the high temperature set point. The controller also required a considerable length of time to damp out oscillations after each disturbance (Graph 7).

The results for the bilinear controller show precise control over the full range. Some overshoot occurred, due to the presence of deadtime. The nature of the plant response remained the same at all set points (Graph 8).

The results of these trials confirmed those from the simulation. Gain scheduled PID control resulted in large overshoot, and oscillation after disturbances. Bilinear control was precise and rapidly recovered from disturbances.

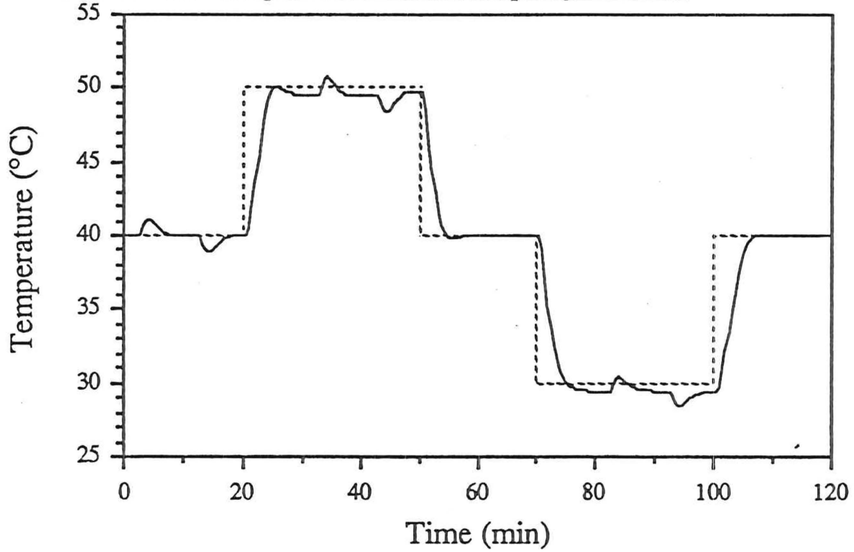
Conclusions

Bilinear control is particularly suited to constant volume chemical plant such as the steam heated tank used in this study.

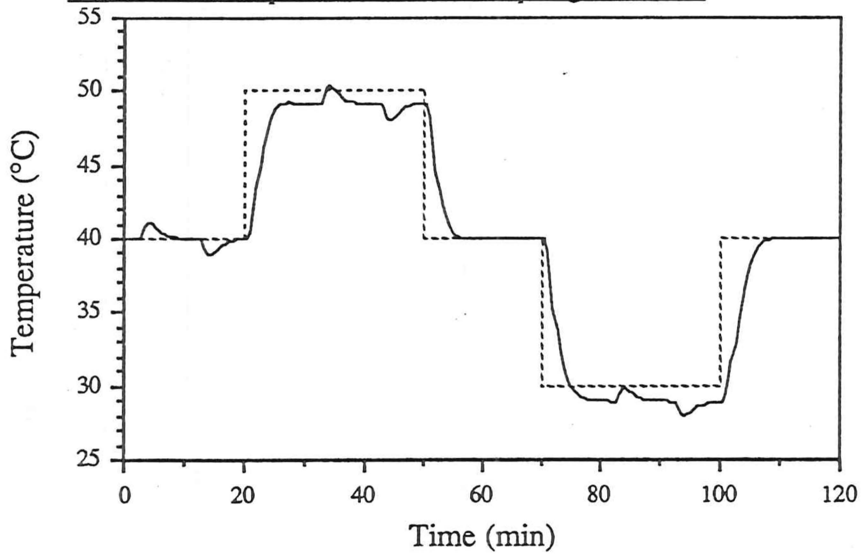
The bilinear controller provides precise control over the entire plant operating range. Stability was maintained even when the process contained small amounts of uncompensated dead-time. Computational requirements are approximately twice those for a linear controller of the same order, easily achieved with most programable devices.

Dead-time compensation was successfully employed by using the bilinear model of the plant. A more cautious controller resulted but the effects of any process modeling errors were multiplied causing controller process variable offset.

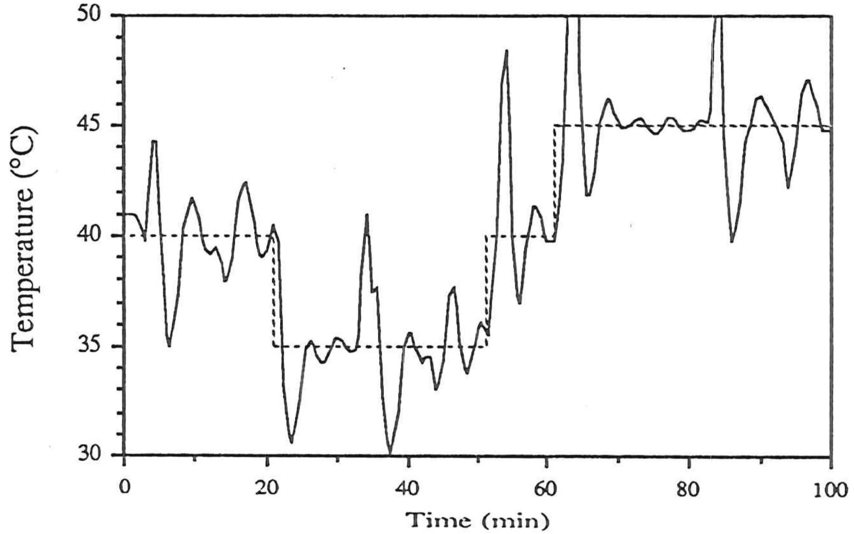
Graph 5 Simulation Response with Bilinear Optimal Control
Deadtime Compensation of 1 Sampling Interval



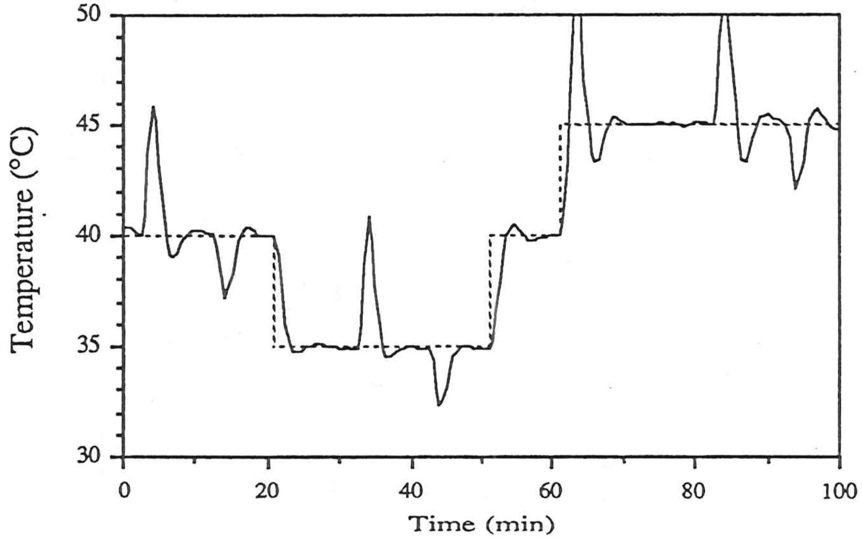
Graph 6 Simulation Response with Bilinear Optimal Control
Deadtime Compensation of 2 Sampling Intervals



Graph 7 Plant Response with Gain Scheduled PID Control



Graph 8 Plant Response with Bilinear Optimal Control



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Nomenclature

	Continuous System		Discrete System
A	State coefficient matrix	α	State coefficient matrix
b_i	Coefficient vector for input u_i	β_i	Coefficient vector for input u_i
C_i	Bilinear coefficient matrix for input u_i	γ_i	Bilinear coefficient matrix for input u_i
$d_i(x)$	Overall coefficient vector for input u_i	$\delta_i(x)$	Overall coefficient vector for input u_i
$D(x)$	Overall input coefficient matrix	$\Delta(x)$	Overall input coefficient matrix
	Heated Tank System		**General**
V	Tank volume (litres)	x	State vector
F	Flowrate (litres / minute)	u	Input vector
C_p	Specific heat ()	n	Number of states
ρ	Density (kg / litre)	m	Number of inputs
Q	Heat input (W)	$\mathbf{P}$	Overall state weighting matrix
t	Time (minutes)	$\mathbf{R}$	Input weighting matrix
		$\mathbf{Q}, \mathbf{S}$	Weighting matrices
		h	Sampling interval

Appendix II

Steady State Decoupling

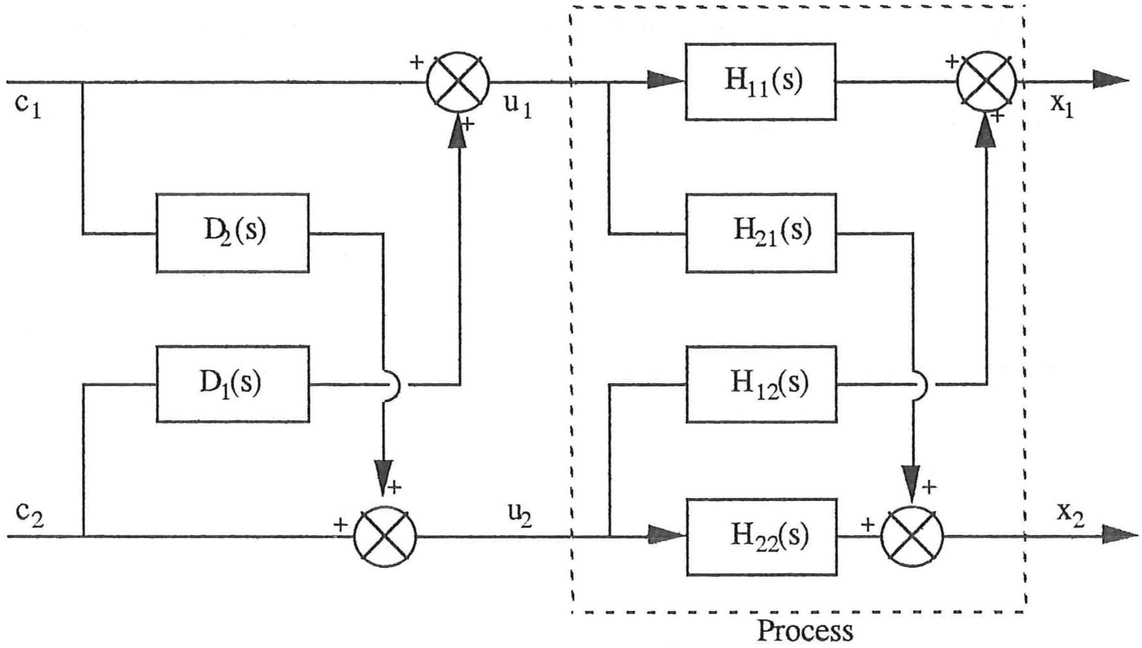

Figure 1 Interacting Process with Decoupler

A system with interaction is shown in figure 1. The standard decoupler arrangement uses two decoupling elements with transfer functions defined by Stephanopoulos (1984) as

$$D_1(s) = -\frac{H_{12}(s)}{H_{11}(s)} \text{ and } D_2(s) = -\frac{H_{21}(s)}{H_{22}(s)} \tag{1}$$

For steady state decoupling the transfer functions $H_{ij}(s)$ are replaced by their gains, giving :

$$D_1 = -\frac{G_{12}}{G_{11}} \text{ and } D_2 = -\frac{G_{21}}{G_{22}} \tag{2}$$

The inputs after the decouplers are

$$u_1 = c_1 - \frac{G_{12}}{G_{11}} c_2 \text{ and } u_2 = c_2 - \frac{G_{21}}{G_{22}} c_1 \tag{3}$$

or in matrix form

$$u = \begin{bmatrix} 1 & -\frac{G_{12}}{G_{11}} \\ -\frac{G_{21}}{G_{22}} & 1 \end{bmatrix} c \tag{4}$$

The corresponding steady state values of the process variables are found by multiplying by the gain matrix.

$$x = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} 1 & -\frac{G_{12}}{G_{11}} \\ -\frac{G_{21}}{G_{22}} & 1 \end{bmatrix} c$$

$$= \begin{bmatrix} G_{11} - \frac{G_{12}G_{21}}{G_{22}} & 0 \\ 0 & G_{22} - \frac{G_{12}G_{21}}{G_{11}} \end{bmatrix} c \qquad (5)$$

Post multiplying the decoupler matrix by the inverse of the matrix in equation 5 results in the inverse of the process gain matrix. Therefore the inverse of the process gain matrix is a valid steady state decoupler. A consequence of using the inverse gain matrix for decoupling is that the open loop gain of the process with respect to the controller outputs $\boldsymbol{c}$ is the identity matrix.

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