



Minimal pseudo-Anosov stretch factors on nonoriented surfaces

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We determine the smallest stretch factor among pseudo-Anosov maps with an orientable invariant foliation on the closed nonorientable surfaces of genus 4, 5, 6, 7, 8, 10, 12, 14, 16, 18 and 20. We also determine the smallest stretch factor of an orientation-reversing pseudo-Anosov map with orientable invariant foliations on the closed orientable surfaces of genus 1, 3, 5, 7, 9 and 11. As a byproduct, we obtain that the stretch factor of a pseudo-Anosov map on a nonorientable surface or an orientation-reversing pseudo-Anosov map on an orientable surface does not have Galois conjugates on the unit circle. This shows that the techniques that were used to disprove Penner's conjecture on orientable surfaces are ineffective in the nonorientable cases.

57M20; 11C08, 37E30, 57M99

1 Introduction

Let S be a surface of finite type. A homeomorphism f of S is *pseudo-Anosov* if there are transverse singular measured foliations \mathcal{F}^u and \mathcal{F}^s and a real number $\lambda > 1$ such that $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$; see Thurston [32]. The number λ is called the *stretch factor* of f .

Denote by N_g the closed nonorientable surface of genus g (the connected sum of g projective planes) and by $\delta^+(N_g)$ the minimal stretch factor among pseudo-Anosov homeomorphisms of N_g with an orientable invariant foliation. (Only one of the foliations can be orientable, otherwise the surface would have to be orientable as well.) The number $\delta^+(N_g)$ exists because, for any surface S , the set of pseudo-Anosov stretch factors on S is a discrete set; see Arnoux and Yoccoz [5] and Ivanov [12].

Theorem 1.1 *The values and minimal polynomials of $\delta^+(N_g)$ for $g = 4, 5, 6, 7, 8, 10, 12, 14, 16, 18$ and 20 are as follows:*

g	$\delta^+(N_g) \approx$	minimal polynomial of $\delta^+(N_g)$	singularity type
4	1.83929	$x^3 - x^2 - x - 1$	(6)
5	1.51288	$x^4 - x^3 - x^2 + x - 1$	(4, 4, 4)
6	1.42911	$x^5 - x^3 - x^2 - 1$	(10)
7	1.42198	$x^6 - x^5 - x^3 + x - 1$	(4, 4, 4, 4, 4)
8	1.28845	$x^7 - x^4 - x^3 - 1$	(14)
10	1.21728	$x^9 - x^5 - x^4 - 1$	(18)
12	1.17429	$x^{11} - x^6 - x^5 - 1$	(22)
14	1.14551	$x^{13} - x^7 - x^6 - 1$	(26)
16	1.12488	$x^{15} - x^8 - x^7 - 1$	(30)
18	1.10938	$x^{17} - x^9 - x^8 - 1$	(34)
20	1.09730	$x^{19} - x^{10} - x^9 - 1$	(38)

The table also contains the singularity type of the minimizing pseudo-Anosov map. For example, (4, 4, 4) means that the pseudo-Anosov map has three 4-pronged singularities.

Based on this result, we conjecture the following:

Conjecture 1.2 For all $k \geq 2$, $\delta^+(N_{2k})$ is the largest root of

$$x^{2k-1} - x^k - x^{k-1} - 1.$$

We think that the minimal stretch factors in genus 9 and 11 are as follows. We will discuss supporting evidence in Section 5.4.

Conjecture 1.3 The approximate values and minimal polynomials of $\delta^+(N_g)$ for $g = 9, 11$ are as follows:

g	$\delta^+(N_g) \approx$	minimal polynomial of $\delta^+(N_g)$	singularity type
9	1.35680	$x^8 - x^5 - x^4 - x^3 - 1$	(16)
11	1.22262	$\frac{x^{12} - x^7 - x^6 - x^5 - 1}{x^2 + x + 1}$	(8, 8, 8)

For our second main result, denote by $\delta_{\text{rev}}^+(S_g)$ the minimal stretch factor among orientation-reversing pseudo-Anosov homeomorphisms of the closed orientable surface S_g of genus g that have orientable invariant foliations.

Theorem 1.4 The values and minimal polynomials of $\delta_{\text{rev}}^+(S_g)$ for $g = 1, 3, 5, 7, 9$ and 11 are as follows:

g	$\delta_{\text{rev}}^+(S_g) \approx$	minimal polynomial of $\delta_{\text{rev}}^+(S_g)$	singularity type
1	1.61803	$x^2 - x - 1$	no singularities
3	1.25207	$\frac{x^8 - x^5 - x^3 - 1}{x^2 + 1}$	(4, 4, 4, 4)
5	1.15973	$\frac{x^{12} - x^7 - x^5 - 1}{x^2 + 1}$	(6, 6, 6, 6)
7	1.11707	$\frac{x^{16} - x^9 - x^7 - 1}{x^2 + 1}$	(8, 8, 8, 8)
9	1.09244	$\frac{x^{20} - x^{11} - x^9 - 1}{x^2 + 1}$	(10, 10, 10, 10)
11	1.07638	$\frac{x^{24} - x^{13} - x^{11} - 1}{x^2 + 1}$	(12, 12, 12, 12)

Moreover, we have

$$\delta_{\text{rev}}^+(S_g) \geq \delta_{\text{rev}}^+(S_{g-1})$$

for $g = 2, 4, 6, 8$ and 10 .

Based on these results, it is natural to conjecture the following:

Conjecture 1.5 For all $k \geq 2$, $\delta_{\text{rev}}^+(S_{2k-1})$ is the largest root of

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1.$$

Theorem 1.4 shows that $\delta_{\text{rev}}^+(S_g)$ fails to be strictly decreasing at every other step for small values of g . We conjecture that in fact the value of $\delta_{\text{rev}}^+(S_g)$ strictly increases at every other step. We will discuss evidence for this after Proposition 5.6.

Conjecture 1.6 For all $k \geq 1$, we have

$$\delta_{\text{rev}}^+(S_{2k}) > \delta_{\text{rev}}^+(S_{2k-1}).$$

Motivation

One motivation for studying $\delta(S)$, the smallest stretch factor of an orientation-preserving pseudo-Anosov map on an orientable surface, is that the shortest closed geodesic (in the Teichmüller metric) on the moduli space of algebraic curves homeomorphic to S has length $\log \delta(S)$.

Another motivation for studying small stretch factors comes from 3–manifold theory. The mapping torus of a pseudo-Anosov map f is a hyperbolic 3–manifold M_f and the stretch factor of f is related to the hyperbolic volume of M_f ; see Kin, Kojima and Takasawa [13] and Kojima and McShane [15]. This relates small-volume hyperbolic manifolds — see Agol [3], Agol, Storm and Thurston [4], Gabai, Meyerhoff and Milley [9] and Milley [24] — to small stretch factor pseudo-Anosov maps.

One reason to study minimal stretch factors specifically in nonorientable settings is the following connection to the conjecture of Schinzel and Zassenhaus [28] that asserts the existence of a universal constant $c > 0$ such that for any algebraic integer that is not a root of unity, the largest modulus among its Galois conjugates is bounded from below by $1 + c/d$, where d is the degree of the algebraic integer. Using a result due to Breusch [7], the first author [19] has shown that this conjecture has an equivalent reformulation that compares, for each genus, the minimal spectral radius > 1 among homological actions of orientation-preserving mapping classes with the minimal spectral radius > 1 among homological actions of orientation-reversing mapping classes. For instance, if for all but finitely many genera one can obtain smaller spectral radii by orientation-reversing mapping classes, the conjecture of Schinzel and Zassenhaus is true.

By studying the stretch factors of pseudo-Anosov mapping classes with an orientable invariant foliation, we restrict to a certain class of homological actions; however, for this class, our results seem to suggest that indeed smaller spectral radii can typically be obtained by orientation-reversing mapping classes; compare Conjecture 1.8 with (1-1) below.

Previous results

The value of $\delta(S_g)$ for hyperbolic S_g is only known for $g = 2$; see Cho and Ham [8]. This value is the largest root of $x^4 - x^3 - x^2 - x + 1$, which is approximately 1.72208. More is known about $\delta^+(S_g)$, the minimal stretch factor of orientation-preserving pseudo-Anosov maps on S_g with orientable invariant foliations. The known values are summarized in Table 1.

Initially, the pseudo-Anosov maps realizing the stretch factors in Table 1 were constructed in different ways. The construction is due to Zhirov [36] for $g = 2$, Lanneau and Thiffeault [17] for $g = 3, 4$, Leininger [18] for $g = 5$, Kin and Takasawa [14] and Aaber and Dunfield [1] for $g = 7$, and Hironaka [10] for $g = 8$. Hironaka [10] then showed that all of the examples above except the $g = 7$ example arise from

g	$\delta^+(S_g) \approx$	minimal polynomial of $\delta^+(S_g)$
1	2.61803	$x^2 - 3x + 1$
2	1.72208	$x^4 - x^3 - x^2 - x + 1$
3	1.40127	$x^6 - x^4 - x^3 - x^2 + 1$
4	1.28064	$x^8 - x^5 - x^4 - x^3 + 1$
5	1.17628	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = \frac{x^{12} - x^7 - x^6 - x^5 + 1}{x^2 - x + 1}$
7	1.11548	$x^{14} + x^{13} - x^9 - x^8 - x^7 - x^6 - x^5 + x + 1$
8	1.12876	$x^{16} - x^9 - x^8 - x^7 + 1$

Table 1: The known values of $\delta^+(S_g)$.

the fibration of a single hyperbolic 3-manifold, the mapping torus of the “simplest hyperbolic braid”. The fact that the values in Table 1 are indeed the minimal stretch factors was shown by Laneeau and Thiffeault [17] by a systematic way of narrowing down the set of possible minimal polynomials of the minimal stretch factors. For the larger values of g , their proof is computer-assisted.

Asymptotics

The rough asymptotic behavior of $\delta(S)$ is well understood: Penner [27] showed that $\log \delta(S_g) \sim 1/g$. For other constructions of small stretch factors and asymptotics for different sequences of surfaces, see Bauer [6], McMullen [22], Minakawa [25], Hironaka and Kin [11], Tsai [33], Valdivia [34] and Yazdi [35].

Since the larger root of the polynomial $x^2 - 2x - 1$ is $1 + \sqrt{2}$, Conjectures 1.2 and 1.5 would imply the following conjectures on the exact limits of the normalized minimal stretch factors:

Conjecture 1.7 $\lim_{\substack{g \rightarrow \infty \\ g \text{ even}}} (\delta^+(N_g))^g = (1 + \sqrt{2})^2 = (\text{silver ratio})^2.$

Conjecture 1.8 $\lim_{\substack{g \rightarrow \infty \\ g \text{ odd}}} (\delta_{\text{rev}}^+(S_g))^g = 1 + \sqrt{2} = \text{silver ratio}.$

In order to compare Conjectures 1.7 and 1.8 to the orientation-preserving case, we recall that Hironaka asked in [10, Question 1.12] whether

$$(1-1) \quad \lim_{g \rightarrow \infty} (\delta^+(S_g))^g = \left(\frac{1}{2}(1 + \sqrt{5})\right)^2 = (\text{golden ratio})^2.$$

Since any pseudo-Anosov map on N_{g+1} can be lifted to a pseudo-Anosov map on S_g with the same stretch factor, it is natural that the limit in Conjecture 1.7 is larger than the limit in (1-1). The fact that the limit in Conjecture 1.8 is smaller than the limit in (1-1) is consistent with the fact that nonorientable hyperbolic 3-manifolds can have smaller volume than orientable ones. For example, the smallest-volume noncompact hyperbolic 3-manifold is the Gieseking manifold, a nonorientable manifold; see Adams [2]. Since the stretch factor is related to the volume of the mapping torus — see Kojima and McShane [15] — on a fixed surface one can expect to find orientation-reversing pseudo-Anosov maps with smaller stretch factor than orientation-preserving ones.

Asymptotics along other genus sequences

We expect the limits in Conjectures 1.7 and 1.8 to be different for other genus sequences. For example, we conjecture the following:

Conjecture 1.9 $\liminf_{\substack{g \rightarrow \infty \\ g \text{ odd}}} (\delta^+(N_g))^g > (1 + \sqrt{2})^2 = (\text{silver ratio})^2$.

This conjecture is supported by the following result in [21]. In that paper, we show that if $\delta_{\mathcal{P}}(N_g)$ denotes the minimal stretch factor among pseudo-Anosov mapping classes on N_g obtained from Penner's construction, then the sequence $\delta_{\mathcal{P}}(N_g)$ has exactly two accumulation points as $g \rightarrow \infty$. One accumulation point, $(1 + \sqrt{2})^2$, is the limit for the sequence restricted to even g . The other accumulation point, conjectured to be the largest root of $x^4 - 8x^3 + 13x^2 - 8x + 1$, which is strictly greater than $(1 + \sqrt{2})^2$, is the limit for the sequence restricted to odd g . We expect this dichotomy to be indicative of how the sequence $(\delta^+(N_g))^g$ behaves for odd and even genus sequences, respectively, since so far no pseudo-Anosov mapping class of a nonorientable surface is known to not have a power arising from Penner's construction (compare with Question 1.11 below).

Uniformity of minimizing examples

In the orientation-preserving case, the concrete descriptions of the examples are all very different. For $g = 2$, Zhurov describes the example by the induced homomorphism $\pi_1(S) \rightarrow \pi_1(S)$. Laneeau and Thiffeault [17, Appendix C] describe the same example as a product of the Humphries generators. For $g = 3, 4$, Laneeau and Thiffeault use Rauzy–Veech induction, and for $g = 5$, Leininger uses Thurston's construction. While Hironaka gives a unified construction in [10] using fibered face theory, her work does not give an explicit description of the maps.

In contrast, the descriptions of our examples are explicit and uniform: all of our examples are constructed as a composition of a Dehn twist and a finite-order mapping class. As we will explain shortly, such constructions cannot work in the orientable setting.

We remark that it is also possible to construct the examples in Theorems 1.1 and 1.4 by studying fibrations of certain small volume nonorientable hyperbolic 3-manifolds, although we will not discuss this construction in this paper.

Galois conjugates and Penner's construction

All of our examples have a power that arises from Penner's construction of pseudo-Anosov mapping classes. In sharp contrast, none of the classical minimal stretch factor examples have a power that arises from Penner's construction. This is because these stretch factors have Galois conjugates on the unit circle. However, Shin and the second author showed in [29] that examples with this property do not have a power arising from Penner's construction.

One may wonder what the reason for this discrepancy is. A heuristic reason for why Galois conjugates of small stretch factors *should* lie on the unit circle is that every pseudo-Anosov stretch factor λ is a bi-Perron algebraic unit, that is, a real number larger than 1 whose Galois conjugates lie in the annulus $\lambda^{-1} \leq |z| \leq \lambda$. If λ is close to 1, this annulus is a thin neighborhood of the unit circle, so it seems natural for the Galois conjugates to lie on the unit circle.

However, in Section 4 we will prove the following theorem, which explains why the nonorientable cases are different:

Theorem 1.10 *If f is a pseudo-Anosov map on a nonorientable surface or an orientation-reversing pseudo-Anosov map on an orientable surface, then the stretch factor of f does not have Galois conjugates on the unit circle.*

Penner's conjecture on nonorientable surfaces

Penner asked in [26] whether every pseudo-Anosov map has a power that arises from his construction.¹ This was answered in the negative by Shin and the second author

¹The conjecture that this is true is known colloquially as Penner's conjecture. However, from the writing in [26, page 195], it is unclear whether Penner intended to pose this as a question or a conjecture, or even whether he conjectured the opposite.

in [29] by providing the obstruction mentioned earlier: if the stretch factor has a Galois conjugate on the unit circle, the pseudo-Anosov map cannot have a power arising from Penner's construction.

However, Theorem 1.10 demonstrates that this obstruction is vacuous for nonorientable surfaces and for orientation-reversing maps. Since there are no other known obstructions, it is possible that the answer to Penner's question is in fact "yes" in these settings. Some evidence for this is provided by the fact that all the minimal stretch factor examples we give in Theorems 1.1 and 1.4 have a power arising from Penner's construction. Some evidence against is provided by the failure of the second author in [30, Section 7] to construct certain pseudo-Anosov maps on nonorientable surfaces using Penner's construction.

Question 1.11 *Does every pseudo-Anosov map on a nonorientable surface have a power arising from Penner's construction?*

Question 1.12 *Does every orientation-reversing pseudo-Anosov map on an orientable surface have a power arising from Penner's construction?*

Outline of the paper

In Sections 2 and 3, we construct the examples for Theorems 1.1 and 1.4. This is done by a generalization of the construction we gave for the Arnoux–Yoccoz pseudo-Anosov maps in [20].

In Section 4, we give various properties that the characteristic polynomials of the action on homology have to satisfy for maps on nonorientable surfaces and orientation-reversing maps. We also give the proof of Theorem 1.10 here.

To show that our examples have minimal stretch factor, we follow Lanneau and Thiffeault's approach for orientable surfaces [17; 16]: we run a brute-force search for integral polynomials whose largest root is smaller than our candidate for the minimal stretch factor and hope that we do not find any. Aside from some low-genus cases, this search is computer-assisted. Our code can be found at <https://github.com/b5strbal/polynomial-filtering>.

In Section 5, we describe this polynomial elimination process and prove Theorem 1.1 without computer assistance in the case $g = 3$. This elimination process ends up being significantly cleaner for us than it was for Lanneau and Thiffeault. In their case, the

restrictions on the polynomials alone are not sufficient to rule out all polynomials, so they were left with a few polynomials that needed to be ruled out by studying the possible singularity structures of the pseudo-Anosov maps and by using Lefschetz number arguments. For us, no arguments like these are necessary.

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2 Construction of pseudo-Anosov maps on nonorientable surfaces

In this and the next section, we use Penner's construction to construct pseudo-Anosov mapping classes. We briefly recall Penner's construction below, stating it in a way that works both for orientable and for nonorientable surfaces. For more details, see [26, Section 4] or [30, Section 2].

In Penner's construction, we have a collection of two-sided simple closed curves $C = \{c_1, \dots, c_n\}$ that fill the surface (the complement of the curves is a union of disks and once-punctured disks), that pairwise intersect minimally, and that are *marked inconsistently*. This means that there is a small regular neighborhood $N(c_i)$ for each curve c_i and an orientation of each annulus $N(c_i)$ such that the orientations of $N(c_i)$ and $N(c_j)$ are different at each intersection whenever $i \neq j$. Penner showed that any product of the Dehn twists T_{c_i} is pseudo-Anosov assuming that

- each twist T_{c_i} is right-handed according to the orientation of $N(c_i)$,
- each twist T_{c_i} is used in the product only with positive powers,
- each twist T_{c_i} is used in the product at least once.

Note that if the surface is oriented, then the above conditions in Penner's construction say that the collection of curves is a union of two multicurves Γ_1 and Γ_2 , and the Dehn twists along the curves in Γ_1 are all right-handed, whereas the Dehn twists along the curves in Γ_2 are all left-handed with respect to the orientation of the surface.

We will present the construction of our examples as follows. First we define the rotationally symmetric graphs that will be the intersection graphs of the collections of curves. Then we describe the rotationally symmetric surfaces and curves on these surfaces whose intersection matrices realize the given graphs. Finally, we define our mapping classes as a composition of a Dehn twist and a rotation.

2.1 The graphs

Let k and n be integers of different parity such that $n \geq 3$ and $1 \leq k \leq n-1$. Let $G_{n,k}$ be the graph whose vertices are the vertices of a regular n -gon and every vertex v is connected to the k vertices that are the farthest away from v in the cyclic order of the vertices.

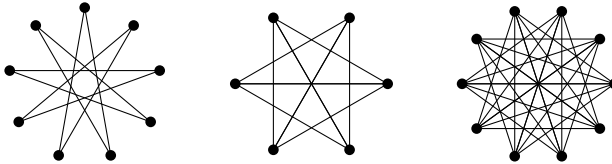


Figure 1: The graphs $G_{9,2}$, $G_{6,3}$ and $G_{10,5}$.

2.2 The surfaces

For each $G_{n,k}$, we will construct a nonorientable surface $\Sigma_{n,k}$ that contains a collection of curves with intersection graph $G_{n,k}$. To construct $\Sigma_{n,k}$, start with a disk with one crosscap. By this, we mean that we cut a smaller disk out of the disk and identify the antipodal points of the boundary of the small disk. We indicate this identification with a cross inside the small disk; see Figure 2. The resulting surface is homeomorphic to the Möbius strip.

Next, we consider $2n$ disjoint intervals on the boundary of the disk and label the intervals with integers from 1 to n so that each label is used exactly twice. In the cyclic order, the labels are $1, s, 2, s+1, \dots, n, s+n$, where $s = \frac{1}{2}(n+k+3)$ and all labels are understood modulo n .

For each label, the corresponding two intervals are connected by a twisted strip, as on Figure 2.

Lemma 2.1 *The Euler characteristic of $\Sigma_{n,k}$ is $-n$.*

Proof The disk with a crosscap has zero Euler characteristic (it is homeomorphic to a Möbius strip), and each attached twisted strip has contribution -1 . \square

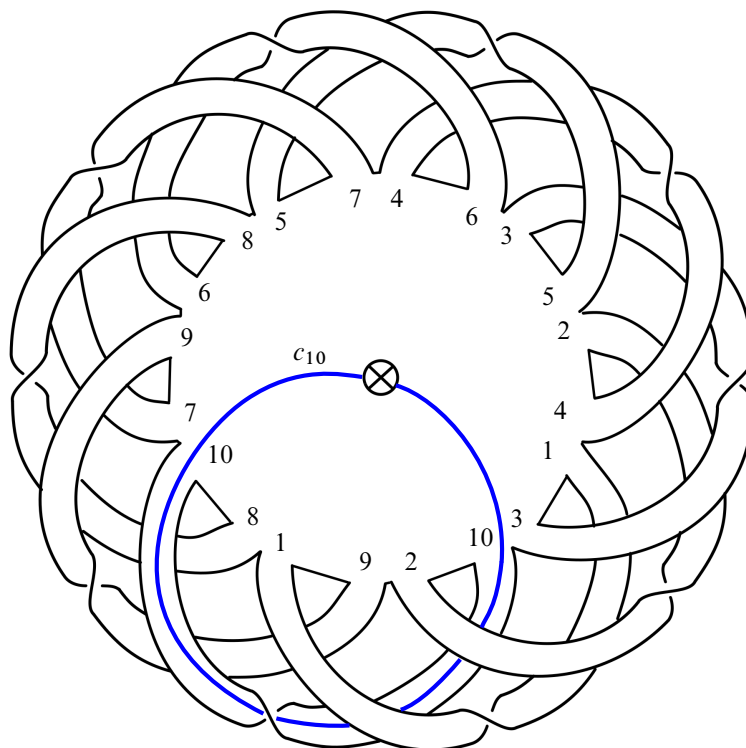


Figure 2: The surface $\Sigma_{10,5}$ and the curve c_{10} .

Lemma 2.2 *The number of boundary components of $\Sigma_{n,k}$ is $\gcd(n, k)$.*

Proof We will show that the number of boundary components of $\Sigma_{n,k}$ is the same as the number of orbits of the dynamical system $x \mapsto x + n - k$ in the group $\mathbb{Z}/2n\mathbb{Z}$. The number of such orbits is $\gcd(n - k, 2n) = \gcd(k, n)$, since $n - k$ is odd.

To prove our claim, we identify $\mathbb{Z}/2n\mathbb{Z}$ with the $2n$ intervals in the cyclic order. We claim that the right endpoint of the interval at position i lies on the same boundary component as the right endpoint of the interval at position $i + n - k$. One can see this by induction. In the case $k = n - 1$, the cyclic order of labels is $1, 1, \dots, n, n$, so the twisted strips identify the right endpoint of every interval with the right endpoint of the next interval. When $k = n - 3$, the cyclic order is $1, n, 2, 1, \dots, n, n - 1$, in which case every third right endpoint is on the same boundary component, and so on. \square

Proposition 2.3 *The surface $\Sigma_{n,k}$ is homeomorphic to the nonorientable surface of genus $n - \gcd(k, n) + 2$ with $\gcd(k, n)$ boundary components.*

Proof The Euler characteristic of the nonorientable surface of genus g with b boundary components is $2 - g - b$. By Lemmas 2.1 and 2.2, we obtain the equation $2 - g - \gcd(k, n) = -n$. Rearranging, we obtain $g = n - \gcd(k, n) + 2$. \square

2.3 The curves

We construct a two-sided curve c_i for each label $i = 1, \dots, n$ as follows. Each curve consists of two parts. One part of each curve is the core of the strip corresponding to the label. The other part is an arc inside the disk that passes through the crosscap and connects the corresponding two intervals. The curve c_{10} is shown on Figure 2.

Note that every pair of curves intersects either once or not at all. The curves c_i and c_j are disjoint if and only if the two i labels and the two j labels *link* in the cyclic order. In other words, if the two i labels separate the two j labels.

Lemma 2.4 *The intersection graph of the curves c_i on $\Sigma_{n,k}$ is $G_{n,k}$.*

Proof We prove the lemma by induction. If $k = n - 1$, then $s = 1$, so the cyclic order is $1, 1, 2, 2, \dots, n, n$. Since no two labels link, all pairs of curves intersect and the intersection graph is the complete graph $G_{n,n-1}$.

Now suppose k is decreased by 2. Then s is decreased by 1, and we obtain the cyclic order $1, n, 2, 1, 3, 2, \dots, n, n - 1$. As a consequence, 1 becomes linked with 2 and n . Hence the intersection graph is indeed $G_{n,k}$.

It is easy to see that every time k is decreased by two each label is linked with two more labels, hence the intersection graph is always $G_{n,k}$. \square

Lemma 2.5 *The curves c_i can be marked so that all intersections are inconsistent.*

Proof Choose markings for the c_i which are invariant under the rotational symmetry; see Figure 3. The marking of the curves is indicated by the coloring as follows. Consider the orientable surface obtained by removing the crosscap and cutting the strips attached to the disk in the middle. Choose an orientation of this surface. Then color the arcs composing the curves using red and blue depending on whether the orientation of the neighborhood of the curve matches the orientation of the surface or not. Note that the color of a curve changes when it goes through the crosscap or the middle of a strip.

Since blue and red meet at every intersection, the marking is inconsistent. \square

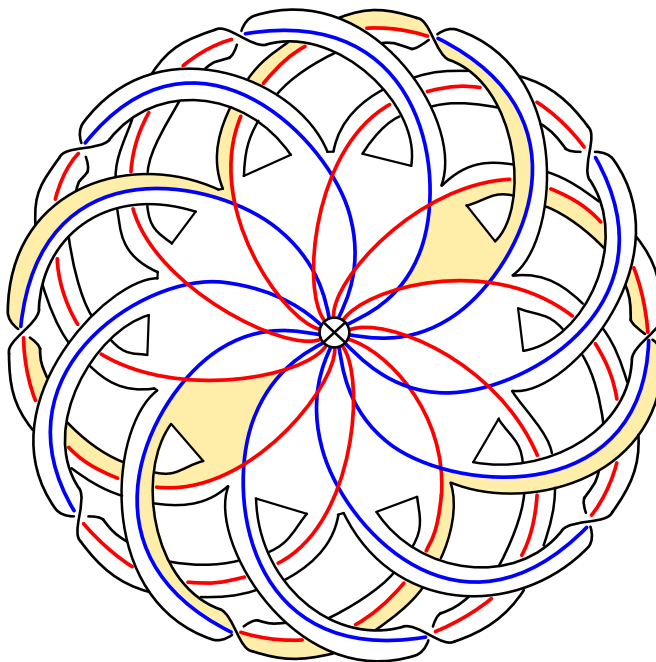


Figure 3: A collection of filling inconsistently marked curves.

2.4 The mapping classes

Denote by r the rotation of $\Sigma_{n,k}$ by one click in the clockwise direction. Define the mapping class

$$f_{n,k} = r \circ T_{c_1},$$

where T_{c_1} is a Dehn twist about the curve c_1 . (There are two possible directions for the Dehn twist, but either choice works for our purposes.) Note that

$$f_{n,k}^n = T_{c_n} \circ \cdots \circ T_{c_1},$$

so $f_{n,k}^n$ arises from Penner’s construction. In particular, $f_{n,k}^n$ is pseudo-Anosov and so is $f_{n,k}$.

We remark that for $k = n - 1$, the mapping class $f_{n,k}$ coincides with the nonorientable Arnoux–Yoccoz mapping class h_{n-1} , described as a product of a Dehn twist and a finite-order mapping class by the authors in [20].

Proposition 2.6 *The stretch factor of $f_{n,k}$ is the largest root of $x^n - x^{n-r} - x^{n-r-1} - \cdots - x^{r+1} - x^r - 1$, where $r = \frac{1}{2}(n - k + 1)$.*

Proof To compute the stretch factor, we use Penner’s approach in the section titled “An upper bound by example” in [27]. Penner constructed an invariant bigon track by smoothing out the intersections of the curves c_i . Each c_i defines a characteristic measure μ_i on this bigon track, defined by assigning 1 to the branches traversed by c_i and zero to the rest. The cone generated by the μ_i is invariant under both T_{c_1} and r , hence it contains the unstable foliation, and the stretch factor is given by the largest eigenvalue of the action of $r \circ T_{c_1}$ on this cone. The rotation r acts by a permutation matrix and the matrix corresponding to T_{c_1} is the sum of the identity matrix and the matrix obtained from the intersection matrix $i(C, C)$ by zeroing out all rows except the first row. The product of these two matrices takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

This particular matrix belongs to $f_{10,5}$.

This matrix is the companion matrix of the polynomial in the statement of the proposition. Hence the characteristic polynomial of this matrix is indeed that polynomial. \square

Our next goal is to determine the singularity structure of the mapping classes $f_{n,k}$. For this, first we need a lemma.

Consider the complementary regions of the curves $\{c_1, \dots, c_n\}$. There are two types of regions, depending on whether a region contains a boundary component of $\Sigma_{n,k}$ (type 1) or not (type 2). A region of type 1 is an annulus that is bounded by a boundary component β of $\Sigma_{n,k}$ on one side and by a polygonal path consisting of arcs of the curves c_i on the other side. The shaded region on Figure 3 illustrates a region of type 1.

Lemma 2.7 *The length of these polygonal paths is $4n/\text{gcd}(k, n)$.*

Proof This follows from the observation that every point in the orbit in $\mathbb{Z}/2n\mathbb{Z}$ corresponding to the boundary component β (see the proof of Lemma 2.2) has two

associated arcs. Since the number of orbits is $\gcd(k, n)$, the length of each orbit is $2n/\gcd(k, n)$, and hence the length of the polygonal path is twice this quantity. \square

Proposition 2.8 *The pseudo-Anosov mapping class $f_{n,k}$ has $\gcd(k, n)$ singularities, one for each boundary component. The number of prongs of each singularity is $2n/\gcd(k, n)$.*

Proof Each complementary region of the curves $\{c_1, \dots, c_n\}$ contains either one singularity or none. The number of prongs of a singularity equals the number of cusps of the bigon track obtained by the smoothing process that are contained in the same region as the singularity. If the number of cusps is 2, then the region does not contain a singularity. If the number of cusps is $k > 2$, then it contains a k -pronged singularity.

g	n	k	$\lambda(f_{n,k})$	minimal polynomial	singularity type
4*	3	2	1.83929	$x^3 - x^2 - x - 1$	(6)
5*	6	3	1.51288	$x^4 - x^3 - x^2 + x - 1$	(4, 4, 4)
6*	5	2	1.42911	$x^5 - x^3 - x^2 - 1$	(10)
7*	10	5	1.42198	$x^6 - x^5 - x^3 + x - 1$	(4, 4, 4, 4, 4)
8*	7	2	1.28845	$x^7 - x^4 - x^3 - 1$	(14)
9	8	3	1.35680	$x^8 - x^5 - x^4 - x^3 - 1$	(16)
10*	9	2	1.21728	$x^9 - x^5 - x^4 - 1$	(18)
11	12	3	1.22262	$\frac{x^{12} - x^7 - x^6 - x^5 - 1}{x^2 + x + 1}$	(8, 8, 8)
12*	11	2	1.17429	$x^{11} - x^6 - x^5 - 1$	(22)
13	22	11	1.27635	$x^{12} - x^{11} - x^6 + x - 1$	(4 ¹¹)
14*	13	2	1.14551	$x^{13} - x^7 - x^6 - 1$	(26)
15	14	3	1.18750	$x^{14} - x^8 - x^7 - x^6 - 1$	(28)
16*	15	2	1.12488	$x^{17} - x^9 - x^8 - 1$	(30)
17	18	3	1.14259	$\frac{x^{18} - x^{10} - x^9 - x^8 - 1}{x^2 + x + 1}$	(12, 12, 12)
18*	17	2	1.10938	$x^{19} - x^{10} - x^9 - 1$	(34)
19	18	5	1.20514	$x^{18} - x^{11} - x^{10} - x^9 - x^8 - x^7 - 1$	(36)
20*	19	2	1.09730	$x^{23} - x^{12} - x^{11} - 1$	(38)

Table 2

Regions of type 2 are rectangles (bounded by four subarcs of the curves c_i), and hence contain two cusps. So they do not correspond to singularities.

The lengths of the polygonal paths bounding regions of type 1 are $4n/\gcd(k, n)$ by Lemma 2.7, so the number of cusps in these regions is $2n/\gcd(k, n)$. Therefore the singularities have that many prongs. By Lemma 2.2, the number of such regions is $\gcd(k, n)$, so that is also the number of the singularities. \square

As a corollary of Propositions 2.6, 2.8 and 2.3, we have the following:

Corollary 2.9 *There exist pseudo-Anosov mapping classes with an orientable invariant foliation on the surfaces N_g with the data given in Table 2. All of these examples belong to the family $f_{n,k}$ for the n and k shown in Table 2. (4^{11} means that there are 11 singularities with 4 prongs.)*

In each genus, the family $f_{n,k}$ contains several examples. In Table 2, we have listed only the example with the smallest stretch factor. In the starred cases, we will be able to certify that the given stretch factors are not only minimal in the family $f_{n,k}$ but among all pseudo-Anosov maps with an orientable invariant foliation.

3 Orientation-reversing pseudo-Anosov mapping classes on odd genus surfaces

In this section, we construct an orientation-reversing pseudo-Anosov mapping class with small stretch factor on every odd genus orientable surface. The construction is analogous to the construction in the previous section, but simpler. As in the previous section, we separate the construction of the surfaces, the curves and finally the mapping classes.

3.1 The surfaces

For every $k \geq 2$, consider the surface Σ_k obtained by chaining together $2k$ annuli in a cycle as on Figure 4.

Proposition 3.1 *The number of boundary components of Σ_k is 4 if k is even and 2 if k is odd.*

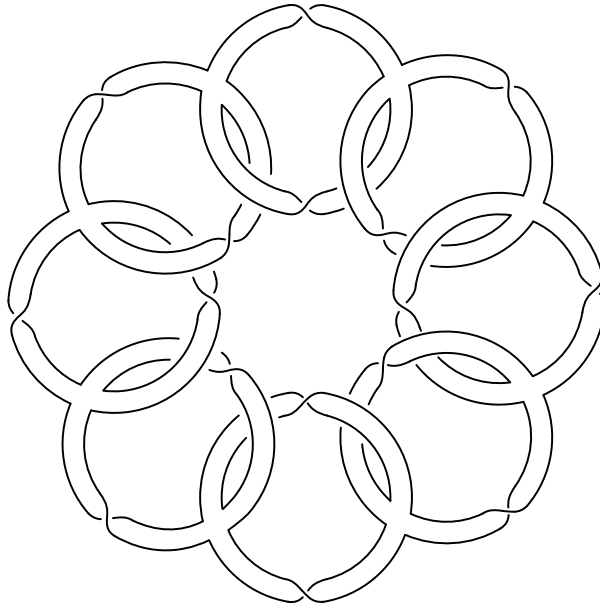


Figure 4: The surface Σ_k .

Proof The boundary of Σ_k is composed of $8k$ arcs, 4 arcs for each annulus. Our goal is to determine which of them belong to the same boundary component.

Denote by r the rotation of Σ_k by one click. By tracing the boundary, one can see that every boundary point x lies on the same boundary component as $r^4(x)$. Moreover, the path between x and $r^4(x)$ traverses each of the 4 types of arcs exactly once. Therefore it suffices to pick any boundary point x and determine into how many equivalence classes the set $\{x, r(x), \dots, r^{2k-1}(x)\}$ falls apart. The number of such equivalence classes is 4 if k is even and 2 if k is odd. \square

Proposition 3.2 *The surface Σ_k is homeomorphic to $S_{k-1,4}$ if k is even and $S_{k,2}$ if k is odd.*

Proof We have $\chi(\Sigma_k) = 2k$. From the equation $\chi = 2 - 2g - b$, where g is the genus and b is the number of boundary components, it follows that $g = k + 1 - \frac{1}{2}b$. The statement now follows from Proposition 3.1. \square

As a consequence, the construction only produces odd genus examples.

3.2 The curves

From now on, suppose that k is even. Consider the set $C = \{c_1, \dots, c_{2k}\}$ of core curves of the $2k$ annuli. Our numbering will differ from the standard cyclic numbering; we will explain this shortly. As in Section 2.3, any rotationally symmetric marking of the curves is an inconsistent marking.

The intersection graph of C is a cycle of length $2k$. We draw this cycle as on Figure 5: the vertices are the vertices of a regular polygon and every vertex is connected to the two vertices that are the second furthest in the cyclic order. We number the curves according to the cyclic orientation induced by this picture.

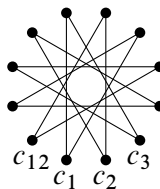


Figure 5: Our unusual way of numbering the curves. For example, the curve c_1 intersects c_k and c_{k+2} , not c_2 and c_{2k} .

3.3 The mapping classes

Denote by r the rotation of Σ_k (see Figure 4) by one click in the clockwise direction. Since c_i and $r(c_i)$ intersect for all i , the rotation r induces a rotation of the cycle on Figure 5 by $k - 1$ clicks. So r^{k-1} rotates the cycle by $(k - 1)^2 = k^2 - 2k + 1$ clicks, which is congruent to 1 modulo $2k$ if k is even. Therefore r^{k-1} induces rotating the cycle on Figure 5 by one click (in the clockwise direction, assuming that we have chosen the numbering of the curves accordingly). In particular, we have $r^{k-1}(c_{i+1}) = c_i$.

We are now ready to define the mapping class:

$$\psi_k = r^{k-1} \circ T_{c_1}.$$

Note that

$$\psi_k^{2k} = T_{c_{2k}} \circ \dots \circ T_{c_1},$$

so ψ_k^{2k} arises from Penner’s construction. In particular, ψ_k^{2k} is pseudo-Anosov and so is ψ_k . Note that while ψ_k^{2k} is orientation-preserving, ψ_k is orientation-reversing. This follows from the observation that T_{c_1} is orientation-preserving and r is orientation-reversing.

Proposition 3.3 *The stretch factor of ψ_k is the largest root of $x^{2k} - x^{k+1} - x^{k-1} - 1$.*

Proof The proof is similar to the proof of Proposition 2.6. We have

$$i(C, C) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where M is the matrix of the action of ψ_k on the cone of measures (the product of a permutation matrix and the sum of the identity matrix and the first row of $i(C, C)$). The matrices above illustrate the case $k = 4$. The matrix M is the companion matrix of the polynomial in the proposition. □

Proposition 3.4 *The pseudo-Anosov mapping class ψ_k has four k -pronged singularities.*

Proof By Proposition 3.1 and its proof, each of the four boundary components of Σ_k consists of $2k$ arcs if k is even. There is a prong for every second corner of the boundary path, therefore there are k prongs for each singularity. □

Corollary 3.5 *There exist orientation-reversing pseudo-Anosov mapping classes with orientable invariant foliations on the surfaces S_g with the data below. All of these examples belong to the family ψ_k for the k shown in the table:*

g	k	$\lambda(\psi_k)$	largest root of	singularity type
1	2	1.61803	$x^2 - x - 1 = \frac{x^4 - x^3 - x - 1}{x^2 + 1}$	no singularities
3	4	1.25207	$x^8 - x^5 - x^3 - 1$	(4, 4, 4, 4)
5	6	1.15973	$x^{12} - x^7 - x^5 - 1$	(6, 6, 6, 6)
7	8	1.11707	$x^{16} - x^9 - x^7 - 1$	(8, 8, 8, 8)
9	10	1.09244	$x^{20} - x^{11} - x^9 - 1$	(10, 10, 10, 10)
11	12	1.07638	$x^{24} - x^{13} - x^{11} - 1$	(12, 12, 12, 12)

Proof The statement follows from Propositions 3.3, 3.4 and 3.2. The reason we have no singularities in the genus 1 case is that by Proposition 3.4 the “singularities” have two prongs, so they are not actually singularities. \square

We remark that in genus 1, there is an alternative, simpler construction that yields the same stretch factor. Consider the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with determinant -1 . The corresponding linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ maps \mathbb{Z}^2 to \mathbb{Z}^2 , hence it descends to an Anosov diffeomorphism f of the torus $\mathbb{R}^2/\mathbb{Z}^2$. Its stretch factor is the largest root of $x^2 - x - 1$, the characteristic polynomial of M .

4 Restrictions on polynomials

Pseudo-Anosov stretch factors are roots of integral polynomials. The properties of these integral polynomials are similar, but slightly different depending on whether a pseudo-Anosov mapping class is an orientation-preserving or orientation-reversing mapping class on an orientable surface or a mapping class on a nonorientable surface. In this section, we discuss these properties for nonorientable surfaces and orientation-reversing mapping classes.

A polynomial $p(x)$ of degree n is called *reciprocal* if $p(x) = \pm x^n p(x^{-1})$, in other words, when its coefficients are the same in reverse order up to sign. Analogously, we define $p(x)$ to be *skew-reciprocal* if $p(x) = \pm x^n p(-x^{-1})$.

Proposition 4.1 *Let $\psi: N_g \rightarrow N_g$ be a pseudo-Anosov map with a transversely orientable invariant foliation on the closed nonorientable surface N_g of genus g . Then its stretch factor λ is a root of a (not necessarily irreducible) polynomial $p(x) \in \mathbb{Z}[x]$ with the following properties:*

- (1) $\deg(p) = g - 1$.
- (2) $p(x)$ is monic and its constant coefficient is ± 1 .
- (3) The absolute values of the roots of $p(x)$ other than λ lie in the open interval (λ^{-1}, λ) . In particular, $p(x)$ is not reciprocal or skew-reciprocal.
- (4) $p(x)$ is reciprocal mod 2.

Proof Note that exactly one of the stable and unstable foliations is transversely orientable (otherwise the surface itself would be orientable). We will assume that it is the stable foliation, otherwise we replace ψ by its inverse.

Consider the action $\psi^*: H^1(N_g; \mathbb{R}) \rightarrow H^1(N_g; \mathbb{R})$ defined by pullback on cohomology with real coefficients. Since the stable foliation is transversely orientable, it is represented by a closed real 1-form, that is, an element of $H^1(N_g; \mathbb{R})$. The stable foliation \mathcal{F}^s is the one whose leaves are contracting and hence the surface is expanding in the transverse direction. Therefore the measure of a transverse arc in the pullback $\psi^*(\mathcal{F}^s)$ is λ times its measure in \mathcal{F}^s . Hence \mathcal{F}^s is an eigenvector of the map ψ^* with eigenvalue λ or $-\lambda$.

Let $p(x)$ be the characteristic polynomial of ψ^* . Note that $\dim(H^1(N_g, \mathbb{R})) = g - 1$, hence $\deg(p) = g - 1$. This proves (1).

The polynomial $p(x)$ has integral coefficients, since ψ^* restricts to an action $H^1(N_g; \mathbb{Z}) \rightarrow H^1(N_g; \mathbb{Z})$. This restriction is invertible, since the action of $\text{Mod}(N_g)$ on $H^1(N_g; \mathbb{Z})$ is a group representation, so the determinant of ψ^* is ± 1 . Therefore the constant coefficient of $p(x)$ is ± 1 . Also, as a characteristic polynomial, $p(x)$ is monic. This proves (2).

It is a standard fact from the theory of orientation-preserving pseudo-Anosov mapping classes on orientable surfaces that stretch factors are strictly maximal among their Galois conjugates. Moreover, if the invariant foliations are orientable, the spectral radius of the action induced on the first homology equals the stretch factor, and every other eigenvalue is strictly smaller; see for example [23, Theorem 5.3(1)]. Applying this result to the orientation-preserving lift $\tilde{\psi}: S_{g-1} \rightarrow S_{g-1}$ of ψ to the orientable double cover S_{g-1} of N_g , we obtain that any root λ' of $p(x)$ other than $\pm\lambda$ satisfies $|\lambda'| < |\lambda|$. Applying the same theorem for $\tilde{\psi}^{-1}$, we conclude that any root λ' of $p(x)$ other than $\pm\lambda^{-1}$ satisfies $|\lambda'^{-1}| < |\lambda|$. Therefore absolute values of the roots of $p(x)$ other than $\pm\lambda$ and possibly $\pm\lambda^{-1}$ lie in the open interval (λ^{-1}, λ) . However, it was shown in the proof of [31, Proposition 2.3] that if λ or $-\lambda$ is a root of $p(x)$, then λ^{-1} and $-\lambda^{-1}$ cannot be roots of $p(x)$, hence mentioning the edge case $\pm\lambda^{-1}$ in the previous sentence is not necessary.

If $p(x)$ were reciprocal, then λ and λ^{-1} or $-\lambda$ and $-\lambda^{-1}$ would have to be roots. If it were skew-reciprocal, then λ and $-\lambda^{-1}$ or $-\lambda$ and λ^{-1} would have to be roots. As we have just shown, these scenarios are impossible, because $\pm\lambda^{-1}$ is not a root of $p(x)$. This proves (3).

The fact that $p(x)$ is reciprocal mod 2 was shown in [31, Proposition 4.2]. This justifies (4).

Finally, notice that we have not guaranteed that λ is a root of $p(x)$ — we have only shown that either λ or $-\lambda$ is a root. If it is $-\lambda$, then the polynomial $p(-x)$ or $-p(-x)$ satisfies all the required properties. \square

We call a $2n \times 2n$ matrix A *antisymplectic* if the corresponding linear transformation sends the standard symplectic form on \mathbb{R}^{2n} to its negative. Formally, this can be written as

$$AJA^T = -J$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $n \times n$ identity matrix.

Proposition 4.2 *The characteristic polynomial $p(x)$ of a $2n \times 2n$ antisymplectic matrix is skew-reciprocal.*

Proof Let A be a $2n \times 2n$ antisymplectic matrix. Since A^2 is symplectic, we have $\det(A^2) = 1$ and $\det(A) = \pm 1$. Since $\det(J) = 1$, we have

$$\begin{aligned} p(x) &= \det(A - xI) = \det(AJ - xJ) = \det(AJ + xAJA^T) \\ &= \det(A) \det(J) \det(I + xA^T) = \pm \det(I + xA) \\ &= \pm x^{2n} \det(A + x^{-1}I) = \pm x^{2n} p(-x^{-1}), \end{aligned}$$

hence $p(x)$ is skew-reciprocal. \square

The proof above is a straightforward modification of the standard proof of the fact that the characteristic polynomials of symplectic matrices are reciprocal.

Proposition 4.3 *Let $\psi: S_g \rightarrow S_g$ be an orientation-reversing pseudo-Anosov map with transversely orientable invariant foliations. Then its stretch factor λ is a root of a (not necessarily irreducible) polynomial $p(x) \in \mathbb{Z}[x]$ with the following properties:*

- (1) $\deg(p) = 2g$.
- (2) $p(x)$ is monic and its constant coefficient is $(-1)^g$.
- (3) $p(x) = (-1)^g x^{2g} p(-x^{-1})$.
- (4) $p(-\lambda^{-1}) = 0$.
- (5) *The absolute values of the roots of $p(x)$ other than λ and $-\lambda^{-1}$ lie in the open interval (λ^{-1}, λ) .*

Proof Let $p(x)$ be the characteristic polynomial of $\psi_*: H_1(S_g) \rightarrow H_1(S_g)$. Clearly, (1) holds. Similarly to the proof of Proposition 4.1, we obtain (5) by a reduction to the known statement for orientation-preserving pseudo-Anosov mapping classes. This time, we directly obtain (5) by applying [23, Theorem 5.3] to the square of ψ .

An orientation-reversing homeomorphism sends the intersection form on $H_1(S_g)$ to its negative. Proposition 4.2 implies that $p(x) = \pm x^{2g} p(-x^{-1})$. To decide which sign is right, we only need to compute the sign of the constant coefficient of $p(x)$. If the constant coefficient is 1, then the sign is positive. If the constant coefficient is -1 , then the sign is negative. To put this in another way, we have

$$(4-1) \quad p(x) = p(0)x^{2g} p(-x^{-1}).$$

For orientation-preserving homeomorphisms, the action on homology is symplectic, hence its determinant is $+1$. It follows that, for fixed g , the determinant is either $+1$ for all orientation-reversing homeomorphisms of S_g or -1 for all orientation-reversing homeomorphisms of S_g . It is sufficient to check only one homeomorphism to decide which one. For example, consider the reflection $i: S_g \rightarrow S_g$ about the plane containing the curves b_i on Figure 6.

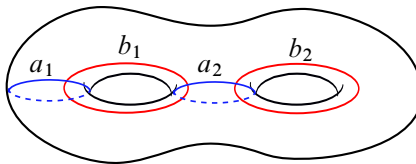


Figure 6: The standard homology basis for S_2 .

The curves $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ form a homology basis. We have $i(b_i) = b_i$ and $i(a_i) = -a_i$ for all i . So the matrix of A is a diagonal matrix whose diagonal contains 1's and -1 's, g of each. Hence the determinant is $(-1)^g$ and we have shown (2).

Item (3) follows from (4-1) and the fact $p(0) = (-1)^g$ we have just shown.

Either λ or $-\lambda$ is a root of $p(x)$. If it is $-\lambda$, we may replace $p(x)$ with $p(-x)$; the previously proven properties remain true. Finally, (4) follows from (3) by setting $x = \lambda$. □

We are now ready to prove Theorem 1.10. We emphasize that, unlike in the previous propositions, in this theorem we are not assuming that the surface is closed or that the pseudo-Anosov mapping class has a transversely orientable invariant foliation.

Proof of Theorem 1.10 An irreducible polynomial $p(x) \in \mathbb{Z}[x]$ with a (complex) root α on the unit circle is reciprocal. This is because then α^{-1} is also a root of $p(x)$, therefore α is a root of the polynomial $x^d p(x^{-1})$, where d is the degree of $p(x)$. But the minimal polynomial is unique up to constant factor, so $x^d p(x^{-1}) = \pm p(x)$. Hence $p(x)$ is indeed reciprocal. So if a stretch factor λ has a Galois conjugate on the unit circle, then the minimal polynomial of λ is reciprocal and λ^{-1} is also a root of the minimal polynomial.

However, by [31, Proposition 2.3], λ and λ^{-1} are not Galois conjugates if λ is a stretch factor of a pseudo-Anosov map (possibly with no orientable invariant foliations) on a nonorientable surface (possibly with punctures). This completes the proof in the case when the pseudo-Anosov map is supported on a nonorientable surface.

We now prove the orientation-reversing case. If our surface is closed, then, by Proposition 4.3, λ and λ^{-1} are not Galois conjugates if λ is a stretch factor of an orientation-reversing pseudo-Anosov map with orientable invariant foliations. If the foliations are not orientable, we can lift the map to the orientation double cover of the foliations to obtain an orientation-reversing pseudo-Anosov map with orientable invariant foliations and with the same stretch factor. Therefore λ and λ^{-1} are not Galois conjugates in this case, either.

If our surface has punctures, then we can fill in the punctures after making the foliations orientable to obtain a pseudo-Anosov map with the same stretch factor on a closed surface, reducing to the closed case discussed in the previous paragraph. This completes the proof in the case when the pseudo-Anosov map is orientation-reversing. \square

5 Elimination of polynomials

In this section we first prove bounds on the sum of k^{th} powers of roots of a polynomial when the absolute values of the roots are bounded by some $r > 1$. These bounds are improved versions of Lemma A.1 of [17], using the special properties of the polynomials in Propositions 4.1 and 4.3.

Then we describe how we use this lemma and Propositions 4.1 and 4.3 in order to systematically narrow down the set of possible minimal polynomials of the minimal stretch factors.

5.1 Power sum bounds

We begin by proving two elementary lemmas.

Lemma 5.1 Suppose $r > 1$ and $r^{-1} \leq a_1, \dots, a_d \leq r$ are positive real numbers such that $a_1 \cdots a_d = 1$. Then

$$\sum_{i=1}^d a_i \leq \begin{cases} n(r + r^{-1}) & \text{if } d = 2n \text{ is even,} \\ n(r + r^{-1}) + 1 & \text{if } d = 2n + 1 \text{ is odd.} \end{cases}$$

Proof The function $x \mapsto x + x^{-1}$ is increasing on the interval $x \geq 1$. So if there are $i \neq j$ such that $r^{-1} < a_i, a_j < r$, then we can increase the sum by moving a_i and a_j away from each other while keeping their product unchanged, until at least one of them is r^{-1} or r . After every such operation, the number of a_i that are equal to r^{-1} or r increases. So eventually we get to a point where at most one a_i is not r^{-1} or r . When $d = 2n$, no such a_i can exist, and exactly half of the a_i equal r , the other half r^{-1} , otherwise their product would not be 1. When $d = 2n + 1$, exactly one such a_i exists, it equals 1 and exactly half of the remaining a_i equal r , the other half r^{-1} . \square

Lemma 5.2 Suppose $r > 1$ and a_1, \dots, a_d are positive real numbers such that $r^{-1} \leq a_1 \leq \dots \leq a_d \leq r$ and $a_1 \cdots a_d = 1$ and $a_1 \geq a_d^{-1}$. Then

$$a_d - \sum_{i=1}^{d-1} a_i \geq \begin{cases} \min\{2 - 2n, -(n - 2)r - nr^{-1}\} & \text{if } d = 2n \text{ is odd,} \\ \min\{1 - 2n, -(n - 2)r - 1 - nr^{-1}\} & \text{if } d = 2n + 1 \text{ is odd.} \end{cases}$$

Moreover, the inequalities are strict if $a_1 > a_d^{-1}$.

Proof Similarly to the proof of Lemma 5.1, our approach is to change the numbers a_1, \dots, a_{d-1} to increase $\sum_{i=1}^{d-1} a_i$ as much as possible while keeping the hypotheses true. Whenever there are $i \neq j$ such that $a_d^{-1} < a_i, a_j < a_d$, we push a_i and a_j apart until at least one of them equals a_d^{-1} or a_d . The end result is the same as before, so we have

$$a_d - \sum_{i=1}^{d-1} a_i \geq a_d - ((n - 1)a_d + na_d^{-1}) = -(n - 2)a_d - na_d^{-1}$$

when $d = 2n$ and

$$a_d - \sum_{i=1}^{d-1} a_i \geq a_d - ((n - 1)a_d + 1 + na_d^{-1}) = -(n - 2)a_d - 1 - na_d^{-1}$$

when $d = 2n + 1$. Since the function $x \mapsto -(n - 2)x - nx^{-1}$ is concave, its minimum on the interval $[1, r]$ is taken at one of the endpoints.

The inequalities in the case $a_1 > a_d^{-1}$ are strict, since in the optimal distribution there has to be an a_i that takes the value a_d^{-1} . □

Now we apply Lemmas 5.1 and 5.2 for roots of polynomials.

Corollary 5.3 *Suppose $P(x)$ is a monic polynomial of degree d with constant coefficient ± 1 . Let z_1, \dots, z_d be the roots of $P(x)$ and let*

$$p_k = z_1^k + \dots + z_d^k$$

be the k^{th} power sum of the roots.

Suppose there is a root $\lambda > 1$ such that all the other roots have absolute values in the interval $[\lambda^{-1}, \lambda]$. For any $r > \lambda$, we have

$$\min\{2 - 2n, -(n - 2)r^k - nr^{-k}\} \leq p_k \leq n(r^k + r^{-k})$$

if $d = 2n$ is even and

$$\min\{1 - 2n, -(n - 2)r^k - 1 - nr^{-k}\} \leq p_k \leq n(r^k + r^{-k}) + 1$$

if $d = 2n + 1$ is odd.

Moreover, strict inequality holds in the lower bound when no eigenvalue equals λ^{-1} .

Proof Let z_1, \dots, z_d be the roots of $P(x)$ and let $a_i = |z_i|^k$ for every i . Note that $a_1 \cdots a_d = 1$ and $r^{-k} \leq a_1, \dots, a_d \leq r^k$. Assuming $a_1 \leq \dots \leq a_d$, we have $a_d = \lambda^k$. Since

$$a_d - \sum_{i=1}^{d-1} a_i \leq p_k = z_1^k + \dots + z_{d-1}^k + \lambda^k \leq \sum_{i=1}^d a_i,$$

the bounds follow from Lemmas 5.1 and 5.2 □

5.2 Newton’s formulas

In this section, we recall Newton’s formulas that relate power sums of the roots to the coefficients of the polynomial.

We will use the notation

$$P(x) = x^d - c_1x^{d-1} - \dots - c_{d-1}x \pm 1$$

for the coefficients of monic polynomials of degree d . As in the statement of Corollary 5.3, we denote by p_k the k^{th} power sum of the roots of $P(x)$.

Newton's formulas relating power sums and symmetric polynomials can be stated either as

$$(5-1) \quad p_k = -c_1 p_{k-1} - c_2 p_{k-2} - \cdots - c_{k-1} p_1 - k c_k$$

or as

$$(5-2) \quad c_k = \frac{-c_1 p_{k-1} - c_2 p_{k-2} - \cdots - c_{k-1} p_1 - p_k}{k}$$

for all $1 \leq k \leq d-1$.

As Lanneau and Thiffeault point out in Section A.1 of [17], it is more computationally efficient to bound the power sums p_i and use Newton's formulas to compute the coefficients c_k from the p_i than to bound the coefficients directly. This is because many scenarios get ruled out just because the numerator in (5-2) is not divisible by k .

5.3 The polynomial elimination algorithm

We give a lower bound on the minimal stretch factor $\delta^+(N_g)$ by a systematic elimination of polynomials. We describe this process below. In order to illustrate the effect of each step in the algorithm, we give the number of candidate polynomials left after each step when $g = 12$ (when the degree is 11).

Algorithm 5.4 Let $d \geq 4$, $g = d - 1$ and $r > 1$ be such that $\delta^+(N_g) < r$. Perform the following steps in order to obtain a small set of polynomials of degree g that include one polynomial whose root is $\delta^+(N_g)$:

- (1) Compute the possible values of p_1, \dots, p_{d-1} using the bounds of Corollary 5.3. For $d = 11$, the total number of combinations is $20 \cdot 20 \cdot 21 \cdot 23 \cdot 24 \cdot 27 \cdot 30 \cdot 34 \cdot 38 \cdot 43 = 10\,641\,541\,131\,648\,000$.
- (2) Compute the coefficients c_1, \dots, c_{d-1} using (5-2), keeping only the cases when all c_i are integers. 57 643 952 cases remain.
- (3) Discard all cases where the polynomial is not reciprocal mod 2. 1 808 922 cases remain.
- (4) Try ± 1 for the constant coefficient. We have now doubled the number of cases to 3 617 844.
- (5) Consider the reciprocal polynomial $P^*(x) = \pm x^d P(x^{-1})$ (with the sign chosen so that the polynomial is monic), and use (5-1) to compute the power sums p_1^*, \dots, p_{d-1}^* of this polynomial from the reversed sequence $\pm c_{d-1}, \dots, \pm c_1$ of coefficients, where the signs here depend on the sign chosen in the previous

step. Discard the cases that do not satisfy the bounds of Corollary 5.3. 5075 cases remain.

- (6) Test the remaining polynomials by Newton's method for finding roots. Start with the upper bound for the Perron root. Since the polynomial is increasing and convex in $[\lambda, \infty)$, we should get a decreasing sequence of x -values larger than λ . Discard the cases when this fails. Stop when two consecutive x -values are very close to each other. 421 cases remain.
- (7) Discard the polynomials where the largest eigenvalue in absolute value is not real. 86 cases remain.
- (8) Discard the cases where the multiplicity of the largest eigenvalue is larger than 1. 54 cases remain.
- (9) Discard the cases where there is a root with absolute value less than or equal to λ^{-1} . 33 cases remain.
- (10) Discard the cases where the largest eigenvalue is larger than our upper bound. 1 case remains.

In practice, the first three steps are implemented in a more sophisticated way. Our computers cannot handle as many as 10 641 541 131 648 000 cases, so the implementation does not actually consider all those combinations. It first chooses a value for p_1 and sets $c_1 = -p_1$ by (5-2). Then it has to choose a value for p_2 of the same parity as p_1 , since $c_2 = -\frac{1}{2}(c_1 p_1 - p_2)$. Similarly, the value chosen for p_3 is then determined mod 3, therefore a huge number of combinations for the p_i are never considered. Also, once more than half of the c_i are computed, we obtain additional constraints on the p_i , since our polynomial has to be reciprocal mod 2. Since these divisibility checks are done early, and not after the whole polynomial is constructed, a huge number of cases gets eliminated early.

The idea of bounding the coefficients using power sums as in steps (1) and (2) instead of using symmetric polynomials to express the coefficients in terms of the roots is due to Lanneau and Thiffeault [17]. As the numbers suggest, these are the steps that are responsible for bringing the size of the set of possible polynomials down from an astronomical size to one that is approachable by computers.

Step (3) is special to nonorientable surfaces and is also crucial. Without this step, not only would the searching process be much slower, but there are quite a few polynomials that pass all the other tests but this is the only step that eliminates them. Perhaps this

step is the main reason why we do not need Lefschetz number tests unlike Lanneau and Thiffeault in the orientable case.

Step (5) is also special to nonorientable surfaces, since in the orientable case the polynomials are reciprocal, so the reciprocal polynomial does not contain any additional information. This was one of the last tests we added, and this reduced the running time of the algorithm for the $d = 11$ case from several hours to a few minutes. The reason this works so effectively is that in most of the coefficient sequences at this point, the last few coefficients (c_{d-1} , c_{d-2} , etc) are much bigger than the required bounds.

Step (7) is another computationally inexpensive test that quickly eliminates a large fraction of the polynomials. The idea of this test is also due to Lanneau and Thiffeault.

The most computationally expensive part is computing the roots. We only compute the roots after Step (7), only in 421 cases. So, in terms of total time, actually steps (1)–(7) take more than 99% of the running time.

We use a very similar algorithm in order to give a lower bound for the minimal stretch factor $\delta_{\text{rev}}^+(S_g)$ among orientation-reversing pseudo-Anosov maps. The difference is that we use the properties from Proposition 4.3 instead of the ones from Proposition 4.1. Our implementation of these algorithms can be found at <https://github.com/b5strbal/polynomial-filtering>.

5.4 Minimal stretch factors

We are now ready to single out the minimal stretch factor $\delta^+(N_g)$ among pseudo-Anosov homeomorphisms with an orientable invariant foliation for certain nonorientable closed surfaces N_g . Theorem 1.1 is a direct consequence of Corollary 2.9 and Proposition 5.5 below.

Proposition 5.5 *Let g and r be as in one of the rows in Table 3. Let f be a pseudo-Anosov mapping class with an orientable invariant foliation on N_g whose stretch factor λ is smaller than r . Then λ must be a root of the polynomial shown in Table 3. In the cases where no polynomial is given, we have indicated to how many polynomials we were able to restrict the list of candidate polynomials.*

Proof The proof consists of running Algorithm 5.4 and is computer-assisted. However, we will prove the proposition by hand in genus 4. We follow the first four steps explicitly, then (since only a handful of polynomials remain) we finish the proof with an ad hoc but simple argument.

g	r	polynomial candidates	largest root
4	1.84	$x^3 - x^2 - x - 1$	1.83929
5	1.52	$x^4 - x^3 - x^2 + x - 1$	1.51288
6	1.43	$x^5 - x^3 - x^2 - 1$	1.42911
7	1.422	$x^6 - x^5 - x^3 + x - 1$	1.42198
8	1.2885	$x^7 - x^4 - x^3 - 1$	1.28845
9	1.3568	18 candidates	
10	1.2173	$x^9 - x^5 - x^4 - 1$	1.21728
11	1.22262	5 candidates	
12	1.1743	$x^{11} - x^6 - x^5 - 1$	1.17429
13	1.2764	288 candidates	
14	1.14552	$x^{13} - x^7 - x^6 - 1$	1.14551
15	1.1875	84 candidates	
16	1.1249	$x^{15} - x^8 - x^7 - 1$	1.12488
17	1.1426	16 candidates	
18	1.10939	$x^{17} - x^9 - x^8 - 1$	1.10938
20	1.09731	$x^{19} - x^{10} - x^9 - 1$	1.09730

Table 3

Step (1) We have

$$-1 = \min\{-1, 1.84 - 1 - 1/1.84\} < p_1 < 1.84 + 1 + 1/1.84 \approx 3.38,$$

therefore the possible values for p_1 are 0, 1, 2 and 3. We have

$$-1 = \min\{-1, 1.84^2 - 1 - 1/1.84^2\} < p_2 < 1.84^2 + 1 + 1/1.84^2 \approx 4.68,$$

so the possible values for p_2 are 0, 1, 2, 3 and 4.

Step (2) By (5-2), we have $c_1 = -p_1$ and $c_2 = \frac{1}{2}(p_1^2 - p_2)$, therefore p_1 and p_2 have the same parity. Hence the possible pairs are (0, 0), (0, 2), (0, 4), (1, 1), (1, 3), (2, 0), (2, 2), (2, 4), (3, 1) and (3, 3).

Step (3) The pair $(p_1, \frac{1}{2}(p_1^2 - p_2))$ also has the same parity, since our polynomial is reciprocal mod 2. That leaves the choices (0, 0), (0, 4), (1, 3), (2, 0), (2, 4), (3, 3) for (p_1, p_2) .

Step (4) We construct the list of possible polynomials:

- (1) $x^3 \pm 1$.
- (2) $x^3 - 2x \pm 1$.
- (3) $x^3 - x^2 - x \pm 1$.
- (4) $x^3 - 2x^2 + 2x \pm 1$.
- (5) $x^3 - 2x^2 \pm 1$.
- (6) $x^3 - 3x^2 + 3x \pm 1$.

The polynomial has to be irreducible, since the degree of a stretch factor on a non-orientable surface is at least three [30, Proposition 8.7.]. The polynomials where neither 1 nor -1 are roots are $x^3 - x^2 - x - 1$, $x^3 - 2x^2 + 2x + 1$, $x^3 - 2x^2 - 1$ and $x^3 - 3x^2 + 3x + 1$. The second and fourth polynomial do not have a positive real root, and the third polynomial has a root that is approximately 2.2. That leaves us with $x^3 - x^2 - x - 1$. \square

We have stopped at genus 20 because of computational difficulties. The genus 18 case took about half a day to run on a single computer. In the genus 20 case the algorithm took about a day to complete when run parallel on 30 computers. We estimate that the genus 22 case would need to run for a few months on the same cluster of computers.

In the odd genus cases, the issue is not the running time, but the fact that our tests are not good enough to eliminate all polynomials that should be eliminated. In the hope of dealing with more odd genus cases, we have also implemented the Lefschetz number tests used by Lanneau and Thiffeault [17, Section 2.3]. These tests help eliminate a large percentage of the remaining polynomials, but, unfortunately, not all. Table 4 shows the polynomials that we could not eliminate in the genus 9, 11 and 13 cases, in addition to the polynomials that we have constructed in Corollary 2.9.

Most of the polynomials that we are not able to eliminate are products of polynomials that appear in some lower genus and cyclotomic polynomials. We think that these polynomials should be possible to eliminate, but we do not know how. In particular, we think that in the genus 9 and 11 cases all three remaining polynomials could be eliminated, and we conjecture that the examples constructed in Corollary 2.9 are the minimal stretch factor examples (see Conjecture 1.3).

Similarly, in the orientation-reversing case, Theorem 1.4 follows directly from Corollary 3.5 and Proposition 5.6 below.

g	polynomial	stratum	stretch factor
9	$(x^7 - x^4 - x^3 - 1)(x - 1)$	(4^7)	1.28845
	$(x^7 - x^5 - x^2 - 1)(x - 1)$	(4^7)	1.30740
11	$(x^9 - x^5 - x^4 - 1)(x - 1)$	(4^9)	1.21728
13	18 polynomials		

Table 4: The polynomials and possible strata in genus 9 and 11 that we cannot rule out using Algorithm 5.4 and Lefschetz arguments. The notation a^b means an orbit of length b consisting of a -pronged singularities.

Proposition 5.6 *Let g and r be as in one of the rows in the table below. Let f be an orientation-reversing pseudo-Anosov mapping class with orientable invariant foliations on S_g whose stretch factor λ is smaller than r . Then λ must be a root of the polynomial shown in the table:*

g	r	polynomial candidates	largest root
1	1.62	$x^2 - x - 1$	1.61803
2	1.62	$x^2 - x - 1$	1.61803
3	1.253	$x^8 - x^5 - x^3 - 1$	1.25207
4	1.253	$x^8 - x^5 - x^3 - 1$	1.25207
5	1.16	$x^{12} - x^7 - x^5 - 1$	1.15973
6	1.16	$x^{12} - x^7 - x^5 - 1$	1.15973
7	1.1171	$x^{16} - x^9 - x^7 - 1$	1.11707
8	1.1171	$x^{16} - x^9 - x^7 - 1$	1.11707
9	1.0925	$x^{20} - x^{11} - x^9 - 1$	1.09244
10	1.0925	$x^{20} - x^{11} - x^9 - 1$	1.09244
11	1.0764	$x^{24} - x^{13} - x^{11} - 1$	1.07638

Proof Analogously to the proof of Proposition 5.5, the proof of this statement is also computer-assisted. The algorithm used is a slight modification of Algorithm 5.4 as mentioned at the end of Section 5.3.

The polynomials in the table for $g \geq 3$ are not irreducible: they are products of $x^2 + 1$ and an irreducible factor. When $g \geq 3$ is odd, the polynomial we get as a result of the elimination process is this irreducible factor. When $g \geq 4$ is even, the only polynomial left is the product of that irreducible factor and $x^2 - 1$. In either case, the stretch factor has to be a root of the irreducible factor, therefore a root of the polynomials in the table. (The reasons for why we have not listed the irreducible factors in the table is that they

have many more terms and they do not show such a clear pattern as the polynomials in the table. Moreover, the polynomials in the table appear also in Corollary 3.5.)

Similarly, in genus 2, the polynomial remaining after the elimination process factors as $(x^2 - x - 1)(x^2 - 1)$, so the stretch factor would have to be a root of $x^2 - x - 1$. \square

By using the Lefschetz number arguments of Lanneau and Thiffeault, we think it is possible to show that in genus 2 the only remaining polynomial, $(x^2 - x - 1)(x^2 - 1)$, cannot actually be the characteristic polynomial of the action on the first homology for an orientation-reversing pseudo-Anosov map with orientable invariant foliations. This would imply that $\delta_{\text{rev}}^+(S_2) > \delta_{\text{rev}}^+(S_1)$. Since Proposition 5.6 shows a very clear pattern, we conjecture that the stretch factor candidates in Proposition 5.6 cannot be realized for any even genus, leading to Conjecture 1.6.

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