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Measurements, quantum discord, and parity in spin-1 systems

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We consider the evaluation of the quantum discord and other related measures of quantum correlations in a system formed by a spin-1 and a complementary spin system. A characterization of general projective measurements in such system in terms of spin averages is thereby introduced, which allows one to easily visualize their deviation from standard spin measurements. It is shown that the measurement optimizing these measures corresponds in general to a nonspin measurement. The important case of states that commute with the total S_z spin-parity is discussed in detail, and the general stationary measurements for such states (parity preserving measurements) are identified. Numerical and analytical results for the quantum discord, the geometric discord, and the one way information deficit in the relevant case of a mixture of two aligned spin-1 states are also presented.

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I. INTRODUCTION

There is presently a great interest in the investigation of quantum correlations and "quantumness" in mixed states of composite quantum systems. While in the case of pure states such correlations can be identified with entanglement, the situation in mixed states is more complex, as separable (i.e., nonentangled) mixed states, defined as convex mixtures of product states [1], can still exhibit signatures of quantum correlations, as the different products may not commute. The interest has been further enhanced by the existence of mixed state based quantum algorithms, such as that of Knill and Laflamme (KL) [2], able to achieve an exponential speedup over the classical algorithms with no entanglement [3]. In contrast, entanglement is essential for achieving exponential speedup in pure state based quantum computation [4].

Consequently, alternative measures of quantum correlations for mixed states, such as the quantum discord [5–7], have recently received much attention. Though coinciding with entanglement in pure states, discord differs essentially from the latter in mixed states, being nonzero in most separable states and vanishing just for "classically correlated" states, i.e., states which are diagonal in a standard or conditional product basis. The existence of a finite discord in the KL algorithm [8] further increased the interest on this measure. Other measures with similar properties were also recently introduced [9–16], including in particular the geometric discord [11], which allows an easier evaluation. Various fundamental properties [10–15,17,18] and operational interpretations [14,15,19–26] of these measures were recently unveiled. For instance, from the results of [27] it follows that in a pure tripartite system $|\Psi_{ABC}\rangle$, the quantum discord between C and A (as obtained due to a measurement in C) is the entanglement of formation [28] between A and B plus the conditional entropy $S(A|B)$ [18,20,21]. This entails that such discord provides the entanglement consumption in the extended quantum state merging scheme from A to B [20,21]. Besides, states with nonzero discord can be used, even if separable, to generate entanglement in the protocols of [14] or [22], with the quantum discord and the one-way information deficit [12,14] (a closely related quantity) providing the minimum partial and total distillable entanglement between the measurement apparatus and the system after a von Neumann measurement on the latter [14]. Operational interpretations of the geometric discord were also recently provided [25,26]. See Ref. [15] for a recent review.

A common feature of discord type measures is that they involve a difficult minimization over a general local measurement on one of the system constituents. Consequently, most evaluations were so far restricted to two qubits (two spins $1/2$) or a qubit plus a complementary system, where the most general projective measurement in the local qubit reduces to a standard spin measurement and is hence easy to parametrize [5,8,11,29–32]. Closed evaluations in Gaussian systems with Gaussian type measurements were also achieved [33,34]. Nonetheless, even for two qubits, general analytic expressions are available just for the geometric discord [11] and some related measures [32]. Here we will examine the evaluation of the quantum discord (and related measures) between a spin-1 and a complementary spin system. This requires first a convenient characterization of measurements in a spin-1 system (a qutrit), since they are no longer restricted to standard spin measurements as in the spin $s = 1/2$ case, even when considering just standard projective measurements. We provide in Sec. II a simple description of such measurements in terms of spin averages, and show that spin measurements are not optimum in general for spin $s \geqslant 1$, even if the state is described in terms of basic spin observables.

We then analytically identify, in Sec. III, the stationary projective measurements for states exhibiting S_z parity symmetry, an ubiquitous symmetry present for instance in any nondegenerate eigenstate of spin arrays with XYZ couplings of arbitrary range in a transverse field [35] (for a pair of qubits such symmetry leads to the well-known X states [29]). This allows a considerable simplification of the problem of discord evaluation in parity conserving systems. As application, we present analytical results for the quantum discord, the geometric discord and the one-way information deficit in the important case of a mixture of two aligned spin-1 states. Such mixture represents the reduced state of *any* spin pair in the ground state of XYZ spin-1 chains in the immediate vicinity of the transverse factorizing field [35,36], so that the present results represent the universal limit of these quantities at such point. We also explicitly determine the projective measurements minimizing these quantities for this state and show that they exhibit important differences. Conclusions are finally given in Sec. IV.

II. MEASUREMENTS IN SPIN SYSTEMS

A. General case

We first consider a spin s system, where we will denote with $\boldsymbol{S}=(S_x,S_y,S_z)=\boldsymbol{\mathcal{S}}/\hbar$ the dimensionless angular momentum and $|m\rangle$ the eigenstates of S_z (standard basis). Spin measurements are measurements in a basis of eigenstates $|m_{\boldsymbol{k}}\rangle = e^{-i\boldsymbol{\theta}_k\cdot\boldsymbol{S}}|m\rangle$ of the spin component $\boldsymbol{k}\cdot\boldsymbol{S}$ along the direction of a unit vector $\boldsymbol{k}$, and are then specified by just two real parameters which determine its orientation. For $s\geqslant 1$ these measurements are, however, only a particular case of complete projective measurement (von Neumann measurement), i.e., those defined by a complete set of rank 1 orthogonal projectors. The latter are determined by a general unitary transformation of the S_z eigenstates,

$$|m_U\rangle = U|m\rangle, \tag{1}$$

and depend therefore on $d(d-1)$ real parameters, with $d=2s+1$ ($U=e^{iH}$, with H Hermitian, depends on d^2 real parameters, but just d^2-d are sufficient to determine the set of projectors $\{\Pi^U_m=|m_U\rangle\langle m_U|\}$ defining the measurement, as the phase of each $|m_U\rangle$ is irrelevant). The states (1) are the eigenstates of the operator $S_z^U=US_zU^\dagger$, which in general is no longer a linear combination of the original S_μ ($\mu=x,y,z$). Such measurements can, nonetheless, be regarded as measurements of a generalized spin S_z^U (the algebra $[S^U_\mu,S^U_\nu]=i\epsilon_{\mu\nu\sigma}S^U_\sigma$ still holds), and can be implemented as measurements in the standard S_z basis preceded by a single *qudit* gate $U^\dagger$.

A first glimpse into the nature of these measurements can be attained through the set of vectors

$$\langle\boldsymbol{S}\rangle_{m_U}=\langle m_U|\boldsymbol{S}|m_U\rangle, \tag{2}$$

which, in contrast with the case of a spin measurement ($\langle\boldsymbol{S}\rangle_{m_k}=m\boldsymbol{k}$), (i) may have any length between 0 and s and (ii) are not necessarily collinear. Nonetheless, since $\boldsymbol{S}$ is traceless, they always sum to zero:

$$\sum_m\langle\boldsymbol{S}\rangle_{m_U}=\boldsymbol{0}. \tag{3}$$

While not fully identifying the measurement, the set of averages (2) allow a rapid visualization of its deviation from a standard spin measurement: if $\langle\boldsymbol{S}\rangle_{m_U}=m\boldsymbol{k}$ for $m=-s,\ldots,s$, it is clearly a spin measurement along $\boldsymbol{k}$ due to the orthogonality of the basis.

B. Spin-1 systems

In the case of a spin-1 system ($d=3$), Eq. (3) entails that the three vectors (2) are *coplanar*. Moreover, the operators S_z^U are at most quadratic functions of the S_μ, as any operator in such a system can be written as a linear combination of the three S_μ and the six operators $(S_\mu S_\nu+S_\nu S_\mu)/2$. For example, a nonspin measurement in such system is provided by the states $|m_\alpha\rangle=e^{-i\alpha(S_xS_y+S_yS_x)}|m\rangle$, i.e.,

$$|\pm 1_\alpha\rangle=\cos\alpha\,|\pm 1\rangle\pm\sin\alpha\,|\mp 1\rangle,\quad |0_\alpha\rangle=|0\rangle, \tag{4}$$

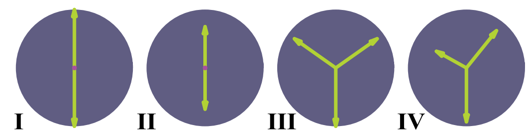

FIG. 1. (Color online) Representation of measurements in a spin-1 system through the spin expectation values in the basis states. I: Spin measurement along z. II: Collinear measurement, determined by the definite parity states (4) or (5) [$\beta=0$ in Eq. (7)]. III: Y-type measurement, determined by the basis (8)–(10) [$\beta=\pi/4$ in (7)]. IV: General measurement, determined by basis (7) and (8).

which satisfy $S_z^\alpha|m_\alpha\rangle=m|m_\alpha\rangle$ with

$$S_z^\alpha=S_z\cos 2\alpha+\left(S_x^2-S_y^2\right)\sin 2\alpha.$$

They lead to

$$\langle\boldsymbol{S}\rangle_{\pm 1_\alpha}=(0,0,\pm\cos 2\alpha),\quad \langle\boldsymbol{S}\rangle_{0_\alpha}=\boldsymbol{0},$$

and hence to the second plot in Fig. 1: the vectors $\langle\boldsymbol{S}\rangle_{m_\alpha}$ are still collinear but $|\langle\boldsymbol{S}\rangle_{\pm 1_\alpha}|\leqslant 1$. Moreover, for $\alpha=\pi/4$, $\langle\boldsymbol{S}\rangle_{m_\alpha}=\boldsymbol{0}\ \forall m$, showing that the average spin may vanish in *all* elements of the basis: in this case $|\pm 1_\alpha\rangle=(|\pm 1\rangle\pm|\mp 1\rangle)/\sqrt{2}$ become the zero eigenstates of S_y and S_x, respectively, which form together with $|0\rangle$ an orthonormal basis.

The most general basis (disregarding global phases and permutations) leading to collinear averages along z for $s=1$ can be obtained by rotating the states (4) around the z axis, which leads to states

$$\left|m_\alpha^\phi\right\rangle=e^{-i\phi S_z}|m_\alpha\rangle. \tag{5}$$

These are the most general states with *definite* S_z *parity*:

$$P_z\left|m_\alpha^\phi\right\rangle=(-1)^{m+1}\left|m_\alpha^\phi\right\rangle,\quad P_z\equiv e^{i\pi(S_z+1)}. \tag{6}$$

We now show that the six parameters specifying a general projective measurement in a spin-1 system can be decomposed into three angles (α,β,γ) which determine the "intrinsic" plot of vectors $\langle\boldsymbol{S}\rangle_{m_U}$ (and hence the type of measurement), plus three angles (ψ,θ,ϕ) which determine the orientation of this plot and of the ensuing states. Assuming first $\langle\boldsymbol{S}\rangle_{m_U}\neq\boldsymbol{0}$ for some m, we choose the intrinsic z axis in the direction of this vector. A state $a|1\rangle+b|0\rangle+c|-1\rangle$ giving rise to $\langle\boldsymbol{S}\rangle=(0,0,\langle S_z\rangle)$ should satisfy $b\bar{a}+\bar{b}c=0$, which implies $b=0$ if $\langle S_z\rangle\neq 0$ ($|a|\neq|c|$). Discarding total phases, the most general orthonormal basis containing such state is then

$$\left|{}^{1_r}_{0_r}\right\rangle=\tfrac{\cos\beta}{\sin\beta}\left(e^{-i\phi_0}\cos\alpha|1\rangle+e^{i\phi_0}\sin\alpha|-1\rangle\right)\mp\tfrac{\sin\beta}{\cos\beta}e^{-i\gamma}|0\rangle, \tag{7}$$

$$|-1_r\rangle=-e^{-i\phi_0}\sin\alpha|1\rangle+e^{i\phi_0}\cos\alpha|-1\rangle, \tag{8}$$

where $\boldsymbol{r}\equiv(\alpha,\beta,\gamma)$. These states lead in general to noncollinear spin averages of different lengths (plot IV in Fig. 1). Choosing ϕ_0 such that the diagram lies in the intrinsic x,z plane ($\langle S_y\rangle_{m_r}=0\ \forall m$), we obtain $\tan\phi_0=\tan\gamma\tan(\pi/4-\alpha)$ and

$$\begin{aligned}\langle\boldsymbol{S}\rangle_{{}^{1_r}_{0_r}}&=\left(\mp\sin 2\beta\sqrt{(1+\cos 2\gamma\sin 2\alpha)/2},0,\tfrac{\cos^2\beta}{\sin^2\beta}\cos 2\alpha\right),\\ \langle\boldsymbol{S}\rangle_{-1_r}&=(0,0,-\cos 2\alpha).\end{aligned} \tag{9}$$

Hence, (α,β,γ) determine respectively $\langle \boldsymbol{S}\rangle_{-1_r}$ and the components of $\langle \boldsymbol{S}\rangle_{0_r}$ parallel and orthogonal to $\langle \boldsymbol{S}\rangle_{-1_r}$. Equations (9) also show that the angle between vectors $\langle \boldsymbol{S}\rangle_{m_r}$ always *exceeds* $\pi/2$: $\langle \boldsymbol{S}\rangle_{m_r}\cdot\langle \boldsymbol{S}\rangle_{m'_r}\leqslant 0$ if $m\neq m'$, vanishing just if one average is zero. The states (7) and (8) can be written as $|m_r\rangle = e^{i(\gamma(S_z^2-1)-\phi_0 S_z)}e^{-i\alpha(S_xS_y+S_yS_x)}e^{i\frac{\beta}{\sqrt{2}}(S_y+S_yS_z+S_zS_y)}|m\rangle$.

The most general orthonormal basis is then obtained by applying a general rotation $e^{-i\psi S_z}e^{-i\theta S_y}e^{-i\phi S_z}$ to this basis. This also includes the case $\langle \boldsymbol{S}\rangle_{m_U}=\boldsymbol{0}\,\forall m$, since such basis are always formed by the zero eigenstates of the components of $\boldsymbol{S}$ along three orthogonal directions: for a state $a|1\rangle+b|0\rangle+c|-1\rangle$, the condition $\langle \boldsymbol{S}\rangle=\boldsymbol{0}$ implies $b\bar{a}+\bar{b}c=0$ and $|a|=|c|$. It is then the eigenstate with zero eigenvalue of $\boldsymbol{k}\cdot\boldsymbol{S}$, with (assuming, with no loss of generality, b real and $c=-\bar{a}$) $\boldsymbol{k}=(-\sqrt{2}\,\mathrm{Re}(a),\sqrt{2}\,\mathrm{Im}(a),b)$. Orthogonality of the basis states then implies that of the associated vectors $\boldsymbol{k}$ (as $\boldsymbol{k}\boldsymbol{k}'=a\bar{a}'+bb'+\bar{a}a'$). Hence, these bases can be obtained, for instance, through a suitable rotation of the intrinsic case $\alpha=\pi/4,\beta=\gamma=0$, where Eqs. (7) and (8) reduce to the zero eigenstates of S_y, S_z, and S_x.

We may then set $\alpha,\beta\in[0,\pi/4]$ and $\gamma\in(-\pi/2,\pi/2]$ in Eqs. (7) and (8), as other values can be mapped to these ranges after suitable rotations (disregarding total phases). Notice that if $\gamma\in(0,\pi/2)$ and $\beta\neq 0$, the values $\pm\gamma$ lead to inequivalent and conjugate basis [as $\phi_0(-\gamma)=-\phi_0(\gamma)$], but the same set of spin averages. The definite parity states (4) are recovered for $\beta=\gamma=0$.

Another relevant case is $\beta=\pi/4$ in Eq. (7), where

$$\left|{}^{1_r}_{0_r}\right\rangle=\frac{1}{\sqrt{2}}(e^{-i\phi_0}\cos\alpha|1\rangle+e^{i\phi_0}\sin\alpha|-1\rangle\mp e^{i\gamma}|0\rangle), \qquad (10)$$

satisfy $P_z\left|{}^{1_r}_{0_r}\right\rangle=\left|{}^{0_r}_{1_r}\right\rangle$, with $P_z|-1_r\rangle=|-1_r\rangle$, such that parity also leaves this basis (i.e., the set of states) *invariant*. This also implies $\langle 1_r|S_z|1_r\rangle=\langle 0_r|S_z|0_r\rangle$, entailing a *symmetric* Y-type spin diagram (plot III in Fig. 1). For $\alpha=\pi/4$, the Y reduces to an horizontal line and the states (8)–(10) become, for $\gamma=\phi_0=0$, eigenstates of S_x [as $\langle \boldsymbol{S}\rangle_{{}^{0_r}_{1_r}}=(\pm 1,0,0)$]. The *fully symmetric case* $\beta=\pi/4$, $\gamma=0$, $\sin 2\alpha=1/3$, where $|\langle \boldsymbol{S}\rangle_{m_r}|^2=8/9\ \forall m$, leads to the *maximum* total squared spin length: $L_S^2=\sum_m|\langle \boldsymbol{S}\rangle_{m_U}|^2=8/3$, larger than the value $L_S^2=2$ obtained for a spin measurement.

III. EVALUATION OF QUANTUM DISCORD AND RELATED MEASURES

A. General case

Let us now consider the evaluation of the quantum discord between $n-1$ arbitrary spins S^i (system A) and a spin s (system B), as obtained due to a local complete projective measurement $M_B=\{\Pi_m^U\}$ on system B. If initially in a state ρ_{AB}, the state of the total system after an unread measurement M_B becomes

$$\rho'_{AB}=\sum_m\left(I_A\otimes\Pi_m^U\right)\rho_{AB}\left(I_A\otimes\Pi_m^U\right). \qquad (11)$$

For a local measurement of this type, the quantum discord [5,6] can be expressed in terms of Eq. (11) as

$$D^B(\rho_{AB})=\min_{M_B}[S(\rho'_{AB})-S(\rho'_B)]-[S(\rho_{AB})-S(\rho_B)], \qquad (12)$$

where $S(\rho)=-\mathrm{Tr}\rho\log\rho$ is the von Neumann entropy and $\rho_B=\mathrm{Tr}_A\rho_{AB}$ the reduced state of B. It can then be considered as the minimum increase of the conditional entropy $S(A|B)=S(A,B)-S(B)$ due to such measurements, and is a non-negative quantity [5,6]. For a pure state ($\rho_{AB}^2=\rho_{AB}$) it becomes the entanglement entropy $S(\rho_B)=S(\rho_A)$, as in this case $S(\rho_{AB})=0$ and $S(\rho'_{AB})=S(\rho'_A)\ \forall M_B$ of this form. However, for a mixed state $D^B(\rho_{AB})$ vanishes just for classically correlated states with respect to B, i.e., states of the form (11) (a particular case of separable state), which are diagonal in a conditional product basis $\{|\nu_m\rangle\otimes|m_U\rangle\}$ and remain hence unchanged under a particular von Neumann measurement in B. Equation (12) actually provides an upper bound to the quantum discord obtained with general POVM measurements, although results for two-qubits indicate that the difference is very small [15].

We will also consider here the minimum generalized information loss due an unread local measurement of the previous type [13,32],

$$I_f^B(\rho_{AB})=\min_{M_B}S_f(\rho'_{AB})-S_f(\rho_{AB}), \qquad (13)$$

where $S_f(\rho)=\mathrm{Tr}\,f(\rho)$ denotes a general entropic form, with f a smooth strictly concave function satisfying $f(0)=f(1)=0$ [37]. Like D^B, it can be shown [13] that $I_f^B(\rho_{AB})\geqslant 0$ for any such f and ρ_{AB}, with $I_f^B(\rho_{AB})$ becoming the generalized entanglement entropy $S_f(\rho_B)=S_f(\rho_A)$ for a pure state, while for a general mixed state it vanishes just for states of the general form (11), i.e., states diagonal in a conditional product basis. Other properties, including the evaluation of I_f^B for any f in some specific states (mixture of a pure state with a maximally mixed state, Bell-diagonal states, etc.), were discussed in [13,32].

Equation (13) contains as particular cases two important measures: if $f(\rho)=-\rho\log\rho$, $S_f(\rho)$ is the von Neumann entropy and Eq. (13) becomes [13] the *one way information deficit* from B to A [12,14]. This quantity is closely related to the quantum discord (12), coinciding with it when the minimizing measurement is the same for both quantities and such that $\rho'_B=\rho_B$ (this occurs for instance when ρ_B is maximally mixed, as in Bell diagonal states). It also reduces to the standard entanglement entropy $S(\rho_A)=S(\rho_B)$ for pure states. The one-way information deficit has been interpreted as the amount of information that cannot be localized through a classical communication channel from B to A [12,14] and, as previously stated, an operational interpretation as the minimum distillable entanglement between the system and the measurement apparatus was recently provided [14].

On the other hand, if $f(\rho)=f_2(\rho)\equiv\rho(1-\rho)$, $S_f(\rho)$ becomes the so-called *linear entropy* $S_2(\rho)=1-\mathrm{Tr}\rho^2$ and Eq. (13) becomes

$$I_2^B(\rho_{AB})=\min_{M_B}\mathrm{Tr}\left(\rho_{AB}^2-\rho'^2_{AB}\right).$$

This quantity is identical [13] with the *geometric measure of discord* [11], the latter defined as the minimum squared Hilbert-Schmidt distance from ρ_{AB} to a classically correlated state: $I_2^B=\min_{\rho'_{AB}}\|\rho_{AB}-\rho'_{AB}\|^2$, where $\|O\|^2=\mathrm{Tr}O^\dagger O$ and ρ'_{AB} is a state diagonal in a conditional product basis with respect to B. In comparison with the previous measures, the geometric discord offers the advantage of an easier evaluation (yet vanishing for the same type of states), as the calculation of $\mathrm{Tr}\,\rho^2$ does not require the explicit knowledge of the eigenvalues of ρ. An analytic expression for general two qubit states was in fact provided in [11], while its extension to $2 \otimes d$ systems was given in [16]. An operational interpretation related with the fidelity and performance of remote state preparation [38] (a variant of the teleportation protocol) has also been recently provided [25,26]. Besides, the geometric discord for a $2 \otimes d$ system can be measured or estimated with direct nontomographic methods [16,39,40], which provide an experimentally accessible scheme. For pure states ρ_{AB}, the geometric discord becomes proportional to the square of the concurrence [41] $C_{AB} = \sqrt{2(1 - \mathrm{Tr}\rho_B^2)}$.

The general stationary condition for Eq. (13) (a necessary condition for the minimizing measurement) reads [32]

$$\Delta_f^B \equiv \mathrm{Tr}_A[f'(\rho'_{AB}),\rho_{AB}] = 0, \tag{14}$$

where f' denotes the derivative of f. In the case of the quantum discord (12), an additional term $-[f'(\rho'_B),\rho_B]$ should be added to Eq. (14) to account for the local terms in Eq. (12), leading to the modified equation [32]

$$\Delta_D^B \equiv \mathrm{Tr}_A[f'(\rho'_{AB}),\rho_{AB}] - [f'(\rho'_B),\rho_B] = 0, \tag{15}$$

where $f(\rho) = -\rho\log\rho$. Since Δ_f^B and Δ_D^B are anti-Hermitian local operators with zero diagonal elements in the measured basis [32], they lead to $d(d-1)/2$ complex equations, which determine suitable values of the $d(d-1)$ real parameters defining the measurement in a d-dimensional system B. They can be solved, for instance, with the gradient method. It is then clear that standard spin measurements, defined by just two real parameters, will not satisfy in general Eq. (14) or (15) for $s > 1/2$, and hence cannot be minimum in general. In the spin-1 case, Eqs. (14) and (15) lead to six real equations which determine suitable values of (α,β,γ) and the three rotation angles.

B. States with S_z parity symmetry and parity preserving measurements

Let us now examine the important case where ρ_{AB} commutes with the total S_z parity,

$$\left[\rho_{AB}, P_z^{AB}\right] = 0, \quad P_z^{AB} = P_z^A \otimes P_z^B, \tag{16}$$

where $P_z^A = \otimes_{i=1}^{n-1} e^{i\pi(S_z^i - S^i)}$. This is an ubiquitous symmetry. For instance, general XYZ type couplings of arbitrary range between spins in a transverse field, not necessarily uniform, lead to a Hamiltonian

$$H = \sum_i b_i S_z^i - \sum_{i,j} \sum_{\mu=x,y,z} J_{ij}^{\mu} S_\mu^i S_\mu^j, \tag{17}$$

which clearly satisfies $[H, P_z^{AB}] = 0$, irrespective of the geometry and dimension of the array. The same holds even if terms $\propto S_x^i S_y^j$ are also present. Hence any nondegenerate eigenstate of H, as well as the thermal state $\rho_{AB} \propto \exp[-\beta H]$, will fulfill Eq. (16). Moreover, if Eq. (16) holds, parity is also preserved at the *local* level, i.e., $[\rho_B, P_z^B] = 0$, as the partial trace involves just diagonal elements in the complementary system A. The reduced state of any subgroup of spins will then also commute with the corresponding local S_z parity.

We also add that any system described by a Hamiltonian containing just quadratic terms $\propto P_i P_j$, $Q_i Q_j$, and $Q_i P_j$ in standard coordinates and momenta $Q_i = \frac{b_i + b_i^\dagger}{\sqrt{2}}$, $P_i = \frac{b_i - b_i^\dagger}{\sqrt{2}i}$, with $b_i, b_i^\dagger$ boson operators ($[b_i, b_j^\dagger] = \delta_{ij}$, $[b_i, b_j] = 0$), does commute with the boson number parity $P_N = e^{i\pi N}$, where $N = \sum_i b_i^\dagger b_i$. Hence, when restricted to a finite N subspace (i.e., $b_i^\dagger b_i \leqslant N_{\max}$), such system is equivalent to a spinlike system whose Hamiltonian commutes with the corresponding S_z parity, defining $S_z^i = b_i^\dagger b_i - N_{\max}/2$.

For an arbitrary ρ_{AB} satisfying Eq. (16), parity will be preserved by the measurement M_B, i.e.,

$$\left[\rho'_{AB}, P_z^{AB}\right] = 0, \tag{18}$$

when $P_z^B \Pi_m^U P_z^B = \Pi_m^U \forall m$ *and also when* $P_z^B \Pi_m^U P_z^B = \Pi_{m'}^U$, where $\Pi_{m'}^U$ is another element of the set of local projectors; as in both cases the set will remain invariant: $\{P_z^B \Pi_m^U P_z^B\} = \{\Pi_m^U\}$. The last case corresponds to $P_z^B|m_U\rangle \propto |m'_U\rangle$ and, since $(P_z^B)^2 = I_B$, such basis can contain just pairs permuted by P_z^B and isolated eigenstates of P_z^B. For a spin-1 system, parity will then be preserved for type II as well as type III measurements, i.e., those based on the states (4) and (5) or (8)–(10).

If Eqs. (16)–(18) hold, the commutator in Eq. (14) will also commute with P_z^{AB}, implying

$$\left[\Delta_f^B, P_z^B\right] = 0, \quad \left[\Delta_D^B, P_z^B\right] = 0. \tag{19}$$

This ensures the existence of parity preserving measurements satisfying Eq. (14) or (15), as the number of independent elements which have to vanish is reduced by Eq. (19), matching exactly the reduced number of free parameters defining such measurements [essentially $\approx d(d-1)/2$]. For instance, in the spin-1 case and for type II measurements, Eq. (19) implies $(\Delta_f^B)_{m,0} = 0$ in the measurement basis and Eq. (14) reduces to a single complex equation $[(\Delta_f^B)_{-1,1} = 0]$ determining α,ϕ. For type III measurements, Eq. (19) implies $(\Delta_f^B)_{0,1}$ imaginary and $(\Delta_f^B)_{-1,0} = (\Delta_f^B)_{-1,1}$ in the measured basis, and Eq. (14) leads to one real and one complex equation, which determine α,γ,ϕ. As there is a maximum and a minimum of I_f^B within these measurements, solutions are ensured. Moreover, if Δ_f^B is *real* in the standard basis, as occurs for instance when ρ_{AB} and all Π_m^U are real in such basis ($\phi = \gamma = 0$), Eq. (19) *reduces to a single real equation in both measurements:*

$$\left(\Delta_f^B\right)_{-1,1} = 0, \tag{20}$$

which determines the optimum α. These arguments also apply for Δ_D^B, leading to $(\Delta_D^B)_{-1,1} = 0$ in the real case.

Parity preserving measurements are then strong candidates for providing the actual minimum of D^B or I_f^B, although "parity breaking" solutions of Eq. (14) may also exist. The latter are *degenerate*, as the sets $\{\Pi_m^U\}$ and $\{P_z^B \Pi_m^U P_z^B\}$ will lead to the *same* values of D^B and I_f^B when Eq. (16) holds. Note also that parity preserving spin measurements are just those along z or an axis perpendicular to z (where $P_z|m_{\boldsymbol{k}}\rangle \propto |-m_{\boldsymbol{k}}\rangle$) and do not have enough parameters for satisfying Eq. (14) if $s \geqslant 1$. In the real case, just those along x, y, or z will lead in general to a real ρ'_{AB} and no continuous free parameter is left.

C. Application

As illustration, we consider a bipartite state formed by the mixture of two aligned spin-1 states,

$$\rho_{AB} = \tfrac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta-\theta\rangle\langle-\theta-\theta|), \tag{21}$$

where $|\theta\rangle \equiv e^{-i\theta S_y}|1\rangle = |1_{\boldsymbol{k}}\rangle$ is the state with maximum spin along $\boldsymbol{k} = (\sin\theta,0,\cos\theta)$ (a coherent state). As $P_z|\theta\rangle = |-\theta\rangle$, Eq. (21) fulfills Eq. (16). This state arises, for instance, as the reduced state of *any* spin pair in the fixed parity states,

$$|\Psi_\pm\rangle = \frac{|\theta\cdots\theta\rangle \pm |-\theta\cdots-\theta\rangle}{\sqrt{2(1\pm\langle-\theta|\theta\rangle^n)}}, \tag{22}$$

if small overlap terms $\propto \langle-\theta|\theta\rangle^{n-2}$ are neglected ($\langle-\theta|\theta\rangle = \cos^{2s}\theta$) [30]. Such states are the *exact* ground states of an XYZ spin chain described by Eq. (17) in the immediate vicinity of the transverse factorizing field [35], existing in the case of fixed anisotropy $\chi = \frac{J^y_{ij}-J^z_{ij}}{J^x_{ij}-J^z_{ij}} = \cos^2\theta\,\forall_{i,j}$ for $|J^y_{ij}| \leqslant J^x_{ij}$, *irrespective* of the geometry or coupling range.

The state (21) is *separable* (a convex mixture of product states [1]) $\forall\theta$, but classically correlated just for $\theta = 0$ or $\pi/2$ (where $\langle-\theta|\theta\rangle = 0$). Accordingly, $D^B(\rho_{AB})$ and $I^B_f(\rho_{AB})$ will be nonzero just for $\theta \in (0,\pi/2)$. As the state is symmetric, we have $D^B = D^A \equiv D$ and $I^B_f = I^A_f \equiv I_f$. As before, we will consider just von Neumann type local projective measurements M_B.

Results for the quantum discord D, the geometric discord I_2, and the one-way information deficit (denoted here as I_1) are shown in Fig. 2. It is first confirmed that minimization over spin measurements provides just an upper bound to the actual value of these quantities, being nonetheless a good approximation for small θ. The qualitative behavior of these three quantities is similar (they are all maximum for θ slightly below $\pi/4$), but important differences in the minimizing measurement do arise. While D is minimized by a real ($\phi = \gamma = 0$) type III measurement $\forall\theta \in (0,\pi/2)$, leading to a smooth curve, I_2 prefers a real type II (III) measurement for $\theta < \theta_c$ ($> \theta_c$), exhibiting a II–III "transition"and hence a cusp maximum at $\theta = \theta_c$. The same holds for I_1 except that the transition between the collinear and Y-type measurements is smoothed through an intermediate region ($0.19\pi \lesssim \theta \lesssim 0.24\pi$) where a *parity breaking* measurement [$\gamma = 0, 0 < \beta < \pi/4$ in Eq. (7)] is preferred. These features resemble then the $s = 1/2$ case [30,32], where D preferred a spin measurement along x $\forall\theta$ [30], whereas I_2 exhibited a sharp $z \to x$ transition, with I_1 selecting a parity breaking axis in a small intermediate interval [32]. Hence, for $s = 1$, parity preserving type II and III measurements play the role of the z and x measurements respectively of the $s = 1/2$ case.

Remarkably, the minimizing value of α, obtained from Eq. (20), *is the same* for D, I_2, and I_1 in all previous cases $\forall\theta$ (i.e., for both type II and III measurements):

$$\tan\alpha = \tan^2\theta/2. \tag{23}$$

At this value the largest eigenvalue of ρ'_{AB} is maximum and ρ'_{AB} attains certain majorizing properties. The evaluation of these measures becomes then analytic. For instance, the

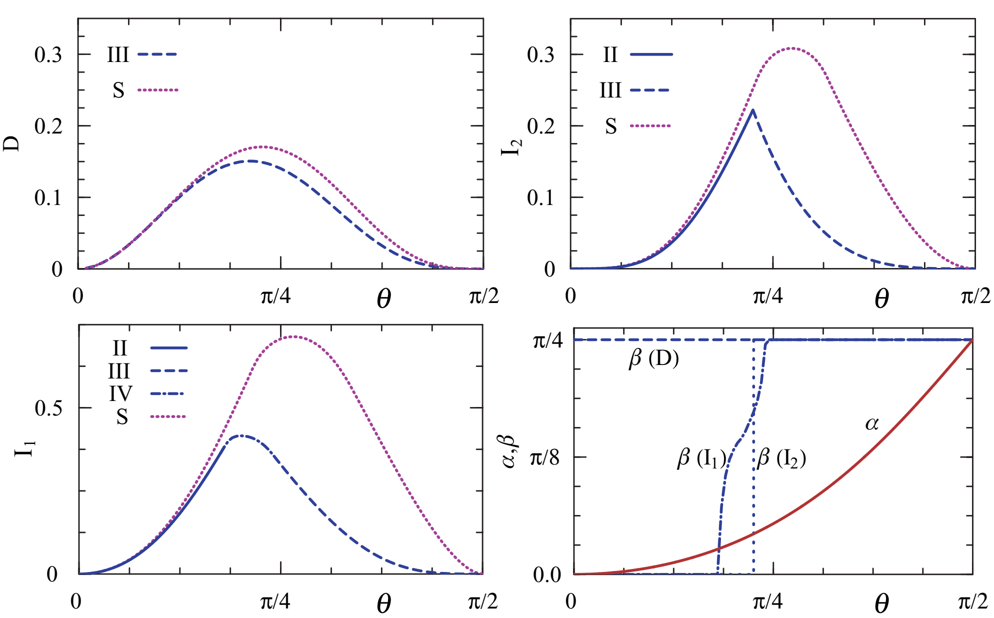

FIG. 2. (Color online) Quantum discord D (top left), the geometric discord I_2 (top right), and the one way information deficit I_1 (bottom left) of the mixture of aligned states (21) as a function of θ for spin $s = 1$. The dotted lines depict the result obtained with a spin measurement, the other curves the actual minimum, obtained with the indicated measurement (see Fig. 1). The bottom right panel depicts the angles characterizing the minimizing measurement. D is minimized by a parity preserving type III measurement $\forall\theta$, whereas I_2 changes from type II to III at $\theta = \theta_c$, and I_1 changes from II to III through a small crossover region where a parity breaking type IV measurement is preferred. The angle α is the same for all quantities Eq. (23). Normalization is such that $D = I_2 = I_1 = 1$ for a Bell state.

quantum and geometric discords read

$$D = 2h_{\frac{1}{2}}(p_\theta) - 1 - h_1[2q_\theta(1-q_\theta)] + h_1(q_\theta)\,, \tag{24}$$

$$I_2 = \begin{cases} \frac{1}{8}\sin^4\theta(3+\cos 2\theta)^2, & \theta < \theta_c \\ \frac{1}{16}\cos^4\theta(11+4\cos 2\theta+\cos 4\theta), & \theta > \theta_c, \quad \cos\theta_c = \frac{1}{\sqrt[4]{3}} \end{cases}, \tag{25}$$

where $h_\nu(x) = -x\log_2 x - (\nu - x)\log_2(\nu - x)$, $p_\theta = \frac{1}{4} - \frac{1}{16}(\frac{115}{8} - \cos 2\theta + \frac{3}{2}\cos 4\theta + \cos 6\theta + \frac{1}{8}\cos 8\theta)^{1/2}$, and $q_\theta = \frac{1}{2}\sin^2\theta$. Remarkably, $\theta_c \approx 0.23\pi$ in I_2 is determined by the overlap condition $\langle -\theta|\theta\rangle^2 = 1/3$, as in the $s = 1/2$ case [32], with $I_2 = 2/9$ at $\theta = \theta_c$ (the same value as for $s = 1/2$). For $\theta \to 0$, $D \approx \theta^2$ while $I_2 \approx 2\theta^4$ (similar to the $s = 1/2$ case [30,32]), whereas for $\theta \to \pi/2$, $D \approx [\frac{1}{2} - \frac{\log_2 e}{4} - \log_2(\frac{\pi}{2} - \theta)](\frac{\pi}{2} - \theta)^4$ while $I_2 \approx \frac{1}{2}(\frac{\pi}{2} - \theta)^4$. In this limit D and I_2 are then proportional to the overlap $\langle -\theta|\theta\rangle^2 = \cos^{4s}\theta$. We also mention that, for small θ, the difference between the approximate value of D obtained with spin measurements and the actual D is very small $[O(-\theta^6\log_2\theta)]$, while in the case of I_2 and I_1, such difference is $O(\theta^4)$ (i.e., of leading order in I_2).

IV. CONCLUSIONS

We have first provided a simple characterization of orthogonal projective measurements in spin-1 systems, which can be extended to arbitrary spin and allows a rapid visualization of the (projective) measurements optimizing discord-type measures of quantum correlations. Standard spin measurements are not optimum in general for minimizing such measures for spin $s \geqslant 1$. Instead, we have shown that for the relevant case of states with parity symmetry, parity preserving measurements provide stationary solutions for all these measures. We have identified such measurements for spin 1, where they are described by just two or three parameters (or one in the real case) allowing to considerably simplify the variational problem associated with discord. Results for the mixture (21), which represents the state of *any* spin pair in an XYZ chain in the immediate vicinity of the factorizing field, confirm the optimality of such measurements in most cases. They also confirm the distinct behavior of the minimizing measurement in the quantum discord as compared to that in the geometric discord [or other measures of type (13) like the information deficit]. The latter are more sensible to changes in the nature of the state and hence more suitable for identifying transitions between different regimes.

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