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Hadamard Well-Posedness for two Nonlinear Structure Acoustic Models

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HADAMARD WELL-POSEDNESS FOR TWO NONLINEAR STRUCTURE ACOUSTIC MODELS

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This dissertation focuses on the Hadamard well-posedness of two nonlinear structure acoustic models, each consisting of a semilinear wave equation defined on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ strongly coupled with a Berger plate equation acting only on a flat portion of the boundary of Ω. In each case, the PDE is of the following form:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + u_t|_\Gamma = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), \end{cases}$$

where the initial data reside in the finite energy space, i.e.,

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma).$$

The chief assumption of the first model is in taking $f(u) = -u|u|^{p-1}$, i.e., f is a restoring source, where $p \geq 1$ is arbitrary. A standard Galerkin approximation

scheme is used to establish a rigorous proof of the existence of local weak solutions. In addition, under some conditions on the parameters in the system, it is shown that such solutions exist globally in time and depend continuously on the initial data.

For the second model, f is taken to be an energy building source, and in particular it is allowed to have a *supercritical* exponent, in the sense that its associated Nemytskii operators is not locally Lipschitz from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$. By employing nonlinear semigroups and the theory of monotone operators, several results on the existence of local and global weak solutions are obtained. Moreover, it is proven that such solutions depend continuously on the initial data, and uniqueness is obtained in two different scenarios.

DEDICATION

To my parents, Arnie and June.

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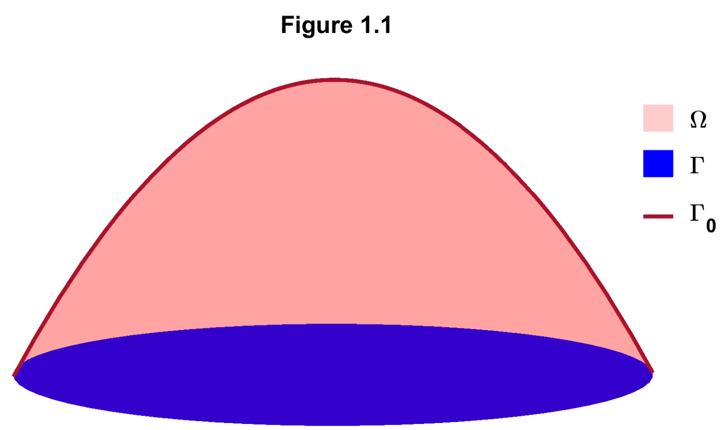
Chapter 1

Introduction

The focus of this dissertation is on the analysis of the standard Structure Acoustic Model with the addition of nonlinear source and damping terms. While classical linear models are more understood, many questions regarding nonlinear models remain unanswered. In particular, answering questions regarding the well-posedness of such models has been the driving force, and the work that follows addresses conditions for existence and uniqueness of local solutions, global solutions, and continuous dependence of solutions on the initial data.

The structure-acoustic model under consideration is comprised of a semilinear wave equation defined on a bounded smooth domain Ω in $\mathbb{R}^3$ coupled strongly with Euler-Bernoulli's plate equation acting only on Γ, a *flat* subset of $\mathbb{R}^2$, where Γ is a portion of the boundary of Ω. This kind of model arises in the context of modeling gas pressure in an acoustic chamber which is surrounded by a combination of rigid and elastic walls. Γ in this case is the elastic wall and we additionally define a surface Γ_0 to be the rigid wall. In particular, note that $\partial\Omega = \overline{\Gamma \cup \Gamma_0}$. A general illustration of Ω, Γ, and Γ_0 is provided in Figure 1.1.



The pressure in the chamber is described by the solution to a wave equation, while vibrations of the elastic wall are described by the solution to a plate equation. One is also often interested in making sense of the acoustic pressure as it appears on the elastic wall, and in fact this acoustic pressure term is the trace of the time derivative of the solution to the wave equation.

Two distinct iterations of this model will be examined, one containing an energy restoring source on the acoustic medium of arbitrary order (Chapter 2), and one with a wave source of bad sign up to supercritical order (Chapter 3).

1.1 Literature Overview

Structural acoustic interaction models have rich history. These models are well known in both the physical and mathematical literature and go back to the canonical models considered in [11, 35]. In the context of stabilization and controllability of structural acoustic models there is a very large body of literature. We refer the reader to the monograph by Lasiecka [41] which provides a comprehensive overview and quotes many works on these topics. Other related contributions worthy of mention include [2, 3, 4, 5, 17, 29, 30, 40].

However, the presence of nonlinear damping has been recognized in the literature as a source of many technical difficulties. Over the years, there has been some novel progress in this area, particularly for wave equations influenced by nonlinear damping [26, 27, 42, 50]. In [28], Georgiev and Todorova considered a semilinear wave equation with frictional damping and a subcritical source term. The paper [28] provided the local and global solvability of the equation, and also provided a blow up result which ignited considerable interest in the area. For structural acoustic we mention the work by Chueshov et al [20, 21, 22, 23, 24].

Consequent results on wave equations with subcritical sources were established in [1, 18, 49, 52, 58]. We also would like to mention the works [8, 9, 10] on wave equations influenced by *degenerate* damping and source terms. Well-posedness results for wave equations with supercritical sources include the breakthrough papers by Bociu and Lasiecka [13, 14] and the papers on systems of wave equations [31, 32, 33]. For other related results on wave equations involving supercritical sources we mention [34, 37, 38, 47, 48] and the references therein.

In this dissertation two iterations of the Structure Acoustic model are examined. In the first, we follow an approach similar to Lions [45] to establish the existence of local weak solutions. For the case of a critical source acting on the wave equation, we prove such solutions depend continuously on the initial data, and so these solutions are unique in the finite energy space. In the second, we use the powerful theory of monotone operators and nonlinear semigroups (Kato's Theorem [6, 55]). Our strategy is similar to the one used by Bociu [12] and our proofs draw substantially from important ideas in [13, 14, 33].

1.2 The Principal Model

In this dissertation, we study a structure acoustic model influenced with nonlinear forces. Precisely, we study the coupled system of PDEs:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + u_t|_\Gamma = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), \end{cases} \tag{1.2.1}$$

where the initial data reside in the finite energy space, i.e.,

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma).$$

In this model, $\Omega \subset \mathbb{R}^3$ is a smooth, bounded, open, connected domain with boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma}$, where Γ_0 and Γ are two disjoint, open, connected sets of positive Lebesgue measure. Moreover, Γ is a *flat* portion of the boundary of Ω and is referred to as the elastic wall. The part Γ_0 of the boundary $\partial\Omega$ describes a rigid wall, while the coupling takes place on the flexible wall Γ. Models such as (1.2.1) arise in the context of modeling gas pressure in an acoustic chamber Ω which is surrounded by a combination of rigid and flexible walls. The pressure in the chamber is described by the solution to a wave equation, while vibrations of the flexible wall are described by the solution to a Berger plate equation. We refer the reader to [25] and the references quoted therein for more details on the Berger model.

The nonlinearities f and h represent interior sources acting on the wave and plate equations respectively. In addition, the system is influenced by two other competing forces, namely $g_1(u_t)$ and $g_2(w_t)$ representing frictional damping terms acting on the wave and plate equations, respectively. The vectors ν and ν_Γ denote the outer normals to Γ and $\partial\Gamma$; respectively.

1.3 Notation

Throughout the work the following notational conventions for L^p space norms and standard inner products will be used:

$$||u||_p = ||u||_{L^p(\Omega)}, \qquad (u,v)_\Omega = (u,v)_{L^2(\Omega)},$$
$$|u|_p = ||u||_{L^p(\Gamma)}, \qquad (u,v)_\Gamma = (u,v)_{L^2(\Gamma)}.$$

We also use the notation γu to denote the *trace* of u on Γ and we write $\frac{d}{dt}(\gamma u(t))$ as γu_t or $\gamma u'$. Occasionally, we also use the notation $u_{|_\Gamma}$ to mean γu. As is customary, C shall always denote a positive constant which may change from line to line.

Further, we put

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}.$$

It is well-known that the standard norm $\|u\|_{H^1_{\Gamma_0}(\Omega)}$ is equivalent to $\|\nabla u\|_2$. Thus, we put:

$$\|u\|_{H^1_{\Gamma_0}(\Omega)} := \|\nabla u\|_2\,,\ (u,v)_{H^1_{\Gamma_0}(\Omega)} = (\nabla u, \nabla v)_\Omega.$$

For a similar reason, we put:

$$\|w\|_{H^2_0(\Gamma)} = |\Delta w|_2\,,\ (w,z)_{H^2_0(\Gamma)} = (\Delta w, \Delta z)_\Gamma.$$

For convenience and brevity, we shall frequently use the notation:

$$\|u\|_{1,\Omega} = \|\nabla u\|_2 \,, \ \ \|w\|_{2,\Gamma} = |\Delta w|_2 \,.$$

Relevant to this work we define the Banach space X and its norm by:

$$X = H^1_{\Gamma_0}(\Omega) \cap L^{p+1}(\Omega), \qquad\qquad \|u\|_X = \|\nabla u\|_2 + \|u\|_{p+1}.$$

With Y is a Banach space, we denote the duality pairing between the dual space Y' and Y by $\langle \psi, y \rangle_{Y',Y}$, or simply by $\langle \cdot, \cdot \rangle$. That is,

$$\langle \psi, y \rangle = \psi(y) \text{ for } y \in Y, \, \psi \in Y'.$$

The following Sobolev imbeddings will be used often without mention:

$$\begin{cases} H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega) \text{ for } \epsilon \in [0,1], \\ H^{1-\epsilon}(\Omega) \xrightarrow{\gamma} H^{\frac{1}{2}-\epsilon}(\Gamma) \hookrightarrow L^{\frac{4}{1+2\epsilon}}(\Gamma) \text{ for } \epsilon \in [0, \frac{1}{2}], \\ H^1(\Gamma) \hookrightarrow L^q(\Gamma) \text{ for all } 1 \le q < \infty. \end{cases}$$

Finally, we remind the reader with the following interpolation inequality:

$$\|u\|^2_{H^\theta(\Omega)} \le \epsilon \, \|u\|^2_{1,\Omega} + C(\epsilon, \theta) \, \|u\|^2_2 \,, \tag{1.3.1}$$

for all $0 \le \theta < 1$ and $\epsilon > 0$.

Chapter 2

Energy Restoring Source

2.1 The Model

In this iteration of the model, further assume the following on (1.2.1) that: $f(u) = -u|u|^{p-1}$, $g_1(u_t) = 0$, and $g_2(w_t) = w_t$. This yields the following system of PDEs:

$$\begin{cases} u_{tt} - \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + w_t + u_t|_\Gamma = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), \end{cases} \tag{2.1.1}$$

where the initial data still reside in the finite energy space, i.e.,

$$u_0 \in H^1_{\Gamma_0}(\Omega) \cap L^{p+1}(\Omega),\ u_1 \in L^2(\Omega),\ \text{and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma).$$

The sign of $f(u)$ creates what is referred to in the literature as an energy restoring source, and in this chapter it will be taken to be of arbitrary power.

2.2 Main Results

Throughout this chapter, we study (2.1.1) under the following general assumptions:

Assumption 2.2.1. *We assume that the sources in (2.1.1) are $\mathbb{R}$-valued functions satisfying:*

- $1 \leq p < \infty$,
- $h \in C^1(\mathbb{R})$ *such that* $|h'(u)| \leq C(|u|^{q-1} + 1)$ *with* $1 \leq q < \infty$.

Remark 2.2.2. As the following bounds will be used often throughout the chapter it is worthy of note that the above assumption implies that

$$\begin{cases} \Big||u|^{p-1}u - |v|^{p-1}v\Big| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|, \\ |h(u)| \leq C(|u|^q + 1), \quad |h(u) - h(v)| \leq C(|u|^{q-1} + |v|^{q-1} + 1)|u - v|. \end{cases}$$

△

We begin by introducing the definition of a suitable weak solution for (2.1.1).

Definition 2.2.3. A pair of functions (u, w) is said to be a weak solution of (2.1.1) on the interval $[0, T]$ provided:

(i) $u \in C_w([0, T]; X)$, $u_t \in C_w([0, T]; L^2(\Omega))$,

(ii) $w \in C_w([0, T]; H_0^2(\Gamma))$, $w_t \in C_w([0, T]; L^2(\Gamma))$,

(iii) $(u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$,

(iv) $(w(0), w_t(0)) = (w_0, w_1) \in H_0^2(\Gamma) \times L^2(\Gamma)$,

(v) The functions u and v satisfy the following variational identities for all $t \in [0, T]$:

$$\begin{aligned}(u_t(t), \phi(t))_\Omega - (u_1, \phi(0))_\Omega - \int_0^t (u_t(\tau), \phi_t(\tau))_\Omega d\tau \\ + \int_0^t (\nabla u(\tau), \nabla \phi(\tau))_\Omega d\tau - \int_0^t (w_t(\tau), \gamma \phi(\tau))_\Gamma d\tau \\ + \int_0^t \int_\Omega |u(\tau)|^{p-1} u(\tau) \phi(\tau) dx d\tau = 0,\end{aligned} \tag{2.2.1}$$

$$\begin{aligned}(w_t(t) + \gamma u(t), \psi(t))_\Gamma - (w_1 + \gamma u(0), \psi(0))_\Gamma - \int_0^t (w_t(\tau), \psi_t(\tau))_\Gamma d\tau \\ - \int_0^t (\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau + \int_0^t (\Delta w(\tau), \Delta \psi(\tau))_\Gamma d\tau \\ + \int_0^t (w_t(\tau), \psi(\tau))_\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau)) \psi(\tau) d\Gamma d\tau,\end{aligned} \tag{2.2.2}$$

for all test functions $\phi \in C_w([0, T]; X)$ with $\phi_t \in L^2(0, T; L^2(\Omega))$, and $\psi \in C_w([0, T]; H_0^2(\Gamma))$ with $\psi_t \in L^2(0, T; L^2(\Gamma))$.

Remark 2.2.4. In Definition 2.2.3 above, $C_w([0, T]; X)$ denotes the space of weakly continuous (often called scalarly continuous) functions from $[0, T]$ into a Banach space X. That is, for each $u \in C_w([0, T]; X)$ and $f \in X'$ the map $t \mapsto \langle f, u(t) \rangle_{X', X}$ is continuous on $[0, T]$. △

Our principal result is the existence of local solutions of problem (2.1.1) in the following sense.

Theorem 2.2.5. *Under the validity of Assumption 2.2.1, problem* (2.1.1) *possesses a local weak solution,* (u, w)*, in the sense of Definition 2.2.3 on a non-degenerate interval* $[0, T]$*, where* T *depends upon the initial positive energy* $\mathscr{E}(0)$ *(where* $\mathscr{E}(t)$ *is* *defined below). Furthermore, if in addition* $1 \le p \le 3$*, then the said solution* (u, w) *satisfies the following energy identity for all* $t \in [0, T]$*:*

$$\mathscr{E}(t) + \int_0^t |w_t(\tau)|_2^2 \, d\tau = \mathscr{E}(0) + \int_0^t \int_\Gamma h(w) w_t d\Gamma d\tau, \tag{2.2.3}$$

where

$$\mathscr{E}(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + |w_t(t)|_2^2 + |\Delta w(t)|_2^2\right) + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}. \tag{2.2.4}$$

If $p > 3$*, then the solution* (u, w) *satisfies the energy inequality:*

$$\mathscr{E}(t) + \int_0^t |w_t(\tau)|_2^2 \, d\tau \le \mathscr{E}(0) + \int_0^t \int_\Gamma h(w) w_t d\Gamma d\tau \quad a.e. \ [0, T]. \tag{2.2.5}$$

Equivalently, (2.2.5) *can also be written as*

$$E(t) + \int_0^t |w_t(\tau)|_2^2 \, d\tau \le E(0) \quad a.e. \ [0, T], \tag{2.2.6}$$

with $E(t) = \mathscr{E}(t) - \int_\Gamma H(w(t)) d\Gamma$*, where* H *is the primitive of* h*, i.e.,* $H(w) = \int_0^w h(s) ds$*.*

Although the source term acting on the plate equation can have a “bad” sign which may cause blow up in finite time, our next result states that solutions established by Theorem 2.2.5 are indeed global solutions, provided the plate source term is essentially linear.

Theorem 2.2.6. *In addition to Assumption 2.2.1, assume* $q = 1$*. Then any solution* (u, w) *furnished by Theorem 2.2.5 is a global weak solution and the existence time* T *may be taken arbitrarily large.*

Theorem 2.2.7. *In addition Assumption 2.2.1, assume $p \leq 3$ and $U_0 = (u_0, w_0, u_1, w_1) \in H$ is an initial data with a corresponding weak solution (u, w) of (2.1.1), where $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$. If $U_0^n = (u_0^n, w_0^n, u_1^n, w_1^n)$ is a sequence of initial data such that $U_0^n \longrightarrow U_0$ in H, as $n \longrightarrow \infty$, then the corresponding weak solutions (u^n, w^n) with initial data U_0^n satisfy:*

$$(u^n, w^n, u_t^n, w_t^n) \longrightarrow (u, w, u_t, w_t) \text{ in } L^\infty(0, T; H), \text{ as } n \longrightarrow \infty,$$

where $0 < T < \infty$ is chosen to be independent of $n \in \mathbb{N}$.

Corollary 2.2.8. *In addition to Assumptions 2.2.1, assume $p \leq 3$. Then, weak solutions of (2.1.1) (in the sense of Definition 2.2.3) are unique.*

2.3 Existence of Local Solutions

2.3.1 Approximate solutions

We begin by selecting a sequence $\{e_j\}_1^\infty \subset X = H^1_{\Gamma_0}(\Omega) \cap L^{p+1}(\Omega)$ with the following properties:

$$\begin{cases} e_1, \cdots, e_N \text{ are linearly independent for every } N \in \mathbb{N}, \text{and} \\ \text{The set of all finite linear combinations of the form:} \\ \left\{ \sum_{j=1}^N c_j e_j : c_j \in \mathbb{R},\ N \in \mathbb{N} \right\} \text{ is dense in } X. \end{cases} \tag{2.3.1}$$

Let $B = \Delta^2$ with its domain $\mathscr{D}(B) = H^4(\Gamma) \cap H^2_0(\Gamma)$. It is well known that B is positive, self-adjoint, and B is the inverse of a compact operator. Moreover, B has the infinite sequence of positive eigenvalues $\{\mu_n : n \in \mathbb{N}\}$ and a corresponding sequence of eigenfunctions $\{\sigma_n : n \in \mathbb{N}\}$ which can be normalized to form an orthonormal basis for $H_0^2(\Gamma)$ while remaining an orthogonal basis for $L^2(\Gamma)$. In particular it is well known that the standard inner product $(w,z)_{H_0^2(\Gamma)}$ is equivalent to $(\Delta w, \Delta z)_\Gamma$, and in turn $|\Delta w|_2$ is equivalent to the standard norm on $H_0^2(\Gamma)$. Thus, we put:

$$(w,z)_{H_0^2(\Gamma)} = (\Delta w, \Delta z)_\Gamma, \quad \|w\|_{H_0^2(\Gamma)} = |\Delta w|_2. \tag{2.3.2}$$

For given initial data $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ we can find for each $N \in \mathbb{N}$ sequences of real numbers $\{u^0_{N,j}\}^\infty_{N,j=1}$, $\{u^1_{N,j}\}^\infty_{N,j=1}$ such that

$$\begin{cases} \sum_{j=1}^N u^0_{N,j} e_j \to u_0 \text{ strongly in } X, \text{ as } N \to \infty, \\ \sum_{j=1}^N u^1_{N,j} e_j \to u_1 \text{ strongly in } L^2(\Omega), \text{ as } N \to \infty. \end{cases} \tag{2.3.3}$$

Similarly, for given initial data $(w_0, w_1) \in H_0^2(\Gamma) \times L^2(\Gamma)$, we may find sequences of scalars $\{w_j^0 = (\Delta w_0, \Delta \sigma_j)_\Gamma : j \in \mathbb{N}\}$ and $\{w_j^1 = \frac{1}{|\sigma_j|_2}(w_1, \sigma_j)_\Gamma : j \in \mathbb{N}\}$ such that

$$\begin{cases} \sum_{j=1}^N w_j^0 \sigma_j \to w_0 \text{ strongly in } H_0^2(\Gamma) \text{ as } N \to \infty, \\ \sum_{j=1}^N w_j^1 \sigma_j \to w_1 \text{ strongly in } L^2(\Gamma), \text{ as } N \to \infty. \end{cases} \tag{2.3.4}$$

We now seek to construct a sequence of approximate solutions in the form

$$\begin{cases} u_N(x,t) = \sum_{j=1}^N u_{N,j}(t) e_j(x), \\ w_N(x,t) = \sum_{j=1}^N w_{N,j}(t) \sigma_j(x), \end{cases} \tag{2.3.5}$$

that satisfy the system of ODEs:

$$\begin{cases} (u_N'', e_j)_\Omega + (\nabla u_N, \nabla e_j)_\Omega - (w_N', \gamma e_j)_\Gamma + \int_\Omega |u_N|^{p-1} u_N e_j dx = 0, \\ (w_N'', \sigma_j)_\Gamma + (\Delta w_N, \Delta \sigma_j)_\Gamma + (w_N', \sigma_j)_\Gamma + (\gamma u_N', \sigma_j)_\Gamma = \int_\Gamma h(w_N)\sigma_j d\Gamma, \end{cases} \tag{2.3.6}$$

with initial data

$$\begin{cases} u_{N,j}(0) = u_{N,j}^0, \;\; u_{N,j}'(0) = u_{N,j}^1, \\ w_{N,j}(0) = w_j^0, \;\; w_{N,j}'(0) = w_j^1. \end{cases} \tag{2.3.7}$$

where $j = 1, \ldots, N$.

We note here that (2.3.6)–(2.3.7) is an initial-value problem for a second order $2N \times 2N$ system of ordinary differential equations with continuous nonlinearities in the unknown functions $u_{N,j}$ and $w_{N,j}$ and their time derivatives. Therefore, it follows from the Cauchy-Peano theorem that for every $N \geq 1$, (2.3.6)–(2.3.7) has a solution $u_{N,j}$, $w_{N,j} \in C^2([0, T_N])$, $j = 1, \ldots N$, for some $T_N > 0$.

2.3.2 A priori estimates

We aim to demonstrate that each of the approximate solutions (u_N, w_N) exists on a non-degenerate interval $[0, T]$, where T is independent of N.

Proposition 2.3.1. *Each approximate solution (u_N, w_N) exists on a non-degenerate interval $[0, T]$, where T depends on the initial positive energy $\mathscr{E}(0)$ and other generic constants. Further, the sequences of approximate solutions $\{u_N\}_1^\infty$ and $\{w_N\}_1^\infty$ satisfy*

$$\{u_N\}_1^\infty \textit{ is a bounded sequence in } L^\infty(0,T;X), \tag{2.3.8a}$$

$$\{u'_N\}_1^\infty \textit{ is a bounded sequence in } L^\infty(0,T;L^2(\Omega)), \tag{2.3.8b}$$

$$\{w_N\}_1^\infty \textit{ is a bounded sequence in } L^\infty(0,T;H_0^2(\Gamma)), \tag{2.3.8c}$$

$$\{w'_N\}_1^\infty \textit{ is a bounded sequence in } L^\infty(0,T;L^2(\Gamma)). \tag{2.3.8d}$$

Proof. Multiplying the first equation of (2.3.6) by $u'_{N,j}$ and summing over $j = 1, \ldots, N$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|u'_N(\tau)\|_2^2 + \|\nabla u_N(\tau)\|_2^2\Big) - (w'_N(\tau), u'_N(\tau))_\Gamma \\ + \int_\Omega |u_N(\tau)|^{p-1} u_N(\tau) u'_N(\tau) dx = 0,\end{aligned} \tag{2.3.9}$$

for each $\tau \in [0, T_N]$. Similarly, multiplying the second equation of (2.3.6) by $w'_{N,j}$ and summing over $j = 1, ..., N$, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(|w'_N(\tau)|_2^2 + |\Delta w_N(\tau)|_2^2\Big) + (u'_N(\tau), w'_N(\tau))_\Gamma + |w'_N(\tau)|_2^2 \\ = \int_\Gamma h(w_N(\tau)) w'_N(\tau) d\Gamma,\end{aligned} \tag{2.3.10}$$

for each $\tau \in [0, T_N]$.

By adding (2.3.9) and (2.3.10) and integrating with respect to τ over $[0, t]$, we obtain

$$\mathscr{E}_N(t) + \int_0^t |w'_N(\tau)|_2^2 d\tau = \mathscr{E}_N(0) + \int_0^t \int_\Gamma h(w_N(\tau)) w'_N(\tau) d\Gamma d\tau, \tag{2.3.11}$$

where $\mathscr{E}_N(t)$ is the positive energy of the system given by:

$$\begin{aligned}\mathscr{E}_N(t) =& \frac{1}{2}\left(\|u_N'(t)\|_2^2 + \|\nabla u_N(t)\|_2^2 + |w_N'(t)|_2^2 + |\Delta w_N(t)|_2^2\right) \\ &+ \frac{1}{p+1}\|u_N(t)\|_{p+1}^{p+1}.\end{aligned} \tag{2.3.12}$$

Let us note here that due to the strong convergence in (2.3.3) and (2.3.4), $\mathscr{E}_N(0) \leq C$ for some positive constant C independent of N, but depends upon $\mathscr{E}(0)$. In order to produce a suitable bound on $\mathscr{E}_N(t)$ we shall estimate the term involving $h(w_N)$ as follows. By the assumption imposed on h, we have

$$\begin{aligned}\left|\int_\Gamma h(w_N(\tau))w_N(\tau)' d\Gamma\right| &\leq C\Big||w_N(\tau)|^q + 1\Big|_2 |w_N'(\tau)|_2 \\ &\leq C\left(|w_N|_{2q}^{2q} + |w_N'|_2^2 + 1\right) \\ &\leq C_1\left(|\Delta w_N|_2^{2q} + |w_N'|_2^2 + 1\right),\end{aligned} \tag{2.3.13}$$

where we have used Hölder's and Young's inequalities, and the positive constant C_1 in (2.3.13) is independent of N.

Combining (2.3.11) and (2.3.13) yields:

$$\begin{aligned}\mathscr{E}_N(t) + \int_0^t |w_N'(\tau)|_2^2 d\tau &\leq C + C_1 \int_0^t \left(|\Delta w_N(\tau)|_2^{2q} + |w_N'(\tau)|_2^2 + 1\right) d\tau \\ &\leq C + C_1 \int_0^t \left(\mathscr{E}_N(\tau) + 1\right)^q d\tau.\end{aligned} \tag{2.3.14}$$

By putting $y_N(t) = 1 + \mathscr{E}_N(t)$, then (2.3.14) yields

$$y_N(t) \leq C + C_1 \int_0^t y_N(\tau)^q d\tau. \tag{2.3.15}$$

If $q = 1$, then it follows by Gronwall's inequality that $y_N(t) \leq Ce^{C_1 t}$, for all $t \geq 0$ and all $N \in \mathbb{N}$. However, if $q > 1$, then by using a standard comparison theorem, (2.3.15) yields that $y_N(t) \leq z(t)$, where $z(t) = (C^{1-q} - C_1(q-1)t)^{\frac{-1}{q-1}}$ is the solution of the Volterra integral equation

$$z(t) = C + C_1 \int_0^t z(\tau)^q d\tau. \tag{2.3.16}$$

Although $z(t)$ blows up in finite time, nonetheless, there exists a time $0 < T < T_N$ depending on q and $\mathscr{E}(0)$ such that $y_N(t) \leq z(t) \leq C_0$ for all $t \in [0, T]$, where C_0 is independent of N, but depending on q and $\mathscr{E}(0)$. Hence, for all $N \geq 1$ and any $q \geq 1$, one has $y_N(t) \leq C_0$ for all $t \in [0, T]$, establishing the proposition. $\square$

An immediate consequence of Proposition 2.3.1 along with the Banach-Alaoglu theorem and the well-known Aubin-Lions-Simon Compactness Theorem (e.g., [15, Thm. II.5.16]) is the following:

Corollary 2.3.2. *For all sufficiently small $\epsilon > 0$ there exists a function u and a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$) such that*

$$u_N \to u \text{ weak}^* \text{ in } L^\infty(0,T;X), \tag{2.3.17a}$$

$$u_N' \to u' \text{ weak}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{2.3.17b}$$

$$w_N \to w \text{ weak}^* \text{ in } L^\infty(0,T;H_0^2(\Gamma)), \tag{2.3.17c}$$

$$w_N' \to w' \text{ weak}^* \text{ in } L^\infty(0,T;L^2(\Gamma)), \tag{2.3.17d}$$

$$u_N \to u \text{ strongly in } C([0,T];H^{1-\epsilon}(\Omega)), \tag{2.3.17e}$$

$$w_N \to w \text{ strongly in } C([0,T];H_0^1(\Gamma)), \tag{2.3.17f}$$

$$\gamma u_N \to \gamma u \text{ strongly in } C([0,T];L^{\frac{4}{1+2\epsilon}}(\Gamma)). \tag{2.3.17g}$$

for all $\epsilon \in (0, \frac{1}{2}]$.

2.3.3 Passage to the limit and verification of (2.2.1)

We begin by considering the wave portion of (2.3.6), and after integrating over $[0,t]$, we obtain:

$$(u_N'(t), e_j)_\Omega - (u_N'(0), e_j)_\Omega + \int_0^t (\nabla u_N(\tau), \nabla e_j)_\Omega d\tau - \int_0^t (w_N'(\tau), \gamma e_j)_\Gamma d\tau$$
$$+ \int_0^t \int_\Omega |u_N(\tau)|^{p-1} u_N(\tau) e_j dx d\tau = 0, \tag{2.3.18}$$

where $j = 1, ..., N$.

We first note that (2.3.17b) implies that

$$(u_N'(t), e_j)_\Omega \longrightarrow (u'(t), e_j)_\Omega \text{ weak}^* \text{ in } L^\infty(0,T). \tag{2.3.19}$$

Also, from (2.3.17a) we see

$$u_N \longrightarrow u \text{ weak}^* \text{ in } L^\infty(0,T; H^1_{\Gamma_0}(\Omega)) = \left(L^1(0,T; (H^1_{\Gamma_0}(\Omega))')\right)',$$

and as a result we conclude that:

$$(\nabla u_N(\tau), \nabla e_j)_\Omega \longrightarrow (\nabla u(\tau), \nabla e_j)_\Omega \text{ weak}^* \text{ in } L^\infty(0,T). \tag{2.3.20}$$

Since $e_j \in X$ and by the continuity of the trace map $H^1_{\Gamma_0}(\Omega) \xrightarrow{\gamma} L^4(\Gamma)$, then it follows from (2.3.17d) that

$$(w_N'(\tau), \gamma e_j)_\Gamma \longrightarrow (w'(\tau), \gamma e_j)_\Gamma \text{ weak}^* \text{ in } L^\infty(0,T). \tag{2.3.21}$$

Proposition 2.3.3. *On a subsequence, which is still labeled as* $\{u_N\}_1^\infty$*, we have:*

$$|u_N|^{p-1}u_N \longrightarrow |u|^{p-1}u \text{ weakly in } L^{\frac{p+1}{p}}(\Omega\times(0,T)). \tag{2.3.22}$$

Proof. By invoking (2.3.17e), then there is a subsequence, labeled as $\{u_N\}_{N=1}^\infty$, such that $u_N \longrightarrow u$ pointwise a.e. in $\Omega\times(0,T)$, which implies that $|u_N|^{p-1}u_N \to |u|^{p-1}u$ pointwise a.e. in $\Omega\times(0,T)$. Since the sequence $\{u_N\}_{N=1}^\infty$ is bounded $L^\infty(0,T;L^{p+1}(\Omega))$ from Proposition 2.3.1, and so $\{|u_N|^{p-1}u_N\}_{N=1}^\infty$ is bounded in $L^{\frac{p+1}{p}}(\Omega\times(0,T))$. Then, (2.3.22) follows immediately from a standard result in analysis. □

Remark 2.3.4. Proposition 2.3.3 easily implies the following convergence:

$$\int_0^t\int_\Omega |u_N(\tau)|^{p-1}u_N(\tau)e_j\,dxd\tau \longrightarrow \int_0^t\int_\Omega |u(\tau)|^{p-1}u(\tau)e_j\,dxd\tau, \text{ for } t\in[0,T]. \tag{2.3.23}$$

△

By noting that $\chi_{[0,t]}\in L^1(0,T)$ for $t\in[0,T]$, and recalling the strong convergence of $u_N'(0)$ in (2.3.3), then by combining (2.3.19)-(2.3.23), we are justified in passing to the limit in (2.3.18) to obtain:

$$(u'(t),e_j)_\Omega-(u_1,e_j)_\Omega+\int_0^t(\nabla u(\tau),\nabla e_j)_\Omega d\tau-\int_0^t(w'(\tau),\gamma e_j)_\Gamma d\tau + \int_0^t\int_\Omega |u(\tau)|^{p-1}u(\tau)e_j\,dxd\tau=0, \tag{2.3.24}$$

where (2.3.24) is valid for all $j\in\mathbb{N}$ and a.e. $t\in[0,T]$.

Now, for any $\phi\in X$, there exists a sequence $\phi_k=\sum_{j=1}^k a_{k,j}e_j$ which converges to ϕ strongly in X. By linearity, one can replace e_j in (2.3.24) with ϕ_k, and then pass to the limit as $k \to \infty$ to obtain:

$$(u'(t), \phi)_\Omega - (u_1, \phi)_\Omega + \int_0^t (\nabla u(\tau), \nabla \phi)_\Omega d\tau - \int_0^t (w'(\tau), \gamma\phi)_\Gamma d\tau$$
$$+ \int_0^t \int_\Omega |u(\tau)|^{p-1} u(\tau)\phi dx d\tau = 0, \tag{2.3.25}$$

for all $\phi \in X$ and a.e. $t \in [0, T]$.

Before proceeding further, we pause to verify that u'' has the desired additional regularity.

Lemma 2.3.5. *The limit function u identified in Corollary* (2.3.2) *verifying identity* (2.3.25) *satisfies $u'' \in L^\infty(0, T; X')$.*

Proof. Let us first note the inclusions $X \subset L^2(\Omega) \subset X'$, where the injections are continuous with dense ranges. In addition,

$$\langle f, \phi \rangle_{X',X} = (f, \phi)_\Omega, \text{ for all } f \in L^2(\Omega) \text{ and all } \phi \in X.$$

Thus, given any $\phi \in X$ we obtain from (2.3.25) that

$$\langle u'(t), \phi \rangle_{X',X} = (u'(t), \phi)_\Omega = (u_1, \phi)_\Omega - \int_0^t (\nabla u(\tau), \nabla \phi)_\Omega d\tau$$
$$+ \int_0^t (w'(\tau), \gamma\phi)_\Gamma d\tau - \int_0^t \int_\Omega |u(\tau)|^{p-1} u(\tau)\phi dx d\tau, \tag{2.3.26}$$

wherein it is clear from (2.3.26) that $\langle u'(t), \phi \rangle_{X',X}$ coincides with an absolutely continuous function on $[0, T]$ with

$$\langle u''(t), \phi \rangle_{X',X} = \frac{d}{dt}(u'(t), \phi)_\Omega = -(\nabla u(t), \nabla \phi)_\Omega + (w'(t), \gamma\phi)_\Gamma$$
$$- \int_\Omega |u(t)|^{p-1} u(t)\phi dx. \tag{2.3.27}$$

By employing Hölder's inequality and the Sobolev Imbedding Theorem, we obtain

$$\begin{aligned} |\langle u''(t), \phi\rangle_{X',X}| &\leq |(\nabla u(t), \nabla\phi)_\Omega| + |(w'(t), \gamma\phi)_\Gamma| + \int_\Omega |u(t)|^p|\phi|dx \\ &\leq \|\nabla u(t)\|_2\|\nabla\phi\|_2 + |w'(t)|_2|\gamma\phi|_2 + \|u(t)\|_{p+1}^p\|\phi\|_{p+1} \\ &\leq C\Big(\|\nabla u(t)\|_2 + |w'(t)|_2 + \|u(t)\|_{p+1}^p\Big)\|\phi\|_X. \end{aligned} \tag{2.3.28}$$

By the regularity enjoyed by u and w as stated in Corollary 2.3.2, we conclude that $u'' \in L^\infty(0,T;X')$. □

2.3.4 Proper verification of (2.2.1)

We now must show that the limit function u satisfies the variational identity (2.2.1) which permits time dependent test functions. By a density arguemnt as in [48, Prop. A.1] it can be shown that the regularity afforded by Lemma 2.3.5 implies the following: For any test function $\phi \in C_w([0,T];X)$ with $\phi_t \in L^2(0,T;L^2(\Omega))$, the function $(u'(t), \phi(t))_\Omega$ coincides with an absolutely continuous function on $[0,T]$ and one has the following product rule in the distributional sense:

$$\frac{d}{dt}(u'(t), \phi(t))_\Omega = \langle u''(t), \phi(t)\rangle_{X',X} + (u'(t), \phi'(t))_\Omega. \tag{2.3.29}$$

With this at hand and noting that the function ϕ in (2.3.25) is time independent, we may express (2.3.25) equivalently as

$$\begin{aligned} \int_0^t \langle u''(\tau), \phi\rangle_{X',X} d\tau + \int_0^t (\nabla u(\tau), \nabla\phi)_\Omega d\tau - \int_0^t (w'(\tau), \gamma\phi)_\Gamma d\tau \\ + \int_0^t \int_\Omega |u(\tau)|^{p-1}u(\tau)\phi dx d\tau = 0, \end{aligned} \tag{2.3.30}$$

for all $\phi \in X$.

As each term in (2.3.30) is absolutely continuous we may differentiate in time and then replace ϕ with $\phi(\tau)$ where the time dependent test function $\phi(\tau)$ satisfying $\phi \in C_w([0,T];X)$ with $\phi_t \in L^2(0,T;L^2(\Omega))$. Integrating the resulting identity on $[0,t]$ and again utilizing the product rule (2.3.29) we obtain the desired identity, namely:

$$\overbrace{(u_t(t),\phi(t))_\Omega - (u_1,\phi(0))_\Omega - \int_0^t (u'(\tau),\phi'(\tau))_\Omega\, d\tau}^{\int_0^t \langle u''(\tau),\phi(\tau)\rangle_{X',X}\, d\tau} + \int_0^t (\nabla u(\tau), \nabla\phi(\tau))_\Omega d\tau$$
$$- \int_0^t (w'(\tau),\gamma\phi(\tau))_\Gamma d\tau + \int_0^t \int_\Omega |u(\tau)|^{p-1} u(\tau)\phi(\tau) dx d\tau = 0, \quad (2.3.31)$$

which is exactly (2.2.1), i.e., the limit function u satisfies the variational identity (3.2.2) in Definition 3.2.3.

2.3.5 Passage to the limit and verification of (2.2.2)

Upon integrating the plate equation in (2.3.6) on $[0,t]$, we obtain:

$$(w_N'(t),\sigma_j)_\Gamma - (w_N'(0),\sigma_j)_\Gamma + \int_0^t (w_N'(\tau),\sigma_j)_\Gamma d\tau + (\gamma u_N(t),\sigma_j)_\Gamma$$
$$- (\gamma u_N(0),\sigma_j)_\Gamma + \int_0^t (\Delta w_N(\tau),\Delta\sigma_j)_\Gamma d\tau = \int_0^t \int_\Gamma h(w_N(\tau))\sigma_j d\Gamma d\tau, \quad (2.3.32)$$

for all $j = 1, \ldots, N$. It follows easily from (2.3.17c)-(2.3.17g) that:

$$\begin{cases} (w_N'(t),\sigma_j)_\Gamma \longrightarrow (w'(t),\sigma_j)_\Gamma \text{ weak}^* \text{ in } L^\infty(0,T) \\ (\Delta w_N(\tau),\Delta\sigma_j)_\Gamma \longrightarrow (\Delta w(\tau),\Delta\sigma_j)_\Gamma \text{ weak}^* \text{ in } L^\infty(0,T) \\ (w_N(t),\sigma_j)_\Gamma \longrightarrow (w(t),\sigma_j)_\Gamma \text{ strongly in } C([0,T]), \\ (\gamma u_N(t),\sigma_j)_\Gamma \longrightarrow (\gamma u(t),\sigma_j)_\Gamma \text{ strongly in } C([0,T]). \end{cases} \quad (2.3.33)$$

for all $j \in \mathbb{N}$.

For the source term in (2.3.32), we show that

$$\int_\Gamma h(w_N(\tau))\sigma_j d\Gamma \longrightarrow \int_\Gamma h(w(\tau))\sigma_j d\Gamma \text{ strongly in } C([0,T]), \text{ as } N \to \infty, \quad (2.3.34)$$

for all $j \in \mathbb{N}$. Indeed, for all $\tau \in [0,T]$ we have

$$\begin{aligned}
&\Big| \int_\Gamma h(w_N(\tau))\sigma_j d\Gamma - \int_\Gamma h(w(\tau))\sigma_j d\Gamma \Big| \\
&\quad \le C \int_\Gamma (|w_N(\tau)|^{q-1} + |w(\tau)|^{q-1} + 1)|w_N(\tau) - w(\tau)||\sigma_j| d\Gamma \\
&\quad \le C(|w_N(\tau)|^{q-1}_{6(q-1)} + |w(\tau)|^{q-1}_{6(q-1)} + 1)|w_N(\tau) - w(\tau)|_2 |\sigma_j|_3 \\
&\quad \le C \sup_{\tau\in[0,T]} |\nabla w_N(\tau) - \nabla w(\tau)|_2 \to 0, \text{ as } N \to \infty, \qquad (2.3.35)
\end{aligned}$$

where we have used in (2.3.35) Hölder's inequality, the Sobolev Imbedding Theorem, and (2.3.17f). Therefore, (2.3.34) follows.

By noting that $\chi_{[0,t]} \in L^1(0,T)$ for $t \in [0,T]$, the strong convergences in (2.3.3)-(2.3.4), and using convergences in (2.3.33)- (2.3.34), we can now pass to the limit as $N \to \infty$ in (2.3.32) to obtain the identity:

$$\begin{aligned}
&(w'(t), \sigma_j)_\Gamma - (w_1, \sigma_j)_\Gamma + \int_0^t (w'(\tau), \sigma_j)_\Gamma d\tau + (\gamma u(t), \sigma_j)_\Gamma \\
&- (\gamma u_0, \sigma_j)_\Gamma + \int_0^t (\Delta w(\tau), \Delta\sigma_j)_\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\sigma_j d\Gamma d\tau, \qquad (2.3.36)
\end{aligned}$$

for all $j \in \mathbb{N}$ and a.e. [0,T].

Since $\{\sigma_n : n \in \mathbb{N}\}$ is an orthonormal basis for $H_0^2(\Gamma)$, then (2.3.36) yields:

$$(w'(t)+\gamma u(t),\eta)_\Gamma - (w_1+\gamma u_0,\eta)_\Gamma + \int_0^t (w'(\tau),\eta)_\Gamma d\tau + \int_0^t (\Delta w(\tau),\Delta\eta)_\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\eta d\Gamma d\tau, \qquad (2.3.37)$$

for all $\eta \in H_0^2(\Gamma)$ and a.e. $t \in [0,T]$.

Before proceeding further, we pause briefly to verify that $\frac{d}{dt}(w'+\gamma u)$ has a desired additional regularity. Namely, we have the following.

Lemma 2.3.6. *The limit functions u and w identified in Corollary* (2.3.2) *verifying identity* (2.3.37) *satisfies* $\frac{d}{dt}\Big(w'+\gamma u\Big) \in L^\infty(0,T;H^{-2}(\Gamma))$.

Proof. In what follows, we shall use the notation $\langle\cdot,\cdot\rangle$ to denote the duality pairing between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$. We first note that $H_0^2(\Gamma) \subset L^2(\Gamma) \subset H^{-2}(\Gamma)$, where the injections are continuous with dense ranges. In addition,

$$\langle f,\eta\rangle = (f,\eta)_\Gamma, \text{ for all } f \in L^2(\Gamma) \text{ and all } \eta \in H_0^2(\Gamma).$$

So, for any $\eta \in H_0^2(\Gamma)$ we obtain from (2.3.37) that

$$\langle w'(t)+\gamma u(t),\eta\rangle = (w'(t)+\gamma u(t),\eta)_\Gamma = (w_1+gu_0,\eta)_\Gamma - \int_0^t (w'(\tau),\eta)_\Gamma d\tau - \int_0^t (\Delta w(\tau),\Delta\eta)_\Gamma d\tau + \int_0^t \int_\Gamma h(w(\tau))\eta d\Gamma d\tau. \qquad (2.3.38)$$

It is evident from (2.3.38) that $\langle w'(t)+\gamma u(t),\eta\rangle$ coincides with an absolutely continuous function on $[0,T]$ with

$$\frac{d}{dt}(w'(t)+\gamma u(t),\eta)_\Gamma = -(w'(t),\eta)_\Gamma - (\Delta w(t),\Delta\eta)_\Gamma + \int_\Gamma h(w(t))\eta d\Gamma, \qquad (2.3.39)$$

for almost all $t \in [0,T]$. In particular, one has

$$\begin{aligned}\Big|\langle\frac{d}{dt}(w'(t)+\gamma u(t)),\eta\rangle\Big| &\le |w'(t)|_2|\eta|_2+|\Delta w(t)|_2|\Delta\eta|_2+C\int_\Gamma(|w(t)|^q+1)|\eta|d\Gamma\\ &\le C\Big(|w'(t)|_2+|\Delta w(t)|_2+|\Delta w(t)|_2^q+1\Big)|\Delta\eta|_2,\end{aligned}\tag{2.3.40}$$

for all $\eta \in H_0^2(\Gamma)$ and for almost all $t \in [0,T]$. By the regularity enjoyed by w as stated in Corollary 2.3.2, we conclude that $\frac{d}{dt}(w'+\gamma u) \in L^\infty(0,T;H^{-2}(\Gamma))$. □

2.3.6 Proper verification of (2.2.2)

We now must show that the limit function w satisfies the variational identity (2.2.2) which permits time dependent test functions. Again, by using [48, Prop. A.1] it can be shown that the regularity afforded by Lemma 2.3.6 implies the following: For any test function $\psi \in C_w([0,T];H_0^2(\Gamma))$ with $\psi_t \in L^2(0,T;L^2(\Gamma))$, the function $(w'(t)+\gamma u(t),\psi(t))_\Gamma$ coincides with an absolutely continuous function on $[0,T]$ and one has the following product rule in the distributional sense:

$$\frac{d}{dt}(w'(t)+\gamma u(t),\psi(t))_\Gamma=\langle\frac{d}{dt}(w'(t)+\gamma u(t)),\psi(t)\rangle+(w'(t)+\gamma u(t),\psi'(t))_\Gamma.\tag{2.3.41}$$

With the validity of (2.3.41) and noting that the function η in (2.3.37) is time independent, we may express (2.3.37) equivalently as

$$\begin{aligned}\int_0^t\langle\frac{d}{d\tau}(w'(\tau)+\gamma u(\tau)),\eta\rangle d\tau+\int_0^t(w'(\tau),\eta)_\Gamma d\tau&\\ +\int_0^t(\Delta w(\tau),\Delta\eta)_\Gamma d\tau&=\int_0^t\int_\Gamma h(w(\tau))\eta d\Gamma d\tau,\end{aligned}\tag{2.3.42}$$

for all $\eta \in H_0^2(\Gamma)$ and all $t \in [0,T]$.

As each term in (2.3.42) is absolutely continuous we may differentiate in time and then replace η with $\psi(\tau)$ where the time dependent test function $\psi(\tau)$ satisfying $\psi \in C_w([0,T]; H_0^2(\Gamma))$ with $\psi_t \in L^2(0,T; L^2(\Gamma))$. Upon integrating the resulting identity on $[0,t]$ and again utilizing the product rule (2.3.41) we obtain the desired identity, namely:

$$\overbrace{(w_t(t)+\gamma u(t), \psi(t))_\Gamma - (w_1+\gamma u(0), \psi(0))_\Gamma - \int_0^t (w_t(\tau)+\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau}^{\int_0^t \langle \frac{d}{d\tau}(w'(\tau)+\gamma u(\tau)), \psi(t)\rangle d\tau}$$
$$+ \int_0^t (\Delta w(\tau), \Delta\psi(\tau)_\Gamma d\tau + \int_0^t (w_t(\tau), \psi(\tau)_\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\psi(\tau) d\Gamma d\tau, \quad (2.3.43)$$

which is precisely (2.2.2).

2.3.7 Additional regularity of solutions

In order to complete the proof of the existence statement of Theorem 2.2.5, we need to verify that the limit functions u and w identified in Corollary 2.3.2 satisfy the additional regularity as stated in of Definition 2.2.3. For this purpose, we shall use a well-known result which often attributed to Lions and Magenes, as in [46, Lem. 8.1].

Proposition 2.3.7. *Up to possible modification on a set of measure zero, the limit functions u and w identified in Corollary 2.3.2 satisfy:*

$$\begin{cases} u \in C_w([0,T]; X),\ u_t \in C_w([0,T]; L^2(\Omega)), \\ w \in C_w([0,T]; H_0^2(\Gamma)),\ w_t \in C_w([0,T]; L^2(\Gamma)). \end{cases} \quad (2.3.44)$$

Proof. As the proofs of both parts in (2.3.44) are similar, we only present the proof of the second statement. We note here that $H_0^2(\Gamma) \subset L^2(\Gamma) \subset H^{-2}(\Gamma)$ where the injections are continuous with dense ranges, then by [46, Lem. 8.1, p. 275]

$$L^\infty(0,T;H_0^2(\Gamma)) \cap C_w([0,T];L^2(\Gamma)) = C_w([0,T];H_0^2(\Gamma)). \tag{2.3.45}$$

Since we know $w \in L^\infty(0,T;H_0^2(\Gamma))$ and $w_t \in L^\infty(0,T;L^2(\Gamma))$, then after a possible modification on a set of measure zero, $w \in C([0,T];L^2(\Gamma))$. It follows from (2.3.45) that $w \in C_w([0,T];H_0^2(\Gamma))$.

Also, we recall from Lemma 2.3.6 that $\frac{d}{dt}(w' + \gamma u) \in L^\infty(0,T;H^{-2}(\Gamma))$ and since $w' + \gamma u \in L^\infty(0,T;L^2(\Gamma))$, then up to possible modification on a set of measure zero, we conclude that $w' + \gamma u \in C([0,T];H^{-2}(\Gamma))$. However, we know from (2.3.17g) that $\gamma u \in C([0,T];L^2(\Gamma))$, and so it must be the case that $w_t \in C([0,T];H^{-2}(\Gamma))$. Hence, by a similar reasoning as in (2.3.45) above, it follows that $w_t \in C_w([0,T];L^2(\Gamma))$, completing the proof. □

2.3.8 Hidden regularity of $u_t|_{\Gamma\times(0,t)}$

We first note that u satisfies the problem:

$$\begin{cases} u_{tt} - \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1) \in \left(H^1_{\Gamma_0}(\Omega) \cap L^{p+1}(\Omega)\right) \times L^2(\Omega), & \end{cases} \tag{2.3.46}$$

where $w_t \in C_w([0,T];L^2(\Gamma))$ is the solution of the plate equation in (3.1.1) which is regarded as a boundary feedback for u.

We define the Dirichlét-Neumann map: $R : H^s(\Gamma) \longrightarrow H^{s+\frac{3}{2}}(\Omega) \cap H^1_{\Gamma_0}(\Omega)$; $s \geq 0$

by:

$$q = Rp \iff q \text{ is the weak solution of the problem } \begin{cases} \Delta q = 0 \text{ in } \Omega, \\ q = 0 \text{ on } \Gamma_0, \\ \partial_\nu q = p \text{ on } \Gamma. \end{cases} \tag{2.3.47}$$

It is well-known that R is continuous from $H^s(\Gamma)$ to $H^{s+\frac{3}{2}}(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, for $s \geq 0$ (see for instance Lasiecka and Triggiani [43, 44]). We also introduce the Dirichlét-Neumann Laplacian: $A = -\Delta : D(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$, with its domain $D(A) = \{u \in H^2(\Omega) : u|_{\Gamma_0} = 0,\ \partial_\nu u|_\Gamma = 0\}$. In addition, recall the sine and cosine operators $S(t)$ and $C(t)$ associated with A. Specifically, $S(t), C(t) : L^2(\Omega) \to L^2(\Omega)$ which are given by $S(t) = A^{-1/2}\sin(A^{1/2}t)$ and $C(t) = \cos(A^{1/2}t)$, $t \geq 0$. We refer the reader to [43, 44]) for more details on the sine and cosine operators.

With these operators at hand, then u must satisfy the integral equation:

$$u(t) = C(t)u_0 + S(t)u_1 + (Kf)(t) + (Lw_t)(t), \tag{2.3.48}$$

where $f(u(\tau)) = -|u(\tau)|^{p-1}u(\tau)$ and

$$\begin{cases} S(t)u_1 = \int_0^t C(\tau)u_1 d\tau, \\ (Kf)(t) = \int_0^t S(t-\tau)f(u(\tau))d\tau, \\ (Lw_t)(t) = \int_0^t AS(t-\tau)Rw_t(\tau)d\tau. \end{cases}$$

Our goal in this section is to obtain better regularity for $u_t|_{\Gamma\times(0,t)}$ than what has been stated in Lemma 2.3.6. In order to do so, we restrict the values of p to the range $1 \leq p \leq 3$. Since we already know that $u \in C_w([0,T]; H^1_{\Gamma_0}(\Omega))$ and $u_t \in C_w([0,T]; L^2(\Omega))$, then it follows easily that $f(u) \in H^1(0,T;L^2(\Omega))$, for $1 \leq p \leq 3$. Then, by Corollary 5.3 in [44] it follows that

$$\begin{cases} (Kf) \in C([0,T],D(A)), \\ \frac{d}{dt}(Kf) \in C([0,T],H^1(\Omega))). \end{cases}$$

In particular, one has

$$\gamma \circ (\frac{d}{dt}(Kf)) \in C([0,T],H^{1/2}(\Gamma))). \tag{2.3.49}$$

On the other hand, by the recent results by Triggiani (see Theorem 1.1 in [57], see also [56] for similar results), and since we already know that $w_t \in C_w([0,T];L^2(\Gamma))$, then the best trace regularity of Lw_t on Σ is given by:

$$Lw_t|_{\Sigma} \in H^{1/3}(\Sigma), \text{ where } \Sigma = \Gamma \times (0,T).$$

Therefore,

$$(\frac{d}{dt}Lw_t)|_{\Sigma} \in H^{-2/3}(\Sigma). \tag{2.3.50}$$

In view of (2.3.49)-(2.3.50) and the well-known properties of $C(t)$ and $S(t)$ [43, 44, 57], then by assuming $u_0 = u_1 = 0$; or else assume $u_0 \in H^1_{\Gamma_0}(\Omega) \cap H^{1+\theta}(\Omega)$, $u_1 \in H^{\theta}(\Omega)$, for some $\theta > 0$, then it follows from (2.3.48) that the best hidden regularity of u_t on Σ is:

$$u_t|_{\Sigma} \in H^{-2/3}(\Sigma). \tag{2.3.51}$$

Moreover, the action of $u_t|_{\Gamma\times(0,t)}$ on a test function ψ, (where ψ is as described in Definition 2.2.3), is given by:

$$\langle u_t|_{\Gamma\times(0,t)}, \psi\rangle = (\gamma u(t), \psi(t))_\Gamma - (\gamma u(0), \psi(0))_\Gamma - \int_0^t (\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau, \qquad (2.3.52)$$

where the terms on the right hand side of (2.3.52) are precisely the corresponding terms appearing in (2.2.2) (see Definition 2.2.3).

We emphasize here that the regularity obtained in (2.3.50) for $u_t|_\Sigma$ is achieved with the additional assumptions:

$$\begin{cases} 1 \le p \le 3, \\ u_0 = u_1 = 0, \text{ or } u_0 \in H^1_{\Gamma_0}(\Omega) \cap H^{1+\theta}(\Omega), \; u_1 \in H^\theta(\Omega), \text{ for some } \theta > 0. \end{cases} \qquad (2.3.53)$$

2.4 Energy Identity and Energy Inequality

This section is devoted to derive the energy identity (2.2.3) in Theorem 2.2.5 in the case $1 \le p \le 3$. One is tempted to test (2.2.1) with u_t and (2.2.2) with w_t, and carry out standard calculations to obtain energy identity. However, this procedure is only *formal*, since u_t and w_t are not regular enough and cannot be used as test functions in (2.2.1) and (2.2.2). In order to overcome this technicality we shall use the difference quotients $D_h u$ and $D_h w$ and their well-known properties that appeared in [39] and later in [33, 52, 54]. We remind the reader that the space $X = H^1_{\Gamma_0}(\Omega) \cap L^{p+1}(\Omega)$ will be replaced simply by $X = H^1_{\Gamma_0}(\Omega)$, since $1 \le p \le 3$ in this section. Properties of the Difference Quotient for general Banach Spaces are outlined in the Appendix.

2.4.1 Proof of Energy Identity

Throughout the proof, we fix $t \in (0,T)$ and let (u,w) be a weak solution of the system (2.1.1) on $[0,T]$ in the sense of Definition 2.2.3. Recall the regularity of u and w, namely: $u \in C_w([0,T]; H^1_{\Gamma_0}(\Omega))$, $u_t \in C_w([0,T]; L^2(\Omega))$, $w \in C_w([0,T]; H^2_0(\Gamma))$, and $w_t \in C_w([0,T]; L^2(\Gamma))$. As such, we can define the difference quotient $D_h u(\tau)$ on $[0,t]$ as in (A.0.1), i.e., $D_h u(\tau) = \frac{1}{2h}[u_e(\tau+h) - u_e(\tau-h)]$, where $u_e(\tau)$ extends $u(\tau)$ from $[0,t]$ to $\mathbb{R}$ as in (A.0.2); and with a similar definition of the difference quotient $D_h w(\tau)$ on $[0,t]$. In what follows, we may abuse notation by writing $u(\tau)$, $w(\tau)$ in place of $u_e(\tau)$, $w_e(\tau)$, and in particular we remind the reader here that $u'(\tau) = w'(\tau) = 0$ outside the segment $[0,t]$.

We aim to first show that $D_h u(\tau)$ and $D_h w(\tau)$ satisfy the required regularity conditions to be suitable test functions in Definition 2.2.3. Indeed, since $u \in C_w([0,t]; H^1_{\Gamma_0}(\Omega))$ and $w \in C_w([0,t]; H^2_0(\Gamma))$, then clearly

$$D_h u \in C_w([0,t]; H^1_{\Gamma_0}(\Omega)) \text{ and } D_h w \in C_w([0,t]; H^2_0(\Gamma)). \tag{2.4.1}$$

In addition, for $0 < h < \frac{t}{2}$ we note:

$$(D_h u)_t(\tau) = \begin{cases} \frac{1}{2h}[u_t(\tau+h) - u_t(\tau-h)], & \text{if } h < \tau < t-h, \\ -\frac{1}{2h} u_t(\tau-h), & \text{if } t-h < \tau < t, \\ \frac{1}{2h} u_t(\tau+h), & \text{if } 0 < \tau < h, \end{cases}$$

with a similar definition for $(D_h w)_t(\tau)$.

Since $u_t \in C_w([0,t];L^2(\Omega))$ and $w_t \in C_w([0,t];L^2(\Gamma))$, then it follows that:

$$(D_h u)_t \in L^2(0,t;L^2(\Omega)) \text{ and } (D_h w)_t \in L^2(0,t;L^2(\Gamma)). \tag{2.4.2}$$

Thus, (2.4.1)-(2.4.2) show that $D_h u$ and $D_h w$ satisfy the required regularity conditions to be suitable test functions in Definition 2.2.3. Therefore, by taking $\phi = D_h u$ in (2.2.1) and $\psi = D_h w$ in (2.2.2), we obtain (the variable τ is being suppressed within the following integrals):

$$\begin{aligned}(u_t(t), D_h u(t))_\Omega - (u_1, D_h u(0))_\Omega - \int_0^t (u_t, (D_h u)_t)_\Omega d\tau + \int_0^t (\nabla u, \nabla D_h u)_\Omega d\tau \\ - \int_0^t (w_t, \gamma D_h u)_\Gamma d\tau + \int_0^t \int_\Omega |u|^{p-1} u D_h u dx d\tau = 0,\end{aligned} \tag{2.4.3}$$

$$\begin{aligned}(w_t(t) + \gamma u(t), D_h w(t))_\Gamma - (w_1 + \gamma u(0), D_h w(0))_\Gamma - \int_0^t (w_t, (D_h w)_t)_\Gamma d\tau \\ - \int_0^t (\gamma u, (D_h w)_t)_\Gamma d\tau + \int_0^t (\Delta w, \Delta D_h w)_\Gamma d\tau + \int_0^t (w_t, D_h w)_\Gamma d\tau \\ = \int_0^t \int_\Gamma h(w) D_h w d\Gamma d\tau.\end{aligned} \tag{2.4.4}$$

We now justify passing to the limit as $h \longrightarrow 0$ in (2.4.3)-(2.4.4) as follows:

By using Proposition A.0.4 with $Y = Z = L^2(\Omega)$, then as $h \to 0$,

$$\begin{cases} D_h u \longrightarrow u_t \text{ in } L^2(\Omega \times (0,t)), \\ D_h w \longrightarrow w_t \text{ in } L^2(\Gamma \times (0,t)). \end{cases} \tag{2.4.5}$$

Since u, $u_t \in C_w([0,t];L^2(\Omega))$ and w, $w_t \in C_w([0,t];L^2(\Gamma))$, then as $h \to 0$, it follows from (A.0.6) that

$$D_h u(0) \longrightarrow \frac{1}{2}u_t(0) \text{ and } D_h u(t) \longrightarrow \frac{1}{2}u_t(t) \text{ weakly in } L^2(\Omega),$$
$$D_h w(0) \longrightarrow \frac{1}{2}w_t(0) \text{ and } D_h w(t) \longrightarrow \frac{1}{2}w_t(t) \text{ weakly in } L^2(\Gamma).$$

Therefore,

$$\begin{cases} \lim_{h\to 0}\Big((u_t(t), D_h u(t))_\Omega - (u_1, D_h u(0))_\Omega\Big) = \frac{1}{2}\Big(\|u_t(t)\|_2^2 - \|u_t(0)\|_2^2\Big), \\ \lim_{h\to 0}(w_t(t) + \gamma u(t), D_h w(t))_\Gamma = \frac{1}{2}|w_t(t)|_2^2 + \frac{1}{2}(\gamma u(t), w_t(t))_\Gamma, \\ \lim_{h\to 0}(w_1 + \gamma u(0), D_h w(0))_\Gamma = \frac{1}{2}|w_t(0)|_2^2 + \frac{1}{2}(\gamma u(0), w_t(0))_\Gamma. \end{cases} \tag{2.4.6}$$

Also, by (A.0.4)

$$\int_0^t (u_t, (D_h u)_t)_\Omega d\tau = \int_0^t (w_t, (D_h w)_t)_\Gamma d\tau = 0. \tag{2.4.7}$$

In addition, since $u \in C_w([0,t]; H^1_{\Gamma_0}(\Omega))$ and $w \in C_w([0,t]; H^2_0(\Gamma))$, then (A.0.3) yields:

$$\begin{cases} \lim_{h\to 0}\int_0^t (\nabla u, \nabla D_h u)_\Omega d\tau = \frac{1}{2}\Big(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2\Big), \\ \lim_{h\to 0}\int_0^t (\Delta w, \Delta D_h w)_\Gamma d\tau = \frac{1}{2}\Big(|\Delta w(t)|_2^2 - |\Delta w(0)|_2^2\Big). \end{cases} \tag{2.4.8}$$

An immediate consequence of (2.4.5) is that

$$\lim_{h\longrightarrow 0}\int_0^t (w_t, D_h w)_\Gamma d\tau = \int_0^t |w_t(\tau)|_2^2 d\tau. \tag{2.4.9}$$

Also, since $u \in C_w([0,T]; H^1_{\Gamma_0}(\Omega))$, then $u \in L^\infty(0,T; L^6(\Omega))$, by the Sobolev

Imbedding Theorem. The assumption $1 \leq p \leq 3$ yields,

$$\left\| |u(t)|^{p-1}u(t) \right\|_2 = \|u(t)\|_{2p}^p \leq C \|u\|_{L^\infty(0,T;H^1_{\Gamma_0}(\Omega))} < \infty.$$

Consequently, $|u|^{p-1}u \in L^2(\Omega \times (0,t))$, and from (2.4.5) we have

$$\lim_{h\to 0} \int_0^t \int_\Omega |u|^{p-1} u D_h u dx d\tau = \int_0^t \int_\Omega |u|^{p-1} u u_t dx d\tau. \tag{2.4.10}$$

In addition, since $w \in C_w([0,T]; H_0^2(\Gamma))$, then $w \in L^\infty(0,T; L^{2q}(\Gamma))$ for all $1 \leq q \leq \infty$. Thus, the bound imposed on h in Remark 2.2.2 implies $h(w) \in L^2(\Gamma \times (0,T))$. As such, (2.4.5) implies

$$\lim_{h\longrightarrow 0} \int_0^T \int_\Gamma h(w) D_h w d\Gamma d\tau = \int_0^T \int_\Gamma h(w) w_t d\Gamma d\tau. \tag{2.4.11}$$

The trouble terms $\int_0^t (\gamma u(\tau), D_h w_t(\tau))_\Gamma d\tau$ and $\int_0^t (w_t(\tau), \gamma D_h u(\tau))_\Gamma d\tau$ are handled as follows. For all sufficiently small $h > 0$, we have

$$\begin{aligned}
&\int_0^t (\gamma u(\tau), D_h w_t(\tau))_\Gamma d\tau \\
&\qquad = \frac{1}{2h}\Big(\int_0^t (\gamma u(\tau), w_t(\tau + h))_\Gamma d\tau - \int_0^t (\gamma u(\tau), w_t(\tau - h))_\Gamma d\tau \Big) \\
&\qquad = \frac{1}{2h}\Big(\int_h^t (\gamma u(\tau - h), w_t(\tau))_\Gamma d\tau - \int_0^{t-h} (\gamma u(\tau + h), w_t(\tau))_\Gamma d\tau \Big),
\end{aligned} \tag{2.4.12}$$

where we have used a change of variables in (2.4.12) and the fact that $w_t = 0$ outside the interval $[0,t]$. By rearranging the terms in (2.4.12), we obtain

$$\begin{aligned}
\int_0^t (\gamma u(\tau), D_h w_t(\tau))_\Gamma d\tau &= -\int_0^t (\gamma D_h u(\tau), w_t(\tau)) d\tau \\
&\quad - \frac{1}{2h}\Big(\int_0^h (\gamma u(\tau - h), w_t(\tau))_\Gamma d\tau - \int_{t-h}^t (\gamma u(\tau + h), w_t(\tau))_\Gamma d\tau \Big)
\end{aligned} \tag{2.4.13}$$

We now utilize the weak continuity of w_t in the last two term in (2.4.13) as follows.

$$\begin{aligned}
\frac{1}{2h}\int_0^h (\gamma u(\tau-h), w_t(\tau))_\Gamma d\tau &= \frac{1}{2h}\int_0^h (\gamma u(0), w_t(\tau))_\Gamma d\tau \\
&= \frac{1}{2h}\int_0^h (\gamma u(0), w_t(\tau)-w_t(0))_\Gamma d\tau + \frac{1}{2h}\int_0^h (\gamma u(0), w_t(0))_\Gamma d\tau \\
&\longrightarrow \frac{1}{2}(\gamma u(0), w_t(0))_\Gamma, \text{ as } h \longrightarrow 0. \qquad (2.4.14)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{2h}\int_{t-h}^t (\gamma u(\tau+h), w_t(\tau))_\Gamma d\tau &= \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(\tau))_\Gamma d\tau \\
&= \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(\tau)-w_t(t))_\Gamma d\tau + \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(t))_\Gamma d\tau \\
&\longrightarrow \frac{1}{2}(\gamma u(t), w_t(t))_\Gamma, \text{ as } h \longrightarrow 0. \qquad (2.4.15)
\end{aligned}$$

Finally, by adding (2.4.3)-(2.4.4) and by combining the results established in (2.4.6)-(2.4.15) we can pass to the limit as $h \longrightarrow 0$ to obtain the energy identity (2.2.3).

2.4.2 Energy Inequality

In order to complete the proof of Theorem 2.2.5 in the case where $p > 3$ it remains only to establish the energy inequalities (2.2.5)-(2.2.6) which are given in Proposition 2.4.4 below. But, we first shall need some ancillary results regarding the the sequences of approximate solutions $\{u_N\}_1^\infty$ and $\{w_N\}_1^\infty$ which satisfy the conclusions of Corollary 2.3.2.

Proposition 2.4.1. *Let $\{u_N\}_1^\infty$ be the sequence of approximate solutions satisfying the conclusions of Corollary 2.3.2. Then, there is a subsequence, still labeled as*

$\{u_N\}_1^\infty$, *such that:*

$$u'_N(t) \to u'(t) \text{ weakly in } L^2(\Omega), \text{ as } N \to \infty, \text{ for all } t \in [0,T]. \tag{2.4.16}$$

Proof. Let us first note that the boundedness of the sequence $\{u_N\}_1^\infty$ in $L^\infty(0,T;X)$ implies that, the sequence $\{|u_N|^{p-1}u_N\}_1^\infty$ is bounded in $L^\infty(0,T;L^{\frac{p+1}{p}}(\Omega))$. Thus, on a subsequence labeled by $\{u_N\}_1^\infty$, we have

$$|u_N|^{p-1}u_N \longrightarrow \xi \quad \text{weak}^* \text{ in } L^\infty(0,T;L^{\frac{p+1}{p}}(\Omega)).$$

However, from the strong convergence in (2.3.17e) we conclude (on a subsequence) that

$$|u_N|^{p-1}u_N \longrightarrow |u|^{p-1}u \text{ point-wise a.e. in } \Omega \times (0,T).$$

Hence, $\xi = |u|^{p-1}u$ a.e. in $\Omega \times (0,T)$. That is,

$$|u_N|^{p-1}u_N \longrightarrow |u|^{p-1}u \quad \text{weak}^* \text{ in } L^\infty(0,T;L^{\frac{p+1}{p}}(\Omega)). \tag{2.4.17}$$

From the first equation in (2.3.6) along with (2.3.20)-(2.3.21) and (2.4.17), we obtain, as $N \longrightarrow \infty$,

$$\begin{aligned}&(u''_N, e_j)_\Omega \\ &\quad \to -(\nabla u, \nabla e_j)_\Omega + (w', \gamma e_j)_\Gamma - \int_\Omega |u|^{p-1}ue_j dx \text{ weak}^* \text{ in } L^\infty(0,T),\end{aligned} \tag{2.4.18}$$

for all $j \in \mathbb{N}$. By comparing (2.4.18) with (2.3.27), it follows that

$$\frac{d}{dt}(u'_N, e_j)_\Omega \longrightarrow \frac{d}{dt}(u', e_j)_\Omega \quad \text{weak}^* \text{ in } L^\infty(0,T), \text{ for all } j \in \mathbb{N}. \tag{2.4.19}$$

Since $\chi_{[0,t]} \in L^1(0,T)$ for $t \in [0,T]$, then by integrating (2.4.19) over $[0,t]$, we obtain

$$(u_N'(t), e_j)_\Omega - (u_N'(0), e_j)_\Omega \longrightarrow (u'(t), e_j)_\Omega - (u'(0), e_j)_\Omega, \text{ as } N \longrightarrow \infty,$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$. By the strong convergence in (2.3.3), it follows that

$$(u_N'(t), e_j)_\Omega \longrightarrow (u'(t), e_j)_\Omega, \text{ as } N \longrightarrow \infty, \tag{2.4.20}$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$.

Now, for any $\phi \in X$, there exists a sequence $\phi_k = \sum_{j=1}^k a_{k,j} e_j$ such that $\phi_k \to \phi$ strongly in X. By linearity, one can replace e_j in (2.4.20) with ϕ_k to obtain

$$(u_N'(t), \phi_k)_\Omega \longrightarrow (u'(t), \phi_k)_\Omega, \text{ as } N \longrightarrow \infty, \text{ for all } t \in [0,T]. \tag{2.4.21}$$

Thus, by using (2.4.21) and the strong convergence of $\{\phi_k\}_{k=1}^\infty$ in X, we have for all $t \in [0,T]$:

$$\begin{aligned}
\left|(u_N'(t), \phi)_\Omega - (u'(t), \phi)_\Omega\right| &\leq \left|(u_N'(t), \phi)_\Omega - (u_N'(t), \phi_k)_\Omega\right| \\
&\quad + \left|(u_N'(t), \phi_k)_\Omega - (u'(t), \phi_k)_\Omega\right| + \left|(u'(t), \phi_k)_\Omega - u'(t), \phi)_\Omega\right| \\
&\leq \|u_N'(t)\|_2 \|\phi - \phi_k\|_2 + \left|(u_N'(t) - u'(t), \phi_k)_\Omega\right| + \|u'(t)\|_2 \|\phi - \phi_k\|_2 \\
&\leq C \|\phi - \phi_k\|_2 + \left|(u_N'(t) - u'(t), \phi_k)_\Omega\right| \longrightarrow 0, \text{ as } N, k \longrightarrow \infty.
\end{aligned} \tag{2.4.22}$$

That is, for all $\phi \in X$,

$$(u_N'(t), \phi)_\Omega \longrightarrow (u'(t), \phi)_\Omega, \text{ as } N \longrightarrow \infty, \text{ for all } t \in [0,T]. \tag{2.4.23}$$

Since the space X is dense in $L^2(\Omega)$, then by a similar density argument as in (2.4.22), we conclude that (2.4.23) remains valid for all $\phi \in L^2(\Omega)$, which completes the proof of the proposition. $\square$

Proposition 2.4.2. *The sequence of approximate solutions* $\{w_N\}_1^\infty$ *satisfying the conclusions of Corollary 2.3.2 also satisfies:*

$$w'_N(t) \to w'(t) \text{ weakly in } L^2(\Gamma), \text{ as } N \to \infty, \text{ for all } t \in [0,T]. \tag{2.4.24}$$

Proof. From the second equation in (2.3.6) along with (2.3.33)-(2.3.34) and (2.3.17a), we have, as $N \longrightarrow \infty$,

$$\begin{aligned}(w''_N + \gamma u'_N, \sigma_j)_\Gamma \longrightarrow &- (\Delta w, \Delta\sigma_j)_\Gamma - (w', \sigma_j)_\Gamma \\ &+ \int_\Gamma h(w)\sigma_j d\Gamma \text{ weak}^* \text{ in } L^\infty(0,T),\end{aligned} \tag{2.4.25}$$

for all $j \in \mathbb{N}$. By comparing (2.4.25) with (2.3.39), we conclude that

$$\begin{aligned}&\frac{d}{dt}(w'_N + \gamma u_N, \sigma_j)_\Gamma \\ &\longrightarrow \frac{d}{dt}(w' + \gamma u, \sigma_j)_\Gamma \quad \text{weak}^* \text{ in } L^\infty(0,T), \text{ for all } j \in \mathbb{N}.\end{aligned} \tag{2.4.26}$$

Again, as $\chi_{[0,t]} \in L^1(0,T)$ for $t \in [0,T]$, then (2.4.26) implies that

$$\begin{aligned}(w'_N(t) + \gamma u_N(t), \sigma_j)_\Gamma - (w'_N(0) + \gamma u_N(0), \sigma_j)_\Gamma &\longrightarrow (w'(t) + \gamma u(t), \sigma_j)_\Gamma \\ &- (w'(0) + \gamma u(0), \sigma_j)_\Gamma, \text{ as } N \longrightarrow \infty,\end{aligned} \tag{2.4.27}$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$. By the strong convergence in (2.3.3) and the continuity of trace operator γ, it follows that

$$(w_N'(t) + \gamma u_N(t), \sigma_j)_\Gamma \longrightarrow (w'(t) + \gamma u(t), \sigma_j)_\Gamma, \text{ as } N \longrightarrow \infty,$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$. However, the strong convergence in (2.3.17g) yields,

$$(w_N'(t), \sigma_j)_\Gamma \longrightarrow (w'(t), \sigma_j)_\Gamma, \text{ as } N \longrightarrow \infty, \tag{2.4.28}$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$. Now, the rest of the proof goes exactly as in the proof of Proposition 2.4.1 by using a density argument. □

Proposition 2.4.3. *Let $\{u_N\}_1^\infty$ and $\{w_N\}_1^\infty$ be the sequences of approximate solutions satisfying the conclusions of Corollary 2.3.2. Then, there are subsequences, still labeled as $\{u_N\}_1^\infty$ and $\{w_N\}_1^\infty$, such that, as $N \longrightarrow \infty$*

$$\begin{cases} u_N(t) \longrightarrow u(t) \text{ weakly in } L^{p+1}(\Omega), \text{ a.e. } [0,T], \\ u_N(t) \longrightarrow u(t) \text{ weakly in } H^1_{\Gamma_0}(\Omega), \text{ a.e. } [0,T], \\ w_N(t) \longrightarrow w(t) \text{ weakly in } H^2_0(\Gamma), \text{ a.e. } [0,T]. \end{cases} \tag{2.4.29}$$

Proof. Since the sequence $\{u_N\}_1^\infty$ is bounded in $L^\infty(0,T;X)$, then in particular it is bounded in $L^1(0,T;L^{p+1}(\Omega))$. Thus, on a subsequence, it follows that

$$u_N \longrightarrow u \quad \text{weakly in } L^1(0,T;L^{p+1}(\Omega)), \text{ as } N \longrightarrow \infty. \tag{2.4.30}$$

Thanks to the strong convergence in (2.3.17e) which implies

$$u_N \longrightarrow u \quad \text{strongly in } L^1(0,T;L^2(\Omega)), \text{ as } N \longrightarrow \infty. \tag{2.4.31}$$

Since $L^{p+1}(\Omega) \subset L^2(\Omega) \subset L^{\frac{p+1}{p}}(\Omega)$, then the first convergence in (2.4.29) follows from Proposition A.0.2 in the Appendix. The other two convergences in (2.4.29) are also routine conclusions of Proposition A.0.2. □

Proposition 2.4.4. *The limit functions u and w identified in Corollary 2.3.2 satisfy the energy inequalities* (2.2.5) *and* (2.2.6) *in the statement of Theorem 2.2.5.*

Proof. From (2.3.11) in the course of establishing the a priori estimates it was shown that each u_N satisfies for all $t \in [0,T]$:

$$\mathscr{E}_N(t) + \int_0^t |w_N'(\tau)|_2^2 d\tau = \mathscr{E}_N(0) + \int_0^t \int_\Gamma h(w_N(\tau))w_N'(\tau)d\Gamma d\tau, \tag{2.4.32}$$

where $\mathscr{E}_N(t)$ is the positive energy of the system given by:

$$\mathscr{E}_N(t) = \frac{1}{2}\left(\|u_N'(t)\|_2^2 + \|\nabla u_N(t)\|_2^2 + |w_N'(t)|_2^2 + |\Delta w_N(t)|_2^2\right) + \frac{1}{p+1}\|u_N(t)\|_{p+1}^{p+1}.$$

By taking $H(w) = \int_0^w h(s)\,ds$ as the primitive of h, then (2.4.32) becomes

$$\mathscr{E}_N(t) + \int_0^t |w_N'(\tau)|_2^2 d\tau = \mathscr{E}_N(0) + \int_\Gamma H(w_N(t))d\Gamma - \int_\Gamma H(w_N(0))d\Gamma. \tag{2.4.33}$$

By defining the total energy by

$$E_N(t) = \mathscr{E}_N(t) - \int_\Gamma H(w_N(t))d\Gamma,$$

we may recast (2.4.33) as

$$E_N(t) + \int_0^t |w_N'(\tau)|_2^2 d\tau = E_N(0). \tag{2.4.34}$$

From the mean value theorem and the polynomial bound for h in Remark 2.2.2, we have

$$\Big|\int_\Gamma \Big(H(w_N(t)) - H(w(t))\Big)d\Gamma\Big| \le C\int_G (1 + |w_N(t)|^q + |w(t)|^q)|w_N(t) - w(t)|d\Gamma$$
$$\le C(1 + |w_N(t)|_{2q}^q + |w(t)|_{2q}^q)|w_N(t) - w(t)|_2$$
$$\le C \sup_{t\in[0,T]} |\nabla w_N(t) - \nabla w(t)|_2 \longrightarrow 0, \text{ as } N \to \infty, \tag{2.4.35}$$

where we have used in (2.4.35) Hölder's inequality, the Sobolev Imbedding Theorem, and (2.3.17f). Hence,

$$\lim_{N\longrightarrow\infty} \int_\Gamma H(w_N(t))d\Gamma = \int_\Gamma H(w(t))d\Gamma, \text{ for all } t \in [0,T]. \tag{2.4.36}$$

Now, by taking the "$\liminf_{N\to\infty}$" in (2.4.34), we obtain

$$\liminf_{N\to\infty} E_N(t) + \liminf_{N\to\infty} \int_0^t |w_N'(\tau)|_2^2 d\tau \le \liminf_{N\to\infty} E_N(0) = E(0), \tag{2.4.37}$$

were we have used (2.4.36) and the strong convergence in (2.3.3)-(2.3.4).

Using the weak lower-semicontinuity of norms, Fatou's Lemma, and (2.4.36) along with Proposition 2.4.1-Proposition 2.4.3, we obtain for almost all $t \in [0,T]$,

$$\liminf_{N\to\infty} E_N(t) + \liminf_{N\to\infty} \int_0^t |w_N'(\tau)|_2^2 d\tau \ge \liminf_{N\to\infty} \mathscr{E}_N(t) + \int_0^t |w'(\tau)|_2^2 d\tau$$
$$- \lim_{N\to\infty} \int_\Gamma H(w_N(t))d\Gamma \ge \mathscr{E}(t) + \int_0^t |w'(\tau)|_2^2 d\tau - \int_\Gamma H(w(t))d\Gamma. \tag{2.4.38}$$

Combining (2.4.37) with (2.4.38), we obtain

$$\mathscr{E}(t) + \int_0^t |w'(\tau)|_2^2 d\tau - \int_\Gamma H(w(t))d\Gamma \le E(0) \text{ a.e. } [0,T], \tag{2.4.39}$$

which is precisely the desired energy inequality (2.2.6).

Finally, the energy inequality (2.2.5) is easily obtained after showing

$$\lim_{N\longrightarrow\infty}\int_0^t\int_\Gamma h(w_N(\tau))w_N'(\tau)d\Gamma d\tau=\int_0^t\int_\Gamma h(w(\tau))w'(\tau)d\Gamma d\tau. \tag{2.4.40}$$

The proof of (2.4.40) is similar to the proof of (2.4.36), and thus it is omitted. □

2.5 Global Existence

This section is devoted to prove the existence of global solutions as described in Theorem 2.2.6. As in [1, 33, 48] and other works, it is the case here that either a given solution (u, w) must exist globally in time or else one may find a value of T_0 with $0 < T_0 < \infty$ so that

$$\limsup_{t\to T_0^-}\left(\mathscr{E}(t)+\int_0^t|w_t(\tau)|_2^2d\tau\right)=\infty, \tag{2.5.1}$$

where, $\mathscr{E}(t)=\frac{1}{2}\left(\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2+|w_t(t)|_2^2+|\Delta w(t)|_2^2\right)+\frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}$.

By demonstrating a bound on the energy

$$\mathscr{E}(t)+\int_0^t|w_t(\tau)|_2^2d\tau$$

on every interval $[0, T]$ which is dependent only upon T and the positive initial energy $\mathscr{E}(0)$, we shall show that the scenario in (2.5.1) cannot occur as the argument is bounded on any finite interval. This bound is possible provided the source term acting on the plate is essentially linear. Indeed, this assertion is contained in the following proposition.

Proposition 2.5.1. *Let (u, w) be a weak solution of (2.1.1) on $[0, T]$ as furnished by Theorem 2.2.5.*

- *If $q = 1$, then for all $t \in [0, T]$, (u, w) satisfies*

$$\mathscr{E}(t) + \int_0^t |w(\tau)|_2^2 d\tau \leq C(T, \mathscr{E}(0)), \tag{2.5.2}$$

where $0 < T < \infty$ is aribitrary.

- *If $q > 1$, then the bound in (2.5.2) holds for all $0 \leq t \leq T'$, where $0 < T' \leq T$ and T' depending upon T and $\mathscr{E}(0)$.*

Proof. Recall the energy inequality in (2.2.5):

$$\mathscr{E}(t) + \int_0^t |w_t(\tau)|_2^2 d\tau \leq \mathscr{E}(0) + \int_0^t \int_\Gamma h(w) w_t d\Gamma d\tau. \tag{2.5.3}$$

By noting the polynomial bound on h in Remark 2.2.2 with $q = 1$ along with Hölder's and Young's inequalities, we have:

$$\begin{aligned} \left| \int_0^t \int_\Gamma h(w) w_t d\Gamma d\tau \right| &\leq C \int_0^t \int_\Gamma (|w(\tau)| + 1) w_t(\tau) d\Gamma d\tau \\ &\leq C \int_0^t (|w(\tau)|_2 + 1)|w_t(\tau)|_2 d\tau \\ &\leq \frac{1}{2} \int_0^t |w_t(\tau)|_2^2 d\tau + C \int_0^t |w(\tau)|_2^2 d\tau + CT \\ &\leq \frac{1}{2} \int_0^t |w_t(\tau)|_2^2 d\tau + C \int_0^t \mathscr{E}(\tau) d\tau + CT, \end{aligned} \tag{2.5.4}$$

where the constant C in (2.5.4) depends on $|\Gamma|$, the Lebesgue measure of Γ. Combining (2.5.3) and (2.5.4) yields,

$$\mathscr{E}(t) + \frac{1}{2} \int_0^t |w_t(\tau)|_2^2 d\tau \leq \mathscr{E}(0) + CT + C \int_0^t \mathscr{E}(\tau) d\tau. \tag{2.5.5}$$

In particular,

$$\mathscr{E}(t) \leq \mathscr{E}(0) + CT + C\int_0^t \mathscr{E}(\tau)d\tau \text{ for } t \in [0,T]. \tag{2.5.6}$$

By Gronwall's inequality, we conclude that

$$\mathscr{E}(t) \leq (\mathscr{E}(0) + CT)e^{CT} \text{ for } t \in [0,T], \tag{2.5.7}$$

where $T > 0$ is arbitrary. Combining (2.5.5) and (2.5.7), the desired result in (2.5.2) follows.

Now, if $q > 1$, we appeal to the polynomial bonud on h in Remark 2.2.2 along with Hölder's and Young's inequalities to obtain:

$$\begin{aligned}
\left|\int_0^t\int_\Gamma h(w)w_t d\Gamma d\tau\right| &\leq C\int_0^t\int_\Gamma (|w(\tau)|^q + 1)w_t(\tau)d\Gamma d\tau \\
&\leq C\int_0^t (|w(\tau)|_{2q}^q + 1)|w_t(\tau)|_2 d\tau \\
&\leq \frac{1}{2}\int_0^t |w_t(\tau)|_2^2 d\tau + C\int_0^t |\Delta w(\tau)|_2^{2q} d\tau + CT \\
&\leq \frac{1}{2}\int_0^t |w_t(\tau)|_2^2 d\tau + C\int_0^t \mathscr{E}(\tau)^q d\tau + CT.
\end{aligned} \tag{2.5.8}$$

Combining (2.5.3) and (2.5.8) yields

$$\mathscr{E}(t) + \frac{1}{2}\int_0^t \|w_t(\tau)\|_2^2 d\tau \leq \mathscr{E}(0) + CT + C\int_0^t \mathscr{E}(\tau)^q d\tau. \tag{2.5.9}$$

In particular,

$$\mathscr{E}(t) \leq \mathscr{E}(0) + CT + C\int_0^t \mathscr{E}(\tau)^q d\tau. \tag{2.5.10}$$

By using a standard comparison theorem, (2.5.10) yields that $\mathscr{E}(t) \leq z(t)$, where $z(t) = [(\mathscr{E}(0) + CT)^{1-q} - C(q-1)t]^{\frac{-1}{q-1}}$ is the solution of the Volterra integral equation

$$z(t) = \mathscr{E}(0) + CT + C\int_0^t z(s)^q ds.$$

Since $q > 1$, $z(t)$ blows up at the finite time $T_1 = \frac{1}{C(q-1)}(\mathscr{E}_0 + CT)^{1-q}$. Note that T_1 depends on initial energy $\mathscr{E}(0)$ and the original existence time, T. Nonetheless, if we choose $T' = \min\{T, \frac{1}{2}T_1\}$, then

$$\mathscr{E}(t) \leq z(t) \leq C_0 := \left[(\mathscr{E}(0) + C_T)^{1-q} - C(q-1)T'\right]^{\frac{-1}{q-1}} < \infty, \tag{2.5.11}$$

for all $t \in [0, T']$. Finally, we combine (2.5.9) and (2.5.11) to conclude the second statement of the proposition. □

2.6 Continuous Dependence on Initial Data

In this section, we provide the proof to Theorem 2.2.7 in the case $1 \leq p \leq 3$, where the bound (2.5.2) is crucial in the proof.

Proof. Let $U_0 = (u_0, w_0, u_1, w_1) \in H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$. Assume that $\{U_0^n = (u_0^n, w_0^n, u_1^n, w_1^n) : n \in \mathbb{N}\}$ is a sequence of initial data that satisfies:

$$U_0^n \longrightarrow U_0 \text{ in } H \text{ strongly as } n \longrightarrow \infty. \tag{2.6.1}$$

Let $\{(u^n, w^n)\}$ and (u, w) be the weak solutions to (2.1.1) defined on $[0, T]$ in the sense of Definition 2.2.3, corresponding to the initial data $\{U_0^n\}$ and $\{U_0\}$, respectively. First, we show that the local existence time T can be taken independent of $n \in \mathbb{N}$.

To see this, we recall that the local existence time provided by Theorem 2.2.5 for the solution (u,w) depends on the initial energy $\mathscr{E}(0)$. Due to the strong convergence of $U_0^n \longrightarrow U_0$, then the local existence time T for the solutions $\{(u^n,w^n)\}$ and (u,w) can be chosen independent of $n \in \mathbb{N}$. Moreover, in view of (2.5.2), T can be taken arbitrarily large in the case when $q = 1$. However, in the case when $q > 1$, we select the local existence time to be $T = T'$, where T' is as given in Proposition 2.5.1 (which is also uniform in n). In either case, it follows from (2.5.2) that there exists $R > 0$ such that, for all $n \in \mathbb{N}$ and all $t \in [0,T]$ (where $T > 0$ is independent of n):

$$\begin{cases} \mathscr{E}(t) + \int_0^t |w(\tau)|_2^2 d\tau \leq R, \\ \\ \mathscr{E}^n(t) + \int_0^t |w^n(\tau)|_2^2 d\tau \leq R, \end{cases} \tag{2.6.2}$$

where $\mathscr{E}^n(t) = \frac{1}{2}\left(\|u_t^n(t)\|_2^2 + \|\nabla u^n(t)\|_2^2 + |w_t^n(t)|_2^2 + |\Delta w^n(t)|_2^2\right) + \frac{1}{p+1}\|u^n(t)\|_{p+1}^{p+1}$.

Now, put $y^n(t) = u(t) - u^n(t)$, $z^n(t) = w(t) - w^n(t)$, and

$$\tilde{\mathscr{E}}^n(t) = \frac{1}{2}\left(\|y_t^n(t)\|_2^2 + \|\nabla y^n(t)\|_2^2 + |z_t^n(t)|_2^2 + |\Delta z^n(t)|_2^2\right), \tag{2.6.3}$$

for $t \in [0,T]$. We aim to show $\tilde{\mathscr{E}}_n(t) \longrightarrow 0$ uniformly on $[0,T]$.

From Definition 2.2.3, then y^n and z^n satisfy:

$$\begin{aligned} &(y_t^n(t), \phi(t))_\Omega - (y_t^n(0), \phi(0))_\Omega - \int_0^t (y_t^n(\tau), \phi_t(\tau))_\Omega d\tau \\ &\quad + \int_0^t (\nabla y^n(\tau), \nabla \phi(\tau))_\Omega d\tau - \int_0^t (z_t^n(\tau), \gamma\phi(\tau))_\Gamma d\tau \\ &\quad + \int_0^t \int_\Omega \Big(|u(\tau)|^{p-1}u(\tau) - |u^n(\tau)|^{p-1}u^n(\tau)\Big)\phi(\tau) dx d\tau = 0, \end{aligned} \tag{2.6.4}$$

$$
\begin{aligned}
&(z_t^n(t)+\gamma y^n(t),\psi(t))_\Gamma-(z_t^n(0)+\gamma y^n(0,\psi(0))_\Gamma-\int_0^t(z_t^n(\tau),\psi_t(\tau))_\Gamma d\tau\\
&\quad-\int_0^t(\gamma y^n(\tau),\psi_t(\tau))_\Gamma d\tau+\int_0^t(\Delta z^n(\tau),\Delta\psi(\tau))_\Gamma d\tau\\
&\quad+\int_0^t(z_t^n(\tau),\psi(\tau))_\Gamma d\tau=\int_0^t\int_\Gamma\Big(h(w(\tau))-h(w^n(\tau))\Big)\psi(\tau)d\Gamma d\tau, \qquad (2.6.5)
\end{aligned}
$$

where ϕ and ψ are proper test functions as described in Definition 2.2.3.

As we demonstrated in the proof of the energy identity in Section 2.4, we can replace $\phi(\tau)$ by $D_hy(\tau)$ in (2.6.4) and $\psi(\tau)$ by $D_hz(\tau)$ in (2.6.5), for any $\tau\in[0,T]$. By using similar arguments as in the proof of the energy identity (2.2.3), we can pass to the limit as $h\longrightarrow 0$ to deduce the identity:

$$
\begin{aligned}
&\tilde{\mathscr{E}}^n(t)+\int_0^t|z^n(\tau)|_2^2d\tau+\int_0^t\int_\Omega\Big(|u(\tau)|^{p-1}u(\tau)-|u^n(\tau)|^{p-1}u^n(\tau)\Big)y_t^n(\tau)dxd\tau\\
&\quad=\tilde{\mathscr{E}}^n(0)+\int_0^t\int_\Gamma\Big(h(w(\tau))-h(w^n(\tau))\Big)z_t^n(\tau)d\Gamma d\tau. \qquad (2.6.6)
\end{aligned}
$$

We first estimate the term coming from the source acting on the wave equation. By recalling the bounds in Remark 2.2.2 and by using Hölder's and Young's Inequalities, one has

$$
\begin{aligned}
&\Big|\int_0^t\int_\Omega\Big(|u(\tau)|^{p-1}u(\tau)-|u^n(\tau)|^{p-1}u^n(\tau)\Big)y_t^n(\tau)dxd\tau\Big|\\
&\qquad\le C\int_0^t\int_\Omega\left(|u(\tau)|^{p-1}+|u^n(\tau)|^{p-1}\right)|u(\tau)-u^n(\tau)||y_t^n(\tau)|dxd\tau\\
&\qquad\le C\int_0^t(\|u(\tau)\|_{3(p-1)}^{p-1}+\|u^n(\tau)\|_{3(p-1)}^{p-1}\|u(\tau)-u^n(\tau)\|_6\|y_t^n(\tau)\|_2d\tau\\
&\qquad\le C_R\int_0^t(\|\nabla y^n(\tau)\|_2^2+\|y_t^n(\tau)\|_2^2)d\tau\le C_R\int_0^t\tilde{\mathscr{E}}^n(\tau)d\tau, \qquad (2.6.7)
\end{aligned}
$$

where we have used in (2.6.7) the assumption $1\le p\le 3$, the Sobolev Imbedding Theorem, and the bounds in (2.6.2).

In a similar manner, we can estimate the term coming from the source acting on the plate and obtain

$$\Big|\int_0^t \int_\Gamma \Big(h(w(\tau)) - h(w^n(\tau))\Big) z_t^n(\tau) d\Gamma d\tau\Big| \leq \; C_R \int_0^t |\Delta z^n(\tau)|_2^2 d\tau$$
$$\leq C_R \int_0^t \tilde{\mathscr{E}}^n(\tau) d\tau. \tag{2.6.8}$$

By combining (2.6.6)-(2.6.8), we conclude

$$\tilde{\mathscr{E}}^n(t) + \int_0^t |z^n(\tau)|_2^2 d\tau \leq \tilde{\mathscr{E}}^n(0) + C_R \int_0^t \tilde{\mathscr{E}}^n(\tau) d\tau. \tag{2.6.9}$$

In particular, Gronwall's inequality yields

$$\tilde{\mathscr{E}}^n(t) \leq \tilde{\mathscr{E}}^n(0) e^{C_R T}, \text{ for all } t \in [0,T]. \tag{2.6.10}$$

Since $\tilde{\mathscr{E}}^n(0) \longrightarrow 0$, as $n \longrightarrow \infty$, then $\tilde{\mathscr{E}}_n(t) \longrightarrow 0$ uniformly on $[0,T]$, completing the proof. □

Remark 2.6.1. Corollary 2.2.8 follows immediately from Theorem 2.2.7. Its proof is outlined below. △

Proof. Let (u,w) and $(\hat{u},\hat{w})$ be two weak solutions to (2.1.1) defined on $[0,T]$ in the sense of Definition 2.2.3 with the same initial data $U_0 = (u_0, w_0, u_1, w_1) \in H$, where $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$. Put: $\hat{y}(t) = u(t) - \hat{u}(t)$, $\hat{z}(t) = w(t) - \hat{w}(t)$, and

$$\hat{\mathscr{E}}(t) = \frac{1}{2}\left(\hat{y}'(t)\|_2^2 + \|\nabla \hat{y}(t)\|_2^2 + |\hat{z}'(t)|_2^2 + |\Delta \hat{z}(t)|_2^2\right). \tag{2.6.11}$$

Then, in the same manner in obtaining the identity (2.6.6), we have

$$\begin{aligned}\hat{\mathscr{E}}(t) + \int_0^t |\hat{z}(\tau)|_2^2 d\tau + \int_0^t \int_\Omega \Big(|u(\tau)|^{p-1}u(\tau) - |\hat{u}(\tau)|^{p-1}\hat{u}(\tau)\Big) y_t^n(\tau) dx d\tau \\ \leq \int_0^t \int_\Gamma \Big(h(w(\tau)) - h(\hat{w}(\tau))\Big) z_t^n(\tau) d\Gamma d\tau \qquad (2.6.12)\end{aligned}$$

Similar estimates as in (2.6.7)-(2.6.8) yield,

$$\hat{\mathscr{E}}(t) + \int_0^t |\hat{z}(\tau)|_2^2 d\tau \leq C \int_0^t \hat{\mathscr{E}}(\tau) d\tau, \qquad (2.6.13)$$

which implies by Gronwall's inequality that $\hat{\mathscr{E}}(t) = 0$ for all $t \in [0, T]$. Hence, $(u, w) = (\hat{u}, \hat{w})$. □

Chapter 3

Supercritical Source and Damping Terms

3.1 The Model

In this iteration of the model we leave the statement of 1.2.1 intact, with more general assumptions placed on the nonlinearities. As a reminder, this leaves us with:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + u_t|_{\Gamma} = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), \end{cases} \tag{3.1.1}$$

where the initial data reside in the finite energy space, i.e.,

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma).$$

3.2 Main Results

Throughout the chapter, we study (3.1.1) under the following assumptions.

Assumption 3.2.1.

Damping *g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing functions with $g_1(0) = g_2(0) = 0$. In addition, the following growth conditions at infinity hold: there exist positive constants α and β such that, for $|s| \geq 1$,*

$$\alpha|s|^{m+1} \leq g_1(s)s \leq \beta|s|^{m+1}, \text{ with } m \geq 1,$$
$$\alpha|s|^{r+1} \leq g_2(s)s \leq \beta|s|^{r+1}, \text{ with } r \geq 1.$$

Interior sources *f and h are functions in $C^1(\mathbb{R})$ such that*

$$|f'(s)| \leq C(|s|^{p-1} + 1), \text{ with } 1 \leq p < 6,$$
$$|h'(s)| \leq C(|s|^{q-1} + 1), \text{ with } 1 \leq q < \infty.$$

Parameters $p\frac{m+1}{m} < 6$.

Remark 3.2.2. As the following bounds will be used often throughout the work it is worthy of note that the above assumption implies that

$$\begin{cases} |f(u) \leq C(|u|^p + 1), \quad |f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)|u - v|, \\ |h(w)| \leq C(|w|^q + 1), \quad |h(w) - h(z)| \leq C(|w|^{q-1} + |z|^{q-1} + 1)|w - z|. \end{cases} \tag{3.2.1}$$

△

We begin by introducing the definition of a suitable weak solution for (3.1.1).

Definition 3.2.3. A pair of functions (u, w) is said to be a weak solution of (3.1.1) on the interval $[0, T]$ provided:

(i) $u \in C([0,T]; H^1_{\Gamma_0}(\Omega))$, $u_t \in C([0,T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$,

(ii) $w \in C([0,T]; H^2_0(\Gamma))$, $w_t \in C([0,T]; L^2(\Gamma)) \cap L^{r+1}(\Gamma \times (0,T))$,

(iii) $(u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$,

(iv) $(w(0), w_t(0)) = (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma)$,

(v) The functions u and w satisfy the following variational identities for all $t \in [0, T]$:

$$
\begin{aligned}
(u_t(t), \phi(t))_\Omega - (u_1, \phi(0))_\Omega - \int_0^t (u_t(\tau), \phi_t(\tau))_\Omega d\tau + \int_0^t (\nabla u(\tau), \nabla \phi(\tau))_\Omega d\tau \\
- \int_0^t (w_t(\tau), \gamma\phi(\tau))_\Gamma d\tau + \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau) dx d\tau \\
= \int_0^t \int_\Omega f(u(\tau))\phi(\tau) dx d\tau,
\end{aligned}
\tag{3.2.2}
$$

$$
\begin{aligned}
(w_t(t) + \gamma u(t), \psi(t))_\Gamma - (w_1 + \gamma u(0), \psi(0))_\Gamma - \int_0^t (w_t(\tau), \psi_t(\tau))_\Gamma d\tau \\
- \int_0^t (\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau + \int_0^t (\Delta w(\tau), \Delta \psi(\tau))_\Gamma d\tau \\
+ \int_0^t \int_\Gamma g_2(w_t(\tau))\psi(\tau) d\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\psi(\tau) d\Gamma d\tau,
\end{aligned}
\tag{3.2.3}
$$

for all test functions ϕ and ψ satisfying: $\phi \in C([0,T]; H^1_{\Gamma_0}(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$, $\psi \in C([0,T]; H^2_0(\Gamma))$ with $\phi_t \in L^1(0,T; L^2(\Omega))$, and $\psi_t \in L^1(0,T; L^2(\Gamma))$.

Our first theorem establishes the existence of a local weak solution to (3.1.1). Specifically, we have the following result.

Theorem 3.2.4. *(Local weak solutions) Under the validity of Assumption 3.2.1, then there exists a local weak soluition* (u, w) *to* (3.1.1) *defined on* $[0, T_0]$ *for some* $T_0 > 0$ *depending on the initial energy* $E(0)$*, where*

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + |w_t(t)|_2^2 + |\Delta w(t)|_2^2\right).$$

In addition, the following energy identity holds for all $t \in [0, T_0]$*:*

$$\begin{aligned} E(t) &+ \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau \\ &= E(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau. \end{aligned} \tag{3.2.4}$$

Our next theorem states that weak solutions furnished by Theorem 3.2.4 are global solutions, provided the exponents of damping are more dominant than the exponents of the corresponding sources.

Theorem 3.2.5. *(Global in time weak solutions) In addition to Assumption 3.2.1, further assume* $u_0 \in L^{p+1}(\Omega)$*. If* $p \le m$ *and* $q \le r$*, then the said solution* (u, w) *in Theorem 3.2.4 is a global weak solution and* T_0 *can be taken arbitrarily large.*

In order to state the next theorem, we need additional assumptions on the source acting on the wave equation.

Assumption 3.2.6. *For* $p > 3$*, assume that* $f \in C^2(\mathbb{R})$ *with* $|f''(u)| \le C(|u|^{p-2} + 1)$ *for all* $u \in \mathbb{R}$.

Theorem 3.2.7. *(Continuous dependence on initial data) Assume the validity of Assumptions 3.2.1 and 3.2.6 and an initial data* $U_0 = (u_0, w_0, u_1, w_1) \in X$ *where the function space* X *is given by* $X = (H^1_{\Gamma_0}(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$*. If*

$U_0^n = (u_0^n, w_0^n, u_1^n, w_1^n)$ is a sequence of initial data such that, as $n \longrightarrow \infty$,

$$U_0^n \longrightarrow U_0 \text{ in } X,$$

then, the corresponding weak solutions (u^n, w^n) and (u, w) of (3.1.1) satisfy:

$$(u^n, w^n, u_t^n, w_t^n) \longrightarrow (u, w, u_t, w_t) \text{ in } C([0,T]; H), \text{ as } n \longrightarrow \infty,$$

where $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$.

Remark 3.2.8. If $p \leq 5$, then the assumption of $u_0 \in L^{p+1}(\Omega)$ in Theorem 3.2.5 is redundant as $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$. Simlarily, if $p \leq 5$, then the spaces X and H in Theorem 3.2.7 are indentical. △

Our final two theorems address uniqueness of weak solutions.

Theorem 3.2.9. *(Uniqueness of weak solutions–Part 1) In addition to Assumptions 3.2.1 and 3.2.6, we further assume that $u_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$. Then, weak solutions of (3.1.1) are unique.*

Remark 3.2.10. It is often the case that the wave source f fail to satisfy Assumption 3.2.6 for the values $3 < p \leq 5$, i.e, $f \notin C^2(\mathbb{R})$. To ensure uniqueness of weak solutions in such a case, we require the exponent of m of the wave damping to be sufficiently large. More precisely, our final result resolves this issue. △

Theorem 3.2.11. *(Uniqueness of weak solutions–Part 2) Under Assumption 3.2.1 we additionally assume that $u_0 \in L^{3(p-1)}(\Omega)$ and $m \geq 3p - 4$ if $p > 3$. Then weak solutions of (3.1.1) are unique.*

3.3 Local Existence

3.3.1 Operator Theoretic Formulation

Our first goal is to put problem (3.1.1) in an operator theoretic form. In order to do so, we introduce the Dirichlét-Neumann Laplacian, given by: $A = -\Delta : \mathscr{D}(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$, with its domain $\mathscr{D}(A) = \{u \in H^2(\Omega) : u|_{\Gamma_0} = 0,\ \partial_\nu u|_\Gamma = 0\}$. We note that A can be extended to a continuous map $A : H^1_{\Gamma_0}(\Omega) \longrightarrow (H^1_{\Gamma_0}(\Omega))'$, where $\langle Au, \phi\rangle = \int_\Omega \nabla u \cdot \nabla\phi dx = (\nabla u, \nabla\phi)_\Omega$, for all $u, \phi \in H^1_{\Gamma_0}(\Omega)$.

We define the Dirichlét-Neumann map: $R : H^s(\Gamma) \longrightarrow H^{s+\frac{3}{2}}(\Omega) \cap H^1_{\Gamma_0}(\Omega)$; $s \geq 0$ by:

$$q = Rp \iff q \text{ is the weak solution of the problem } \begin{cases} \Delta q = 0 \text{ in } \Omega, \\ q = 0 \text{ on } \Gamma_0, \\ \partial_\nu q = p \text{ on } \Gamma. \end{cases} \tag{3.3.1}$$

It is well-known that R is continuous from $H^s(\Gamma)$ to $H^{s+\frac{3}{2}}(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, for $s \geq 0$ (see for instance Lasiecka and Triggiani [43, 44]). Let us note here that (3.3.1) and a straightforward computation yields the following useful identity:

$$\langle ARp, \phi\rangle = (\nabla Rp \cdot \nabla\phi)_\Omega = (p, \gamma\phi)_\Gamma, \tag{3.3.2}$$

for all $p \in L^2(\Gamma)$ and $\phi \in H^1_{\Gamma_0}(\Omega)$.

Also, the biharmonic operator $\Delta^2 : \mathscr{D}(\Delta^2) \subset L^2(\Gamma) \longrightarrow L^2(\Gamma)$ with its domain $\mathscr{D}(\Delta^2) = H^4(\Gamma) \cap H^2_0(\Gamma)$ can be extended as a continuous mapping from $H^2_0(\Gamma)$ to $H^{-2}(\Gamma)$, where $\langle \Delta^2 w, \phi\rangle = (\Delta w, \Delta\phi)_\Gamma$, for all $w, \phi \in H^2_0(\Gamma)$.

By using the operators above, then (3.1.1) can be formally casted as:

$$\begin{cases} u_{tt} + A(u - Rw_t) + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + \gamma u_t = h(w) & \text{in } \Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \\ (w(0), w_t(0)) = (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma). \end{cases} \tag{3.3.3}$$

Now, we introduce the state space $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$ with the natural norm:

$$|U|_H^2 = \|\nabla u\|_2^2 + |\Delta w|_2^2 + \|y\|_2^2 + |z|_2^2, \text{ for all } U = (u, w, y, z) \in H,$$

and define the nonlinear operator

$$\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset H \longrightarrow H$$

by

$$\mathscr{A} \begin{bmatrix} u \\ w \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ A(u - Rz) + g_1(y) - f(u) \\ \Delta^2 w + g_2(z) + \gamma y - h(w) \end{bmatrix}^{tr} \tag{3.3.4}$$

where

$$\mathscr{D}(\mathscr{A}) = \Big\{(u,w,y,z) \in \big(H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)\big)^2 : A(u - Rz) + g_1(y) - f(u) \in L^2(\Omega),$$
$$g_1(y) \in \big(H^1_{\Gamma_0}(\Omega)\big)' \cap L^1(\Omega),\ \Delta^2 w \in L^2(\Gamma)\Big\}.$$

By putting $U = (u, w, u_t, w_t)$, then the system (3.3.3) is equivalent to

$$U_t + \mathscr{A}U = 0 \text{ with } U(0) = (u_0, w_0, u_1, w_1) \in H. \tag{3.3.5}$$

3.3.1.1 Globally Lipschitz Sources

First, we deal with the case where the sources are globally Lipschitz. In this case, we have the following lemma.

Lemma 3.3.1. *Assume that $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ and $h : H^2_0(\Gamma) \longrightarrow L^2(\Gamma)$ are globally Lipschitz continuous. Then, system* (3.3.3) *has a unique global strong solution $U \in W^{1,\infty}(0,T;H)$ for arbitrary $T > 0$; provided the initial datum $U_0 \in \mathscr{D}(\mathscr{A})$.*

Proof. In order to prove Lemma 3.3.1 it suffices to show that the operator $\mathscr{A} + \omega I$ is m-accretive for some positive ω. We say an operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset H \longrightarrow H$ is accretive if $(\mathscr{A}x_1 - \mathscr{A}x_2, x_1 - x_2)_H \geq 0$, for all $x_1, x_2 \in \mathscr{D}(\mathscr{A})$, and it is m-accretive if, in addition, $\mathscr{A} + I$ maps $\mathscr{D}(\mathscr{A})$ onto H. In fact, by Kato's Theorem [55], if $\mathscr{A} + \omega I$ is m-accretive for some positive ω, then for each $U_0 \in \mathscr{D}(\mathscr{A})$ there is a unique strong solution U of (3.3.5), i.e., $U \in W^{1,\infty}(0,T;H)$ such that $U(0) = U_0$, $U(t) \in \mathscr{D}(\mathscr{A})$ for all $t \in [0,T]$, and equation (3.3.5) is satisfied a.e. $[0,T]$, where $T > 0$ is arbitrary.

Step 1: Proof for $\mathscr{A} + \omega I$ is accretive for some positive ω. Let $U =$

(u, w, y, z), $\hat{U} = (\hat{u}, \hat{w}, \hat{y}, \hat{z}) \in \mathscr{D}(\mathscr{A})$. We aim to find $\omega > 0$ such that

$$((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H \geq 0.$$

By straightforward calculations we obtain:

$$\begin{aligned}
((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H &= (\mathscr{A}(U) - \mathscr{A}(\hat{U}), U - \hat{U})_H + \omega|U - \hat{U}|_H^2 \\
= &- (\nabla(y - \hat{y}), \nabla(u - \hat{u}))_\Omega - (\Delta(z - \hat{z}), \Delta(w - \hat{w}))_\Gamma + \langle A(u - \hat{u}), y - \hat{y}\rangle \\
&- \langle AR(z - \hat{z}), y - \hat{y}\rangle + \langle g_1(y) - g_1(\hat{y}), y - \hat{y}\rangle - (f(u) - f(\hat{u}), y - \hat{y})_\Omega \\
&+ \langle \Delta^2(w - \hat{w}), z - \hat{z}\rangle + \langle g_2(z) - g_2(\hat{z}), z - \hat{z}\rangle + (\gamma(y - \hat{y}), z - \hat{z})_\Gamma \\
&- (h(w) - h(\hat{w}), z - \hat{z})_\Gamma + \omega\|\nabla(u - \hat{u})\|_2^2 + \omega\,|\Delta(w - \hat{w})|_2^2 \\
&+ \omega\|y - \hat{y}\|_2^2 + \omega|z - \hat{z}|_2^2.
\end{aligned} \tag{3.3.6}$$

We note here that

$$\begin{cases}
\langle A(u - \hat{u}), y - \hat{y}\rangle = (\nabla(u - \hat{u}), \nabla(y - \hat{y})_\Omega, \\
\\
\langle \Delta^2(w - \hat{w}), z - \hat{z}\rangle = (\Delta(w - \hat{w}), \Delta(z - \hat{z}))_\Gamma.
\end{cases} \tag{3.3.7}$$

Moreover, since $g_1(y) - g_1(\hat{y}) \in \left(H^1_{\Gamma_0}(\Omega)\right)' \cap L^1(\Omega)$ and $y - \hat{y} \in H^1_{\Gamma_0}(\Omega)$ satisfying $(g_1(y(x)) - g_1(\hat{y}(x)))(y(x) - \hat{y}(x)) \geq 0$, for all $x \in \Omega$, then by Lemma 2.2 (p.89) in [6], we have $(g_1(y) - g_1(\hat{y}))(y - \hat{y}) \in L^1(\Omega)$ and

$$\langle g_1(y) - g_1(\hat{y}), y - \hat{y}\rangle = \int_\Omega (g_1(y) - g_1(\hat{y}))(y - \hat{y})dx \geq 0. \tag{3.3.8}$$

We also immediately have

$$\langle g_2(z) - g_2(\hat{z}), z - \hat{z} \rangle \rangle = \int_{\Omega} (g_2(z) - g_2(\hat{z}))(z - \hat{z}) dx \geq 0. \tag{3.3.9}$$

Additionally, (3.3.2) yields

$$\langle AR(z - \hat{z}), y - \hat{y} \rangle = (z - \hat{z}, \gamma(y - \hat{y}))_{\Gamma} \tag{3.3.10}$$

By the assumption on the mappings f and h, let L_f and L_h be their globally Lipschitz constants, and select $L = \max\{L_f, L_h\}$. Then, one has

$$\begin{aligned} -(f(u) - f(\hat{u}), y - \hat{y})_{\Omega} - (h(w) - h(\hat{w}), z - \hat{z})_{\Gamma} \\ \geq -L\|\nabla(u - \hat{u})\|_2 \|y - \hat{y}\|_2 - L|\Delta(w - \hat{w})|_2 |z - \hat{z}|_2 \\ \geq -\frac{L}{2} \left(\|\nabla(u - \hat{u})\|_2^2 + |\Delta(w - \hat{w})|_2^2 + \|y - \hat{y}\|_2^2 + |z - \hat{z}|_2^2 \right), \end{aligned} \tag{3.3.11}$$

where we have used Young's inequality. By combining (3.3.6)-(3.3.11), we have

$$\begin{aligned} ((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H \\ \geq \left(\omega - \frac{L}{2} \right) \left(\|\nabla(u - \hat{u})\|_2^2 + |\Delta(w - \hat{w})|_2^2 + \|y - \hat{y}\|_2^2 + |z - \hat{z}|_2^2 \right). \end{aligned}$$

Therefore, by choosing $\omega > \frac{L}{2}$, then $\mathscr{A} + \omega I$ is accretive.

Step 2: Proof for $\mathscr{A} + \lambda I$ is m-accretive for some $\lambda > 0$. It suffices to show that the range of $\mathscr{A} + \lambda I$ is all of H, for some $\lambda > 0$.

So, let $(a, b, c, d) \in H$. We have to show that there exists $(u, w, y, z) \in \mathscr{D}(\mathscr{A})$ such that $(\mathscr{A}+\lambda I)(u,w,y,z)=(a,b,c,d)$, for some $\lambda>0$, that is,

$$\begin{cases} -y+\lambda u=a \\ -z+\lambda w=b \\ A(u-Rz)+g_1(y)-f(u)+\lambda y=c \\ \Delta^2 w+g_2(z)+\gamma y-h(w)+\lambda z=d. \end{cases} \tag{3.3.12}$$

Note, (3.3.12) is equivalent to

$$\begin{cases} \frac{1}{\lambda}Ay-ARz+g_1(y)-f(\frac{a+y}{\lambda})+\lambda y=c-\frac{1}{\lambda}Aa \\ \frac{1}{\lambda}\Delta^2 z+g_2(z)+\gamma y-h(\frac{b+z}{\lambda})+\lambda z=d-\frac{1}{\lambda}\Delta^2 b. \end{cases} \tag{3.3.13}$$

Let $V=H^1_{\Gamma_0}(\Omega)\times H^2_0(\Gamma)$ and notice that the right hand side of (3.3.13) belongs to V'. Thus, we define the operator $\mathscr{B}:\mathscr{D}(\mathscr{B})\subset V\longrightarrow V'$ by:

$$\mathscr{B}\begin{bmatrix} y \\ z \end{bmatrix}^{tr}=\begin{bmatrix} \frac{1}{\lambda}Ay-ARz+g_1(y)-f(\frac{a+y}{\lambda})+\lambda y \\ \frac{1}{\lambda}\Delta^2 z+g_2(z)+\gamma y-h(\frac{b+z}{\lambda})+\lambda z \end{bmatrix}^{tr}$$

where $\mathscr{D}(\mathscr{B})=\{(y,z)\in V: g_1(y)\in \big(H^1_{\Gamma_0}(\Omega)\big)'\cap L^1(\Omega)\}$. Therefore, the issue reduces to proving that $\mathscr{B}:\mathscr{D}(\mathscr{B})\subset V\longrightarrow V'$ is surjective. By Corollary 1.2 (p.45) in [6], it is enough to show that $\mathscr{B}$ is maximal monotone and coercive. To do this we will split $\mathscr{B}$ into two operators:

$$\mathscr{B}_1\begin{bmatrix} y \\ z \end{bmatrix}^{tr}=\begin{bmatrix} \frac{1}{\lambda}Ay-ARz-f(\frac{a+y}{\lambda})+\lambda y \\ \frac{1}{\lambda}\Delta^2 z+\gamma y-h(\frac{b+z}{\lambda})+\lambda z \end{bmatrix}^{tr},$$

and

$$\mathscr{B}_2 \begin{bmatrix} y \\ \\ z \end{bmatrix}^{tr} = \begin{bmatrix} g_1(y) \\ \\ g_2(z) \end{bmatrix}^{tr}.$$

$\mathscr{B}_1$ **is maximal monotone and coercive:** Clearly, $\mathscr{D}(\mathscr{B}_1) = V$. We first show that $\mathscr{B}_1 : V \longrightarrow V'$ is strongly monotone. So, let $Y, \hat{Y} \in V$, where $Y = (y, z)$ and $\hat{Y} = (\hat{y}, \hat{z})$. By straightforward calculations, we obtain

$$\begin{aligned} &\langle \mathscr{B}_1(Y) - \mathscr{B}_1(\hat{Y}), Y - \hat{Y}\rangle_{V',V} = \frac{1}{\lambda}\langle A(y-\hat{y}), y-\hat{y}\rangle - \langle AR(z-\hat{z}), y-\hat{y}\rangle \\ &- \left(f(\frac{a+y}{\lambda}) - f(\frac{a+\hat{y}}{\lambda}), y-\hat{y}\right)_\Omega + \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}\langle \Delta^2(z-\hat{z}), z-\hat{z}\rangle \\ &+ (\gamma(y-\hat{y}), z-\hat{z})_\Gamma - \left(h(\frac{b+z}{\lambda}) - h(\frac{b+\hat{z}}{\lambda}), z-\hat{z}\right)_\Gamma + \lambda|z-\hat{z}|_2^2. \end{aligned} \tag{3.3.14}$$

Thanks to (3.3.2), we have $\langle AR(z-\hat{z}), y-\hat{y}\rangle = (z-\hat{z}, \gamma(y-\hat{y}))_\Gamma$. Therefore, it follows from (3.3.14) and Young's inequality (with an $\eta > 0$) that:

$$\begin{aligned} &\langle \mathscr{B}_1(Y) - \mathscr{B}_1(\hat{Y}), Y - \hat{Y}\rangle_{V',V} \geq \frac{1}{\lambda}\|\nabla(y-\hat{y})\|_2^2 - \frac{L}{\lambda}\|\nabla(y-\hat{y})\|_2\|y-\hat{y}\|_2 \\ &+ \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}|\Delta(z-\hat{z})|_2^2 - \frac{L}{\lambda}|\Delta(z-\hat{z})|_2|z-\hat{z}|_2 + \lambda|z-\hat{z}|_2^2 \\ &\geq \frac{1}{\lambda}\|\nabla(y-\hat{y})\|_2^2 - \frac{L^2}{4\eta\lambda}\|\nabla(y-\hat{y})\|_2^2 - \frac{\eta}{\lambda}\|y-\hat{y}\|_2^2 + \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}|\Delta(z-\hat{z})|_2^2 \\ &- \frac{L^2}{4\eta\lambda}|\Delta(z-\hat{z})|_2^2 - \frac{\eta}{\lambda}|z-\hat{z}|_2^2 + \lambda|z-\hat{z}|_2^2. \end{aligned} \tag{3.3.15}$$

Therefore, we conclude from (3.3.15) that

$$\begin{aligned} \langle \mathscr{B}_1(Y) - \mathscr{B}_1(\hat{Y}), Y - \hat{Y}\rangle_{V',V} &\geq \Big(\frac{1}{\lambda} - \frac{L^2}{4\eta\lambda}\Big)\|\nabla(y-\hat{y})\|_2^2 + \Big(\lambda - \frac{\eta}{\lambda}\Big)\|y-\hat{y}\|_2^2 \\ &+ \Big(\frac{1}{\lambda} - \frac{L^2}{4\eta\lambda}\Big)|\Delta(z-\hat{z})|_2^2 + \Big(\lambda - \frac{\eta}{\lambda}\Big)|z-\hat{z}|_2^2. \end{aligned} \tag{3.3.16}$$

By selecting η large enough, say $\eta = \frac{L^2}{2}$, and then by selecting $\lambda = \frac{L}{\sqrt{2}}$, it follows from (3.3.16) that

$$\langle \mathscr{B}_1(Y) - \mathscr{B}_1(\hat{Y}), Y - \hat{Y} \rangle_{V',V} \geq \frac{1}{2\lambda}\Big(\|\nabla(y-\hat{y})\|_2^2 + |\Delta(z-\hat{z})|_2^2 \Big) = \frac{1}{2\lambda}\|Y - \hat{Y}\|_V^2,$$

proving $\mathscr{B}_1$ is strongly monotone. It is easy to see that strong monotonicity implies coercivity of $\mathscr{B}_1$. Next, we show that $\mathscr{B}_1$ is continuous. It is clear that the mappings $A : H^1_{\Gamma_0}(\Omega) \longrightarrow (H^1_{\Gamma_0}(\Omega))'$, $\Delta^2 : H^2_0(\Gamma) \longrightarrow H^{-2}(\Gamma)$, and $\gamma : H^1_{\Gamma_0}(\Omega) \longrightarrow H^{-2}(\Gamma)$ are continuous. Moreover, as $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ and $h : H^2_0(\Gamma) \longrightarrow L^2(\Gamma)$ are globally Lipschitz continuous, then the mapping $y \mapsto f(\frac{a+y}{\lambda})$ is continuous from $H^1_{\Gamma_0}(\Omega)$ to $(H^1_{\Gamma_0}(\Omega))'$ and the mapping $z \mapsto h(\frac{b+z}{\lambda})$ is also continuous from $H^2_0(\Gamma)$ to $H^{-2}(\Gamma)$. In addition, by the properties of the Dirichlét-Neumann map R, we deduce that $AR : H^2_0(\Gamma) \longrightarrow (H^1_{\Gamma_0}(\Omega))'$ is continuous.

It follows that $\mathscr{B}_1 : V \longrightarrow V'$ is continuous, and along with the monotonicity of $\mathscr{B}_1$, we conclude that $\mathscr{B}_1$ is maximal monotone.

$\mathscr{B}_2$ **is maximal monotone:** We first note that $\mathscr{D}(\mathscr{B}_2) = \{(y,z) \in V : g_1(y) \in \big(H^1_{\Gamma_0}(\Omega)\big)' \cap L^1(\Omega)\}$. We will study first the operator $g_1(y)$, and in order to do so we define the functional $J_1 : H^1_{\Gamma_0}(\Omega) \longrightarrow [0,\infty]$ by

$$J_1(y) = \int_\Omega j_1(y(x))dx$$

where $j_1 : \mathbb{R} \longrightarrow [0,\infty)$ is the convex function defined by

$$j_1(s) = \int_0^s g_1(\tau)d\tau.$$

Clearly J_1 is proper, convex, and lower semi-continuous. Moreover, by Corollary 2.3

in [7] we know that $\partial J_1 : H^1_{\Gamma_0}(\Omega) \longrightarrow (H^1_{\Gamma_0}(\Omega))'$ satisfies

$$\partial J_1(y) = \{\mu \in (H^1_{\Gamma_0}(\Omega))' \cap L^1(\Omega) : \mu = g_1(y) \text{ a.e. in } \Omega\} \tag{3.3.17}$$

This implies that $\mathscr{D}(\partial J_1) = \{y \in H^1_{\Gamma_0}(\Omega) : g_1(y) \in (H^1_{\Gamma_0}(\Omega))' \cap L^1(\Omega)\}$, and for all $y \in \mathscr{D}(\partial J_1)$, $\partial J_1(y)$ is a singleton such that $\partial J_1(y) = \{g_1(y)\}$. Since any subdifferential is maximal monotone, we obtain the maximal monotonicity of the operator $g_1(\cdot) : \mathscr{D}(\partial J_1) \subset H^1_{\Gamma_0}(\Omega) \longrightarrow (H^1_{\Gamma_0}(\Omega))'$. Using the same approach, we define the functional $J_2 : H^2_0(\Gamma) \longrightarrow [0, \infty]$ by $J_2(z) = \int_\Gamma j_2(z(x))d\Gamma$, where $j_2(s) = \int_0^s g_2(\tau)d\tau$. By the same argument above and using a result by Brézis [16], we obtain $\partial J_2 : H^2_0(\Gamma) \longrightarrow H^{-2}(\Gamma)$ with $\mathscr{D}(\partial J_2) = H^2_0(\Gamma)$, and for all $z \in \mathscr{D}(\partial J_2)$, $\partial J_2(z)$ is a singleton such that $\partial J_2(z) = \{g_2(z)\}$. Therefore, the operator $g_2(\cdot) : \mathscr{D}(\partial J_2) \subset H^2_0(\Gamma) \longrightarrow H^{-2}(\Gamma)$ is maximal monotone. Hence, by Proposition 7.1 in [33], it follows that $\mathscr{B}_2 : \mathscr{D}(\mathscr{B}_2) \subset V \longrightarrow V'$ is maximal monotone. Now, since both $\mathscr{B}_1$ and $\mathscr{B}_2$ are both maximal monotone and $\mathscr{D}(\mathscr{B}_1) = V$, then by a well-known Theorem in [6], $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is maximal monotone.

Finally, since $\mathscr{B}_2$ is monotone and $\mathscr{B}_2(0) = 0$, it follows that $\langle \mathscr{B}_2 Y, Y\rangle \geq 0$ for all $Y \in \mathscr{D}(\mathscr{B}_2)$, and along with the fact $\mathscr{B}_1$ is coercive, we obtain $\mathscr{B}$ is coercive as well. Then, the surjectivity of $\mathscr{B}$ follows immediately by Corollary 1.2 (p.45) in [6]. Thus, we proved the existence of $(y, z) \in \mathscr{D}(\mathscr{B}) \subset V = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$ such that (y, z) satisfies (3.3.13). So by (3.3.12), $(u, v) = \left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) \in H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$. In addition, one can easily see that $(u, w, y, z) \in \mathscr{D}(\mathscr{A})$. Thus, the proof of maximal accretivity is completed and so is the proof of Lemma 3.3.1. $\square$

3.3.1.2 Locally Lipschitz Sources

In this subsection, we loosen the restrictions on sources, namely allow f to be locally Lipschitz continuous.

Lemma 3.3.2. *Assume that g_1 and g_2 are satisfying the conditions in Assumption 3.2.1. Further assume that $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ is locally Lipschitz continuous. Then, system* (3.3.3) *has a unique local strong solution $U \in W^{1,\infty}(0, T_0; H)$ for some $T_0 > 0$; provided the initial datum $U_0 \in \mathscr{D}(\mathscr{A})$.*

Proof. As in [14, 19, 33], we use standard truncation of the sources. Recall, $V = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$ and define

$$f^K(u) = \begin{cases} f(u) & \text{if } \|\nabla u\|_2 \le K, \\ f\left(\frac{Ku}{\|\nabla u\|_2}\right) & \text{if } \|\nabla u\|_2 > K, \end{cases}$$

$$h^K(w) = \begin{cases} h(w) & \text{if } |\Delta w|_2 \le K, \\ h\left(\frac{Kw}{|\Delta w|_2}\right) & \text{if } |\Delta w|_2 > K, \end{cases}$$

where K is a positive constant such that $K^2 \ge 4E(0) + 1$, where the energy $E(t)$ is given by

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2\right).$$

With the truncated sources above, we consider the following K problem:

$$(K)\begin{cases} u_{tt}+A(u-Rw_t)+g_1(u_t)=f^K(u) & \text{in } \Omega\times(0,T), \\ w_{tt}+\Delta^2 w+g_2(w_t)+\gamma u_t=h^K(w) & \text{in } \Gamma\times(0,T), \\ (u(0),u_t(0))=(u_0,u_1)\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega), \\ (w(0),w_t(0))=(w_0,w_1)\in H^2_0(\Gamma)\times L^2(\Gamma). \end{cases}$$

We note here that for each such K, the operators $f^K : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ and $h^K : H^2_0(\Gamma) \longrightarrow L^2(\Gamma)$ are globally Lipschitz continuous (see for instance [19, 36]). Therefore, by Lemma 3.3.1, the (K) problem has a unique global strong solution $U_K \in W^{1,\infty}(0,T;H)$ for any $T>0$ provided the initial datum $U_0 \in \mathscr{D}(\mathscr{A})$.

In what follows, we shall express $(u_K(t), w_K(t))$ as $(u(t), w(t))$. Since $u_t \in H^1_{\Gamma_0}(\Omega)$ and $v_t \in H^2_0(\Gamma)$, such that $g_1(u_t) \in (H^1_{\Gamma_0}(\Omega))' \cap L^1(\Omega)$, then by (3.3.8) we may use the multiplier u_t and v_t on the (K) problem and obtain the following energy identity:

$$\begin{aligned} E(t) &+ \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau \\ &= E(0) + \int_0^t \int_\Omega f^K(u)u_t dx d\tau + \int_0^t \int_\Gamma h^K(w)w_t d\Gamma d\tau. \end{aligned} \tag{3.3.18}$$

In addition, since $m, r \geq 1$, we know $\tilde{m} = \frac{m+1}{m}, \tilde{r} = \frac{r+1}{r} \leq 2$. Hence, by our assumptions on the sources, it follows that $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ and $h : H^2_0(\Gamma) \longrightarrow L^{\tilde{r}}(\Gamma)$ are locally Lipschitz continuous with Lipschitz constants $L_f(K)$ and $L_h(K)$, respectively, on the ball $\{(u,w) \in V : \|(u,w)\|_V \leq K\}$. Put $L_K = \max\{L_f(K), L_h(K)\}$.

We now estimate the terms due to the sources in the energy identity (3.3.18). By

Hölder's and Young's inequalities, we have

$$\begin{aligned}\int_0^t \int_\Omega f^K(u) u_t dx d\tau &\le \int_0^t \|f^K(u)\|_{\tilde{m}} \|u_t\|_{m+1} d\tau \\ &\le \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \|f^K(u)\|_{\tilde{m}}^{\tilde{m}} d\tau \\ &\le \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \Big(\|f^K(u) - f^K(0)\|_{\tilde{m}}^{\tilde{m}} + \|f^K(0)\|_{\tilde{m}}^{\tilde{m}} \Big) d\tau \\ &\le \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon L_K^{\tilde{m}} \int_0^t \|u\|_{1,\Omega}^{\tilde{m}} d\tau + C_\epsilon t |f(0)|^{\tilde{m}} |\Omega|, \end{aligned} \tag{3.3.19}$$

where $\epsilon > 0$ will be chosen below. Likewise, we deduce

$$\int_0^t \int_\Gamma h^K(w) w_t d\Gamma d\tau \le \epsilon \int_0^t |w_t|_{r+1}^{r+1} d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t \|w\|_{2,\Gamma}^{\tilde{r}} d\tau + C_\epsilon t |h(0)|^{\tilde{r}} |\Gamma|. \tag{3.3.20}$$

By the assumptions on damping, it follows that

$$g_1(s)s \ge \alpha(|s|^{m+1} - 1) \quad \text{and} \quad g_2(s)s \ge \alpha(|s|^{r+1} - 1), \tag{3.3.21}$$

for all $s \in \mathbb{R}$. Therefore,

$$\begin{cases} \int_0^t \int_\Omega g_1(u_t) u_t dx d\tau \ge \alpha \int_0^t \|u_t\|_{m+1}^{m+1} d\tau - \alpha t |\Omega|, \\ \\ \int_0^t \int_\Gamma g_2(w_t) w_t d\Gamma d\tau \ge \alpha \int_0^t |w_t|_{r+1}^{r+1} d\tau - \alpha t |\Gamma|. \end{cases} \tag{3.3.22}$$

By combining (3.3.19)-(3.3.22) in the energy identity (3.3.18), one has

$$
\begin{aligned}
E(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}) d\tau - \alpha t(|\Omega| + |\Gamma|) \\
\leq E(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}) d\tau \\
+ C_\epsilon L_K^{\tilde{m}} \int_0^t \|u\|_{1,\Omega}^{\tilde{M}} d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t \|w\|_{2,\Gamma}^{\tilde{r}} d\tau \\
+ C_\epsilon t(|f(0)|^{\tilde{m}}|\Omega| + |h(0)|^{\tilde{r}}|\Gamma|). \qquad (3.3.23)
\end{aligned}
$$

If $\epsilon \leq \alpha$, then (3.3.23) implies

$$
\begin{aligned}
E(t) \leq E(0) + C_\epsilon L_K^{\tilde{m}} \int_0^t \|u\|_{1,\Omega}^{\tilde{m}} d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t \|w\|_{2,\Gamma}^{\tilde{r}} d\tau \\
+ C_\epsilon t(|f(0)|^{\tilde{m}}|\Omega| + |h(0)|^{\tilde{r}}|\Gamma|) + \alpha t(|\Omega| + |\Gamma|). \qquad (3.3.24)
\end{aligned}
$$

Since $\tilde{m}, \tilde{r} \leq 2$, then by Young's inequality,

$$
\int_0^t \|u\|_{1,\Omega}^{\tilde{m}} d\tau \leq \int_0^t (\|u\|_{1,\Omega}^2 + \tilde{C}) d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C}t,
$$

$$
\int_0^t \|w\|_{2,\Gamma}^{\tilde{r}} d\tau \leq \int_0^t (\|w\|_{2,\Gamma}^2 + \tilde{C}) d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C}t,
$$

where $\tilde{C}$ is a positive constant that depends on m and r. Therefore, if we set $C(L_K) = 2C_\epsilon(L_K^{\tilde{m}} + L_K^{\tilde{r}})$ and $C_0 = C_\epsilon(|f(0)|^{\tilde{m}}|\Omega| + |h(0)|^{\tilde{r}}|\Gamma|) + \alpha(|\Omega| + |\Gamma|) + 2\tilde{C}$, then it follows from (3.3.24) that

$$
E(t) \leq (E(0) + C_0 T_0) + C(L_K) \int_0^t E(\tau) d\tau, \text{ for all } t \in [0, T_0],
$$

where T_0 will be chosen below. By Gronwall's inequality, one has

$$E(t) \leq (E(0) + C_0 T_0) e^{C(L_K)t} \text{ for all } t \in [0, T_0]. \tag{3.3.25}$$

We select

$$T_0 = \min\left\{\frac{1}{4C_0}, \frac{1}{C(L_K)} \log 2\right\}, \tag{3.3.26}$$

and recall our assumption that $K^2 \geq 4E(0) + 1$. Then, it follows from (3.3.25) that

$$E(t) \leq 2\left(E(0) + \frac{1}{4}\right) \leq \frac{K^2}{2}, \tag{3.3.27}$$

for all $t \in [0, T_0]$. This implies that $\|(u(t), w(t))\|_V \leq K$, for all $t \in [0, T_0]$, and therefore, $f^K(u) = f(u)$ and $h^K(w) = h(w)$ on the time interval $[0, T_0]$. Because of the uniqueness of solutions for the (K) problem, the solution to the truncated problem (K) coincides with the solution to the system (3.3.3) for $t \in [0, T_0]$, completing the proof of Lemma 3.3.2. $\square$

Remark 3.3.3. In Lemma 3.3.2, the local existence time T_0 depends on L_K, which is the local Lipschitz constant of $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ and $h : H^2_0(\Gamma) \longrightarrow L^{\tilde{r}}(\Gamma)$. The advantage of this result is that T_0 does not depend on the locally Lipschitz constants for the mappings $f : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ and $h : H^2_0(\Gamma) \longrightarrow L^2(\Gamma)$. This fact is critical for the remaining parts of the proof of the local existence statement in Theorem 3.2.4 $\triangle$

3.3.1.3 Lipschitz Approximations of the Sources.

This subsection is devoted for constructing Lipschitz approximations of the wave source. We begin with essential propositions.

Proposition 3.3.4. *Assume* $1 \le p < 6$, $m \ge 1$, *and* $p\frac{m+1}{m} \le \frac{6}{1+2\epsilon}$ *for some* $\epsilon > 0$. *Further assume* $f \in C^1(\mathbb{R})$ *such that* $|f'(s)| \le C(|s|^{p-1} + 1)$, *for all* $s \in \mathbb{R}$. *Then* $f : H^{1-\epsilon}_{\Gamma_0}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ *is locally Lipschitz continuous, where* $\tilde{m} = \frac{m+1}{m}$.

Remark 3.3.5. Since $H^1_{\Gamma_0}(\Omega) \hookrightarrow H^{1-\epsilon}_{\Gamma_0}(\Omega)$, then it follows from Proposition 3.3.4 that f is locally Lipschitz from $H^1_{\Gamma_0}(\Omega)$ into $L^{\tilde{m}}(\Omega)$. In particular, if $1 \le p \le 3$, then it is easy to verify that f is locally Lipschitz from $H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$. △

Proof. Let u, $\hat{u} \in H^{1-\epsilon}_{\Gamma_0}(\Omega)$ such that $\|u\|_{H^{1-\epsilon}_{\Gamma_0}(\Omega)}, \|\hat{u}\|_{H^{1-\epsilon}_{\Gamma_0}(\Omega)} \le R$, for some $R > 0$. It follows from (3.2.1) that

$$\|f(u) - f(\hat{u})\|^{\tilde{m}}_{\tilde{m}} \le C \int_\Omega |u - \hat{u}|^{\tilde{m}} \left(|u|^{\tilde{m}(p-1)} + |\hat{u}|^{\tilde{m}(p-1)} + 1\right) dx. \tag{3.3.28}$$

All terms in (3.3.28) are estimated in the same manner. In particular, for a typical term in (3.3.28), we estimate it by Hölder's inequality and the Sobolev imbedding $H^{1-\epsilon}_{\Gamma_0}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$ together with the assumption $p\tilde{m} \le \frac{6}{1+2\epsilon}$ and $\|u\|_{H^{1-\epsilon}_{\Gamma_0}(\Omega)} \le R$. For instance,

$$\begin{aligned}\int_\Omega |u-\hat{u}|^{\tilde{m}}|u|^{(p-1)\tilde{m}} dx &\le \|u - \hat{u}\|^{\tilde{m}}_{p\tilde{m}} \|u\|^{(p-1)\tilde{m}}_{p\tilde{m}} \\ &\le C\|u - \hat{u}\|^{\tilde{m}}_{H^{1-\epsilon}_{\Gamma_0}(\Omega)} \|u\|^{(p-1)\tilde{m}}_{H^{1-\epsilon}_{\Gamma_0}(\Omega)} \le CR^{(p-1)\tilde{m}} \|u - \hat{u}\|^{\tilde{m}}_{H^{1-\epsilon}_{\Gamma_0}(\Omega)}.\end{aligned}$$

Hence, we obtain

$$\|f(u) - f(\hat{u})\|_{\tilde{m}} \le C(R)\|u - \hat{u}\|_{H^{1-\epsilon}_{\Gamma_0}(\Omega)},$$

for some constant $C(R) > 0$ depending on R. $\square$

Recall that for the values $3 < p < 6$, the source $f(u)$ is not locally Lipschitz continuous from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$. So, in order to apply Lemma 3.3.2 to prove Theorem 3.2.4, we shall construct Lipschitz approximations of the source f. In particular, we shall use smooth cutoff functions $\eta_n \in C_0^\infty(\mathbb{R})$, similar to those used in Bociu [12] and later by [33] and others, such that each η_n satisfies:
$0 \le \eta_n \le 1$; $\eta_n(u) = 1$ if $|u| \le n$; $\eta_n(u) = 0$ if $|u| \ge 2n$; and $|\eta_n'(u)| \le \frac{C}{n}$. Put

$$f^n(u) = f(u)\eta_n(u), \ u \in \mathbb{R}, \ n \in \mathbb{N}, \tag{3.3.29}$$

where f satisfies Assumption 3.2.1. The following proposition summarizes important properties of f^n. Its proof has appeared in [14] and it is also similar to the one appeared in [33], thus its proof is omitted.

Proposition 3.3.6. *For each $n \in \mathbb{N}$, the function f^n defined in* (3.3.29) *satisfies:*

- *$f^n(u) : H^1_{\Gamma_0}(\Omega) \longrightarrow L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant depending on n.*
- *There exists $\epsilon > 0$ such that $f^n : H^{1-\epsilon}_{\Gamma_0}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz continuous where the local Lipschitz constant is independent of n and where $\tilde{m} = \frac{m+1}{m}$.*

The following proposition deals with the plate source h and its proof is trivial.

Proposition 3.3.7. *Assume $1 \le q < \infty$ and $r \ge 1$. If $h \in C^1(\mathbb{R})$ such that $|h'(s)| \le C(|s|^{q-1} + 1)$, then h is Locally Lipschitz from $H_0^{2-\epsilon}(\Gamma)$ into $L^{\frac{r+1}{r}}(\Gamma)$, where $\epsilon > 0$ is sufficiently small.*

3.3.2 Approximate Solutions and Passage to the Limit.

We complete the proof of the local existence statement in Theorem 3.2.4 in the following four steps.

Step 1: Approximate system.

Recall $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$, and the approximate source f^n which was introduced in (3.3.29). Now, we define the nonlinear operator $\mathscr{A}^n : \mathscr{D}(\mathscr{A}^n) \subset H \longrightarrow H$ by:

$$\mathscr{A}^n \begin{bmatrix} u \\ w \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ A(u - Rz) + g_1(y) - f^n(u) \\ \Delta^2 w + g_2(z) + \gamma y - h(w) \end{bmatrix}^{tr}, \tag{3.3.30}$$

where $\mathscr{D}(\mathscr{A}^n) = \{(u, w, y, z) \in \left(H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)\right)^2 : A(u - Rz) + g_1(y) - f^n(u) \in L^2(\Omega),\ g_1(y) \in \left(H^1_{\Gamma_0}(\Omega)\right)' \cap L^1(\Omega),\ \Delta^2 w \in L^2(\Gamma)\}$.

Clearly, the space of test functions $\mathscr{D}(\Omega)^4 \subset \mathscr{D}(\mathscr{A}^n)$, and since $\mathscr{D}(\Omega)^4$ is dense in H, for each $(u_0, w_0, u_1, w_1) \in H$ there exists a sequence of functions $U^n_0 = (u^n_0, w^n_0, u^n_1, w^n_1) \in \mathscr{D}(\Omega)^4$ such that $U^n_0 \longrightarrow U_0$ in H.

Put $U = (u, w, u_t, w_t)$ and consider the approximate system:

$$U_t + \mathscr{A}^n U = 0 \text{ with } U(0) = (u^n_0, w^n_0, u^n_1, w^n_1) \in \mathscr{D}(\Omega)^4. \tag{3.3.31}$$

Step 2: Approximate solutions.

Since f^n satisfies the assumptions of Lemma 3.3.2, then for each n, the approximate problem (3.3.31) has a strong local solution $U^n = (u^n, w^n, u_t^n, w_t^n) \in W^{1,\infty}(0,T_0;H)$ such that $U^n(t) \in \mathscr{D}(\mathscr{A}^n)$ for $t \in [0,T_0]$. It is important to note here that T_0 is totally independent of n. Although in (3.3.26) T_0 does depend on the local Lipschitz constant of the mapping $f^n : H^1_{\Gamma_0}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, however, according to Proposition 3.3.6 the said Lipschitz constant is independent of n. Also, recall that in the proof of Lemma 3.3.2, T_0 depends on K which itself depends on the initial data, and since $U_0^n \longrightarrow U_0$ in H, we can choose K sufficiently large such that one K is uniform for all n. Thus, we will only emphasize the dependence of T_0 on K.

Now, by (3.3.27), we know $E^n(t) \leq \frac{K^2}{2}$ for all $t \in [0,T_0]$, which implies that,

$$\|U^n(t)\|_H^2 = \|u^n(t)\|_{1,\Omega}^2 + \|w^n(t)\|_{2,\Gamma}^2 + \|u_t^n(t)\|_2^2 + |w_t^n(t)|_2^2 \leq K^2, \tag{3.3.32}$$

for all $t \in [0,T_0]$. In addition, by letting $0 < \epsilon \leq \frac{\alpha}{2}$ in (3.3.23) and by the fact $\tilde{m}$, $\tilde{r} \leq 2$ and the bound (3.3.32), we deduce that,

$$\int_0^{T_0} \|u_t^n\|_{m+1}^{m+1} dt + \int_0^{T_0} |w_t^n|_{r+1}^{r+1} dt < C(K), \tag{3.3.33}$$

for some constant $C(K) > 0$ (independent of n). Since $|g_1(s)| \leq \beta |s|^m$ for all $|s| \geq 1$, and g_1 is increasing with $g_1(0) = 0$, then $|g_1(s)| \leq \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$. Hence, it follows from (3.3.33) that

$$\int_0^{T_0} \int_\Omega |g_1(u_t^n)|^{\tilde{m}} dx dt \leq \beta^{\tilde{m}} \int_0^{T_0} \int_\Omega (|u_t^n|^{m+1} + 1) dx dt < C(K). \tag{3.3.34}$$

Similarly, one has

$$\int_0^{T_0}\int_\Gamma |g_2(w_t^n)|^{\tilde{r}} d\Gamma dt \le \beta^{\tilde{r}} \int_0^{T_0}\int_\Gamma (|w_t^n|^{r+1}+1) d\Gamma dt < C(K). \tag{3.3.35}$$

Now recall that $U^n = (u^n, w^n, u_t^n, w_t^n) \in \mathscr{D}(\mathscr{A}^n)$ is a strong solution of (3.3.31). If ϕ and ψ satisfy the conditions imposed on test functions in Definition 3.2.3, then by (3.3.34) and (3.3.35) we can test the approximate system (3.3.31) with ϕ and ψ to obtain:

$$\begin{aligned}(u_t^n(t), \phi(t))_\Omega - (u_t^n(0), \phi(0))_\Omega - \int_0^t (u_t^n(\tau), \phi_t(\tau))_\Omega d\tau + \int_0^t (\nabla u^n(\tau), \nabla\phi(\tau))_\Omega d\tau \\ - \int_0^t (w_t^n(\tau), \gamma\phi(\tau))_\Gamma d\tau + \int_0^t\int_\Omega g_1(u_t^n(\tau))\phi(\tau) dx d\tau \\ = \int_0^t\int_\Omega f^n(u^n(\tau))\phi(\tau) dx d\tau,\end{aligned} \tag{3.3.36}$$

$$\begin{aligned}(w_t^n(t) + \gamma u^n(t), \psi(t))_\Gamma - (w_t^n(0) + \gamma u^n(0), \psi(0))_\Gamma - \int_0^t (w_t^n(\tau), \psi_t(\tau))_\Gamma d\tau \\ - \int_0^t (\gamma u^n(\tau), \psi_t(\tau))_\Gamma d\tau + \int_0^t (\Delta w^n(\tau), \Delta\psi(\tau))_\Gamma d\tau \\ + \int_0^t\int_\Gamma g_2(w_t^n(\tau))\psi(\tau) d\Gamma d\tau = \int_0^t\int_\Gamma h(w^n(\tau))\psi(\tau) d\Gamma d\tau,\end{aligned} \tag{3.3.37}$$

for all $t \in [0, T_0]$.

Step 3: Passage to the limit.

We aim to prove that there exists a subsequence of $\{U^n\}$, labeled again as $\{U^n\}$, that converges to a solution of the original problem (3.1.1). In what follows, we focus on passing to the limit in (3.3.36) and (3.3.37).

First, we note that (3.3.32) shows $\{U^n\}$ is bounded in $L^\infty(0, T_0; H)$. So by Alaoglu's

Theorem, there exists a subsequence, labeled as $\{U^n\}$, such that

$$U^n \longrightarrow U \text{ weakly}^* \text{ in } L^\infty(0,T_0;H). \tag{3.3.38}$$

Also, by (3.3.32), we know $\{u^n\}$ is bounded in $L^\infty(0,T_0;H^1_{\Gamma_0}(\Omega))$. In addition, by (3.3.33), we know $\{u^n_t\}$ is also bounded in $L^{m+1}(\Omega\times(0,T_0))$, and since $m\geq 1$, we see that $\{u^n_t\}$ is also bounded in $L^{\tilde m}(\Omega\times(0,T_0)) = L^{\tilde m}(0,T_0;L^{\tilde m}(\Omega))$. We note here that for sufficiently small $\epsilon>0$, the imbedding $H^1_{\Gamma_0}(\Omega)\hookrightarrow H^{1-\epsilon}_{\Gamma_0}(\Omega)$ is compact, and $H^{1-\epsilon}_{\Gamma_0}(\Omega)\hookrightarrow L^{\tilde m}(\Omega)$. Then, by Aubin-Lions-Simon Compactness Theorem (e.g., [15, Thm. II.5.16]), there exists a subsequence, reindexed by $\{u^n\}$, such that

$$u^n \longrightarrow u \text{ strongly in } C([0,T_0];H^{1-\epsilon}_{\Gamma_0}(\Omega)). \tag{3.3.39}$$

Similarly, we deduce that there exists a subsequence such that

$$w^n \longrightarrow w \text{ strongly in } C([0,T_0];H^{2-\epsilon}_0(\Gamma)). \tag{3.3.40}$$

Now since $H^{1-\epsilon}(\Omega)\hookrightarrow L^2(\Gamma)$ for sufficiently small $\epsilon>0$, it follows from (3.3.39) that

$$\gamma u^n \longrightarrow \gamma u \text{ strongly in } C([0,T_0];L^2(\Gamma)). \tag{3.3.41}$$

Now, fix $t\in[0,T_0]$. Since $\phi\in C([0,t];H^1_{\Gamma_0}(\Omega))$ and $\phi_t\in L^1(0,t;L^2(\Omega))$, then by (3.3.38) we obtain

$$\lim_{n\longrightarrow\infty}\int_0^t(\nabla u^n(\tau),\nabla\phi(\tau))_\Omega d\tau = \int_0^t(\nabla u(\tau),\nabla\phi(\tau))_\Omega d\tau, \tag{3.3.42}$$

and

$$\lim_{n\longrightarrow\infty}\int_0^t (u_t^n(\tau),\phi_t(\tau))_\Omega d\tau = \int_0^t (u_t(\tau),\phi_t(\tau))_\Omega d\tau. \tag{3.3.43}$$

Similarly, we obtain

$$\lim_{n\longrightarrow\infty}\int_0^t (\Delta w^n(\tau),\Delta\psi(\tau))_\Gamma d\tau = \int_0^t (\Delta w(\tau),\Delta\psi(\tau))_\Gamma d\tau, \tag{3.3.44}$$

$$\lim_{n\longrightarrow\infty}\int_0^t (w_t^n(\tau),\psi_t(\tau))_\Gamma d\tau = \int_0^t (w_t(\tau),\psi_t(\tau))_\Gamma d\tau, \tag{3.3.45}$$

and

$$\lim_{n\longrightarrow\infty}\int_0^t (\gamma u^n(\tau),\psi_t(\tau))_\Gamma d\tau = \int_0^t (\gamma u(\tau),\psi_t(\tau))_\Gamma d\tau. \tag{3.3.46}$$

From (3.3.32) we know $\{w_t^n\}$ is bounded in $L^\infty(0,T_0;L^2(\Gamma))$. Now since $\phi\in C([0,t];H^1_{\Gamma_0}(\Omega))$ and $H^1_{\Gamma_0}(\Omega)\hookrightarrow L^2(\Gamma)$, then by (3.3.38) we obtain

$$\lim_{n\longrightarrow\infty}\int_0^t (w_t^n,\gamma\phi)_\Gamma d\tau = \int_0^t (w_t,\gamma\phi)_\Gamma d\tau. \tag{3.3.47}$$

Now by (3.3.34), on a subsequence,

$$g_1(u_t^n)\longrightarrow g_1^* \text{ weakly in } L^{\tilde m}(\Omega\times(0,t)) \tag{3.3.48}$$

for some $g_1^*\in L^{\tilde m}(\Omega)$. Similarly, by (3.3.35), on a subsequence,

$$g_2(w_t^n)\longrightarrow g_2^* \text{ weakly in } L^{\tilde r}(\Gamma\times(0,t)) \tag{3.3.49}$$

for some $g_2^* \in L^{\tilde{r}}(\Gamma)$. Our goal is to show that $g_1^* = g_1(u_t)$ and $g_2^* = g_2(w_t)$. In order to do so, we consider two solutions to the approximate problem (3.3.31), U^n and U^j. For sake of simplifying the notation, put $\tilde{u} = u^n - u^j$ and $\tilde{w} = w^n - w^j$. Since $U^n, U^j \in W^{1,\infty}(0,T_0;H)$ and $U^n(t), U^j(t) \in \mathscr{D}(\mathscr{A}^n)$, then $\tilde{u}_t \in W^{1,\infty}(0,T_0;L^2(\Omega))$ with $\tilde{u}_t(t) \in H^1_{\Gamma_0}(\Omega)$ and $\tilde{w}_t \in W^{1,\infty}(0,T_0;L^2(\Gamma))$ with $\tilde{w}_t(t) \in H^2_0(\Gamma)$. Moreover, by (3.3.33) we know $\tilde{u}_t \in L^{m+1}(\Omega \times (0,T_0))$ and $\tilde{w}_t \in L^{r+1}(\Gamma \times (0,T_0))$. Hence, we may consider the difference of the approximate problems corresponding to the parameters n and j, and then use the multiplier $\tilde{u}_t$ on the first equation. By performing integration by parts in the first equation, one has the following energy identity (in what follows, we suppress the variable τ and use the abbreviated notation for norms and inner product):

$$\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2 + \|\tilde{u}(t)\|_{1,\Omega}^2\right) - \int_0^t (\tilde{w}_t, \gamma\tilde{u}_t)_\Gamma d\tau + \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))\tilde{u}_t dx d\tau$$
$$= \frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2 + \|\tilde{u}(0)\|_{1,\Omega}^2\right) + \int_0^t \int_\Omega (f^n(u^n) - f^j(u^j))\tilde{u}_t dx d\tau. \tag{3.3.50}$$

Similarly, we may consider the difference of the approximate problems corresponding to the parameters n and j, and then use the multiplier $\tilde{w}_t$ on the second equation. By performing integration by parts in the second equation, one has:

$$\frac{1}{2}\left(|\tilde{w}_t(t)|_2^2 + \|\tilde{w}(t)\|_{2,\Gamma}^2\right) + \int_0^t \int_\Gamma (g_2(w^n) - g_2(w^j))\tilde{w}_t d\Gamma d\tau + \int_0^t (\gamma\tilde{u}_t, \tilde{w}_t)_\Gamma d\tau$$
$$= \frac{1}{2}\left(|\tilde{w}_t(0)|_2^2 + \|\tilde{w}(0)\|_{2,\Gamma}^2\right) + \int_0^t \int_\Gamma (h(w^n) - h(w^j))\tilde{w}_t d\Gamma d\tau. \tag{3.3.51}$$

Now adding (3.3.50) and (3.3.51) we obtain:

$$\begin{aligned}
&\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2+\|\tilde{u}(t)\|_{1,\Omega}^2+|\tilde{w}_t(t)|_2^2+\|\tilde{w}(t)\|_{2,\Gamma}^2\right)\\
&\quad+\int_0^t\int_\Omega (g_1(u_t^n)-g_1(u_t^j))\tilde{u}_t dxd\tau+\int_0^t\int_\Gamma (g_2(w^n)-g_2(w^j))\tilde{w}_t d\Gamma d\tau\\
&\quad\leq\frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2+\|\tilde{u}(0)\|_{1,\Omega}^2+|\tilde{w}_t(0)|_2^2+\|\tilde{w}(0)\|_{2,\Gamma}^2\right)\\
&\quad+\int_0^t\int_\Omega |f^n(u^n)-f^j(u^j)||\tilde{u}_t|dxd\tau+\int_0^t\int_\Gamma |h(w^n)-h(w^j)||\tilde{w}_t|d\Gamma d\tau. \quad (3.3.52)
\end{aligned}$$

We will show that each term on the right hand side of (3.3.52) converges to 0 as $n,j\longrightarrow\infty$. First, since $\lim_{n\longrightarrow\infty}\|u_0^n-u_0\|_{1,\Omega}=\lim_{n\longrightarrow\infty}\|u_1^n-u_1\|_2=\lim_{n\longrightarrow\infty}\|w_0^n-w_0\|_{2,\Gamma}=\lim_{n\longrightarrow\infty}|w_1^n-w_1|_2=0$, we obtain

$$\begin{aligned}
&\lim_{n,j\longrightarrow\infty}\|\tilde{u}(0)\|_{1,\Omega}=\lim_{n,j\longrightarrow 0}\|u_0^n-u_0^j\|_{1,\Omega}=0,\\
&\lim_{n,j\longrightarrow\infty}\|\tilde{u}_t(0)\|_2=\lim_{n,j\longrightarrow 0}\|u_1^n-u_1^j\|_2=0,\\
&\lim_{n,j\longrightarrow\infty}\|\tilde{w}(0)\|_{2,\Gamma}=\lim_{n,j\longrightarrow 0}\|w_0^n-w_0^j\|_{2,\Gamma}=0,\\
&\lim_{n,j\longrightarrow\infty}|\tilde{w}_t(0)|_2=\lim_{n,j\longrightarrow 0}|w_1^n-w_1^j|_2=0. \quad (3.3.53)
\end{aligned}$$

Next, we look at the second term on the right-hand side of (3.3.52). We have,

$$\begin{aligned}
&\int_0^t\int_\Omega |f^n(u^n)-f^j(u^j)||\tilde{u}_t|dxd\tau\\
&\quad\leq\int_0^t\int_\Omega |f^n(u^n)-f^n(u)||\tilde{u}_t|dxd\tau+\int_0^t\int_\Omega |f^n(u)-f(u)||\tilde{u}_t|dxd\tau\\
&\quad+\int_0^t\int_\Omega |f(u)-f^j(u)||\tilde{u}_t|dxd\tau+\int_0^t\int_\Omega |f^j(u)-f^j(u^j)||\tilde{u}_t|dxd\tau. \quad (3.3.54)
\end{aligned}$$

We now estimate each term on the right-hand side of (3.3.54) as follows. Recall, by Proposition 3.3.6, $f^n: H_{\Gamma_0}^{1-\epsilon}\longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz where the local Lipschitz constant is independent of n. By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega |f^n(u^n) - f^n(u)||\tilde{u}_t| dx d\tau \\
&\quad \leq \left(\int_0^t \int_\Omega |f^n(u^n) - f^n(u)|^{\tilde{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\tilde{u}_t|^{m+1} dx d\tau \right)^{\frac{1}{m+1}} \\
&\quad \leq C(K) \left(\int_0^t \|u^n - u\|^{\tilde{m}}_{H^{1-\epsilon}_{\Gamma_0}(\Omega)} d\tau \right)^{\frac{m}{m+1}} \longrightarrow 0, \qquad (3.3.55)
\end{aligned}$$

as $n \longrightarrow \infty$, where we have used the convergence (3.3.39) and the uniform bound in (3.3.32). To handle the second term on the right-hand side of (3.3.54), we shall show

$$f^n(u) \longrightarrow f(u) \text{ in } L^{\tilde{m}}(\Omega \times (0, T_0)). \qquad (3.3.56)$$

Indeed, by (3.3.38), we know $U \in L^\infty(0, T_0; H)$, thus $u \in L^\infty(0, T_0; H^1_{\Gamma_0}(\Omega))$. In addition, by (3.3.29) and the definition of f^n, we have

$$\|f^n(u) - f(u)\|^{\tilde{m}}_{L^{\tilde{m}}(\Omega \times (0,T_0))} = \int_0^{T_0} \int_\Omega \left(|f(u)||\eta_n(u) - 1| \right)^{\tilde{m}} dx dt. \qquad (3.3.57)$$

Since $0 \leq \eta_n(u) \leq 1$, it follows that $\left(|f(u)||\eta_n(u) - 1| \right)^{\tilde{m}} \leq |f(u)|^{\tilde{m}}$. To see $|f(u)|^{\tilde{m}} \in L^1(\Omega \times (0, T_0))$, we use the assumptions $|f(u)| \leq C\left(|u|^p + 1\right)$ and $p\tilde{m} < 6$ along with the imbedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$. Indeed,

$$\begin{aligned}
\int_0^{T_0} \int_\Omega |f(u)|^{\tilde{m}} dx dt &\leq C \int_0^{T_0} \int_\Omega \left(|u|^{p\tilde{m}} + 1 \right) dx dt \\
&\leq C \int_0^{T_0} \left(\|u\|^{p\tilde{m}}_{H^1_{\Gamma_0}(\Omega)} + |\Omega| \right) dt < \infty.
\end{aligned}$$

Clearly, $\eta_n(u) \longrightarrow 1$ a.e. on Ω as $n \longrightarrow \infty$. By applying the Lebesque Dominated Convergence Theorem on (3.3.57), then (3.3.56) follows, as desired. Now, by using

Hölder's inequality and the limit (3.3.56), one has

$$\int_0^t \int_\Omega |f^n(u) - f(u)||\tilde{u}_t| dx d\tau$$
$$\leq \left(\int_0^t \int_\Omega |f^n(u) - f(u)|^{\tilde{m}} dx d\tau\right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\tilde{u}_t|^{m+1} dx d\tau\right)^{\frac{1}{m+1}} \longrightarrow 0, \quad (3.3.58)$$

as $n \longrightarrow \infty$, where we have used the uniform bound in (3.3.32). Combining (3.3.55) and (3.3.58) in (3.3.54) gives us the desired result, namely,

$$\lim_{n,j \longrightarrow \infty} \int_0^t \int_\Omega |f^n(u^n) - f^j(u^j)||\tilde{u}_t| dx d\tau = 0. \quad (3.3.59)$$

Next we show

$$\lim_{n,j \longrightarrow \infty} \int_0^t \int_\Gamma |h(w^n) - h(w^j)||\tilde{w}_t| d\Gamma d\tau = 0. \quad (3.3.60)$$

To see this, we write

$$\int_0^t \int_\Gamma |h(w^n) - h(w^j)||\tilde{w}_t| d\Gamma d\tau$$
$$\leq \int_0^t \int_\Gamma |h(w^n) - h(w)||\tilde{w}_t| d\Gamma d\tau + \int_0^t \int_\Gamma |h(w) - h(w^j)||\tilde{w}_t| d\Gamma d\tau. \quad (3.3.61)$$

By Proposition 3.3.7, $h : H_0^{2-\epsilon}(\Gamma) \longrightarrow L^{\tilde{r}}(\Gamma)$ is locally Lipschitz. Therefore, by Hölder's inequality

$$\int_0^t \int_\Gamma |h(w^n) - h(w)||\tilde{w}_t| d\Gamma d\tau$$
$$\leq \left(\int_0^t \int_\Gamma |h(w^n) - h(w)|^{\tilde{r}} d\Gamma d\tau\right)^{\frac{r}{r+1}} \left(\int_0^t \int_\Gamma |\tilde{w}_t|^{r+1} d\Gamma d\tau\right)^{\frac{1}{r+1}}$$
$$\leq C(K) \left(\int_0^t \|w^n - w\|^{\tilde{r}}_{H_0^{2-\epsilon}(\Gamma)}\right)^{\frac{r}{r+1}} \longrightarrow 0, \quad (3.3.62)$$

as $n \longrightarrow \infty$, where we have used the convergence (3.3.40) and the uniform bound in (3.3.33). This is enough to yield the desired result (3.3.60).

Now, by using the fact that g_1 and g_2 are monotone increasing and using (3.3.53), (3.3.59), and (3.3.60), we can take the limit as $n, j \longrightarrow \infty$ in (3.3.52) to deduce

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))(u_t^n - u_t^j)dxd\tau = 0, \tag{3.3.63}$$

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Gamma (g_2(w_t^n) - g_2(w_t^j))(w_t^n - w_t^j)d\Gamma d\tau = 0. \tag{3.3.64}$$

In addition, it follows from (3.3.33) that, on relabeled subsequences $u_t^n \longrightarrow u_t$ weakly in $L^{m+1}(\Omega\times(0,T_0))$ and $w_t^n \longrightarrow w_t$ weakly in $L^{r+1}(\Gamma\times(0,T_0))$. Therefore, Lemma 1.3 (p.49) [6] along with (3.3.48), (3.3.49), (3.3.63), and (3.3.64) assert that $g_1^* = g_1(u_t)$ and $g_2^* = g_2(w_t)$; provided we show that

$$g_1 : L^{m+1}(\Omega \times (0,T_0)) \longrightarrow L^{\tilde{m}}(\Omega \times (0,T_0))$$

and

$$g_2 : L^{r+1}(\Gamma \times (0,T_0)) \longrightarrow L^{\tilde{r}}(\Gamma \times (0,T_0))$$

are maximal monotone. Indeed, since g_1 and g_2 are monotone increasing, it is easy to see g_1 and g_2 are monotone operators. Thus, we need to verify that g_1 and g_2 are hemi-continuous, i.e., in the case for g_1 we have to show that

$$\lim_{\lambda\longrightarrow\infty} \int_0^t \int_\Omega g_1(u+\lambda v)zdxd\tau = \int_0^t \int_\Omega g_1(u)zdxd\tau, \tag{3.3.65}$$

for all u, v, and $z \in L^{m+1}(\Omega \times (0,t))$. Indeed, since g_1 is continuous, then $g_1(u + \lambda v)z \longrightarrow g_1(u)z$ point-wise as $\lambda \longrightarrow 0$. Moreover, since $|g_1(s)| \leq \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$, we know if $|\lambda| \leq 1$, then $|g_1(u + \lambda v)z| \leq \beta(|u + \lambda v|^m + 1)|z| \leq C(|u|^m|z| + |v|^m|z| + |z|) \in L^1(\Omega \times (0,t))$, by Hölder's inequality. Thus, (3.3.65) follows from the Lebesque Dominated Convergence Theorem. Hence, g_1 is maximal monotone and we conclude $g_1^* = g_1(u_t)$, i.e.,

$$g_1(u_t^n) \longrightarrow g_1(u_t) \text{ weakly in } L^{\tilde{m}}(\Omega \times (0,t)). \tag{3.3.66}$$

In a similar way, one can show that g_2 is indeed maximal monotone, and in turn we deduce $g_2^* = g_2(w_t^n)$, that is

$$g_2(w_t^n) \longrightarrow g_2(w_t) \text{ weakly in } L^{\tilde{r}}(\Gamma \times (0,t)). \tag{3.3.67}$$

Now as $\phi \in L^{m+1}(\Omega \times (0,t))$, it follows from (3.3.66) that

$$\lim_{n \longrightarrow \infty} \int_0^t \int_\Omega g_1(u_t^n)\phi dx d\tau = \int_0^t \int_\Omega g_1(u_t)\phi dx d\tau \tag{3.3.68}$$

Similarly, we obtain

$$\lim_{n \longrightarrow \infty} \int_0^t \int_\Gamma g_2(w_t^n)\psi d\Gamma d\tau = \int_0^t \int_\Gamma g_2(w_t)\psi d\Gamma d\tau. \tag{3.3.69}$$

Next we wish to show that

$$\lim_{n \longrightarrow \infty} \int_0^t \int_\Omega f^n(u^n)\phi dx d\tau = \int_0^t \int_\Omega f(u)\phi dx d\tau \tag{3.3.70}$$

To prove (3.3.70), we write

$$\left|\int_0^t \int_\Omega (f^n(u^n) - f(u))\phi dx d\tau\right|$$
$$\leq \int_0^t \int_\Omega |f^n(u^n) - f^n(u)||\phi| dx d\tau + \int_0^t \int_\Omega |f^n(u) - f(u)||\phi| dx d\tau. \qquad (3.3.71)$$

Since $\phi \in L^{m+1}(\Omega \times (0,t))$, then by replacing $\tilde{u}_t$ by ϕ in (3.3.55), we deduce

$$\lim_{n\longrightarrow\infty} \int_0^t \int_\Omega |f^n(u^n) - f^n(u)||\phi| dx d\tau = 0. \qquad (3.3.72)$$

In addition, (3.3.56) yields

$$\lim_{n\longrightarrow\infty} \int_0^t \int_\Omega |f^n(u) - f(u)||\phi| dx d\tau = 0. \qquad (3.3.73)$$

Hence, (3.3.70) is verified. In a similar manner, one can deduce

$$\lim_{n\longrightarrow\infty} \int_0^t \int_\Gamma h(w^n)\psi d\Gamma d\tau = \int_0^t \int_\Gamma h(w)\psi d\Gamma d\tau. \qquad (3.3.74)$$

Step 4: Completion of the proof.

Lastly, since $t \in [0, T_0]$ and g_1, g_2 are monotone increasing on $\mathbb{R}$, then (3.3.52), along with (3.3.53), (3.3.59), and (3.3.60) imply

$$\lim_{n,j\longrightarrow\infty} \|u^n(t) - u^j(t)\|_{1,\Omega}^2 = \lim_{n,j\longrightarrow\infty} \|\tilde{u}(t)\|_{1,\Omega}^2 = 0 \text{ uniformly in } t \in [0, T_0];$$

$$\lim_{n,j\longrightarrow\infty} \|u_t^n(t) - u_t^j(t)\|_2^2 = \lim_{n,j\longrightarrow\infty} \|\tilde{u}_t(t)\|_2^2 = 0 \text{ uniformly in } t \in [0, T_0];$$

$$\lim_{n,j\longrightarrow\infty} \|w^n(t) - w^j(t)\|_{2,\Gamma}^2 = \lim_{n,j\longrightarrow\infty} \|\tilde{w}(t)\|_{2,\Gamma}^2 = 0 \text{ uniformly in } t \in [0, T_0];$$

$$\lim_{n,j\longrightarrow\infty} |w_t^n(t) - w_t^j(t)|_2^2 = \lim_{n,j\longrightarrow\infty} |\tilde{w}_t(t)|_2^2 = 0 \text{ uniformly in } t \in [0, T_0].$$

Hence,

$$\begin{cases} u^n(t) \longrightarrow u(t) \text{ in } H^1_{\Gamma_0}(\Omega) \text{ uniformly on } [0,T_0], \\ u^n_t(t) \longrightarrow u_t(t) \text{ in } L^2(\Omega) \text{ uniformly on } [0,T_0], \\ w^n(t) \longrightarrow w(t) \text{ in } H^2_0(\Gamma) \text{ uniformly on } [0,T_0], \\ w^n_t(t) \longrightarrow w_t(t) \text{ in } L^2(\Gamma) \text{ uniformly on } [0,T_0]. \end{cases} \tag{3.3.75}$$

Since $U^n \in W^{1,\infty}(0,T_0;H)$, by (3.3.75), we conclude

$$u \in C([0,T_0];H^1_{\Gamma_0}(\Omega)), \quad u_t \in C([0,T_0];L^2(\Omega)),$$
$$w \in C([0,T_0];H^2_0(\Gamma)), \quad w_t \in C([0,T_0];L^2(\Gamma)).$$

Moreover, (3.3.75) shows $u^n(0) \longrightarrow u(0)$ in $H^1_{\Gamma_0}(\Omega)$. Since $u^n(0) = u^n_0 \longrightarrow u_0$ in $H^1_{\Gamma_0}(\Omega)$, then the initial condition $u(0) = u_0$ holds. Also, since $u^n_t(0) \longrightarrow u_t(0)$ in $L^2(\Omega)$ and $u^n_t(0) = u^n_1 \longrightarrow u_1$ in $L^2(\Omega)$, we obtain $u_t(0) = u_1$. Similarly, we find that $w(0) = w_0$ and $w_t(0) = w_1$. Finally, by using (3.3.38)-(3.3.47), (3.3.68)-(3.3.70), (3.3.74)-(3.3.75), we can pass to the limit in (3.3.36) and (3.3.37) to obtain (3.2.2) and (3.2.3) with the imposed regularity on u and w. This completes the proof of the local existence statement in Theorem 3.2.4.

3.4 Energy Identity

This section is devoted to derive the energy identity (3.2.4) in Theorem 3.2.4. As with chapter 2, one must utilize difference quotients due to limitations in regularity.

3.4.1 Proof of the Energy Identity.

Throughout the proof, we fix $t \in (0, T_0]$ and let (u, w) be a weak solution of the system (3.1.1) in the sense of Definition 3.2.3. Recall the regularity of u and w. We can define the difference quotient $D_h u(\tau)$ on $[0, t]$ as (A.0.1), i.e., $D_h u(\tau) = \frac{1}{2h}[u_e(\tau + h) - u_e(\tau - h)]$, where $u_e(\tau)$ extends $u(\tau)$ from $[0, t]$ to $\mathbb{R}$ as in (A.0.2). By Proposition A.0.4, with $X = L^{m+1}(\Omega)$ and $Y = L^2(\Omega)$, we have

$$D_h u \in L^{m+1}(\Omega \times (0,t)) \text{ and } D_h u \longrightarrow u_t \text{ in } L^{m+1}(\Omega \times (0,t)). \tag{3.4.1}$$

Similarly, we have

$$D_h w \in L^{r+1}(\Gamma \times (0,t)) \text{ and } D_h w \longrightarrow w_t \text{ in } L^{r+1}(\Gamma \times (0,t)). \tag{3.4.2}$$

Moreover, since $u \in C([0,t]; H^1_{\Gamma_0}(\Omega))$ and $w \in C([0,t]; H^2_0(\Gamma))$, then

$$D_h u \in C([0,t]; H^1_{\Gamma_0}(\Omega)) \text{ and } D_h w \in C([0,t]; H^2_0(\Gamma)). \tag{3.4.3}$$

We now show that

$$(D_h u)_t \in L^1(0,t; L^2(\Omega)) \text{ and } (D_h w)_t \in L^1(0,t; L^2(\Gamma)). \tag{3.4.4}$$

Indeed, for $0 < h < \frac{t}{2}$, we note that

$$(D_h u)_t(\tau) = \begin{cases} \frac{1}{2h}[u_t(\tau+h) - u_t(\tau-h)], \text{ if } h < \tau < t-h, \\ \frac{-1}{2h} u_t(\tau - h), \text{ if } t-h < \tau < t, \\ \frac{1}{2h} u_t(\tau+h), \text{ if } 0 < \tau < h, \end{cases}$$

and since $u_t \in C([0,t];L^2(\Omega))$, we conclude $(D_h u)_t \in L^1(0,t;L^2(\Omega))$. Similarly, $(D_h w)_t \in L^1(0,t;L^2(\Gamma))$. Thus, (3.4.1)-(3.4.4) show that $D_h u$ and $D_h w$ satisfy the required regularity conditions to be suitable test functions in Definition 3.2.3. Therefore, by taking $\phi = D_h u$ in (3.2.2) and $\psi = D_h w$ in (3.2.3), we obtain

$$\begin{aligned}
&(u_t(t), D_h u(t))_\Omega - (u_1, D_h u(0))_\Omega - \int_0^t (u_t(\tau), (D_h u)_t(\tau))_\Omega d\tau \\
&\quad + \int_0^t (u(\tau), D_h u(\tau))_{1,\Omega} d\tau - \int_0^t (w_t(\tau), \gamma D_h u(\tau))_\Gamma d\tau \\
&\quad + \int_0^t \int_\Omega g_1(u_t(\tau)) D_h u(\tau) dx d\tau = \int_0^t \int_\Omega f(u(\tau)) D_h u(\tau) dx d\tau, \qquad (3.4.5)
\end{aligned}$$

and

$$\begin{aligned}
&(w_t(t) + \gamma u(t), D_h w(t))_\Gamma - (w_1 + \gamma u(0), D_h w(0))_\Gamma - \int_0^t (w_t(\tau), (D_h w)_t(\tau)_\Gamma d\tau \\
&\quad - \int_0^t (\gamma u(\tau), (D_h w)_t(\tau))_\Gamma d\tau + \int_0^t (w(\tau), D_h w(\tau))_{2,\Gamma} d\tau \\
&\quad + \int_0^t \int_\Gamma g_2(w_t(\tau)) D_h w(\tau) d\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau)) D_h w(\tau) d\Gamma d\tau. \qquad (3.4.6)
\end{aligned}$$

We will justify passing to the limit as $h \longrightarrow 0$ in both (3.4.5) and (3.4.6). Since u, $u_t \in C([0,t];L^2(\Omega))$ and w, $w_t \in C([0,t];L^2(\Gamma))$, then as $h \to 0$, it follows from (A.0.6) that

$$\begin{aligned}
&D_h u(0) \longrightarrow \frac{1}{2} u_t(0) \text{ and } D_h u(t) \longrightarrow \frac{1}{2} u_t(t) \text{ weakly in } L^2(\Omega), \\
&D_h w(0) \longrightarrow \frac{1}{2} w_t(0) \text{ and } D_h w(t) \longrightarrow \frac{1}{2} w_t(t) \text{ weakly in } L^2(\Gamma).
\end{aligned}$$

Therefore,

$$\begin{cases} \lim_{h\to 0}\Big((u_t(t), D_h u(t))_\Omega - (u_1, D_h u(0))_\Omega\Big) = \frac{1}{2}\Big(\|u_t(t)\|_2^2 - \|u_t(0)\|_2^2\Big), \\ \lim_{h\to 0}(w_t(t) + \gamma u(t), D_h w(t))_\Gamma = \frac{1}{2}|w_t(t)|_2^2 + \frac{1}{2}(\gamma u(t), w_t(t))_\Gamma, \\ \lim_{h\to 0}(w_1 + \gamma u(0), D_h w(0))_\Gamma = \frac{1}{2}|w_t(0)|_2^2 + \frac{1}{2}(\gamma u(0), w_t(0))_\Gamma. \end{cases} \tag{3.4.7}$$

Also, by (A.0.4)

$$\int_0^t (u_t, (D_h u)_t)_\Omega d\tau = \int_0^t (w_t, (D_h w)_t)_\Gamma d\tau = 0. \tag{3.4.8}$$

In addition, since $u \in C([0,t]; H^1_{\Gamma_0}(\Omega))$, then (A.0.3) yields

$$\lim_{h\longrightarrow 0} \int_0^t (u, D_h u)_{1,\Omega} d\tau = \frac{1}{2}\left(\|u(t)\|_{1,\Omega}^2 - \|u(0)\|_{1,\Omega}^2\right). \tag{3.4.9}$$

Similarly, we obtain

$$\lim_{h\longrightarrow 0} \int_0^t (w, D_h w)_{2,\Gamma} d\tau = \frac{1}{2}\left(\|w(t)\|_{2,\Gamma}^2 - \|w(0)\|_{2,\Gamma}^2\right). \tag{3.4.10}$$

Since $u_t \in L^{m+1}(\Omega \times (0,t))$ and $|g_1(s)| \leq \beta|s|^m$ whenever $|s| \geq 1$, then clearly $g_1(u_t) \in L^{\tilde{m}}(\Omega \times (0,t))$, where $\tilde{m} = \frac{m+1}{m}$. Hence, by (3.4.1)

$$\lim_{h\longrightarrow 0} \int_0^t \int_\Omega g_1(u_t) D_h u dx d\tau = \int_0^t \int_\Omega g_1(u_t) u_t dx d\tau. \tag{3.4.11}$$

Similarly, as $w_t \in L^{r+1}(\Gamma \times (0,t))$ and $|g_2(s)| \leq \beta |s|^r$ whenever $|s| \geq 1$, then clearly $g_2(w_t) \in L^{\tilde{r}}(\Gamma \times (0,t))$, where $\tilde{r} = \frac{r+1}{r}$. Then by (3.4.2)

$$\lim_{h \longrightarrow 0} \int_0^t \int_\Gamma g_2(w_t) D_h w d\Gamma d\tau = \int_0^t \int_\Gamma g_2(w_t) w_t d\Gamma d\tau. \tag{3.4.12}$$

In order to handle the wave and plate sources, we note that since $u \in C([0,t]; H^1_{\Gamma_0}(\Omega))$, then there exists $M_0 > 0$ such that $\|u(\tau)\|_6 \leq M_0$ for all $\tau \in [0,t]$. Also, since $|f(u)| \leq C(|u|^p + 1)$, then

$$\int_\Omega |f(u(\tau))|^{\frac{6}{p}} dx \leq C \int_\Omega (|u(\tau)|^6 + 1) dx \leq C(M_0),$$

for all $\tau \in [0,t]$. Hence, $f(u) \in L^\infty\left(0,t; L^{\frac{6}{p}}(\Omega)\right)$, and so, $f(u) \in L^{\frac{6}{p}}(\Omega \times (0,t))$. Since $\frac{6}{p} > \tilde{m}$, then $f(u) \in L^{\tilde{m}}(\Omega \times (0,t))$. Therefore, it follows from (3.4.1) that

$$\lim_{h \longrightarrow 0} \int_0^t \int_\Omega f(u) D_h u dx d\tau = \int_0^t \int_\Omega f(u) u_t dx d\tau. \tag{3.4.13}$$

Note that since $w \in C([0,t]; H^2_0(\Gamma))$, then for any $s > 1$, then there exists $M_1 > 0$ such that $\|w(\tau)\|_s \leq M_1$ for all $\tau \in [0,t]$, and all $1 \leq s < \infty$. In particular, $h(w) \in L^{\tilde{r}}(\Gamma \times (0,t))$. Therefore, it follows from (3.4.2) that

$$\lim_{h \longrightarrow 0} \int_0^t \int_\Gamma h(w) D_h w d\Gamma d\tau = \int_0^t \int_\Gamma h(w) w_t d\Gamma d\tau. \tag{3.4.14}$$

The trouble terms, namely $\int_0^t (w_t(\tau), \gamma D_h u(\tau))_\Gamma d\tau$ and $\int_0^t (\gamma u(\tau), (D_h w)_t(\tau))_\Gamma d\Gamma d\tau$, are handled as follows. For all sufficiently small $h > 0$, we have

$$\begin{aligned}
&\int_0^t (\gamma u(\tau), (D_h w)_t(\tau))_\Gamma d\tau \\
&\quad = \frac{1}{2h}\left(\int_0^t (\gamma u(\tau), w_t(\tau+h))_\Gamma d\tau - \int_0^t (\gamma u(\tau), w_t(\tau-h))_\Gamma d\tau\right) \\
&\quad = \frac{1}{2h}\left(\int_h^t (\gamma u(\tau-h), w_t(\tau))_\Gamma d\tau - \int_0^{t-h} (\gamma u(\tau+h), w_t(\tau))_\Gamma d\tau\right), \qquad (3.4.15)
\end{aligned}$$

where we have used a change of variables in (3.4.15) and the fact that $w_t = 0$ outside of the interval $[0, t]$. By rearranging the terms in (3.4.15), we obtain

$$\begin{aligned}
&\int_0^t (\gamma u(\tau), (D_h w)_t(\tau))_\Gamma d\tau = -\int_0^t (\gamma D_h u(\tau), w_t(\tau))_\Gamma d\tau \\
&\quad - \frac{1}{2h}\left(\int_0^h (\gamma u(\tau-h), w_t(\tau))_\Gamma d\tau - \int_{t-h}^t (\gamma u(\tau+h), w_t(\tau))_\Gamma d\tau\right). \qquad (3.4.16)
\end{aligned}$$

We now utilize the continuity of w_t in the last two terms of (3.4.16) as follows.

$$\begin{aligned}
&\frac{1}{2h}\int_0^h (\gamma u(\tau-h), w_t(\tau))_\Gamma d\tau = \frac{1}{2h}\int_0^h (\gamma u(0), w_t(\tau))_\Gamma d\tau \\
&\quad = \frac{1}{2h}\int_0^h (\gamma u(0), w_t(\tau) - w_t(0))_\Gamma d\tau + \frac{1}{2h}\int_0^h (\gamma u(0), w_t(0))_\Gamma d\tau \\
&\quad \longrightarrow \frac{1}{2}(\gamma u(0), w_t(0))_\Gamma, \qquad (3.4.17)
\end{aligned}$$

as $h \longrightarrow 0$. Similarly, we have

$$\begin{aligned}
&\frac{1}{2h}\int_{t-h}^t (\gamma u(\tau+h), w_t(\tau))_\Gamma d\tau = \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(\tau))_\Gamma d\tau \\
&\quad = \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(\tau) - w_t(t))_\Gamma d\tau + \frac{1}{2h}\int_{t-h}^t (\gamma u(t), w_t(t))_\Gamma d\tau \\
&\quad \longrightarrow \frac{1}{2}(\gamma u(t), w_t(t))_\Gamma, \qquad (3.4.18)
\end{aligned}$$

as $h \longrightarrow 0$. Finally, by adding (3.4.5) and (3.4.6) and combining (3.4.7)-(3.4.18) we can pass to the limit $h \longrightarrow 0$ to obtain the energy identity 3.2.4 in Theorem 3.2.4.

3.5 Global Existence

This section is devoted to prove the existence of global solutions as described in Theorem 3.2.5. As in [1, 33, 48] and other works, it is the case here that either a given solution (u, w) must exist globally in time or else one may find a value of T_0 with $0 < T_0 < \infty$, so that

$$\limsup_{t \to T_0^-} E_1(t) = +\infty, \tag{3.5.1}$$

where $E_1(t)$ is modified energy given by:

$$\begin{aligned} E_1(t) = &\frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + |w_t(t)|_2^2 + |\Delta w(t)|_2^2\right) \\ &+ \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} + \frac{1}{q+1}|w(t)|_{q+1}^{q+1}, \end{aligned} \tag{3.5.2}$$

where, without any loss of generality, we may assume that $E_1(t) \geq 1$. We aim to show that (3.5.1) cannot happen under the assumptions of Theorem 3.2.5. Indeed, this assertion is contained in the following proposition.

Proposition 3.5.1. *Let (u, w) be a weak solution of (3.1.1) on $[0, T_0]$ as furnished by Theorem 3.2.4. Assume $u_0 \in L^{p+1}(\Omega)$, if $p > 5$. We have:*

- *If $p \leq m$ and $q \leq r$, then for all $t \in [0, T_0]$, (u, w) satisfies*

$$E_1(t) + \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau \leq C(T_0, E_1(0)), \tag{3.5.3}$$

where $0 < T_0 < \infty$ is being arbitrary.

- *If $p > m$ or $q > r$, then the bound in* (3.5.3) *holds for $0 \le t < T'$, for some $T' > 0$ depending on $E_1(0)$ and T_0.*

Proof. With the modified energy as given in (3.5.2), the energy identity (3.2.4) now reads,

$$\begin{aligned}
&E_1(t) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau \\
&\quad = E_1(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau \\
&\quad + \frac{1}{p+1}\int_\Omega \left(|u(t)|^{p+1} - |u(0)|^{p+1}\right) dx + \frac{1}{q+1}\int_\Gamma \left(|w(t)|^{q+1} - |w(0)|^{q+1}\right) d\Gamma \\
&\quad = E_1(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau \\
&\quad + \int_0^t \int_\Omega |u|^{p-1}uu_t dx d\tau + \int_0^t \int_\Gamma |w|^{q-1}ww_t d\Gamma d\tau. \qquad (3.5.4)
\end{aligned}$$

To estimate the source terms on the right-hand side of (3.5.4), we recall the assumptions f, h, and (3.2.1). By employing Hölder's and Young's inequalities, we find

$$\begin{aligned}
\left|\int_0^t \int_\Omega f(u)u_t dx d\tau\right| &\le C\int_0^t \int_\Omega (|u|^p + 1)|u_t| dx d\tau \\
&\le C\int_0^t \|u_t\|_{p+1}\left(\|u\|_{p+1}^p + |\Omega|^{\frac{p}{p+1}}\right) d\tau \\
&\le \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p+1}^{p+1} + |\Omega|\right) d\tau \\
&\le \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon |\Omega| T_0. \qquad (3.5.5)
\end{aligned}$$

Similarly, we deduce

$$\left|\int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau\right| \le \epsilon \int_0^t |w_t|_{q+1}^{q+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon |\Gamma| T_0. \qquad (3.5.6)$$

By adopting similar estimates as in (3.5.5), we obtain

$$\begin{aligned}\left|\int_0^t\int_\Omega |u|^{p-1}uu_t dx d\tau\right| &\leq \int_0^t\int_\Omega |u|^p|u_t| dx d\tau \\ &\leq \epsilon\int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon\int_0^t E_1(\tau)d\tau. \end{aligned} \tag{3.5.7}$$

Likewise, we deduce

$$\left|\int_0^t\int_\Gamma |w|^{q-1}ww_t d\Gamma d\tau\right| \leq \epsilon\int_0^t |w_t|_{q+1}^{q+1} d\tau + C_\epsilon\int_0^t E_1(\tau)d\tau. \tag{3.5.8}$$

By recalling (3.3.22), one has

$$\begin{aligned}&\int_0^t\int_\Omega g_1(u)u_t dx d\tau + \int_0^t\int_\Gamma g_2(w)w_t d\Gamma d\tau \\ &\quad \geq \alpha\int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau - \alpha T_0(|\Omega| + |\Gamma|). \end{aligned} \tag{3.5.9}$$

Now, if $p \leq m$ and $q \leq r$, it follows from (3.5.5)-(3.5.9) and the energy identity (3.5.4) that, for all $t \in [0, T_0]$,

$$\begin{aligned}&E_1(t) + \alpha\int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau \\ &\quad \leq E_1(0) + \epsilon\int_0^t \left(\|u_t\|_{p+1}^{p+1} + |w_t|_{q+1}^{q+1}\right) d\tau + C_\epsilon\int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon} \\ &\quad \leq E_1(0) + \epsilon\int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau + C_\epsilon\int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon}, \end{aligned} \tag{3.5.10}$$

where we have used Hölder's and Young's inequalities in the last line of (3.5.10). By choosing $0 < \epsilon \leq \frac{\alpha}{2}$, then (3.5.10) yields

$$E_1(t) + \frac{\alpha}{2}\int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau \leq C_\epsilon\int_0^t E_1(\tau)d\tau + E_1(0) + C_{T_0,\epsilon}. \tag{3.5.11}$$

In particular,

$$E_1(t) \leq C_\epsilon \int_0^t E_1(\tau)d\tau + E_1(0) + C_{T_0,\epsilon}. \tag{3.5.12}$$

By Gronwall's inequality, we conclude that

$$E_1(t) \leq (E_1(0) + C_{T_0,\epsilon})e^{C_\epsilon T_0} \text{ for } t \in [0, T_0], \tag{3.5.13}$$

where $T_0 > 0$ is arbitrary, and by combining (3.5.11) and (3.5.13), the desired result in (3.5.3) follows.

Now, if $p > m$ or $q > r$, then we slightly modify estimate (3.5.5) by using different Hölder's conjugates. Specifically, we apply Hölder's inequality with $m+1$ and $\tilde{m} = \frac{m+1}{m}$ followed by Young's inequality to obtain

$$\begin{aligned}\left|\int_0^t \int_\Omega f(u)u_t dx d\tau\right| &\leq C \int_0^t \int_\Omega (|u|^p + 1)|u_t| dx d\tau \\ &\leq C \int_0^t \|u_t\|_{m+1} \left(\|u\|_{p\tilde{m}}^p + |\Omega|^{\frac{m}{m+1}}\right) d\tau \\ &\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p\tilde{m}}^{p\tilde{m}} + |\Omega|\right) d\tau. \end{aligned} \tag{3.5.14}$$

Since $p\tilde{m} < 6$ and $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$, then

$$\left|\int_0^t \int_\Omega f(u)u_t dx d\tau\right| \leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}} d\tau + C_\epsilon |\Omega| T_0. \tag{3.5.15}$$

Likewise, we may deduce

$$\left|\int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau\right| \leq \epsilon \int_0^t |w_t|_{r+1}^{r+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{q\tilde{r}}{2}} d\tau + C_\epsilon |\Gamma| T_0. \tag{3.5.16}$$

In addition, by employing similar estimates as in (3.5.14) and (3.5.15), we have

$$\left|\int_0^t \int_\Omega |u|^{p-1} u u_t dx d\tau\right| \leq \int_0^t \int_\Omega |u|^p |u_t| dx d\tau$$
$$\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}} d\tau. \tag{3.5.17}$$

Likewise, we deduce

$$\left|\int_0^t \int_\Gamma |w|^{q-1} w w_t d\Gamma d\tau\right| \leq \epsilon \int_0^t |w_t|_{r+1}^{r+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{q\tilde{r}}{2}} d\tau. \tag{3.5.18}$$

By using (3.5.15)-(3.5.18) along with (3.5.9), we obtain from the energy identity (3.5.4) that

$$E_1(t) + \alpha \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau$$
$$\leq E_1(0) + \epsilon \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau + C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon}, \tag{3.5.19}$$

where $\sigma = \max\{\frac{p\tilde{m}}{2}, \frac{q\tilde{r}}{2}\}$. Notice, the assumption $p > m$ or $q > r$ implies that $\sigma > 1$. By choosing $0 < \epsilon < \frac{\alpha}{2}$, then it follows that

$$E_1(t) + \frac{\alpha}{2} \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau \leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon}, \tag{3.5.20}$$

for $t \in [0, T_0]$. In particular,

$$E_1(t) \leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon}, \tag{3.5.21}$$

for $t \in [0, T_0]$. By using a standard comparison theorem, then (3.5.21) yields that

$E_1(t) \leq z(t)$, where

$$z(t) = \left[(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)t \right]^{\frac{-1}{\sigma-1}}$$

is the solution of the Volterra integral equation

$$z(t) = C_\epsilon \int_0^t z(s)^\sigma ds + E_1(0) + C_{T_0,\epsilon}.$$

Since $\sigma > 1$, then clearly $z(t)$ blows up at the finite time $T_1 = \frac{1}{C_\epsilon(\sigma-1)}(E_1(0)+C_{T_0,\epsilon})^{1-\sigma}$, i.e., $z(t) \longrightarrow \infty$, as $t \longrightarrow T_1^-$. Note that T_1 depends on the initial energy $E_1(0)$ and the original existence time T_0. Nonetheless, if we choose $T' = \min\left\{T_0, \frac{1}{2}T_1\right\}$, then

$$E_1(t) \leq z(t) \leq C_0 := \left[(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)T' \right]^{\frac{-1}{\sigma-1}}, \tag{3.5.22}$$

for all $t \in [0, T']$. Finally, we may combine (3.5.20) and (3.5.22) to obtain

$$E_1(t) + \frac{\alpha}{2} \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1} \right) d\tau \leq C_\epsilon T' C_0^\sigma + E_1(0) + C_{T_0,\epsilon}, \tag{3.5.23}$$

for all $t \in [0, T']$, which completes the proof of the proposition. □

3.6 Continuous Dependence on Initial Data

In this section, we provide the proof of Theorem 3.2.7.

Proof. Let $U_0 = (u_0, w_0, u_1, w_1) \in X$, where

$$X = \left(H^1_{\Gamma_0}(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega) \right) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma).$$

Assume that $\{U_0^n = (u_0^n, w_0^n, u_1^n, w_1^n)\}$ is a sequence of initial data that satisfies

$$U_0^n \longrightarrow U_0 \text{ in } X, \text{ as } n \longrightarrow \infty. \tag{3.6.1}$$

Notice that in Remark 3.2.8, we have pointed out that if $p \leq 5$, then the space X is identitcal to $H = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma) \times L^2(\Omega) \times L^2(\Gamma)$. Let $\{(u^n, w^n)\}$ and (u, w) be the weak solutions to (3.1.1) defined on $[0, T_0]$ in the sense of Definition 3.2.3, corresponding to the initial data $\{U_0^n\}$ and $\{U_0\}$, respectively. First, we show that the local existence time T_0 can be taken independent of $n \in \mathbb{N}$. To see this, we recall that the local existence time T_0 provided by Theorem 3.2.4 depends on the initial energy $E(0)$. In addition, since $U_0^n \longrightarrow U_0$ in X, then $u_0^n \longrightarrow u_0$ in $L^{p+1}(\Omega)$, if $p > 5$. Hence, we may assume $E_1^n(0) \leq E_1(0) + 1$, for all $n \in \mathbb{N}$, where $E_1(t)$ is defined in (3.5.2) and $E_1^n(t)$ is defined by:

$$E_1^n(t) = E^n(t) + \frac{1}{p+1}\|u^n(t)\|_{p+1}^{p+1} + \frac{1}{q+1}|w^n(t)|_{q+1}^{q+1}$$

where $E^n(t) = \frac{1}{2}\left(\|u^n(t)\|_{1,\Omega}^2 + \|w^n(t)\|_{2,\Gamma}^2 + \|u_t^n(t)\|_2^2 + |w_t^n(t)|_2^2\right)$. Therefore, we can choose K, as in (3.3.26), sufficiently large, say $K^2 \geq 4E_1(0) + 5$, such that the local existence time T_0 for the solutions $\{(u^n, w^n)\}$ and (u, w) can be chosen independent of $n \in \mathbb{N}$. Moreover, in view of (3.5.3), T_0 can be taken arbitrarily large in the case when $p \leq m$ and $q \leq r$. However, in the case when $p > m$ or $q > r$, we select the local existence time to be $T = T'$ where T' is given in Proposition 3.5.1 (which is also uniform in n). In either case, it follows from (3.5.3) that there exists $R > 0$ such that, for all $n \in \mathbb{N}$ and all $t \in [0,T]$,

$$\begin{cases} E_1(t) + \int_0^t \left(\|u_t\|_{m+1}^{m+1} + |w_t|_{r+1}^{r+1}\right) d\tau \leq R, \\ \\ E_1^n(t) + \int_0^t \left(\|u_t^n\|_{m+1}^{m+1} + |w_t^n|_{r+1}^{r+1}\right) d\tau \leq R, \end{cases} \tag{3.6.2}$$

where T can be arbitrarily large if $p \leq m$ and $q \leq r$, or T sufficiently small if $p > m$ or $q > r$. From here on, the proof will be carried out in four steps.

Step 1: Put $y^n(t) = u(t) - u^n(t)$, $z^n(t) = w(t) - w^n(t)$, and

$$\tilde{E}_n(t) = \frac{1}{2}\left(\|y^n(t)\|_{1,\Omega}^2 + \|z^n(t)\|_{2,\Gamma}^2 + \|y_t^n(t)\|_2^2 + |z_t^n(t)|_2^2\right), \tag{3.6.3}$$

for $t \in [0,T]$. We aim to show $\tilde{E}_n(t) \longrightarrow 0$ uniformly on $[0,T]$ for sufficiently small T. Now by construction, y^n and z^n satisfy:

$$\begin{aligned} &(y_t^n(t), \phi(t))_\Omega - (y_t^n(0), \phi(0))_\Omega - \int_0^t (y_t^n(\tau), \phi_t(\tau))_\Omega d\tau + \int_0^t (y^n(\tau), \phi(\tau))_{1,\Omega} d\tau \\ &\quad - \int_0^t (z_t^n(\tau), \gamma\phi(\tau))_\Gamma d\tau + \int_0^t \int_\Omega (g_1(u_t(\tau)) - g_1(u_t^n(\tau)))\phi(\tau) dx d\tau \\ &\quad = \int_0^t \int_\Omega (f(u(\tau)) - f(u^n(\tau)))\phi(\tau) dx d\tau, \end{aligned} \tag{3.6.4}$$

and

$$
\begin{aligned}
&(z_t^n(t),\psi(t))_\Gamma - (z_t^n(0),\psi(0))_\Gamma - \int_0^t (z_t^n(\tau),\psi_t(\tau))_\Gamma d\tau + (\gamma y^n(t),\psi(t))_\Gamma \\
&- (\gamma y^n(0),\psi(0))_\Gamma - \int_0^t (\gamma y^n(\tau),\psi_t(\tau)_\Gamma d\tau + \int_0^t (z^n(\tau),\psi(\tau))_{2,\Gamma} d\tau \\
&+ \int_0^t \int_\Gamma (g_2(w_t(\tau)) - g_2(w_t^n(\tau)))\psi(\tau) d\Gamma d\tau \\
&= \int_0^t \int_\Gamma (h(w(\tau)) - h(w^n(\tau)))\psi(\tau) d\Gamma d\tau, \qquad (3.6.5)
\end{aligned}
$$

for all $t \in [0,T]$ and for all test functions ϕ and ψ as described in Definition 3.2.3. Let $\phi(\tau) = D_h y^n(\tau)$ in (3.6.4) and $\psi(\tau) = D_h z^n(\tau)$ in (3.6.5) for $\tau \in [0,t]$ where the difference quotients $D_h y^n$ and $D_h z^n$ are defined in (A.0.1). Using a similar argument as in obtaining the energy identity (3.2.4), we can pass to the limit as $h \longrightarrow 0$ and deduce

$$
\begin{aligned}
\tilde{E}_n(t) &+ \int_0^t \int_\Omega (g_1(u_t(\tau)) - g_1(u_t^n(\tau))) y_t^n(\tau) dx d\tau \\
&+ \int_0^t \int_\Gamma (g_2(w_t(\tau)) - g_2(w_t^n(\tau))) z_t^n(\tau) d\Gamma d\tau \\
&= \tilde{E}_n(0) + \int_0^t \int_\Omega (f(u(\tau)) - f(u^n(\tau))) y_t^n(\tau) dx d\tau \\
&+ \int_0^t \int_\Gamma (h(w(\tau)) - h(w^n(\tau))) z_t^n(\tau) d\Gamma d\tau. \qquad (3.6.6)
\end{aligned}
$$

Employing the monotonicity propoerties of g_1 and g_2 to (3.6.6) yields

$$
\begin{aligned}
\tilde{E}_n(t) &\le \tilde{E}_n(0) + \int_0^t \int_\Omega (f(u(\tau)) - f(u^n(\tau))) y_t^n(\tau) dx d\tau \\
&+ \int_0^t \int_\Gamma (h(w(\tau)) - h(w^n(\tau))) z_t^n(\tau) d\Gamma d\tau, \qquad (3.6.7)
\end{aligned}
$$

for all $t \in [0,T]$. We will now estimate each term on the right hand side of (3.6.7).

Step 2: "Estimate for the wave source term."

Put:

$$R_f^n = \int_0^t \int_\Omega (f(u(\tau)) - f(u^n(\tau)))y_t^n(\tau)dxd\tau.$$

First we note that, if $1 \leq p \leq 3$, then by Remark 3.3.5 we know f is locally Lipschitz from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$. In this case, the estimate for R_f^n is straightforward, as follows:

$$\begin{aligned}
&\left|\int_0^t \int_\Omega (f(u) - f(u^n))y_t^n dxd\tau\right| \\
&\quad \leq \left(\int_0^t \int_\Omega |f(u) - f(u^n)|^2 dxd\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega |y_t^n|^2 dxd\tau\right)^{\frac{1}{2}} \\
&\quad \leq C(R) \left(\int_0^t \|y^n\|_{1,\Omega}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|y_t^n\|_2^2 d\tau\right)^{\frac{1}{2}} \leq C(R) \int_0^t \tilde{E}_n(\tau)d\tau. \qquad (3.6.8)
\end{aligned}$$

Therefore, for $1 \leq p \leq 3$, we have that

$$\left|R_f^n\right| \leq C(R) \int_0^t \tilde{E}_n(\tau)d\tau. \qquad (3.6.9)$$

For the case $3 < p < 6$, f is not locally Lipschitz from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$, and therefore the computation in (3.6.8) does not work. To overcome this difficulty, we shall use a clever idea by Bociu and Lasiecka [13, 14] which involves integration by parts. In order to do so, we require the assumption $f \in C^2(\mathbb{R})$, with $|f''(u)| \leq C(|u|^{p-2} + 1)$, as in Assumption 3.2.6. We also remind the reader with Assumption 3.2.1 and (3.2.1).

Now, we evaluate R_f^n in the case $3 < p < 6$. By integration by parts in time, one has

$$
\begin{aligned}
R_f^n &= \int_0^t \int_\Omega (f(u(\tau)) - f(u^n(\tau)))y_t^n(\tau)dxd\tau \\
&= \int_\Omega (f(u(t)) - f(u^n(t)))y^n(t)dx - \int_\Omega (f(u(0)) - f(u^n(0)))y^n(0)dx \\
&\quad - \int_0^t \int_\Omega \Big(f'(u)u_t - f'(u^n)u_t^n\Big) y^n dxd\tau \\
&= \int_\Omega (f(u(t)) - f(u^n(t)))y^n(t)dx - \int_\Omega (f(u(0)) - f(u^n(0)))y^n(0)dx \\
&\quad - \int_0^t \int_\Omega (f'(u) - f'(u^n))u_t^n y^n dxd\tau - \int_0^t \int_\Omega f'(u)y_t^n y^n dxd\tau \\
&:= P_1 + P_2 + P_3 + P_4, \qquad (3.6.10)
\end{aligned}
$$

respectively. By using the assumptions on f, we obtain

$$
\begin{cases}
|P_1| \le C \int_\Omega (|u(t)|^{p-1} + |u^n(t)|^{p-1} + 1)\, |y^n(t)|^2 dx, \\
|P_2| \le C \int_\Omega (|u(0)|^{p-1} + |u^n(0)|^{p-1} + 1)\, ||y^n(0)|^2 dx, \\
|P_3| \le C \int_0^t \int_\Omega (|u|^{p-2} + |u^n|^{p-2} + 1)\, |u_t^n||y^n|^2 dxd\tau.
\end{cases} \qquad (3.6.11)
$$

As for P_4, we integrate by parts one more time to obtain

$$
\begin{aligned}
-P_4 &= \int_0^t \int_\Omega f'(u)y_t^n y^n dxd\tau \\
&= \frac{1}{2}\int_\Omega f'(u(t))(y^n(t))^2 dx - \frac{1}{2}\int_\Omega f'(u(0))(y^n(0))^2 dx \\
&\quad - \frac{1}{2}\int_0^t \int_\Omega f''(u)u_t(y^n)^2 dxd\tau. \qquad (3.6.12)
\end{aligned}
$$

By employing the assumptions on f, we deduce

$$|P_4| \le C \int_\Omega \left(|u(t)|^{p-1}+1\right)|y^n(t)|^2dx + C\int_\Omega \left(|u(0)|^{p-1}+1\right)|y^n(0)|^2dx \\ + C\int_0^t\int_\Omega \left(|u|^{p-2}+1\right)|u_t||y^n|^2dxd\tau. \tag{3.6.13}$$

It follows from (3.6.10)-(3.6.11), and (3.6.13) that

$$|R_f^n| \le C\int_\Omega (|y^n(t)|^2+|y^n(0)|^2)dx + C\int_0^t\int_\Omega (|u_t|+|u_t^n|)|y^n|^2dxd\tau \\ + C\int_0^t\int_\Omega \left(|u|^{p-2}+|u^n|^{p-2}\right)(|u_t|+|u_t^n|)|y^n|^2dxd\tau \\ + C\int_\Omega \left(|u(t)|^{p-1}+|u^n(t)|^{p-1}\right)|y^n(t)|^2dx \\ + C\int_\Omega \left(|u(0)|^{p-1}+|u^n(0)|^{p-1}\right)||y^n(0)|^2dx. \tag{3.6.14}$$

Now, we estimate each term on the right-hand side of (3.6.14) as follows.

1. Estimate for $I_1 = \int_\Omega |y^n(t)|^2dx$:

Since y^n, $y_t^n \in C([0,T];L^2(\Omega))$, we obtain with Young's inequality

$$I_1 = \int_\Omega |y^n(t)|^2dx = \int_\Omega \left|y^n(0)+\int_0^t y_t^n(\tau)d\tau\right|^2 dx \\ \le 2\int_\Omega |y^n(0)|^2dx + 2\int_\Omega \left|\int_0^t y_t^n(\tau)d\tau\right|^2dx \\ \le C\left(\|y^n(0)\|_{1,\Omega}^2dx + t\int_0^t \|y_t^n(\tau)\|_2^2d\tau\right) \\ \le C\left(\tilde{E}_n(0)+T\int_0^t \tilde{E}_n(\tau)d\tau\right). \tag{3.6.15}$$

2. Estimate for $I_2 = \int_0^t\int_\Omega (|u_t|+|u_t^n|)|y^n|^2dxd\tau$:

Both terms in I_2 are estimated in the same manner, for instance we have

$$\int_0^t \int_\Omega |u_t||y^n|^2 dx d\tau \leq \int_0^t \|y^n\|_6^2 \|u_t\|_{\frac{3}{2}} d\tau$$
$$\leq C \int_0^t \|y^n\|_{1,\Omega}^2 \|u_t\|_2 d\tau \leq C(R) \int_0^t \tilde{E}_n(\tau) d\tau, \tag{3.6.16}$$

where we have used the fact that $\|u_t(t)\|_2^2 \leq R$, for all $t \in [0,T]$ (see (3.6.2)). Therefore,

$$I_2 \leq C(R) \int_0^t \tilde{E}_n(\tau) d\tau. \tag{3.6.17}$$

3. Estimate for $I_3 = \int_0^t \int_\Omega \left(|u|^{p-2} + |u^n|^{p-2}\right)(|u_t| + |u_t^n|)|y^n|^2 dx d\tau$:

A typical term in I_3 is estimated as follows. Recall the assumption $p\frac{m+1}{m} < 6$ which implies $\frac{6}{6-p} < m+1$. Thus, by using Hölder's inequality and (3.6.2), one has

$$\int_0^t \int_\Omega |u|^{p-2}|u_t||y^n|^2 dx d\tau \leq \int_0^t \|u\|_6^{p-2} \|u_t\|_{\frac{6}{6-p}} \|y^n\|_6^2$$
$$\leq C \int_0^t \|u\|_{1,\Omega}^{p-2} \|u_t\|_{m+1} \|y^n\|_{1,\Omega}^2 d\tau$$
$$\leq C(R) \int_0^t \tilde{E}_n(\tau) \|u_t\|_{m+1} d\tau. \tag{3.6.18}$$

Therefore,

$$I_3 \leq C(R) \int_0^t \tilde{E}_n(\tau) \left(\|u_t\|_{m+1} + \|u_t^n\|_{m+1}\right) d\tau. \tag{3.6.19}$$

4. Estimate for $I_4 = \int_\Omega (|u(t)|^{p-1} + |u^n(t)|^{p-1})|y^n(t)|^2) dx$:

As the first term in I_4 is a little easier to estimate, we shall focus on the second term $\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx$ in the following two cases for the exponent $p \in (3,6)$.

<u>Case 1: $3 < p < 5$.</u> In this case, we have

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \leq \int_\Omega |y^n(t)|^2 dx + \int_{\{x\in\Omega:\,|u^n(t)|>1\}} |u^n(t)|^{p-1}|y^n(t)|^2 dx \quad (3.6.20)$$

The first term on the right-hand side of (3.6.20) has been already estimated in (3.6.15). For the second term, we notice if $0 < \sigma < 5 - p$, then $|u^n(t)|^{p-1} \leq |u^n(t)|^{4-\sigma}$, since $|u^n(t)| > 1$. Again, by using Hölder's inequality, (3.6.2), and (1.3.1), it follows that

$$\begin{aligned}\int_{\{x\in\Omega:\,|u^n(t)|>1\}} |u^n(t)|^{p-1}|y^n(t)|^2 dx &\leq \int_\Omega |u^n(t)|^{4-\sigma}|y^n(t)|^2 dx \\ &\leq \|u^n(t)\|_6^{4-\sigma}\|y^n(t)\|^2_{\frac{6}{1+\sigma/2}} \\ &\leq C\|u^n(t)\|_{1,\Omega}^{4-\sigma}\|y^n(t)\|^2_{H^{1-\sigma/4}_{\Gamma_0}(\Omega)} \\ &\leq C(R)\left(\epsilon\|y^n(t)\|^2_{1,\Omega} + C_\epsilon\|y^n(t)\|_2^2\right), \qquad (3.6.21)\end{aligned}$$

where $\epsilon > 0$ that will be selected below. By utilizing (3.6.15) and (3.6.21), then from (3.6.20) it follows that

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \leq C(R)\epsilon\tilde{E}_n(t) + C(R,\epsilon)\left(\tilde{E}_n(0) + T\int_0^t \tilde{E}_n(\tau)d\tau\right) \quad (3.6.22)$$

in the case $3 < p < 5$.

<u>Case 2: $5 \leq p < 6$.</u>

In this case, the assumption $p\frac{m+1}{m} < 6$ implies $m > 5$. Recall that in Theorem 3.2.7 we required a higher regularity of initial datum u_0, namely, $u_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$. By density of $C_0(\Omega)$ in $L^{\frac{3(p-1)}{2}}(\Omega)$, then for any $\epsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that

$\|u_0 - \phi\|_{\frac{3(p-1)}{2}} < \epsilon^{\frac{1}{p-1}}$. Therefore,

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \le C\Big(\int_\Omega |u^n(t) - u_0^n|^{p-1}|y^n(t)|^2 dx + \int_\Omega |u_0^n - u_0|^{p-1}|y^n(t)|^2 dx + \int_\Omega |u_0 - \phi|^{p-1}|y^n(t)|^2 dx + \int_\Omega |\phi|^{p-1}|y^n(t)|^2 dx\Big). \tag{3.6.23}$$

Since $p < \frac{6m}{m+1}$ and $m > 5$, then $\frac{3(p-1)}{2(m+1)} < 1$. So, by using Hölder's inequality and the bound $\int_0^T \|u_t^n\|_{m+1}^{m+1} \le R$, one has

$$\begin{aligned}\int_\Omega |u^n(t) - u_0^n|^{p-1}|y^n(t)|^2 dx &\le \left(\int_\Omega |u^n(t) - u_0^n|^{\frac{3(p-1)}{2}} dx\right)^{\frac{2}{3}} \|y^n(t)\|_6^2 \\ &\le C\left(\int_\Omega \left|\int_0^t u_t^n(\tau) d\tau\right|^{\frac{3(p-1)}{2}} dx\right)^{\frac{2}{3}} \|y^n(t)\|_{1,\Omega}^2 \\ &\le CT^{\frac{m(p-1)}{m+1}} \left[\int_\Omega \left(\int_0^t |u_t^n(\tau)|^{m+1} d\tau\right)^{\frac{3(p-1)}{2(m+1)}} dx\right]^{\frac{2}{3}} \tilde{E}_n(t) \\ &\le C(R)T^{\frac{m(p-1)}{m+1}} \tilde{E}_n(t),\end{aligned} \tag{3.6.24}$$

where we have used the important fact that $\frac{3(p-1)}{2(m+1)} < 1$. Also, by using Hölder's inequality and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\int_\Omega |u_0^n - u_0|^{p-1}|y^n(t)|^2 dx \le \|u_0^n - u_0\|_{\frac{3(p-1)}{2}}^{p-1} \|y^n(t)\|_6^2 \le \epsilon \tilde{E}_n(t), \tag{3.6.25}$$

for all sufficiently large n, since $u_0^n \longrightarrow u_0$ in $L^{\frac{3(p-1)}{2}}(\Omega)$.

The third term on the right-hand side of (3.6.23) is easily estimated as follows:

$$\int_\Omega |u_0 - \phi|^{p-1}|y^n(t)|^2 dx \le \|u_0 - \phi\|_{\frac{3(p-1)}{2}}^{p-1} \|y^n(t)\|_6^2 \le C\epsilon \tilde{E}_n(t). \tag{3.6.26}$$

Since $\phi \in C_0(\Omega)$ then $|\phi(x)| \leq C(\epsilon)$, for all $x \in \Omega$. So, by (3.6.15), the last term on the right-hand side of (3.6.23) is estimated as follows:

$$\int_\Omega |\phi|^{p-1}|y^n(t)|^2 dx \leq C(\epsilon) \int_\Omega |y^n(t)|^2 dx \leq C(\epsilon) \left(\tilde{E}_n(0) + T \int_0^t \tilde{E}_n(\tau) d\tau \right). \quad (3.6.27)$$

By combining (3.6.24)-(3.6.27), (3.6.23) yields

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \leq C(R) \left(T^{\frac{m(p-1)}{m+1}} + \epsilon \right) \tilde{E}_n(t) + C(R,\epsilon)\tilde{E}_n(0) + C(\epsilon, T) \int_0^t \tilde{E}(\tau) d\tau, \quad (3.6.28)$$

in the case $5 \leq p < 6$, and all sufficiently large $n \in \mathbb{N}$.

By combining the estimates in (3.6.22) and (3.6.28), then for the case $3 < p < 6$, we conclude

$$\begin{aligned} I_4 &= \int_\Omega (|u(t)|^{p-1} + |u^n(t)|^{p-1})|y^n(t)|^2) dx \\ &\leq C(R) \left(T^{\frac{m(p-1)}{m+1}} + \epsilon \right) \tilde{E}_n(t) + C(R,\epsilon)\tilde{E}_n(0) + C(R,\epsilon,T) \int_0^t \tilde{E}_n(\tau) d\tau. \quad (3.6.29) \end{aligned}$$

5. Estimate for $I_5 = \int_\Omega \left(1 + |u(0)|^{p-1} + |u^n(0)|^{p-1}\right) ||y^n(0)|^2 dx$:

If $1 \leq p \leq 5$, then a typical term I_5 is estimated in the following manner. By using Hölder's inequality and (3.6.2), we have

$$\begin{aligned} \int_\Omega |u_0^n|^{p-1}|y^n(0)|^2 dx &\leq \|u_0^n\|^{p-1}_{\frac{3(p-1)}{2}} \|y^n(0)\|_6^2 \\ &\leq C(R) \|y^n(0)\|^2_{1,\Omega} \\ &\leq C(R)\tilde{E}_n(0). \quad (3.6.30) \end{aligned}$$

For the values $5 < p < 6$, we proceed as in (3.6.23) to obtain

$$\int_\Omega |u^n(0)|^{p-1}|y^n(0)|^2 dx \leq C\epsilon \tilde{E}_n(0). \tag{3.6.31}$$

Finally, by combining the estimates (3.6.15), (3.6.17), (3.6.19), (3.6.29)-(3.6.31) back into (3.6.14), we obtain for $3 < p < 6$:

$$\begin{aligned} \left|R_f^n\right| &\leq C(R,\epsilon)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t) \\ &\quad + C(T,R,\epsilon)\int_0^t \tilde{E}_n(\tau)\left(\|u_t\|_{m+1} + \|u_t^n\|_{m+1} + 1\right)d\tau, \end{aligned} \tag{3.6.32}$$

where $\epsilon > 0$ is sufficiently small. According to (3.6.9), estimate (3.6.32) also holds for $1 \leq p \leq 3$, i.e., (3.6.32) holds for all $1 \leq p < 6$.

Step 3: "Estimate for the plate source term."

Since h is locally Lipschitz from $H_0^2(\Gamma)$ into $L^2(\Gamma)$, then it straightforward to obtain

$$\begin{aligned} |R_h^n| &= \int_0^t \int_\Gamma (h(w(\tau)) - h(w^n(\tau)))z_t^n(\tau) d\Gamma d\tau \\ &\leq C(R)\left(\int_0^t \|z^n\|_{2,\Gamma}^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t |z_t^n|_2^2 d\tau\right)^{\frac{1}{2}} \\ &\leq C(R)\int_0^t \tilde{E}_n(\tau)d\tau. \end{aligned} \tag{3.6.33}$$

Step 4: "Completion of the proof"

By the estimates (3.6.32) and (3.6.33), we obtain from (3.6.7) that

$$\tilde{E}_n(t) \leq C(R,\epsilon)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t)$$
$$+ C(T,R,\epsilon)\int_0^t \tilde{E}_n(\tau)\left(\|u_t\|_{m+1} + \|u_t^n\|_{m+1} + 1\right)d\tau,$$

for all $t \in [0,T]$. Choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right) < 1.$$

By Gronwall's inequality, we obtain

$$\tilde{E}_n(t) \leq C(R,\epsilon,T)\tilde{E}_n(0)\exp\left[\int_0^t \left(\|u_t\|_{m+1} + \|u_t^n\|_{m+1} + 1\right)d\tau\right], \tag{3.6.34}$$

and so by (3.6.2), we have

$$\tilde{E}_n(t) \leq C(R,\epsilon,T)\tilde{E}_n(0), \tag{3.6.35}$$

for all sufficiently large n. Hence, $\tilde{E}_n(t) \longrightarrow 0$ uniformly on $[0,T]$. This concludes the proof of Theorem 3.2.7. □

3.7 Uniqueness of Weak Solutions

The uniqueness results of Theorem 3.2.9 and 3.2.11 will be justified in the following two subsections.

3.7.1 Proof of Theorem 3.2.9

In this section, we provide the proof of Theorem 3.2.9. The strategy here is to adopt the same argument as in the proof of Theorem 3.2.7.

Proof. Let (u, w) and $(\hat{u}, \hat{w})$ be two weak solutions on $[0, T]$, in the sense of Definition 3.2.3 satisfying the same initial conditions. Put $y = u - \hat{u}$ and $z = w - \hat{w}$. The energy corresponding to (y, z) is given by:

$$\tilde{E}(t) = \frac{1}{2}\left(\|y(t)\|_{1,\Omega}^2 + \|z(t)\|_{2,\Gamma}^2 + \|y_t(t)\|_2^2 + |z_t(t)|_2^2\right) \tag{3.7.1}$$

for all $t \in [0, T]$. We aim to show that $\tilde{E}(t) = 0$, and thus $y(t) = z(t) = 0$ for all $t \in [0, T]$. By the regularity imposed by weak solutions in Definition 3.2.3, there exists a constant $R > 0$ such that

$$\begin{cases} \|u(t)\|_{1,\Omega}, \|\hat{u}\|_{1,\Omega}, \|w(t)\|_{2,\Gamma}, \|\hat{w}(t)\|_{2,\Gamma} \leq R, \\ \|u_t(t)\|_2, \|\hat{u}_t(t)\|_2, |w_t(t)|_2, |\hat{w}_t(t)|_2 \leq R, \\ \int_0^T \|u_t\|_{m+1}^{m+1} dt, \int_0^T \|\hat{u}(t)\|_{m+1}^{m+1} \leq R, \\ \int_0^T |w_t|_{r+1}^{r+1} dt, \int_0^T |\hat{w}(t)|_{r+1}^{r+1} \leq R \end{cases} \tag{3.7.2}$$

for all $t \in [0, T]$. We now begin following the proof of Theorem 3.2.7, where u^n, w^n, y^n, z^n, and $\tilde{E}_n$ are now replaced by $\hat{u}$, $\hat{w}$, y, z, and $\tilde{E}$, respectively. In fact, since $y(0) = y_t(0) = z(0) = z_t(0) = 0$, several terms from the proof of Theorem 3.2.7 are simplified or completely eliminated.

First, as in (3.6.7), accounting for $\tilde{E}(0) = 0$ and employing the monotonicity properties of g_1, g_2, we obtain the energy inequality:

$$\tilde{E}(t) \leq R_f + R_h, \tag{3.7.3}$$

where

$$\begin{cases} R_f = \int_0^t \int_\Omega (f(u(\tau)) - f(\hat{u}(\tau)))y_t(\tau)dxd\tau, \\ \\ R_h = \int_0^t \int_\Gamma (h(w(\tau)) - h(\hat{w}(\tau)))z_t(\tau)d\Gamma d\tau. \end{cases}$$

We can follow (3.6.9)-(3.6.33), making the proper replacements outlined in the previous section and recalling that $\tilde{E}(0) = 0$, then we conclude

$$\begin{aligned} \tilde{E}(t) \leq |R_f| + |R_h| \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}(t) \\ + C(T,R,\epsilon)\int_0^t \tilde{E}(\tau)\left(\|u_t\|_{m+1} + \|\hat{u}_t\|_{m+1} + 1\right)d\tau, \end{aligned} \tag{3.7.4}$$

for all $t \in [0,T]$. Again, choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right) < 1.$$

By applying Gronwall's inequality with an L^1-kernel, it follows that $\tilde{E}(t) = 0$ on $[0,T]$. Hence $y(t) = z(t) = 0$ on $[0,T]$. Finally, we note that, it is sufficient to consider a small time interval $[0,T]$, since this process can be reiterated. The proof of Theorem 3.2.9 is now complete. □

3.7.2 Proof of Theorem 3.2.11

We begin by pointing out that the only difference between 3.2.11 and Theorem 3.2.9 is that Assumption 3.2.6 is not imposed in Theorem 3.2.11. Thus, the proof of Theorem

3.2.11 is essentially the same as Theorem 3.2.9 (which itself was only a slight reworking of the proof of Theorem 3.2.7), with the exception of the estimate for R_f. So, we focus on estimating R_f in the case where $p > 3$ and the wave source f is not necessarily a C^2-function. With this scenario in place, the method of integration by parts twice fails. To handle this difficulty, recall the additional restriction on parameters and the initial data in Theorem 3.2.11, namely, $m > 3p - 4$ if $p > 3$, and $u_0 \in L^{(3(p-1)}(\Omega)$.

Proof. Put $y = u - \hat{u}$ and recall (3.2.1). Then, we have

$$\left|\int_0^t \int_\Omega (f(u) - f(\hat{u}))y_t dx d\tau\right| \leq C \int_0^t \int_\Omega (|u|^{p-1} + |\hat{u}|^{p-1} + 1)|y||y_t| dx d\tau. \qquad (3.7.5)$$

Put:

$$I_1 = \int_0^t \int_\Omega |y||y_t| dx d\tau, \quad I_2 = \int_0^t \int_\Omega (|u|^{p-1} + |\hat{u}|^{p-1})|y||y_t| dx d\tau.$$

The estimate for I_1 is straightforward. Invoking Hölder's inequality yields,

$$I_1 \leq C \int_0^t \|y(\tau)\|_6 \|y_t(\tau)\|_2 d\tau \leq C \int_0^t \tilde{E}(\tau)^{\frac{1}{2}} \tilde{E}(\tau)^{\frac{1}{2}} d\tau = C \int_0^t \tilde{E}(\tau) d\tau. \qquad (3.7.6)$$

A typical term in I_2 is estimated as follows:

$$\int_0^t \int_\Omega |u|^{p-1}|y||y_t| dx d\tau$$
$$\leq C \int_0^t \int_\Omega |u - u_0|^{p-1}|y||y_t| dx d\tau + C \int_0^t \int_\Omega |u_0|^{p-1}|y||y_t| dx d\tau. \qquad (3.7.7)$$

By invoking Hölder's inequality,

$$\int_0^t \int_\Omega |u-u_0|^{p-1}|y||y_t|dxd\tau$$
$$\leq \int_0^t \left(\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx\right)^{\frac{1}{3}} \left(\int_\Omega |y(\tau)|^6 dx\right)^{\frac{1}{6}} \left(\int_\Omega |y_t|^2 dx\right)^{\frac{1}{2}} d\tau. \quad (3.7.8)$$

Since u, $u_t \in C([0,T];L^2(\Omega))$, we can write

$$\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx = \int_\Omega \left|\int_0^\tau u_t(s)ds\right|^{3(p-1)} dx$$
$$\leq C(T)\int_\Omega \left(\int_0^\tau |u_t(s)|^{m+1}ds\right)^{\frac{3(p-1)}{m+1}} dx. \quad (3.7.9)$$

Since $m \geq 3p-4$, then $\frac{3(p-1)}{m+1} \leq 1$. Therefore, by using Hölder's inequality and (3.7.2), it follows that

$$\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx \leq C(T)\left(\int_\Omega \int_0^\tau |u_t(s)|^{m+1}dsdx\right)^{\frac{3(p-1)}{m+1}} \leq C(R,T). \quad (3.7.10)$$

So, (3.7.10) and (3.7.8) yield

$$\int_0^t \int_\Omega |u-u_0|^{p-1}|y||y_t|dxd\tau \leq C(R,T)\int_0^t \|y(\tau)\|_6\|y_t(\tau)\|_2 d\tau$$
$$C(R,T) \leq \int_0^t \tilde{E}(\tau)^{\frac{1}{2}}\tilde{E}(\tau)^{\frac{1}{2}}d\tau = C(R,T)\int_0^t \tilde{E}(\tau)d\tau. \quad (3.7.11)$$

By recalling the assumption $u_0 \in L^{3(p-1)}(\Omega)$, then the second term on the right-hand side of (3.7.7) is estimated by:

$$\int_0^t \int_\Omega |u_0|^{p-1}|y||y_t|dxd\tau \leq \int_0^t \|u_0\|_{3(p-1)}^{p-1}\|y(\tau)\|_6\|y_t(\tau)\|_2 d\tau$$
$$\leq C(T)\|u_0\|_{3(p-1)}^{p-1}\int_0^t \tilde{E}(\tau)d\tau. \quad (3.7.12)$$

Combining (3.7.11) and (3.7.12) back into (3.7.7) yields

$$\int_0^t \int_\Omega |u|^{p-1}|y||y_t|dxd\tau \leq C\left(R, T, \|u_0\|_{3(p-1)}\right)\int_0^t \tilde{E}(\tau)d\tau. \tag{3.7.13}$$

The other term in I_2 are estimated in the same manner, and one has

$$I_2 \leq C\left(R, T, \|u_0\|_{3(p-1)}\right)\int_0^t \tilde{E}(\tau)d\tau. \tag{3.7.14}$$

Hence, (3.7.6), (3.7.14), and (3.7.5) yield

$$\left|\int_0^t \int_\Omega (f(u) - f(\hat{u}))y_t dxd\tau\right| \leq C\left(R, T, \|u_0\|_{3(p-1)}\right)\int_0^t \tilde{E}(\tau)d\tau. \tag{3.7.15}$$

Finally, we may use the same argument for the proof of Theorem 3.2.9 and Gronwall's inequality to complete the proof of Theorem 3.2.11. □

Chapter A

Ancillary Results

The following auxiliary results were invoked at various points in the dissertation and appear in various references (we refer the reader to [48, 51, 53, 33, 39] for instance). We list them here for sake of convenience.

Proposition A.0.1 (Prop. A.1 in [48])**.** *Let H be a Hilbert space and X be a Banach space such that $X \subset H \subset X'$ where each injection is continuous with dense range. If*

$$\begin{cases} f \in L^2(0,T;H), \quad f' \in L^2(0,T;X'), \\ g \in L^2(0,T;X), \quad g' \in L^2(0,T;H), \end{cases}$$

then the map $t \mapsto (f(t), g(t))_H$ coincides with an absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(f(t), g(t))_H = \langle f'(t), g(t)\rangle_{X',X} + (f(t), g'(t))_H \ \textit{a.e.}\ [0,T].$$

Proposition A.0.2 (Prop. A.2 in [48])**.** *Let H be a Hilbert space and X be a Banach space such that $X \subset H \subset X'$ where with each injection is continuous with dense range.*

Suppose X' is separable and $\{u_N\}_1^\infty$ is a sequence in $L^1(0,T;X)$ satisfying:

$$\begin{cases} u_N \to u \text{ weakly in } L^1(0,T;X), \\ u_N \to u \text{ strongly in } L^1(0,T;H), \end{cases}$$

as $N \to \infty$. Then, there exists a subsequence of $\{u_N\}_1^\infty$ (again reindexed by N) such that

$$u_N(t) \to u(t) \text{ weakly in } X \text{ a.e. } [0,T], \text{ as } N \to \infty.$$

A.0.1 The Difference Quotient

Let Y be a Banach space. For $u \in C_w([0,T];Y)$ or $C([0,T];Y)$ and $h > 0$, we define its *symmetric difference quotient* by:

$$D_h u(t) = \frac{u_e(t+h) - u_e(t-h)}{2h}, \tag{A.0.1}$$

where u_e denotes the extension of u to $\mathbb{R}$ given by:

$$u_e(t) = \begin{cases} u(0) \text{ for } t \le 0, \\ u(t) \text{ for } t \in (0,T), \\ u(T) \text{ for } t \ge T. \end{cases} \tag{A.0.2}$$

For the reader's convenience, we review the important results of the difference quotient (see for instance [33, 39, 52, 54]).

Proposition A.0.3 ([39])**.** *Let $u \in C_w([0,T];Y)$ where Y is a Hilbert space with* *inner product* $(\cdot,\cdot)_Y$ *. Then,*

$$\lim_{h\longrightarrow 0}\int_0^T (u, D_h u)_Y dt = \frac{1}{2}\left(\|u(T)\|_Y^2 - \|u(0)\|_Y^2\right). \tag{A.0.3}$$

If, in addition, $u_t \in C_w([0,T];Y)$*, then*

$$\int_0^T (u_t, (D_h u)_t)_Y dt = 0, \text{ for each } h > 0, \tag{A.0.4}$$

and, as $h \longrightarrow 0$*,*

$$D_h u(t) \longrightarrow u_t(t) \text{ weakly in } Y, \text{ for every } t \in (0,T), \tag{A.0.5}$$

$$D_h u(0) \longrightarrow \frac{1}{2}u_t(0) \text{ and } D_h u(T) \longrightarrow \frac{1}{2}u_t(T) \text{ weakly in } Y. \tag{A.0.6}$$

Proposition A.0.4 ([33])**.** *Let* Y *and* Z *be Banach spaces. Assume* $u \in L^1([0,T];Y)$ *and* $u_t \in L^1(0,T;Y) \cap L^p(0,T;Z)$*, where* $1 \le p < \infty$*. Then* $D_h u \in L^p(0,T;Z)$ *and* $\|D_h u\|_{L^p(0,T;Z)} \le \|u_t\|_{L^p(0,T;Z)}$*. Moreover,* $D_h u \longrightarrow u_t$ *in* $L^p(0,T;Z)$*, as* $h \longrightarrow 0$*.*

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