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Travel Cost Budget based User Equilibrium in a Bottleneck Model with Stochastic Capacity

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Abstract: This paper studies a bottleneck model in which the capacity of the bottleneck is constant within a day but changes stochastically from day-to-day between a designed value (good condition) and a degraded one (bad condition). The study relates the travel cost variability due to stochastic capacity with commuters' departure time choice behaviors. We postulate that commuters acquire the variability of travel cost based on past experiences and factor such variability into their departure time choice consideration by minimizing their travel cost budget (TCB), defined as a weighted average of mean travel cost and standard deviation of travel cost. We show that the consideration of TCB yields seven possible equilibrium patterns. Closed form solutions to all possible equilibrium patterns and their corresponding parameter ranges are derived. The rationality of the patterns has been investigated. Dependence of travel cost and the duration of peak hours on the commuters' risk attitude has also been derived in each equilibrium pattern. Finally, numerical studies have been conducted to illustrate the properties.

Keywords: Stochastic capacity; Bottleneck model; Travel cost budget

1. Introduction

In many countries, traffic congestion is getting worse and commuting cost is getting higher particularly in urban areas. There exists a generally established belief that traffic congestion threatens urban prosperity as a drain on the economy. For example, the US Department of Transportation (USDOT) stated that "Congestion in 498 metropolitan areas caused urban Americans to travel 5.5 billion hours more and to purchase an extra 2.9 billion gallons of fuel for a congestion cost of $121 billion" (Federal Highway Administration 2013). To make economic analysis of traffic congestion and study the resulting departure patterns of commuters, the well-known Vickrey's bottleneck model (Vickrey, 1969) has become a basic model and was extended by Arnott et al. (1990a, 1993). To capture the dynamic of traffic congestion, a bottleneck with a fixed capacity is considered in the Vickrey's model and commuters with scheduling preferences must pass the bottleneck to arrive at the destination by making a trade-off between the anticipated travel time cost and schedule delay cost. Traffic congestion may arise in equilibrium because of

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the limited bottleneck capacity.

However, uncertainty is unavoidable in real life, especially in the transportation system (Arnott et al., 1999; Lindsey, 2009; Fosgerau, 2008; Lo et al., 2006). Many uncertainties exist in transport systems such as work zones, crash accidents, adverse weather, traffic management and control, etc, which could lead to the stochasticity of road capacity. For these reasons, researchers have investigated the impact of capacity uncertainty on system performance and departure patterns of commuters by extending the deterministic bottleneck model (Li et al., 2009a; Chen et al., 2002; Xiao et al., 2014). For example, Lindsey (1994) extended the Vickrey's bottleneck model with a general distribution of bottleneck capacity to study the properties of no-toll equilibrium and system optimum of the commuting system. Arnott et al. (1999) modeled the commuting system with capacity fluctuations and demand variations, and demonstrated the influence of information on the system performance. Fosgerau (2008) further investigated the bottleneck model with both capacity and demand stochasticity, and derived the expected marginal and total congestion costs mathematically. By considering the heterogeneity of commuters and the stochasticity of traffic arrival, Siu and Lo (2009) investigated the random travel delay using an extended bottleneck model. As traffic incidents may occur at any time during the peak period, Peer et al. (2010) derived the user equilibrium traffic pattern using analytical methods, based on the bottleneck model with time-varying capacities within-day. Under the condition that the within-day capacity is fixed but the day-to-day capacity is stochastic, Xiao et al. (2015) investigated the bottleneck model with a uniformly distributed capacity and designed according toll pricing scheme for higher system performance. Later, Long et al. (2017) extended the model proposed by Xiao et al. (2015) by assuming that the random bottleneck capacity is not restricted to follow any specified distribution. To the extent of our knowledge, in most of the proposed bottleneck models with stochastic capacity, commuters' departure time choice is assumed to follow the User Equilibrium (UE) principle by considering only the mean trip cost.

Notational glossary

α	The unit cost of travel time	$\bar{\pi}$	$\bar{\pi} = \pi + \lambda\sqrt{\pi(1-\pi)}$
β	The unit cost of schedule delay early	N	The total number of commuters
γ	The unit cost of schedule delay late	t^*	The official work start time
$\bar{s}$	The design capacity	t_s	The departure time for the first commuter
θ	The degradation ratio of capacity ($0 < \theta \leq 1$)	t_e	The departure time for the last commuter
π	The degradation probability of capacity ($0 \leq \pi \leq 1$)	t_{ij}	Critical time point between the j^{th} situation and the $(j+1)^{th}$ situation in Pattern i
λ	The risk preference coefficient of commuters	$T(t)$	The travel time at time t
$r_i(t)$	The departure rate in the i^{th} situation at time t	$C(t)$	Travel cost at time t
$B(t)$	The travel cost budget at time t	π_C	$\pi_C = \gamma/(\alpha+\gamma)$
$E(C(t))$	The mean travel cost at time t	π_N	$\pi_N = \beta\theta/((\alpha-\beta)(1-\theta))$
$\sigma(C(t))$	The standard deviation of travel cost at time t	π_S	$\pi_S = \beta\theta/((\alpha+\gamma)(1-\theta))$
π_T	$\pi_T = -\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta)$	π_M	$\pi_M = -\gamma\theta/\big((\alpha+\gamma)(1-\theta)\big)$

In reality, the effects of travel time variability on commuters' choice behaviors are unneglectable, which could be treated as travel time reliability (Senna, 1994; Abdel-Aty and Kitamura, 1995; Lam, 2000; Brownstone et al., 2003; de Palma and Picard, 2005; Hollander and Liu, 2008; Flötteröd and Liu, 2014; Xin and Levinson, 2015; Kou et al., 2017).

To account for the travel choice behaviors under stochastic travel times, a number of models were proposed which could be briefly described as follows. Lo and Tung (2003) proposed a probabilistic user equilibrium (PUE) model, in which travelers select their routes to lower their mean travel time by considering routes' travel time variabilities. Moreover, Lo et al. (2006) further extended the PUE model and introduced the concept of travel time budget, which is defined as a linear combination of expected travel time and standard deviation of travel time, to capture the effect of travel time variation on the travelers' route choice behaviors. Although there are a lot of models which could account for the travel time reliability, such as mean-excess travel time based model (Xu et al., 2013; Zhou and Chen, 2008), late arrival penalty model (Watling, 2006), prospect based model (Xu et al., 2011), etc., there is no doubt that mean and standard deviation of travel time are the two key points to depict the travel time reliability (Nie, 2011; Wang et al., 2014). In a similar way, it is reasonable to assume that commuters consider both mean and standard deviation of travel cost in the bottleneck model with stochastic capacity.

Recently, Lu et al (2020) conducted controlled laboratory experiments examining participants' departure time choices through a single bottleneck with stochastic capacity. The travel costs, including both the travel time and schedule delay costs, are calculated for each departure-time choice. The results show a distinct linear relationship between the mean and the standard deviation of travel cost, see Fig. 4 in Lu et al. (2020). Furthermore, the mean travel cost of each individual is not a constant value for early or late departures. In general, we found that travelers who depart early will have a lower mean travel cost because the standard deviation of travel cost is smaller, see Fig. 8(b) in Lu et al. (2020), which shows an increasing mean cost with departure times from two of the laboratory experiments. The experimental results justify that the participants minimize not only their mean but also variability in travel costs in making their departure-time choices.

Based on the experimental results, this paper investigates the bottleneck model with stochastic capacity, assuming that commuters choose the departure times to minimize the travel cost budget. Closed form solutions are obtained for all the seven possible equilibrium patterns. We show that, depending on the equilibrium patterns, the impact of the commuters' risk behavior on the peak period (including its start time, end time, and length) and on commuters travel costs (queuing and schedule delay costs) vary, making it more difficult for the policy makers to manage morning traffic congestion without deeper understanding of commuters risk behavior.

The rest of the paper is organized as follows. In Section 2, the travel cost budget (TCB) based stochastic bottleneck model is analyzed. All seven possible equilibrium departure patterns and their corresponding parameter ranges are discussed in detail. The rationality of the equilibrium patterns is studied. Section 3 investigates the impact of risk attitude on the equilibrium patterns. Numerical examples are given in Section 4 to illustrate the properties of the model. Finally, Section 5 concludes the paper.

2. Travel cost budget based bottleneck model with stochastic capacity

2.1 The classical bottleneck model with a fixed bottleneck capacity

The classical bottleneck model (Vickrey, 1969) considers a highway with a single bottleneck which connects a residential district with a central business district (CBD). The free flow travel time of the highway is denoted as t_{free} and the bottleneck capacity is denoted as *s*. In the rush hour, there are *N* commuters departing from the residential district with free flow travel speed and may experience queuing delay in front of the bottleneck. Without loss of generality, the free flow travel time on the highway is set to zero, i.e., $t_{free} = 0$. By definition, the cumulative departures *R*(*t*) can be formulated as follows:

$$R(t) = \int_{t_s}^{t} r(x)dx \tag{1}$$

where $r(x)$ is the departure rate at time instant x, and t_s is the departure time for the first commuter.

The highway is congested during the peak period, and the capacity of the bottleneck will have been fully utilized from time instant t_s. The length of the queue can be formulated as,

$$Q(t) = \max\left\{R(t) - s(t - t_s), 0\right\} \tag{2}$$

Under the assumption $t_{free} = 0$, the travel time of commuters departing at time *t* equals the queuing time and can be given as follows:

$$T(t) = \frac{Q(t)}{s} = \max\left\{\frac{R(t)}{s} - (t - t_s), 0\right\} \tag{3}$$

The cost of commuters who travel from the residential district to the CBD may consist of two components: the cost of travel time and the cost of schedule delay early or late. The total cost can be formulated as,

$$C(t) = \alpha T(t) + \begin{cases} \beta\left[t^* - t - T(t)\right], \text{if } t^* \geq t + T(t) \\ \gamma\left[t + T(t) - t^*\right], \text{if } t^* < t + T(t) \end{cases} \tag{4}$$

Here t^* is the work start time; α, β and γ denote the unit cost of travel time, the unit cost of schedule delay early (SDE), and the unit cost of schedule delay late (SDL), respectively. According to empirical results (Small, 1982), the following relationship holds, i.e., $0 < \beta < \alpha < \gamma$.

The UE principle is used to formulate commuters' departure time choice: no commuter can reduce his or her travel cost by unilaterally altering his or her departure time at equilibrium. This equilibrium condition implies $\frac{dC(t)}{dt} = 0$, if $r(t) > 0$, and hence the departure rate can be explicitly expressed as

$$r(t) = \begin{cases} \dfrac{\alpha}{\alpha - \beta} s, \text{if } t_s \leq t \leq t_o \\ \dfrac{\alpha}{\alpha + \gamma} s, \text{if } t_o \leq t \leq t_e \end{cases} \tag{5}$$

where t_s and t_e are the departure time for the first commuter and the last commuter and t_o is the departure time at which a commuter departs and arrives at the destination on time t^*. Meanwhile, as derived in Arnott et al. (1990b), $t_s = t^* - \frac{\gamma N}{((\beta+\gamma)s)}$, $t_e = t^* + \frac{\beta N}{((\beta+\gamma)s)}$, and $t_o = t^* - \frac{\beta\gamma N}{(\alpha s(\beta+\gamma))}$.

2.2 Bottleneck models with stochastic capacity

This subsection reviews the behavioral assumption in previous bottleneck models with stochastic capacity.

2.2.1 Minimizing mean travel cost

As mentioned before, in the previous studies on bottleneck model concerned with uncertainty, commuters are usually assumed to minimize the mean trip cost

$$E\left(C(t)\right)=E\left[\alpha T(t)+\max\left(\beta\left(t^{*}-t-T(t)\right),0\right)+\max\left(\gamma\left(t+T(t)-t^{*}\right),0\right)\right] \tag{6}$$

With this assumption, for commuters always arriving early,

$$E\left(C(t)\right)=(\alpha-\beta)E\left(T(t)\right)+\beta\left(t^{*}-t\right)$$

For commuters always arriving late,

$$E\left(C(t)\right)=(\alpha+\gamma)E\left(T(t)\right)+\gamma\left(t-t^{*}\right)$$

In both situations, the standard deviation of travel time is not involved. Only for commuters either early or late, one can derive (Li et al., 2009b, 2016; Fosgerau, 2010; Fosgerau and Karlström, 2010)

$$E\left(C(t)\right)=\alpha E\left(T(t)\right)+\max\left(\beta\left(t^{*}-t-E\left(T(t)\right)\right),0\right)+\max\left(\gamma\left(t+E\left(T(t)\right)-t^{*}\right),0\right)+\xi_{t}\sigma\left(T(t)\right)$$

where ξ_t depends on the travel time distribution of time instant t and $0 \le \xi_t \le \frac{1}{2}\cdot(\beta+\gamma)$. In this situation, the standard deviation of travel time is involved in the mean trip cost.

2.2.2 Minimizing $u(t)$

In order to consider the standard deviation of travel time in all situations, Li et al. (2008, 2017) assumed that commuters minimize

$$u(t)=\alpha E\left(T(t)\right)+\max\left(\beta\left(t^{*}-t-E\left(T(t)\right)\right),0\right)+\max\left(\gamma\left(t+E\left(T(t)\right)-t^{*}\right),0\right)+\varepsilon\sigma\left(T(t)\right) \tag{7}$$

Here ε is a parameter. Note that for commuters always early or always late,

$$u(t)=E\left(C(t)\right)+\varepsilon\sigma\left(T(t)\right)$$

For commuters either early or late,

$$u(t)=E\left(C(t)\right)+\left(\varepsilon-\xi_{t}\right)\sigma\left(T(t)\right)$$

In other words, Li et al. (2008, 2017) assumed that commuters choose their departure times according to both expected travel cost and the standard deviation of travel time. However, the weight coefficient of the standard deviation of travel time is situation dependent.

Li et al. (2008) analyzed only two situations: always early and always late arrivals. Later they considered whether travelers need to queue or not (Li et al., 2017). Three situations (always experience queuing and always early; always experience queuing and always late; possibly experience queuing and always late) were studied. But according to Lindsey (1994) and Long et al. (2017), there are some more possible situations from the theoretical perspective. Moreover, since a continuous distribution of random capacity is used, the closed form solution for equilibrium departure pattern cannot be derived in Li et al. (2008, 2017).

2.2.3 Minimizing $\bar{u}(t)$

Recently, Jiang and Lo (2016) have extensively considered the incentive of a traveler to choose a specific departure time under random travel conditions. They related the influence of travel cost variability on departure time choice and assumed that commuters minimize

$$\bar{u}(t) = E\big(C(t)\big) + \lambda\tilde{\sigma}(t) \tag{8}$$

in which $\tilde{\sigma}(t)$ denotes the variability of travel cost and is defined as

$$\tilde{\sigma}(t) = \int_{\theta_{\min}}^{\theta_{\max}} \left|C(t) - E\big(C(t)\big)\right| f(\theta) d\theta$$

Here, Jiang and Lo (2016) assumed that travelers have to endure an exogenous random delay $\Theta(\mathrm{t}) = \frac{Q(t)}{s}\theta$ in which θ is a random variable with uniform distribution. $f(\theta)$ is its probability density function. θ_{min} and θ_{max} are lower and upper bound of the random variable, respectively. λ is risk attitude parameter.

However, Jiang and Lo (2016) only discussed three situations, (i) travelers always arrive early, (ii) travelers may arrive early or late, and (iii) travelers always arrive late. The situations that travelers always experience queue or possible experience queue have not been studied.

Moreover, since a continuous distribution of random capacity is used, the closed form solution for equilibrium departure pattern cannot be derived in Jiang and Lo (2016), either.

As pointed out in Lu et al. (2020), the laboratory experiment does not exclude that commuters minimize $\bar{u}(t) = E\big(C(t)\big) + \lambda\tilde{\sigma}(t)$. However, since both $\tilde{\sigma}(t)$ and $\sigma\big(C(t)\big)$ reflect variability of travel cost, and $\sigma\big(C(t)\big)$ is much more frequently used than $\tilde{\sigma}(t)$, we use travel cost budget in the modeling.

2.2.4 Minimizing $\hat{u}(t)$

Finally, we would like to mention that Li et al. (2009a) proposed that a cost function consisting of expected travel cost and variability of travel cost

$$\hat{u}(t) = \chi E\big(C(t)\big) + \lambda\sigma\big(C(t)\big) \tag{9}$$

can be adopted to model travelers' choice behavior under uncertainty. However, they only studied the special case $\lambda = 0$.

2.3 Travel cost budget based User Equilibirum in bottleneck model with stochastic capacity

We adopt the following assumptions in our model:

Assumption A1: Commuters are homogeneous with the same value of time, value of schedule delays and the same risk preference.

Assumption A2: The capacity of bottleneck is constant within a day but changes from day-to-day. For simplicity, here we assume that there are only two values of capacity, which are the designed capacity in good condition $\bar{s}$ with probability $1-\pi$, and the degraded one in bad condition $\theta\bar{s}$ with probability π, where $0 < \theta \le 1$ and $0 \le \pi \le 1$.

Assumption A3: Commuters acquire the variability of travel cost based on past experiences and their departure time choice follows the UE principle in terms of TCB.

Assumption A4: The risk attitude parameter λ defined below is within the range $\frac{-\pi}{\sqrt{\pi(1-\pi)}} < \lambda < \frac{1-\pi}{\sqrt{\pi(1-\pi)}}$. The reason for imposing Assumption A4 will become apparent later.

The TCB associated with the commuters departing at time instant t is expressed as,

$$B(t) = E(C(t)) + \lambda\sigma(C(t)), \quad \forall t \in [t_s, t_e] \tag{10}$$

where λ is a parameter accounting for the risk attitude of commuters. For $\lambda > 0$, commuters are risk averse. For $\lambda < 0$, commuters are risk preferring. For $\lambda = 0$, commuters are risk neutral. If all commuters are risk neutral, the TCB based model degenerates into the model proposed by Xiao et al. (2015) and Long et al. (2017).

Based on the concept of TCB, the UE condition for commuters' departure time choice in a single bottleneck with stochastic capacity could be defined as follows: no commuter can reduce his/her TCB by unilaterally altering his or her departure time at equilibrium. This condition implies that commuters' TCB is constant with respect to the time instant if the departure rate is positive, i.e.,

$$\frac{dB(t)}{dt} = 0, \text{ if } r(t) > 0 \tag{11}$$

The calculation of the TCB relies on the calculations of the mean and standard deviation of total travel time cost. As it is assumed that the capacity of the bottleneck is constant within a day, but fluctuates from day to day, commuters may endure schedule delay early or schedule delay late and may or may not encounter queuing delay in different days even if they depart at the same time of day. Similar to the work by Long et al. (2017), the possible schedule delay and queuing experiences in the stochastic bottleneck are summarized in Table 1. Three types of schedule delay and two types of queuing experience could lead to six combinations/situations faced by travelers. The six situations are shown in Table 2, and their analytical derivations will be presented in Section 2.4. We need to point out that, all six combinations may not exist in any one particular equilibrium pattern simultaneously. In fact, there are seven equilibrium patterns as shown in Table 3 from theoretical perspective, each consists of a combination of some of the six situations. Furthermore, one may also observe that the schedule delay and queuing combinations occur orderly in the listed seven equilibrium patterns. Next, we analyze the equilibrium departure patterns in detail.

Table 1: Schedule delay types and queuing experience types in the stochastic bottleneck.

Schedule delay experience types	Queuing experience types
Experience schedule delay early (SDE)	Always experience queuing (AQ)
Possibly experience schedule delay either early or late (SDE/L)	Possibly experience queuing (PQ)
Experience schedule delay late (SDL)	

Table 2: Six possible situations in the stochastic bottleneck.

Situation	S1	S2	S3	S4	S5	S6
Combination	SDE+AQ	SDE/L+AQ	SDL+AQ	SDL+PQ	SDE/L+PQ	SDE+PQ

Table 3: Seven possible equilibrium departure patterns in the stochastic bottleneck.

Pattern						
1	2	3	4	5	6	7
SDE+AQ	SDE+AQ	SDE+PQ	SDE+AQ	SDE+AQ	SDE+PQ	SDL+PQ
SDE/L+AQ	SDE/L+AQ	SDE/L+PQ	SDE/L+AQ	SDE/L+AQ	SDE/L+PQ	
SDL+AQ	SDE/L +PQ	SDL+PQ	SDL+AQ	SDE/L+PQ		
SDL+PQ	SDL+PQ					

2.4 Equilibria associated with stochastic bottleneck model

2.4.1 Equilibrium departure rates

We studied the equilibrium departure rate of each situation in Table 2. The results are summarized as follows. All departure rates are positive, otherwise the situations would not have occurred.

(S1) Commuters always experience schedule delay early and always experience queuing (SDE+AQ)

In this situation, no matter how the capacity of bottleneck varies, commuters always arrive early and always experience queuing. The travel cost can be formulated as follows:

$$C(t)=\alpha\left(\frac{R(t)}{s}-t+t_s\right)+\beta\left(t^*-t_s-\frac{R(t)}{s}\right),\quad s=\theta\overline{s}\ \text{ or }\ \overline{s} \tag{12}$$

According to the expression of travel cost (12), the mean travel cost and the standard deviation of travel cost can be formulated as follows, respectively,

$$E\big(C(t)\big)=(\alpha-\beta)\left(\frac{\pi}{\theta\overline{s}}+\frac{1-\pi}{\overline{s}}\right)R(t)+\alpha(t_s-t)+\beta\left(t^*-t_s\right) \tag{13}$$

$$\sigma\big(C(t)\big)=(\alpha-\beta)\sqrt{\pi(1-\pi)}\left(\frac{1-\theta}{\theta\overline{s}}\right)R(t) \tag{14}$$

Substituting (13) and (14) into the expression of travel cost budget (10), and using the equilibrium condition (11), i.e., $dB(t)/dt=0$, the equilibrium departure rate can be obtained for this situation, given as follows:

$$r_1(t)=r_1=\theta\overline{s}\frac{\alpha/(\alpha-\beta)}{(1-\theta)\overline{\pi}+\theta} \tag{15}$$

where, $\overline{\pi}=\pi+\lambda\sqrt{\pi(1-\pi)}$ is defined to simplify the notation in this paper. Note that the subscript 1 is introduced into $r(t)$ to indicate that it is the departure rate for situation S1 at time t. For a positive departure rate of $r_1>0$, one derives $\overline{\pi}>-\theta/(1-\theta)$ should be satisfied in S1.

(S2) Commuters possibly experience schedule delay either early or late, and always experience queuing (SDE/L+AQ)

In this situation, whether commuters arrive early or late depends on the capacity of bottleneck, but they always experience queuing. The equilibrium departure rate can be obtained for this situation, given as follows (see Appendix A for the detailed derivation):

$$r_2(t) = r_2 = \theta\overline{s}\frac{\alpha/(\alpha-\beta)}{\left[(\alpha+\gamma)/(\alpha-\beta)-\theta\right]\overline{\pi}+\theta} \tag{16}$$

To satisfy $r_2 > 0$, one derives $\overline{\pi} > -\theta/\left((\alpha+\gamma)/(\alpha-\beta)-\theta\right)$ should be satisfied in S2.

(S3) Commuters experience schedule delay late and always experience queuing (SDL+AQ)

In this situation, no matter how the capacity varies, commuters always experience schedule delay late and always experience queuing. The equilibrium departure rate can be obtained as given as follows (see Appendix A for the detailed derivation):

$$r_3(t) = r_3 = \theta\overline{s}\frac{\alpha/(\alpha+\gamma)}{(1-\theta)\overline{\pi}+\theta} \tag{17}$$

To satisfy $r_3 > 0$, one derives $\overline{\pi} > -\theta/(1-\theta)$ should be satisfied in S3. The condition is the same as that for S1.

(S4) Commuters experience schedule delay late and possibly experience queuing (SDL+PQ)

In this situation, commuters always arrive late, but may experience queuing depending on the capacity of bottleneck. The equilibrium departure rate can be obtained for this situation, given as follows (see Appendix A for the detailed derivation):

$$r_4(t) = r_4 = \theta\overline{s}\left[1-\frac{\gamma}{(\alpha+\gamma)\overline{\pi}}\right] \tag{18}$$

For $r_4 > 0$, one derives $\overline{\pi} > \gamma/(\alpha+\gamma)$ or $\overline{\pi} < 0$ which should be satisfied in S4.

(S5) Commuters possibly experience schedule delay either early or late, and possibly experience queuing (SDE/L+PQ)

In this situation, commuters may experience schedule delay early or late, and may experience queuing depending on the capacity of bottleneck. The equilibrium departure rate can be formulated as follows (see Appendix A for the detailed derivation):

$$r_5(t) = r_5 = \theta\overline{s}\left[\frac{\alpha-\beta}{\alpha+\gamma}+\frac{\beta}{(\alpha+\gamma)\overline{\pi}}\right] \tag{19}$$

For $r_5 > 0$, one derives $\overline{\pi} > 0$ or $\overline{\pi} < -\beta/(\alpha-\beta)$ which should be satisfied in S5.

(S6) Commuters experience schedule delay early, and possibly experience queuing (SDE+PQ)

In this situation, commuters always arrive early, but may experience queuing depending on the capacity of bottleneck. The equilibrium departure rate can be obtained for this situation, given as follows (see Appendix A for the detailed derivation):

$$r_6(t) = r_6 = \theta\overline{s}\left[\frac{\beta}{(\alpha-\beta)\overline{\pi}}+1\right] \tag{20}$$

For $r_6 > 0$, one derives $\bar{\pi} > 0$ or $\bar{\pi} < -\beta/(\alpha-\beta)$ should be satisfied in S6.

2.4.2 Theoretical equilibrium patterns

As presented in Figure 1-5, there are seven theoretical departure patterns in theory. Let t_{ij} denote critical time point that separates the j^{th} situation and $(j+1)^{th}$ situation in Pattern *i*. The details of the seven departure patterns are given as follows:

Pattern 1 in Figure 1: This pattern consists of four situations (i.e., S1, S2, S3 and S4). The corresponding departure rates are r_1, r_2, r_3 and r_4, respectively. From Figure 1, we can see that commuters who depart before t_{11} arrive at the destination early and always experience queuing; those who depart during (t_{11}, t_{12}) arrive at the destination possibly early or late, and always experience queuing; those who depart during (t_{12}, t_{13}) arrive at the destination late and always experience queuing and those who depart after t_{13} arrive at the destination late and possibly experience queuing.

In Figure 1(a), $r_2 > r_3 > r_4$, and the pattern is named as Pattern 1a. In Figure1(b), $r_2 < r_3 < r_4$, and the pattern is named as Pattern 1b. From $r_2 > r_3 > r_4$ ($r_2 < r_3 < r_4$), one can easily derive $\bar{\pi} < 1$ ($\bar{\pi} > 1$) should be satisfied.

Pattern 2 in Figure 2: This pattern consists of four situations (i.e., S1, S2, S5 and S4). The corresponding departure rates are r_1, r_2, r_5 and r_4, respectively. From Figure 2, we can see that commuters who depart before t_{21} arrive at the destination early and always experience queuing; those who depart during (t_{21}, t_{22}) arrive at the destination possibly early or late and always experience queuing; those who depart during (t_{22}, t^*) arrive at the destination possibly early or late and possibly experience queuing and those who depart after t^* always arrive at the destination late and possibly experience queuing.

In Figure 2(a), $r_2 > r_5 > r_4$, and the pattern is named as Pattern 2a. In Figure 2(b), $r_2 < r_5 < r_4$, and the pattern is named as Pattern 2b. From $r_2 > r_5 > r_4$ ($r_2 < r_5 < r_4$), one can easily derive $\bar{\pi} < 1$ ($\bar{\pi} > 1$) should be satisfied.

Pattern 3 in Figure 3: This pattern consists of three situations (i.e., S6, S5 and S4). The corresponding departure rates are r_6, r_5 and r_4, respectively. From Figure 3, we can see that commuters who depart before t_{31} arrive at the destination early and possibly experience queuing; those who depart during (t_{31}, t^*) arrive at the destination possibly early or late and possibly experience queuing and those who depart after t^* arrive at the destination late and possibly experience queuing.

In Figure 3(a), $r_5 > r_4$, and the pattern is named as Pattern 3a. In Figure 3(b), $r_5 < r_4$, and the pattern is named as Pattern 3b. As in Pattern 2a and 2b, $\bar{\pi} < 1$ and $\bar{\pi} > 1$ should be satisfied in Pattern 3a and 3b, respectively.

Pattern 4 in Figure 4: This pattern consists of three situations (i.e., S1, S2 and S3). The corresponding departure rates are r_1, r_2 and r_3, respectively. From Figure 4, we can see that commuters who depart before t_{41} arrive at the destination early and always experience queuing; those who depart during (t_{41}, t_{42}) arrive at the destination possibly early or late and always experience queuing and those who depart during (t_{42}, t_e) arrive at the destination late and always experience queuing.

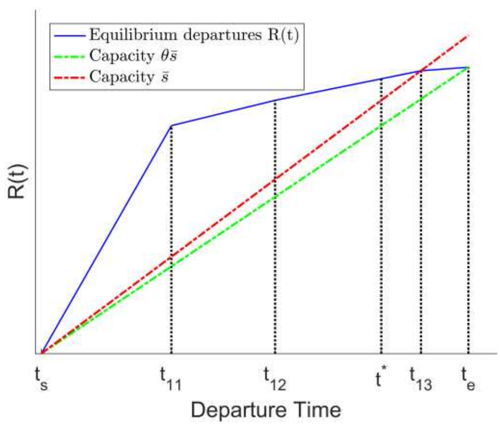
In Figure 4(a), $r_1 > r_2$, and the pattern is named as Pattern 4a. In Figure 4(b), $r_1 < r_2$, and the pattern is named as Pattern 4b. From $r_1 > r_2$ ($r_1 < r_2$), one can easily derive $\bar{\pi} > 0$ ($\bar{\pi} < 0$) should be satisfied.

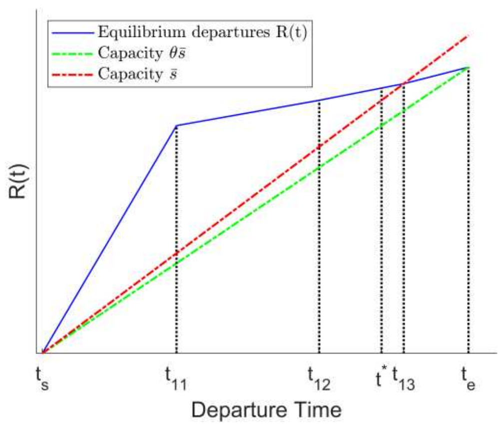
Pattern 5 in Figure 5(a): This pattern consists of three situations (i.e., S1, S2 and S5). The corresponding departure rates are r_1, r_2 and r_5, respectively. From Figure 5(a), we can see that commuters who depart before t_{51} arrive at the destination early and always experience queuing; those who depart during (t_{51}, t_{52}) arrive at the destination possibly early or late and always experience queuing and those who depart after t_{52} arrive at the destination possibly early or late and possibly experience queuing.

Pattern 6 in Figure 5(b): This pattern consists of two situations (i.e., S6 and S5). The corresponding departure rates are r_6 and r_5, respectively. From Figure 5(b), we can see that commuters who depart before t_{61} arrive at the destination early and possibly experience queuing and those who depart after t_{61} arrive at the destination possibly early or late and possibly experience queuing.

Pattern 7 in Figure 5(c): This pattern consists of only one situation (i.e., S4), and the corresponding departure rate is r_4. We can observe from Figure 5(c) that all commuters departing after t^* arrive at the destination late and possibly experience queuing.

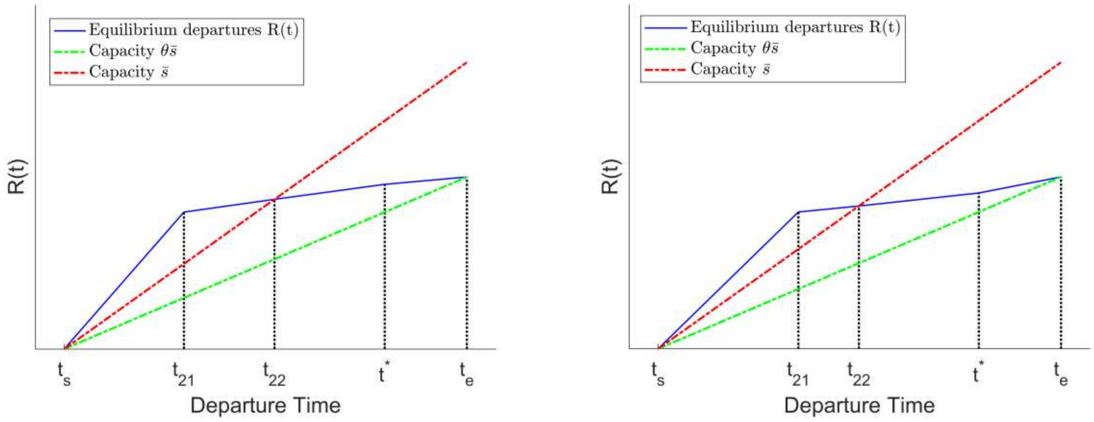
As shown in next section, Patterns 1b, 2b, 3b, 4b, and 7 are implausible in practice despite they are theoretical equilibrium states.





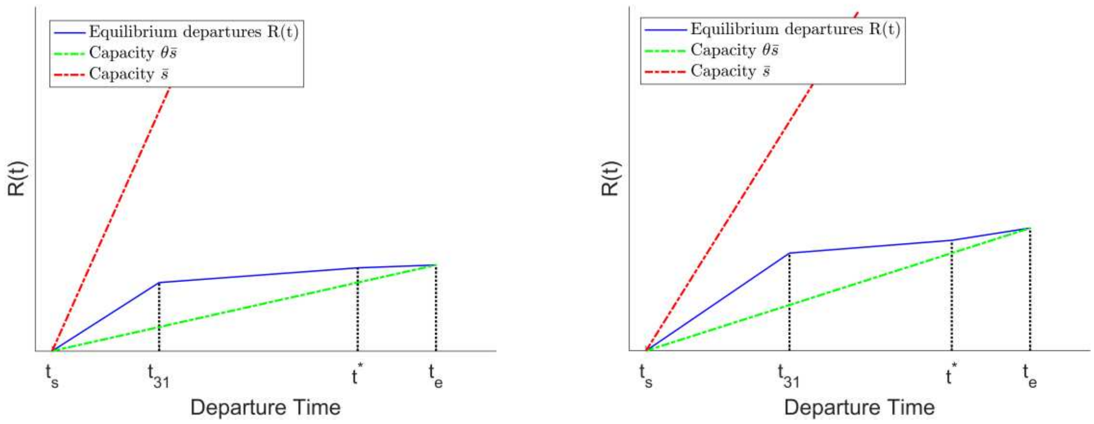
(a) (b)

Figure 1. Equilibrium departure patterns. (a) Pattern 1a, (b) Pattern 1b.



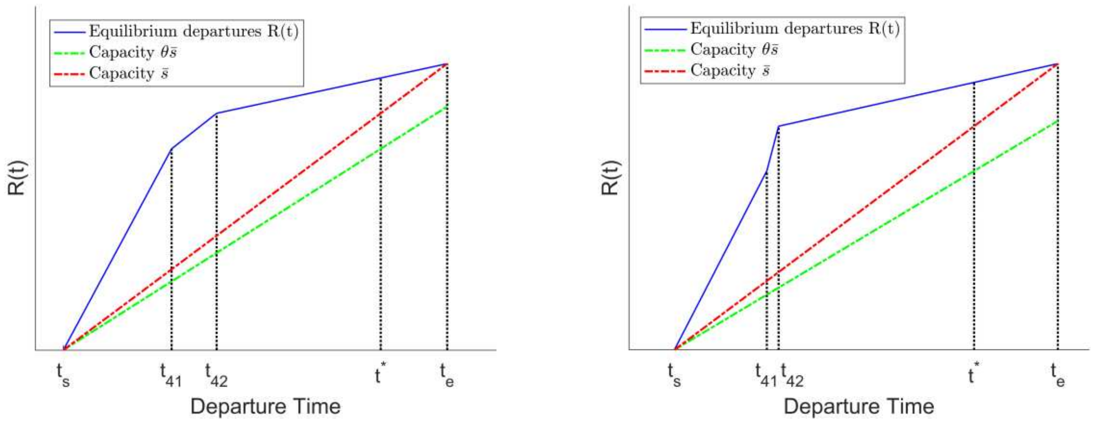
(a) (b)

Figure 2. Equilibrium departure patterns. (a) Pattern 2a, (b) Pattern 2b.



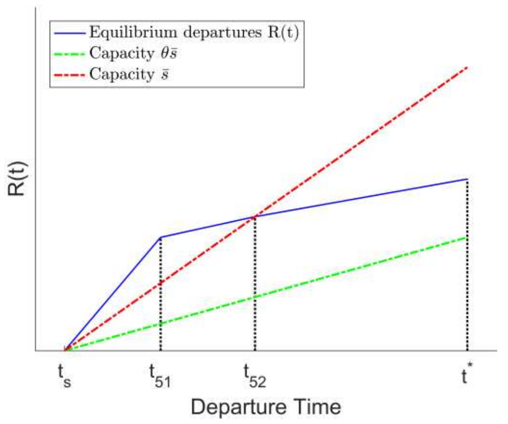
(a) (b)

Figure 3. Equilibrium departure patterns. (a) Pattern 3a, (b) Pattern 3b.

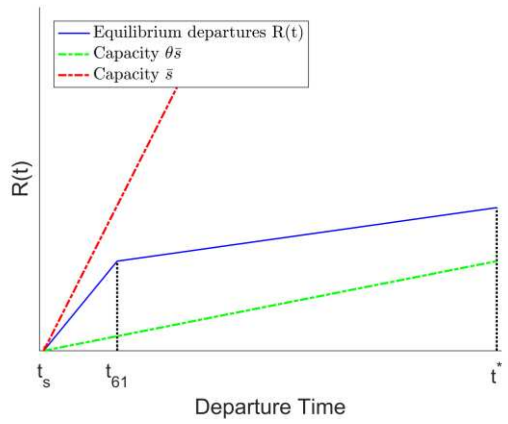


(a) (b)

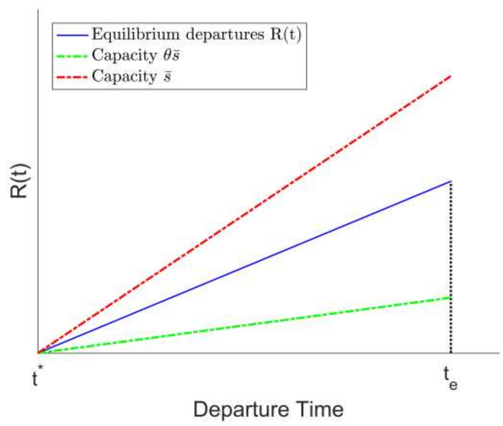
Figure 4. Equilibrium departure patterns. (a) Pattern 4a, (b) Pattern 4b.



(a)



(b)



(c)

Figure 5. Equilibrium departure patterns. (a)-(c) corresponds to Patterns 5-7.

2.4.3 Critical time points in equilibrium departure patterns

Naturally, the first commuter could always avoid queuing in these seven patterns no matter how capacity varies. That is to say, the first commuter only experiences schedule early. Then at equilibrium the condition

$$B(t) = \beta\left(t^* - t_s\right) \tag{21}$$

always holds in all patterns.

In Patterns 1-3, the boundary condition is that commuters departing at t_e could avoid queuing if the capacity is in the bad condition, i.e., $s = \theta\bar{s}$ (see Figure 1-3). Meanwhile, we have the following formula,

$$R\left(t_e\right) = \theta\bar{s}\left(t_e - t_s\right) = N \tag{22}$$

Using condition (21) at $t = t_e$ i.e., $B(t_e) = \beta(t^* - t_s)$, together with condition (22), it could be found that the first commuter and the last commuter depart at the same time in Patterns 1-3, i.e., the value of t_s and t_e are all equal in Patterns 1-3, which could be given as follows (see Appendix B for the detailed derivation):

$$t_s = t^* - \frac{\gamma N}{\theta\bar{s}\left(\beta + \gamma\right)} \tag{23}$$

$$t_e = t^* + \frac{\beta N}{\theta \overline{s}\left(\beta+\gamma\right)} \tag{24}$$

In **Pattern 4**, commuters departing at t_e could avoid queuing, if the capacity is in the good condition, i.e., $s = \overline{s}$ (see Figure 4). Hence, comparing condition (22) with Patterns 1-3, the condition could be formulated as,

$$R\left(t_e\right) = \overline{s}\left(t_e - t_s\right) = N \tag{25}$$

In the same way, using condition (21) at $t = t_e$, then we have (see Appendix B for the detailed derivation):

$$t_s = t^* - N\frac{\gamma + \left(\alpha+\gamma\right)\left(1/\theta - 1\right)\overline{\pi}}{\overline{s}\left(\beta+\gamma\right)} \tag{26}$$

$$t_e = t^* + N\frac{\beta - \left(\alpha+\gamma\right)\left(1/\theta - 1\right)\overline{\pi}}{\overline{s}\left(\beta+\gamma\right)} \tag{27}$$

In **Patterns 5 and 6**, $t_e = t^*$ always holds, i.e., the last commuter departs at the work start time (see Figure 5(a) and 5(b)). Similarly, conditions $R\left(t_e\right) = R\left(t^*\right) = N$ and $B\left(t_e\right) = B\left(t^*\right) = \beta\left(t^* - t_s\right)$ should be also satisfied. Then we have (see Appendix B for the detailed derivation):

$$t_s = t^* - \frac{N}{\theta \overline{s}}\frac{\left(\alpha+\gamma\right)\overline{\pi}}{\beta + \left(\alpha+\gamma\right)\overline{\pi}} \tag{28}$$

In **Pattern 7**, we have $t_s = t^*$, i.e., the first traveler departs at the work start time (see Figure 5(c)). All commuters depart after the work start time with a constant departure rate and have the same travel cost budget which is equal to zero. Therefore, we have

$$R\left(t_e\right) = r_4\left(t_e - t^*\right) = N \tag{29}$$

which yields

$$t_e = t^* + \frac{N}{\theta \overline{s}}\frac{\left(\alpha+\gamma\right)\overline{\pi}}{\left(\alpha+\gamma\right)\overline{\pi} - \gamma} \tag{30}$$

Other critical time points in the first six patterns are given in Table 4. Because Pattern 7 consists of only one situation, there is no critical time point in this pattern. Meanwhile, $\hat{t} = \gamma N / \left(\theta \overline{s}\left(\beta+\gamma\right)\right)$ is defined to simplify expressions in Table 4. The detailed derivation of these critical time points is provided in Appendix B.

2.4.4 Boundary conditions for the seven possible equilibrium departure patterns

In Table 5, the boundary conditions of these seven possible equilibrium patterns are given. The detailed derivation of the boundary conditions is provided in Appendix C. For simplicity, we introduce the composite parameters: $\pi_C = \gamma/(\alpha+\gamma)$; $\pi_N = \beta\theta/((\alpha-\beta)(1-\theta))$; $\pi_S = \beta\theta/((\alpha+\gamma)(1-\theta))$; $\pi_T = -\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta)$ and $\pi_M = -\gamma\theta/((\alpha+\gamma)(1-\theta))$. Obviously, $\pi_N > \pi_S$ and $\pi_M < \pi_T < 0$ are always true.

Figure 6 shows a typical diagram to exhibit the seven equilibrium patterns. When $\bar{\pi} > \pi_C$, we have one of Patterns 1-3; When $\pi_T \le \bar{\pi} \le \pi_C$, we have one of Patterns 4-6. When $\bar{\pi} < \pi_M$, we have Pattern 7. Note that there is a shadow region when $\pi_M < \bar{\pi} < \pi_T$, in which none of Patterns 1-7 exists and no equilibrium solution exists in the region. The travelers would forever change their departure times[1]. This is not plausible. One possible reason is that travelers would not behave so risk-loving. Further behavior data analysis is needed to check this issue.

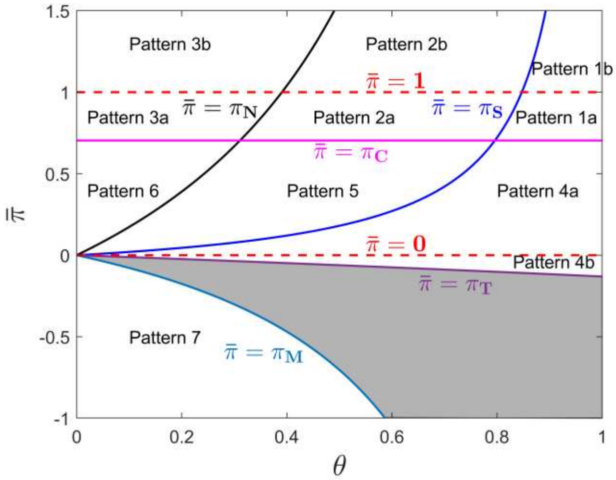


Figure 6. A typical diagram exhibiting the seven equilibrium patterns. The parameters are $\alpha = 6.4$, $\beta = 3.9$, $\gamma = 15.21$, $\bar{s} = 3000$, $N = 5000$.

[1] This, to some extent, is similar to that in Minority Game, in which an odd number of participants enter two rooms and those in the room that has less persons win. In the Minority Game, the participants are forever changing their decisions.

Table 4: Critical time points in six possible equilibrium departure patterns.

Pattern	**Critical time point 1**	**Critical time point 2**	**Critical time point 3**
1	$t_{11}=t^{*}-\hat{t}\left\{1-\frac{(\alpha-\beta)\left[(1-\theta)\bar{\pi}+\theta\right]}{\alpha}\right\}$	$t_{12}=t^{*}-\hat{t}\frac{\beta-\bar{\pi}(\alpha+\gamma)(1/\theta-1)}{\alpha}$	$t_{13}=t^{*}+\hat{t}\frac{\beta-\bar{\pi}(\alpha+\gamma)(1/\theta-1)}{\bar{\pi}(\alpha+\gamma)(1/\theta-1)+\gamma}$
2	$t_{21}=t_{11}$	$t_{22}=t^{*}-\hat{t}\frac{\beta-\bar{\pi}(\alpha+\gamma)(1/\theta-1)}{\beta-\bar{\pi}\left[(\alpha+\gamma)(1/\theta-1)+(\beta+\gamma)\right]}$	$t_{23}=t^{*}$
3	$t_{31}=t^{*}-\hat{t}\frac{\beta}{\beta+(\alpha-\beta)\bar{\pi}}$	$t_{32}=t^{*}$	-
4	$t_{41}=t^{*}-\hat{t}\frac{\bar{\pi}(1-\theta)(\alpha+\gamma)+\theta\gamma}{\gamma}\left[1-\frac{\bar{\pi}(1-\theta)+\theta}{\alpha/(\alpha-\beta)}\right]$	$t_{42}=t^{*}-\hat{t}\frac{\bar{\pi}(1-\theta)(\alpha+\gamma)+\theta\gamma}{\gamma}\frac{\beta-\bar{\pi}(\alpha+\gamma)(1/\theta-1)}{\alpha}$	-
5	$t_{51}=t^{*}-\hat{t}\frac{\bar{\pi}(\beta+\gamma)/\gamma}{\beta/(\alpha+\gamma)+\bar{\pi}}\left[1-\frac{\bar{\pi}(1-\theta)+\theta}{\alpha/(\alpha-\beta)}\right]$	$t_{52}=t^{*}-\hat{t}\frac{\bar{\pi}(\beta+\gamma)/\gamma}{\beta/(\alpha+\gamma)+\bar{\pi}}\frac{\beta-\bar{\pi}(\alpha+\gamma)(1/\theta-1)}{\beta-\bar{\pi}\left[(\alpha+\gamma)(1/\theta-1)+(\beta+\gamma)\right]}$	-
6	$t_{61}=t^{*}-\hat{t}\frac{(\beta+\gamma)/\gamma}{\beta+(\alpha-\beta)\bar{\pi}}\frac{\beta\bar{\pi}}{\beta/(\alpha+\gamma)+\bar{\pi}}$	-	-

Table 5: Boundary conditions for the seven possible equilibrium departure patterns.

Pattern	Condition
1	$\pi_C < \bar{\pi} \le \pi_S$
2	$\max\{\pi_S, \pi_C\} < \bar{\pi} \le \pi_N$
3	$\bar{\pi} > \max\{\pi_N, \pi_C\}$
4	$\pi_T < \bar{\pi} < \min\{\pi_C, \pi_S\}$
5	$\pi_S \le \bar{\pi} \le \min\{\pi_C, \pi_N\}$
6	$\pi_N < \bar{\pi} \le \pi_C$
7	$\bar{\pi} \le \pi_M$

2.4.5 The rationality of patterns

Now we discuss rationality of patterns. It is clear that if the low cost of a departure time (that is achieved on good condition) is larger than the high cost of any other departure time (that is achieved on bad condition), then no one would select this departure time.

Therefore, for an equilibrium departure pattern, we denote the cost of commuters departing at time *t* under good condition as $C^{\bar{s}}(t)$, and that under bad condition as $C^{\theta\bar{s}}(t)$. Clearly, if

$$\max\left(C^{\bar{s}}(t)\right) > \min\left(C^{\theta\bar{s}}(t')\right), \quad t \in [t_s, t_e], \quad t' \in [t_s, t_e] \tag{31}$$

then the pattern is implausible in reality.

Based on condition (31), we can derive that a pattern is implausible if,

$$\bar{\pi} > 1 \quad \text{or} \quad \pi_T < \bar{\pi} < 0 \quad \text{or} \quad \bar{\pi} \le \pi_M \tag{32}$$

The detailed derivation of the condition (32) is provided in Appendix D.

In other words, Patterns 1b, 2b, 3b, 4b, and 7 are implausible. Under the circumstance, one can expect that travelers might choose departure time based on other principles than TCB defined in Eq. (10). There is another possible explanation of the result, i.e., travelers would not behave so risk loving or risk averse. As mentioned in Assumption A4, travelers are rational and the corresponding λ values of them are within the range $0 < \pi + \lambda\sqrt{\pi(1-\pi)} < 1$. Further behavior data analysis is needed to examine the true values of the behavioral variable λ.

3. The impact of risk attitude

This section compares the results with previous study (Lindsey, 1994). To this end, we investigate the impact of risk attitude on equilibrium patterns, because previous studies correspond to the special case $\lambda =$

0. We have derived the following Propositions and Corollary to account for the impact of risk attitude. It is shown that when $\lambda \neq 0$, the beginning time, end time, and length of rush hour might change, and commuters might experience higher or lower queuing cost and schedule delay cost. The proofs of the Propositions are presented in Appendix E.

Proposition 1. *In Patterns 1-3, the departure times of the first commuter and the last commuter are independent of λ. In Pattern 4, with the increase of λ, the departure times of the first commuter and the last commuter become earlier, however the departure time window $(t_e - t_s)$ is independent of λ. In Patterns 5 and 6, with the increase of λ, the departure time of the first commuter becomes earlier, the departure time of the last commuter does not change, the departure time window increases.*

Proposition 1 concerns with the beginning time, the end time, and the length of rush hour. Note that results in Lindsey (1994) correspond to that of risk neutral commuters.

- In Patterns 1-3, no matter the commuters are risk neutral, risk preferring or risk averse, the beginning time, the end time, and the length of rush hour are the same.
- In Pattern 4, comparing with risk neutral commuters, if the commuters are risk averse/preferring, then rush hour begins and ends earlier/later. However, the length of rush hour is the same.
- In Patterns 5 and 6, if the commuters are risk averse/preferring, then the rush hour begins earlier/later. However, the rush hour ends at the same time. As a result, the rush hour becomes longer/shorter.

Corollary 1. *The travel cost budget is independent of λ in Patterns 1-3, and increases with λ in Patterns 4-6.*

Proof. In Patterns 1-6, from Eq. (21), one has

$$\frac{dB(t)}{d\lambda} = -\beta \frac{dt_s}{d\lambda}$$

From Proposition 1, t_s is independent of λ in Patterns 1-3, and decreases (i.e., becomes earlier) with the increase of λ in Patterns 4-6. Therefore, $\frac{dB(t)}{d\lambda} = 0$ in Patterns 1-3 and $\frac{dB(t)}{d\lambda} > 0$ in Patterns 4-6. This completes the proof. □

Corollary 1 indicates that in Patterns 1-3, the travel cost budget of risk averse or risk preferring commuters is the same as the mean travel cost of risk neutral commuters. However, in Patterns 4-6, the travel cost budget of risk averse/preferring commuters is larger/smaller than the mean travel cost of risk neutral commuters.

Proposition 2. *In Patterns 1-3, the total queuing cost decreases with the increase of λ.*

As an external cost, the queueing cost is related to the wasted time, wasted fuel consumption and additional pollution. In Patterns 1-3, comparing with the risk neutral commuters (Lindsey, 1994), more queueing cost will be incurred for risk preferring commuters[2]. Therefore, to better manage traffic congestion, the risk preferring commuters need to be guided to behave less risk preferring (λ is still negative but $|\lambda|$ decreases), which might be achieved by, e.g., providing pre-trip traffic information.

However, in Patterns 4-6, depending on the parameters, the total queueing cost either increases or decreases with the increase of λ, as shown in the numerical example in Section 5.

Proposition 3. *The total early arrival cost is independent of λ in Pattern 1, decreases with increasing λ*

[2] Note that Patterns 1-3 might include also areas (commuters) with negative value of λ, see Fig.7(b).

in Patterns 2 and 3, while increases with increasing λ *in Patterns 4 and 5. The total late arrival cost increases with increasing* λ *in Patterns 1-3, and decreases with the increasing* λ *in Patterns 4-6.*

Proposition 3 indicates that in Pattern 1, commuters of all risk levels including those of risk neutral have the same total early arrival cost. Comparing to risk neutral commuters (e.g. Lindsey, 1994), In Patterns 2 and 3, the total early arrival cost of risk averse/preferring commuters is smaller/larger than that of risk neutral commuters, while the opposite is true in Patterns 4 and 5. We would like to mention that in Pattern 6, although we cannot prove rigorously, extensive numerical studies indicate that the total early arrival cost also increases with the increase of λ.

Comparing the total late arrival costs of the different risk taking commuters, Proposition 3 suggests that the risk averse/preferring commuters have larger/smaller late arrival costs than that of risk neutral commuters in Patterns 1-3, while the opposite is true in Patterns 4-6.

To some extent, the early cost and late cost can be regarded as utility and disutility from the employers' point of view. Namely, if employees arrive at work earlier/later, they will get more/less work done during the day than they would if they arrived at work on time. Note that the trend that utility (early-arrival cost) changes with λ is always the opposite to that of disutility (late-arrival cost) in Patterns 2-6, and utility does not change in Pattern 1. As a result, one can easily derive the trend of net utility.

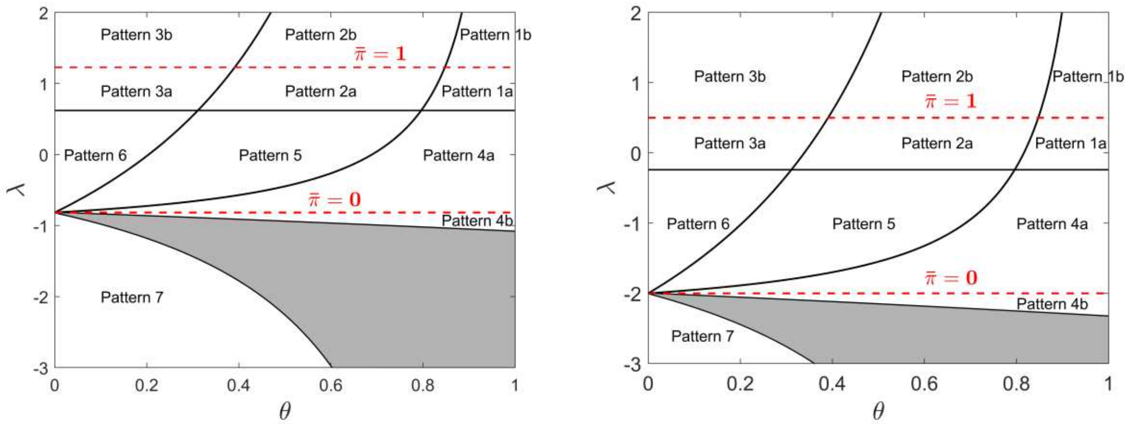
Specifically, comparing with risk neutral commuters (Lindsey, 1994), the net utility decreases/increases if commuters are risk averse/preferring in Patterns 1-3; in contrast, the net utility increases/decreases if commuters are risk averse/preferring in Patterns 4-6. Therefore, from the employers' point of view, they like risk preferring employees in Patterns 1-3, and risk averse employees in Patterns 4-6.

4. Numerical results

To demonstrate the theoretical results presented above, numerical examples are given in this section. Unless otherwise mentioned, the parameters α, β, γ, the design capacity $\bar{s}$, the traffic demand N are the same as in Figure 6. The work start time $t^* = 9$.

Figure 7(a) and 7(b) show the pattern diagram with parameter $\pi = 0.4$ and $\pi = 0.8$, respectively. Note that $\lambda \in [-\infty, +\infty]$, and the diagram shows only $-3 \le \lambda \le 2$. One can see that with the change of λ, a pattern could transform into another pattern. For example, suppose commuters are risk neutral (Lindsey, 1994), then when $\pi = 0.4$, the system might be in Patterns 4a, 5, or 6. However, if commuters are risk averse and $\lambda = 1$, then the system might be in Patterns 1a, 2a, or 3a. Therefore, ignoring the risk attitude might miscalculate the equilibrium pattern.

With the increase of π, the two dashed lines $\bar{\pi} = 1$ and $\bar{\pi} = 0$ move downward. This means that if commuters are risk averse/preferring, then with the increase/decrease of degradation probability of capacity, commuters that originally choose departure time based on TCB might change their departure time choice principle. For example, suppose commuters are risk averse and $\lambda = 1$, then when $\pi = 0.4$, these commuters can choose departure time based on TCB and the system might be in Patterns 1a, 2a, or 3a, see Figure 7(a). However, when π increases to 0.8, then commuters will not choose departure time based on TCB, since the system becomes implausible, see Figure 7(b).



(a) (b)

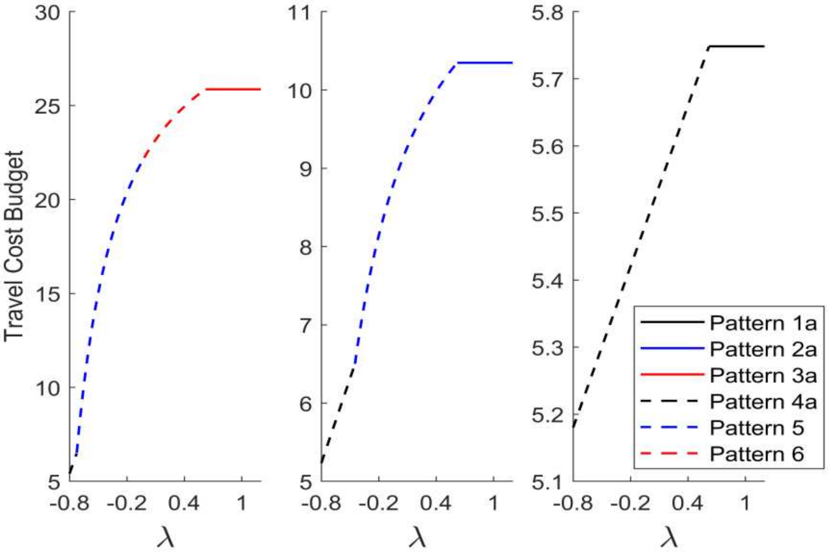
Figure 7. The pattern diagram. (a) $\pi = 0.4$, (b) $\pi = 0.8$

Next, we investigate the impact of parameter λ on travel cost and the duration of peak hour. We only consider the patterns satisfying $0 \le \bar{\pi} \le 1$. Hence, Patterns 1b, 2b, 3b, 4b and 7 are excluded.

Figure 8 shows the variation of travel cost budget with λ under different values of θ and π. It can be seen that given the value of θ, the travel cost budget increases with the increase of λ in Patterns 4-6, but keeps constant in Patterns 1-3. This is consistent with Corollary 1.

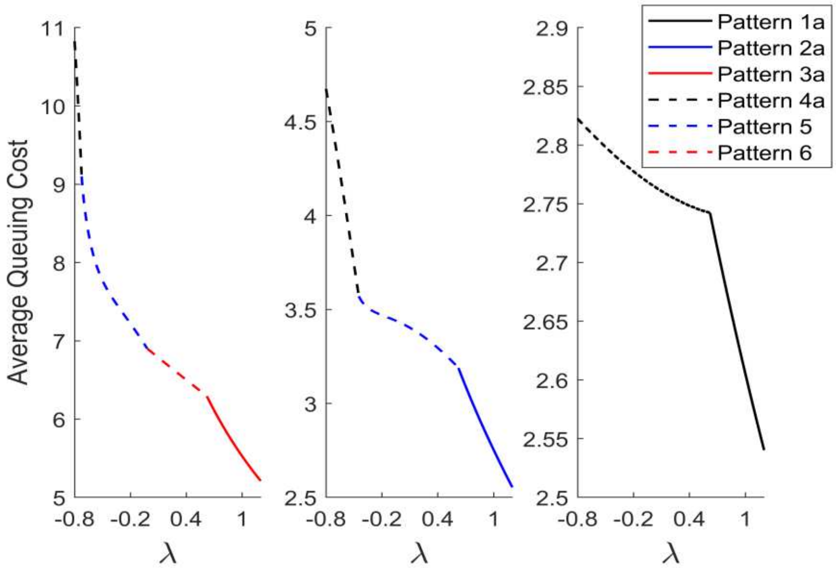
Figure 9 shows the dependence of average queuing cost on λ. One can see that with the increase of λ, the average queuing cost decreases in Patterns 1-3. This is consistent with Proposition 2. However, as mentioned before, in Patterns 4-6, the variation of average queuing cost with λ depends on the parameters. For example, in Figure 9(a)-(c), the average queuing cost decreases with λ in the three patterns. In Figure 9(d), the average queuing cost increases with λ in Pattern 6, and changes nonmonotonically in Pattern 5. In Figure 9(e), the average queuing cost changes nonmonotonically in Pattern 4a.

Figure 10 and 11 present the dependence of average early arrival cost and average late arrival cost on λ, respectively. One can see that except in Pattern 1, the variation of average early cost with λ has opposite trend to that of average late cost. The late cost increases with the increase of λ in Patterns 1-3, and decreases with the increase of λ in Patterns 4-6. In Pattern 1, the average early cost is constant. The numerical results are consistent with Proposition 3. Moreover, as we mentioned before, in Pattern 6, although we cannot prove rigorously, extensive numerical studies indicate that the early arrival cost also increases with the increase of λ.

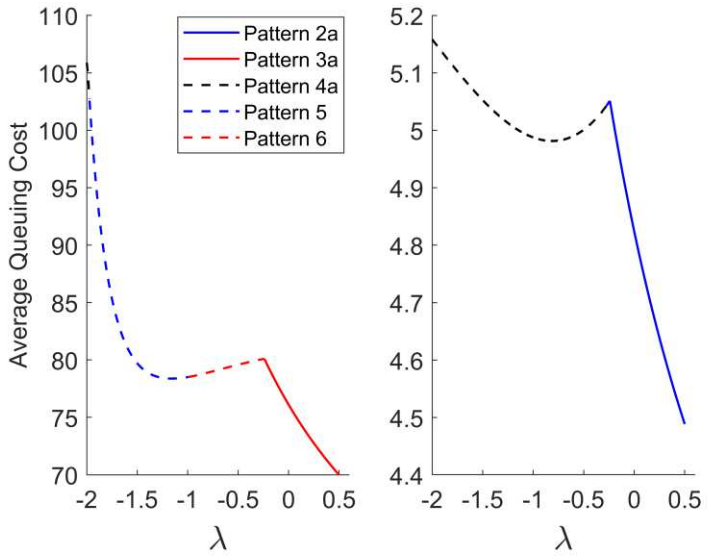


(a) (b) (c)

Figure 8. Travel cost budget, $\pi = 0.4$. (a) $\theta = 0.2$, (b) $\theta = 0.5$, (c) $\theta = 0.9$.

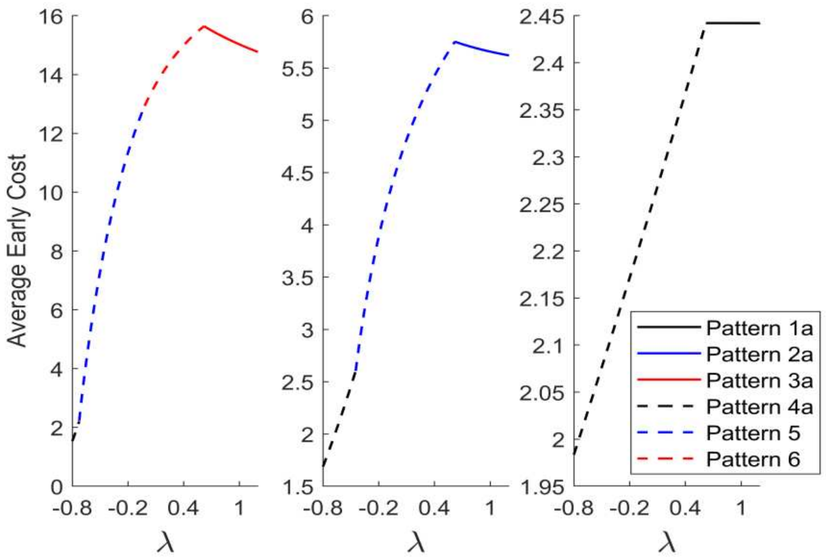


(a) (b) (c)



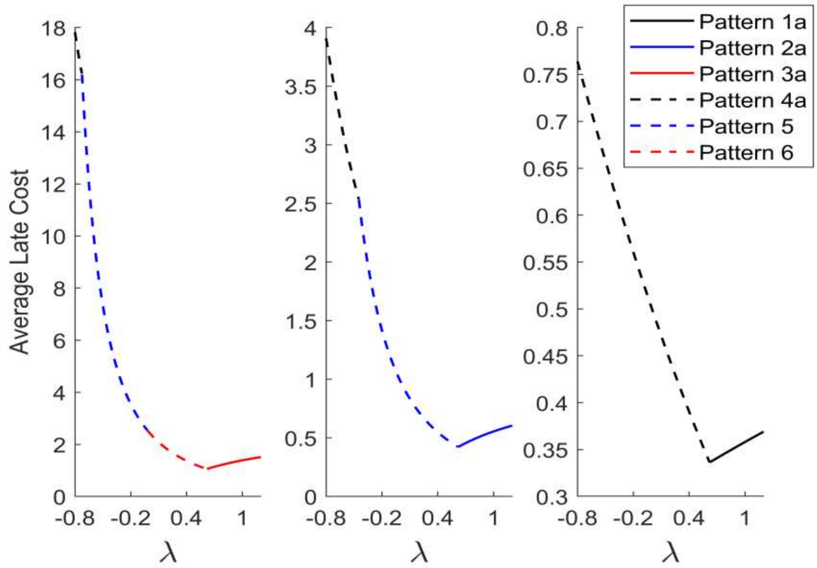
(d) (e)

Figure 9. The average queuing cost. (a)-(c) $\pi = 0.4$. (d)-(e) $\alpha = 6.4$, $\beta = 5.8$, $\gamma = 15.21$, $\pi = 0.8$. (a) $\theta = 0.2$, (b) $\theta = 0.5$, (c) $\theta = 0.9$, (d) $\theta = 0.04$, (e) $\theta = 0.72$.



(a) (b) (c)

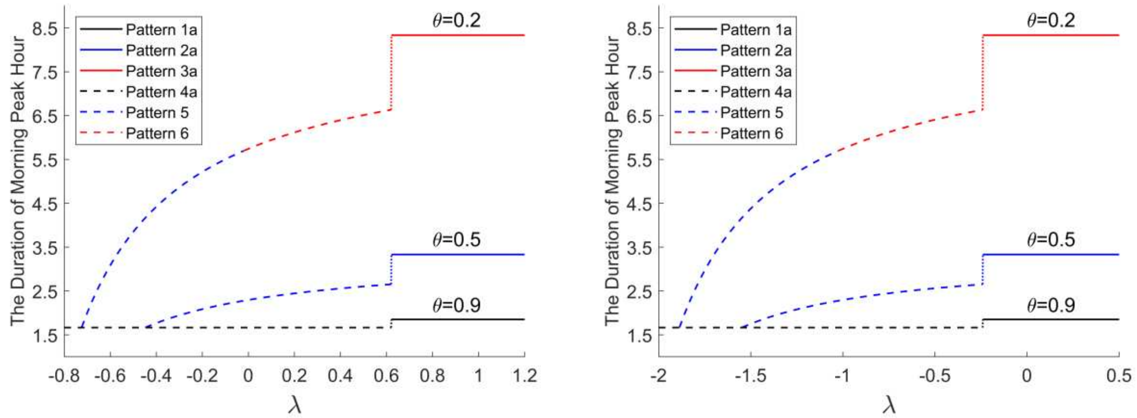
Figure 10. The average early cost, $\pi = 0.4$. (a) $\theta = 0.2$, (b) $\theta = 0.5$, (c) $\theta = 0.9$.



(a) (b) (c)

Figure 11. The average late cost, $\pi = 0.4$. (a) $\theta = 0.2$, (b) $\theta = 0.5$, (c) $\theta = 0.9$.

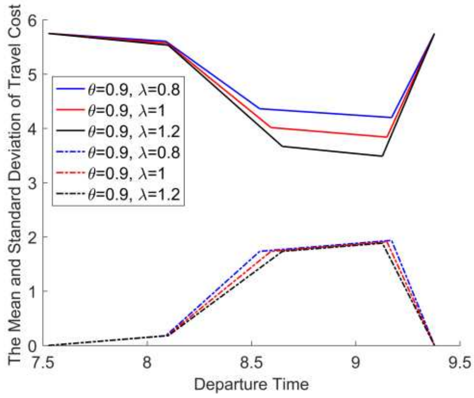
Figure 12 shows the relationship between the duration of peak hour and λ. One can see that, as stated in Proposition 1, the duration of morning peak hour is independent of the risk attitude of commuters in Patterns 1-4, and it increases with λ in Patterns 5 and 6. We also note that at the critical value of λ corresponding to the transition from Patterns 1-3 into Patterns 4-6, the peak hour length suddenly increases. This is because comparing with Patterns 1-3, Situation 4 is absent in Patterns 4-6. Consequently, departure time of the last commuter is significantly earlier than that in Patterns 1-3.



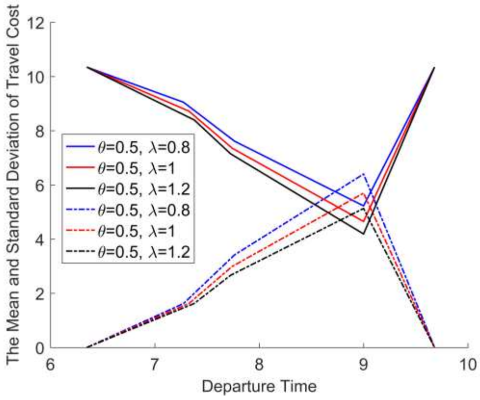
(a) (b)

Figure 12. The duration of morning peak hour. (a) $\pi = 0.4$, (b) $\pi = 0.8$.

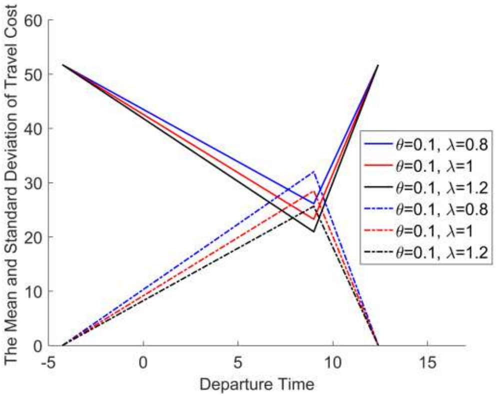
Figure 13 shows examples of the mean travel cost and the standard deviation of travel cost in each pattern. For the risk averse commuters, i.e., $\lambda > 0$, the mean and standard deviation of travel cost have opposite variation trend. For the risk preferring commuters, their variation trend is the same. Note that standard deviation of travel cost is zero for the first commuter and the last commuter in Patterns 1-3. In Patterns 2 and 3, the mean travel cost decreases/increases and the standard deviation of travel cost increases/decreases with the departure time before/after the work start time. In Pattern 7, both the mean and standard deviation of travel cost are zero for the first commuter. Thus, the travel cost budget of commuters equals zero. The mean and standard deviation of the travel cost are larger than zero after work start time, and increases linearly.



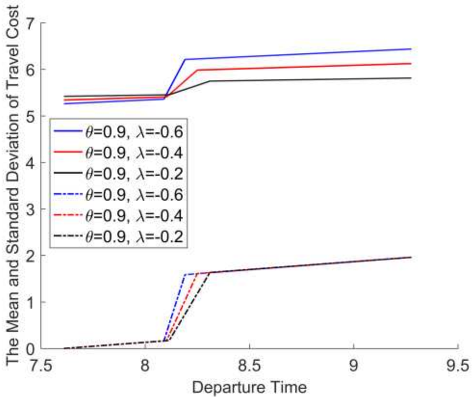
(a)



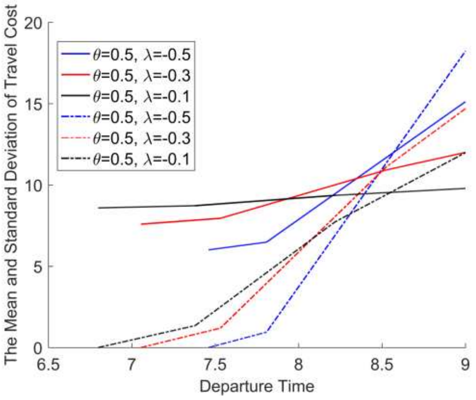
(b)



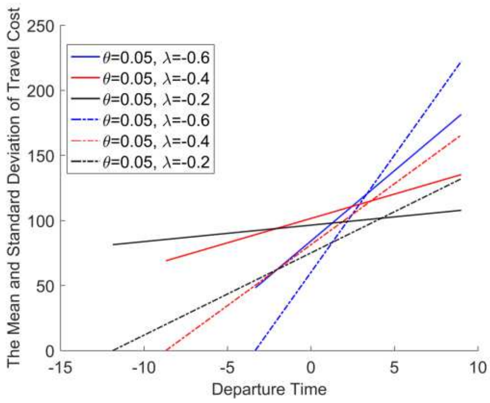
(c)



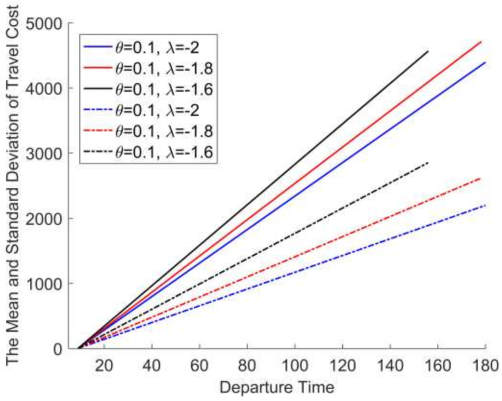
(d)



(e)



(f)



(g)

Figure 13. The mean (solid lines) and standard deviation of travel cost (dashed lines) in each pattern. (a)-(g) corresponds to Patterns 1-7. $\pi = 0.4$.

6. Conclusion

In this paper, we develop a bottleneck model in which the capacity of bottleneck is constant within a day but changes stochastically from day-to-day. The study relates the travel cost variability due to stochastic capacity with commuters' departure time choice behaviors. We assume that commuters acquire the variability of travel cost based on past experiences and they factor such variability into their departure time choice consideration by minimizing their travel cost budget, which is defined as weighted average of mean and standard deviation of travel cost. It is found that there exist seven possible equilibrium patterns. All possible equilibrium departures are analyzed theoretically. The implausible parameter range was also derived, which is related to the capacity degradation probability, the degradation ratio, and the risk attitude of commuters, but independent of the designed capacity.

Furthermore, we study the impact of risk attitude both analytically and numerically. It is shown that depending on the commuters' risk attitude and the equilibrium pattern, (i) the beginning time, end time, and length of rush hour might change. For instance, the duration of peak hour is independent of commuters' risk attitude in Patterns 1-4. However, the duration of peak hour increases when commuters become risk averse in Patterns 5 and 6. (ii) commuters might experience higher or lower queuing cost and schedule delay cost.

Our studies indicate that the risk attitude plays an important role in determining the equilibrium pattern and the travel cost. It is, therefore, important to capture the standard deviation of travel cost so as to better understand the choice behavior of commuters under the stochastic circumstance.

The proposed stochastic bottleneck model can be extended in various directions in the future. Firstly, the bivariate capacity needs to be extended to consider general distribution of stochastic capacity. Actually, we have studied continuous distribution of stochastic capacity and trivariate capacity. However, unfortunately, the closed form solution of equilibrium patterns cannot be derived. Secondly, in real transportation system, the commuters are heterogeneous from perspective of value of time, risk attitude, etc., therefore, it is interesting to study commuter heterogeneity in the stochastic bottleneck model (Xiao et al., 2014). Apart from the supply side, there are uncertainties from the demand side (Sun et al., 2011). To study the influence of the uncertainty from both supply side and demand side synthetically is another interesting topic. The model can also be extended to study the toll pricing and tradeable credit scheme (Lindsey et al. 2012; Nie et al., 2013), the multiple transportation modes (Huang, 2000), the staggered work hour settings (Chu et al., 2005; Zhu et al., 2019), the ridesharing (Ma et al., 2017; Liu et al. 2017), the parking management (Liu. 2018; Zhang et al. 2019), and so on.

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Appendix A. Derivation of the departure rate in each situation

There are six situations and thus six departure rates. In Section 2.5.1, we have provided derivation of departure rate in the first situation. The derivations of other five departure rates are provided here.

(S2) Commuters possibly experience schedule delay either early or late, and always experience queuing (SDE/L+AQ)

In this situation, whether commuters arrive at destination early or late depends on the capacity of bottleneck, but they always experience queuing. From (3), commuters experience schedule delay early if $T(t)+t=R(t)/s-(t-t_s)+t\le t^*$. That is to say, when the condition of bottleneck is good with a large capacity $\overline{s}$, commuters always experience schedule delay early; otherwise, they will experience schedule delay late. The travel cost can be formulated as follows:

$$C(t)=\begin{cases}\alpha\left(\dfrac{R(t)}{s}-t+t_s\right)+\gamma\left(\dfrac{R(t)}{s}+t_s-t^*\right),\ s=\theta\overline{s}\\[2ex] \alpha\left(\dfrac{R(t)}{s}-t+t_s\right)+\beta\left(t^*-t_s-\dfrac{R(t)}{s}\right),\ s=\overline{s}\end{cases}\tag{A.1}$$

According to the expression of travel cost (16), the mean travel cost and the standard deviation of travel cost can be formulated as follows, respectively:

$$E(C(t))=\alpha R(t)\left(\frac{\pi}{\theta\overline{s}}+\frac{1-\pi}{\overline{s}}\right)-\alpha(t-t_s)+\pi\gamma\left(\frac{R(t)}{\theta\overline{s}}+t_s-t^*\right)+(1-\pi)\beta\left(t^*-t_s-\frac{R(t)}{\overline{s}}\right)\tag{A.2}$$

$$\sigma(C(t))=\sqrt{\pi(1-\pi)}\left[\alpha R(t)\left(\frac{1}{\theta\overline{s}}-\frac{1}{\overline{s}}\right)+\gamma\left(\frac{R(t)}{\theta\overline{s}}+t_s-t^*\right)-\beta\left(t^*-t_s-\frac{R(t)}{\overline{s}}\right)\right]\tag{A.3}$$

Substituting (A.2) and (A.3) into the expression of travel cost budget (10) and using the equilibrium condition (11), the equilibrium departure rate can be obtained for this situation, $r_2(t)=\theta\overline{s}\left(\alpha/(\alpha-\beta)\right)\Big/\left(\left((\alpha+\gamma)/(\alpha-\beta)-\theta\right)\overline{\pi}+\theta\right)$.

(S3) Commuters experience schedule delay late and always experience queuing (SDL+AQ)

In this situation, no matter how the capacity varies, all commuters experience schedule delay late and always queuing. The travel cost can be formulated as follows:

$$C(t)=\alpha\left(\frac{R(t)}{s}-t+t_s\right)+\gamma\left(\frac{R(t)}{s}+t_s-t^*\right),\quad s=\theta\overline{s}\ \text{ or }\ \overline{s}\tag{A.4}$$

According to the expression of travel cost (16), the mean travel cost and the standard deviation of travel cost can be formulated as follows, respectively:

$$E(C(t))=(\alpha+\gamma)R(t)\left(\frac{\pi}{\theta\overline{s}}+\frac{1-\pi}{\overline{s}}\right)+\alpha(t_s-t)+\gamma(t_s-t^*)\tag{A.5}$$

$$\sigma(C(t))=(\alpha+\gamma)\sqrt{\pi(1-\pi)}\left(\frac{1-\theta}{\theta\overline{s}}\right)R(t)\tag{A.6}$$

Substituting (A.5) and (A.7) into the expression of travel cost budget (10) and using the equilibrium condition (11), the equilibrium departure rate can be obtained, $r_3(t)=\theta\overline{s}\left(\alpha/(\alpha+\gamma)\right)\Big/\left((1-\theta)\overline{\pi}+\theta\right)$.

(S4) Commuters experience schedule delay late and possibly experience queuing (SDL+PQ)

In this situation, commuters arrive at the destination always late, but may experience queuing depending on the capacity of bottleneck. Based on Eq. (3), commuters experience queuing if $T(t) = R(t)/s - (t - t_s) > 0$. That is to say, when the bottleneck is in bad condition with capacity $\theta\overline{s}$, commuters always experience queuing; otherwise, they will not experience queuing. The travel cost can be formulated as follows:

$$C(t) = \begin{cases} \alpha\left(\dfrac{R(t)}{s} - t + t_s\right) + \gamma\left(\dfrac{R(t)}{s} + t_s - t^*\right), & s = \theta\overline{s} \\ \gamma\left(t - t^*\right), & s = \overline{s} \end{cases} \tag{A.7}$$

According to the expression of travel cost (A.9), the mean travel cost and the standard deviation of travel cost can be formulated as follows, respectively:

$$E(C(t)) = \pi\left[\alpha\left(\frac{R(t)}{\theta\overline{s}} - t + t_s\right) + \gamma\left(\frac{R(t)}{\theta\overline{s}} + t_s - t^*\right)\right] + \gamma(1-\pi)\left(t - t^*\right) \tag{A.8}$$

$$\sigma\left(C(t)\right) = \sqrt{\pi(1-\pi)}\left[\alpha\left(\frac{R(t)}{\theta\overline{s}} - t + t_s\right) + \gamma\left(\frac{R(t)}{\theta\overline{s}} + t_s - t^*\right) - \gamma\left(t - t^*\right)\right] \tag{A.9}$$

Substituting (A.8) and (A.9) into the expression of travel cost budget (10) and using the equilibrium condition (11), the equilibrium departure rate can be obtained for this situation $r_4(t) = \theta\overline{s}\left(1 - \gamma/\left((\alpha+\gamma)\overline{\pi}\right)\right)$.

(S5) Commuters possibly experience schedule delay either early or late, and possibly experience queuing (SDE/L+PQ)

In this situation, all commuters may experience schedule delay early or late, and may experience queuing depending on the capacity of bottleneck. If the condition of bottleneck is good with a large value $\overline{s}$, all commuters arrive at the destination early and do not experience queuing. On the other hand, when the condition of bottleneck is bad with a small value of capacity $\theta\overline{s}$, all commuters arrive at the destination late and always experience queuing. The travel cost can be formulated as follows:

$$C(t) = \begin{cases} \alpha\left(\dfrac{R(t)}{s} - t + t_s\right) + \gamma\left(\dfrac{R(t)}{s} + t_s - t^*\right), & s = \theta\overline{s} \\ \beta\left(t^* - t\right), & s = \overline{s} \end{cases} \tag{A.10}$$

According to the expression of travel cost (A.10), the mean travel cost and the standard deviation of travel cost can be formulated as follows, respectively:

$$E\left(C(t)\right) = \pi\left[\alpha\left(\frac{R(t)}{\theta\overline{s}} - t + t_s\right) + \gamma\left(\frac{R(t)}{\theta\overline{s}} + t_s - t^*\right)\right] + (1-\pi)\beta\left(t^* - t\right) \tag{A.11}$$

$$\sigma\left(C(t)\right) = \sqrt{\pi(1-\pi)}\left[\alpha\left(\frac{R(t)}{\theta\overline{s}} - t + t_s\right) + \gamma\left(\frac{R(t)}{\theta\overline{s}} + t_s - t^*\right) - \beta\left(t^* - t\right)\right] \tag{A.12}$$

Substituting (A.11) and (A.12) into the expression of travel cost budget (10) and using the equilibrium condition (11), the equilibrium departure rate can be obtained for this situation $r_5(t) = \theta\overline{s}\left((\alpha-\beta)/(\alpha+\gamma) + \beta/\left((\alpha+\gamma)\overline{\pi}\right)\right)$.

(S6) Commuters experience schedule delay early, and possibly experience queuing (SDE+PQ)

In this situation, commuters arrive at the destination always early, but may experience queuing depending on the capacity of bottleneck. From (3), commuters experience queuing if $T(t)=R(t)/s-(t-t_s)>0$. That is to say, when the capacity is in bad condition with value $\theta\overline{s}$, commuters always experience queuing. Otherwise, they will not experience queuing. The travel cost can be formulated as follows:

$$C(t)=\begin{cases}\alpha\left(\dfrac{R(t)}{s}-t+t_s\right)+\beta\left(t^*-t_s-\dfrac{R(t)}{s}\right), & s=\theta\overline{s}\\ \beta\left(t^*-t\right), & s=\overline{s}\end{cases} \tag{A.13}$$

According to the expression of travel cost (A.13), the mean travel cost and the standard deviation of travel cost could be formulated as follows, respectively:

$$E\left(C(t)\right)=\pi\left[\alpha\left(\frac{R(t)}{\theta\overline{s}}-t+t_s\right)+\beta\left(t^*-t_s-\frac{R(t)}{\theta\overline{s}}\right)\right]+(1-\pi)\beta\left(t^*-t\right) \tag{A.14}$$

$$\sigma\left(C(t)\right)=\sqrt{\pi(1-\pi)}\left[\alpha\left(\frac{R(t)}{\theta\overline{s}}-t+t_s\right)+\beta\left(t^*-t_s-\frac{R(t)}{\theta\overline{s}}\right)-\beta\left(t^*-t\right)\right] \tag{A.15}$$

Substituting (A.14) and (A.15) into the expression of travel cost budget (10) and using the equilibrium condition (11), the equilibrium departure rate can be obtained for this situation $r_6(t)=\theta\overline{s}\left(\beta/\left((\alpha-\beta)\overline{\pi}\right)+1\right)$.

Appendix B. Derivation of the critical time points in each pattern

B.1. Derivation of the critical time points in Pattern 1

As shown in Figure 1, the boundary condition for time instant t_{11} is that commuters who depart at t_{11} arrive at work start time under the degraded capacity $\theta\overline{s}$. Therefore, the cumulative departures,

$$R\left(t_{11}\right)=r_1\left(t\right)\left(t_{11}-t_s\right)=\theta\overline{s}\left(t^*-t_s\right) \tag{B.1}$$

which yields

$$t_{11}=t_s+\frac{\alpha-\beta}{\alpha}\left[(1-\theta)\overline{\pi}+\theta\right]\left(t^*-t_s\right) \tag{B.2}$$

As shown in Figure 1, the boundary condition for time instant t_{12} is that commuters who depart at t_{12} arrive at work start time under the capacity $\overline{s}$. Therefore, the cumulative departures

$$R\left(t_{12}\right)=\overline{s}\left(t^*-t_s\right) \tag{B.3}$$

Substituting (A.2) and (A.3) into the expression of travel cost budget (10) at time instant t_{12}, and substituting (B.3) into the expression of travel cost budget, we have

$$R(t_{12})=(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)(t^{*}-t_{s})\bar{\pi}+\alpha(t^{*}-t_{12}) \tag{B.4}$$

Since condition (21) always holds in all patterns, i.e., $B(t_{12})=\beta(t^{*}-t_{s})$. Substituting (B.3) into this equation, we can derive

$$t_{12}=t^{*}-\frac{(t^{*}-t_{s})}{\alpha}\left[\beta-(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)\bar{\pi}\right] \tag{B.5}$$

As shown in Figure 1, the boundary condition for time instant t_{13} is that commuters who depart at t_{13} arrive at destination immediately under the capacity $\bar{s}$. Therefore, the cumulative departures

$$R(t_{13})=\bar{s}(t_{13}-t_{s}) \tag{B.6}$$

Similarly, substituting (B.6) into the expression of travel cost budget at time instant t_{13}, we have

$$B(t_{13})=(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)(t_{13}-t_{s})\bar{\pi}+\gamma(t_{13}-t^{*}) \tag{B.7}$$

Substituting (B.7) into (21), we can derive,

$$t_{13}=\frac{(\beta+\gamma)t^{*}+\left[(\alpha+\gamma)(1/\theta-1)\bar{\pi}-\beta\right]t_{s}}{(\alpha+\gamma)(1/\theta-1)\bar{\pi}+\gamma} \tag{B.8}$$

Similarly, substituting $R(t_{e})=N$ into the expression of travel cost budget at time instant t_{e}, we have

$$B(t_{e})=(\alpha+\gamma)\left(\frac{N}{\theta\bar{s}}-t_{e}+t_{s}\right)\bar{\pi}+\gamma(t_{e}-t^{*}) \tag{B.9}$$

The equilibrium condition of the bottleneck model implies $B(t_{e})=\beta(t^{*}-t_{s})$. Substituting Eq. (B.9) into this equation, one can derive

$$t_{s}=t^{*}-\frac{\gamma N}{\theta\bar{s}(\gamma+\beta)} \tag{B.10}$$

Substituting (B.10) into (B.2), (B.5), (B.8) and (22), we can obtain the critical time points respectively, given as follows:

$$t_{11}=t^{*}-\frac{\gamma N}{\theta\bar{s}(\gamma+\beta)}+\frac{\gamma N}{\theta\bar{s}(\gamma+\beta)}\frac{(\alpha-\beta)}{\alpha}\left[(1-\theta)\bar{\pi}+\theta\right] \tag{B.11}$$

$$t_{12}=t^{*}-\frac{\gamma N}{\theta\bar{s}(\gamma+\beta)}\frac{\left[\beta-(\alpha+\gamma)(1/\theta-1)\bar{\pi}\right]}{\alpha} \tag{B.12}$$

$$t_{13}=t^{*}+\frac{\gamma N}{\theta\bar{s}(\gamma+\beta)}\left[\frac{\beta-(\alpha+\gamma)(1/\theta-1)\bar{\pi}}{(\alpha+\gamma)(1/\theta-1)\bar{\pi}+\gamma}\right] \tag{B.13}$$

$$t_e = t^* + \frac{\beta N}{\theta \bar{s}(\gamma+\beta)} \tag{B.14}$$

B.2. Derivation of the critical time points in Pattern 2

In Pattern 2, the critical time points t_s, t_{21} and t_e follow Eqs. (B.9), (B.11) and (B.14), respectively, except that t_{11} is replaced by t_{21}. As shown in Figure 2, the boundary condition for time instant t_{22} is that commuters who depart at t_{22} arrive at destination immediately under the capacity $\bar{s}$. Therefore, the cumulative departures

$$R(t_{22}) = \bar{s}(t_{22} - t_s) \tag{B.15}$$

Substituting (A.11) and (A.12) into the expression of travel cost budget (10) at time instant t_{22}, and substituting (B.15) into the expression of travel cost budget, we have

$$B(t_{22}) = \bar{\pi}\left[(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)(t_{22}-t_s) - (\gamma+\beta)(t^*-t_{22})\right] + \beta(t^*-t_{22}) \tag{B.16}$$

Substituting (B.16) into (21), we can derive

$$t_{22} = t^* - \frac{\gamma N}{\theta \bar{s}(\gamma+\beta)} \frac{\beta - (\alpha+\gamma)(1/\theta - 1)\bar{\pi}}{\beta - (\alpha+\gamma)(1/\theta-1)\bar{\pi} - (\beta+\gamma)\bar{\pi}} \tag{B.17}$$

As shown in Figure 2, $t_{23} = t^*$.

B.3. Derivation of the critical time points in Pattern 3

As shown in Figure 3, t_s and t_e are the same as Pattern 1, following Eqs. (B.9) and (B.14), respectively. The boundary condition for time instant t_{31} is that commuters who depart at t_{31} arrive at work start time under the degraded capacity $\theta \bar{s}$. Therefore, the cumulative departures

$$R(t_{31}) = \theta \bar{s}(t^* - t_s) \tag{B.18}$$

Substituting (A.14) and (A.15) into the expression of travel cost budget (10) at time instant t_{31}, and substituting (B.18) into the expression of travel cost budget, we have

$$B(t_{31}) = \bar{\pi}(\alpha-\beta)(t^*-t_{31}) + \beta(t^*-t_{31}) \tag{B.19}$$

Substituting (B.19) into (21), we can derive

$$t_{31} = t^* - \frac{\gamma N}{\theta \bar{s}(\gamma+\beta)}\left(\frac{\beta}{\beta+(\alpha-\beta)\bar{\pi}}\right) \tag{B.20}$$

As shown in Figure 3, $t_{32} = t^*$.

B.4. Derivation of the critical time points in Pattern 4

As shown in Figure 4, the boundary condition for time instant t_{41} in Pattern 4 is the same as Eq.(B.2) in Pattern 1, except that t_{11} is replaced by t_{41}. Therefore, we can obtain the expression of t_{41}, given as follows:

$$t_{41} = t_s + \frac{\alpha - \beta}{\alpha}\left[(1-\theta)\bar{\pi} + \theta\right]\left(t^* - t_s\right) \tag{B.21}$$

As shown in Figure 4, the boundary condition for time instant t_{42} in Pattern 4 is the same as Eq.(B.5) in Pattern 1, except that t_{12} is replaced by t_{42}. Therefore, we can obtain the expression of t_{42}, given as follows:

$$t_{42} = t^* - \frac{\left(t^* - t_s\right)}{\alpha}\left[\beta - (\alpha + \gamma)\left(\frac{1}{\theta} - 1\right)\bar{\pi}\right] \tag{B.22}$$

Substituting (A.5) and (A.6) into the expression of travel cost budget (10) at time instant t_e, and substituting $R\left(t_e\right) = N$ into the expression of travel cost budget, we have

$$B\left(t_e\right)=\bar{\pi}(\alpha+\gamma)N\frac{1-\theta}{\theta\bar{s}} + \frac{(\alpha+\gamma)N}{\bar{s}} - \alpha\left(t_e - t_s\right) + \gamma\left(t_s - t^*\right) \tag{B.23}$$

The equilibrium condition of the bottleneck model implies $B\left(t_e\right) = \beta\left(t^* - t_s\right)$. Substituting Eq. (B.23) into this equation, we have

$$t_s = t^* - N\frac{(\alpha+\gamma)(1/\theta - 1)\bar{\pi} + \gamma}{\bar{s}(\beta+\gamma)} \tag{B.24}$$

Substituting (B.24) into (B.21), (B.22) and (25), we can obtain the critical time points respectively, given as follows:

$$t_{41} = t^* - N\frac{(\alpha+\gamma)\bar{\pi}(1-\theta) + \theta\gamma}{\theta\bar{s}(\beta+\gamma)}\left[1 - \frac{\theta + (1-\theta)\bar{\pi}}{\alpha/(\alpha-\beta)}\right] \tag{B.25}$$

$$t_{42} = t^* - N\frac{(\alpha+\gamma)(1/\theta - 1)\bar{\pi} + \gamma}{\alpha\bar{s}(\beta+\gamma)}\left[\beta - (\alpha+\gamma)\left(\frac{1}{\theta} - 1\right)\bar{\pi}\right] \tag{B.26}$$

$$t_e = t^* + N\frac{\beta - (\alpha+\gamma)(1/\theta - 1)\bar{\pi}}{\bar{s}(\beta+\gamma)} \tag{B.27}$$

B.5. Derivation of the critical time points in Pattern 5

As shown in Figure 5(a), we have $t_e = t^*$ in Pattern 5. Substituting (29) and (30) into the expression of travel cost budget (10) at time instant t_e, and substituting $R(t_e) = N$ into the expression of travel cost budget, we have

$$B(t_e) = \bar{\pi}(\alpha+\gamma)\left(\frac{N}{\theta\bar{s}} - t^* + t_s\right) \tag{B.28}$$

The equilibrium condition of the bottleneck model implies $B(t_e) = \beta(t^* - t_s)$. Substituting Eq. (B.28) into this equation, we have

$$t_s = t^* - \frac{N}{\theta\bar{s}} \frac{(\alpha+\gamma)\bar{\pi}}{\beta+(\alpha+\gamma)\bar{\pi}} \tag{B.29}$$

The critical time point t_{51} in Pattern 5 follows Eq. (B.2), except that t_{11} is replaced by t_{51}. Substituting (B.29) into that expression, we have

$$t_{51} = t^* - \frac{N}{\theta\bar{s}} \frac{(\alpha+\gamma)\bar{\pi}}{\beta+(\alpha+\gamma)\bar{\pi}} \left[1 - \frac{\theta+(1-\theta)\bar{\pi}}{\alpha/(\alpha-\beta)}\right] \tag{B.30}$$

The critical time point t_{52} in Pattern 5 follows Eq. (B.17), except that t_{22} is replaced by t_{52}. Therefore, we have

$$t_{52} = t^* - \frac{N}{\theta\bar{s}} \frac{(\alpha+\gamma)\bar{\pi}}{\beta+(\alpha+\gamma)\bar{\pi}} \frac{\beta-(\alpha+\gamma)(1/\theta-1)\bar{\pi}}{\beta-(\alpha+\gamma)(1/\theta-1)\bar{\pi}-(\beta+\gamma)\bar{\pi}} \tag{B.31}$$

B.6. Derivation of the critical time points in Pattern 6

t_s and t_e in Pattern 6 are the same as in Pattern 5. Similar to Pattern 3, t_{61} in Pattern 6 follows Eq. (B.20), except that t_{31} is replaced with t_{61}. Therefore, we have

$$t_{61} = t^* - \frac{N}{\theta\bar{s}} \frac{\bar{\pi}\beta/(\alpha-\beta)}{\left[(\beta/\alpha+\gamma)+\bar{\pi}\right]\left[(\beta/\alpha-\beta)+\bar{\pi}\right]} \tag{B.32}$$

Appendix C. Derivation of the boundary conditions in each pattern

C.1. Derivation of boundary condition in Pattern 1

By definition, the first condition $r_1(t) \geq \bar{s}$ must be satisfied in Pattern 1. Substituting (15) into this condition, we have

$$-\frac{\theta}{1-\theta} < \bar{\pi} \le \frac{\beta\theta}{(\alpha-\beta)(1-\theta)} \tag{C.1}$$

The second condition $t_{12} \le t_{13}$ must be satisfied in Pattern 1. Substituting (B.12) and (B.13) into this condition, we have

$$\frac{-\gamma\theta}{(\alpha+\gamma)(1-\theta)} < \bar{\pi} \le \frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.2}$$

The third condition $t_{13} \le t_e$ must be satisfied in Pattern 1. Substituting (B.13) and (B.14) into this condition, we have

$$0 \le \bar{\pi} \le \frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.3}$$

The last condition is to ensure that all departure rates in Pattern 1 are positive. That is to say, (15), (16), (17) and (18) must be positive at the same time, thus we have

$$\bar{\pi} > \frac{\gamma}{\alpha+\gamma} \tag{C.4}$$

Note that because of $0 < \beta < \alpha < \gamma$ and $0 < \theta \le 1$, we have $\beta\theta/((\alpha-\beta)(1-\theta)) > \beta\theta/(\alpha+\gamma)(1-\theta)$.

Therefore, the boundary condition for Pattern 1 can be obtained as the intersection of above four conditions, which reads:

$$\frac{\gamma}{\alpha+\gamma} < \bar{\pi} \le \frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.5}$$

C.2. Derivation of boundary condition in Pattern 2

By definition, the first condition (C.1) can also be applied in Pattern 2. The second condition $t_{12} > t_{13}$ must be satisfied to separate Patterns 1 and 2. Then we have the mutual exclusion condition of (C.2), given as follows:

$$\bar{\pi} > \frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.6}$$

The third condition $t_{22} \le t^*$ must be met in Pattern 2. Then we have

$$\bar{\pi} < \frac{\beta\theta}{(\alpha+\gamma)-\theta(\alpha-\beta)} \quad \text{or} \quad \bar{\pi} > \frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.7}$$

Note that because of $0 < \beta < \alpha < \gamma$ and $0 < \theta \le 1$, we have $\beta\theta/(\alpha-\beta)(1-\theta) > \beta\theta/(\alpha+\gamma)(1-\theta) > \beta\theta/(\alpha+\gamma)-\theta(\alpha-\beta)$.

The last condition is to ensure that all departure rates in Pattern 2 are positive. Then we have

$$\bar{\pi} > \frac{\gamma}{\alpha+\gamma} \tag{C.8}$$

Combining above four conditions, the boundary condition of Pattern 2 can be obtained, given as follows:

$$\max\left\{\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)},\frac{\gamma}{\alpha+\gamma}\right\}<\bar{\pi}\le\frac{\beta\theta}{(\alpha-\beta)(1-\theta)} \tag{C.9}$$

C.3. Derivation of boundary condition in Pattern 3

By definition, the first condition $r_1(t)<\bar{s}$ must be met in Pattern 3. Then the mutual exclusion condition of (C.1) can be applied here, given as follows:

$$\bar{\pi}>\frac{\beta\theta}{(\alpha-\beta)(1-\theta)} \text{ or } \bar{\pi}<-\frac{\theta}{1-\theta} \tag{C.10}$$

The second condition $t_{12}>t_{13}$ and third condition $t_{22}\le t^*$ should also be met in Pattern 3. In other words, (C.6) and (C.7) can also be applied here. Note that the first condition must be satisfied, otherwise Pattern 2 will happen.

The last condition is to ensure that all departure rates in Pattern 3 are positive. Then we have

$$\bar{\pi}>\frac{\gamma}{\alpha+\gamma} \tag{C.11}$$

Combining above four conditions, the boundary condition of Pattern 3 can be obtained, given as follows:

$$\bar{\pi}>\max\left\{\frac{\beta\theta}{(\alpha-\beta)(1-\theta)},\frac{\gamma}{\alpha+\gamma}\right\} \tag{C.12}$$

C.4. Derivation of boundary condition in Pattern 4

By definition, the first condition (C.1) can also be applied in Pattern 4. The second condition is to ensure that the last departure time t_e in Pattern 4 must be later than work start time, i.e., $t_e>t^*$. Then we have

$$\bar{\pi}<\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.13}$$

The difference between Patterns 1 and 4 is the existence of Situation 4. Therefore, condition $r_4(t)\le 0$ must be satisfied, otherwise we have Pattern 1. The third condition is given as follows:

$$\bar{\pi}\le\frac{\gamma}{\alpha+\gamma} \tag{C.14}$$

The last condition is to ensure that all departure rates in Pattern 4 are positive. Then we have

$$\bar{\pi}>\frac{-\theta}{(\alpha+\gamma)/(\alpha-\beta)-\theta} \tag{C.15}$$

Combining above four conditions, the boundary condition of Pattern 4 can be obtained, given as follows:

$$-\frac{\theta}{(\alpha+\gamma)/(\alpha-\beta)-\theta}<\bar{\pi}<\min\left\{\frac{\gamma}{\alpha+\gamma},\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)}\right\} \tag{C.16}$$

C.5. Derivation of boundary condition in Pattern 5

By definition, the first condition (C.1) can also be applied in Pattern 5. The second condition is the mutual exclusion condition of (C.13) to separate Patterns 5 and 4. Then we have

$$\bar{\pi}\geq\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)} \tag{C.17}$$

The third condition (C.14) must also be satisfied, otherwise we have Pattern 2. The last condition is to ensure that all departure rates in Pattern 5 are positive. Then we have

$$\bar{\pi}>0 \tag{C.18}$$

Combining above four conditions, the boundary condition of Pattern 5 can be obtained, given as follows:

$$\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)}\leq\bar{\pi}\leq\min\left\{\frac{\gamma}{\alpha+\gamma},\frac{\beta\theta}{(\alpha-\beta)(1-\theta)}\right\} \tag{C.19}$$

C.6. Derivation of boundary conditions in Pattern 6

By definition, the first condition (C.10) can be applied in Pattern 6. The second condition (C.14) must also be satisfied, otherwise we have Pattern 3. The last condition is to ensure that all departure rates in Pattern 6 are positive. Then we have

$$\bar{\pi}>0 \tag{C.20}$$

Thus, the boundary condition of Pattern 6 can be obtained, given as follows:

$$\frac{\beta\theta}{(\alpha-\beta)(1-\theta)}<\bar{\pi}\leq\frac{\gamma}{\alpha+\gamma} \tag{C.21}$$

C.7. Derivation of boundary condition in Pattern 7

Pattern 7 consists of only Situation 4. This means that all commuters in this pattern always arrive late except the first commuter, but possibly experience queuing. Thus, the condition $\theta\bar{s}\leq r_4(t)\leq\bar{s}$ should be met, given as follows:

$$\bar{\pi}\leq-\frac{\gamma}{(\alpha+\gamma)}\frac{\theta}{(1-\theta)} \tag{C.22}$$

Appendix D. Derivation of the rationality of these patterns

In Pattern 1, the travel cost in each situation can be given as (12), (A.1), (A.4) and (A.7), respectively. Substituting $s=\bar{s}$ into cost function, and differentiating the cost function in each situation with respect to t, one has

$$\frac{dC(t)}{dt}=\begin{cases}-\dfrac{\alpha(1-\theta)\bar{\pi}}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_s,t_{11}]\\ -\dfrac{\alpha[(\alpha+\gamma)/(\alpha-\beta)-\theta]\bar{\pi}}{[(\alpha+\gamma)/(\alpha-\beta)-\theta]\bar{\pi}+\theta}, & t\in[t_{11},t_{12}]\\ -\dfrac{\alpha(1-\theta)\bar{\pi}}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_{12},t_{13}]\\ \gamma, & t\in[t_{13},t_e]\end{cases} \tag{D.1}$$

Because $0<\beta<\alpha<\gamma$ and $0<\theta\le 1$, we have $(\alpha+\gamma)/(\alpha-\beta)-\theta>0$. According to the boundary condition of Pattern 1, cost function monotonically decreases with respect to $t\in[t_s,t_{13}]$ and increases with respect to $t\in[t_{13},t_e]$. As a result, the maximum cost under capacity $\bar{s}$ can be obtained either at $t=t_s$ or at $t=t_e$. Mathematically,

$$C_{\max}^{\bar{s}}=\max\left\{C_{t_s}^{\bar{s}},C_{t_e}^{\bar{s}}\right\} \tag{D.2}$$

where $C_{\max}^{\bar{s}}$ indicates the maximum cost under capacity $\bar{s}$; $C_{t_i}^{\bar{s}}$ indicates the cost under capacity $\bar{s}$ at $t=t_i$. Since neither the first traveler nor the last traveler experiences queuing in Pattern 1, we have

$$C_{\max}^{\bar{s}}=C_{t_s}^{\bar{s}}=C_{t_e}^{\bar{s}}=\beta\left(t_e-t^*\right) \tag{D.3}$$

Similarly, differentiating cost function under $s=\theta\bar{s}$ yields,

$$\frac{dC(t)}{dt}=\begin{cases}\dfrac{\alpha(1-\theta)(1-\bar{\pi})}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_s,t_{11}]\\ \dfrac{\alpha[(\alpha+\gamma)-\theta(\alpha-\beta)](1-\bar{\pi})}{[(\alpha+\gamma)-\theta(\alpha-\beta)]\bar{\pi}+\theta(\alpha-\beta)}, & t\in[t_{11},t_{12}]\\ \dfrac{\alpha(1-\theta)(1-\bar{\pi})}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_{12},t_{13}]\\ \gamma\left(1-\dfrac{1}{\bar{\pi}}\right), & t\in[t_{13},t_e]\end{cases} \tag{D.4}$$

According to the boundary condition of Pattern 1, we cannot determine the monotonicity of cost function with respect to departure time under degraded capacity. As a result of linear travel cost in each situation, the minimum cost under degraded capacity can be obtained at the critical time point. The cost of each critical time point under the degraded capacity can be given as follows,

$$C_{t_s}^{\theta\bar{s}}=\beta\left(t^*-t_s\right) \tag{D.5}$$

$$C_{t_{11}}^{\theta\bar{s}}=\alpha\left(t^*-t_{11}\right) \tag{D.6}$$

$$C_{t_{12}}^{\theta\bar{s}}=(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)\left(t^*-t_s\right)+\alpha\left(t^*-t_{12}\right) \tag{D.7}$$

$$C_{t_{13}}^{\theta\bar{s}} = (\alpha+\gamma)\left(\frac{1}{\theta}-1\right)(t_{13}-t_s)+\gamma\left(t_{13}-t^*\right) \tag{D.8}$$

$$C_{t_e}^{\theta\bar{s}} = \gamma\left(t_e-t^*\right) \tag{D.9}$$

where $C_{t_i}^{\theta\bar{s}}$ indicates the cost at departure time t_i under the degraded capacity. Symbols will be expressed in this way later. Therefore, the condition (31) also can be given as follows,

$$C_{\max}^{\bar{s}} > \left\{C_{t_s}^{\theta\bar{s}}, C_{t_{11}}^{\theta\bar{s}}, C_{t_{12}}^{\theta\bar{s}}, C_{t_{13}}^{\theta\bar{s}}, C_{t_e}^{\theta\bar{s}}\right\} \tag{D.10}$$

Substituting (B.10)-(B.14) into (D.5)-(D.9), then we can obtain from (D.10)

$$\bar{\pi} > 1 \tag{D.11}$$

That is to say, Pattern 1 is implausible if the condition (D.11) holds.

In Pattern 2, substituting $s=\bar{s}$ into the cost function and differentiating the cost function yields,

$$\frac{dC(t)}{dt} = \begin{cases} \dfrac{-\alpha(1-\theta)\bar{\pi}}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_s,t_{21}] \\ \dfrac{-\alpha\left[(\alpha+\gamma)/(\alpha-\beta)-\theta\right]\bar{\pi}}{\left[(\alpha+\gamma)/(\alpha-\beta)-\theta\right]\bar{\pi}+\theta}, & t\in[t_{21},t_{22}] \\ -\beta, & t\in\left[t_{22},t^*\right] \\ \gamma, & t\in\left[t^*,t_e\right] \end{cases} \tag{D.12}$$

Similar to Pattern 1, cost function monotonically decreases with respect to $t\in\left[t_s,t^*\right]$ and increases with respect to $t\in\left[t^*,t_e\right]$ in Pattern 2. As a result, the maximum cost under capacity $\bar{s}$ can be obtained either at $t=t_s$ or at $t=t_e$. Hence, (D.3) also applies here.

Similarly, differentiating cost function under $s=\theta\bar{s}$ yields,

$$\frac{dC(t)}{dt} = \begin{cases} \dfrac{\alpha(1-\theta)(1-\bar{\pi})}{(1-\theta)\bar{\pi}+\theta}, & t\in[t_s,t_{21}] \\ \dfrac{\alpha\left[(\alpha+\gamma)-\theta(\alpha-\beta)\right](1-\bar{\pi})}{\left[(\alpha+\gamma)-\theta(\alpha-\beta)\right]\bar{\pi}+\theta(\alpha-\beta)}, & t\in[t_{21},t_{22}] \\ \beta\left(\dfrac{1}{\bar{\pi}}-1\right), & t\in\left[t_{22},t^*\right] \\ \gamma\left(1-\dfrac{1}{\bar{\pi}}\right), & t\in\left[t^*,t_e\right] \end{cases} \tag{D.13}$$

According to the boundary condition of Pattern 2, we cannot determine the monotonicity of cost function under degraded capacity. Because of the linear cost function, the minimum cost under degraded capacity can be obtained at critical time point as Pattern 1. The cost at t_s, t_{21} and t_e follows Eqs (D.5), (D.6) and (D.9), respectively, except that t_{11} is replaced with t_{21}. The cost at other critical time points can be given as follows,

$$C_{t_{22}}^{\theta\overline{s}}=(\alpha+\gamma)\left(\frac{1}{\theta}-1\right)(t_{22}-t_s)+\gamma\left(t_{22}-t^*\right) \tag{D.14}$$

$$C_{t^*}^{\theta\overline{s}}=(\alpha+\gamma)\left(\frac{(t_{22}-t_s)}{\theta}+\frac{r_5\left(t^*-t_{22}\right)}{\theta\overline{s}}-\left(t^*-t_s\right)\right) \tag{D.15}$$

Therefore, condition (31) also can be given as follows,

$$C_{\max}^{\overline{s}}>\min\left\{C_{t_s}^{\theta\overline{s}},C_{t_{21}}^{\theta\overline{s}},C_{t_{22}}^{\theta\overline{s}},C_{t^*}^{\theta\overline{s}},C_{t_e}^{\theta\overline{s}}\right\} \tag{D.16}$$

From (D.16), we obtain the result, which is the same as (D.11). That is to say, Pattern 2 is implausible if the condition (D.11) holds.

In Pattern 3, substituting $s=\overline{s}$ into cost function and differentiating cost function yields,

$$\frac{dC(t)}{dt}=\begin{cases}-\beta, & t\in\left[t_s,t_{31}\right]\\ -\beta, & t\in\left[t_{31},t^*\right]\\ \gamma, & t\in\left[t^*,t_e\right]\end{cases} \tag{D.17}$$

Therefore, cost function monotonically decreases with respect to $t\in\left[t_s,t^*\right]$ and increases with respect to $t\in\left[t^*,t_e\right]$. The same as Pattern 1, (D.3) can also be applied here.

Similarly, differentiating cost function under $s=\theta\overline{s}$ yields,

$$\frac{dC(t)}{dt}=\begin{cases}\beta\left(\frac{1}{\pi}-1\right), & t\in\left[t_s,t_{31}\right]\\ \beta\left(\frac{1}{\pi}-1\right), & t\in\left[t_{31},t^*\right]\\ \gamma\left(1-\frac{1}{\pi}\right), & t\in\left[t^*,t_e\right]\end{cases} \tag{D.18}$$

We cannot determine the monotonicity of cost function under degraded capacity in Pattern 3. Because of linear cost function in each situation, the cost at t_s and t_e follow (D.5) and (D.9), respectively. Hence, the cost at other critical time points under the degraded capacity can be given as follows,

$$C_{t_{31}}^{\theta\overline{s}}=\alpha\left(t^*-t_{31}\right) \tag{D.19}$$

$$C_{t^*}^{\theta\overline{s}}=(\alpha+\gamma)\frac{r_5\left(t^*-t_{31}\right)}{\theta\overline{s}} \tag{D.20}$$

The condition (31) can also be given as follows,

$$C_{\max}^{\overline{s}}>\min\left\{C_{t_s}^{\theta\overline{s}},C_{t_{31}}^{\theta\overline{s}},C_{t^*}^{\theta\overline{s}},C_{t_e}^{\theta\overline{s}}\right\} \tag{D.21}$$

From (D.21), the result can be obtained, which is the same as (D.11). That is to say, Pattern 3 is implausible if the condition (D.11) holds.

In Pattern 4, the derivation of cost under the capacity $\overline{s}$ follows the first three expressions of (D.1), respectively, except that t_{11} is replaced with t_{41}; t_{12} is replaced with t_{42} and t_{13} is replaced with t_e.

When $\bar{\pi} \geq 0$, the cost function under capacity $\bar{s}$ monotonically decreases, yet when $-\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta) < \bar{\pi} < 0$ the cost function under capacity $\bar{s}$ monotonically increases with respect to $t \in [t_s, t_e]$. As a result, the maximum cost under capacity $\bar{s}$ can be obtained either at $t = t_s$ or at $t = t_e$. We have

$$C_{\max}^{\bar{s}} = \max\left\{C_{t_s}^{\bar{s}}, C_{t_e}^{\bar{s}}\right\} = \max\left\{\beta\left(t^* - t_s\right), \gamma\left(\frac{N}{\bar{s}} + t_s - t^*\right)\right\} \tag{D.22}$$

Similarly, when $s = \theta\bar{s}$, the derivation of cost under the degraded capacity in Pattern 4 follows the first three expressions of (D.4), respectively, except that t_{11} is replaced with t_{41}; t_{12} is replaced with t_{42} and t_{13} is replaced with t_e. When $\bar{\pi} \geq 0$, the cost function under degraded capacity monotonically decreases, yet when $-\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta) < \bar{\pi} < 0$ the cost function under degraded capacity monotonically increases with respect to $t \in [t_s, t_e]$. Hence, the minimum cost under degraded capacity can be obtained either at $t = t_s$ or at $t = t_e$. Then we have

$$C_{\min}^{\theta\bar{s}} = \min\left\{C_{t_s}^{\theta\bar{s}}, C_{t_e}^{\theta\bar{s}}\right\} = \min\left\{\beta\left(t^* - t_s\right), \alpha\left(\frac{N}{\theta\bar{s}} - t_e + t_s\right) + \gamma\left(\frac{N}{\theta\bar{s}} + t_s - t^*\right)\right\} \tag{D.23}$$

From (C.22) and (C.23), we can obtain that $-\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta) < \bar{\pi} < 0$ satisfies condition (31). Thus, when $-\theta/((\alpha+\gamma)/(\alpha-\beta)-\theta) < \bar{\pi} < 0$, Pattern 4 is implausible.

In Pattern 5, the derivation of cost under the capacity $\bar{s}$ follows the first three expressions of (D.12), respectively, except that t_{21} is replaced with t_{51} and t_{22} is replaced with t_{52}. According to the boundary condition of Pattern 5, cost function under capacity $\bar{s}$ monotonically decreases with respect to $t \in [t_s, t_e]$. As a result, the maximum cost under capacity $\bar{s}$ can be obtained at $t = t_s$, which means that (D.22) can also be applied here.

Similarly, the derivation of cost under the degraded capacity in Pattern 5 follows the first three expressions of (D.13), respectively, except that t_{21} is replaced with t_{51} and t_{22} is replaced with t_{52}. Hence, $dC(t)/dt \geq 0$ under degraded capacity always holds in each situation in Pattern 5. In other words, cost function under degraded capacity monotonically increases with respect to $t \in [t_s, t_e]$. Hence, (D.23) can also be applied here. Therefore, $C_{\max}^{\bar{s}} = C_{\min}^{\theta\bar{s}}$ also holds, which means that there is no implausible parameter in Pattern 5.

In Pattern 6, the derivation of cost under the capacity $\bar{s}$ follows the first three expressions of (D.17), except that t_{31} is replaced with t_{61}. According to the boundary condition of Pattern 6, cost function under capacity $\bar{s}$ monotonically decreases with respect to $t \in [t_s, t_e]$. As a result, the maximum cost under capacity $\bar{s}$ can be obtained at $t = t_s$. That is to say, (D.22) can also be applied here.

Similarly, the derivation of cost under the degraded capacity in Pattern 6 follows the first two expressions of (D.18) in Pattern 3, except that t_{31} is replaced with t_{61}. $dC(t)/dt \geq 0$ under degraded capacity always holds in each situation in Pattern 6. In other words, cost function under degraded capacity monotonically increases with respect to departure time $t \in [t_s, t_e]$. Hence, (D.23) can also be applied here. Therefore, $C_{\max}^{\bar{s}} = C_{\min}^{\theta\bar{s}}$ also holds, which means there is no implausible parameter in Pattern 6.

In Pattern 7, the cost under capacity $\bar{s}$ that commuters experience can be given as (24). Therefore, the maximum cost under the capacity $\bar{s}$ is at $t = t_e$, i.e.,

$$C_{\max}^{\bar{s}} = \gamma\left(t_e - t^*\right) \tag{D.24}$$

Similarly, differentiating cost function under $s = \theta\bar{s}$ yields,

$$\frac{dC(t)}{dt} = (\alpha + \gamma)\frac{r_4(t)}{\theta\bar{s}} - \alpha, t \in [t_s, t_e] \tag{D.25}$$

The condition $\theta\bar{s} \leq r_4(t) \leq \bar{s}$ holds in this pattern. That is to say, $r_4/\theta\bar{s} \geq 1$. Then we have $dC(t)/dt \geq 0$. Hence, the minimum cost under the degraded capacity is at $t = t_s$. Then we have

$$C_{\min}^{\theta\bar{s}} = C_{t_s}^{\theta\bar{s}} = \beta\left(t^* - t_s\right) = 0 \tag{D.26}$$

Obviously, $C_{\max}^{\bar{s}} > C_{\min}^{\theta\bar{s}}$ always holds. That means that Pattern 7 is always implausible in reality.

In summary, when parameters satisfied the condition

$$\bar{\pi} > 1 \text{ or } -\frac{\theta}{(\alpha+\gamma)/(\alpha-\beta)-\theta} < \bar{\pi} < 0 \text{ or } \bar{\pi} \leq -\frac{\gamma}{(\alpha+\gamma)}\frac{\theta}{(1-\theta)} \tag{D.27}$$

the pattern is implausible in practice.

Appendix E. Proof of proposition

E.1. Proof of Proposition 1

Proof. The departure time of the first commuter and the last commuters follow (23) and (24) in Patterns 1-3, respectively. Obviously, they are independent of λ. The duration of peak hour in Patterns 1-3 can be expressed as follows,

$$t_e - t_s = t^* + \frac{\beta N}{\theta \bar{s}(\gamma+\beta)} - \left[t^* - \frac{\gamma N}{\theta \bar{s}(\gamma+\beta)} \right] = \frac{N}{\theta \bar{s}} \tag{E.1}$$

Therefore, the duration of peak hour in Patterns 1-3 is independent of λ.

In Pattern 4, the departure time of the first commuter and the last commuters follow (26) and (27) at equilibrium, respectively. Differentiating these two equations with respect to λ yields,

$$\frac{dt_s}{d\lambda} = -\frac{1-\theta}{\theta} \frac{N(\alpha+\gamma)\sqrt{\pi(1-\pi)}}{\bar{s}(\beta+\gamma)} \le 0 \tag{E.2}$$

$$\frac{dt_e}{d\lambda} = -\frac{1-\theta}{\theta} \frac{N(\alpha+\gamma)\sqrt{\pi(1-\pi)}}{\bar{s}(\beta+\gamma)} \le 0 \tag{E.3}$$

Hence, the departure time of the first commuter and the last commuter are both earlier with the increase of λ.

The duration of peak hour in Pattern 4 can be expressed as follows,

$$t_e - t_s = \frac{N}{\bar{s}} \tag{E.4}$$

Therefore, the duration of peak hour is independent of λ.

In Patterns 5 and 6, the departure time of the first commuter follows (28). The derivative of this departure time with respect to λ can be given as follows,

$$\frac{dt_s}{d\lambda} = -\frac{N}{\theta \bar{s}} \frac{\beta(\alpha+\gamma)\sqrt{\pi(1-\pi)}}{\left[\beta + (\alpha+\gamma)\bar{\pi}\right]^2} < 0 \tag{E.5}$$

Hence, the departure time of the first commuter becomes earlier with the increase of λ.

The duration of peak hour in Patterns 5 and 6 can be given as follows,

$$t_e - t_s = \frac{N}{\theta \bar{s}} \frac{(\alpha+\gamma)\bar{\pi}}{\beta + (\alpha+\gamma)\bar{\pi}} \tag{E.6}$$

The derivative of peak hour with respect to λ is given as follows,

$$\frac{d(t_e - t_s)}{d\lambda} = \frac{N}{\theta \bar{s}} \frac{\beta(\alpha+\gamma)\sqrt{\pi(1-\pi)}}{\left[\beta + (\alpha+\gamma)\bar{\pi}\right]^2} > 0 \tag{E.7}$$

Thus, the duration of peak hour increases with the increase of λ. This completes the proof. □

E.2. Proof of Proposition 2

Proof. Firstly, let QC_i denote the total queuing cost in Pattern i and QC_{ij} denote the total queuing cost of the Situation j in Pattern i.

For the first situation, the expected travel cost is given as (13). The total queuing cost with respect to $t \in [t_s, t_{11}]$ is given as follows,

$$QC_{11} = \int_{t_s}^{t_{11}} r_1 \left[\alpha \left(\frac{\pi}{\theta \bar{s}} + \frac{1-\pi}{\bar{s}} \right) R(t) - \alpha (t - t_s) \right] dt \tag{E.8}$$

where

$$R(t) = r_1(t - t_s) \tag{E.9}$$

Substituting (B.1) into (E.8), we have

$$QC_{11} = \frac{1}{2}\alpha\theta\bar{s}\left(t^* - t_s\right)^2\left[\left(\pi + \theta(1-\pi)\right) - \frac{\theta\bar{s}}{r_1}\right] \tag{E.10}$$

For Situation 2, the expected travel cost is given as (A.2). The total queuing cost with respect to $t \in [t_{11}, t_{12}]$ follows (E.8), except that the integral interval $[t_s, t_{11}]$ is replaced with $[t_{11}, t_{12}]$, r_1 is replaced with r_2 and QC_{11} is replaced with QC_{12}. In this time interval, we have

$$R(t) = R(t_{11}) + r_2(t - t_{11}) = \theta\bar{s}\left(t^* - t_s\right) + r_2(t - t_{11}) \tag{E.11}$$

Using the boundary condition (B.3), we know that

$$r_2(t_{12} - t_{11}) = R(t_{12}) - R(t_{11}) = (1-\theta)\bar{s}\left(t^* - t_s\right) \tag{E.12}$$

Substituting (E.12) into QC_{12} and rearranging it, we have

$$QC_{12} = \frac{1}{2}\alpha\bar{s}(1-\theta)\left(t^* - t_s\right)^2\left\{\left[\pi + \theta(1-\pi)\right] + \left[\left(\frac{1}{\theta} - 1\right)\pi + 1\right] - \frac{\bar{s}(1-\theta)}{r_2} - 2\frac{\theta\bar{s}}{r_1}\right\} \tag{E.13}$$

For Situation 3, the expected travel cost is given as (A.5). The total queuing cost with respect to $t \in [t_{12}, t_{13}]$ follows (E.8), except that the integral interval $[t_s, t_{11}]$ is replaced with $[t_{12}, t_{13}]$, r_1 is replaced with r_3 and QC_{11} is replaced with QC_{13}. In this time interval, we have

$$R(t) = R(t_{12}) + r_3(t - t_{12}) = \bar{s}\left(t^* - t_s\right) + r_3(t - t_{12}) \tag{E.14}$$

Substituting (E.14) into QC_{13}, we have

$$QC_{13} = \alpha r_3(t_{13} - t_{12})\left\{\left(t^* - t_s + \frac{r_3(t_{13} - t_{12})}{2\bar{s}}\right)\left(\frac{\pi}{\theta} - \pi + 1\right) - \frac{t_{13} - t_s + t_{12} - t_s}{2}\right\} \tag{E.15}$$

Using the boundary condition (B.6), we know that

$$r_3(t_{13} - t_{12}) = N - \bar{s}\left(t^* - t_s\right) - r_4(t_e - t_{13}) \tag{E.16}$$

Substituting (E.16) into (E.15) and rearranging it, we have

$$\begin{aligned} QC_{13} = &\frac{1}{2}\alpha\left[N - \bar{s}\left(t^* - t_s\right) - r_4(t_e - t_{13})\right] \\ &\left\{\left(\frac{\pi}{\theta} - \pi\right)\left[t^* - t_s - \frac{r_4}{\bar{s}}(t_e - t_{13}) + \frac{N}{\bar{s}}\right] + \left(t^* - t_s\right) - \frac{(1-\theta)\bar{s}\left(t^* - t_s\right)}{r_2} - \frac{\theta\bar{s}\left(t^* - t_s\right)}{r_1}\right\} \end{aligned} \tag{E.17}$$

For Situation 4, the expected travel cost is given as (A.8). The total queuing cost with respect to

$t \in [t_{13}, t_e]$ is given as follows,

$$QC_{14} = \int_{t_{13}}^{t_e} \alpha \pi r_4 \left[\frac{R(t)}{\theta \bar{s}} + t_s - t \right] dt \tag{E.18}$$

In this time interval, we have

$$R(t) = R(t_{13}) + r_4 (t - t_{13}) = \bar{s}(t_{13} - t_s) + r_4 (t - t_{13}) = N - r_4 (t_e - t) \tag{E.19}$$

Using the condition (22) and rearranging the expression of QC_{14}, we have

$$QC_{14} = \frac{1}{2} \alpha \pi r_4 \left(1 - \frac{r_4}{\theta \bar{s}} \right) (t_e - t_{13})^2 \tag{E.20}$$

Then the total queuing cost is the sum of queuing cost in each situation. So, the total queuing cost in Pattern 1 can be given as follows,

$$QC_1 = \frac{1}{2} \alpha \left\{ \begin{array}{l} \left(\frac{1}{\theta} - 1 \right) \frac{N\pi}{\bar{s}} \left[N - 2r_4 (t_e - t_{13}) \right] + (t^* - t_s) \left[N - r_4 (t_e - t_{13}) \right] \\ + \left[1 - \frac{r_4}{\bar{s}} \right] \pi r_4 (t_e - t_{13})^2 - \frac{\theta \bar{s}^2}{r_1} (t^* - t_s)^2 \\ - \bar{s} \left(\frac{(1-\theta)}{r_2} + \frac{\theta}{r_1} \right) (t^* - t_s) \left[\theta \bar{s} (t_e - t^*) - r_4 (t_e - t_{13}) \right] \end{array} \right\} \tag{E.21}$$

Differentiating the total queuing cost with respect to $\bar{\pi}$, we have

$$\frac{dQC_1}{d\bar{\pi}} = \frac{1}{2} \alpha \left\{ \begin{array}{l} \left[\left(\frac{(1-\theta)}{r_2} + \frac{\theta}{r_1} \right) \bar{s} - 1 \right] \left(\frac{dr_4}{d\bar{\pi}} (t_e - t_{13}) + r_4 \frac{d(t_e - t_{13})}{d\bar{\pi}} \right) (t^* - t_s) \\ + \left((1-\theta) \frac{d(1/r_2)}{d\bar{\pi}} + \theta \frac{d(1/r_1)}{d\bar{\pi}} \right) \bar{s} (t^* - t_s) \left[r_4 (t_e - t_{13}) - \theta \bar{s} (t_e - t^*) \right] \\ - \theta \bar{s}^2 (t^* - t_s)^2 \frac{d(1/r_1)}{d\bar{\pi}} - \pi \frac{dr_4}{d\bar{\pi}} (t_e - t_{13})^2 \end{array} \right\} \tag{E.22}$$

Substituting (15) and (19) into the first term of (E.22), we have

$$\left(\frac{1-\theta}{r_2} + \frac{\theta}{r_1} \right) \bar{s} - 1 = \frac{(1-\theta)(\alpha + \gamma)\bar{\pi} - \theta \beta}{\theta \alpha} \tag{E.23}$$

According to the boundary condition of Pattern 1, we have

$$\frac{\gamma}{\alpha + \gamma} < \bar{\pi} \le \frac{\beta \theta}{(\alpha + \gamma)(1-\theta)} \Rightarrow (1-\theta)(\alpha + \gamma)\bar{\pi} - \beta \theta \le 0 \Rightarrow \left(\frac{1-\theta}{r_2} + \frac{\theta}{r_1} \right) \bar{s} - 1 \le 0 \tag{E.24}$$

Substituting (18), (B.13) and (B.14) into the first term of (E.22), we have

$$\frac{dr_4}{d\bar{\pi}} = \theta \bar{s} \frac{\gamma}{(\alpha + \gamma)\bar{\pi}^2} > 0 \tag{E.25}$$

$$\frac{d(t_e - t_{13})}{d\lambda} = -\frac{dt_{13}}{d\lambda} = \frac{\gamma N}{\theta \bar{s}} \frac{(\alpha + \gamma)(1/\theta - 1)}{\left[\bar{\pi} (\alpha + \gamma)(1/\theta - 1) + \gamma \right]^2} \tag{E.26}$$

Hence, the first term of (E.22) is negative. Substituting (15), (16), (18), (B.13) and (B.14) into the second term of (E.22), we have

$$\frac{d\left(1/r_1\right)}{d\overline{\pi}}=\frac{(\alpha-\beta)(1-\theta)}{\alpha\theta\overline{s}}>0 \tag{E.27}$$

$$\frac{d\left(1/r_2\right)}{d\overline{\pi}}=\frac{(\alpha+\gamma)-\theta(\alpha-\beta)}{\alpha\theta\overline{s}}>0 \tag{E.28}$$

$$r_4\left(t_e-t_{13}\right)-\theta\overline{s}\left(t_e-t^*\right)=\gamma N\left\{\frac{(1-\theta)(\alpha+\gamma)\overline{\pi}-\theta\beta-(1-\theta)(\gamma+\beta)}{\theta(\gamma+\beta)\left[\overline{\pi}(\alpha+\gamma)(1/\theta-1)+\gamma\right]}\right\} \tag{E.29}$$

According to (E.24), we have

$$r_4\left(t_e-t_{13}\right)-\theta\overline{s}\left(t_e-t^*\right)<0 \tag{E.30}$$

Hence, the second term of (E.22) is negative. The third term and the forth term of (E.22) is negative according to (E.27) and (E.25), respectively. As a result, we have

$$\frac{dQC_1}{d\overline{\pi}}<0 \tag{E.31}$$

and

$$\frac{d\overline{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)} \tag{E.32}$$

Thereby,

$$\frac{dQC_1}{d\lambda}=\frac{dQC_1}{d\overline{\pi}}\frac{d\overline{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dQC_1}{d\overline{\pi}}<0 \tag{E.33}$$

In Pattern 2, the total queuing cost with respect to $t\in\left[t_s,t_{21}\right]$ follows (E.10), except that QC_{11} is replaced with QC_{21}. The total queuing cost with respect to $t\in\left[t_{21},t_{22}\right]$ follows (E.8), except that the integral interval $\left[t_s,t_{11}\right]$ is replaced with $\left[t_{21},t_{22}\right]$, r_1 is replaced with r_2 and QC_{11} is replaced with QC_{22}. In this time interval, the form of $R(t)$ follows (E.11) except that $R\left(t_{11}\right)$ is replaced $R\left(t_{21}\right)$ and t_{11} is replaced with t_{21}. Using the boundary condition (B.15) we know that

$$r_2\left(t_{22}-t_{21}\right)=R\left(t_{22}\right)-R\left(t_{21}\right)=\overline{s}\left(t_{22}-t_s\right)-\theta\overline{s}\left(t^*-t_s\right) \tag{E.34}$$

Substituting (E.34) and rearranging QC_{22}, we have

$$QC_{22}=\frac{1}{2}\alpha\overline{s}\left[\left(t_{22}-t_s\right)-\theta\left(t^*-t_s\right)\right]\left\{\left[\pi+\theta(1-\pi)\right]\left(t^*-t_s\right)+\pi\left(\frac{1}{\theta}-1\right)\left(t_{22}-t_s\right)-\left(t_{21}-t_s\right)\right\} \tag{E.35}$$

The total queuing cost with respect to $t\in\left[t_{22},t^*\right]$ follows (E.18), except that the integral interval $\left[t_{13},t_e\right]$ is replaced with $\left[t_{22},t^*\right]$, r_4 is replaced with r_5 and QC_{14} is replaced with QC_{23}. In this time interval, we have

$$R(t)=R\left(t_{22}\right)+r_5\left(t-t_{22}\right)=\overline{s}\left(t_{22}-t_s\right)+r_5\left(t-t_{22}\right) \tag{E.36}$$

Using the boundary condition (B.15) and rearranging QC_{23}, we have

$$QC_{23}=\alpha\pi r_5\left(t^*-t_{22}\right)\left[\left(\frac{1}{\theta}-\frac{1}{2}\right)\left(t_{22}-t_s\right)+\frac{1}{2}\frac{r_5}{\theta\bar{s}}\left(t^*-t_{22}\right)-\frac{1}{2}\left(t^*-t_s\right)\right] \tag{E.37}$$

According to the boundary condition we know that

$$r_5\left(t^*-t_{22}\right)=N-r_4\left(t_e-t^*\right)-\bar{s}\left(t_{22}-t_s\right) \tag{E.38}$$

Substituting (E.38) into (E.37) and rearranging it we have

$$QC_{23}=\frac{1}{2}\alpha\pi\left\{\begin{array}{l}\left(1-2\frac{r_4}{\theta\bar{s}}\right)N\left(t_e-t^*\right)-N\left(t_{22}-t_s\right)+\frac{1}{\theta\bar{s}}r_4^{\,2}\left(t_e-t^*\right)^2\\+r_4\left(t_{22}-t_s\right)\left(t_e-t^*\right)+r_4\left(t^*-t_s\right)\left(t_e-t^*\right)\\+\left(1-\frac{1}{\theta}\right)\bar{s}\left(t_{22}-t_s\right)^2+\bar{s}\left(t^*-t_s\right)\left(t_{22}-t_s\right)\end{array}\right\} \tag{E.39}$$

The total queuing cost with respect to $t\in\left[t^*,t_e\right]$ follows (E.20), except that t_{13} is replaced with t^* and QC_{14} is replaced with QC_{24}.

By summing up queuing cost in each situation, the total queuing cost in Pattern 2 can be given as follows,

$$QC_2=\frac{1}{2}\alpha\left\{\begin{array}{l}\pi\theta\bar{s}\left(t_e-t_s\right)\left(t_e-t^*\right)-\pi\theta\bar{s}\left(t_e-t_s\right)\left(t_{22}-t_s\right)+\pi r_4\left(t_e-t^*\right)\left(t_{22}-t_s\right)\\-\pi r_4\left(t_e-t^*\right)\left(t_e-t_s\right)+\pi\bar{s}\left(t^*-t_s\right)\left(t_{22}-t_s\right)+\left(1-\frac{\bar{s}}{r_1}\right)\theta\bar{s}\left(t^*-t_s\right)\left(t_{22}-t_s\right)\end{array}\right\} \tag{E.40}$$

Differentiating the total queuing cost with respect to $\bar{\pi}$, we have

$$\frac{dQC_2}{d\bar{\pi}}=\frac{1}{2}\alpha\left\{\begin{array}{l}\pi\left[\bar{s}\left(t^*-t_s\right)+r_4\left(t_e-t^*\right)-\theta\bar{s}\left(t_e-t_s\right)\right]\frac{d\left(t_{22}-t_s\right)}{d\bar{\pi}}-\pi\left(t_e-t^*\right)\left(t_e-t_{22}\right)\frac{dr_4}{d\bar{\pi}}\\+\left[\left(1-\frac{\bar{s}}{r_1}\right)\frac{d\left(t_{22}-t_s\right)}{d\bar{\pi}}-\bar{s}\left(t_{22}-t_s\right)\frac{d\left(1/r_1\right)}{d\bar{\pi}}\right]\theta\bar{s}\left(t^*-t_s\right)\end{array}\right\} \tag{E.41}$$

Substituting (18), (23), (24) and (B.17) into the first term of (E.41), we have

$$\frac{d\left(t_{22}-t_s\right)}{d\bar{\pi}}=\frac{dt_{22}}{d\bar{\pi}}=-\frac{\gamma N}{\theta\bar{s}}\frac{\beta}{\left[\beta-\bar{\pi}\left(\alpha+\gamma\right)\left(1/\theta-1\right)-\left(\beta+\gamma\right)\bar{\pi}\right]^2}\le 0 \tag{E.42}$$

$$\bar{s}\left(t^*-t_s\right)+r_4\left(t_e-t^*\right)-\theta\bar{s}\left(t_e-t_s\right)=\frac{\gamma N}{\gamma+\beta}\left[\frac{1-\theta}{\theta}-\frac{\beta}{\left(\alpha+\gamma\right)\bar{\pi}}\right] \tag{E.43}$$

According to the boundary condition of Pattern 2, we have

$$\max\left\{\frac{\beta\theta}{\left(\alpha+\gamma\right)\left(1-\theta\right)},\frac{\gamma}{\alpha+\gamma}\right\}<\bar{\pi}\le\frac{\beta\theta}{\left(\alpha-\beta\right)\left(1-\theta\right)}\Rightarrow\frac{\left(\alpha-\beta\right)\left(1-\theta\right)}{\left(\alpha+\gamma\right)\theta}<\frac{\beta}{\left(\alpha+\gamma\right)\bar{\pi}}\le\min\left\{\frac{1-\theta}{\theta},\frac{\beta}{\gamma}\right\} \tag{E.44}$$

Therefore, (E.43) is positive and the first term of (E.41) is negative. According to (E.25), the second term of (E.41) is negative. By definition, condition $r_1\ge\bar{s}$. Hence, $1-\bar{s}/r_1\ge 0$. According to (E.42) and (E.27), the third term of (E.41) is negative. As a result, according to (E.32), we have

$$\frac{dQC_2}{d\lambda}=\frac{dQC_2}{d\overline{\pi}}\frac{d\overline{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dQC_2}{d\overline{\pi}}<0 \tag{E.45}$$

In Pattern 3, the total queuing cost with respect to $t\in[t_s,t_{31}]$ follows (E.18), except that the integral interval $[t_{13},t_e]$ is replaced with $[t_s,t_{31}]$, r_4 is replaced with r_6 and QC_{14} is replaced with QC_{31}. In this time interval, we have

$$R(t)=r_6(t-t_s) \tag{E.46}$$

Substituting (E.46) and the boundary condition (B.18) into QC_{31}, we have

$$QC_{31}=\frac{1}{2}\alpha\pi\theta\overline{s}\left(1-\frac{\theta\overline{s}}{r_6}\right)\left(t^*-t_s\right)^2 \tag{E.47}$$

The total queuing cost with respect to $t\in[t_{31},t^*]$ follows (E.18), except that the integral interval $[t_{13},t_e]$ is replaced with $[t_{31},t^*]$, r_4 is replaced with r_5 and QC_{14} is replaced with QC_{32}. In this time interval, we have

$$R(t)=N-r_4\left(t_e-t^*\right)-r_5\left(t^*-t\right) \tag{E.48}$$

and use the condition (B.18), we have

$$r_5\left(t^*-t_{31}\right)=N-\theta\overline{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right) \tag{E.49}$$

Substituting (E.48) and (E.49) into QC_{32} and rearranging this expression, we have

$$QC_{32}=\frac{1}{2}\alpha\pi\left[N-\theta\overline{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right)\right]\left\{\left(1-\frac{r_4}{\theta\overline{s}}\right)\left(t_e-t^*\right)+\frac{1}{r_5}\left[N-\theta\overline{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right)\right]\right\} \tag{E.50}$$

The total queuing cost with respect to $t\in[t^*,t_e]$ follows (E.20), except that the t_{13} is replaced with t^* and QC_{14} is replaced with QC_{33}.

Summing up queuing cost in each situation, the total queuing cost in Pattern 3 can be given as follows,

$$QC_3=\frac{1}{2}\alpha\pi\left\{\begin{array}{l}N\left(1-\frac{r_4}{\theta\overline{s}}\right)\left(t_e-t^*\right)-\theta\overline{s}\left(1-\frac{r_4}{\theta\overline{s}}\right)\left(t_e-t^*\right)\left(t^*-t_s\right)\\+\theta\overline{s}\left(t^*-t_s\right)\left(\left(t^*-t_s\right)-\frac{\theta\overline{s}\left(t^*-t_s\right)}{r_6}\right)+r_5\left(t^*-t_{31}\right)^2\end{array}\right\} \tag{E.51}$$

Differentiating the total queuing cost with respect to $\overline{\pi}$, we have

$$\frac{dQC_3}{d\overline{\pi}}=\frac{1}{2}\alpha\pi\left\{-\frac{dr_4}{d\overline{\pi}}\left(t_e-t^*\right)^2-\left(\theta\overline{s}\right)^2\left(t^*-t_s\right)^2\frac{d\left(1/r_6\right)}{d\overline{\pi}}+\left(t^*-t_{31}\right)^2\frac{dr_5}{d\overline{\pi}}+2r_5\left(t^*-t_{31}\right)\frac{d\left(t^*-t_{31}\right)}{d\overline{\pi}}\right\} \tag{E.52}$$

According to (E.25), the first term of (E.52) is negative. Substituting (20) into the second term of (E.52), we have

$$\frac{d\left(1/r_6\right)}{d\bar{\pi}}=\frac{1}{\theta\bar{s}}\frac{\beta(\alpha-\beta)}{\left[\beta+(\alpha-\beta)\bar{\pi}\right]^2}>0 \tag{E.53}$$

Therefore, the second term of (E.52) is negative. Substituting (19) into the third term of (E.52), we have

$$\frac{dr_5}{d\bar{\pi}}=-\theta\bar{s}\frac{\beta}{(\alpha+\gamma)\bar{\pi}^2}<0 \tag{E.54}$$

Thereby, the third term of (E.52) is negative. Substituting (B.20) into the fourth term of (E.52), we have

$$\frac{d\left(t^*-t_{31}\right)}{d\bar{\pi}}=-\frac{dt_{31}}{d\bar{\pi}}=-\frac{N\beta\gamma(\alpha-\beta)}{\theta\bar{s}(\gamma+\beta)\left[\beta+(\alpha-\beta)\bar{\pi}\right]^2}<0 \tag{E.55}$$

Then the fourth term of (E.52) is negative. Thus, all terms of (E.52) is negative. According to (E.32), we have

$$\frac{dQC_3}{d\lambda}=\frac{dQC_3}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dQC_3}{d\bar{\pi}}<0 \tag{E.56}$$

According to (E.33), (E.45) and (E.56), the total queuing cost decreases with the increase of λ in Patterns 1-3. This completes the proof. □

E.3. Proof of Proposition 3

Proof. Firstly, the proof of variation of total early cost with λ will be given in Patterns 1-5. Let EC_i denote the total early cost in Pattern i and EC_{ij} denote the early cost of the Situation j in Pattern i.

For the first situation in Pattern 1, the expected travel cost is given as (13). The total early cost with respect to $t\in\left[t_s,t_{11}\right]$ is given as follows,

$$EC_{11}=\int_{t_s}^{t_{11}}r_1\left[-\beta\left(\frac{\pi}{\theta\bar{s}}+\frac{1-\pi}{\bar{s}}\right)R(t)+\beta\left(t^*-t_s\right)\right]dt \tag{E.57}$$

Substituting (E.9) into (E.57) and using the boundary condition (B.1), we have

$$\text{Early Cost}_{11}=\beta\theta\bar{s}\left(t^*-t_s\right)^2\left[-\frac{1}{2}\left(\pi+\theta(1-\pi)\right)+1\right] \tag{E.58}$$

The total early cost with respect to $t\in\left[t_{11},t_{12}\right]$ can be given as

$$EC_{12}=\int_{t_{11}}^{t_{12}}r_2(1-\pi)\beta\left[t^*-t_s-\frac{R(t)}{\bar{s}}\right]dt \tag{E.59}$$

where $R(t)$ is given as (E.11). Substituting (E.12) into the expression of EC_{12} and rearranging this expression, we have

$$EC_{12}=\frac{1}{2}\beta(1-\pi)(1-\theta)^2\bar{s}\left(t^*-t_s\right)^2 \tag{E.60}$$

There is no early cost in Situation 3 and 4, which means that early cost with respect to $t \in [t_{12}, t_e]$ is zero, i.e., $EC_{13} = EC_{14} = 0$. Then the total early cost in Pattern 1 can be given as

$$EC_1 = \left[-\frac{1}{2}\theta\left(\pi + \theta(1-\pi)\right) + \theta + \frac{1}{2}(1-\pi)(1-\theta)^2 \right] \beta \overline{s}\left(t^* - t_s\right)^2 \tag{E.61}$$

According to the Proposition 1, the departure time of the first commuter are independent of λ in Pattern 1. So, the total early cost in Pattern 1 is independent of λ at equilibrium.

In Pattern 2, the early cost with respect to $t \in [t_s, t_{21}]$ follows (E.58), except that EC_{11} is replaced with EC_{21}. The total early cost with respect to $t \in [t_{21}, t_{22}]$ follows (E.59), except that the integral interval $[t_{11}, t_{12}]$ is replaced with $[t_{21}, t_{22}]$ and EC_{12} is replaced with EC_{22}. $R(t)$ in this expression follows (E.11) except that $R(t_{11})$ is replaced $R(t_{21})$ and t_{11} is replaced with t_{21}. Substituting (E.34) into EC_{22} and rearranging the expression, we have

$$EC_{22} = (1-\pi)\beta\overline{s}\left[(t_{22} - t_s) - \theta\left(t^* - t_s\right)\right]\left\{\left(1 - \frac{1}{2}\theta\right)\left(t^* - t_s\right) - \frac{1}{2}(t_{22} - t_s)\right\} \tag{E.62}$$

The total early cost with respect to $t \in \left[t_{22}, t^*\right]$ follows (E.59), except that the integral interval $[t_{11}, t_{12}]$ is replaced with $\left[t_{22}, t^*\right]$ and EC_{12} is replaced with EC_{23}. $R(t)$ in this expression is given as (E.36). Substituting $R(t)$ into EC_{23} and rearranging the expression, we have

$$EC_{23} = \frac{1}{2}\beta r_5 (1-\pi)\left(t^* - t_{22}\right)^2 \tag{E.63}$$

The early cost with respect to $t \in \left[t^*, t_e\right]$ equals to zero. Hence the total early cost in Pattern 2 can be given as follows

$$EC_2 = \frac{1}{2}(1-\pi)\beta\left[2\overline{s}\left(t^* - t_s\right)(t_{22} - t_s) - \overline{s}(t_{22} - t_s)^2 + r_5\left(t^* - t_{22}\right)^2\right] + \frac{1}{2}\pi\beta\theta\overline{s}\left(t^* - t_s\right)^2 \tag{E.64}$$

Differentiating the total early cost in Pattern 2 with respect to $\overline{\pi}$, we have

$$\frac{dEC_2}{d\overline{\pi}} = (\overline{s} - r_5)(1-\pi)\beta\left(t^* - t_{22}\right)\frac{dt_{22}}{d\overline{\pi}} + \frac{1}{2}(1-\pi)\beta\left(t^* - t_{22}\right)^2\frac{dr_5}{d\overline{\pi}} \tag{E.65}$$

According to (E.42), we know that $dt_{22}/d\overline{\pi} < 0$. From the boundary condition in Pattern 2, we have

$$\begin{aligned} &\max\left\{\frac{\beta\theta}{(\alpha+\gamma)(1-\theta)}, \frac{\gamma}{\alpha+\gamma}\right\} < \overline{\pi} \le \frac{\beta\theta}{(\alpha-\beta)(1-\theta)} \\ &\Rightarrow \frac{\alpha-\beta}{\alpha+\gamma} + \frac{(\alpha-\beta)(1-\theta)}{(\alpha+\gamma)\theta} < \frac{\alpha-\beta}{\alpha+\gamma} + \frac{\beta}{(\alpha+\gamma)\overline{\pi}} \le \min\left\{\frac{\alpha-\beta}{\alpha+\gamma} + \frac{1-\theta}{\theta}, \frac{\alpha(\gamma+\beta)}{\gamma(\gamma+\alpha)}\right\} \end{aligned} \tag{E.66}$$

By definition, we have $0<\beta<\alpha<\gamma$. Hence, $\alpha(\gamma+\beta)/(\gamma(\gamma+\alpha))<1$. According to the expression of r_5, we have

$$r_5=\theta\bar{s}\left[\frac{\alpha-\beta}{\alpha+\gamma}+\frac{\beta}{(\alpha+\gamma)\bar{\pi}}\right]<\theta\bar{s} \tag{E.67}$$

Therefore, $\bar{s}-r_5>0$. The first term of expression (E.65) is negative. According to (E.54), i.e., $dr_5/d\bar{\pi}<0$, the second term of expression (E.65) is negative. According to (E.32), we have

$$\frac{dEC_2}{d\lambda}=\frac{dEC_2}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dEC_2}{d\bar{\pi}}<0 \tag{E.68}$$

In Pattern 3, the early cost with respect to $t\in[t_s,t_{31}]$ is given as follows,

$$EC_{31}=\int_{t_s}^{t_{31}}r_6\left[\pi\beta\left(t^*-t_s-\frac{R(t)}{\theta\bar{s}}\right)+(1-\pi)\beta\left(t^*-t\right)\right]dt \tag{E.69}$$

In this time interval, $R(t)$ is given as (E.46). Substituting (E.46) and the boundary condition (B.18) into the expression of EC_{31} and rearranging it , we have

$$EC_{31}=\left[1-\frac{1}{2}\pi-\frac{1}{2}\frac{\theta\bar{s}}{r_6}(1-\pi)\right]\beta\theta\bar{s}\left(t^*-t_s\right)^2 \tag{E.70}$$

The total early cost with respect to $[t_{31},t^*]$ follows (E.63), except that t_{22} is replaced with t_{31} and EC_{23} is replaced with EC_{33}. The early cost with respect to $t\in[t^*,t_e]$ is zero like Pattern 1. Hence the total early cost in Pattern 3 can be given as follows

$$EC_3=\left(1-\frac{1}{2}\pi-\frac{1}{2}\frac{\theta\bar{s}}{r_6}(1-\pi)\right)\beta\theta\bar{s}\left(t^*-t_s\right)^2+\frac{1}{2}(1-\pi)\beta r_5\left(t^*-t_{31}\right)^2 \tag{E.71}$$

Differentiating the total early cost in Pattern 3 with respect to $\bar{\pi}$, we have

$$\frac{dEC_3}{d\bar{\pi}}=\beta(1-\pi)\left\{-\frac{1}{2}\left[\theta\bar{s}\left(t^*-t_s\right)\right]^2\frac{d(1/r_6)}{d\bar{\pi}}+\frac{1}{2}\left(t^*-t_{31}\right)^2\frac{dr_5}{d\bar{\pi}}+r_5\left(t^*-t_{31}\right)\frac{d\left(t^*-t_{31}\right)}{d\bar{\pi}}\right\} \tag{E.72}$$

According to (E.53), (E.54) and (E.55), all terms of (E.72) is negative. According to (E.32), we have

$$\frac{dEC_3}{d\lambda}=\frac{dEC_3}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dEC_3}{d\bar{\pi}}<0 \tag{E.73}$$

According to (E.68) and (E.73), the total early cost decreases with the increase of λ in Patterns 2 and 3.

In Pattern 4, the early cost with respect to $t\in[t_s,t_{41}]$ follows (E.58), except that EC_{11} is replaced with EC_{41} and t_s is given as (26). The early cost with respect to $t\in[t_{41},t_{42}]$ follows (E.60), except that EC_{12} is replaced with EC_{42} and t_s is given as (26). There is no early cost in the Situation 3. Therefore, the total early cost in Pattern 4 can be given as follows,

$$EC_4 = \frac{1}{2}\left[1-(1-\theta)\pi\right]\beta\bar{s}\left(t^* - t_s\right)^2 \tag{E.74}$$

Differentiating the total early cost in Pattern 4 with respect to $\bar{\pi}$, we have

$$\frac{dEC_4}{d\bar{\pi}} = \left[1-(1-\theta)\pi\right]\beta\bar{s}\left(t^* - t_s\right)\frac{d\left(t^* - t_s\right)}{d\bar{\pi}} \tag{E.75}$$

By definition, $0<\theta\le 1$ and $0\le\pi\le 1$. Hence, $1-(1-\theta)\pi>0$. According to (E.107), we have $d\left(t^* - t_s\right)/d\bar{\pi}>0$. Therefore, we have $dEC_4/d\bar{\pi}>0$. According to (E.32), we have

$$\frac{dEC_4}{d\lambda} = \frac{dEC_4}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda} = \sqrt{\pi(1-\pi)}\frac{dEC_4}{d\bar{\pi}} > 0 \tag{E.76}$$

In Pattern 5, the early cost with respect to $t\in\left[t_s, t_{51}\right]$ follows (E.58), except that EC_{11} is replaced with EC_{51}. The early cost with respect to $t\in\left[t_{51}, t_{52}\right]$ follows (E.62), except that t_{22} is replaced with t_{52} and EC_{22} is replaced with EC_{52}. The early cost with respect to $t\in\left[t_{52}, t^*\right]$ follows (E.63), except that t_{22} is replaced with t_{52} and EC_{23} is replaced with EC_{53}. Substituting (E.109) into EC_{53}, we have

$$EC_{53} = \frac{1}{2}\beta(1-\pi)\frac{1}{r_5}\left[N-\bar{s}\left(t_{52}-t_s\right)\right]^2 \tag{E.77}$$

The total early cost in Pattern 5 can be given as follows,

$$\begin{aligned} EC_5 =& \beta(1-\pi)\bar{s}\left(t_{52}-t_s\right)\left[\left(t^*-t_s\right)-\frac{N}{r_5}+\frac{1}{2}\left(t_{52}-t_s\right)\left(\frac{\bar{s}}{r_5}-1\right)\right] \\ &+\frac{1}{2}\beta\left[\pi\theta\bar{s}\left(t^*-t_s\right)^2+(1-\pi)\frac{N^2}{r_5}\right] \end{aligned} \tag{E.78}$$

Differentiating the total early cost in Pattern 5 with respect to $\bar{\pi}$, we have

$$\frac{dEC_5}{d\bar{\pi}} = \beta\bar{s}\left[(1-\pi)\left(t_{52}-t_s\right)+\pi\theta\left(t^*-t_s\right)\right]\frac{d\left(t^*-t_s\right)}{d\bar{\pi}}+\frac{1}{2}\beta(1-\pi)\left(N-\bar{s}\left(t_{52}-t_s\right)\right)^2\frac{d\left(1/r_5\right)}{d\bar{\pi}} \tag{E.79}$$

According to (E.113), we know that $d\left(t^*-t_s\right)/d\bar{\pi}>0$. From (E.54), we know that $d\left(1/r_5\right)/d\bar{\pi}>0$. Hence, we have $dEC_5/d\bar{\pi}>0$. According to (E.32), we have

$$\frac{dEC_5}{d\lambda} = \frac{dEC_5}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda} = \sqrt{\pi(1-\pi)}\frac{dEC_5}{d\bar{\pi}} > 0 \tag{E.80}$$

According to (E.76) and (E.80), the total early cost increase with the increase of λ in Patterns 4 and 5.

Secondly, the proof of variation of total late cost with λ will be given in Patterns 1-6. Let LC_i

denote total late cost in Pattern i and LC_{ij} denote total late cost of the Situation j in Pattern i.

For the first situation in Pattern 1, no matter how the capacity of bottleneck varies, all commuters always arrive early, which means $LC_{11}=0$. The total late cost with respect to $t\in[t_{11},t_{12}]$ is given as follows,

$$LC_{12}=\int_{t_{11}}^{t_{12}}\pi\gamma r_2\left(\frac{R(t)}{\theta\overline{s}}+t_s-t^*\right)dt \tag{E.81}$$

where $R(t)$ is given as (E.11). Substituting (E.12) into LC_{12} and rearranging this expression, we have

$$LC_{12}=\frac{1}{2}\pi\gamma\frac{(1-\theta)^2}{\theta}\overline{s}\left(t^*-t_s\right)^2 \tag{E.82}$$

For Situation 3, the total late cost with respect to $t\in[t_{12},t_{13}]$ is given as follows

$$LC_{13}=\int_{t_{12}}^{t_{13}}r_3\left[\gamma R(t)\left(\frac{\pi}{\theta\overline{s}}+\frac{1-\pi}{\overline{s}}\right)+\gamma\left(t_s-t^*\right)\right]dt \tag{E.83}$$

In this time interval, substituting (E.14) into (E.83) and using condition (E.16), we have

$$LC_{13}=\gamma\left[N-\overline{s}\left(t^*-t_s\right)-r_4\left(t_e-t_{13}\right)\right]\left\{\begin{array}{l}\pi\left(\dfrac{1}{\theta}-1\right)\left(t^*-t_s\right)+\dfrac{1}{2}\left(\dfrac{\pi}{\theta\overline{s}}+\dfrac{1-\pi}{\overline{s}}\right)\\ \left[N-\overline{s}\left(t^*-t_s\right)-r_4\left(t_e-t_{13}\right)\right]\end{array}\right\} \tag{E.84}$$

For Situation 4, the total late cost with respect to $t\in[t_{13},t_e]$ is given as follows,

$$LC_{14}=\int_{t_{13}}^{t_e}r_4\left[\pi\gamma\left(\frac{R(t)}{\theta\overline{s}}+t_s-t^*\right)+(1-\pi)\gamma\left(t-t^*\right)\right]dt \tag{E.85}$$

where $R(t)$ follows (E.19). Using the condition (22) and rearranging LC_{14}, we have

$$LC_{14}=\gamma\left[r_4\left(t_e-t_{13}\right)\left(t_e-t^*\right)-\left(\frac{\pi}{2}\frac{r_4}{\theta\overline{s}}+\frac{1}{2}(1-\pi)\right)r_4\left(t_e-t_{13}\right)^2\right] \tag{E.86}$$

Then the total late cost in Pattern 1 is given as follows,

$$LC_1=\gamma\left\{\begin{array}{l}\dfrac{1}{2}\left(\dfrac{r_4}{\overline{s}}-1\right)(1-\pi)r_4\left(t_e-t_{13}\right)^2+(1-\theta)(1-\pi)r_4\left(t_e-t_s\right)\left(t_e-t_{13}\right)\\ +\dfrac{1}{2}\left(\pi+\theta(1-\pi)\right)\theta\overline{s}\left(t_e-t_s\right)^2\\ +\dfrac{1}{2}\left(1-\pi(1-\theta)\right)\overline{s}\left(t^*-t_s\right)^2-\theta\overline{s}\left(t^*-t_s\right)\left(t_e-t_s\right)\end{array}\right\} \tag{E.87}$$

Differentiating the total late cost with respect to $\overline{\pi}$, we have

$$\frac{dLC_1}{d\bar{\pi}}=\gamma\left\{\begin{aligned}&(1-\pi)\left[\left(\frac{r_4}{\bar{s}}-1\right)(t_e-t_{13})+(1-\theta)(t_e-t_s)\right]\\&\left[(t_e-t_{13})\frac{dr_4}{d\bar{\pi}}+r_4\frac{d(t_e-t_{13})}{d\bar{\pi}}\right]+\frac{1}{2}(1-\pi)(t_e-t_{13})^2\frac{dr_4}{d\bar{\pi}}\end{aligned}\right\} \tag{E.88}$$

Substituting (18) and (B.8) into the first term of (E.88), we have

$$\begin{aligned}&\left(\frac{r_4}{\bar{s}}-1\right)(t_e-t_{13})+(1-\theta)(t_e-t_s)=\frac{r_4(t_e-t_{13})}{\bar{s}}-(t_e-t_{13})+(1-\theta)(t_e-t_s)\\&=\frac{N-\bar{s}(t_{13}-t_s)}{\bar{s}}-(t_e-t_{13})+(1-\theta)(t_e-t_s)\\&=\frac{\theta\bar{s}(t_e-t_s)-\bar{s}(t_{13}-t_s)}{\bar{s}}-(t_e-t_{13})+(1-\theta)(t_e-t_s)\\&=(\theta-1)(t_e-t_s)+(1-\theta)(t_e-t_s)=0\end{aligned} \tag{E.89}$$

Therefore, $dLC_1/d\bar{\pi}=\frac{1}{2}\gamma(1-\pi)(t_e-t_{13})^2\,dr_4/d\bar{\pi}$. According to (E.25), (E.88) is positive. According to (E.32), we have

$$\frac{dLC_1}{d\lambda}=\frac{dLC_1}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dLC_1}{d\bar{\pi}}>0 \tag{E.90}$$

Similar to Pattern 1, the late cost with respect to $t\in[t_s,t_{21}]$ is zero, i.e., $LC_{21}=0$. The late cost with respect to $t\in[t_{21},t_{22}]$ follows (E.81), except that the integral interval $[t_{11},t_{12}]$ is replaced with $[t_{21},t_{22}]$ and LC_{12} is replaced with LC_{22}. $R(t)$ follows (E.11), except that $R(t_{11})$ is replaced $R(t_{21})$ and t_{11} is replaced with t_{21}. Substituting (E.34) into LC_{22} and rearranging the expression, we have

$$LC_{22}=\frac{1}{2}\frac{1}{\theta\bar{s}}\pi\gamma\left[\bar{s}(t_{22}-t_s)-\theta\bar{s}(t^*-t_s)\right]^2 \tag{E.91}$$

The late cost with respect to $t\in[t_{22},t^*]$ follows (E.81), except that the integral interval $[t_{11},t_{12}]$ is replaced with $[t_{22},t^*]$, r_2 is replaced with r_5 and LC_{12} is replaced with LC_{23}. $R(t)$ is given as (E.36). Substituting (E.38) into the expression of LC_{23}, we have

$$LC_{23}=\pi\gamma\left[N-\bar{s}(t_{22}-t_s)-r_4(t_e-t^*)\right]\left[(t_e-t^*)-\frac{1}{2}\frac{r_4}{\theta\bar{s}}(t_e-t^*)-\frac{1}{2}(t_e-t_s)+\frac{1}{2}\frac{(t_{22}-t_s)}{\theta}\right] \tag{E.92}$$

The total late cost with respect to $t\in[t^*,t_e]$ follows (E.86), except that t_{13} is replaced with t^* and LC_{14} is replaced with LC_{24}. Then the total late cost in Pattern 1 is given as follows,

$$LC_2=\pi\gamma\left\{\left[\theta\bar{s}(t_e-t_s)-\frac{r_4}{2}(t_e-t^*)\right](t_e-t^*)+\frac{1}{2}\theta\bar{s}(t^*-t_s)^2-\frac{1}{2}N(t_e-t_s)\right\}+\frac{\gamma r_4}{2}(t_e-t^*)^2 \tag{E.93}$$

Differentiating the total late cost with respect to $\bar{\pi}$, we have

$$\frac{dLC_2}{d\bar{\pi}}=\frac{\gamma}{2}(1-\pi)\left(t_e-t^*\right)^2\frac{dr_4}{d\bar{\pi}} \tag{E.94}$$

According to (E.25), (E.94) is positive. Use the expression (E.32) and we have

$$\frac{dLC_2}{d\lambda}=\frac{dLC_2}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dLC_2}{d\bar{\pi}}>0 \tag{E.95}$$

There is no late cost in the first situation of Pattern 3, i.e., $LC_{31}=0$. The total late cost with respect to $t\in\left[t_{31},t^*\right]$ follows (E.81), except that the integral interval $\left[t_{11},t_{12}\right]$ is replaced with $\left[t_{31},t^*\right]$, r_2 is replaced with r_5 and LC_{12} is replaced with LC_{32}. $R(t)$ is given as (E.48). Using the condition (E.49), we have

$$LC_{32}=\frac{1}{2}\frac{1}{\theta\bar{s}}\pi\gamma\left[N-\theta\bar{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right)\right]^2 \tag{E.96}$$

The late cost with respect to $t\in\left[t^*,t_e\right]$ follows (E.85), except that t_{13} is replaced with t^* and LC_{14} is replaced with LC_{33}. $R(t)$ is given as follows,

$$R(t)=R\left(t^*\right)+r_4\left(t-t^*\right)=\theta\bar{s}\left(t^*-t_s\right)+r_5\left(t^*-t_{31}\right)+r_4\left(t-t^*\right) \tag{E.97}$$

Using the condition (E.49), the late cost with respect to $t\in\left[t^*,t_e\right]$ is given as follows,

$$LC_{33}=\frac{\pi\gamma r_4\left(t_e-t^*\right)}{\theta\bar{s}}\left[N-\theta\bar{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right)\right]+\frac{\gamma r_4\left(t_e-t^*\right)^2}{2}\left[\left(\frac{r_4}{\theta\bar{s}}-1\right)\pi+1\right] \tag{E.98}$$

Then the total late cost in Pattern 3 can be given as follows,

$$LC_3=\frac{\pi\gamma}{2\theta\bar{s}}\left[N-\theta\bar{s}\left(t^*-t_s\right)-r_4\left(t_e-t^*\right)\right]^2+\frac{\gamma r_4\left(t_e-t^*\right)^2}{2}\left[\left(\frac{r_4}{\theta\bar{s}}-1\right)\pi+1\right] \tag{E.99}$$

Differentiating the total late cost in Pattern 3 with respect to $\bar{\pi}$, we have

$$\frac{dLC_3}{d\bar{\pi}}=\frac{\gamma}{2}(1-\pi)\left(t_e-t^*\right)^2\frac{dr_4}{d\bar{\pi}} \tag{E.100}$$

(E.100) is the same as (E.94), which is positive. Use the expression (E.32) and we have

$$\frac{dLC_3}{d\lambda}=\frac{dLC_3}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda}=\sqrt{\pi(1-\pi)}\frac{dLC_3}{d\bar{\pi}}>0 \tag{E.101}$$

According to (E.90), (E.95) and (E.101), the total late cost increases with the increase of λ in Patterns 1-3.

In Pattern 4, the late cost with respect to $t\in\left[t_s,t_{41}\right]$ equals to zero similar to Pattern 1, i.e., $LC_{41}=0$. The late cost with respect to $t\in\left[t_{41},t_{42}\right]$ follows (E.82), except that LC_{12} is replaced with LC_{42} and t_s is given as (26) in Pattern 4. In Situation 3, the late cost with respect to $t\in\left[t_{42},t_e\right]$ follows (E.83), except that the integral interval $\left[t_{12}, t_{13}\right]$ is replaced with $\left[t_{42}, t_e\right]$ and LC_{13} is replaced with LC_{43}. In this time interval, substituting (E.14) into the expression of LC_{43}, we have

$$\text{Late Cost}_{43} = \gamma r_3\left(t_e - t_{42}\right)\left[\pi\left(\frac{1}{\theta}-1\right)\left(t^* - t_s\right) + \frac{1}{2}\left(\frac{\pi}{\theta\bar{s}} + \frac{1-\pi}{\bar{s}}\right) r_3\left(t_e - t_{42}\right)\right] \tag{E.102}$$

According to the boundary condition in Pattern 4, we have

$$r_3\left(t_e - t_{42}\right) = N - \bar{s}\left(t^* - t_s\right) \tag{E.103}$$

Substituting (E.103) into (E.102), we have

$$LC_{43} = \gamma\pi\left(\frac{1}{\theta}-1\right)\left(t^* - t_s\right)\left[N - \bar{s}\left(t^* - t_s\right)\right] + \frac{\gamma}{2}\left(\frac{\pi}{\theta\bar{s}} + \frac{1-\pi}{\bar{s}}\right)\left[N - \bar{s}\left(t^* - t_s\right)\right]^2 \tag{E.104}$$

Then the total late cost in Pattern 4 can be given as follows

$$LC_4 = \frac{\gamma}{2}\left[1 - \pi\left(1-\theta\right)\right]\bar{s}\left(t^* - t_s\right)^2 - \gamma N\left(t^* - t_s\right) + \frac{\gamma}{2}\left(\frac{\pi}{\theta\bar{s}} + \frac{1-\pi}{\bar{s}}\right) N^2 \tag{E.105}$$

Differentiating the total late cost in Pattern 4 with respect to $\bar{\pi}$, we have

$$\frac{dLC_4}{d\bar{\pi}} = \gamma\left\{\left[1 - \pi\left(1-\theta\right)\right]\bar{s}\left(t^* - t_s\right) - N\right\}\frac{d\left(t^* - t_s\right)}{d\bar{\pi}} \tag{E.106}$$

According to (26) in Pattern 4, we have

$$\frac{d\left(t^* - t_s\right)}{d\bar{\pi}} = -\frac{dt_s}{d\bar{\pi}} = N\left(\frac{1}{\theta}-1\right)\frac{\left(\alpha+\gamma\right)}{\bar{s}\left(\beta+\gamma\right)} \tag{E.107}$$

By definition, $0 < \theta \le 1$ and $0 \le \pi \le 1$. Hence, $1 - \pi\left(1-\theta\right) < 1$. Thereby, we have $\left[1 - \pi\left(1-\theta\right)\right]\bar{s}\left(t^* - t_s\right) - N < 0$ always holds. Accordingly, we have $dLC_4/d\bar{\pi} < 0$.

Use the expression (E.32) and we have

$$\frac{dLC_4}{d\lambda} = \frac{dLC_4}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda} = \sqrt{\pi\left(1-\pi\right)}\frac{dLC_4}{d\bar{\pi}} < 0 \tag{E.108}$$

In Pattern 5, the late cost with respect to $t \in \left[t_s, t_{51}\right]$ is zero, i.e., $LC_{51} = 0$. The late cost with respect to $t \in \left[t_{51}, t_{52}\right]$ follows (E.91), except that LC_{22} is replaced with LC_{52} and t_{22} is replaced with t_{52}. The late cost with respect to $t \in \left[t_{52}, t^*\right]$ follows (E.81), except that the integral interval $\left[t_{11}, t_{12}\right]$ is replaced with $\left[t_{52}, t^*\right]$, r_2 is replaced with r_5 and LC_{12} is replaced with LC_{53}. $R(t)$ follows (E.36), except that t_{22} is replaced with t_{52}. According to the boundary condition in Pattern 5, we have

$$r_5\left(t^* - t_{52}\right) = N - \bar{s}\left(t_{52} - t_s\right) \tag{E.109}$$

Substituting (E.109) into the expression of LC_{53}, we have

$$LC_{53} = \pi\gamma\left[N - \bar{s}\left(t_{52} - t_s\right)\right]\left[\frac{\left(t_{52} - t_s\right)}{\theta} + \left(t_s - t^*\right) + \frac{1}{2}\frac{N}{\theta\bar{s}} - \frac{1}{2}\frac{\left(t_{52} - t_s\right)}{\theta}\right] \quad \text{(E.110)}$$

Then the total late cost in Pattern 5 can be given as follows

$$LC_5 = \pi\gamma\left[\frac{1}{2}\left(\theta\bar{s}\right)\left(t^* - t_s\right)^2 - N\left(t^* - t_s\right) + \frac{1}{2}\frac{N^2}{\theta\bar{s}}\right] \quad \text{(E.111)}$$

Differentiating the total late cost in Pattern 5 with respect to $\bar{\pi}$, we have

$$\frac{dLC_5}{d\bar{\pi}} = -\pi\gamma\left[N - \theta\bar{s}\left(t^* - t_s\right)\right]\frac{d\left(t^* - t_s\right)}{d\bar{\pi}} \quad \text{(E.112)}$$

According to (28) in Pattern 5, we have

$$\frac{dt_s}{d\bar{\pi}} = -\frac{N\left(\alpha + \gamma\right)}{\theta\bar{s}}\frac{\beta}{\left(\beta + \left(\alpha + \gamma\right)\bar{\pi}\right)^2} < 0 \quad \text{(E.113)}$$

Hence, we have $d\left(t^* - t_s\right)/d\bar{\pi} > 0$ and $dLC_5/d\bar{\pi} < 0$. Use the expression (E.32) and we have

$$\frac{dLC_5}{d\lambda} = \frac{dLC_5}{d\bar{\pi}}\frac{d\bar{\pi}}{d\lambda} = \sqrt{\pi\left(1 - \pi\right)}\frac{dLC_5}{d\bar{\pi}} < 0 \quad \text{(E.114)}$$

In Pattern 6, the late cost with respect to $t \in \left[t_s, t_{61}\right]$ is zero, i.e., $LC_{61} = 0$. The total late cost with respect to $t \in \left[t_{61}, t^*\right]$ follows (E.81), except that the integral interval $\left[t_{11}, t_{12}\right]$ is replaced with $\left[t_{61}, t^*\right]$, r_2 is replaced with r_5 and LC_{12} is replaced with LC_{62}. $R(t)$ is given as follows,

$$R(t) = R\left(t_{61}\right) + r_5\left(t - t_{61}\right) = \theta\bar{s}\left(t^* - t_s\right) + r_5\left(t - t_{61}\right) \quad \text{(E.115)}$$

Then we have

$$LC_{62} = \frac{1}{2}\frac{1}{\theta\bar{s}}\pi\gamma r_5^2\left(t^* - t_{61}\right)^2 \quad \text{(E.116)}$$

According to the boundary condition, we know that

$$r_5\left(t^* - t_{61}\right) = N - \theta\bar{s}\left(t^* - t_s\right) \quad \text{(E.117)}$$

Substitute (E.117) into (E.116) and the total late cost in Pattern 6 is given as follows

$$LC_6 = \frac{1}{2}\frac{1}{\theta\bar{s}}\pi\gamma\left[N - \theta\bar{s}\left(t^* - t_s\right)\right]^2 \quad \text{(E.118)}$$

Differentiating the total late cost in Pattern 6 with respect to $\bar{\pi}$, we have

$$\frac{dLC_5}{d\bar{\pi}} = -\pi\gamma\left[N - \theta\bar{s}\left(t^* - t_s\right)\right]\frac{d\left(t^* - t_s\right)}{d\bar{\pi}} \quad \text{(E.119)}$$

(E.119) is the same as (E.112). Therefore, $dLC_6/d\lambda < 0$. Hence, according to (E.108), (E.112) and (E.119), the total late cost decreases with the increase of λ in Patterns 4-6. This completes the proof. □