Highlights

- Liquidity risk has been largely ignored for online portfolio selection methods.
- We present an optimal intraday trading algorithm that considers liquidity risk.
- Our algorithm is compatible with any existing online portfolio selection methods.
- We develop a simplified suboptimal solution to increase computational efficiency.
- The suboptimal solution converges when the fund size is over $10 million.

Algorithmic trading for online portfolio selection under limited market liquidity

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We propose an optimal intraday trading algorithm to reduce overall transaction costs by absorbing price shocks when an online portfolio selection (OPS) method rebalances its portfolio. Having considered the real-time data of limit order books (LOB), the trading algorithm optimally splits a sizeable market order into a number of consecutive market orders to minimize the overall transaction costs, including both the liquidity costs and the proportional transaction costs. The proposed trading algorithm, compatible with any OPS methods, optimizes the number of intraday trades and finds an optimal intraday trading path. Backtesting results from the historical LOB data of NASDAQ-traded stocks show that the proposed trading algorithm significantly reduces the overall transaction costs when market liquidity is limited.

Keywords: Investment analysis; Algorithmic trading; Online portfolio selection; Market impact cost; Limit order book

JEL Classification: C61, C63, G11, G23

1. Introduction

Online portfolio selection (OPS henceforth) rebalances a portfolio in every period with the aim of maximising the portfolio's expected terminal wealth as a whole. OPS differs from the previous studies of prediction-based portfolio selection (Freitas *et al.* 2009, Otranto 2010, Ferreira and Santa-Clara 2011, DeMiguel *et al.* 2014, Palczewski *et al.* 2015) which forecast the expected values or covariance matrix of stock returns under the mean-variance optimization framework. Most of the existing OPS methods, see Györfi and Vajda (2008), Kozat and Singer (2011) among others, consider only the proportional transaction costs (hereafter TCs). However, liquidity risk or the market impact costs (MICs), a common feature of financial markets, has been largely ignored in OPS literature. Although Castellano and Cerqueti (2014) consider a mean–variance optimal portfolio selection problem in presence of limited liquidity reflected by the dynamics of low-frequency trading (illiquid) assets following pure-jump processes, they mainly focus on the liquidity condition of financial markets. Ha and Zhang (2018) propose a more practical OPS method that considers both the proportional TCs and MICs, but none compatible optimal intraday trading algorithm has been developed in their framework to reduce the higher liquidity risk involved.

As Bertsimas and Lo (1998) pointed out"the demand for financial securities is not perfectly elastic, and the price impact of current trades, even small trades, can affect the course of future prices", then a big concern for existing OPS methods is its sizeable TCs when frequently sending large market orders to rebalance a large-sized portfolio.Indeed a large market order might not be executed due to limited liquidity at the rebalancing time. Even if it is executed, the transaction normally involves significantly high liquidity risk due to large volume trading, pushing the price up when buying the asset and pushing it down while selling (Damodaran 2012, Chapter 5). Moreover, it might cause a permanent rather temporary impact on asset prices, thus making OPS strategies

Table 1. A 5-level limit order book of Microsoft Corporation, traded on NASDAQ, on 21 Jun 2012 at 16:00:00 (downloaded from https://lobsterdata.com/info/DataSamples.php). Bid-ask spread is USD 0.01, and midpoint price is USD 30.135.

	Level	Price (USD)	Volume (shares)
Asks	5	30.18	110,006
	4	30.17	86,886
	3	30.16	65,399
	2	30.15	80,663
	1	30.14	16,600
Bids	-1	30.13	-50,426
	-2	30.12	-83,306
	-3	30.11	-8,506
	-4	30.10	-43,838
	-5	30.09	-167,371

unprofitable. As the main assumption that portfolio rebalancing by OPS in the current period does not affect the stock prices in the next one is no longer valid should the rebalancing cause a permanent impact.

Existing trading algorithms might be considered for OPS to rebalance a large-sized portfolio, however, almost most of them are impractical as none of which considers the liquidity risk reflected by the limit order book (hereafter LOB; Table 1 provides an example of LOB). For example, Almgren and Chriss (2000) and Kissell *et al.* (2004) mathematically model the market impact but do not consider LOB. Alfonsi *et al.* (2008, 2010) consider the shape of LOB as a block or a continuous function rather than using LOB data. Guéant *et al.* (2012) use LOB data to calibrate the intensity parameters of trading execution, but they do not directly use LOB data for optimal trading.

The aim of this paper, therefore, is to develop an algorithmic trading strategy that splits a very large market order into a number of consecutive market orders to reduce price shocks and minimize the overall TCs when market liquidity is limited. This is straightforward as the market usually absorbs shocks from these smaller slices, resulting in reduced MICs (Kissell *et al.* 2003, p. 196). To our knowledge, this paper is the first attempt to combine OPS with algorithmic trading under limited market liquidity. Propose an optimal intraday trading algorithm by considering real-time LOB data, and being compatible with any OPS method and available for any fund size, is the main contribution of this paper. Moreover, we provide further evidence of the superiority of our method from the backtesting using real-world historical NASDAQ LOB data.

The rest of this paper is organized as follows. Section 2 briefly reviews existing algorithmic trading strategies. Some key mathematical notations used in this paper are summarized in Section 3. Section 4 introduces a liquidity risk measure using LOB and develops a mathematical model of optimal intraday trading for multi-asset portfolios, followed by backtesting in Section 5. Finally, Section 6 concludes. Technical details and further numerical results can be found in Appendices.

2. Literature review of algorithmic trading

Algorithmic trading is the computerised execution of financial instruments following pre-specified rules and guidelines (Kissell 2013, p. 269), and it is classified by Kissell (2013, pp. 17–20) as follows:

(i) **Arrival price algorithm** that optimizes a trading path to balance the trade-off between cost and risk at a user-specified level of risk aversion;
(ii) **Implementation shortfall algorithm**, which is similar to the arrival price algorithm, but incorporates real-time adaptation, i.e. the trading path of implementation shortfall algorithm is updated by real-time data on every intraday trade while that of the arrival price algorithm is determined before trading and does not change during an intraday trade;
(iii) **Black box algorithm** that searches for profitable opportunities and makes investment

decisions based on market signals (e.g. asset prices and trading volume).

Based on the Kissell (2013)'s classification,we briefly review the aforementioned three trading algorithms in the following subsections.

2.1. *Arrival price algorithm*

Almgren and Chriss (2000) and Kissell *et al.* (2004) propose an efficient frontier (akin to the Markowitz efficient frontier in the portfolio theory) in a two-dimensional plane whose axes are: the expected value of MICs arising from the temporary and permanent market impact and its variance, which comes from price volatility. Hence, the efficient frontier allows investors to choose their trading strategy for portfolio management with a user-specified parameter of risk aversion. The difference between the two methods is how to derive the equation of MICs. The former was derived from consecutive trades, whereas the latter was derived from an aggregate trade.

Alfonsi *et al.* (2008) suggest a trading strategy that splits a very large market order for a single-asset portfolio into a number of consecutive market orders to reduce expected overall market impact. The size of the individual orders is determined by a parameter of the resilience rate of a block-shaped LOB (it is assumed that an LOB consists of a continuous price distribution of orders with a constant height). However, the strategy does not consider the risk of price volatility as it assumes that traders are risk-neutral, which is different from the Almgren and Chriss (2000) and Kissell *et al.* (2004)'s trading strategy. Therefore, the Alfonsi *et al.*'s strategy minimises the expected value of MICs regardless of the risk.

Alfonsi *et al.* (2010) extend their previous trading strategy in LOBs of the constant function to one of a general shape function. They modeled discrete data of LOB as a continuous function of LOB density. Both strategies have the same optimal solution of intermediate orders: $\xi_1 = \xi_2 = \cdots = \xi_{N-1}$, where ξ_n is the size of the market order placed at time t_n, and t_N is the ending time of trading. However, the optimal initial market order ξ_0 for the generally-shaped LOBs is expressed as an implicit formula, while that for the block-shaped LOBs is expressed as an explicit formula.

2.2. *Implementation shortfall algorithm*

A path-dependent or dynamic trading strategy by Lorenz (2008, Chapter 2–3), where the trading path is updated by real-time data, is superior in terms of generating a more efficient frontier to the path-independent or static trading strategy by Almgren and Chriss (2000), where the trading path is determined before trading starts. The superiority of the dynamic strategy comes from trading faster and reducing the risk of price volatility for the remaining time in the future if there was a windfall trading gain (i.e. lower trading cost) in the past.

Two other trading strategies as in Almgren and Chriss (2000), Guéant *et al.* (2012) respectively are similar in terms of liquidating a certain quantity of a single-asset portfolio within a given time horizon. However, they are different in terms of order types and optimization methods. The former sends market orders by considering the trade-off between price risk and MICs, whereas the latter sends limit orders by considering both price risk and non-execution risk. Moreover, the Almgren and Chriss's strategy uses quadratic programming to construct an efficient frontier where the trade-off between price risk and MICs is binding, whereas the Guéant *et al.*'s strategy applies the dynamic programming method to solve the stochastic control problem of optimal liquidation.

2.3. *Black box algorithm*

The following types of black box algorithm are reviewed:

- **Pairs trading**
  Statistical arbitrage strategies based on cointegrated pairs of assets;

- **High-frequency trading**
  Short-term trading strategies using sophisticated mathematics and high-speed computers;
- **Artificial intelligence trading**
  Alpha generation strategies using artificial intelligence techniques.

2.3.1. **Pairs trading.** Avellaneda and Lee (2010) construct a statistical arbitrage strategy of a market-neutral long–short portfolio. To be specific, they long one dollar in a stock and short β_j dollars in the j-th factor, where a multi-factor regression model decomposes a stock return R into the sum of systematic components $\sum_{j=1}^{m} \beta_j F_j$ (m is the number of factors) and an idiosyncratic component $\tilde{R}$:

$$R = \sum_{j=1}^{m} \beta_j F_j + \tilde{R}. \tag{1}$$

This method generates trading signals of buy, sell, or close of the long (short) position by using the mean-reverting property of the long–short portfolio's return.

Similar to Avellaneda and Lee (2010), Caldeira and Moura (2013) also consider a market-neutral strategy but use a different long–short portfolio. Avellaneda and Lee's strategy constructs either a pair of one stock and multiple exchange-traded funds or a pair of one stock and factors calculated from the principal component analysis while Caldeira and Moura construct only two stocks as a pair. Moreover, if these two stocks selected are cointegrated, then it is possible to form a mean-reverting stationary process from a linear combination of stock A and B.

Having extended previous pairs trading research in another direction, Huck (2010) incorporates the forecasting techniques (neural networks) into multi-criteria decision making methods (Electre III), the application of which leads to better pairs selection in a highly non-linear environment.

2.3.2. **High-frequency trading.** A Markov chain model of the short-term dynamics of a LOB has been proposed by Cont *et al.* (2010). To be specific, the volume of limit orders (see Table 1) is modeled as a Markov state, where a state transition occurs by a limit order, a market order, or a stop order. Furthermore, this model can be applied to high-frequency trading by making a short-term prediction of the mid-price and making a round-trip transaction. It enters a long position when the probability of the mid-price increasing is high, and it exits the position either when a profit is secured or when a loss of one tick is made.

A high-frequency arbitrage opportunity through an empirical analysis of the LOB resiliency of stocks traded on Shenzhen Stock Exchange was found by (Xu *et al.* 2017). The analysis showed that buy (sell) market orders attract more buy (sell) limit orders especially i) when the bid-ask spread is one tick and ii) when the buy (sell) market order size is less than the best ask (bid) volume, which is the volume of LOB at level 1 (-1) (see Table 1).

2.3.3. **Artificial intelligence trading.** Tan *et al.* (2011) show how to detect stock cycles from historical stock prices. Based on the mean reverting property of stock prices they provide a reinforcement learning framework to trade on the cycles. Specifically, long positions are held after detecting troughs of stock cycles, and short positions are held after detecting peaks of stock cycles. Besides, a dynamic asset switching strategy was proposed to detect buying opportunities (troughs) of all assets in a portfolio.

Mousavi *et al.* (2014) propose a multi-tree genetic programming model that i) extracts profitable trading rule bases for a multi-asset portfolio from historical data (daily closing price and transaction volume) and ii) updates the portfolio weights over time. Even though it generates a distinct decision rule for each stock, the rules for multiple stocks evolve simultaneously, and the correlations among multiple stocks are taken into account. Besides, the proposed model includes liquidity risk and TCs to make the trading system more realistic.

Bendtsen and Peña (2016) develop a single-asset trading algorithm with either a long or closed position (i.e. no short selling) by using technical indicators such as moving average and relative strength index. Its goal is to generate buy or sell signals for trading a stock by learning and predicting the stock movement. A gated Bayesian network has been adopted to create a lower risk investment strategy compared with the buy-and-hold strategy. The network goes back and forth between the buy and sell phases and seeks opportunities to buy or sell shares.

Krauss *et al.* (2017) generate daily trading signals from lagged returns of stocks. They conduct nonparametric nonlinear regression between the lagged returns and one-day-ahead return. In particular, the following nonlinear regression methods were employed: deep neural networks, gradient-boosted trees, and random forests. After the regression, a daily portfolio of either going long for the stocks of higher expected returns or going short for lower expected return stocks has been constructed from the combined signals of the three methods.

3. Notations

The following notations are used in this paper:

- A lower case italic letter x indicates a deterministic scalar value, while a capital italic letter X indicates a random variable. A lower case italic bold letter $\boldsymbol{x}$ indicates a deterministic vector, while a capital italic bold letter $\boldsymbol{X}$ indicates a multivariate random variable (i.e. a random vector). A capital upright bold letter $\mathbf{X}$ denotes a deterministic matrix or a random matrix.
- $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ denotes a positive real number, and $\mathbb{R}_+^d = \left\{ \left[x^{(1)}\ x^{(2)}\ \ldots\ x^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}^d \middle| x^{(j)} > 0, \forall j \in \{1, 2, \ldots, d\} \right\}$ denotes a d-dimensional vector of positive real numbers.
- $\boldsymbol{b}_n = \left[b_n^{(1)}\ b_n^{(2)}\ \ldots\ b_n^{(d)}\right]^{\mathrm{T}}$ is a portfolio vector of d risky assets (there is no risk-free asset in the portfolio) on the n-th day (see Figure 1), where $n \in \{1, 2, \ldots\}$, $b_n^{(j)} \in 0 \cup \mathbb{R}_+$ (i.e. neither short selling nor buying stocks on margin is permitted), and $\sum_{j=1}^{d} b_n^{(j)} = 1$ (i.e. $b_n^{(j)}$ is the proportion of a portfolio invested in asset $j \in \{1, 2, \ldots, d\}$ at the n-th day). Hence, $\boldsymbol{b}_n \in \Delta^{d-1}$, where $\Delta^{d-1} = \left\{ \left[b^{(1)}\ b^{(2)}\ \ldots\ b^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}^d \,\middle|\, \sum_{j=1}^{d} b^{(j)} = 1, b^{(j)} \geq 0 \right\}$ is the standard $(d-1)$-simplex.
- $\boldsymbol{b}_{n,t} \in \Delta^{d-1}$ is an intraday portfolio vector at time t after the end of the n-th day, where $t \in \{0, 1, \ldots, \tau\}$, and τ is the number of intraday trades (see Figure 1; the portfolio rebalancing from $\boldsymbol{b}_{n,\tau}$ to $\boldsymbol{b}_{n+1}$ is not counted as an intraday trade).
- $\boldsymbol{b}_1 = [1/d\ 1/d\ \ldots\ 1/d]^{\mathrm{T}}$ is an initial portfolio vector.
- A deterministic value $m_n^{(j)}$ (if n is a past or present day), or a random variable $M_n^{(j)}$ (if n is a future day) is the mid-price of asset j at the end of the n-th day.
- A deterministic value $m_{n,t}^{(j)}$ (if n, t is in the past or at the present), or a random variable $M_{n,t}^{(j)}$ (if n, t is in the future) is the mid-price of asset j at time t after the end of the n-th day. Technically, $m_{n,0}^{(j)} = m_n^{(j)}$ and $M_{n,0}^{(j)} = M_n^{(j)}$.
- A deterministic value $x_n^{(j)} = \dfrac{m_n^{(j)}}{m_{n-1}^{(j)}}$, or a random variable $X_n^{(j)} = \dfrac{M_n^{(j)}}{M_{n-1}^{(j)}}$ $\left(\text{or } X_n^{(j)} = \dfrac{M_n^{(j)}}{m_{n-1}^{(j)}}\right)$ is the relative price of asset j for one day at the end of the n-th day.
- A deterministic vector $\boldsymbol{x}_n = \left[x_n^{(1)}\ x_n^{(2)}\ \ldots\ x_n^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}_+^d$, or a multivariate random variable $\boldsymbol{X}_n = \left[X_n^{(1)}\ X_n^{(2)}\ \ldots\ X_n^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}_+^d$ is the vector of the relative prices of all assets at the end of the n-th day ($\boldsymbol{x}_n$ or $\boldsymbol{X}_n$ is called a market vector in this paper).

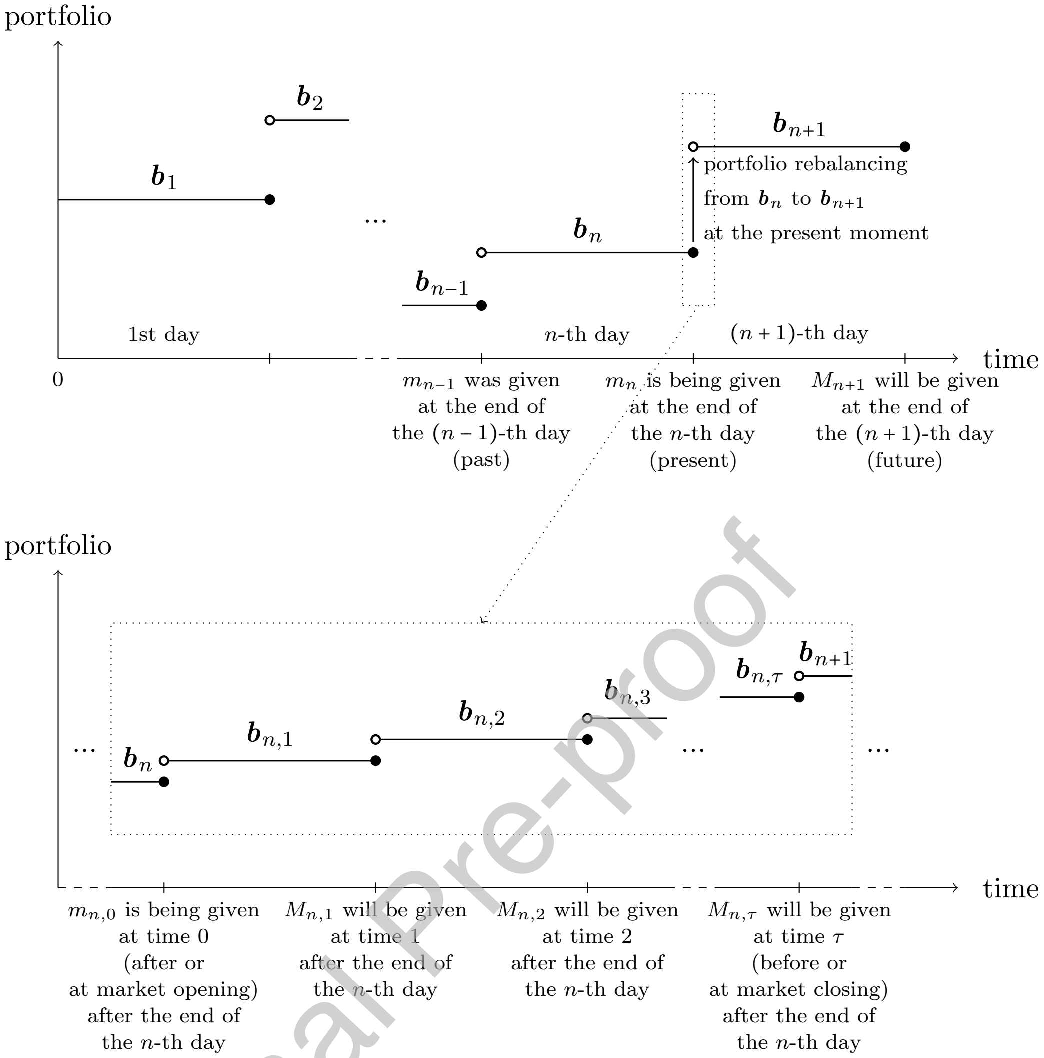


Figure 1. The timeline of intraday trading when the present moment is the end of n-th day. OPS rebalances a portfolio at the end of every trading day (a day ends at time 0, e.g. 9:30 a.m. or 10:00 a.m., not midnight, in this paper), and an algorithmic trading strategy cushions the shock by portfolio rebalancing from $\boldsymbol{b}_n$ to $\boldsymbol{b}_{n+1}$.

- A deterministic value $x_{n,t}^{(j)} = \dfrac{m_{n,t}^{(j)}}{m_{n,t-1}^{(j)}}$, or a random variable $X_{n,t}^{(j)} = \dfrac{M_{n,t}^{(j)}}{M_{n,t-1}^{(j)}}$ $\left(\text{or } X_{n,t}^{(j)} = \dfrac{M_{n,t}^{(j)}}{m_{n,t-1}^{(j)}}\right)$ is the intraday relative price of asset j between time $t-1$ and t after the end of the n-th day, where $m_{n,-1} \stackrel{\text{def}}{=} m_{n-1,\tau}$, and $M_{n,-1} \stackrel{\text{def}}{=} M_{n-1,\tau}$.
- A deterministic vector $\boldsymbol{x}_{n,t} = \left[x_{n,t}^{(1)}\ x_{n,t}^{(2)}\ \ldots\ x_{n,t}^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}_+^d$, or a multivariate random variable $\boldsymbol{X}_{n,t} = \left[X_{n,t}^{(1)}\ X_{n,t}^{(2)}\ \ldots\ X_{n,t}^{(d)}\right]^{\mathrm{T}} \in \mathbb{R}_+^d$ is the vector of the intraday relative prices of all assets, at time t after the end of the n-th day ($\boldsymbol{x}_{n,t}$ or $\boldsymbol{X}_{n,t}$ is called an intraday market vector in this paper).

4. Model Setup

In this section, we aim to minimize the overall TCs including both the proportional transaction costs and liquidity costs for OPS methods when rebalancing a portfolio. We first introduce the measure of liquidity risk using real LOB data and incorporate the liquidity risk measure within the framework of transaction cost factor in Section 4.1. Then we construct a benchmark model, being compatible with existing OPS methods, where there is no intraday trading activity. We further extend our benchmark model to a more general case with intraday trading and propose the *Algorithm 1* to calculate the optimal number of intraday trades. Finally, we develop the *Algorithm 2* for intraday algorithmic trading that considers real LOB data.

To be more specific, we explain in details in the following subsections:

(i) how to incorporate liquidity costs into the framework of existing OPS methods in Section 4.1;
(ii) how to calculate the expected value of the gross wealth of tomorrow in the case of no intraday trading (i.e. $\tau = 0$) in Section 4.2;
(iii) how to obtain an optimal trading path when rebalancing a portfolio from $\boldsymbol{b}_n$ to $\boldsymbol{b}_{n+1}$, given the number of intraday trades $\tau \geq 1$ (see Figure 1) in Section 4.3;
(iv) how to calculate the optimal number of intraday trades τ^* in Section 4.4;
(v) and how to consider real-time LOB data for optimal intraday trading in Section 4.5.

For the simplicity of our proposed intraday trading model, we make the following assumptions:

ASSUMPTION 4.1 *Asset prices follow the multi-dimensional Brownian motion with zero drift (forecasting expected returns is not performed in this paper) during intraday trading;*

- *hence, the increments of asset prices are jointly normally distributed;*
- *and the increments of asset prices are mutually independent for different trading times $t \neq t'$.*

ASSUMPTION 4.2 *LOB at time $t' \in \{t+1, t+2, \ldots, \tau\}$ is the same as LOB at time t on the same day, where the time unit "1" is 30 minutes in our case.*

4.1. *Liquidity risk measure and transaction cost factor*

Following Olsson (2005), we first define the average price per share $\bar{p}(q,\ldots)$ for the order size q. Obviously, the average price per share for specific order size depends on the LOB structure. For example, as in Table 1, if an investor wants to buy 10,000 shares of Microsoft, she pays \$30.14 per share. If she wants to buy 20,000 shares, then she pays \$30.14 for the first 16,600 shares purchased, and for the remaining 3,400 shares she has to pay a higher price (\$30.15 per share). Indeed, the higher the order size in this case, the higher average price per share to be paid by the investor.

The average price per share for the order size q is defined as (Olsson 2005, Chapter 2.3)

$$\bar{p}(q, m, p_1, p_2, \ldots, p_{-1}, p_{-2}, \ldots, v_1, v_2, \ldots, v_{-1}, v_{-2}, \ldots) \stackrel{\text{def}}{=} \begin{cases} \dfrac{\sum_{i=k+1}^{-1} p_i v_i + p_k \left(q - \sum_{i=k+1}^{-1} v_i\right)}{q}, & \text{if } q < v_{-1} \\ p_{-1}, & \text{if } v_{-1} \leq q < 0 \\ m, & \text{if } q = 0 \\ p_1, & \text{if } 0 < q \leq v_1 \\ \dfrac{\sum_{i=1}^{k-1} p_i v_i + p_k \left(q - \sum_{i=1}^{k-1} v_i\right)}{q}, & \text{if } v_1 < q \end{cases}, \tag{2}$$

where

- $m = \frac{p_{-1}+p_1}{2}$ is the midpoint between the best bid and the best ask price, called mid-price,
- positive (negative) q stands for buying (selling) stocks,
- p_i and v_i with positive (negative) i are the quoted ask (bid) price and volume at level i,

respectively (p_i and v_i correspond to the second and third column of Table 1, respectively, where $v_i \geq 0, v_{-i} \leq 0, \forall i \in \{1, 2, \dots\}$),

- and the highest (lowest) trading level k when $q > v_1$ ($q < v_{-1}$) is given by

$$k = \left\{ x \in \mathbb{Z} \;\middle|\; x \geq 2, \sum_{i=1}^{x-1} v_i < q \leq \sum_{i=1}^{x} v_i \right\}, \tag{3a}$$

$$\left(k = \left\{ x \in \mathbb{Z} \;\middle|\; x \leq -2, \sum_{i=x}^{-1} v_i \leq q < \sum_{i=x+1}^{-1} v_i \right\} \right), \tag{3b}$$

Indeed k represents the level in the order book where the q-th share would be executed.

Similar to Györfi and Vajda (2008), we now define the transaction cost factor as the ratio of the net wealth over the gross wealth, which is given by

$$w_{n,t} \stackrel{\text{def}}{=} \frac{\nu_{n,t}}{s_{n,t}}, \tag{4}$$

where $\nu_{n,t} \stackrel{\text{def}}{=} s_{n,t} - \gamma_{n,t}$ is the net wealth, $\gamma_{n,t}$ is transaction cost (TC) at time t after the end of the n-th day, and $s_{n,t}$, the gross wealth, is given by

$$s_{n,t} = \nu_{n,t-1} \sum_{j=1}^{d} b_{n,t}^{(j)} x_{n,t}^{(j)} = \nu_{n,t-1} \langle \boldsymbol{b}_{n,t}, \boldsymbol{x}_{n,t} \rangle. \tag{5}$$

Unlike Györfi and Vajda (2008), our model considers both the proportional transaction costs and market impact costs (MICs) in our transaction cost factor, where MICs, a key measure of liquidity risk when OPS methods rebalance a portfolio, are defined as

$$\text{MICs} \stackrel{\text{def}}{=} \sum_{j=1}^{d} \left(\bar{p}\left(q_{n,t}^{(j)} \right) - m_{n,t}^{(j)} \right) q_{n,t}^{(j)}. \tag{6}$$

Indeed, MICs can be explained as the extra price paid (the lower price received) for stock buyers (stock sellers) in our model.

The gross wealth $s_{n,t}$ is the sum of the net wealth $\nu_{n,t}$, the MICs, the purchase TCs, and the sale TCs (Ha and Zhang 2018, Section 5.2–5.3):

$$s_{n,t} = \nu_{n,t} + \sum_{j=1}^{d} \left(\bar{p}\left(q_{n,t}^{(j)} \right) - m_{n,t}^{(j)} \right) q_{n,t}^{(j)} + c_p \sum_{j=1}^{d} \left(\bar{p}\left(q_{n,t}^{(j)} \right) q_{n,t}^{(j)} \right)^{+} + c_s \sum_{j=1}^{d} \left(-\bar{p}\left(q_{n,t}^{(j)} \right) q_{n,t}^{(j)} \right)^{+}, \tag{7}$$

where $c_p, c_s \in [0, 1)$ denotes the rate of proportional TCs when purchasing and selling stocks respectively and $a^{+} \stackrel{\text{def}}{=} \max(0, a)$. [1] $q_{n,t}^{(j)}$ is an unknown order size of asset j:

$$q_{n,t}^{(j)} = \frac{b_{n,t+1}^{(j)} s_{n,t} w_{n,t} - b_{n,t}^{(j)} x_{n,t}^{(j)} \nu_{n,t-1}}{m_{n,t}^{(j)}}; \tag{8}$$

[1] Some arguments of Equation (2), $m, p_1, p_2, \dots, p_{-1}, p_{-2}, \dots, v_1, v_2, \dots, v_{-1}, v_{-2}, \dots$ of asset j, are omitted in Equation (7) and the subsequent expressions for notational simplicity.

and $b^{(j)}_{n,\tau+1} \stackrel{\text{def}}{=} b^{(j)}_{n+1}$. Equation (7) can be simplified, by the property of $a^+ = a + (-a)^+$, as

$$s_{n,t} = \nu_{n,t} + \sum_{j=1}^{d} \left((1 - c_s)\bar{p}\left(q^{(j)}_{n,t}\right) - m^{(j)}_{n,t} \right) q^{(j)}_{n,t} + (c_p + c_s) \sum_{j=1}^{d} \left(\bar{p}\left(q^{(j)}_{n,t}\right) q^{(j)}_{n,t} \right)^+, \tag{9}$$

and this can be rewritten, using Equation (4), as

$$w_{n,t} = 1 - \frac{\sum_{j=1}^{d} \left((1 - c_s)\bar{p}\left(q^{(j)}_{n,t}\right) - m^{(j)}_{n,t} \right) q^{(j)}_{n,t} + (c_p + c_s) \sum_{j=1}^{d} \left(\bar{p}\left(q^{(j)}_{n,t}\right) q^{(j)}_{n,t} \right)^+}{s_{n,t}}, \tag{10}$$

where $w_{0,t} \stackrel{\text{def}}{=} 1, \forall t \in \{0, 1, \ldots, \tau\}$ (i.e. there are no TCs between time 0 and τ on the 0-th day). Equation (8) and (10) are solvable by using a root-finding algorithm, where $w_{n,t} = w\left(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{x}_{n,t}, \nu_{n,t-1}\right)$ is an unknown variable, c_p and c_s are omitted here for notational simplicity.

4.2. *Benchmark: no intraday trading ($\tau = 0$)*

We first consider a benchmark case where there is no intraday trading activity. This is the worst case of liquidity risk management that existing OPS methods do not optimally minimize the liquidity risk.

Suppose that a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\{1,2,\ldots\}}, \mathbb{P}\right)$ is given, where $\mathcal{F}_n$ denotes the natural filtration of the process $\{\boldsymbol{X}_n\}_{n\in\{1,2,\ldots\}}$ up to day n. If no intraday trading occurs, the conditional expected value of the gross wealth at the end of the $(n+1)$-th day, given the past observations $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, and using Equation (5) and (4), is given by

$$\begin{aligned} \mathbb{E}[S_{n+1}|\mathcal{F}_n] &= \mathbb{E}[N_n\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1}\rangle|\mathcal{F}_n] \\ &= s_n w_n \mathbb{E}[\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1}\rangle], \end{aligned} \tag{11}$$

where

- N_n is stochastic net wealth at the end of the n-th day,
- $S_n = s_0 \prod_{i=1}^{n} \langle \boldsymbol{b}_i, \boldsymbol{X}_i\rangle$ is stochastic gross wealth at the end of the n-th day with an initial wealth s_0,
- $W_n = w(\boldsymbol{b}_n, \boldsymbol{b}_{n+1}, \boldsymbol{X}_n, N_{n-1})$ is stochastic TCF at the end of the n-th day ($S_n W_n$ and $\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1}\rangle$ are mutually independent by Assumption 4.1 that $\boldsymbol{X}_n$ and $\boldsymbol{X}_{n+1}$ are mutually independent),
- $s_n = s_0 \prod_{i=1}^{n} \langle \boldsymbol{b}_i, \boldsymbol{x}_i\rangle$ is deterministic gross wealth at the end of the n-th day,
- $w_n = w(\boldsymbol{b}_n, \boldsymbol{b}_{n+1}, \boldsymbol{x}_n, \nu_{n-1})$ is deterministic TCF at the end of the n-th day ($S_n W_n$ is converted to $s_n w_n$ by the conditional expectation given $\mathcal{F}_n$),
- and the market vector $\boldsymbol{X}_{n+1}$ is jointly normally distributed with the mean vector of all 1's $[1\ 1\ \ldots\ 1]^{\mathrm{T}}$.

By Assumption 4.1, Equation (11) can be further simplified, and the price change $\boldsymbol{M}_{n+1} - \boldsymbol{m}_n$ follows the multi-dimensional Brownian motion with zero drift:

$$(\boldsymbol{M}_{n+1} - \boldsymbol{m}_n) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_n), \tag{12}$$

where $\boldsymbol{M}_{n+1} = \left[M^{(1)}_{n+1}\ M^{(2)}_{n+1}\ \ldots\ M^{(d)}_{n+1}\right]^{\mathrm{T}}$ is the random mid-price vector at the end of the $(n+1)$-th day, $\boldsymbol{m}_n = \left[m^{(1)}_n\ m^{(2)}_n\ \ldots\ m^{(d)}_n\right]^{\mathrm{T}}$ is the deterministic mid-price vector at the end of the n-th day,

$\boldsymbol{0}$ denotes the all-zero vector, and $\boldsymbol{\Sigma}_n$ is the covariance matrix of price changes between the end of the n-th day and the end of the $(n+1)$-th day. Hence, $\boldsymbol{M}_{n+1}$ is jointly normally distributed as

$$\boldsymbol{M}_{n+1} \sim \mathcal{N}(\boldsymbol{m}_n, \boldsymbol{\Sigma}_n). \tag{13}$$

Also, as $\boldsymbol{X}_{n+1}$ can be calculated as

$$\boldsymbol{X}_{n+1} = \mathbf{D}^{-1}\boldsymbol{M}_{n+1}, \tag{14}$$

where $\mathbf{D} = \operatorname{diag}(\boldsymbol{m}_n)$, it is jointly normally distributed as

$$\boldsymbol{X}_{n+1} \sim \mathcal{N}\left(\boldsymbol{1}, \mathbf{D}^{-1}\boldsymbol{\Sigma}_n\mathbf{D}^{-1}\right). \tag{15}$$

Consequently, $\mathbb{E}[S_{n+1}|\mathcal{F}_n]$ in Equation (11) can be simplified as

$$\begin{aligned}\mathbb{E}[S_{n+1}|\mathcal{F}_n] =& s_n w_n \mathbb{E}[\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1}\rangle] \\ =& s_n w_n \left(b_{n+1}^{(1)}\mathbb{E}\left[X_{n+1}^{(1)}\right] + b_{n+1}^{(2)}\mathbb{E}\left[X_{n+1}^{(2)}\right] + \cdots + b_{n+1}^{(d)}\mathbb{E}\left[X_{n+1}^{(d)}\right]\right) = s_n w_n.\end{aligned} \tag{16}$$

4.3. *General intraday trading ($\tau \geq 1$)*

In a general case with intraday trading, then the conditional expected value of the gross wealth at the end of the $(n+1)$-th day, given the past observations $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, and using Equation (5), (4), and (11), is given by

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = s_n w_n \mathbb{E}\left[\left(\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}\right)\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1,0}\rangle\right], \tag{17}$$

where

- $w_n = w(\boldsymbol{b}_n, \boldsymbol{b}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})$,
  - $\boldsymbol{x}_{n,0} = \left[\frac{m_n^{(1)}}{m_{n-1,\tau}^{(1)}} \; \frac{m_n^{(2)}}{m_{n-1,\tau}^{(2)}} \; \cdots \; \frac{m_n^{(d)}}{m_{n-1,\tau}^{(d)}}\right]^{\mathrm{T}}$,
- $\boldsymbol{X}_{n,t} = \begin{cases}\left[\frac{M_{n,1}^{(1)}}{m_n^{(1)}} \; \frac{M_{n,1}^{(2)}}{m_n^{(2)}} \; \cdots \; \frac{M_{n,1}^{(d)}}{m_n^{(d)}}\right]^{\mathrm{T}}, & \text{if } t = 1 \\ \left[\frac{M_{n,t}^{(1)}}{M_{n,t-1}^{(1)}} \; \frac{M_{n,t}^{(2)}}{M_{n,t-1}^{(2)}} \; \cdots \; \frac{M_{n,t}^{(d)}}{M_{n,t-1}^{(d)}}\right]^{\mathrm{T}}, & \text{if } 2 \leq t \leq \tau\end{cases}$,
- $W_{n,t} = w(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{X}_{n,t}, s_n w_n \prod_{t'=1}^{t-1}\langle \boldsymbol{b}_{n,t'}, \boldsymbol{X}_{n,t'}\rangle W_{n,t'})$,
  - $\prod_{t'=1}^{0}(\cdot) \overset{\text{def}}{=} 1$,
  - $\boldsymbol{b}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$,
- and $\boldsymbol{X}_{n+1,0} = \left[\frac{M_{n+1}^{(1)}}{M_{n,\tau}^{(1)}} \; \frac{M_{n+1}^{(2)}}{M_{n,\tau}^{(2)}} \; \cdots \; \frac{M_{n+1}^{(d)}}{M_{n,\tau}^{(d)}}\right]^{\mathrm{T}}$.

By Assumption 4.1 that the random vectors $\boldsymbol{X}_{n,1}, \boldsymbol{X}_{n,2}, \ldots, \boldsymbol{X}_{n,\tau}, \boldsymbol{X}_{n+1,0}$ are mutually independent, Equation (17) can be rewritten as

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = s_n w_n \mathbb{E}\left[\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}\right]\mathbb{E}[\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1,0}\rangle], \tag{18}$$

but $\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle$ and $W_{n,t}$ are mutually dependent because $W_{n,t}$ is a function of $\boldsymbol{X}_{n,t}$. Also, $\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}$ and $\langle \boldsymbol{b}_{n,t'}, \boldsymbol{X}_{n,t'}\rangle W_{n,t'}$ are mutually dependent, where $t \neq t'$, because $W_{n,t}$ is a function of $\boldsymbol{X}_{n,1}, \boldsymbol{X}_{n,2}, \ldots, \boldsymbol{X}_{n,t-1}$ as well as $\boldsymbol{X}_{n,t}$. Finally, by using the property of $\mathbb{E}[\langle \boldsymbol{b}_{n+1}, \boldsymbol{X}_{n+1,0}\rangle] = 1$, as proved in Equation (16), $\mathbb{E}[S_{n+1}|\mathcal{F}_n]$ in Equation (17) can be simplified as

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = s_n w_n \mathbb{E}\left[\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}\right]. \tag{19}$$

Our goal is to find the optimal portfolio vectors $\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \dots, \boldsymbol{b}^*_{n,\tau}$ that maximize $\mathbb{E}[S_{n+1}|\mathcal{F}_n]$, and this is a stochastic programming problem as follows:

$$\begin{aligned}
&\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \dots, \boldsymbol{b}^*_{n,\tau} \\
&= \underset{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}}{\arg\max} \; \mathbb{E}[S_{n+1}|\mathcal{F}_n] \\
&= \underset{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}}{\arg\max} \; w_n \int_{\boldsymbol{x}_{n,1}\in\mathbb{R}^d_+}\int_{\boldsymbol{x}_{n,2}\in\mathbb{R}^d_+}\cdots\int_{\boldsymbol{x}_{n,\tau}\in\mathbb{R}^d_+}\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{x}_{n,t}\rangle w_{n,t} f(\boldsymbol{x}_{n,t}) d\boldsymbol{x}_{n,1} d\boldsymbol{x}_{n,2}\cdots d\boldsymbol{x}_{n,\tau},
\end{aligned} \tag{20}$$

where

- $w_{n,t} = w(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{x}_{n,t}, s_n w_n \prod_{t'=1}^{t-1}\langle \boldsymbol{b}_{n,t'}, \boldsymbol{x}_{n,t'}\rangle w_{n,t'})$,
  - $\boldsymbol{b}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$,
- $f(\boldsymbol{x}_{n,t})$ is the PDF of the multivariate normal distribution $\mathcal{N}\left(\boldsymbol{1}, \mathbf{D}^{-1}_{n,t-1}\boldsymbol{\Sigma}_{n,t}\mathbf{D}^{-1}_{n,t-1}\right)$,
  - and $\mathbf{D}_{n,t-1} = \begin{cases} \operatorname{diag}(\boldsymbol{m}_n), & \text{if } t = 1 \\ \operatorname{diag}(\boldsymbol{M}_{n,t-1}), & \text{if } 2 \le t \le \tau \end{cases}$

and $\arg\max_{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}} s_n w_n \mathbb{E}\left[\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}\right]$ is equivalent to $\arg\max_{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}} w_n \mathbb{E}\left[\prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{X}_{n,t}\rangle W_{n,t}\right]$ since the gross wealth at the end of the n-th day $s_n = s_0\prod_{i=1}^{n}\langle \boldsymbol{b}_i, \boldsymbol{x}_i\rangle$ is independent of $\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n,2}, \dots, \boldsymbol{b}_{n,\tau}$. Therefore, not only forecasting the covariance matrix of price changes between time $t-1$ and t after the end of the n-th day $\boldsymbol{\Sigma}_{n,t}$, but also calculating the Monte Carlo numerical integration is required to obtain $\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \dots, \boldsymbol{b}^*_{n,\tau}$.

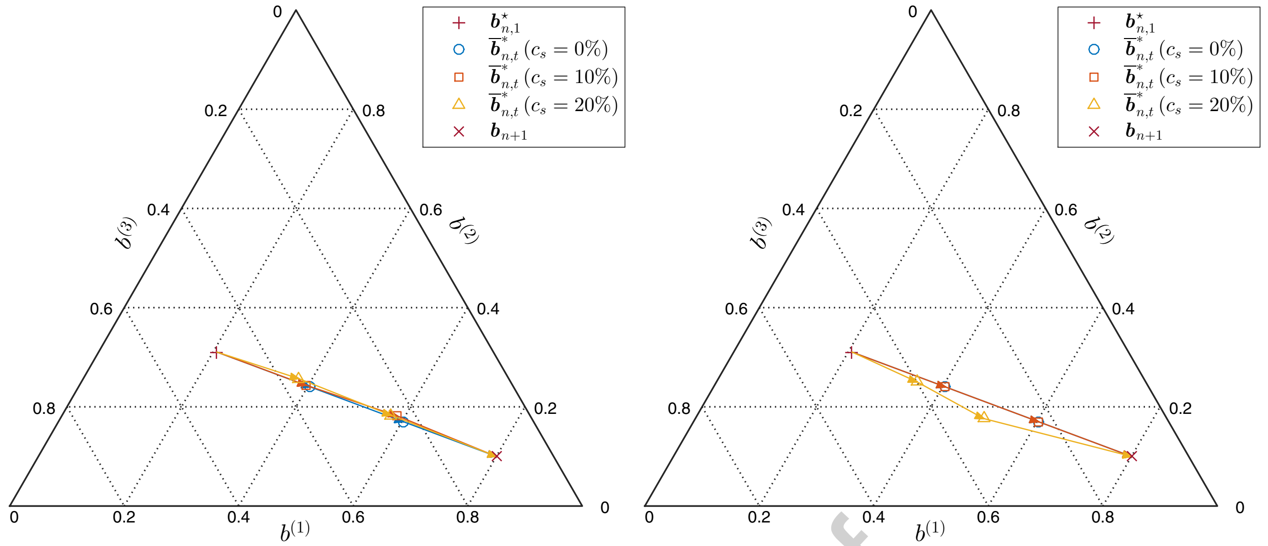
To make the stochastic programming problem in Equation (20) simpler, the *expected value solution* of the stochastic programming (a suboptimal solution of $\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \dots, \boldsymbol{b}^*_{n,\tau}$) is calculated by replacing all the random variables $\boldsymbol{X}_{n,1}, \boldsymbol{X}_{n,2}, \dots, \boldsymbol{X}_{n,\tau}$ in Equation (20) with their expected values $\mathbb{E}[\boldsymbol{X}_{n,1}] = \mathbb{E}[\boldsymbol{X}_{n,2}] = \dots = \mathbb{E}[\boldsymbol{X}_{n,\tau}] = \boldsymbol{1}$ (Birge and Louveaux 2011, p. 165):

$$\begin{aligned}
\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \cdots, \overline{\boldsymbol{b}}^*_{n,\tau} &= \underset{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}}{\arg\max} \; w_n \prod_{t=1}^{\tau}\langle \boldsymbol{b}_{n,t}, \boldsymbol{1}\rangle \overline{w}_{n,t} \\
&= \underset{\boldsymbol{b}_{n,1},\boldsymbol{b}_{n,2},\dots,\boldsymbol{b}_{n,\tau}\in\Delta^{d-1}}{\arg\max} \; w_n \prod_{t=1}^{\tau}\overline{w}_{n,t},
\end{aligned} \tag{21}$$

where $\overline{w}_{n,t} = w\left(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{1}, s_n w_n \prod_{t'=1}^{t-1}\overline{w}_{n,t'}\right)$, and $\boldsymbol{b}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$ (the unimodality of $w_n \prod_{t=1}^{\tau}\overline{w}_{n,t}$ with respect to $\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n,2}, \dots, \boldsymbol{b}_{n,\tau} \in \Delta^{d-1}$ is not proved in this paper; therefore, a local optimum is not guaranteed to be a global optimum). $\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \dots, \overline{\boldsymbol{b}}^*_{n,\tau}$ is always worse than or equal to $\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \dots, \boldsymbol{b}^*_{n,\tau}$ in terms of the *value of the stochastic solution* (VSS), defined as the loss by not considering the random variations (Birge and Louveaux 2011, p. 9):

$$w^*_n \mathbb{E}\left[\prod_{t=1}^{\tau}\left\langle \boldsymbol{b}^*_{n,t}, \boldsymbol{X}_{n,t}\right\rangle W^*_{n,t}\right] - \overline{w}^*_n \mathbb{E}\left[\prod_{t=1}^{\tau}\left\langle \overline{\boldsymbol{b}}^*_{n,t}, \boldsymbol{X}_{n,t}\right\rangle \overline{W}^*_{n,t}\right] \ge 0, \tag{22}$$

where



(a) $\nu_{n-1,\tau}$ is USD 9×10^5. (b) $\nu_{n-1,\tau}$ is USD 3×10^6.

Figure 2. The suboptimal path of two intraday trades is dependent on c_p, c_s, and $\nu_{n-1,\tau}$ ($c_p = 0$, $\boldsymbol{b}_n = [1/3\ 1/3\ 1/3]^{\mathrm{T}}$, $\boldsymbol{b}_{n+1} = [0.8\ 0.1\ 0.1]^{\mathrm{T}}$, and $\boldsymbol{x}_{n,0} = [0.6\ 0.9\ 1.4]^{\mathrm{T}}$). 10-level limit order book data of AAPL ($b^{(1)}$), AMZN ($b^{(2)}$), and GOOG ($b^{(3)}$) on 21 Jun 2012 at 16:00:00 was used.

- $w^*_n = w(\boldsymbol{b}_n, \boldsymbol{b}^{\star}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})$,
- $\overline{w}^*_n = w\left(\boldsymbol{b}_n, \overline{\boldsymbol{b}}^{*}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau}\right)$,
- $W^*_{n,t} = w\left(\boldsymbol{b}^*_{n,t}, \boldsymbol{b}^*_{n,t+1}, \boldsymbol{X}_{n,t}, s_n w^*_n \prod_{t'=1}^{t-1} \left\langle \boldsymbol{b}^*_{n,t'}, \boldsymbol{X}_{n,t'} \right\rangle W^*_{n,t'}\right)$,
  - $\boldsymbol{b}^*_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$,
- $\overline{W}^*_{n,t} = w\left(\overline{\boldsymbol{b}}^*_{n,t}, \overline{\boldsymbol{b}}^*_{n,t+1}, \boldsymbol{X}_{n,t}, s_n \overline{w}^*_n \prod_{t'=1}^{t-1} \left\langle \overline{\boldsymbol{b}}^*_{n,t'}, \boldsymbol{X}_{n,t'} \right\rangle \overline{W}^*_{n,t'}\right)$,
  - and $\overline{\boldsymbol{b}}^*_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$.

Meanwhile, forecasting intraday returns can be performed to find a more optimal solution than $\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \ldots, \overline{\boldsymbol{b}}^*_{n,\tau}$. Firstly, if forecasting intraday expected returns is possible, Equation (21) can be rewritten as the following equation by replacing $\boldsymbol{1}$ in Equation (21) with $\mathbb{E}[\boldsymbol{X}_{n,t}]$:

$$\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \cdots, \overline{\boldsymbol{b}}^*_{n,\tau} = \underset{\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n,2}, \ldots, \boldsymbol{b}_{n,\tau} \in \Delta^{d-1}}{\arg\max}\ w_n \prod_{t=1}^{\tau} \langle \boldsymbol{b}_{n,t}, \mathbb{E}[\boldsymbol{X}_{n,t}] \rangle \overline{w}_{n,t}, \tag{23}$$

where $\overline{w}_{n,t} = w\left(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \mathbb{E}[\boldsymbol{X}_{n,t}], s_n w_n \prod_{t'=1}^{t-1} \langle \boldsymbol{b}_{n,t'}, \mathbb{E}[\boldsymbol{X}_{n,t'}] \rangle \overline{w}_{n,t'}\right)$, and $\boldsymbol{b}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$. The computational burden of Equation (23) is the same as that of Equation (21) because the numerical integration is not performed in both the cases. Secondly, intraday (each day is divided into 10-minute intervals) volatility forecasting can be employed by using the univariate GARCH model ($\boldsymbol{\Sigma}_{n,t}$ is assumed as a diagonal matrix) by Engle and Sokalska (2012). However, calculating integrations in Equation (20) is required, which in turn causes a heavy computational burden. Lastly, the optimal solution $\boldsymbol{b}^*_{n,1}, \boldsymbol{b}^*_{n,2}, \ldots, \boldsymbol{b}^*_{n,\tau}$ in Equation (20) can be obtained by forecasting both intraday expected returns and volatilities.

$\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \ldots, \overline{\boldsymbol{b}}^*_{n,\tau}$ is a function of c_p, c_s, and $\nu_{n-1,\tau}$ as shown in Figure 2 because w_n and $\overline{w}_{n,t}, \forall t \in \{1, 2, \ldots, \tau\}$ are functions of c_p, c_s, and $\nu_{n-1,\tau}$. To be specific, each arrow indicates a suboptimal trading path from the starting point $\boldsymbol{b}^{\star}_{n,1} = \frac{\boldsymbol{b}_n \odot \boldsymbol{x}_{n,0}}{\langle \boldsymbol{b}_n, \boldsymbol{x}_{n,0} \rangle}$ to the end point $\boldsymbol{b}_{n+1}$.

A recommended initial value of $\boldsymbol{b}_{n,t}$ for the optimization of Equation (21) is

$$\boldsymbol{b}^{\dagger}_{n,t} = \boldsymbol{b}^{\star}_{n,1} + \frac{t}{\tau+1}(\boldsymbol{b}_{n+1} - \boldsymbol{b}^{\star}_{n,1}), \tag{24}$$

where $t \in \{1, 2, \ldots, \tau\}$ (i.e. $\boldsymbol{b}^{\dagger}_{n,t}, \forall t$ is linearly located between $\boldsymbol{b}^{\star}_{n,1}$ and $\boldsymbol{b}_{n+1}$ with the same distance). This is because the suboptimal portfolio vector $\overline{\boldsymbol{b}}^{*}_{n,t}, \forall t$ is not far from $\boldsymbol{b}^{\dagger}_{n,t}, \forall t$ as shown in Figure 2. As a result, using the initial value $\boldsymbol{b}^{\dagger}_{n,t}$ will reduce computation time searching for the solution $\overline{\boldsymbol{b}}^{*}_{n,1}, \overline{\boldsymbol{b}}^{*}_{n,2}, \cdots, \overline{\boldsymbol{b}}^{*}_{n,\tau}$ in Equation (21).

4.4. *Optimal number of intraday trades*

The optimal number of intraday trades τ^* can be written as

$$\tau^* = \underset{\tau \in \{0,1,\ldots\}}{\arg\max}\, \overline{s}_{n+1}(\tau), \tag{25}$$

where $\overline{s}_{n+1}(\tau)$ is the gross wealth at the end of the $(n+1)$-th day i) when a portfolio is rebalanced through the suboptimal trading path $\overline{\boldsymbol{b}}^{*}_{n,1}, \overline{\boldsymbol{b}}^{*}_{n,2}, \ldots, \overline{\boldsymbol{b}}^{*}_{n,\tau}$, given the number of intraday trades τ, and ii) when LOBs does not change between time 0 and time τ after the end of the n-th day (this corresponds to Assumption 4.2). $\overline{s}_{n+1}(\tau)$ can be written, using Equation (16) and (21), as

$$\overline{s}_{n+1}(\tau) = \begin{cases} s_n w(\boldsymbol{b}_n, \boldsymbol{b}_{n+1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau}), & \text{if } \tau = 0 \\ s_n \overline{w}^{*}_{n} \prod_{t=1}^{\tau} \overline{w}^{*}_{n,t}, & \text{if } \tau \geq 1 \end{cases}, \tag{26}$$

where

- $\overline{w}^{*}_{n} = w\left(\boldsymbol{b}_n, \overline{\boldsymbol{b}}^{*}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau}\right)$,
- $\overline{w}^{*}_{n,t} = w\left(\overline{\boldsymbol{b}}^{*}_{n,t}, \overline{\boldsymbol{b}}^{*}_{n,t+1}, \boldsymbol{1}, s_n \overline{w}^{*}_{n} \prod_{t'=1}^{t-1} \overline{w}^{*}_{n,t'}\right)$,
    - and $\overline{\boldsymbol{b}}^{*}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$.

Also, $\overline{s}_{n+1}(\tau)$ can be rewritten, using Equation (5), as

$$\overline{s}_{n+1}(\tau) = s_n - \sum_{t=0}^{\tau} \gamma_{n,t}(\tau), \tag{27}$$

where $\gamma_{n,t}(\tau)$ is overall TCs at time t after the end of the n-th day, given the number of intraday trades τ. By using Equation (9) and the no price change of Assumption 4.2 (i.e. $m^{(j)}_{n,t} = m^{(j)}_{n}, \forall t \in \{1, 2, \ldots, \tau\}$), Equation (25) can be further simplified as:

$$\begin{aligned} &\tau^* \\ &= \underset{\tau \in \{0,1,\ldots\}}{\arg\min} \sum_{t=0}^{\tau} \gamma_{n,t}(\tau) \\ &= \underset{\tau \in \{0,1,\ldots\}}{\arg\min} \sum_{t=0}^{\tau} \left[\sum_{j=1}^{d} \left((1-c_s)\bar{p}\left(\overline{q}^{(j)*}_{n,t}(\tau)\right) - m^{(j)}_{n} \right) \overline{q}^{(j)*}_{n,t}(\tau) + (c_p + c_s) \sum_{j=1}^{d} \left(\bar{p}\left(\overline{q}^{(j)*}_{n,t}(\tau)\right) \overline{q}^{(j)*}_{n,t}(\tau) \right)^{+} \right], \end{aligned} \tag{28}$$

where $\overline{q}^{(j)*}_{n,t}(\tau)$ is the order size of asset j when rebalancing a portfolio by following the suboptimal trading path from $\overline{\boldsymbol{b}}^{*}_{n,t}$ to $\overline{\boldsymbol{b}}^{*}_{n,t+1}$, given the number of intraday trades τ.

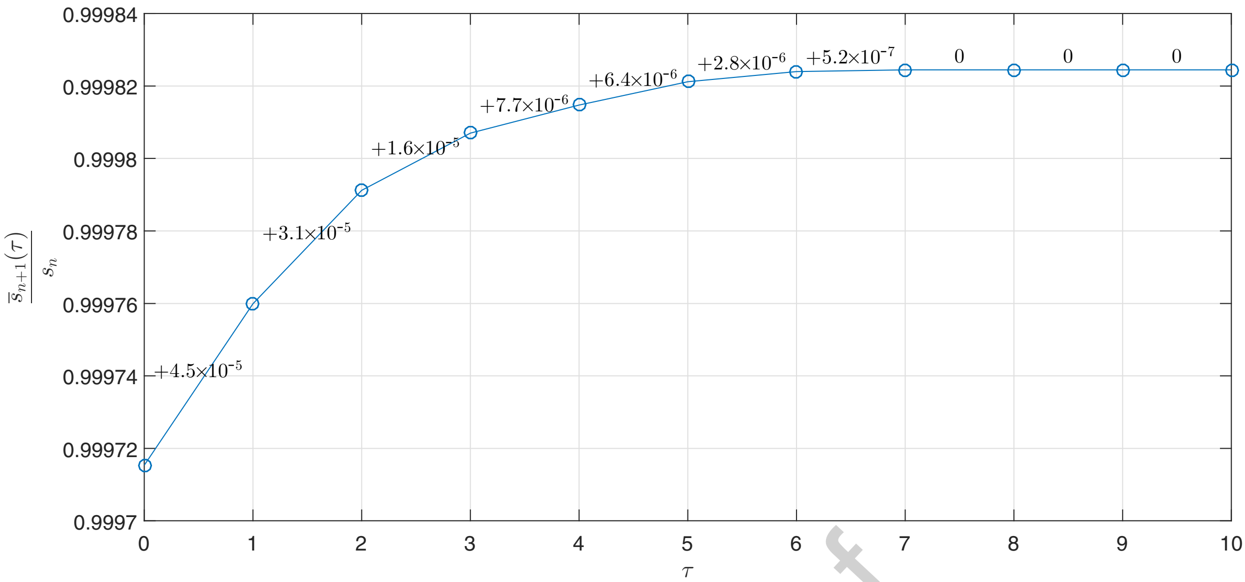


Figure 3. The optimal number of intraday trades τ^* is not unique ($c_p = 0$, $c_s = 0.00218\%$, $\boldsymbol{b}_n = [1/3\ 1/3\ 1/3]^{\mathrm{T}}$, $\boldsymbol{b}_{n+1} = [0.8\ 0.1\ 0.1]^{\mathrm{T}}$, $\boldsymbol{x}_{n,0} = [0.6\ 0.9\ 1.4]^{\mathrm{T}}$, and $\nu_{n-1,\tau} = 2 \times 10^6$ USD). 10-level limit order book data of AAPL ($b^{(1)}$), AMZN ($b^{(2)}$), and GOOG ($b^{(3)}$) on 21 Jun 2012 at 16:00:00 was used (each value above the line is the change amount of $\frac{\bar{s}_{n+1}(\tau)}{s_n}$).

Equation (28) implies that the overall TCs $\sum_{t=0}^{\tau} \gamma_{n,t}(\tau)$, consisting of proportional TCs and MICs, can be minimised if τ is large enough to make trading order size $\left|\bar{q}_{n,t}^{(j)^*}(\tau)\right|$ small for all j and all $t \in \{0, 1, \ldots, \tau\}$. Also, the small trading order is equivalent to that where all assets are traded at the best ask price $p_1^{(j)}$ or the best bid price $p_{-1}^{(j)}$ for all $t \in \{0, 1, \ldots, \tau\}$: $\bar{p}\left(\bar{q}_{n,t}^{(j)^*}(\tau^*)\right) = \begin{cases} p_1^{(j)}, & \text{if } \bar{q}_{n,t}^{(j)^*}(\tau^*) > 0 \\ p_{-1}^{(j)}, & \text{if } \bar{q}_{n,t}^{(j)^*}(\tau^*) < 0 \\ m_n^{(j)}, & \text{otherwise} \end{cases}$, as $v_{-1}^{(j)} \le \bar{q}_{n,t}^{(j)^*}(\tau^*) \le v_1^{(j)}$, $\forall t$. Consequently, the overall TCs can be minimised when $\tau = \tau^*$ as

$$\sum_{t=0}^{\tau^*} \gamma_{n,t}(\tau^*) = \sum_{j=1}^{d} \left[\left((1 + c_p)p_1^{(j)} - m_n^{(j)}\right) \sum_{t=0}^{\tau^*} \left(\bar{q}_{n,t}^{(j)^*}(\tau^*)\right)^+ + \left(m_n^{(j)} - (1 - c_s)p_{-1}^{(j)}\right) \sum_{t=0}^{\tau^*} \left(-\bar{q}_{n,t}^{(j)^*}(\tau^*)\right)^+ \right] \tag{29}$$

The optimal number of intraday trades τ^* in Equation (28) is not a unique number, it can be several numbers as $\tau^* \in \{\tau \in \mathbb{Z} | \tau \ge \tau_{\min}^*\}$, where $\tau_{\min}^*$ is the minimum optimal number of intraday trades. This is because both $\sum_{t=0}^{\tau} \left(\bar{q}_{n,t}^{(j)^*}(\tau)\right)^+$ and $\sum_{t=0}^{\tau} \left(-\bar{q}_{n,t}^{(j)^*}(\tau)\right)^+$ in Equation (29) are constants if $\tau \ge \tau_{\min}^*$. As a result, $\frac{\bar{s}_{n+1}(\tau)}{s_n}$ is a monotonically increasing function of τ as shown in Figure 3, and $\frac{\bar{s}_{n+1}(\tau)}{s_n}$ does not change after $\tau = \tau_{\min}^*$ ($\tau_{\min}^*$ is 7 in the case of Figure 3).

Algorithm 1 describes how to obtain the minimum optimal number of intraday trades $\tau_{\min}^*$ from the property of $v_{-1}^{(j)} \le \bar{q}_{n,t}^{(j)^*}(\tau^*) \le v_1^{(j)}$, $\forall j \in \{1, 2, \ldots, d\}$, $\forall t \in \{0, 1, \ldots, \tau\}$. This algorithm increases the number of intraday trades τ from 0 until either i) when τ equals the upper limit $\tau_{\max}$ (see the 5th line of Algorithm 1), where $\tau_{\max}$ is a user parameter decided by trading hours and an intraday trading interval, or ii) when trading all assets at the best ask or best bid price is possible (see the 7th line of Algorithm 1).

Algorithm 1: How to obtain the minimum optimal number of intraday trades.

Input: $\tau_{\max}$, $\nu_{n-1,\tau}$, $\boldsymbol{x}_{n,0}$, and limit order book data at the end of the n-th day.
Output: the minimum optimal number of intraday trades $\tau^*_{\min}$.

calculate order size $q^{(j)}_{n,0}, \forall j$ when there is no intraday trading by using Equation (8) and (10);
if $v^{(j)}_{-1} \le q^{(j)}_{n,0} \le v^{(j)}_{1}, \forall j$ **then**
  $\tau^*_{\min} \leftarrow 0$;
else
  for $\tau \leftarrow 1$ **to** $\tau_{\max}$ **do**
    `// ` τ `is the number of intraday trades`
    $\overline{\boldsymbol{b}}^*_{n,1}, \overline{\boldsymbol{b}}^*_{n,2}, \ldots, \overline{\boldsymbol{b}}^*_{n,\tau} \leftarrow \underset{\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n,2}, \ldots, \boldsymbol{b}_{n,\tau} \in \Delta^{d-1}}{\arg\max} \, w_n \prod_{t=1}^{\tau} \overline{w}_{n,t}$, where $w_n = w(\boldsymbol{b}_n, \boldsymbol{b}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})$,
      $\overline{w}_{n,t} = w\left(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{1}, s_n w_n \prod_{t'=1}^{t-1} \overline{w}_{n,t'}\right)$, and $\boldsymbol{b}_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$; `// from Equation (21)`
    if $v^{(j)}_{-1} \le \overline{q}^{(j)^*}_{n,t}(\tau) \le v^{(j)}_{1}, \forall j \in \{1, 2, \ldots, d\}, \forall t \in \{0, 1, \ldots, \tau\}$ `// where` $\overline{q}^{(j)^*}_{n,t}(\tau)$`, calculable from Equation (8) and (10), is the order size of asset` j `when rebalancing a portfolio from` $\overline{\boldsymbol{b}}^*_{n,t}$ `to` $\overline{\boldsymbol{b}}^*_{n,t+1}$ `(`$\overline{\boldsymbol{b}}^*_{n,0} \overset{\text{def}}{=} \boldsymbol{b}_n$ `and` $\overline{\boldsymbol{b}}^*_{n,\tau+1} \overset{\text{def}}{=} \boldsymbol{b}_{n+1}$`), given the number of intraday trades` τ
    then
      break;
    end
  end
  $\tau^*_{\min} \leftarrow \tau$;
end

4.5. *Algorithmic trading using real-time LOB data*

The proposed method described in Algorithm 2 is an implementation shortfall algorithm (an intraday trading strategy determined by real-time LOB data at every intraday trading time t) for a multi-asset portfolio. It performs intraday trading by sending market orders in the following order.

(i) The portfolio vector of next day $\boldsymbol{b}_{n+1}$ is obtained from an OPS algorithm (see the 2nd line of Algorithm 2) at the end of every trading day (the end of the trading day is the market opening, not midnight; see Figure 1).

(ii) The current LOBs of all assets in the portfolio are taken into account at every intraday trading time t until either the time that satisfies $t = \tau_{\max}$ (see the 3rd line of Algorithm 2) or the time that satisfies $\left(v^{(j)}_{-1}\right)_{n,t} \le q^{(j)}_{n,t} \le \left(v^{(j)}_{1}\right)_{n,t}, \forall j$ (i.e. whether trading all assets in the current best ask or best bid price is possible or not is checked on every t; see the 10th line of Algorithm 2), whichever happens first.

(iii) If any of the inequalities at the 10th line of Algorithm 2 are false, then the minimum optimal number of intraday trades $\tau^*_{\min}$ is obtained by considering the current LOBs and by using Algorithm 1 (see between the 15th line and the 21st line of Algorithm 2).

(iv) Among the suboptimal path components $\overline{\boldsymbol{b}}^*_{n,t+1}, \overline{\boldsymbol{b}}^*_{n,t+2}, \ldots, \overline{\boldsymbol{b}}^*_{n,\tau}$, calculated at the 16th line of Algorithm 2, only one component $\overline{\boldsymbol{b}}^*_{n,t+1}$ is used for the rebalancing at time t (see the 22nd line of Algorithm 2), but the other components of the suboptimal path $\overline{\boldsymbol{b}}^*_{n,t+2}, \overline{\boldsymbol{b}}^*_{n,t+3}, \ldots, \overline{\boldsymbol{b}}^*_{n,\tau}$ are ignored. This is because new LOBs will be given at time $t+1$ (see the 8th line of Algorithm 2).

However, Algorithm 2 ignores the risk of intraday price volatility as Alfonsi *et al.* (2008, 2010) did (i.e. the volatility of the random intraday market vectors $\boldsymbol{X}_{n,t+1}, \boldsymbol{X}_{n,t+2}, \ldots, \boldsymbol{X}_{n,\tau}$ is not considered

Algorithm 2: Proposed method of optimal intraday trading.

Input: s_0, μ, $\tau_{\max}$, where s_0 is an initial wealth, and $(\mu - 1)$ is the number of rebalancing days by online portfolio selection.

for $n \leftarrow 1$ **to** $\mu - 1$ **do**
  obtain $\boldsymbol{b}_{n+1}$ from an online portfolio selection algorithm;
  for $t \leftarrow 0$ **to** $\tau_{\max}$ **do**
    if $t = \tau_{\max}$ **then**
      rebalance a portfolio from $\boldsymbol{b}_{n,t}$ to $\boldsymbol{b}_{n+1}$;
      break; `// the for-loop of` t
    end
    receive real-time limit order book data at time t after the end of the n-th day;
    calculate order size $q_{n,t}^{(j)}$, $\forall j$ for rebalancing a portfolio from $\boldsymbol{b}_{n,t}$ to $\boldsymbol{b}_{n+1}$ by using Equation (8) and (10);
    if $\left(v_{-1}^{(j)}\right)_{n,t} \le q_{n,t}^{(j)} \le \left(v_{1}^{(j)}\right)_{n,t}, \forall j$ `// where` $\left(v_{i}^{(j)}\right)_{n,t}$ `is the quoted volume of limit order book of asset` j `at level` i `at time` t `after the end of the` n`-th day`
    then
      rebalance a portfolio from $\boldsymbol{b}_{n,t}$ to $\boldsymbol{b}_{n+1}$;
      break; `// the for-loop of` t
    else
      for $\tau \leftarrow t+1$ **to** $\tau_{\max}$ **do**

$$\overline{\boldsymbol{b}}^{*}_{n,t+1}, \overline{\boldsymbol{b}}^{*}_{n,t+2}, \ldots, \overline{\boldsymbol{b}}^{*}_{n,\tau} \leftarrow \underset{\boldsymbol{b}_{n,t+1}, \boldsymbol{b}_{n,t+2}, \ldots, \boldsymbol{b}_{n,\tau} \in \Delta^{d-1}}{\arg\max} w_{n,t} \prod_{t'=t+1}^{\tau} \overline{w}_{n,t'},$$

        where $w_{n,t} = w\left(\boldsymbol{b}_{n,t}, \boldsymbol{b}_{n,t+1}, \boldsymbol{x}_{n,t}, \nu_{n,t-1}\right)$,
        $\overline{w}_{n,t'} = w\left(\boldsymbol{b}_{n,t'}, \boldsymbol{b}_{n,t'+1}, \boldsymbol{1}, s_{n,t'-1} w_{n,t'-1} \prod_{t''=t+1}^{t'-1} \overline{w}_{n,t''}\right)$, and $\boldsymbol{b}_{n,\tau+1} \stackrel{\text{def}}{=} \boldsymbol{b}_{n+1}$;
        `// from Equation (21)`
        if $\left(v_{-1}^{(j)}\right)_{n,t'} \le \overline{q}_{n,t'}^{(j)\,*} \le \left(v_{1}^{(j)}\right)_{n,t'}, \forall j \in \{1, 2, \ldots, d\}, \forall t' \in \{t, t+1, \ldots, \tau\}$ `// where` $\overline{q}_{n,t'}^{(j)\,*}$`, calculable from Equation (8) and (10), is the order size of asset` j `when rebalancing a portfolio from` $\overline{b}^{*}_{n,t'}$ `to` $\overline{b}^{*}_{n,t'+1}$ `(`$\overline{b}^{*}_{n,t} \stackrel{\text{def}}{=} b_{n,t}$ `and` $\overline{b}^{*}_{n,\tau+1} \stackrel{\text{def}}{=} b_{n+1}$`)`
        then
          break; `// the for-loop of` τ
        end
      end
      rebalance a portfolio from $\boldsymbol{b}_{n,t}$ to $\overline{\boldsymbol{b}}^{*}_{n,t+1}$;
    end
  end
end

under the assumption that traders are risk-neutral).

5. Simulations

Monte Carlo (MC) simulations that consisting of independent trials of random stock selection have been conducted to compare the performance between OPS without and with the intraday trading via the proposed method.

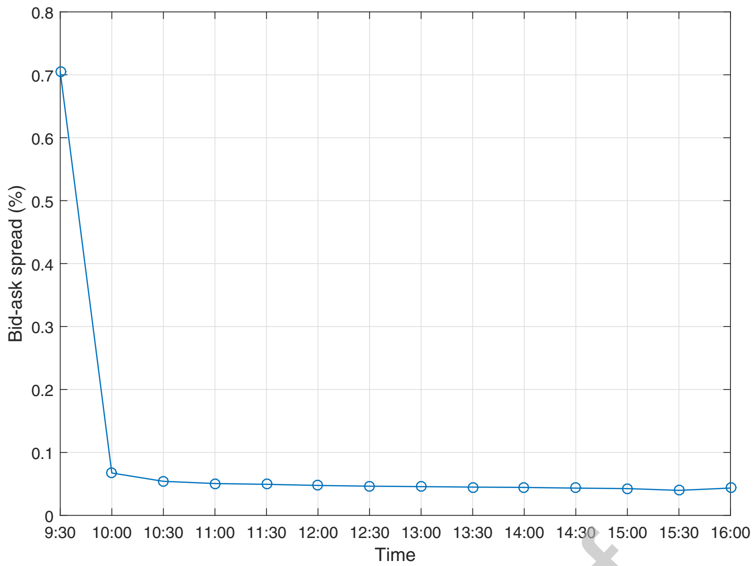


Figure 4. The median values of intraday bid-ask spreads of NASDAQ 100 Index Components (between 1 Jan 2008 and 31 Mar 2016).

The following assumptions were made for simplicity:

- Assets are arbitrarily divisible (i.e. $q_{n,t}^{(j)} \in \mathbb{R}$ instead of $q_{n,t}^{(j)} \in \mathbb{Z}$) to avoid mixed-integer nonlinear programming.
- Hidden limit orders (HLOs), invisible in limit order books, are never submitted.
- The execution of market orders by OPS at the current time t does not affect the LOBs at the next time $t+1$.
- The computation time to calculate $b_{n,t+1}$ is zero. Neither the running time between the 9th line and the 11th line in Algorithm 2 nor that between the 15th line and the 21st line in Algorithm 2 is considered in our simulation.

Trading at the market opening (9:30 a.m.) was not conducted in this experiment because bid-ask spreads are much higher at the opening than mid-day or closing, as shown in Figure 4 [2]. Therefore, we assume trading starts at 10:00 a.m. in this experiment, which corresponds to time 0 ($t = 0$; see Figure 1). The trading interval was fixed at 30 minutes under an assumption that LOBs reverts to its normal shape within 30 minutes after the execution of market orders by OPS (Xu *et al.* 2017). Consequently, $\tau_{\max}$ is set at 12 because trading starts at 10:00 a.m. instead of 9:30 a.m. and because NASDAQ regular market hours end at 4:00 p.m.

5.1. *The source of the backtesting data*

10-level historical LOB data of NASDAQ 100 Index Components from 1 Jan 2008 to 31 Mar 2016 (total 2076 trading days) was downloaded from the Limit Order Book System: The Efficient Reconstructor (LOBSTER). The LOB data was sampled in periods of 30 minutes during NASDAQ regular market hours. The number of stock candidates used in this experiment is 83 because 17 companies were delisted before 31 Mar 2016.

In addition, if accessing LOB data at greater than level 10 is required, ask price and volume at level $i \in \{11, 12, \ldots\}$ are estimated as $p_i = p_{10} + \frac{p_{10}-p_{-1}}{10}(i-10)$ and $v_i = \frac{\sum_{k=1}^{10} v_k}{10}$, respectively. Similarly, if accessing LOB data at less than level -10 is required, bid price and volume at level $i \in \{\ldots, -12, -11\}$ are estimated as $p_i = p_{-10} + \frac{p_{-10}-p_1}{10}(-i-10)$ and $v_i = \frac{\sum_{k=1}^{10} v_{-k}}{10}$, respectively. When calculating the

[2] Figure 4 corresponds to the empirical analysis by Kissell (2013, pp. 67–69) that bid-ask spreads decrease and level out after about the first 15–30 minutes for large cap stocks and after about 30–60 minutes for small cap stocks.

Table 2. A list of online portfolio selection (OPS) strategies.

Category	Author(s)	OPS strategy	Abbreviation
Follow the winner	Cover (1991)	Universal portfolio	UP
	Helmbold *et al.* (1998)	Exponential gradient	EG
	Borodin *et al.* (2000)	Markov of order zero	M0, T0[a]
	Agarwal *et al.* (2006)	Online Newton step	ONS
	Kozat and Singer (2011)	Universal semi-constant rebalanced portfolio	USCRP
Follow the loser	Borodin *et al.* (2004)	Anti-correlation	ANTICOR, ANTICOR_ANTICOR[b]
	Li and Hoi (2012)	Online moving average reversion	OLMAR1, OLMAR2[c]
	Li *et al.* (2012)	Passive aggressive mean reversion	PAMR, PAMR1, PAMR2[d]
	Li *et al.* (2013)	Confidence weighted mean reversion	CWMR_VAR, CWMR_STDEV[e]
Pattern matching	Györfi *et al.* (2006)	Nonparametric kernel-based log-optimal	BK
	Györfi *et al.* (2008)	Nonparametric nearest neighbour log-optimal	BNN
	Li *et al.* (2011)	Correlation-driven nonparametric learning	CORN, CORNU, CORNK[f]

[a]T0 considers historical market vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ fully, while M0 does not.

[b]ANTICOR_ANTICOR is the twice compounded algorithm of ANTICOR.

[c]OLMAR1 uses a simple (equally weighted) moving average, while OLMAR2 uses an exponential (exponentially weighted) moving average.

[d]PAMR1 added a slack variable ξ to the objective function of PAMR, and PAMR2 added ξ^2.

[e]CWMR_STDEV is a modified algorithm of CWMR_VAR to obtain convex constraints (Crammer *et al.* 2008).

[f]CORN: each expert has a weight proportional to its historical performance; CORNU: all experts have the same weight; CORNK: only the K-best experts have weights.

relative price of asset j between time τ after the end of the $(n-1)$-th day and the end of the n-th day, cash dividends, stock dividends, and stock splits were considered as

$$x_{n,0}^{(j)} = \frac{m_{n,0}^{(j)} \dfrac{a_{n,\tau_{\max}}^{(j)}}{g_{n,\tau_{\max}}^{(j)}}}{m_{n-1,\tau}^{(j)} \dfrac{a_{n-1,\tau_{\max}}^{(j)}}{g_{n-1,\tau_{\max}}^{(j)}}}, \tag{30}$$

where $m_{n,t}^{(j)}$ is the mid-price of asset j at time t after the end of the n-th day from LOBSTER, and $\left\{g_{n,\tau_{\max}}^{(j)},\ a_{n,\tau_{\max}}^{(j)}\right\}$ (the subscript $n, \tau_{\max}$ indicates time $\tau_{\max}$ after the end of the n-th day) is the {closing, adjusted closing} price of asset j of the $(n+1)$-th day, not the n-th day (a day ends after or at the market opening in this paper; see Figure 1), from Yahoo Finance.

5.2. *Performance comparison between OPS methods with and without the proposed method*

A list of OPS strategies has been summarized in Table 2. Based on the p-values of the t-tests in Table C1, T0, UP, and ONS have been selected as the best three OPS methods for the comparison. We compare the performance of each OPS method without and with the proposed method (PM henceforth) using Algorithm 2. All of these three methods increase the relative weights of more successful assets in the past periods (i.e. *following the winner*), as the portfolio vector of each algorithm in the next period $\boldsymbol{b}_{n+1}$ (or $b_{n+1}^{(j)}$) is

- T0: $b_{n+1}^{(j)} = \dfrac{c_n^{(j)} + \beta}{d\beta + \sum_{j'=1}^{d} c_n^{(j')}}$, where $c_n^{(j)} = c_{n-1}^{(j)} + \log_2\left(1 + x_n^{(j)}\right)$, and $\beta \in [0, \infty)$ is a parameter;

- UP: $\boldsymbol{b}_{n+1} = \dfrac{\int_{\Delta^{d-1}} \boldsymbol{b} s_n(\boldsymbol{b}, \boldsymbol{x}_{1:n}) d\boldsymbol{b}}{\int_{\Delta^{d-1}} s_n(\boldsymbol{b}, \boldsymbol{x}_{1:n}) d\boldsymbol{b}}$, where, $s_n(\boldsymbol{b}, \boldsymbol{x}_{1:n}) = s_0 \prod_{i=1}^{n} \langle \boldsymbol{b}, \boldsymbol{x}_i \rangle$ is wealth at the end of the n-th period with an initial wealth s_0 (the integral can be calculated numerically by using MC methods (Ishijima 2001));
- ONS: $\boldsymbol{b}_{n+1} = \underset{\boldsymbol{b} \in \Delta^{d-1}}{\arg\max} \left(\sum_{i=1}^{n} \ln \langle \boldsymbol{b}, \boldsymbol{x}_i \rangle - \dfrac{\beta}{2} \|\boldsymbol{b}\|^2 \right)$, where $\beta \in [0, \infty)$ is a trade-off parameter between the *follow the winner* term $\sum_{i=1}^{n} \ln \langle \boldsymbol{b}, \boldsymbol{x}_i \rangle$ and the regularisation term $\|\boldsymbol{b}\|^2$.

Table 3. Statistics of annualised returns of different online portfolio selection methods without and with proposed method (PM) ($c_p = 0$, $s_0 = 10^5$ USD)

c_s (%)	B&H	0.00218 T0 w/o PM	0.00218 T0 w/ PM	0.00218 UP w/o PM	0.00218 UP w/ PM	0.00218 ONS w/o PM	0.00218 ONS w/ PM	0.16667 T0 w/o PM	0.16667 T0 w/ PM	0.16667 UP w/o PM	0.16667 UP w/ PM	0.16667 ONS w/o PM	0.16667 ONS w/ PM
P-value of JB test	0.956	0.594	0.617	0.509	0.503	0.076	0.082	0.603	0.585	0.519	0.514	0.073	0.079
Standard deviation (%)	1.26	1.53	1.53	1.50	1.50	10.93	11.22	1.53	1.53	1.50	1.50	10.88	11.12
Mean (%)	9.3	11.3	11.3	11.2	11.2	21.3	22.1	11.0	11.0	11.0	11.0	20.1	20.7
Difference of means[a] (%)	-	1.97***	1.97***	1.92***	1.92***	11.99***	12.73***	1.69***	1.69***	1.65***	1.65***	10.72***	11.36***
P-value of t-test	-	4.6×10^{-19}	4.9×10^{-19}	1.3×10^{-18}	1.3×10^{-18}	8.9×10^{-19}	1.3×10^{-19}	4.9×10^{-15}	4.0×10^{-15}	8.2×10^{-15}	8.2×10^{-15}	2.4×10^{-16}	4.1×10^{-17}
Difference of means[b] (%)	-	-0.00		0.00		0.73		0.01		0.00		0.63	
P-value of t-test	-	0.990		0.999		0.641		0.976		0.999		0.684	

c_s (%)	0.33333 T0 w/o PM	0.33333 T0 w/ PM	0.33333 UP w/o PM	0.33333 UP w/ PM	0.33333 ONS w/o PM	0.33333 ONS w/ PM	0.5 T0 w/o PM	0.5 T0 w/ PM	0.5 UP w/o PM	0.5 UP w/ PM	0.5 ONS w/o PM	0.5 ONS w/ PM
P-value of JB test	0.612	0.595	0.529	0.524	0.070	0.075	0.621	0.606	0.539	0.534	0.067	0.073
Standard deviation (%)	1.53	1.53	1.50	1.50	10.83	11.08	1.53	1.53	1.50	1.50	10.78	11.02
Mean (%)	10.7	10.7	10.7	10.7	18.8	19.4	10.4	10.4	10.4	10.4	17.5	18.1
Difference of means[a] (%)	1.40***	1.40***	1.38***	1.38***	9.45***	10.08***	1.11***	1.11***	1.11***	1.11***	8.18***	8.78***
P-value of t-test	3.0×10^{-11}	2.6×10^{-11}	3.2×10^{-11}	3.2×10^{-11}	7.3×10^{-14}	1.1×10^{-14}	7.3×10^{-8}	6.3×10^{-8}	5.4×10^{-8}	5.3×10^{-8}	2.0×10^{-11}	3.2×10^{-12}
Difference of means[b] (%)	0.01		0.00		0.63		0.01		0.00		0.60	
P-value of t-test	0.977		0.999		0.686		0.977		0.999		0.699	

[a]Difference equals average annualised return of corresponding OPS method minus that of buy-and-hold (B&H). $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

[b]Difference equals average annualised return of corresponding OPS method with proposed method (PM) minus that without PM. $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

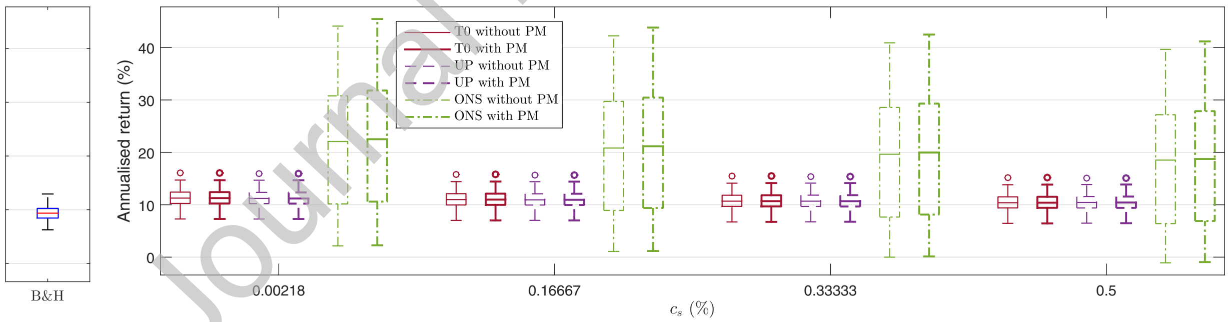


Figure 5. Box plots of annualised returns of different online portfolio selection methods without and with the proposed method (PM) ($c_p = 0$, $s_0 = 10^5$ USD).

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Table 4. Statistics of annualised returns of different online portfolio selection methods without and with proposed method (PM) ($c_p = 0$, $s_0 = 10^6$ USD)

c_s (%)	BnH	0.00218 T0 w/o PM	0.00218 T0 w/ PM	0.00218 UP w/o PM	0.00218 UP w/ PM	0.00218 ONS w/o PM	0.00218 ONS w/ PM	0.16667 T0 w/o PM	0.16667 T0 w/ PM	0.16667 UP w/o PM	0.16667 UP w/ PM	0.16667 ONS w/o PM	0.16667 ONS w/ PM
P-value of JB test	0.956	0.643	0.552	0.511	0.487	0.059	0.090	0.651	0.564	0.520	0.503	0.057	0.086
Standard deviation (%)	1.26	1.53	1.54	1.50	1.51	10.12	11.23	1.53	1.53	1.50	1.50	10.10	11.17
Mean (%)	9.3	11.3	11.3	11.3	11.3	18.6	22.1	11.0	11.0	11.0	11.0	17.5	20.8
Difference of means[a] (%)	-	1.95***	2.00***	1.93***	1.93***	9.24***	12.74***	1.66***	1.71***	1.66***	1.66***	8.12***	11.51***
P-value of t-test	-	$9.2{\times}10^{-19}$	$2.0{\times}10^{-19}$	$8.5{\times}10^{-19}$	$8.8{\times}10^{-19}$	$9.8{\times}10^{-15}$	$1.3{\times}10^{-19}$	$9.2{\times}10^{-15}$	$2.4{\times}10^{-15}$	$5.5{\times}10^{-15}$	$5.9{\times}10^{-15}$	$2.2{\times}10^{-12}$	$2.5{\times}10^{-17}$
Difference of means[b] (%)	-	0.05		0.00		3.50**		0.05		-0.00		3.39**	
P-value of t-test	-	0.810		0.995		0.022		0.825		1.000		0.025	

c_s (%)	0.33333 T0 w/o PM	0.33333 T0 w/ PM	0.33333 UP w/o PM	0.33333 UP w/ PM	0.33333 ONS w/o PM	0.33333 ONS w/ PM	0.5 T0 w/o PM	0.5 T0 w/ PM	0.5 UP w/o PM	0.5 UP w/ PM	0.5 ONS w/o PM	0.5 ONS w/ PM
P-value of JB test	0.659	0.578	0.530	0.513	0.055	0.082	0.667	0.591	0.540	0.524	0.054	0.079
Standard deviation (%)	1.53	1.53	1.50	1.50	10.08	11.11	1.52	1.53	1.50	1.50	10.06	11.06
Mean (%)	10.7	10.8	10.7	10.7	16.3	19.6	10.4	10.5	10.5	10.4	15.2	18.4
Difference of means[a] (%)	1.37***	1.42***	1.39***	1.39***	6.99***	10.27***	1.09***	1.12***	1.12***	1.11***	5.86***	9.02***
P-value of t-test	$5.4{\times}10^{-11}$	$1.7{\times}10^{-11}$	$2.2{\times}10^{-11}$	$2.4{\times}10^{-11}$	$4.9{\times}10^{-10}$	$5.5{\times}10^{-15}$	$1.2{\times}10^{-7}$	$4.9{\times}10^{-8}$	$3.9{\times}10^{-8}$	$4.4{\times}10^{-8}$	$8.3{\times}10^{-8}$	$1.2{\times}10^{-12}$
Difference of means[b] (%)	0.04		-0.00		3.28**		0.04		-0.00		3.16**	
P-value of t-test	0.841		0.993		0.030		0.856		0.987		0.036	

[a]Difference equals average annualised return of corresponding OPS method minus that of buy-and-hold (B&H). $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

[b]Difference equals average annualised return of corresponding OPS method with proposed method (PM) minus that without PM. $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

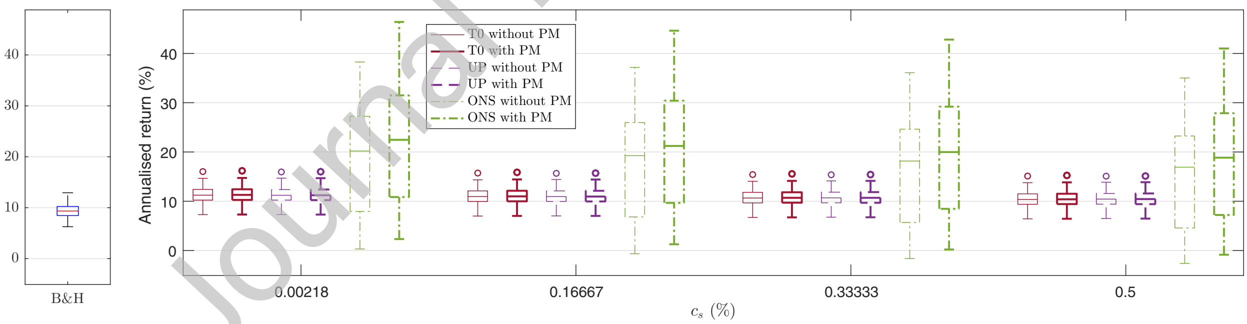

Figure 6. Box plots of annualised returns of different online portfolio selection methods without and with the proposed method (PM) ($c_p = 0$, $s_0 = 10^6$ USD).

The overall TCs, consisting of both the proportional TCs and MICs, has been calculated whenever a portfolio was rebalanced for each case. More specifically, for the case of OPS without using the PM, TCs is calculated at 10:00 a.m. on every trading day, whereas TCs is calculated not only at 10:00 a.m. but also between 10:30 a.m. and 4:00 p.m. 30 minutes on every trading day for the case of OPS using the PM.

If initial wealth s_0 is as small as USD 100,000, the performance difference between OPS without and with the PM is not statistically significant, as shown in Table 3 and Figure 5, reflecting the PM is not useful for small-sized funds. However, the PM is useful when initial wealth s_0 is as large as USD 1,000,000 as shown in Table 4 and Figure 6. In particular, the performance difference between ONS without and with the PM is statistically significant. Indeed the performance difference between without and with the PM varies by OPS method. ONS makes a greater performance gap than T0 and UP. This is because ONS is a more dynamic investment strategy than T0 and UP as shown in Figure C2; i.e. the PM has more opportunities to reduce TCs when OPS tries to cause the higher TCs.

The performance difference between without and with the PM (the difference of means in Table 3 and Table 4) is less significant when c_s is higher. This is because proportional TCs, $c_p \sum_{j=1}^{d} \left(\bar{p}\left(q_{n,t}^{(j)}\right) q_{n,t}^{(j)}\right)^{+} + c_s \sum_{j=1}^{d} \left(-\bar{p}\left(q_{n,t}^{(j)}\right) q_{n,t}^{(j)}\right)^{+}$ in Equation (7), are more dominant compared to MICs, $\sum_{j=1}^{d} \left(\bar{p}\left(q_{n,t}^{(j)}\right) - m_{n,t}^{(j)}\right) q_{n,t}^{(j)}$ in Equation (7), when c_p or c_s is the greater. Even if a large market order is divided into consecutive intraday market orders by the PM, proportional TCs with the PM are as large as those without the PM. Consequently, the PM is more useful with the lower c_p and c_s.

6. Conclusion

As claimed in the introduction, this paper is, to the best of authors' knowledge, the first attempt to combine OPS with an algorithmic trading under limited market liquidity. We develop a mathematical framework within which the optimal intraday trading strategy minimises overall TCs, consisting of both the proportional TCs and liquidity risk, when rebalancing a multi-asset portfolio.

By considering the real-time LOBs when rebalancing the portfolio, the strategy optimally splits very large market orders into small sequential market orders, which significantly cushions the shock especially for large volume trading. Moreover, the proposed intraday trading algorithm is applicable to any portfolio rebalancing strategy, including all the OPS methods regardless of capital size.

It is also worth noting that the applications of our PM algorithm is not limited to OPS methods mainly discussed in this paper. One could easily apply the PM algorithm to other aspects of financial trading such as optimal control of execution costs of trading a portfolio.

Using historical NASDAQ LOB data, the numerical experiments and backtesting results have demonstrated that:

- The proposed algorithm is much effective for large capital investment as it generally leads to higher TCs;
- Our intraday trading algorithm generates more benefit (higher TCs reduction) for OPS methods which rebalance more frequently such as ONS;
- The lower the proportional transaction fees rate is, the more benefit the proposed algorithm creates as the overall TCs is much sensitive to MICs other than the proportional TCs.

The backtesting results are very promising, however, the heavy computation, analysed in Section C.3, should be reduced further for real-time algorithmic trading.

Acknowledgements

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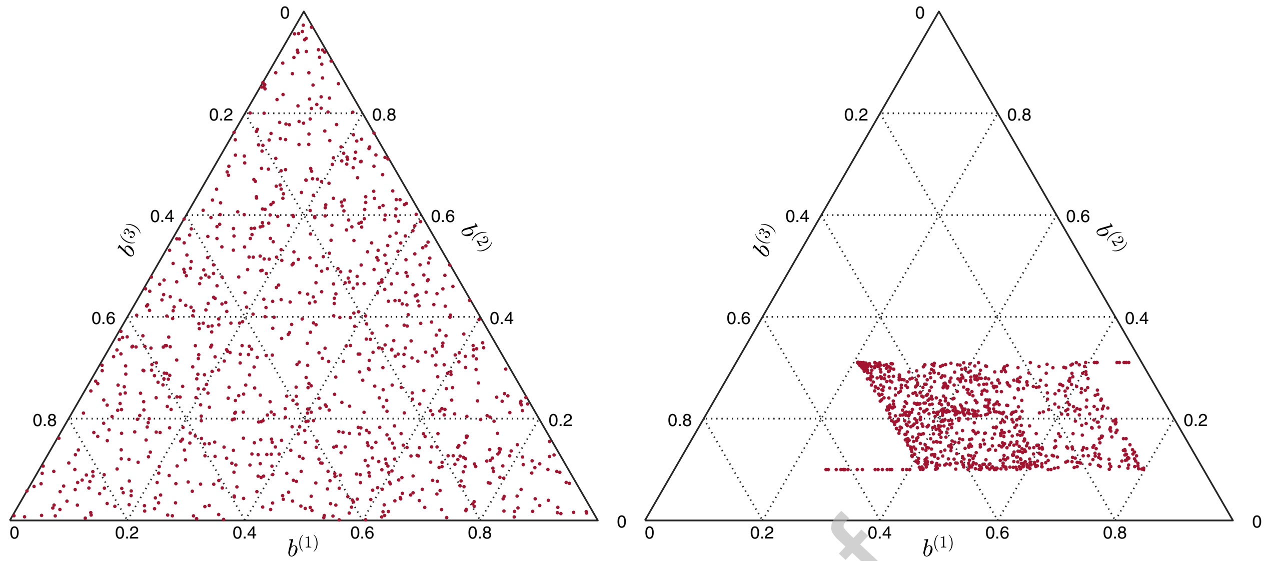
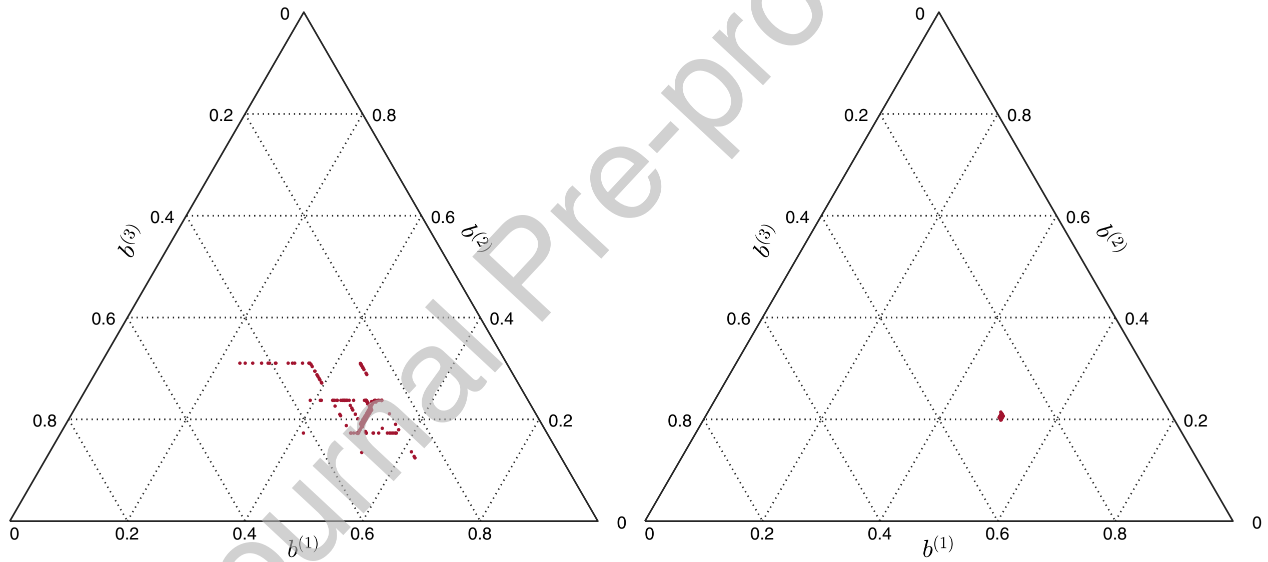


(a) 1,000 random initial guesses from the Dirichlet distribution with $\boldsymbol{\alpha} = [1\ 1\ 1]^{\mathrm{T}}$.

(b) The suboptimal solutions by the proposed method when $\nu_{n-1,1}$ is USD 10^5.



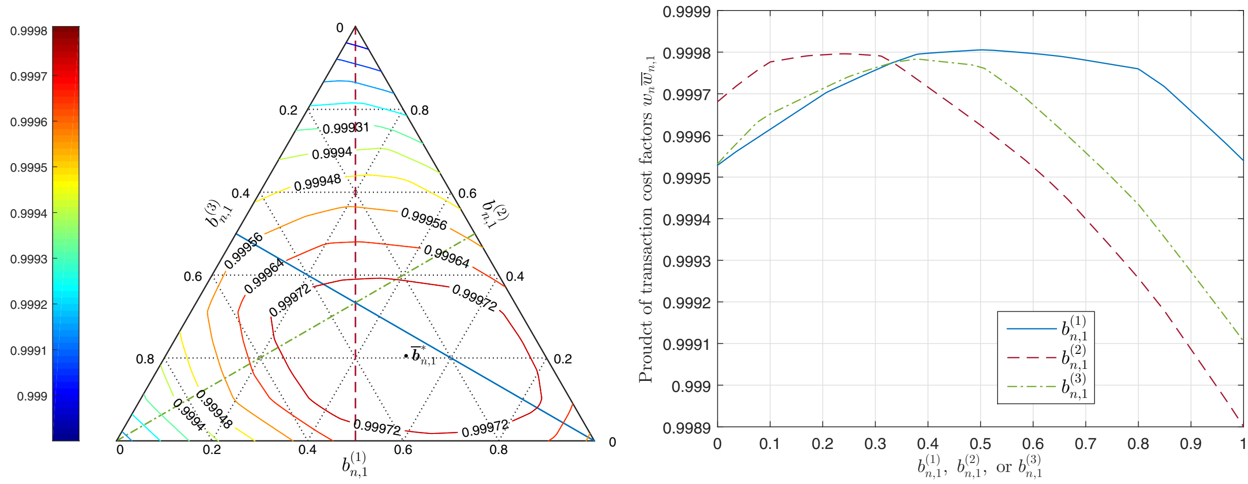
(c) The suboptimal solutions by the proposed method when $\nu_{n-1,1}$ is USD 10^6.

(d) The suboptimal solutions by the proposed method when $\nu_{n-1,1}$ is USD 10^7.

Figure A1. Ternary scatter plots of single intrady trading with the fixed values: $\boldsymbol{b}_n = [1/3\ 1/3\ 1/3]^{\mathrm{T}}$, $\boldsymbol{x}_{n,0} = [0.6\ 0.9\ 1.4]^{\mathrm{T}}$, $\boldsymbol{b}_{n+1} = [0.8\ 0.8\ 0.1]^{\mathrm{T}}$, $c_p = 0$, and $c_s = 0.00218\%$. 10-level limit order book data of AAPL ($b^{(1)}$), AMZN ($b^{(2)}$), and GOOG ($b^{(3)}$) on 21 Jun 2012 at 16:00:00 was used.

Appendix A: The convergence of the proposed method with random initial guesses

The convergence of the proposed method can be visually estimated by scatter plots as shown in Figure A1. 1,000 random initial guesses were chosen to solve the optimization problem in Equation (21), as shown in Figure 1(a). The suboptimal solutions $\boldsymbol{b}_{n,1}$ are not converged for small-sized funds as shown in Figure 1(b) and Figure 1(c). This is because small-sized funds need a small amount of intraday trading, and $\boldsymbol{b}_{n,1}$ is not critical to minimise transaction costs. In contrast, the suboptimal solutions $\boldsymbol{b}_{n,1}$ are converged for large-sized funds as shown in Figure 1(d). This is because large-sized funds need a large amount of intraday trading, and $\boldsymbol{b}_{n,1}$ is critical to minimise transaction costs. Consequently, the proposed method is more effective for large-sized funds than



(a) Ternary contour plot of $w_n\overline{w}_{n,1}$.

(b) 1D plots of $w_n\overline{w}_{n,1}$ (each line matches each straight line in the left plot).

Figure B1. Ternary contour plots and 1D plots of the product of transaction cost factors $w_n\overline{w}_{n,1}$ with the variable $\boldsymbol{b}_{n,1}$ and the fixed values: $\boldsymbol{b}_n = [1/3\ 1/3\ 1/3]^{\mathrm{T}}$, $\boldsymbol{x}_{n,0} = [0.6\ 0.9\ 1.4]^{\mathrm{T}}$, $\nu_{n-1,\tau} = 10^6$ USD, $\boldsymbol{b}_{n+1} = [0.8\ 0.8\ 0.1]^{\mathrm{T}}$, $c_p = 0$, and $c_s = 0.00218\%$. 10-level limit order book data of AAPL ($b^{(1)}$), AMZN ($b^{(2)}$), and GOOG ($b^{(3)}$) on 21 Jun 2012 at 16:00:00 was used.

small-sized funds.

Appendix B: An example of unimodality of $w_n\overline{w}_{n,1}$

The product of TCFs $w_n\overline{w}_{n,1}$ can be rewritten as a function of $\boldsymbol{b}_{n,1}$ as

$$\begin{aligned} w_n\overline{w}_{n,1} =& w(\boldsymbol{b}_n, \boldsymbol{b}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})w(\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n+1}, \boldsymbol{1}, s_n w_n) \\ =& w(\boldsymbol{b}_n, \boldsymbol{b}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})w(\boldsymbol{b}_{n,1}, \boldsymbol{b}_{n+1}, \boldsymbol{1}, \nu_{n-1,\tau}\langle \boldsymbol{b}_n, \boldsymbol{x}_{n,0}\rangle w(\boldsymbol{b}_n, \boldsymbol{b}_{n,1}, \boldsymbol{x}_{n,0}, \nu_{n-1,\tau})), \end{aligned} \tag{B1}$$

and it is plotted as a unimodal function of $\boldsymbol{b}_{n,1}$ (i.e. $w_n\overline{w}_{n,1}$ strictly decreases as $\boldsymbol{b}_{n,1}$ goes away from the maximum point $\overline{\boldsymbol{b}}^*_{n,1}$) as shown in Figure B1. Therefore, $w_n\overline{w}_{n,1}$ in Figure B1 is a unimodal function of $\boldsymbol{b}_{n,1} \in \Delta^2$. However, this is only one example from the given values ($\boldsymbol{b}_n$, $\boldsymbol{x}_{n,0}$, $\nu_{n-1,\tau}$, $\boldsymbol{b}_{n+1}$, c_p, and c_s) and the LOBs of the three stocks. The mathematical proof of the unimodality of $w_n\overline{w}_{n,1}$ is not provided in this paper.

Appendix C: Further Results

C.1. *Benchmark case: performance comparison among online portfolio selection methods without TCs*

The annualised return of the OPS methods in Table 2 without TCs ($c_p = 0$, $c_s = 0$, and zero MICs) is compared in Table C1 and Figure C4 [3], where all the OPS methods rebalanced a portfolio at 10:00 a.m. on every U.S. trading day. In particular, the unpaired two-sample t-tests with unequal variances (hereafter we simply referred to as t-tests), whose null hypothesis is that the data in two groups comes from independent random samples from normal distributions with equal means but different variances, were performed to compare the performance between i) a buy-and-hold

[3] MATLAB programs of OPS by Li and Hoi (2015, Appendix A) were used to obtain Table C1 and Figure C4.

(B&H) strategy with the initial portfolio $\boldsymbol{b}_1 = [1/d \; 1/d \; \ldots \; 1/d]^{\mathrm{T}}$ and ii) the OPS methods listed in Table 2. Also, the normality assumption of the t-test was confirmed by the Jarque–Bera (JB) test with the significance level of 0.05 as shown in Table C1 except OLMAR1, OLMAR2, and BK. The standard deviation in Table C1 can be interpreted as the sensitivity of the annualised return to the random stock selection. USCRP and OLMAR1 are the least and the most sensitive, respectively.

The t-tests provide the answer as to whether the performance difference of the two methods is significant or whether it is due to random fluctuations (Simon 2013, p. 631). To be specific, the p-value of the t-test is interpreted as the probability that a difference in the mean values would be obtained, given that the population means of the two methods are equivalent (the p-value is not equal to the probability that the population means are equivalent) (Simon 2013, p. 635). Hence, if the p-value of the t-test is less than a significance level, the performance difference is significant. All the OPS methods of *follow the winner* are highly superior to B&H as shown in Table C1. In contrast, all the OPS methods of *follow the loser* and *pattern matching* except BNN are not superior to B&H, and some of them are inferior to B&H (i.e. the differences of means are negative with low p-values in Table C1). However, this should not be interpreted that the OPS methods of *follow the loser* and *pattern matching* are always inferior to those of *follow the winner*, as the NASDAQ bull market between 2009 and 2016 (see Figure C1) was unfavourable to the OPS methods of *follow the loser*.

The performance of each OPS method without TCs and intraday trading in Table C1 and Figure C4 can be considered as the upper limit of the performance with TCs and intraday trading. This is because OPS carries out a long-term investment for every period at time $t = 0$, while the proposed method of intraday trading absorbs the shock to the market whenever OPS rebalances a portfolio. In other words, the proposed method minimises the performance gap between OPS without TCs and OPS with TCs, but no additional profits made.

C.2. *Graphical comparisons*

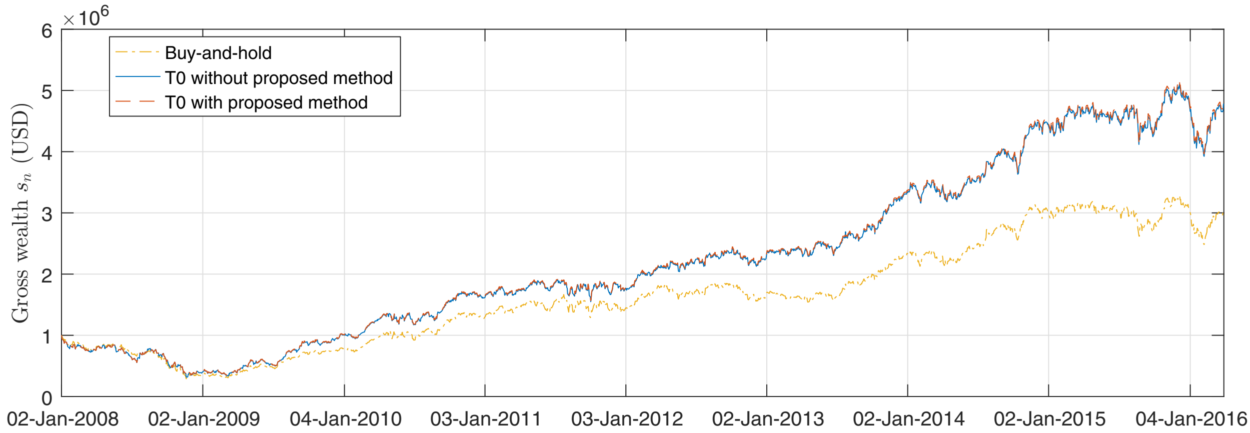
Figure C1 shows gross wealth s_n of a portfolio consisting of six stocks with an initial wealth of a million USD. The PM of intraday trading has little value in the case of T0 and UP as shown in Figure 1(a) and Figure 1(b). However, the PM works well when ONS is used as shown in Figure 1(c). These correspond to the performance difference without and with the PM in Table 4 and Figure 6.

Figure C2 shows the proportion of the portfolio (the portfolio vector $\boldsymbol{b}_n$) that made the gross wealth plots in Figure C1, in the form of area plots ($\boldsymbol{b}_n$ is independent of the usage of an intraday trading algorithm). The portfolio vector of B&H changes over time as the prices of assets also change over time as shown in Figure 2(a). T0 generates almost constant portfolio $\boldsymbol{b}_n = \boldsymbol{b}_1$ as shown in Figure 2(b), and UP generates rougher portfolio weights over time than those of T0 but smoother changes than those of B&H as shown in Figure 2(c). ONS makes the most abrupt changes of portfolio weights over time as shown in Figure 2(d).

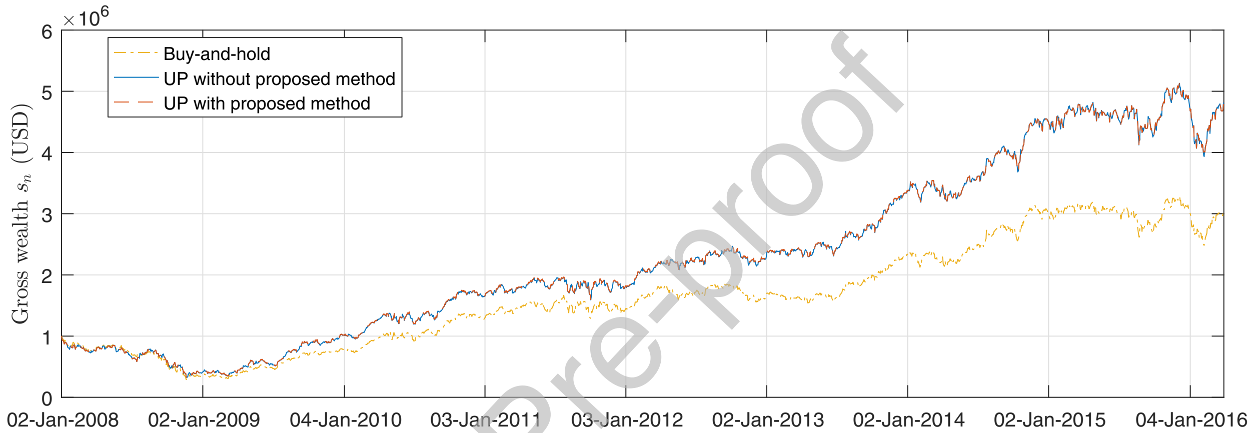
Figure C3 shows how much the PM can decrease TCs c_n, consisting of both proportional TCs and MICs, when following the gross wealth in Figure C1. TC reduction by the PM is not significant in the case of T0 and UP as shown in Figure 3(a) and Figure 3(b) since both T0 and UP do not trade stocks too much. On the contrary, TCs are reduced greatly in the case of a high-volume trading algorithm like ONS as shown in Figure 3(c).

C.3. *Computation time*

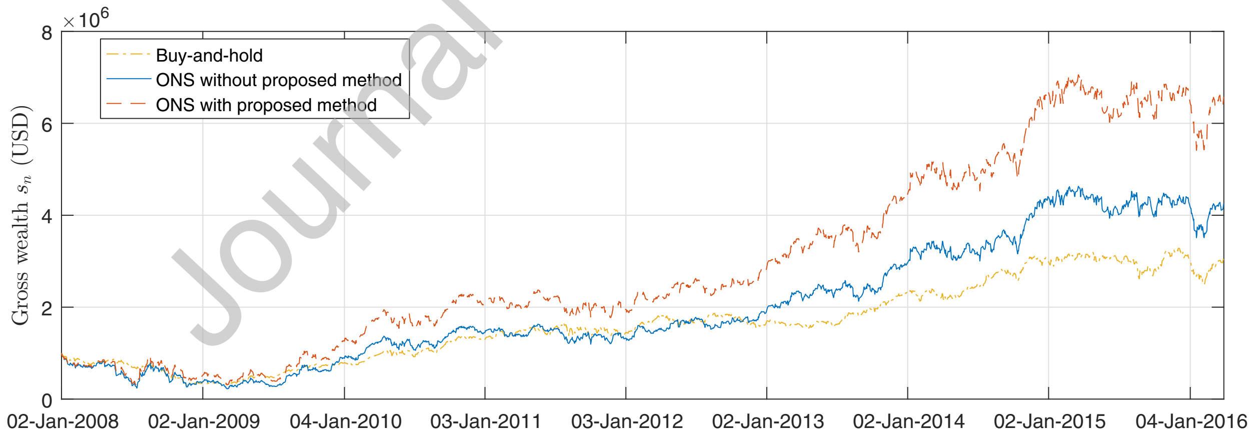
The mean computation time of the PM depends on the OPS strategy, initial wealth s_0, and intraday trading time t as shown in Figure C5. Firstly, the PM requires more computation time for the higher-volume trading algorithms (e.g. ONS) than the lower-volume trading algorithms (e.g. T0 and UP) as shown in both Figure 5(a) and Figure 5(b). Secondly, the PM requires more computation time for the bigger-sized funds (see Figure 5(b)) than the smaller-sized funds (see Figure 5(a)).



(a) T0.

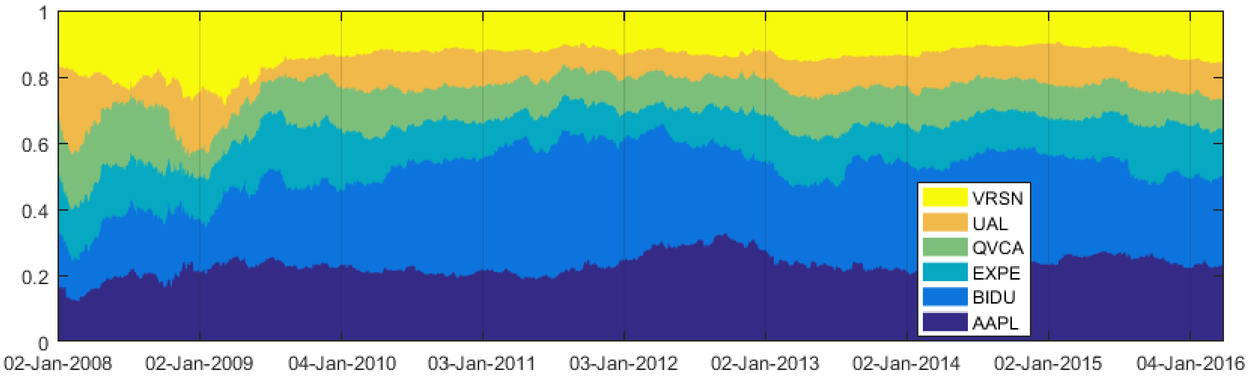


(b) UP.

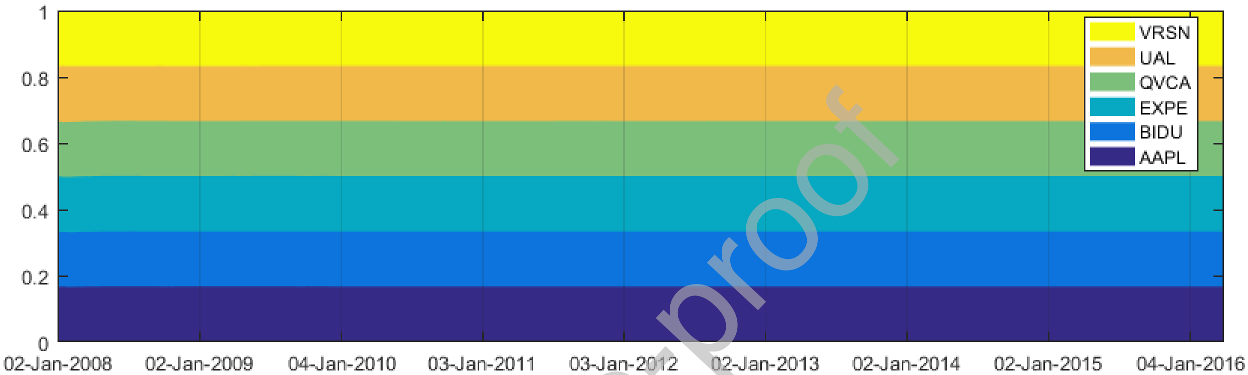


(c) ONS.

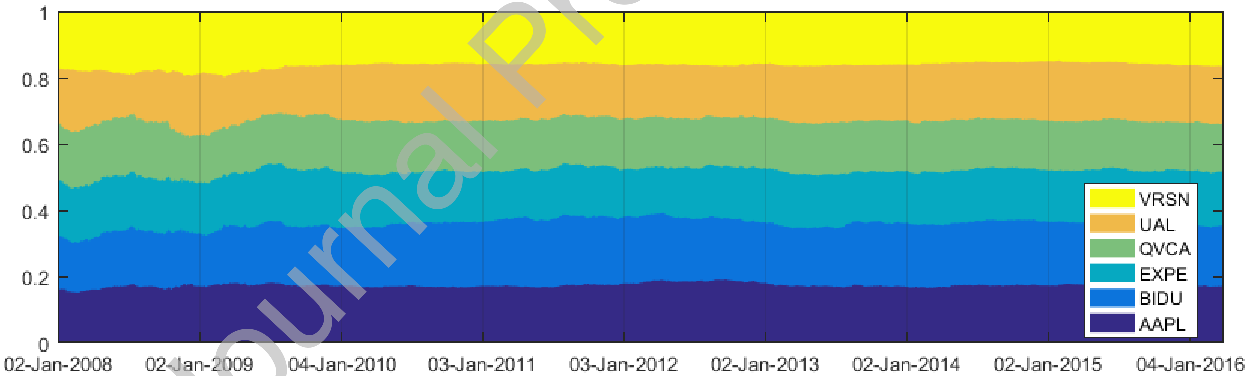
Figure C1. Gross wealth over time when the portfolio consists of AAPL, BIDU, EXPE, QVCA, UAL, and VRSN ($s_0 = 10^6$ USD, $c_p = 0$, $c_s = 0.00218\%$).



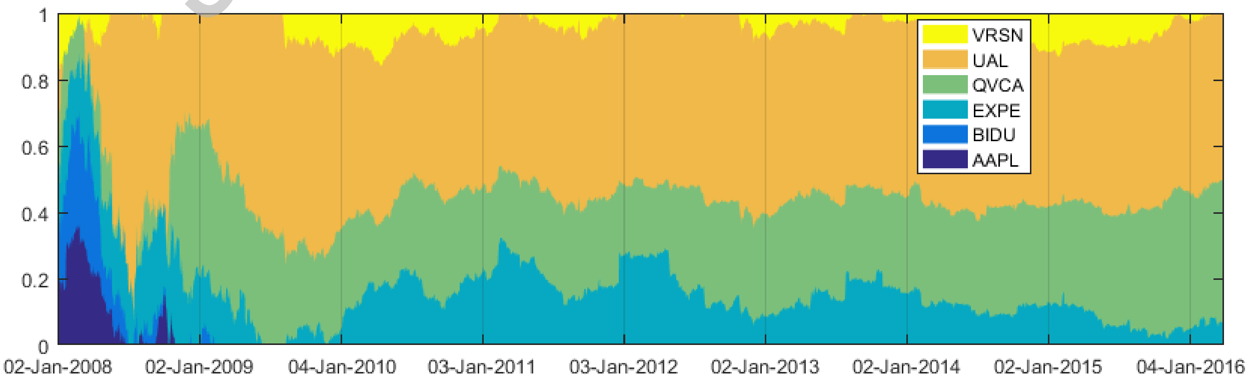
(a) Buy-and-hold (B&H).



(b) T0.



(c) UP.



(d) ONS.

Figure C2. The proportion of portfolio over time.

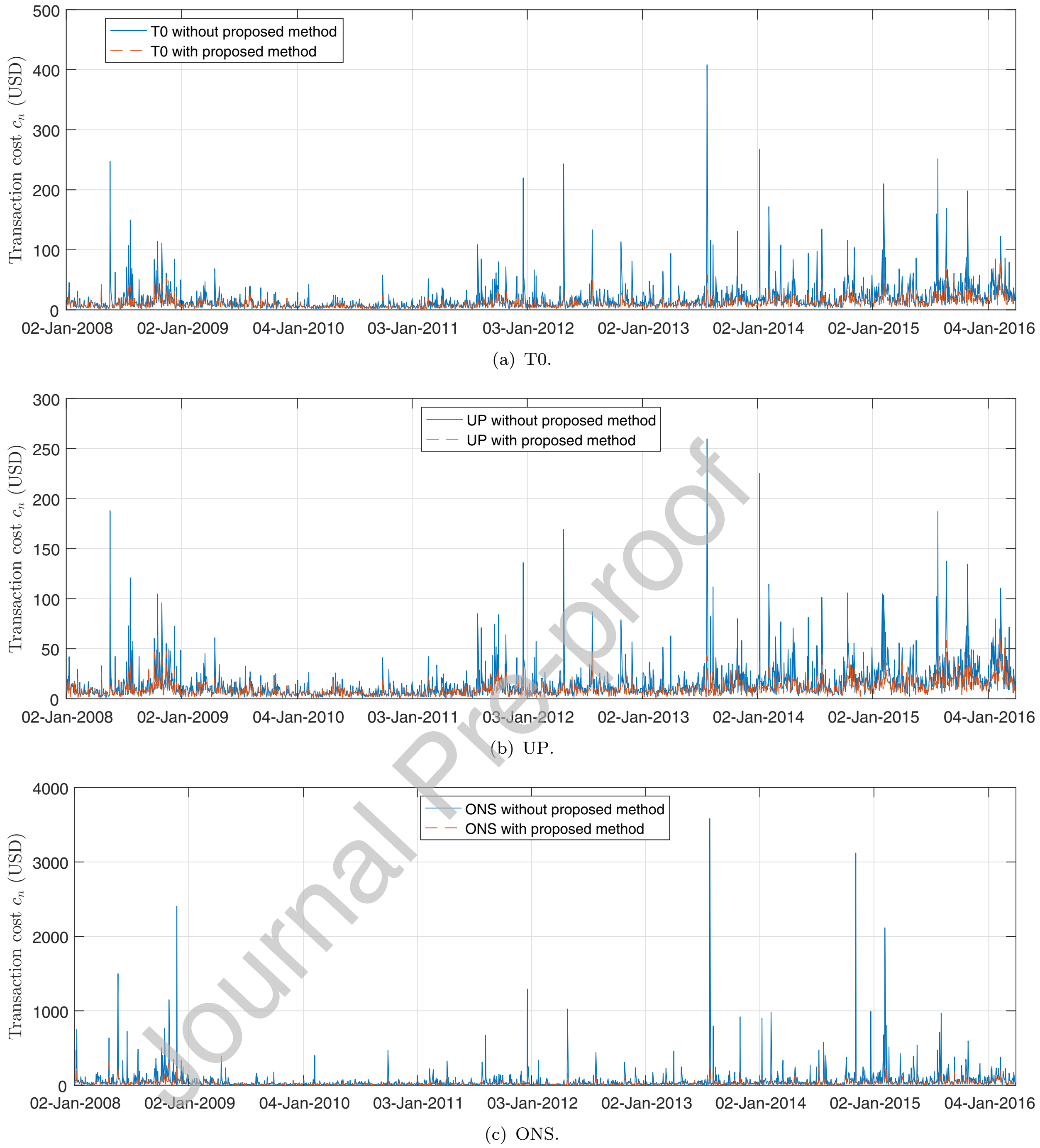


Figure C3. Transaction costs, consisting of proportional costs and market impact costs, over time ($s_0 = 10^6$ USD, $c_p = 0$, and $c_s = 0.00218\%$).

Table C1. Statistics of annualised returns of different online portfolio selection methods without transaction costs ($c_p = 0$, $c_s = 0$, and zero market impact costs).

	B&H	UP	EG	ONS	USCRP	ANTICOR	ANTICOR_ANTICOR	OLMAR1	OLMAR2	PAMR	PAMR1
P-value of JB test	0.956	0.450	0.530	0.077	0.570	0.330	0.766	0.048	0.034	0.332	0.332
Standard deviation (%)	1.26	1.50	1.51	11.43	1.32	8.29	11.93	17.37	17.18	16.58	16.58
Mean (%)	9.3	11.4	11.4	23.1	10.4	8.4	6.6	6.2	9.6	9.2	9.2
Difference of means[a] (%)	-	2.11***	2.10***	13.81***	1.08***	−0.88	−2.75**	−3.11*	0.24	−0.12	−0.12
P-value of *t*-test	-	$2.0{\times}10^{-21}$	$3.4{\times}10^{-21}$	$3.3{\times}10^{-21}$	$1.3{\times}10^{-8}$	$2.9{\times}10^{-1}$	$2.4{\times}10^{-2}$	$7.8{\times}10^{-2}$	$8.9{\times}10^{-1}$	$9.4{\times}10^{-1}$	$9.4{\times}10^{-1}$

	PAMR2	CWMR_VAR	CWMR_STDEV	BK	BNN	CORN	CORNU	CORNK	M0	T0
P-value of JB test	0.357	0.233	0.231	0.002	0.127	0.672	0.629	0.729	0.445	0.533
Standard deviation (%)	16.53	16.07	15.99	5.39	8.05	9.46	6.34	8.90	3.36	1.53
Mean (%)	9.5	11.1	11.3	8.6	12.2	5.3	10.0	6.0	12.8	11.5
Difference of means[a] (%)	0.16	1.80	1.96	−0.73	2.85***	−3.99***	0.68	−3.30***	3.43***	2.18***
P-value of *t*-test	$9.2{\times}10^{-1}$	$2.7{\times}10^{-1}$	$2.2{\times}10^{-1}$	$1.9{\times}10^{-1}$	$7.1{\times}10^{-4}$	$6.0{\times}10^{-5}$	$3.0{\times}10^{-1}$	$3.8{\times}10^{-4}$	$1.3{\times}10^{-16}$	$3.9{\times}10^{-22}$

[a]Difference equals average annualised return of the corresponding OPS method minus that of buy-and-hold (B&H). $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

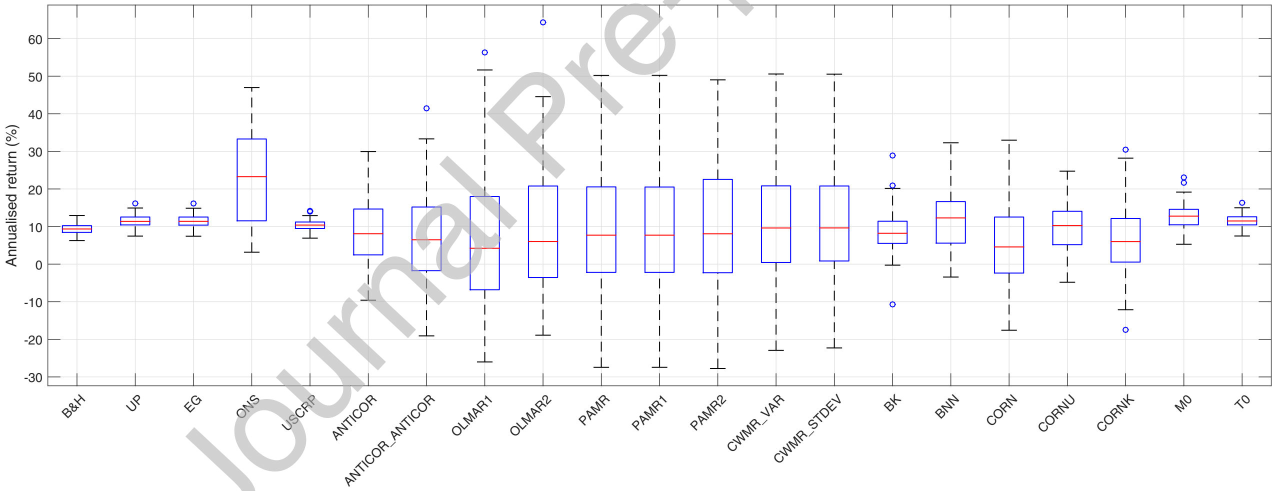


Figure C4. Box plots of annualised returns of different online portfolio selection methods without transaction costs ($c_p = 0$, $c_s = 0$, and zero market impact costs).

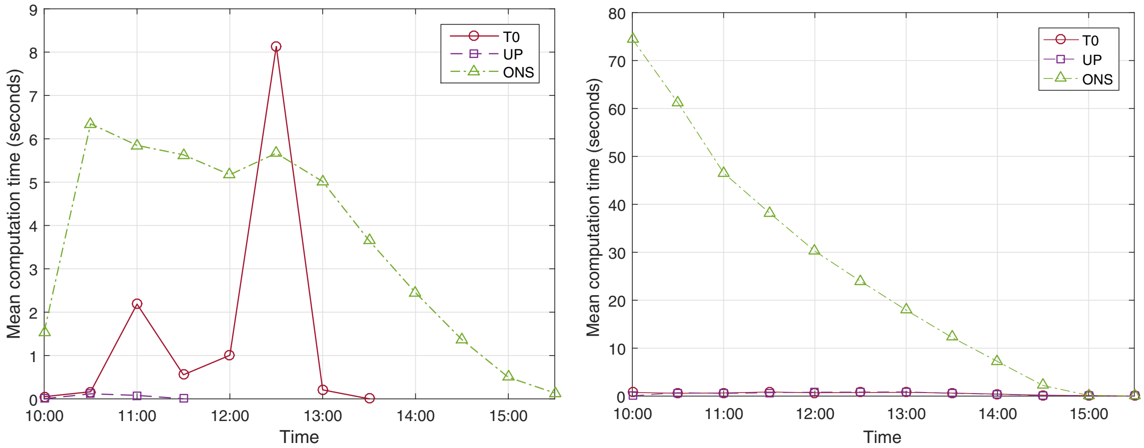


Figure C5. The mean computation time of the proposed method when the initial wealth s_0 is USD 10^5 and 10^6 respectively.

Thirdly, the computation time changes over time during the day. To be specific, the computation time of the PM with ONS decreases as time goes by during NASDAQ trading hours as shown in Figure 5(b). This is because Algorithm 2 has the iteration for $(\tau_{\max} - t)$ times for each t (see the 15th line of Algorithm 2) to calculate the minimum optimal number of intraday trades. Consequently, the time complexity of the PM is $O(\tau_{\max})$.

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