A NOTE ON THE SMALL-TIME BEHAVIOUR OF THE LARGEST BLOCK SIZE OF BETA N-COALESCENTS

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Abstract. We study the largest block size of Beta n-coalescents at small times as n tends to infinity, using the paintbox construction of Beta-coalescents and the link between continuous-state branching processes and Beta-coalescents established in Birkner et al [4] and Berestycki et al [2]. As a corollary, a limit result on the largest block size at the coalescence time of the individual/block $\{1\}$ is provided.

1. Introduction and main results

Beta n-coalescents form a class of partition-valued coagulating Markov chains. This family was introduced by Schweinsberg ([20]) following pioneer works of Pitman ([17]), Sagitov([18]) and Möhle and Sagitov ([16]). Formally, a Beta n-coalescent $(\Pi^{(n)}(t), t \geq 0)$ is a continuous-time Markov chain with values in partitions of $[n] := \{1, 2, \ldots, n\}$ starting at $\Pi^{(n)}(0) = \{\{1\}, \{2\}, \ldots, \{n\}\}$. As n-coalescents can be used as models for the genealogy of a sample of n individuals, we refer to $[n]$ as the set of (labels of) individuals. Its dynamics are determined by a parameter $\alpha \in (0, 2)$: when $\Pi^{(n)}$ has b blocks, any k-tuple of them merges into one block at rate

$$\lambda_{b,k} := \frac{\beta(k - \alpha, b - k + \alpha)}{\beta(\alpha, 2 - \alpha)} \tag{1}$$

where $\beta(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the Beta function. In this paper, we are only interested in the case $\alpha \in (1, 2)$.

Equation (1) induces exchangeability and consistency of these processes. Exchangeability means that if we permute the labels of individuals, the law of $\Pi^{(n)}$ stays unchanged. Consistency refers to that for any couple of integers $n < m$, the projection of $\Pi^{(m)}$ on $[n]$ has the same law as $\Pi^{(n)}$. By Kolmogorov's extension theorem ([17]), we can construct the so-called Beta-coalescent process $(\Pi(t), t \geq 0)$ taking values in partitions of $\mathbb{N}$ such that the projection of Π on $[n]$ is equal in distribution to $\Pi^{(n)}$. When $\alpha \in (1, 2)$ the Beta-coalescent has proper frequency (i.e., almost surely for any $t > 0$, Π has no singletons, see [17]) and comes down from infinity (i.e., almost surely for any $t > 0$, Π has a finite number of blocks, see [19]).

Berestycki et al ([2]) provided many results on the behaviour of functionals of $\Pi(t)$ as t tends to 0, such as the number of blocks, the ranked sequence of asymptotic frequencies of those blocks and the asymptotic frequency of the largest block. For the latter, they establish the following result in Proposition 1.6:

Proposition 1.1. *let $X(t)$ be the asymptotic frequency of the largest block of Π at time t, then*

$$(\alpha\Gamma(\alpha)\Gamma(2 - \alpha))^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}} X(t) \xrightarrow{d} X, \text{ as } t \text{ goes to } 0 \tag{2}$$

where X is a Fréchet random variable with parameter α, i.e., $\mathbb{P}(X \leq x) = e^{-x^{-\alpha}}$, for any $x \geq 0$, and "$\xrightarrow{d}$" stands for the convergence in law.

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This is a result in the infinite coalescent for $t \to 0$. Often, especially when used as a genealogy model, we are actually more interested in the n-coalescents and their asymptotic behaviour, since we can then interpret results in terms of the finite models (as in [7, 8, 9, 12, 13, 15, 21, 22]). Proposition 1.1 would in this sense be first taking $n \to \infty$, then $t \to 0$, while we would like a simultaneous limit $(t_n, n) \to (0, \infty)$. In this case, we could look at specific, interpretable/interesting small times t_n.

Such time is the external branch length of individual 1 (studied in [9], and with further extensions given recently in [22] and [24]), denoted by $T_1^{(n)}$ and defined by

$$T_1^{(n)} := \sup\{t, \{1\} \in \Pi^{(n)}(t)\}.$$

This can be seen as seeing the coalescent from the eyes of individual 1 and measuring its "distance" to the rest of the sample or its *genetic uniqueness* ([6]). Here individual 1 represents a randomly chosen individual of the sample thanks to exchangeability. Observe that, since the Beta-coalescent has proper frequency when $\alpha \in (1, 2)$, this variable vanishes as we let n tend to infinity. We are now curious how the block structure of the coalescent looks like at this specific time (asymptotically).

One possible tool for this study is the minimal clade size, studied in [22] for $\alpha \in (1, 2)$ (see also [11] for $\alpha = 1$ and [5] for $\alpha = 2$). This is the size of the block containing 1 at time $T_1^{(n)}$. The size of the minimal clade gives the information of how many individuals share the genealogy with individual 1 after he merges. It was shown in [22] that the minimal clade size converges in law, without any renormalization, to a heavy-tailed random variable of index $(\alpha - 1)^2$.

Now we would like to compare this minimal clade size to the size of the largest block at time $T_1^{(n)}$, denoted by $\tilde{W}^{(n)}$. This comparison gives a first picture of the inhomogeneity of the block structure of the Beta n-coalescent at small times. To study $\tilde{W}^{(n)}$, we first consider the size of the largest block at any time t, denoted by $W^{(n)}(t)$. Hence, we have

$$\tilde{W}^{(n)} = W^{(n)}(T_1^{(n)}).$$

We obtain an asymptotic result for $W^{(n)}$ at the $n^{1-\alpha}t$ scale.

Theorem 1.2. *For a Beta n-coalescent with $1 < \alpha < 2$, as n tends to infinity*

$$(\alpha\Gamma(\alpha)\Gamma(2-\alpha))^{\frac{1}{\alpha}}(nt)^{-\frac{1}{\alpha}}W^{(n)}(n^{1-\alpha}t) \xrightarrow{d} X, \tag{3}$$

where X is a Fréchet random variable with parameter α.

Rewriting (3) as

$$\alpha\Gamma(\alpha)\Gamma(2-\alpha))^{\frac{1}{\alpha}}(n^{1-\alpha}t)^{-\frac{1}{\alpha}}\frac{W^{(n)}(n^{1-\alpha}t)}{n} \xrightarrow{d} X,$$

the reader can observe the similarity with (2).

To study the behaviour of $\tilde{W}^{(n)}$, we shall consider the restriction of $\Pi^{(n)}$ on $\{2, \dots, n\}$, denoted by $\Pi^{(n,2)} = (\Pi^{(n,2)}(t), t \geq 0)$. By consistency, the latter is equal in law to $\Pi^{(n-1)}$ modulo notations of the labels of individuals. Then $\tilde{W}^{(n)}$ is actually the largest block size of $\Pi^{(n,2)}(T_1^{(n)})$ plus 1, if $\{1\}$ coalesces with the largest block of $\Pi^{(n,2)}(T_1^{(n)})$ or plus 0 otherwise.

It has been established in the proof of Theorem 5.2 of [9] that conditional on $\Pi^{(n,2)}$, $n^{\alpha-1}T_1^{(n)}$ converges in law to a random variable T. More precisely,

$$\mathbb{P}(n^{\alpha-1}T_1^{(n)} \geq t | \Pi^{(n,2)}) \xrightarrow{d} \mathbb{P}(T \geq t) = (1 + \frac{t}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}}. \tag{4}$$

This shows that in the decomposition of $\tilde{W}^{(n)} = W^{(n)}(T_1^{(n)})$, the terms $(W^{(n)}(n^{1-\alpha}t), t \geq 0)$ and $n^{\alpha-1}T_1^{(n)}$ are asymptotically independent. Combining (4) together with Theorem 1.2, we can describe the limit of $\tilde{W}^{(n)}$ as a mixture.

Corollary 1.3. *As n tends to infinity,*

$$\frac{\tilde{W}^{(n)}}{n^{\frac{1}{\alpha}}} \xrightarrow{d} \tilde{W}, \tag{5}$$

where $\tilde{W}$ is a positive random variable such that for any $x \geq 0$,

$$\mathbb{P}(\tilde{W} \leq x) = \int_0^\infty \frac{\exp(-x^{-\alpha}\frac{t}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)})}{(\alpha-1)\Gamma(\alpha)}(1+\frac{t}{\alpha\Gamma(\alpha)})^{-\frac{2\alpha-1}{\alpha-1}}dt.$$

This note is organised as follows. In Section 2, we introduce the main tools such as the construction of Beta-coalescents via continuous-state branching processes and the paintbox construction of exchangeable coalescents. Section 3 is devoted to the proofs of Theorem 1.2.

2. Preliminaries

2.1. **Ranked coalescent and paintbox construction.** Assume all along the rest of the paper that $1 < \alpha < 2$. Let $\Pi = (\Pi(t), t \geq 0)$ be the Beta-coalescent and denote by $K = (K(t), t > 0)$ the block-counting process of Π. In words, $K(t)$ stands for the number of blocks of $\Pi(t)$. It is known that Π is coming down from infinity: for any $t > 0$, $K(t)$ is finite almost surely ([19]). Also recall that for any $t \geq 0$, $\Pi(t)$ is an exchangeable random partition of $\mathbb{N}$. This means that if we permute finitely many integers in $\Pi(t)$, the law of $\Pi(t)$ is unchanged. Applying Kingman's paintbox theorem on exchangeable random partitions ([14]), almost surely for every block $B \in \Pi(t)$, the following limit, called the *asymptotic frequency* of B, exists:

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \mathbf{1}_{\{i\in B\}}.$$

Furthermore, when $t > 0$, the sum of all asymptotic frequencies equals 1 since Π is of proper frequency ([17]). Hence, one can reorder all the asymptotic frequencies in a non-increasing way to define a sequence $\Theta(t) = \{\theta_1(t), \theta_2(t), \cdots, \theta_{K(t)}(t)\}$ where $\theta_1(t) \geq \theta_2(t) \geq \cdots \geq \theta_{K(t)}(t)$ and $\sum_{i=1}^{K(t)} \theta_i(t) = 1$. At time $t = 0$, every block is a singleton and then has asymptotic frequency 0. Hence one can naturally set $\Theta(0) = \{0, 0, \ldots\}$. Then the process $\Theta = (\Theta(t), t \geq 0)$ is well defined. We call it the *ranked coalescent*.

Given $\Theta(t)$ for some $t > 0$, one can recover the distribution of $\Pi(t)$ using again Kingman's paintbox theorem. Let us at first divide $[0, 1]$ into $K(t)$ subintervals such that their lengths are equal one to one to the values of elements of $\Theta(t)$. Then we throw individuals $1, 2, \cdots$ uniformly and independently into $[0, 1]$. Finally, all individuals within one interval form a block and this procedure provides a random exchangeable partition which has the same law as $\Pi(t)$. Thanks to the consistency property, the restricted partition $\Pi^{(n)}(t)$ can be obtained using the same procedure but throwing n particles instead of infinitely many.

2.2. **Beta-coalescents and stable continuous-state branching processes.** To prove Theorem 1.2, we will use classical relations between Beta-coalescents and continuous-state branching processes (CSBPs) developed in [4] (see also Section 2 of [2]). We give a short summary to provide a minimal set of tools. A continuous-state branching process $(Z(t), t \geq 0)$ is a $[0,\infty]$-valued Markov process (in continuous time) whose transition semigroup $p_t(x, \cdot)$ satisfies the branching property

$$p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot), \quad \text{for all } x, y \geq 0.$$

For each $t \geq 0$, there exists a function $u_t : [0,\infty) \to \mathbb{R}$ such that

$$\mathbb{E}[e^{-\lambda Z(t)}|Z(0) = a] = e^{-au_t(\lambda)}. \tag{6}$$

If, almost surely, the process has no instantaneous jump to infinity, the function u_t satisfies the following differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} = -\Psi(u_t(\lambda)),$$

where $\Psi : [0,\infty) \longrightarrow \mathbb{R}$ is a function of the form

$$\Psi(u) = \gamma u + \beta u^2 + \int_0^\infty (e^{-xu} - 1 + xu\mathbf{1}_{\{x\leq 1\}})\pi(dx),$$

where $\gamma \in \mathbb{R}, \beta \geq 0$ and π is a Lévy measure on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge x^2)\pi(dx) < \infty$. The function Ψ is called the *branching mechanism* of the CSBP.

As explained in [3], a CSBP can be extended to a two-parameter random process $(Z(t,a), t \geq 0, a \geq 0)$ with $Z(0,a) = a$. For fixed t, $(Z(t,a), a \geq 0)$ turns out to be a subordinator with Laplace exponent $\lambda \mapsto u_t(\lambda)$ thanks to (6).

There exists a measure-valued process $(M_t, t \geq 0)$ taking values in the set of finite measures on $[0,1]$ which characterises $(Z(t,a), t \geq 0, 0 \leq a \leq 1)$. More precisely, $(M_t([0,a]), t \geq 0, 0 \leq a \leq 1)$ has the same finite-dimensional distributions as $(Z(t,a), t \geq 0, 0 \leq a \leq 1)$. Hence $(M_t([0,a]), 0 \leq a \leq 1)$ is a subordinator with Laplace exponent $\lambda \mapsto u_t(\lambda)$ and $Z(t,1) = M_t([0,1])$ is a CSBP with branching mechanism Ψ started at $M_0([0,1]) = 1$. In particular, if the branching mechanism is $\Psi(\lambda) = \lambda^\alpha$, its Lévy measure is given by $\pi(dx) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} x^{-1-\alpha} dx$ and, for all $t > 0$, M_t consists only of a finite number of atoms. For the construction of $(M_t([0,a]), t \geq 0, 0 \leq a \leq 1)$, we refer to [1, 4, 10].

A deep relation has been revealed in [4] between the Beta-coalescent and the CSBP with branching mechanism $\Psi(\lambda) = \lambda^\alpha$. It is described by the following two lemmas which are respectively Lemma 2.1 and 2.2 of [2]. To save notations, from now on, $(Z(t), t \geq 0)$ will always denote a continuous-state branching process $(Z(t,1), t \geq 0)$.

Lemma 2.1. *Assume that $(Z(t), t \geq 0)$ is a CSBP with branching mechanism $\Psi(\lambda) = \lambda^\alpha$ and let $(M_t, t \geq 0)$ be its associated measure-valued process. If $(\Pi(t), t \geq 0)$ is a Beta-coalescent and $(\Theta(t), t \geq 0)$ is the associated ranked coalescent, then for all $t > 0$, the distribution of $\Theta(t)$ is the same as the distribution of the sizes of the atoms of the measure $\frac{M_{R^{-1}(t)}}{Z(R^{-1}(t))}$, ranked in decreasing order. Here $R(t) = (\alpha-1)\alpha\Gamma(\alpha)\int_0^t Z(s)^{1-\alpha} ds$ and $R^{-1}(t) = \inf\{s : R(s) > t\}$.*

Let μ denote the Slack's probability distribution on $[0,\infty)$ (see [23]) characterised by its Laplace transform

$$\mathcal{L}_\mu(\lambda) = \int_0^\infty e^{-\lambda x} \mu(dx) = 1 - (1 + \lambda^{1-\alpha})^{-\frac{1}{\alpha-1}}, \quad \lambda \geq 0. \tag{7}$$

Lemma 2.2. *Assume $\Psi(\lambda) = \lambda^\alpha$. For any $t \geq 0$, let $D(t)$ be the number of atoms of M_t, and let $J(t) = (J_1(t), \cdots, J_{D(t)}(t))$ be the sizes of the atoms of M_t, ranked in decreasing order. Then $D(t)$ is Poisson with mean $\gamma(t) = ((\alpha-1)t)^{-\frac{1}{\alpha-1}}$. Moreover, conditional on $D(t) = k$, the distribution of $J(t)$ is the same as the distribution of $(\gamma(t)^{-1}X_1, \cdots, \gamma(t)^{-1}X_k)$ where $X_1, \cdots, X_k$ are obtained by picking k i.i.d. random variables with distribution μ and then ranking them in decreasing order.*

Remark 2.1. From the relation between $(M_t, t \geq 0)$ and $(Z(t,a), t \geq 0, 0 \leq a \leq 1)$ and also the fact that for all $t > 0$, M_t has a finite number of atoms $D(t)$, we can deduce that for a given $t > 0$, there exist $0 \leq a_1, \cdots, a_{D(t)} \leq 1$ such that $\{Z(t,a_1) - Z(t,a_1-), \cdots, Z(t,a_{D(t)}) - Z(t,a_{D(t)}-)\}$ are exactly the sizes of the atoms of M_t. Markov property of $(Z(t,a), t \geq 0, 0 \leq a \leq 1)$ implies that for $s \geq t$, discontinuity points of the subordinator $(Z(s,a), 0 \leq a \leq 1)$ must be part (or all) of the points $a_1, \cdots, a_{D(t)}$. Therefore, $t \mapsto D(t)$ is almost surely non-increasing.

3. Proofs

In this section, we aim to prove Theorem 1.2 and Corollary 1.3. From now on, we will use the notations $t_n = n^{1-\alpha}t$ and $t'_n = \frac{t_n}{(\alpha-1)\alpha\Gamma(\alpha)}$. Lemma 2.1 entails that $\Theta(t_n)$ has the same law as $\frac{M_{R^{-1}(t_n)}}{Z(R^{-1}(t_n))}$. Moreover, Lemma 4.2 of [2] states that $\frac{R^{-1}(t_n)}{t'_n} \xrightarrow{P} 1$, as n goes to ∞. From this arises the idea of approximating the block sizes of the coalescent at time t_n by the atoms of the renormalized measure-valued process at time t'_n. The advantage of this approximation is that the time is no longer random. This idea will be executed through three steps. First, we will study the size of the largest atom of the rescaled measure M/Z at deterministic time t'_n, using tools of the theory of CSBPs. Second we show that the paintbox construction of an exchangeable partition can also be provided by using a different paintbox and by modifying it according to the differences between the paintboxes. In the third step, we use this construction to approximate the partition $\Pi^{(n)}$ at time t_n from partitions built from the rescaled atoms of M/Z at time $(1 \pm \varepsilon)t'_n$ for small ε.

3.1. **The largest atom size of M/Z at a fixed time.** We start with a technical lemma associated to the measure μ. We write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Recall from Equation (33) of [2] that

$$\mu([x,\infty)) \sim \frac{x^{-\alpha}}{\Gamma(2-\alpha)} \tag{8}$$

when x goes to ∞.

Lemma 3.1. *Let $k > 0$ and X be a random variable distributed according to μ. Define $\mathcal{X}$ such that conditional on X, $\mathcal{X}$ is a Poisson variable with parameter $\frac{X}{k}$. Then for any $x > 0$,*

$$\lim_{n\to\infty} n\mathbb{P}(\mathcal{X} \geq xn^{\frac{1}{\alpha}}) = \frac{(kx)^{-\alpha}}{\Gamma(2-\alpha)}.$$

Proof. Let $M = \lfloor xn^{\frac{1}{\alpha}} \rfloor$. We start the proof with two claims. First, using Stirling's formula for $M!$ and a change of variable, we get that for any $0 < \beta < 1$,

$$\begin{aligned}\int_0^{M\beta} e^{-t}\frac{t^M}{M!}dt &= \int_0^{M\beta} e^{M-t}\left(\frac{t}{M}\right)^M (2\pi M)^{-\frac{1}{2}}(1+O(M^{-1}))dt \\ &= \int_0^{\beta} e^{M(1-t+\ln t)}(\frac{M}{2\pi})^{\frac{1}{2}}(1+O(M^{-1}))dt \\ &= O(e^{M(1-\beta+\ln\beta)}M^{\frac{1}{2}}).\end{aligned} \tag{9}$$

The last equality is due to the fact that $1-t+\ln t$ is negative and increasing for $t \in (0,1)$. Second, if $\beta > 1$, then

$$\int_{M\beta}^{\infty} e^{-t}\frac{t^M}{M!}dt = \int_{\beta}^{\infty} e^{M(1-t+\ln t)}(\frac{M}{2\pi})^{\frac{1}{2}}(1+O(M^{-1}))dt.$$

Notice that $1-t+\ln t$ is strictly decreasing and concave over $[\beta,\infty]$. Then there exists a positive number ε such that $1-t+\ln t \leq -\varepsilon t$ for any $t \geq \beta$. Therefore,

$$\int_{M\beta}^{\infty} e^{-t}\frac{t^M}{M!}dt \leq \int_{\beta}^{\infty} e^{-\varepsilon Mt}(\frac{M}{2\pi})^{1/2}(1+O(M^{-1}))dt = O(e^{-\varepsilon M\beta}M^{-1/2}). \tag{10}$$

Now we can turn to $\mathcal{X}$. Thanks to successive integrations by parts,

$$\mathbb{P}(\mathcal{X} \geq M+1) = \mathbb{E}[\int_0^{\frac{X}{k}} e^{-t}\frac{t^M}{M!}dt]. \tag{11}$$

Let $0 < \beta_1 < 1$ and $\beta_2 > 1$, then we have

$$\mathbb{P}(\mathcal{X} \geq M+1) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}I_1 &= \mathbb{E}[\int_0^{\frac{X}{k}} e^{-t}\frac{t^M}{M!}dt\mathbf{1}_{\{X<kM\beta_1\}}], \\ I_2 &= \mathbb{E}[\int_0^{\frac{X}{k}} e^{-t}\frac{t^M}{M!}dt\mathbf{1}_{\{kM\beta_1\leq X\leq kM\beta_2\}}], \\ I_3 &= \mathbb{E}[\int_0^{\frac{X}{k}} e^{-t}\frac{t^M}{M!}dt\mathbf{1}_{\{X>kM\beta_2\}}].\end{aligned}$$

Now let n tend to infinity. By (9), we get

$$0 \leq nI_1 \leq n\mathbb{P}(X < kM\beta_1)\int_0^{M\beta_1} e^{-t}\frac{t^M}{M!}dt \longrightarrow 0,\, n\to\infty. \tag{12}$$

It is easy to verify that $\int_0^\infty e^{-t}\frac{t^M}{M!}dt = 1$ for any integer $M \geq 0$. Then using together (8) and (10), we obtain

$$\lim_{n\to\infty} nI_3 = \lim_{n\to\infty} n\mathbb{P}(X > kM\beta_2) = \frac{(kx\beta_2)^{-\alpha}}{\Gamma(2-\alpha)}. \tag{13}$$

In the same way, we have

$$0 \le nI_2 \le n\mathbb{P}(kM\beta_1 \le X \le kM\beta_2) \longrightarrow \frac{(kx\beta_1)^{-\alpha}}{\Gamma(2-\alpha)} - \frac{(kx\beta_2)^{-\alpha}}{\Gamma(2-\alpha)},\ n\to\infty. \tag{14}$$

If β_1 and β_2 are close enough to 1, nI_2 can be bounded by an arbitrarily small positive number for n large enough. The proof is finished by combining (12), (13) and (14). □

Fix $t > 0$. If $D(t) \neq 0$, let $\bar{J}_i(t) = \frac{J_i(t)}{Z(t)}$ for $1 \le i \le D(t)$. Let $\{d_1(t), \cdots, d_{D(t)}(t)\}$ be an interval partition of $[0,1]$ such that the Lebesgue measure of $d_i(t)$ is $\bar{J}_i(t)$. Build a partition of $[n]$ thanks to a paintbox associated with $\{d_1(t), \cdots, d_{D(t)}(t)\}$. Let $N_i(t)$ be the number of integers in the i-th interval and $N(t) = \max\{N_i(t) : 1 \le i \le D(t)\}$. This random variable stands for the size of the largest block of a partition of $[n]$ obtained by a paintbox construction from the atoms of M/Z at time t.

Lemma 3.2. *Let $x > 0$. Then*

1)

$$\lim_{n\to\infty} \mathbb{P}(N(t'_n) \le xn^{\frac{1}{\alpha}}) = \exp(-\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}).$$

2) *Let $0 < y < x$. Then*

$$\lim_{n\to\infty} \mathbb{P}(\exists i : J_i(t'_n) < n^{\frac{1-\alpha}{\alpha}} y, N_i(t'_n) \ge xn^{\frac{1}{\alpha}}) = 0. \tag{15}$$

Proof. 1) Let us throw a Poisson number of integers on $[0,1]$ with parameter $nZ(t'_n)$. Then, conditional on $\{J_i(t'_n) : 1 \le i \le D(t'_n)\}$, the number of integers falling in $d_i(t'_n)$, denoted by $\mathcal{N}_i$, is a Poisson variable with parameter $nJ_i(t'_n)$ and $\{\mathcal{N}_i : 1 \le i \le D(t'_n)\}$ forms a family of (conditional) independent random variables. Let $\mathcal{N}$ be the maximum of all $\mathcal{N}_i$'s. Then, using Lemmas 3.1 and 2.2, as n tends to infinity,

$$\mathbb{P}(\mathcal{N} \le xn^{\frac{1}{\alpha}}) = \mathbb{E}[\Pi_{i=1}^{D(t'_n)} \mathbb{P}(\mathcal{N}_i \le xn^{\frac{1}{\alpha}})] \longrightarrow \exp(-\gamma(\frac{t}{(\alpha-1)\alpha\Gamma(\alpha)})^{1-\alpha}\frac{x^{-\alpha}}{\Gamma(2-\alpha)}) = \exp(-\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}).$$

Lemma 2.2 implies that $Z(t'_n)$ tends in probability to 1 as n goes to infinity. Hence N and $\mathcal{N}$ are close in the limit and standard comparison techniques allow to conclude.

2) As $Z(t'_n)$ converges to 1, it is easy to show that (15) is equivalent to

$$\lim_{n\to\infty} \mathbb{P}(\exists i : J_i(t'_n) < n^{\frac{1-\alpha}{\alpha}} y, \mathcal{N}_i \ge xn^{\frac{1}{\alpha}}) = 0.$$

Let $\tilde{\mathcal{N}} = \max\{\mathcal{N}_i : J_i(t'_n) < n^{\frac{1-\alpha}{\alpha}} y\}$. It is necessary and sufficient to show that $\lim_{n\to\infty} \mathbb{P}(\tilde{\mathcal{N}} \ge xn^{\frac{1}{\alpha}}) = 0$. Notice that conditional on $J_i(t'_n)$, $\mathcal{N}_i$ is a Posson variable with paprameter $nJ_i(t'_n)$. Let $\{P_1(yn^{\frac{1}{\alpha}}), P_2(yn^{\frac{1}{\alpha}}), \cdots\}$ be a sequence of i.i.d. Poisson variables with parameter $yn^{\frac{1}{\alpha}}$ and also independent of $D(t'_n)$. Then

$$\mathbb{P}(\tilde{\mathcal{N}} \ge xn^{\frac{1}{\alpha}}) \le \mathbb{P}\left(\max\{P_i(yn^{\frac{1}{\alpha}}) : 1 \le i \le D(t'_n)\} \ge xn^{\frac{1}{\alpha}}\right) = 1 - \mathbb{E}[(\mathbb{P}(P_1(yn^{\frac{1}{\alpha}}) < xn^{\frac{1}{\alpha}}))^{D(t'_n)}].$$

Using (11) and (9), one gets

$$\mathbb{P}(P_1(yn^{\frac{1}{\alpha}}) < xn^{\frac{1}{\alpha}}) = 1 - o(\frac{1}{n}).$$

Meanwhile, Lemma 2.2 tells that $\frac{D(t'_n)}{n}$ converges in probability to $\gamma(\frac{t}{((\alpha-1)\alpha\Gamma(\alpha)})$ as n goes to infinity. Hence the proof is finished. □

Remark 3.1. The key points to prove (15) is that $Z(t'_n)$ converges to 1 in probability and $\frac{D(t'_n)}{n}$ is asymptotically bounded by a positive value from above. The distribution of $\{J_i(t'_n)\}_{1\le i\le D(t'_n)}$ is not necessary to know. Actually (15) remains true if t'_n is random and conditions on $Z(t'_n)$ and $D(t'_n)$ are still satisfied. This fact will be used in the proof of Theorem 1.2.

3.2. **Alternative paintbox construction.** Let $(A_1, \cdots, A_k)$ and $(B_1, \cdots, B_k)$ be two partitions of $[0, 1]$ with $k \geq 1$. We throw n particles uniformly and independently on $[0, 1]$ and group those within the same intervals of $(B_1, \cdots, B_k)$, which gives a sequence of k numbers $(N_{B_1}, \cdots, N_{B_k})$ such that N_{B_i} is the number of particles located in B_i. We can obtain the law of this sequence in another way using $(A_1, \cdots, A_k)$. Throw n particles uniformly and independently on $[0, 1]$. Let $I := \{i : 1 \leq i \leq n, l(A_i) \leq l(B_i)\}$, where $l(\cdot)$ denotes the Lebesgue measure. If a particle falls in A_i with $i \in I$, then move this particle to B_i. If a particle falls in A_i with $i \in I^c$, then associate to this particle an independent Bernoulli variable with parameter $\frac{l(B_i)}{l(A_i)}$. If the Bernoulli variable gives 1, then the particle is put into B_i. Otherwise, this particle will be put into B_j for $j \in I$ with probability

$$\frac{l(B_j) - l(A_j)}{\sum_{h \in I}(l(B_h) - l(A_h))}. \tag{16}$$

We denote by $N^A_{B_i}$ the new amount of particles in B_i. We have the the following result.

Lemma 3.3. *The following identity in law holds.*

$$(N^A_{B_1}, \cdots, N^A_{B_k}) \overset{(d)}{=} (N_{B_1}, \cdots, N_{B_k}).$$

Proof. Notice that only the Lebesgue measure of each element of $(A_1, \cdots, A_k)$ and $(B_1, \cdots, B_k)$ matters. So one can always assume that $[0, 1]$ is divided in a way that A_i is contained in B_i for $i \in I$ and B_i is contained in A_i for $i \in I^c$. Then if a particle is located in A_i for $i \in I$, it is also located in B_i. But if a particle is located in A_i for $i \in I^c$, with probability $\frac{l(B_i)}{l(A_i)}$ it is located in B_i. Assume that this particle is not located in B_i, then it must be in $\cup_{h \in I} B_h \backslash A_h$. Using the uniformity of the throws, this particle falls in B_j with probability (16). □

3.3. **Proof of Theorem 1.2.** Let us first recall some technical results from [2]. Let $\varepsilon > 0$, $t > 0$ and recall t_n and t'_n. Let $t_- = (1-\varepsilon)t'_n$ and $t_+ = (1+\varepsilon)t'_n$. Define the event $B_{1,t} := \{t_- \leq R^{-1}(t_n) \leq t_+\}$. It can be found in Lemma 4.2 of [2] that there exists a constant C_{17} such that

$$\mathbb{P}(B_{1,t}) \geq 1 - C_{17} t_n \varepsilon^{-\alpha}. \tag{17}$$

Also from Lemma 5.1 of [2], there exists a constant C_{18} such that for all $a > 0$, $t > 0$ and $\eta > 0$,

$$\mathbb{P}(\sup_{0 \leq s \leq t} |Z(s, a) - a| > \eta) \leq C_{18}(a + \eta) t \eta^{-\alpha}. \tag{18}$$

Thus, if we define $B_{2,t} := \{1 - n^{\frac{1-\alpha}{2\alpha}} \leq Z(s) \leq 1 + n^{\frac{1-\alpha}{2\alpha}},\ \forall s \in [t_-, t_+]\}$, we obtain that

$$\mathbb{P}(B_{2,t}) \geq 1 - C_{19} t(1 + \varepsilon)(1 + n^{\frac{1-\alpha}{2\alpha}}) n^{\frac{1-\alpha}{2}} \tag{19}$$

where $C_{19} = C_{18}/(\alpha - 1)\alpha\Gamma(\alpha)$.

Fix any $s \geq 0$ and let π be the random partition of $[n]$ obtained from a paintbox associated with $\frac{M_{R^{-1}(s)}}{Z(R^{-1}(s))}$. Then $\pi \overset{d}{=} \Pi^{(n)}(s)$. Observe that if $R^{-1}(s) \geq t_-$, we can as well at first build a partition from a paintbox associated with $\frac{M_{t_-}}{Z(t_-)}$ and then use Lemma 3.3 to obtain that associated with $\frac{M_{R^{-1}(s)}}{Z(R^{-1}(s))}$ which has the same law as π.

By Markov and branching properties of CSBPs, for any $s \geq t_-$, we can consider the CSBP as the sum of $D(t_-)$ independent CSBP's which we denote by $m_i(s) = Z_i(s - t_-, J_i(t_-))$. Notice that $m_i(s)$ can be 0 while $J_i(t_-)$ is always positive. Let us then build a partition $V^{(n)}(s) = (V^{(n)}_1(s), V^{(n)}_2(s), \ldots, V^{(n)}_{D(t_-)}(s))$ of $[n]$ from a paintbox associated with $(\frac{m_i(s)}{Z(s)}, 1 \leq i \leq D(t_-))$. Let $I^{(n)}_i(s)$ be the number of particles in $V^{(n)}_i(s)$. and $I^{(n)}_+(s) = \sup\{I^{(n)}_i(s), 1 \leq i \leq D(t_-)\}$. Fix $x > 0$ and define $B_{3,t} = \{\exists k : I^{(n)}_k(t_-) \geq x n^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}, \sup_{t_- \leq s \leq t_+} |m_k(s) - J_k(t_-)| \leq \varepsilon J_k(t_-)\}$.

On the event $B_{3,t}$, we have that $I^{(n)}_+(t_-) \geq x n^{\frac{1}{\alpha}}$. Conditional on $B_{1,t}$ we can also build the partition $V^{(n)}(R^{-1}(t_n))$ from a paintbox associated to the partition $Z(t_-)^{-1}(J_1(t_-), \ldots, J_{D(t_-)}(t_-))$

and Lemma 3.3. Let $B(m,p)$ be a binomial variable with parameters $m \geq 2$ and $0 \leq p \leq 1$. Lemma 3.3 implies that

$$\begin{aligned}
&\mathbb{P}\left(I_+^{(n)}(R^{-1}(t_n)) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \\
\geq&\mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}}\rceil, \frac{m_k(R^{-1}(t_n))Z(t_-)}{J_k(t_-)Z(R^{-1}(t_n))} \wedge 1\right) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \\
\geq&\mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}}\rceil, (1-\varepsilon)\frac{1-n^{\frac{1-\alpha}{2\alpha}}}{1+n^{\frac{1-\alpha}{2\alpha}}}\right) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}}\right) \\
=&\mathbb{P}\left((xn^{\frac{1}{\alpha}})^{-1}B\left(\lceil xn^{\frac{1}{\alpha}}\rceil, (1-\varepsilon)\frac{1-n^{\frac{1-\alpha}{2\alpha}}}{1+n^{\frac{1-\alpha}{2\alpha}}}\right) \geq (1-\varepsilon)-\varepsilon\right).
\end{aligned}$$

A law of large numbers argument implies that

$$\mathbb{P}\left(I_+^{(n)}(R^{-1}(t_n)) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \geq 1-\varepsilon \tag{20}$$

for n large enough. Now observe from (18) that

$$\begin{aligned}
\mathbb{P}(B_{3,t}) = &\mathbb{P}(\exists k : I_k^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) \\
&\times \mathbb{P}(\sup_{t_- \leq s \leq t_+} |m_k(s) - J_k(t_-)| \leq \varepsilon J_k(t_-) | \exists k : I_k^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) \\
\geq &\mathbb{P}(\exists k : I_k^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}})(1 - 2tC_{19}n^{\frac{(1-\alpha)(2-\alpha)}{\alpha}}(1+\varepsilon)\varepsilon^{1-\alpha}).
\end{aligned}$$

By Lemma 3.2, we obtain that

$$\begin{aligned}
\mathbb{P}(\exists k : I_k^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) &\sim \mathbb{P}(\exists k : I_k^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}) = \mathbb{P}(I_+^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}) \\
&\sim 1 - \exp(-(1-\varepsilon)\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}).
\end{aligned}$$

In consequence,

$$\liminf_{n\to\infty} \mathbb{P}(B_{3,t}) \geq 1 - \exp(-(1-\varepsilon)\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)})$$

when n tends to ∞. Then, thanks to (17) and (19), we deduce that

$$\liminf_{n\to\infty} \mathbb{P}(B_{1,t} \cap B_{2,t} \cap B_{3,t}) \geq 1 - \exp(-(1-\varepsilon)\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}).$$

Combining the latter with (20), we obtain

$$\liminf_{n\to\infty} \mathbb{P}\left(I_+^{(n)}(R^{-1}(t_n)) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}}\right) \geq 1 - \exp(-(1-\varepsilon)\frac{tx^{-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}). \tag{21}$$

Next, we seek to find an upper bound for $\mathbb{P}\left(I_+^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}}\right)$. Conditional on $B_{1,t}$, we construct $V^{(n)}(t_+)$ from $V^{(n)}(R^{-1}(t_n))$ using the method in Lemma 3.3. Let

$$B_{4,t} = B_{1,t} \cap \{\exists k : I_k^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}}, m_k(R^{-1}(t_n)) \geq n^{\frac{2(1-\alpha)}{\alpha}}, \sup_{R^{-1}(t_n) \leq s \leq t_+} \frac{|m_k(s) - m_k(R^{-1}(t_n))|}{m_k(R^{-1}(t_n))} \leq \varepsilon\}.$$

Similarly as for the lower bound,

$$\begin{aligned}
&\mathbb{P}\left(I_+^{(n)}(t_+) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}}|B_{2,t}\cap B_{4,t}\right)\\
\geq&\mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}}\rceil, \frac{Z(R^{-1}(t_n))m_k(t_+)}{Z(t_+)m_k(R^{-1}(t_n))}\wedge 1\right) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}}|B_{2,t}\cap B_{4,t}\right)\\
\geq&\mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}}\rceil, (1-\varepsilon)\frac{1-n^{(1-\alpha)/\alpha}}{1+n^{(1-\alpha)/\alpha}}\right) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}}\right) \longrightarrow 1. \qquad (22)
\end{aligned}$$

Using the strong Markov property of the CSBP and (18), we have

$$\begin{aligned}
\mathbb{P}(B_{4,t}) = \mathbb{P}(B_{1,t}\cap\{\exists k : I_k^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}}, m_k(R^{-1}(t_n)) \geq n^{\frac{2(1-\alpha)}{\alpha}}\}) \qquad (23)\\
\times(1-2tC_{19}n^{\frac{(1-\alpha)(2-\alpha)}{\alpha}}(1+\varepsilon)\varepsilon^{1-\alpha}) \qquad (24)
\end{aligned}$$

Notice that using (18), in the sense of convergence of probability

$$\lim_{n\to\infty}\sup_{t_-\leq s\leq t_+} Z(s) = \lim_{n\to\infty}\inf_{t_-\leq s\leq t_+} Z(s) = 1$$

Together with (17), we get the following convergence in probability

$$\lim_{n\to\infty} Z(R^{-1}(t_n)) = 1.$$

Recall Remark 2.1 where it is deduced that $t\mapsto D(t)$ is non-increasing. Thus, on the event $B_{1,t}$, we have $D(t_-)\leq D(R^{-1}(t_n))\leq D(t_+)$. It is then easy to see that $\frac{D(R^{-1}(t_n))}{n}$ is asymptotically bounded from above by a certain positive number. Now we can apply Remark 3.1 and get

$$\mathbb{P}(B_{4,t}) = \mathbb{P}(\exists k : I_k^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}}) + o(1) = \mathbb{P}(I_+^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}}) + o(1). \qquad (25)$$

Using (22), (19) and (25), we get that

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty}\mathbb{P}(I_+^{(n)}(R^{-1}(t_n)) \geq xn^{\frac{1}{\alpha}})\\
\leq&\lim_{n\longrightarrow\infty}\mathbb{P}(I_+^{(n)}(t_+) \geq (1-2\varepsilon)xn^{\frac{1}{\alpha}})\\
=&1-\exp(-(x(1-2\varepsilon))^{-\alpha}\frac{t(1+\varepsilon)}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}). \qquad (26)
\end{aligned}$$

Since ε can be arbitrarily small, (21) and (26) allow to conclude.

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