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Information, VARs and DSGE Models

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Information, VARs and DSGE Models*

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Abstract

How informative is a time series representation of a given vector of observables about the structural shocks and impulse response functions in a DSGE model? In this paper we refer to this econometrician's problem as "E-invertibility" and consider the corresponding information problem of the agents in the assumed DGP, the DSGE model, which we refer to as "A-invertibility" We consider how the general nature of the agents' signal extraction problem under imperfect information impacts on the econometrician's problem of attempting to infer the nature of structural shocks and associated impulse responses from the data. We also examine a weaker condition of recoverability. A general conclusion is that validating a DSGE model by comparing its impulse response functions with those of a data VAR is more problematic when we drop the common assumption in the literature that agents have perfect information as an endowment. We develop measures of approximate fundamentalness for both perfect and imperfect information cases and illustrate our results using analytical and numerical examples.

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Keywords: Invertibility/Fundamentalness, VARs, agent perfect versus imperfect information, recoverability

Contents

1 Introduction

How informative is time series representation of a given vector of observables about the structural shocks and impulse response functions (IRFs) in a DSGE model? This invertibility/fundamentalness problem, first pointed out in the economics literature by Hansen and Sargent (1980), is often described in the literature as one of missing information on the part the econometrician. It occurs when she is faced with a number of observables that is less than the number of shocks; some observable variables of the system are observed with a lag; models feature anticipated shocks with a delayed effect on the system such as "news" shocks; and even with square systems with a particular choice of observables observed with neither delayed effects, nor a lag. In this paper we refer to this econometrician's problem as "E-invertibility"; our contribution is to consider the corresponding information problem of the agents in the assumed DGP, the DSGE model, which we refer to as "A-invertibility".

Agents may or may not have perfect information, an assumption we argue that must be justifiable in terms of the underlying structure of the model. We study how the general nature of the agents' signal extraction problem under imperfect information impacts on the econometrician's problem of attempting to infer the nature of structural shocks and associated impulse responses from the data. While the agents' problem under imperfect information is in many respects analogous to that of a standard signal extraction problem, it has an additional, and crucial, added complication: the solution to their signal extraction problem will in general feed back, via optimising behaviour, into the behaviour of any endogenous states. As a direct result the filtering process itself thus increases the state space relative to the benchmark case of perfect information. We show that this in turn has significant effects on the econometrian's problem.

We start in Section 2 by briefly considering the nature of informational imperfection. While there is a growing literature on the impact of imperfect information in DSGEs[1]

[1]Imperfect information (II) in representative agent models was initiated by Minford and Peel (1983) and generalised by Pearlman *et al.* (1986) - henceforth PCL - with major contributions by Collard and Dellas (2004) and Collard and Dellas (2006), who showed that II can act as an endogenous persistence mechanism in the business cycle. Ellison and Pearlman (2011) incorporates II into a statistical learning environment. Applications with estimation were made by Collard *et al.* (2009), Neri and Ropele (2012) and Levine *et al.* (2012). II models with heterogenous agents distinguish local (idiosyncratic) information and (aggregate) information e.g. Nimark (2008), Nimark (2014), Ilut and Saijo (2016) and Graham and Wright (2010). Recent papers close to ours are Blanchard *et al.* (2013) and Forni *et al.* (2017) who examine invertibility in a simple model with II and heterogeneous agents who have noisy observations of news shocks. Our paper

many (indeed most) models of the macro-economy are still solved and/or estimated on the assumption that agents are simply provided with perfect information, effectively as an endowment. We argue that these perfect or imperfect information assumptions should be consistent with choice of a complete vs incomplete markets structure of the model.

The main results of the paper then focus on the econometric implications of agents having imperfect information. We start by showing (Theorem 1) how to map a very general class of models with imperfect information into a form that allows us to apply the solution technique originally introduced by PCL. We then show (Theorem 2) the necessary link between "A-invertibility" (agents can infer the structural shocks from their information set: a history of some set of observables, $I_t^A = \left\{m_i^A\right\}_{i=-\infty}^{t}$, which is assumed to be of strictly lower dimension than under (endowed) perfect information) and "E-invertibility" (the econometrician can do the same based on their own information set, $I_t^E = \left\{m_i^E\right\}_{i=-\infty}^{t}$). Having established the (restrictive) conditions under which E-invertibility can occur (a generalisation of the "Poor Man's Invertibility Condition" (PMIC) of Fernandez-Villaverde *et al.* (2007)), we then consider the nature of the econometrician's problem when these conditions are not satisfied, due to a failure of A-invertibility.[2]

When A-invertibility fails the true dynamics of the system's response to structural shocks will, as noted above, in general have a state space dimension strictly greater than under perfect information. But we show (Theorem 3) that the *observable* dynamics will always have the *same* state space dimension as under perfect information. Equivalently, the data generating process (DGP) in the absence of A-invertibility is a non-minimal spectral factorization of the spectrum of the agents' information set, incorporating a set of Blaschke factors that map the true structural shocks to observable white noise innovations. In the terminology of Lippi and Reichlin (1994), this means that true time series representation of the observables is both nonfundamental[3] and "nonbasic" (i.e., of higher VARMA order).

This implies two closely related results that arise from the features of the true DGP:

provides a general treatment of the issues explored in these two papers.

[2]Excellent recent surveys of invertibility/fundamentalness and the relationship between VAR and DSGE models are provided by Sims (2012) and Giacomini (2013). However, in common with the literature, these surveys explore the issue without examining the information assumptions of the agents in the underlying structural model.

[3]In the context of the class of models we examine, E-invertibility is equivalent to fundamentalness of the time series representation, so when E-invertibility fails, the structural shocks are nonfundamental.

1. Any fundamental time series representation of the observables (typically via a VAR approximation), is at best an approximation to a *minimal* spectral factorization. It cannot therefore possibly generate the true impulse response functions of the system.

2. A recent important paper, Chahrour and Jurado (2017), has argued that even when E-invertibility fails, nonfundamental structural shocks may be "recoverable": i.e., the econometrician may be able to estimate the $t-$dated shocks and their associated impulse responses, from a dataset of T observations, with arbitrary precision, for $t \in (\tau, T-\tau)$ for sufficiently large values of τ. But we show (Theorem 4) that the non-minimal nature of the true process means that any shocks that are recoverable from an a-theoretical (hence minimal) time series representation cannot be linearly related to the true structural shocks.

While both these features imply pessimism about econometric inference when A-invertibility fails, we suggest three key reasons to temper this pessimism:

- In many applications A-invertibility does *not* fail and we provide examples including a standard RBC model with the appropriate choice of observables and the estimated model of Smets and Wouters (2007).

- In the context of a structural DSGE model, the Blaschke factors that generate the non-minimality of the GDP are *not* arbitrary, since they can be related back to the underlying structure of the model Thus, subject to identification of the appropriate parameters[4] that generate the Blaschke factors an econometrician will, at least in principle, be able to recover structural shocks even from E-non-invertible systems, using full sample information.

- We derive measures of approximate fundamentalness which allow us to diagnose whether at least some structural shocks can be derived perfectly from the data, and, if not, whether they can at least be derived to some chosen degree of precision.[5] We also consider other possible ways that have been proposed to circumvent the non-fundamentalness of structural shocks.

[4]We do not address issues of parameter identification in this paper, since these are clearly endemic to all DSGE estimation, whether under API or AII.

[5]This provides a generalization of Beaudry *et al.* (2016), Forni *et al.* (2017) and Forni *et al.* (2019) to a DGP where agents have imperfect information.

2 Information Assumptions, the Agents' Problem and A-Invertibility

This section first discusses the relationship between our informational assumptions and the market environment (complete vs incomplete markets). We then show that a general class of linear rational expectations models can always be transformed into the form utilized by PCL to generalize the solution of Blanchard and Kahn (1980) where agents have imperfect rather than perfect information. We provide outline RE solutions in these two cases, and provide a definition of A-invertibility.

2.1 Information in Macroeconomic Models

Many (indeed most) models of the macro-economy are solved on the assumption that agents have perfect information. Under certain circumstances this assumption can be justified by the assumed market environment. Radner (1979) established that under conditions of complete markets, market equilibrium must usually imply revelation of perfect information. However, a wide range of macroeconomic models developed over the past two or three decades have been predicated on some element of market incompleteness. Market incompleteness need not *necessarily* imply imperfect information; but perfect information is only consistent with market completeness (as in the representative agent framework)[6] and with incomplete markets becomes an assumed endowment. Graham and Wright (2010), building on earlier analysis by King (1983), argue that assumptions on information sets in any macroeconomic model should consistent with the underlying market structure (complete versus incomplete markets). They propose a concept of "market-consistent information", i.e., agents only use market prices to infer the underlying states of the economy, and show that, in a heterogeneous agent simple linear RBC model, impulse responses based on a perfect information solution are highly misleading.[7]

[6]Svensson and Woodford (2003) make the same point in a different language when they write:"It does not make sense that any state variables should matter for the determination of economically relevant quantities ... if they are not known to anyone in the private sector. But if all private agents are to have a common information set, they must then have full information about the relevant state variables".

[7]Angeletos and Lian (2016) examine these issues in the context of what they refer to as incomplete information literature. Here a comment on terminology is called for. Our use of perfect/imperfect information corresponds to the standard use in dynamic game theory when describing the information of the history of play driven by draws by Nature from the distributions of exogenous shocks. Complete/incomplete information refers to agent's beliefs regarding each other's payoffs and information sets. In our set-up this informational friction is absent.

We show in Section 2.4 that our model with imperfect information as set out in the following sections can be shown, in general, to be consistent with a limiting case of an incomplete markets model where in linear form agents are ex ante identical but ex post heterogeneous. Then the set-up (2) with which we start is expressed only in terms of aggregates.

The following simple filtering problem illustrates this point. Agent i in the economy observes an exogenous shock process $x_{i,t}$ which is the sum of an aggregate component x_t and a n.i.i.d idiosyncratic component $\epsilon_{i,t}$; i.e.,

$$x_{i,t} = x_t + \epsilon_{i,t}\,; \quad \epsilon_{i,t} \sim N(0, \sigma_{\epsilon_i}) \tag{1}$$

Then assuming an AR1 process $x_t = \rho_x x_{t-1} + \epsilon_{x,t}; \quad \epsilon_{x,t} \sim N(0, \sigma_{\epsilon_x})$, from the Kalman Filter for agent i we have the RE up-dating:

$$\begin{aligned}
\mathbb{E}_i[x_t|x_{i,t}] &= \mathbb{E}_i[x_t|x_{i,t-1}] + J_x(x_{i,t} - \mathbb{E}_i[x_t|x_{i,t-1}]) \\
&= (1 - J_x)\rho_x x_{t-1} + J_x x_{i,t}
\end{aligned}$$

where the Kalman gain in this case is

$$J_x = \frac{\sigma_{\epsilon_x}^2}{\sigma_{\epsilon_x}^2 + \sigma_{\epsilon_{x_i}}^2}$$

To arrive at the model with II or PI considered in the rest of the paper we then make a crucial assumption. We consider the (empirically plausible) limit as the signal for x_t becomes very noisy ($\frac{\sigma_{\epsilon_{x_i}}}{\sigma_{\epsilon_x}} \to \infty$), $J_x \to 0$ and the idiosyncratic component provides no information. Then the model with incomplete markets can be set up purely in terms of aggregates as in the rest of the paper and resembles a representative agent model that allows for the possibility of perfect or imperfect information. However we shall also show, by example, that our II solution procedure gives the same equilibrium as the high idiosyncratic shock volatility limiting case of hierarchy models.[8]

In the general case without the limiting assumption heterogeneity (incomplete markets)s induces a solution via a hierarchy of expectations (Townsend (1983), Nimark (2008),

[8]See Appendix B.

Graham and Wright (2010)), the state space increases and in principle becomes infinite.[9] However recent work by Rondina and Walker (2017) and Huo and Takayama (2018) have shown how one can solve these models by completely different methods. Our paper can be viewed as a stepping-stone to extending the results of Rondina and Walker (2017) to a more general framework of heterogeneous information, which will add to the issues we raise in this paper about the validity of using VAR estimation to generate impulse response functions when information is imperfect.

2.2 The Agents' Problem

We begin by writing a linearized RE model in the following general form

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t \qquad m_t^E = L^E Y_t \qquad m_t^A = L^A Y_t \tag{2}$$

where matrix A_0 may be singular, Y_t is an $n \times 1$ vector of macroeconomic variables; and ε_t is a $k \times 1$ vector of Gaussian white noise structural shocks.[10] We assume that the structural shocks are normalized such that their covariance matrix is given by the identity matrix i.e., $\varepsilon_t \sim N(0, I)$.

We define $Y_{t,s} \equiv \mathbb{E}\left[Y_t | I_s^A\right]$ where I_t^A is information available at time t to economic agents, given by $I_t^A = \{m_s^A : s \leq t\}$. We assume that all agents have the same information set about some strict subset of the elements of Y_t, hence information is in general imperfect. Note that measurement errors can be accounted for by including them in the vector ε_t. In the special case that agents are endowed with perfect information, $L^A = I$ (the identity matrix). At this stage we focus solely on the agents' informational problem: we specify the properties of $m \times 1$ vector m_t^E where $m \leq k$, the vector of observables available to the econometrician, in Sections 3.1 and 3.2 below.

This section is structured so that we first show how (2) can be transformed into a state space form utilised by PCL, a generalization of the Blanchard-Kahn form (Theorem 1), and then provide the RE saddle-path stable solution to the agents' problem under perfect and imperfect information.

[9]Numerical solution methods rely on the perceived convergence of expectational hierarchies, but there is as yet no theoretical justification for this.

[10]The Gaussian framework is adopted throughout our paper, but see Gouriéroux *et al.* (2019) for a relaxation of this assumption in their examination of both identification and fundamentalness issues.

2.3 Conversion to PCL Form

Anderson (2008) lists a selection of methods that can be used to solve (2) for the case when agents have perfect information. The most well-known of these are Sims (2002) and Blanchard and Kahn (1980) - henceforth BK - but as the former points out, it is not always obvious how to write a system of the form (2) in BK form even under perfect information.

We shall be using a generalized version of the BK form that was utilised by PCL, which provided a solution under imperfect information. In order to move seamlessly from (2) to results that are based on PCL, we introduce our first key result, which appears to be novel in the literature: [11]

Theorem 1. *For any information set,* (2) *can always be converted into the following form, as used by PCL*

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{3}$$

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{4}$$

where z_t, x_t are vectors of backward and forward-looking variables, respectively.

Proof of Theorem 1. See Appendix A.1. □

The expressions involving $z_{t,t}, x_{t,t}$ arise from rewriting the model in PCL form (3). This transformation (outlined in Appendix A.1) involves a novel iterative stage which replaces any forward-looking expectations with the appropriate model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But at the same time it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv \mathbb{E}[z_t|I_t^A]\right)$ and estimates of forward-looking variables, $x_{t,t}$. The presence of the latter is the key feature that distinguishes our results on invertibility from those of

[11] The nearest to our construction for perfect information only is found in Boucekkine *et al.* (1996), but for a less general set-up than (2).

Baxter *et al.* (2011) - henceforth BGW - the applicability of which is restricted to cases where all forward-looking variables are directly observable.

For later convenience we define matrices G and H conformably with z_t and x_t and define two more structural matrices F and J

$$G \equiv \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H \equiv \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{5}$$

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2G_{22}^{-1}G_{21} \tag{6}$$

F and J capture intrinsic dynamics in the system, that are invariant to expectations formation (i.e., by substituting from the second block of equations in (3) we can write $z_t = Fz_{t-1}+$ additional terms; $m_t^A = Jz_t+$ additional terms).

The reason for transforming the equations of the model from (2) is that the corresponding solution method of Sims (2002) does not extend easily to imperfect information.[12]

2.4 II as a Limiting Case of Incomplete Markets

A follow-up paper by the authors will show in the time domain how a variation of the solution (16)–(17) can be implemented that will match the results generated by Rondina and Walker (2017) for heterogeneous agents using the Wiener-Kolmogorov prediction formulae. Here we show that in the limit as the idiosyncratic measurement error variance for the heterogeneous agents tends to infinity, the solution is indeed given by (16)–(17).

We start for convenience with a variation of the model setup of (3)–(4), which takes account of the decisions made by each agent i:

$$z_{t+1} = G_{11}z_t + G_{12}x_t + \int (\mathbb{E}_{it}[H_{11}z_t + H_{12}x_t])di + B\epsilon_{t+1} \tag{7}$$

$$x_{it} = G_{22}^{-1}(-G_{21} + \mathbb{E}_{it}[x_{t+1} - H_{21}z_t - H_{22}x_t]) \tag{8}$$

where $x_t = \int x_{it}di$ is the average of all the $\{x_{it}\}$. The information set of for each agent i

[12] We give an indication of how to modify the latter method to extend to imperfect information in Appendix C, and the implication is that to complete the task requires techniques no less complicated than those used in PCL.

is made up of past and current m_{it}^A, given in the Rondina and Walker (2017) case by

$$m_{it}^A = M_1 z_t + M_2 x_t + \mathbb{E}_{it}[M_3 z_t + M_4 x_t] + v_{it} \qquad v_{it} \sim N(0, V) \tag{9}$$

This particular setup is not very useful for describing the limit as the diagonal elements of V tend to infinity; even if V is not of full rank, the number of observations from m_{it}^A that are not ignored by agents will now be less than the number of shocks ϵ_t. This would not therefore be a limiting case relevant to invertibility and VARs.

Instead, we assume a variant of the simple example in Nimark (2008); all agents i observe both (a variant of) m_t^A as in (4), and in addition observe some or all of the shocks ϵ_t with noise:

$$m_{i1t}^A = M_1 z_t + M_2 x_t + \mathbb{E}_{it}[M_3 z_t + M_4 x_t] \tag{10}$$

$$m_{i2t}^A = \epsilon_t + v_{it} \tag{11}$$

where $v_{it} \sim N(0, V)$. As the diagonal terms of V tend to infinity, the private information m_{i2t}^A becomes worthless, and the expectations of all the agents are identical. As a result the system can be described by (3)–(4).

2.5 The Agents' Solution under Perfect Information (API)

Here we assume that agents directly observe all elements of Y_t, hence of (z_t, x_t). Hence $z_{t,t} = z_t$, $x_{t,t} = x_t$, and using the standard BK solution method there is a saddle path satisfying

$$x_t + N z_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{12}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G + H)$ is the same as the dimension of x_t, then the system will be determinate.[13]

To find N, consider the matrix of eigenvectors W satisfying

$$W(G + H) = \Lambda^U W \tag{13}$$

[13] Note that in general, as Sims (2002) has pointed out, the dimension of x_t will not match the number of expectational variables in (2), as we see in the algorithm for the proof of Theorem 1 (see Appendix A.1).

Then, as for G and H, partitioning W conformably with z_t and x_t, from PCL we have

$$N = -W_{22}^{-1}W_{21} \tag{14}$$

From the saddle-path relationship (14), the saddle-path stable RE solution under API is

$$z_t = Az_{t-1} + B\varepsilon_t \qquad x_t = -Nz_t \tag{15}$$

where $A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N$.

2.6 The Agents' Solution under Imperfect Information (AII)

For the general case, in which agents have imperfect information (AII), the transformation of (2) into the form (3) and (4) in Theorem 1 allows us to apply the solution techniques originally derived in PCL. We briefly outline this solution method below.

Following Pearlman *et al.* (1986), we apply the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K \left[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right]$$

The Kalman filter was developed in the context of backward-looking models, but extends here to forward-looking models. The agents' best estimate of $\{z_t, x_t\}$ based on current information is a weighted average of their best estimate using last period's information and the new information m_t^A. Thus the best estimator of (z_t, x_t) at time $t-1$ is updated by the "Kalman gain" K of the error in the predicted value of the measurement[14].

Using the Kalman filter, the solution as derived by PCL[15] can be expressed in terms of the impact of the structural shocks on the processes $z_{t,t-1}$ (the predictions of z_t) and $\tilde{z}_t$ (the unobservable prediction errors $z_t - z_{t,t-1}$), which describe the pre-determined $z_t = \tilde{z}_t + z_{t,t-1}$ and non-predetermined variables x_t:

$$\text{Predictions}: \quad z_{t+1,t} \quad = \quad Az_{t,t-1} + AP^AJ'(JP^AJ')^{-1}J\tilde{z}_t \tag{16}$$

[14] K is solved endogenously as $K = \begin{bmatrix} P^AJ' \\ -NP^AJ' \end{bmatrix} [(M_1 - M_2N)P^AJ']^{-1}$, where P^A is defined below, but is not directly incorporated into the solution for x_t, z_t.

[15] Now implemented in Dynare, together with associated estimation software - see Appendix J.

$$\begin{aligned}
\text{Non-predetermined :} \quad x_t &= -Nz_{t,t-1} - G_{22}^{-1}G_{21}\tilde{z}_t - (N - G_{22}^{-1}G_{21})P^A J'(JP^A J')^{-1}J\tilde{z}_t \\
\text{Prediction Errors :} \quad \tilde{z}_t &= F[I - P^A J'(JP^A J')^{-1}J]\tilde{z}_{t-1} + B\varepsilon_t \qquad (17) \\
\text{Measurement Equation:} \quad m_t^A &= Ez_{t,t-1} + EP^A J'(JP^A J')^{-1}J\tilde{z}_t \qquad (18)
\end{aligned}$$

where $E \equiv M_1 + M_3 - (M_2 + M_4)N$ and we recall definitions of matrices F and J in (6). The matrix A, as defined after (15), is the autoregressive matrix of the states z_t under API; B captures the direct (but unobservable) impact of the structural shocks ε_t; F, as defined after Theorem 1, captures the intrinsic dynamics of z_t. $P^A = \mathbb{E}[\tilde{z}_t\tilde{z}_t']$ is the solution of a Riccati equation given by

$$P^A = Q^A P^A Q^{A'} + BB' \qquad \text{where } \; Q^A = F - FP^A J'(JP^A J')^{-1}J \tag{19}$$

To ensure stability of the solution P^A, we also need to satisfy the convergence condition, that Q^A has all eigenvalues in the unit circle.[16] Since there is a unique solution of the Riccati equation that satisfies this condition, it follows that the solution (16)–(17) is also unique thereby extending this property of the perfect information BK solution to the imperfect information case.

2.7 A-Invertibility: When Imperfect Information Replicates Perfect Information

By inspection of equations (16) to (17) is evident that for the general case, imperfect information introduces nontrivial additional dynamics into the responses to structural shocks - a contrast which is crucial to much of our later analysis. However there is a special case of the general problem under imperfect information, in which, despite agents' information set being a subset of the the information set under API, the solution to the agents' problem still approaches the complete information solution, and indeed asymptotically replicates it.

Definition 2.1. *A-Invertibility. The system in (2) is A-invertible if agents can infer the true values of the shocks ε_t from the history of their observables, $\{m_s^A : s \leq t\}$, or*

[16]To explain this, we note that the iterative version of this over time is given by $P_{t+1}^A = Q_t^A P_t^A Q_t^{A'} + BB'$, where $Q_t^A = F - FP_t^A J'(JP_t^A J')^{-1}J$; For small deviations $\Delta P_t^A = P_t^A - P^A$ from steady state, one can show that $\Delta P_{t+1}^A = Q^A \Delta P_t^A Q^{A'}$ and this will only converge to 0 if Q^A is a stable matrix.

equivalently, if $P^A = BB'$ is a stable fixed point of the agents' Ricatti equation, (19).

Setting $P^A = BB'$ in (19), it follows that the condition for A-invertibility[17] is that $Q^A = F - FB(JB)^{-1}B$ is a stable matrix, and we shall see below that A-invertibility is a crucial determinant of whether an econometrician can derive the structural shocks from the history of the observables.

3 Background Results

There are several fairly standard results in Kalman filtering, invertibility, spectral analysis and recoverability that are essential to understand the theorems below, and we first cover these briefly. The reader familiar with this literature can skip some if not all the sub-sections below and proceed to Section 4 for the main results of the paper.

3.1 The ABCD (and E) of VARs

We first note the general feature of state-space representations of the type that arise naturally from our solution method in Section 2.

Consider an econometrician's representation of the general form

$$s_t = \tilde{A}s_{t-1} + \tilde{B}\varepsilon_t \qquad m_t^E = \tilde{E}s_t \tag{20}$$

This "ABE" representation form is the form usually found in the statistics literature. In contrast the following "ABCD" form is often but not exclusively used in the economics literature, e.g., Fernandez-Villaverde *et al.* (2007)

$$s_t = \tilde{A}s_{t-1} + \tilde{B}\varepsilon_t \qquad m_t^E = \tilde{C}s_{t-1} + \tilde{D}\varepsilon_t \tag{21}$$

It is straightforward to show that any ABE form implies an ABCD form, with $\tilde{C} = \tilde{E}\tilde{A}$ and $\tilde{D} = \tilde{E}\tilde{B}$. Appendix D shows that (less obviously) the reverse also applies; it also shows that all of the state-space models that are used in the statistics, control theory and econometrics literature can be rewritten in terms of one another.

[17] BGW refer to A-invertibility as "Asymptotic Invertibility", in contrast to the case where the t-dated state variables can be derived directly from the t-dated observables (which requires L^A to be square and invertible). In this paper, we use the term invertibility in the time series sense, which does not distinguish direct from asymptotic invertibility.

3.2 E-Invertibility

Definition 3.1. *E-Invertibility. The system in (2) is E-invertible if the values of the shocks* ε_t *can be deduced from the history of the econometrician's observables,* $\{m_s^E : s \leq t\}$.

The condition for the system (20) to be E-invertible is a version of the PMIC of Fernandez-Villaverde *et al.* (2007),[18] which is obtained by some algebraic manipulation of (21): they assume a 'square system' with $m = k$ (an assumption we relax when we consider the innovations representation and when we come to Section 6 on measures of approximate invertibility/fundamentalness). They also assume that $\tilde{D}$ (now a square matrix) is non-singular. Then from (21) we have $\varepsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$ where L is the lag operator. Hence from (21) we have

$$(I - \tilde{A}L)s_t = \tilde{B}\varepsilon_t = \tilde{B}\tilde{D}^{-1}(m_t^E - \tilde{C}Ls_t)$$

from which we obtain $s_t = [I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_t^E$ and hence

$$\varepsilon_t = \tilde{D}^{-1}(m_t^E - \tilde{C}s_{t-1}) = \tilde{D}^{-1}(m_t^E - \tilde{C}[I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]^{-1}\tilde{B}\tilde{D}^{-1}m_{t-1}^E) \tag{22}$$

Expanding $(I - X)^{-1} = I + X + X^2 + \cdots$) we then have

$$\varepsilon_t = \tilde{D}^{-1}\left(m_t^E - \tilde{C}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j\tilde{B}\tilde{D}^{-1}m_{t-j}^E\right) \tag{23}$$

A necessary and sufficient condition for the summation to converge is that $\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$ has stable eigenvalues (eigenvalues within the unit circle in the complex plane).[19]

The PMIC transforms into ABE notation as follows: we note that the following term in (23) can be written in two equivalent ways

$$\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^j = \tilde{E}\tilde{A}(\tilde{A} - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}\tilde{A})^j = \tilde{E}\tilde{A}^j(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})^j\tilde{A} \tag{24}$$

so that the PMIC requirements are that $\tilde{E}\tilde{B}$ is invertible and that $\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E})$ has

[18] This result appears to date back at least to the work of Brockett and Mesarovic (1965).

[19] A slightly weaker condition than invertibility is fundamentalness which allows some eigenvalues to be on the unit circle. However we use the two terms interchangeably and in fact, if we restrict our models to have only stationary variables, then the two concepts are equivalent.

stable eigenvalues.

A final observation is that invertibility does not require the ABE representation to be in minimal (i.e. controllable and observable) form; we mention this since the ABE representation of the imperfect information solution below might not be minimal[20].

3.3 The Spectrum of a Stochastic Process and Blaschke Factors

The spectrum of a stochastic process is a representation of all its second moments - auto, cross and auto-cross covariances, so that a VAR with sufficient lags will pick up all of these moments to a high degree of accuracy.

The spectrum (or spectral density) $\Phi_y(L)$ of a stochastic process y_t of dimension r is defined to be $\Phi_y(L) = \sum_{k=-\infty}^{\infty} cov(y_t, y_{t-k})L^k$, and this is a rational function of L if y_t can be expressed as a state space system with finite dimension. It is a standard result that the spectrum of the ABE system above is given by $\tilde{E}(I - \tilde{A}L)^{-1}\tilde{B}\tilde{B}'(I - \tilde{A}'L)^{-1}\tilde{E}'$.

Definition 3.2. *A rational spectral density $\Phi_y(L)$ admits a spectral factorization of the form $\Phi_y(L) = W(L)W'(L^{-1})$. A* minimal spectral factorization *(Baggio and Ferrante (2016)) is one where the McMillan degree of $W(L)$ is a minimum.* [21]

Of importance for our main results below is the Blaschke factor $b(L) = (1-aL)/(L-a)$, which has the easily verifiable property that $b(L)b(L^{-1}) = 1$. This implies that if $y_{1t} = \epsilon_t$ is a scalar white noise process, with spectrum given by $\Phi_{y_1}(L) = var(\epsilon_t)$, then $y_{2t} = b(L)\epsilon_t$ has the same spectrum. The second-moment properties of y_{1t} and y_{2t} are therefore identical; however although there is a minimal realization of y_{2t} in ABCD form ($x_t = \frac{1}{a}x_{t-1} + (a - \frac{1}{a})\varepsilon_t,\ y_t = x_{t-1} - a\varepsilon_t$), it is not a minimal *spectral* factorization of the process, which is given by the fundamental representation $y_{2t} = \eta_t$, where $var(\eta_t) = var(\epsilon_t)$. Crucially the IRFs of y_{1t} and y_{2t} in response to a shock to ϵ_t are completely different, with the latter being non-zero at all lags.

[20] To show this, suppose that $(\tilde{A}, \tilde{B})$ is not controllable; then there exists an eigenvalue-eigenvector pair (λ, x) such that $x'\tilde{A} = \lambda x', x'\tilde{B} = 0$. It is then trivial to show that $x'\tilde{A}(I - \tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}) = \lambda x'$. But we have assumed that $\tilde{A}$ is a stable matrix, so an uncontrollable mode cannot be the source of non-invertibility. The same conclusion can be drawn for non-observability, for which there exists an eigenvalue-eigenvector pair (μ, y) such that $\tilde{A}y = \mu y, \tilde{E}y = 0$.

[21] The Smith-McMillan representation (Youla (1961)) of a rational matrix function $W(L)$ is given by $W(L) = \Gamma(L)diag(\frac{n_1(L)}{d_1(L)}, ..., \frac{n_r(L)}{d_r(L)})\Theta(L)$, where $\Gamma(L), \Theta(L)$ have determinants equal to a constant, $d_k(L)$ divides $d_{k+1}(L)$ and $n_k(L)$ divides $n_{k-1}(L)$. The McMillan degree of $W(L)$ is the highest power of L in $d_1(L)d_2(L)...d_r(L)$.

More generally, for the scalar case, suppose $W(L) = n(L)/d(L)$. Now use a Blaschke factor to define $W_1(L) = (1-aL)/(L-a)W(L)$, so that $W_1(L)W_1(L^{-1}) = W(L)W(L^{-1})$. This changes $n(L)$ to $n(L)(1-aL)$ and $d(L)$ to $d(L)(L-a)$. The degree of the latter is obviously greater than that of $d(L)$, so that $W_1(L)$ is a non-minimal spectral factorization. To reiterate the point raised earlier, if $y_t = W_1(L)\epsilon_t$ represents the true response to the structural shock, then a VAR econometrician will estimate a very good approximation to $W(L)$ but would have no way of inferring the correct impulse response.

3.4 Recoverability and Agents' Information sets

In the absence of E-invertibility, the best the econometrician can do, given the history of the observations, is to estimate the innovations representation (see below) of the true model. However a recent literature, initiated by Chahrour and Jurado (2017), has raised the possibility that non-invertible structural shocks may be *recoverable*, in a finite sample of length T, from the full sample history $\left\{m_i^E : i = 1..T\right\}$ for $t \in (\tau, T-\tau)$ for τ sufficiently large. Analogously to invertibility, recoverability is an asymptotic concept: the shock ε_t is recoverable if it can be written as a convergent sum of both past *and* future observables, in which case the impact of both initial and terminal conditions on any observation in the interior of the sample becomes vanishingly small as $T \to \infty$.

Recoverability, reviewed more didactically in Appendix E, makes the assumption that a vector process can be represented as a finite order VARMA: whether by direct estimation, or as an approximation, based on a finite order VAR.[22] A fundamental VARMA representation is a minimal spectral factorization; but there is a finite set of alternative nonfundamental representations of the same order that have an identical autocovariance (Lippi & Reichlin (1994): each of these is also a *minimal* spectral factorization of the same process.

Thus a VAR econometrician who is well enough informed can reconstruct an alternative minimal spectral factorization that can approximate a true minimal spectral factorization, and the shocks to any such representation are recoverable. However, the VAR econometrician *cannot* reconstruct a *non*-minimal spectral factorization; we show below that this arises under imperfect information, in the absence of A-invertibility. Key to this is the

[22] See Appendix E.

following lesser-known result due to Lindquist and Picci (2015) in their Corollary 16.5.7 and Lemma 16.5.8:

Lemma 3.4. Let (20) be a minimal representation of the spectral factor of a stationary stochastic process. There is a one-to-one correspondence between symmetric solutions of the Riccati equation (27) $P = \tilde{A}P\tilde{A}' - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}P\tilde{A}' + \tilde{B}\tilde{B}'$ and minimal spectral factors that retain stationarity; this correspondence is defined via the state space representation

$$w_t = \tilde{A}w_{t-1} + P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\eta_t \qquad m_t^E = \tilde{E}w_t \qquad \eta_t \sim N(0, \tilde{E}P\tilde{E}') \tag{25}$$

Thus for a square system, these alternative solutions for P lead to transfer functions from shocks to observables that differ by one or more Blaschke factors. However what we need subsequently is a result that we can deduce from this lemma, which derives from the PMIC matrices associated with (25) that arise from the general solution for P and the particular solution $P = \tilde{B}\tilde{B}'$, namely $\tilde{A} - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}$ and $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$:

Corollary 3.4. If P is a symmetric solution of (27), then the eigenvalues of $\tilde{A} - \tilde{A}P\tilde{E}'(\tilde{E}P\tilde{E}')^{-1}\tilde{E}$ and $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ are either identical or reciprocals of one another.

3.5 The Econometrician's Innovations Process

We now consider the general nature of the time series representation of the system that the econometrician can extract from the history of the observables. At this stage we do need to make any assumptions about the number of observables vs shocks, other than to assume that $m \leq k$.

For any given set of observables, m_t^E, the econometrician's updating equation for state estimates, assuming convergence of the Kalman filtering matruces, is

$$\mathbb{E}_t s_{t+1} = \tilde{A}\mathbb{E}_{t-1}s_t + \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t, \qquad e_t = m_t^E - \tilde{E}\mathbb{E}_{t-1}s_t \qquad e_t \sim N(0, \tilde{E}P^E\tilde{E}') \tag{26}$$

where $\mathbb{E}_s$ denotes expectations conditioned on the econometrician's information set at time s, and $e_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$, the innovations to the observables in period t, conditional upon information in period $t-1$.

The Riccati matrix $P^E = cov(s_t - \mathbb{E}_{t-1}s_t)$ for this Kalman filter is given in the limit by

$$P^E = Q^E P^E Q^{E'} + \tilde{B}\tilde{B}' \qquad \text{where } Q^E = \tilde{A} - \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}\tilde{E} \tag{27}$$

To ensure stability of the solution P^E, it must satisfy the convergence condition that Q^E is a stable matrix, analogous to the requirement for Q^A above; a sufficient condition is either that $\tilde{A}$ is a stable matrix, or else the controllability of $(\tilde{A}, \tilde{B})$ and observability[23] of $(\tilde{E}, \tilde{A})$.

Note that if we subtract the first equation of (26) from the first equation of (20), we are able to evaluate $cov(s_{t+1} - \mathbb{E}_t s_{t+1},\ \varepsilon_{t+1}) = \tilde{B}$, from which it follows that the covariance between the innovations process and the shocks is given by $cov(e_t, \varepsilon_t) = cov(E(s_t - \mathbb{E}_{t-1}s_t),\ \varepsilon_t) = \tilde{E}\tilde{B}$. We shall use this property later to evaluate how correlated are the residuals from a VAR to the structural shocks.

The Kalman Filter updated expectation of the state s_t given the extra information at time t is given by $\mathbb{E}_t s_t = \mathbb{E}_{t-1}s_t + P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t$, and a little manipulation of (26) enables us to obtain the alternative steady state innovations representation as

$$\mathbb{E}_t s_t = \tilde{A}\mathbb{E}_{t-1}s_{t-1} + P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}e_t \qquad m_t^E = \tilde{E}\mathbb{E}_t s_t \tag{28}$$

This representation will be our main focus, but the representation of the innovations process in (26) is important in proving some of our theoretical results because it provides a means of evaluating the innovations process, and is essential for addressing approximate fundamentalness.

The innovations e_t to this representation have a dimension m equal to the number of observables, and the representation is valid given our general assumption as stated above that $m \leq k$.

The discussion up to now then leads to the following Lemma which applies for any general information set:

Lemma 3.5. The innovations representation (28) applies for $m \leq k$ iff $\tilde{A}$ and Q^E has stable eigenvalues. Sufficient conditions for this to hold are the observability and controllability of $(\tilde{A}, \tilde{B}, \tilde{E})$.

[23]Reduction to minimal form with these properties is fairly straightforward.

3.6 The Innovations Representation under E-invertibility

When the structural shock system (20) is E-invertible, this means that $P^E = \tilde{B}\tilde{B}'$ is a stable solution to the Riccati equation, which in turn requires $Q^E = \tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ to be a stable matrix. This is identical to the PMIC requirement and implies that the innovations process e_t from the filtering problem converges to $\tilde{E}\tilde{B}\varepsilon_t$ as $t \to \infty$. As a result, the state vector s_t is observable asymptotically by the econometrician.

4 The Econometrician's Problem, E-Invertibility and Recoverability

This section shows how the econometrician's problem relates to the solution of the agents' problem presented in subsections 2.5 and 2.6. It also provides the main theoretical results of the paper. It establishes that for square systems (when the number of shocks is equal to the number of observables $m = k$), and if E-invertibility holds when agents in the model have perfect information, an additional condition must be satisfied for the system to still be E-invertible for the imperfect information case. Only if this holds do the solutions under perfect and imperfect information coincide. For possibly non-square systems and $m \leq k$ it examines the dynamic properties of the innovations representation when this extra condition fails and shows that in this case recoverability of structural shocks from an atheoretic time series representation is impossible. All these results raise questions about the appropriateness of comparing impulse responses of VARs with those of a DSGE model.

4.1 Informational Assumptions

In our central case we assume that the econometrician always has the same information set, which is the same as the information set available to the agents under AII. Thus under AII, we assume that $m_t^E = m_t^A$. Under API, we assume that the econometrician's observations m_t^E are the same as under AII; then, using (15), it follows that under API $m_t^E = Ez_t$, where E is defined above after (18).

Having derived two key results below (Theorems 2 and 4) under this assumption, we consider the implications of the econometrician's information set being a strict subset of that of the agents.

Both the API and AII representations of the previous section are in the ABE form of (20). In particular for API, given the informational assumptions set out above, we have $s_t = z_t, \tilde{A} = A, \tilde{B} = B, \tilde{E} = E$, while for AII, we have

$$s_t = \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} \tag{29}$$

$$\tilde{A} \equiv \begin{bmatrix} A & AP^A J'(JP^A J')^{-1} J \\ 0 & F[I - P^A J'(JP^A J')^{-1} J] \end{bmatrix} \tag{30}$$

$$\tilde{B} \equiv \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{31}$$

$$\tilde{E} \equiv [E \quad EP^A J'^A J'^{-1} J] \tag{32}$$

The advantages of using the ABE state-space form in what follows are (i) the Riccati equation is simpler than for any of the other formulations (ii) the solution under imperfect information is much simpler to express and, most usefully, (iii) the representation of the model using the innovations process (see Section 3.5 above) has the same structure as the original model.

4.2 E-invertibility When Agents Have Perfect Information (API)

The conditions for API+E-invertibility are straightforward, and merely mimic the PMIC requirements of the previous section, but with $\tilde{A} = A, \tilde{B} = B, \tilde{E} = E, s_t = z_t$. Hence:
Lemma 4.2. If agents have perfect information (API), the conditions for E-invertibility (as in Definition 3.1) are: the square matrix EB is of full rank and $A(I - B(EB)^{-1}E)$ is a stable matrix.

4.3 E-Invertibility When Agents Have Imperfect Information (AII)

We now consider the more general case of E-invertibility when agents have imperfect information.

Theorem 2. *Assume that the number of observables equals the number of shocks* $(m = k)$.

Assume further that the PMIC conditions in Lemma 4.2 hold (so the system would be E-invertible under API) but agents do not have perfect information. Then each of the following conditions is necessary and sufficient for each of the other two (i.e., the three conditions are equivalent):

a) AII is E-invertible (see Definition 3.1)

b) The square matrix JB is of full rank, and $F(I - B(JB)^{-1}J)$ is a stable matrix.

c) AII is A-invertible (see Definition 2.1)

The counter-intuitive feature of this Theorem is that it is derived under the assumption that the econometrician has the same information set as the agents under imperfect information (AII). If the conditions for API+E-invertibility are satisfied, then if the econometrician had an identical information set, and agents had perfect information (API) then the system would be E-invertible. But Theorem 2 states these conditions are necessary but *not* sufficient for E-invertibility under AII: crucially, E-invertibility is only possible under AII if the solution to the agents informational problem replicates perfect information: that is the conditions that satisfy A-invertibility must *also* hold.

While there is a clear mathematical parallel between the conditions for API+E-invertibility in Lemma 4.2 and the conditions in part (b) of Theorem 2, the crucial difference is that the former depend on the nature of the saddlepath solution (i.e., on the matrices N and hence A), while those in part (b) of Theorem 2 do not. In Sections 5 and 7 below we illustrate Theorem 2 with examples of information sets that satisfy the PMIC conditions in Lemma 4.2 but do or do not satisfy the *extra* conditions (b) in Theorem 2.

From the authors' experience with numerous RE models, the most common reason (other than the obvious ones that observations are lagged or noisy) for AII not to be equivalent to API is associated with:

Corollary 2.1. *Suppose that EB is of full rank and invertible, but J is not of full row rank, then A-invertibility fails.*

To explain this result, consider the case where $J = M_1 - M_2G_{22}^{-1}G_{21}$ is not of full rank. Let U be a matrix that satisfies $UJ = 0$ i.e., $UM_1 = UM_2G_{22}^{-1}G_{21}$. Define V as the orthogonal complement of U (i.e., $UV' = 0$). Then rewriting the set of measurements m_t as their linear transformation $m_t^U = Um_t$ and $m_t^V = Vm_t$ we have a further corollary:

Corollary 2.2. *m_t^V contains all the available information about current shocks whereas m_t^U is unaffected by these shocks and is redundant of information about them.*

4.4 The Innovations Process for AII without E-invertibility

In the absence of E-invertibility, Lemma 3.5 showed that there is still an innovations representation under mild conditions. The counterpart to the innovations representation in (28) is, in population, a finite order fundamental[24] VARMA (or VAR(∞)) in the observables, m_t^E, with innovations e_t. This can either be directly estimated via its state space representation (using DYNARE, for example), or, more commonly, it may be approximated by a finite-order VAR(p) approximation. When the conditions stated in Theorem 2 do not hold, the VARMA or VAR approximation will generate a series of reduced-form residuals that are a linear transformation of e_t in (28) but *not* of the structural shocks ε_t.

We now examine the properties of the innovations representation as in (28) under general conditions when a failure of A-invertibility leads to a failure of E-invertibility.

Theorem 3. *Consider the case where there is a failure of A-invertibility under AII, and hence (from Theorem 2) of E-invertibility. The state space process that generates the impulse response functions of the structural shocks* (16)–(18)*, is of a higher dimension than the innovations representation of the RE saddle-path solution, where the latter is of the same dimension as API and is given by:*

$$\xi_{t+1} = A\xi_t + ZE'(EZE')^{-1}e_{t+1} \qquad m_t^E = E\xi_t \qquad e_t \sim N(0, EZE') \tag{33}$$

where ξ_t is a vector process of precisely half the dimension of s_t in (28) *and*

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1}JP^A \tag{34}$$

Remarkably, this result tells us that even though the dynamics of the RE saddle-path solution under imperfect information are considerably more complex and add more inertia than under perfect information (and hence have a state space representation of twice the dimension), the innovations process e_t is generated by equations that are of the same

[24] We deliberately use the term fundamental here, rather than invertible, to reflect the fact that estimated VARs may contain stationary transformations of unit root processes.

dimension as under API.[25]

The implication of this result is profound, of major significance for empirical work, and one of the main results of this paper.

Corollary 3.1. *Since the spectrum of* (33) *must be identical to that of* (16)-(17)*, it follows that in the absence of A-invertibility the latter is a non-minimal spectral factorization. It therefore incorporates a set of Blaschke factors (Lippi and Reichlin (1994)), whose presence cannot be detected by an estimated a-theoretical representation. Hence the statistical properties of data as generated by the model under AII and represented by a fundamental VARMA or VAR approximation cannot in general generate the true impulse response functions*

In empirical work, a common approach (in the tradition of, for example, Christiano *et al.* (2005)) is to compare impulse responses by applying a structural identification scheme to the estimated VAR(p) with the impulse responses implied by their structural DSGE model. In contrast Kehoe (2006), advocates the approach of Sims (1989) and Cogley and Nason (1995) which compares impulse responses of a finite order, finite sample structural VAR estimated on the data with a VAR with the same structure, run on artificially generated data from the model.

However, for both approaches in the absence of E-invertibility, the reduced form residuals in the data VAR are not a linear transformation of the structural shocks ε_t (even with correct choice of identification matrix), but are instead a finite-order, finite-sample estimate of e_t in (28). In the absence of E-invertibility, e_t is not a linear transformation of ε_t and it follows that comparisons of impulse response functions may be seriously misleading.

4.5 The Innovations Representation When the Econometrician Knows Strictly Less Than the Agents

A criticism of the imperfect information approach that we have been using thus far is that it is possible that agents will have more information about the variables of the model than the econometrician has, although this does not necessarily imply that agents have perfect information. This would imply that the properties of the model solution still embody

[25] This result is a generalisation of BGW, Corollary 1, p302, but without relying on their assumption that all forward-looking variables are observable.

those of Theorem 3, i.e., that a VAR estimation by agents would not be able to replicate the IRFs of shocks.

Corollary 3.2. *If the econometrician's information set is a subset of that of the agents and the system is not A-invertible, then the innovations process as estimated by the econometrician will again be of the same dimension as under API, and thus will be of lower dimension than the true system in* (16)–(18)*.*

The implication therefore is that with any failure of A-invertibility, then provided the econometrician is no better informed than the agents, one should be wary of using an unrestricted VAR (or indeed VARMA) to generate the IRFs of the structural shocks.

4.6 Are the Structural Shocks Recoverable when E-invertibilty Fails?

In Section 3.4 we noted that, in the absence of E-invertibility, there is a finite set of nonfundamental representations of the observables, the shocks to which are not invertible, but *are* recoverable. But the key feature of such representations, that makes recoverability possible, is that all such representations admit a minimal spectral factorisation of the spectrum of the observables. Yet we have shown that, when AII fails, the true data generating process implies a *non*-minimal spectral factorisation[26] due to the presence of Blatschke factors that map the true structural shocks, ε_t to e_t, the innovations to the observables. Thus we immediately have the following further result:

Theorem 4. *If the model has intrinsic (saddle-path) dynamics (i.e., the saddle-path matrix $N \neq 0$) and the system is not A-invertible then the true data generating process is a non-minimal spectral factorization of the spectrum of the agents' information set. Hence the structural shocks are not recoverable from any atheoretic time series representation of the observables (or VAR approximation thereof), which must imply a minimal spectral factorization of the data.*

Thus when *AII* is not E-invertible, and there are saddle-path dynamics, when converting the innovations process representation of the former into any non-invertible representation, such alternative representations will always retain the dimension of the innovations

[26] Or equivalently, in Lippi & Reichlin's (1994) terminology, the implied nonfundamental VARMA representation is also non-*basic* (ie., is of higher order).

process. Since the latter, as we have seen, is of dimension lower than that of the state space describing the effect of individual shocks under II, it follows that the two representations can never be equivalent. Hence the non-E-invertible structural shocks are not recoverable from any stochastically minimal representation,, whether fundamental or nonfundamental.[27]

Thus, at least in this form, recoverability cannot provide an alternative means of using VARs for deriving impulse response functions of structural shocks under imperfect information in the absence of E-invertibility

Does this mean that recoverability has no applicability at all to such models? On the contrary, Theorem 3 and Corollary 3.1 showed that the true model has a non-minimal stochastic representation, incorporating a set of Blaschke factors. From an atheoretic perspective, while any such factors may exist in principle, they can be of arbitrary form. However, in the context of a structural model with AII these Blaschke factors are not arbitrary, since they can be related back to the underlying structure of the model Thus, subject to identification of the appropriate parameters[28] that generate Blaschke factors an econometrician may, at least in principle be able to recover structural shocks even to E-non-invertible systems. We illustrate this possibility in a simple analytical example in the next section.

5 An Analytical Example: A Simple RBC Model

We can illustrate Theorems 2 to 4 with a simple analytical example:[29] that extends the one-variable example of BGW, that uses the linearised ‘stochastic growth’ model of Campbell (1994), with a single observable, the real interest rate. BGW note that this example can be derived as a limiting case of the model of Graham and Wright (2010) which assumes that the agents information set is “market-consistent”: agents also have information on their own wage, which contains both idiosyncratic and aggregate effects;

[27] Note that Forni *et al.* (2017) have an example where recoverability does hold but their very simple model (See Appendix I) lacks the intrinsic dynamics referred to in the Theorem.

[28] We do not address issues of parameter identification in this paper, since these are clearly endemic to all DSGE estimation, whether under API or AII.

[29] Theorem 1 does not apply to this example, given its simplicity, but is applied in a wide range of more complex models in Section 7 below. See also Appendix B for another example also used by Nimark (2008) to illustrate Theorems 2 and 3. This provides an example of the failure of A-invertibility owing to lagged observations.

but the simple case below can be shown to represent a limiting case as the variance of the idiosyncratic component goes to infinity. But BGW consider only the informational problem of the agents; they do not address the econometricians' problem.

From Appendix G.2, the model is a special case of the full RBC model considered in the Section 7.1 below. In linearized form it has the following structure

$$\begin{aligned}
\text{Capital}: \quad k_{t+1} &= \lambda_1 k_t + \lambda_2 \varepsilon_{a,t} + (1-\lambda_1-\lambda_2)c_t && (35)\\
\text{Consumption}: \quad c_{t+1,t} - c_t &= -\kappa k_{t+1,t} && (36)\\
\text{Measurement: Interest Rate} \quad m_t^A = m_t^E &= \varepsilon_{a,t} - k_t \propto r_{t-1} && (37)
\end{aligned}$$

where, under the assumption of a zero-growth steady state, $\lambda_1 \equiv 1 + r$ and $\lambda_2 \equiv \alpha\,(r+\delta)\,/\,(1-\alpha) > 0$, where r is the average real interest rate, δ is the depreciation rate, and α is the exponent on labour in a Cobb-Douglas production function; parameter $\kappa \equiv \frac{\sigma\alpha(1-\alpha)}{1+r}\left(\frac{r+\delta}{1-\alpha}\right)^{\alpha}$ where σ is the elasticity of inter-temporal substitution.

The single shock $\varepsilon_{a,t}$ is a technology shock, which raises the marginal product of capital and hence the return (the single observable), while an increase in capital reduces it. The informational problem for agents thus arises from the ambiguity of the signal when there is a rise in returns: while it could indicate an improvement in technology, it could also indicate that capital is lower than was previously estimated.

To simplify the algebra, technology itself is assumed to have zero persistence,[30] and the return has been normalised such that the constant that would usually multiply the terms in the interest rate equation has been set to 1.

This system can be set up in the form of (3) as

$$\begin{bmatrix} \varepsilon_{a,t+1} \\ k_{t+1} \\ c_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & 1-\lambda_1-\lambda_2 \\ 0 & 0 & 1+\kappa(\lambda_1+\lambda_2-1) \end{bmatrix} \begin{bmatrix} \varepsilon_{a,t} \\ k_t \\ c_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\kappa\lambda_2 & -\kappa\lambda_1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{a,t,t} \\ k_{t,t} \\ c_{t,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{a,t+1} \tag{38}$$

$$m_t^A = m_t^E = [1 \quad -1 \quad 0] \begin{bmatrix} \varepsilon_{a,t} \\ k_t \\ c_t \end{bmatrix} \qquad \varepsilon_{a,t} \sim N(0,\sigma^2) \tag{39}$$

[30]If technology is an AR(1), as in BGW, this introduces an aditional AR root into the representation below, and complicates the algebra somewhat, but without changing any of the substantive results.

Using our earlier notation, we obtain (after a little effort for matrix A)

$$F = \begin{bmatrix} 0 & 0 \\ \lambda_2 & \lambda_1 \end{bmatrix} \qquad J = E = [1 \quad -1] \qquad A = \begin{bmatrix} 0 & 0 \\ \frac{\lambda_2}{\lambda_1}\mu & \mu \end{bmatrix} \tag{40}$$

where 0 and μ are the stable eigenvalues of the system.

It then follows that if agents have perfect information (API) it is straightforward to show that the L-operator representation of the interest rate is an ARMA(1,1) given by

$$m_t^E = E(I - AL)^{-1}B\epsilon_t = \left(\frac{1 - \frac{(\lambda_1+\lambda_2)\mu L}{\lambda_1}}{1 - \mu L} \right) \epsilon_{a,t} \tag{41}$$

It is possible to show (by exploiting the properties of the linearisation constants and the stable eigenvalue, μ) that the MA parameter is $\frac{(\lambda_1+\lambda_2)\mu}{\lambda_1}$ is non-negative, but, for different values of the elasticity of intertemporal substitution, σ and hence κ may lie either below or above unity. Thus the representation may, at least, be fundamental. If this is the case it follows directly that under API the PMIC is satisfied, and hence the system is E-invertible (Lemma 4.2).

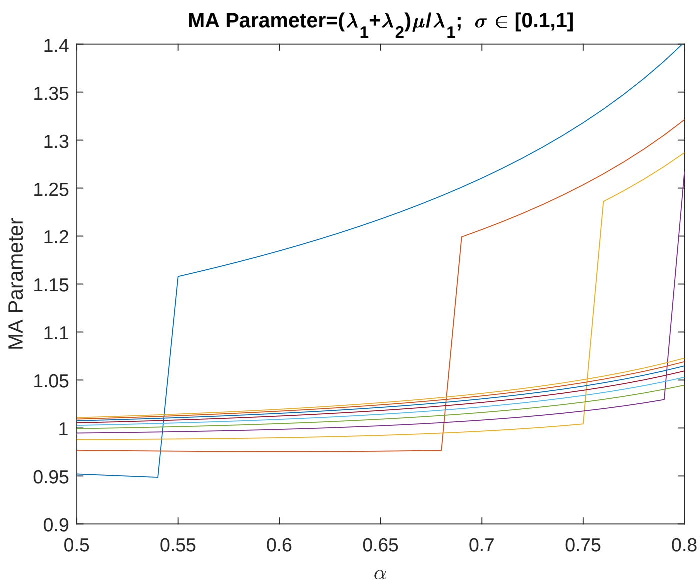

Figure 1: **E-invertibility for the RE Solution of Campbell (1994)'s RBC Model**

Note: Using the analytical example this looks for suitable combinations of α and σ for which E-invertibility holds where we requires the inverse of the root of the MA component from the ARMA(1,1) representation to be less than 1 (this is the MA parameter on y-axis). The grid for $\alpha \in [0.5, 0.8]$ (x-axis) and $\sigma \in [0.1, 1]$ (the bottom blue curve is for $\sigma = 0.1$ with a step size of 0.1).

However, under AII, the stable solution to the Ricatti equation is given by $P^A = \sigma^2 diag(1,\ \ (\lambda_1+\lambda_2)^2 - 1)$ and the Kalman gain is given by

$$P^A J'(JP^A J')^{-1} J = \begin{bmatrix} \frac{1}{(\lambda_1+\lambda_2)^2} \\ \frac{1}{(\lambda_1+\lambda_2)^2} - 1 \end{bmatrix} [1\ \ -1] \tag{42}$$

so, as noted above, any positive shock to the interest rate is ascribed in part to an estimated positive shock to technology, but also in part to a downward adjustment to the estimate of the capital stock. Stability of the solution to the Ricatti equation is given by the stability of

$$Q^A = F(I - P^A J'(JP^A J')^{-1} J) = \begin{bmatrix} 0 & 0 \\ \lambda_1+\lambda_2 - \frac{1}{\lambda_1+\lambda_2} & \frac{1}{\lambda_1+\lambda_2} \end{bmatrix} \tag{43}$$

which is a stable matrix since $1 < (\lambda_1+\lambda_2)$.[31]

Thus, despite the fact that the PMIC may be sometimes be satisfied under API, the system can *never* be A-invertible: AII does not replicate API. Hence, from Theorem 2, the system is not E-invertible.

It is easy to show that the L-operator representation of the interest rate under AII is then given by

$$\begin{aligned} m_t^E &= E(I-AL)^{-1} P^A J'(JP^A J')^{-1} J (I - Q^A L)^{-1} B\epsilon_t \\ &= -(\lambda_1+\lambda_2)\left(\frac{1 - \frac{\mu L}{(\lambda_1+\lambda_2)\lambda_1}}{1-\mu L}\right)\left(\frac{1-(\lambda_1+\lambda_2)L}{L-(\lambda_1+\lambda_2)}\right)\epsilon_{a,t} \qquad (44) \\ &= -(\lambda_1+\lambda_2)\left(\frac{1 - \frac{\mu L}{(\lambda_1+\lambda_2)\lambda_1}}{1-\mu L}\right) e_t \qquad (45) \end{aligned}$$

Note that in the second line, the third term in brackets is a Blaschke factor which ensures that e_t, the innovation to the observable, is white noise, conditional upon the information set (assumed symmetric for both agents and the econometrician). Thus under both AII and API the interest rate has an ARMA(1,1) representation, which must imply that the innovations representation of the system under AII is of the same dimension as under API (illustrating Theorem 3 and Corollary 3.1).

Figure 2 compares the impulse responses to the technology shock $\epsilon_{a,t}$ in the API, AII

[31] The alternative solution of the Riccati equation is $P^A = diag(1, 0)$ but this is not a stable solution since it implies that $Q^A = diag(0, \lambda_1)$, which is an unstable matrix.

cases given by (41) and (44) with the that of the innovation e_t given by (45).

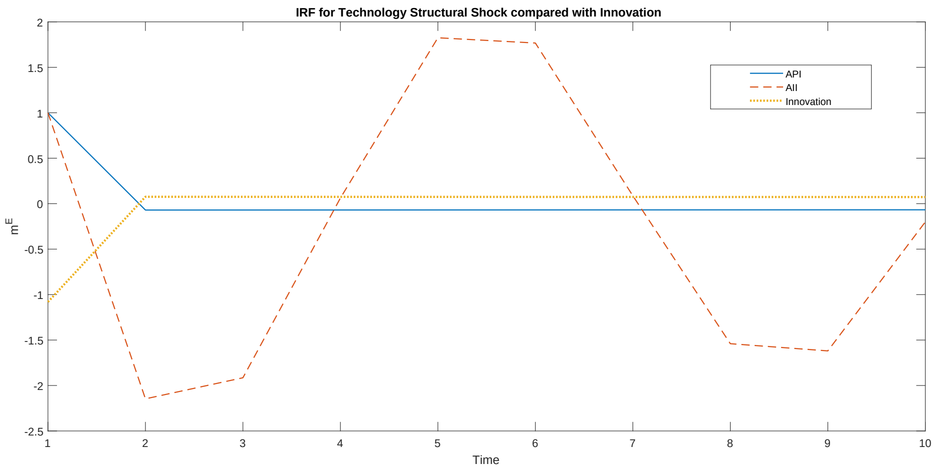

Figure 2: Simple RBC Model. Impulse Responses to a Technology Shock for API and AII compare with Innovation.

Finally, this example also clearly illustrates Theorem 4. If the econometrician is purely data-driven, then the model that is estimated using interest rate data would be the ARMA(1,1) representation in the last line of (45). The non-fundamental counterpart is $m_t^E = -(\lambda_1 + \lambda_2)\left(\frac{\frac{\mu}{(\lambda_1+\lambda_2)\lambda_1} - L}{1-\mu L}\right)\eta_t$. Both e_t and η_t are recoverable. In an atheoretic application of recoverability, either one of these representations could be assumed to be the correct one, but *neither* is a scaling of the true structural shock, thus illustrating Theorem 4.

The failure of this atheoretic application of recoverability arises because the atheoretic econometrician would have no idea that the true representation involved a Blaschke factor. In contrast the DSGE econometrician estimating the system under AII would estimate using the same innovations process. But, taking a structural approach, on the identifying assumption that $\lambda_1 = 1 + r$ is known, it follows that λ_2 and μ are identified, and it would then be possible to completely characterize the correct representation as (45). As a result ε_t would be recoverable to the DSGE econometrician.

6 Approximate Fundamentalness

This section examines, for possibly non-square systems, measures of approximate fundamentalness when invertibility fails for both perfect and imperfect information cases.[32]

Two methods are notable in this regard: Beaudry *et al.* (2016) recommend using the difference in variances between the innovations process and the structural shocks, motivated by the perfect information case (28) which can be written as

$$e_t = m_t - Ez_{t,t-1} = E(z_t - z_{t,t-1}) = EA(z_{t-1} - z_{t-1.t-1}) + EB\varepsilon_t \tag{46}$$

Under invertibility, $z_{t-1} - z_{t-1.t-1}$ has a value of 0, so that regressing the innovations process e_t on this latter term yields (in the scalar case) a perfect lack of fit $R^2 = 0$. For the univariate case, in general we have $R^2 = 1 - var(\varepsilon_t)/var(e_t)$. In the multivariate case, $\text{cov}(e_t) = EP^E E'$, so that the departure of this from $\text{cov}(EB\varepsilon_t)$ yields a measure of how similar the innovations process is to the structural shocks.

However in the empirical literature using VARs it is common to focus on just one shock such as in the examination of the hours-technology question in Gali (1999). To address fundalmentalness on a shock-by-shock basis, one requires the Choleski decomposition of $EP^E E' = VV'$, or else a decomposition that depends for example on long run effects of each shock i.e., an SVAR decomposition. The corresponding R_i^2 for each shock is then given by

$$R_i^2 = 1 - u_{ii} \qquad U = V^{-1}EBB'E'(V')^{-1} = (u_{ij}) \tag{47}$$

The further is R_i^2 from 0, the worse is the fit.

6.1 A Multivariate Measure with Perfect Information

An obvious multivariate version of this is $R = I - V^{-1}EBB'E'(V')^{-1}$, and the maximum eigenvalue of R would then be a measure of the overall fit of the innovations to the fundamentals. In addition one can check whether any fundamentals can be perfectly identified by examining the eigenvalues of the difference between the variances of the

[32] See also Canova and Ferroni (2018) for a treatment of (what we call) E-invertibility and the interpretation of SVAR where the number of structural shocks exceed the number of observables.

innovations and and the fundamentals

$$\mathbb{B}^{PI} = EP^E E' - EBB'E' \tag{48}$$

Any zero eigenvalues coupled with the corresponding eigenvector will provide a means of decomposing the covariance matrix of the innovations $EP^E E'$.

Forni *et al.* (2019) suggest that one can use VARs as well for 'short systems', where the number of observables is smaller than the number of shocks. Utilising the underlying VARMA model, they suggest regressing the structural shocks against the innovations process, i.e., for the structural shock i, choose the least-squares vector m_i by minimizing the sum of squares of $\varepsilon_{i,t} - m_i' e_t$. Clearly, the theoretical value of this is

$$\hat{m}_i = cov(e_t)^{-1} cov(e_t, \varepsilon_{i,t}) = (EP^E E')^{-1} (EB)_i \tag{49}$$

where $(EB)_i$ denotes the ith column of EB. A measure of goodness of fit is then

$$\mathbb{F}_i^{PI} = cov(\varepsilon_{i,t}) - cov(\varepsilon_{i,t}, e_t) cov(e_t)^{-1} cov(e_t, \varepsilon_{i,t}) = 1 - (EB)_i' (EP^E E')^{-1} (EB)_i \tag{50}$$

Thus one can as usual define a linear transformation of the Me_t (where M is made up of the rows m_i') as representing the structural shocks, but only take serious note of those shocks where the goodness of fit is close to 0. Once again, one can use the multivariate measure of goodness of fit

$$\mathbb{F}^{PI} = I - B'E'(EP^E E')^{-1} EB \tag{51}$$

where the diagonal terms then correspond to the terms $\mathbb{F}_i$ of (50). In (51) we note that $EP^E E' = cov(e_t)$ from the steady state of (27), and $(EB)_i = cov(e_t, \varepsilon_{i,t})$.

If the number of measurements is equal to the number of shocks, and if $\mathbb{F}_i = 0$ for all i, then since $\mathbb{F}^{PI}$ is by definition a positive definite matrix, it must be identically equal to 0. Of course, it may be the case that none of the $\mathbb{F}_i$ are zero, but that a linear combination of the structural shocks are exactly equal to a linear combination of the residuals. In addition, we might specify a particular value of the R^2 (e.g. $R_s^2 = 0.9$) fit of residuals to fundamentals such that we are happy to approximate the fundamental by the best fit of residuals.[33]

The maximum eigenvalue of $\mathbb{F}^{PI}$ then provides a measure of overall non-fundamentalness. It must of course be emphasised that none of these measures can be obtained directly from the data. The papers cited above all provide details of how simulations on the underlying VARMA models can indicate how to make appropriate inferences on the structural shocks using just the data and a VAR estimation.

6.2 A Multivariate Measure with Imperfect Information

Collard and Dellas (2004) and Collard and Dellas (2006) provide examples where there are large differences in the impulse response functions under imperfect and perfect information, and indeed Theorem 3 appears to indicate that this may be a major issue. In addition, Levine *et al.* (2012), for an estimated DSGE model, find that such differences are quite large as well.

As we have seen for the perfect information case above, it is quite straightforward to obtain goodness of fit measures for the individual shocks from the multivariate measures, so for convenience we only list the latter. Firstly, the Beaudry *et al.* (2016) measure, which can be abbreviated to the difference between the variances of the innovations and the fundamentals, is given by

$$\mathbb{B}^{II} = EZE' - EBB'E' \tag{52}$$

where Z is given by (A.24).

Likewise, the multivariate Forni *et al.* (2019) measure can, after some effort, be written as

$$\mathbb{F}^{II} = I - B'J'(JP^AJ')^{-1}JP^AE'(EZE')^{-1}EP^AJ'(JP^AJ')^{-1}JB \tag{53}$$

Analogously to the perfect information case, $EZE' = cov(e_t)$, with $EP^AJ'(JP^AJ')^{-1}JB = cov(e_t, \varepsilon_t)$. The latter follows firstly because from (18) and (A.26) we can write $e_t = E(z_{t,t-1} - \bar{s}_{1t}) + EP^AJ'(JP^AJ')^{-1}J\tilde{z}_t$. The first term is clearly independent of ε_t, while the covariance of the second term with ε_t is obtained by calculating $\mathbb{E}[\tilde{z}_{t+1}\varepsilon'_{t+1}]$ in (16).

We can bring together (51) and (53) in the following final Theorem of the paper.

[33] A perfect fit in the Forni *et al.* (2019) case is $\mathbb{F}_i = 0,\ R_i^2 = 1$.

Theorem 5. *Consider the more general case with the number of structural shocks possibly greater than the number of measurements. (a) All zero eigenvalues of* $\mathbb{F}^{PI}$ *or* $\mathbb{F}^{II}$*, for the perfect or imperfect information cases respectively, correspond to a perfect fit between a linear combination of fundamentals and a best regression fit of residuals; (b) The number of eigenvalues of* $\mathbb{F}^{PI}$ *or* $\mathbb{F}^{II}$ *that are less than* $1 - R_s^2$*, where* R_s^2 *is the chosen threshold for* R^2*, correspond to the number of linear combinations of fundamentals that can be obtained approximately from the residuals.*

In addition $\mathbb{F}_i^{II}$ corresponds to a measure of goodness of fit of the innovations residuals to the structural shocks, and provides information as to how well the VAR residuals correspond to the fundamentals. Note however that these measures correspond to the case when all observables are of current variables. While it is not difficult to perform the appropriate calculations in the case when some variables are current and others are lagged, it is not straightforward to write down a mathematical expression in such a case. Nevertheless we can apply the ideas above when all variables are lagged. In particular, the theoretical value of $\mathbb{F}^{II,lagged}$ can now be defined as

$$\mathbb{F}^{II,lagged} = cov(\varepsilon_t) - cov(\varepsilon_t, e_{t-1})cov(e_{t-1})^{-1}cov(e_{t-1}, \varepsilon_t) \tag{54}$$

$cov(e_{t-1})$ is of course equal to $cov(e_t) = EZE'$, so the only change is to $cov(e_{t-1}, \varepsilon_t)$, which after a little effort can be derived as

$$\begin{aligned} cov(e_{t-1}, \varepsilon_t) = {} & EAP^AJ'(JP^AJ')^{-1}JB - EAZE'(EZE')^{-1}EP^AJ'(JP^AJ')^{-1}JB \\ & + EP^AJ'(JP^AJ')^{-1}JFB - EP^AJ'(JP^AJ')^{-1}JFP^AJ'(JP^AJ')^{-1}JB \end{aligned} \tag{55}$$

Then the fit $\mathbb{F}_i^{II,lagged}$ to the ith shock is just given by the ith main diagonal term of $\mathbb{F}^{II,lagged}$.

In a later section we compare numerically these perfect and imperfect information multivariate measures of the fit of the innovations to the fundamentals for a DSGE model.

7 Applications to RBC and NK Models

This section illustrates our theoretical results using numerical solutions of RBC and NK models. We consider and implement invertibility conditions of Theorem 2 and the multivariate measure of goodness of fit set out in Section 6. For the latter our focus is on (51) and (53), the corresponding measures of correlation between e_t and ε_t, for the perfect and imperfect information cases, respectively, where $cov(e_t) = EP^E E'$ and $cov(e_t) = EZE'$ are the covariance matrices of the innovation processes for the two cases, and $cov(\varepsilon_t)$ of the structural shocks in the model. As noted, the maximum eigenvalue provides a measure of overall non-fundamentalness. In addition, any zero eigenvalues provide information as to which structural shocks can be satisfactorily identified (i.e., evidence of partial sufficiency of individual shocks in the system).

7.1 Example 1: Invertibility and Fundamentalness Measures for RBC Model

We first consider a standard RBC model set out in Appendix G. Example 1 presents a simplified non-linear RBC model without investment adjustment costs and variable hours (i.e. $H_t = \bar{H} = 1$ and $\varrho = 0$), in line with the linearized model of Campbell (1994).[34] With two shock processes, A_t and G_t (normalized such that $cov(\varepsilon_t) = I$) the following combinations of two observables (from a set of observables: $(Y_t, C_t, I_t, W_t, R_t, R_{K,t})$) result in A-invertibility: ($m_t^E = m_t^A = Y_t, C_t$), (Y_t, I_t), (Y_t, W_t) and (C_t, W_t). Since $m_t^E = m_t^A =$ these combinations also imply E-invertibility. On the other hand for the following combinations A-invertibility fails: (Y_t, R_t), (W_t, R_t) and (C_t, R_t).

Table 1 below summarises a complete set of combinations of two observables for this model, i.e., $c = \frac{6!}{(6-2)!2!} = 15$, based on the rank and stability conditions of Theorem 2. Table 1 also checks the difference between perfect and imperfect information in terms of identifying the fundamentals from the perspective of VARs via the eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$, assuming that the RBC Model is the true DGP. Figure 3 shows the E- and A-invertibility regions for the RBC model with $R_{K,t}$ the only observable and one shock, A_t. For E-invertibility under API, it requires the risk parameter $\sigma_c \ll 1$ and this completely agrees with the numerical results reported in Table 2. As we now have a complete agree-

[34]The results for the richer model are reported in Appendix G.

ment between the numerical and analytical results with $R_{K,t}$ observable and one shock in Section 5 and Table 2, respectively, we turn to Table 1 for the RBC invertibility checks, examining two cases for $(\sigma_c, \alpha) = (0.3, 0.6)$ and $(\sigma_c, \alpha) = (2, 0.6)$, respectively.

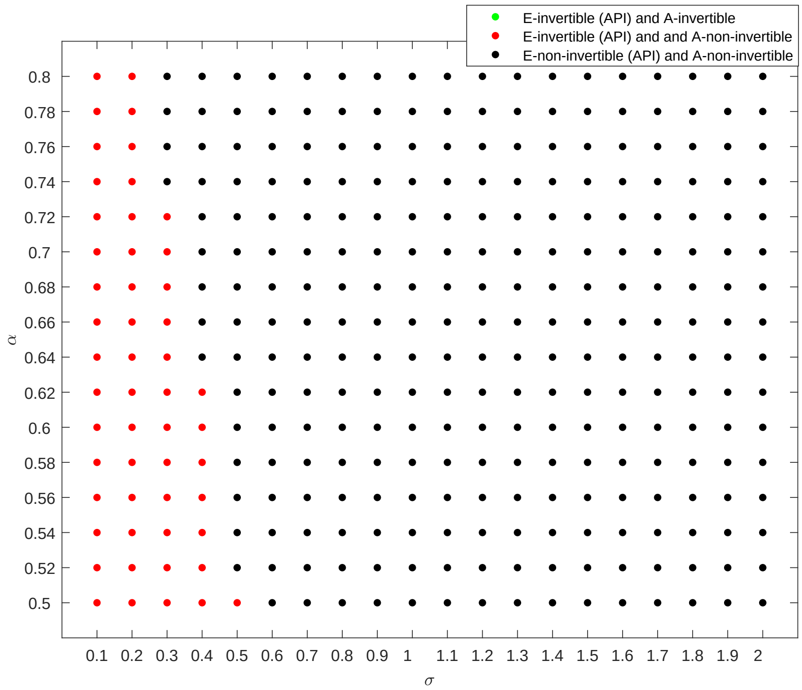

Figure 3: **E- and A-invertibility Regions over Parameters σ_c and α**

Note: This shows the E- and A-invertibility regions for the linearized model of Campbell (1994) set out as an analytical example in Section 5 and in Appendix G.2, and a simplified non-linear RBC model presented in Table 2. In line with Figure 1, $\sigma_c \in [0.1, 2]$ and $\alpha \in [0.5, 0.8]$.

The most common non-obvious reason for A-invertibility to fail for both cases is indicated from the second to fourth columns of the table, where J is not of full row rank.[35] Theorem 2 also establishes an extra condition, given that models API are E-invertible, that the square matrix JB is of full rank, *and* $F(I - B(JB)^{-1}J)$ is a stable matrix (has all eigenvalues inside the unit circle), for AII to be E-invertible too. In Table 1, we report the only cases with (C_t, I_t) and $(C_t, R_{K,t})$ when this eigenvalue condition for AII is not satisfied, despite J being full rank. Another interesting special case is the model with observable set $(W_t, R_{K,t})$, where API is not E-invertible and is therefore not equivalent to AII; even though EB is of full rank $A\left(I - B(EB)^{-1}E\right)$ is not a stable matrix.

For the case of the RE saddle-path solution being A-invertible, the solution (19) is $P^A = BB'$ and, from which it follows that $\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$, and the two processes are

[35]See Corollary 2.1.

Information Set	E-Invertibility under API?	A-Invertibility?	Notes	Eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{PI}$	Diagonal values of $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$
RBC Case 1: $\sigma_c = 0.3$ and $\alpha = 0.6$					
$(Y_t, C_t), (Y_t, I_t)$ $(Y_t, W_t), (C_t, W_t)$ $(I_t, W_t), (I_t, R_{K,t})$	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0, 0]$	$\mathbb{F}_i^{PI} = \mathbb{F}_i^{II} = [0, 0]$
(Y_t, R_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0007, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0007, 1]$
(C_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.4488, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.4499, 0.9989]$
(I_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0132, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0425, 0.9707]$
(W_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0007, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0007, 1]$
(C_t, I_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0, 0.2279]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0228, 0.2051]$
$(Y_t, R_{K,t})$	NO	NO	EB is **rank deficient** JB is **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 1]$ $eig(\mathbb{F}^{II}) = [0, 1]$	$\mathbb{F}_i^{PI} = [0, 1]$ $\mathbb{F}_i^{II} = [0, 1]$
$(W_t, R_{K,t})$	NO	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 1]$ $eig(\mathbb{F}^{II}) = [0, 1]$	$\mathbb{F}_i^{PI} = [0, 1]$ $\mathbb{F}_i^{II} = [0, 1]$
$(C_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0, 0.9771]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0006, 0.9765]$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.2399, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.2399, 1]$
RBC Case 2: $\sigma_c = 2$ and $\alpha = 0.6$					
$(Y_t, C_t), (Y_t, I_t)$ $(Y_t, W_t), (C_t, W_t)$ $(I_t, W_t), (C_t, I_t), (I_t, R_{K,t})$	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0, 0]$	$\mathbb{F}_i^{PI} = \mathbb{F}_i^{II} = [0, 0]$
(Y_t, R_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0051, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0051, 1]$
(C_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0392, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0392, 0.9999]$
(I_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0051, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.1602, 0.8411]$
(W_t, R_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0051, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0051, 1]$
$(C_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0, 0.984]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0001, 0.9839]$
$(Y_t, R_{K,t})$	NO	NO	EB is **rank deficient** JB is **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 1]$ $eig(\mathbb{F}^{II}) = [0, 1]$	$\mathbb{F}_i^{PI} = [0.0008, 0.9992]$ $\mathbb{F}_i^{II} = [0.0005, 0.9995]$
$(W_t, R_{K,t})$	NO	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 1]$ $eig(\mathbb{F}^{II}) = [0, 1]$	$\mathbb{F}_i^{PI} = [0, 1]$ $\mathbb{F}_i^{II} = [0, 1]$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) = [0.0954, 1]$	$\mathbb{F}_i^{PI} = [0, 0]$ $\mathbb{F}_i^{II} = [0.0954, 1]$

Table 1: **Exact and Approximate Invertibility Checks for RBC Model (Order of Shocks: A_t, G_t)**

Note: Check Conditions in Lemma 4.2 and Theorem 2. This is the simplified RBC model without investment adjustment costs and variable hours (i.e. $H_t = \bar{H} = 1$ and $\varrho = 0$). We consider two cases for $(\sigma_c, \alpha) = (0.3, 0.6)$ and $(\sigma_c, \alpha) = (2, 0.6)$.

Information Set	E-Invertibility under API?	A-Invertibility?	Notes	Eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{PI}$
RBC Case 1: $\sigma_c = 0.3$ and $\alpha = 0.6$				
(C_t)	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0]$
(R_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [1]$
$(R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [1]$
(Y_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [1]$
(I_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [0.9847]$
(W_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [1]$
RBC Case 2: $\sigma_c = 2$ and $\alpha = 0.6$				
$(C_t), (I_t)$	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0]$
(R_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [1]$
$(R_{K,t})$	NO	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0.0579]$ $eig(\mathbb{F}^{II}) = [0.9972]$
(Y_t)	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [0.9668]$
(W_t)	YES	NO	Ditto	$eig(\mathbb{F}^{PI}) = [0]$ $eig(\mathbb{F}^{II}) = [0.9668]$

Table 2: **Exact and Approximate Invertibility Checks for RBC Model with One Shock:** A_t

Note: Check Conditions in Lemma 4.2 and Theorem 2. This is the simplified RBC model without investment adjustment costs and variable hours (i.e. $H_t = \bar{H} = 1$ and $\varrho = 0$). We consider two cases for $(\sigma_c, \alpha) = (0.3, 0.6)$ and $(\sigma_c, \alpha) = (2, 0.6)$.

perfectly correlated across the API and AII cases. This shows that invertibility or fundamentalness can allow for innovations to exactly approximate structural shocks. For the case of non-invertibility, the further is $\mathbb{F}^{II}$ from 0, the worse is the fit. Examples $(Y_t, R_t), (C_t, R_t), (I_t, R_t), (W_t, R_t)$ and $(C_t, R_{K,t})$ in Table 1 show the cases while the perfect information solution is invertible (or there is complete fundamentalness, i.e., $\mathbb{F}^{PI} = 0$) the imperfect information counterparts are not (i.e., $\mathbb{F}^{II} > 0$ in the positive definite sense). With the same observables, solving the system under perfect information, the steady state solution of (27) gives $P^E - BB' = 0$, from which this means $\mathbb{F}^{PI} = 0$. Solving the steady state Riccati equation (19) for our case of imperfect information, we have $P^A > BB'$ and it automatically follows that $\mathbb{F}^{II} > \mathbb{F}^{PI}$. Therefore, interestingly, we show that the simple RBC model introduces non-fundamentalness with the same measurements under AII as under API.

The only way to decide the overall fit of the RBC model approximating the fundamentals by the innovations process is to determine the maximum eigenvalue of $\mathbb{F}^{II}$. From the fifth column of Table 1, it is not surprising to find that the fit of the innovations to the structural shocks under AII is very poor as the maximum eigenvalues are all far from 0, when J and JB are not of full row rank or the eigenvalue condition fails. However, in some cases, the first eigenvalue being very close to 0 (e.g. with (Y_t, R_t) and $(Y_t, R_{K,t})$) indicates partial fundamentalness or that one of the two shocks may be satisfactorily identified in this model. Assuming that a simple baseline RBC model is the DGP from which potentially VARs and SVARs are identified, this diagnostic result remarkably and strongly underlines our Theorem 3. When there are large differences in the impulse response functions under imperfect and perfect information, non-fundamentalness may be quantitatively severe, indeed according to Theorem 3, the simulation appears to indicate that this may be a major issue.

The last column of Table 1 reports the diagonal values of the (non-zero) $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$ matrices. These tell us explicitly about the goodness of fit of the residuals to the structural shocks. Any zero values reported in the diagonal matrices indicate an exact fit for the corresponding individual shocks in the models (for example, the shock A_t in many cases).[36]

[36] Appendix H carries out a further illustrative exercise on a RBC model with a news shock.

7.2 Example 2: Invertibility and Fundamentalness Measures for SW Model

Testing for non-fundamentalness for non-square systems as a number of structural shocks increases can be achieved by looking at a richer model to which we now turn.

We run our simulation exercise using a version of Smets and Wouters (2007) model (henceforth SW). This model is selected because it features a number of nominal and real frictions in order to closely mimic the pattern of real aggregate variables, inflation and interest rate. There are seven structural shocks in SW. The model has five AR(1) processes, for the shocks on government spending, technology, preference, investment specific, monetary policy, and two ARMA(1,1) processes, for price and wage markup. In this exercise, we skip the description of the model and slightly modify the model by gradually adding more shocks. The SW model is estimated based on seven quarterly macroeconomic time series. When we assume that this exactly coincides with the agents' limited information set so in effect the number of measurements is equal to the number of shocks and EB is non-singular (Case 1: Original SW). In the modified versions of the model, the only changes we make are that (1) we add an inflation target shock so the number of shocks exceeds the number of observables (Case 2: SW with 8 shocks); (2) we further add measurement errors to the observations of real variables and inflation (Case 3: SW with 13 shocks). Table 3 summarises the key results from the simulation, based on Theorems 2 and 5 and the test for non-fundamentalness.

As before, the models are solved and simulated through Theorem 1 and the conversion procedure set out in Appendix A.1. We find that the original system with the original sets of measurements and shocks is exactly invertible according to Theorem 2, the eigenvalue measures and indeed produces exactly the same simulated moments across the perfect and imperfect information assumptions. As expected, when we add the additional shock in Case 2, compared to non-invertibility of API the eigenvalues are larger for AII ($\mathbb{F}^{II} > \mathbb{F}^{PI}$), introducing non-fundamentalness into the model. The overall fit for fundamentalness under AII is much improved from the baseline results (the RBC model), but with a larger-sized model (e.g. Case 2) the difference between API and AII is less marked. Based on Theorem 3 again, this means that the differences between IRFs for API and AII, from the perspective of identifying VARs, are less marked. This result clearly depends on the size

	Case 1: Original SW	Case 2: SW with Inflation Obj.		Case 3: SW with MEs	
	Measurements = Shocks (=7)	8 Shocks		13 Shocks	
Theorem 2 Corollary 2.1	E, EB are full row rank (=7) J, JB are full row rank (=7) $A(I-B(EB)^{-1}E)$ is stable $F(I-B(JB)^{-1}J)$ is stable	E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I-B(EB)^{-1}E)$ is non-existent $F(I-B(JB)^{-1}J)$ is non-existent		E, EB are rank deficient (=7) J, JB are rank deficient (=7) $A(I-B(EB)^{-1}E)$ is non-existent $F(I-B(JB)^{-1}J)$ is non-existent	
Goodness of Fit	$\mathbb{F}^{PI}=\mathbb{F}^{II}=0$	$\mathbb{F}^{PI}_{(8\times8)}$	$\mathbb{F}^{II}_{(8\times8)}$	$\mathbb{F}^{PI}_{(13\times13)}$	$\mathbb{F}^{II}_{(13\times13)}$
Eigenvalues	$eig(\mathbb{F}^{PI})=eig(\mathbb{F}^{II})=0$	$\begin{bmatrix}1.0000\\0.0013\\0\\0\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1.0000\\0.0016\\0.0009\\0.0001\\0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0.0971\\0.0454\\0.0138\\0.0001\\0.0019\\0.0058\\0.0100\\1\\1\\1\\1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0.5404\\0.3627\\0.2975\\0.0302\\0.0011\\0.0044\\0.8182\\1\\1\\1\\1\\1\\1\end{bmatrix}$
Diagonal values		$\begin{bmatrix}-0.0000\\0.0006\\0.0000\\0.0005\\0.0245\\0.0000\\0.0001\\0.9756\end{bmatrix}$	$\begin{bmatrix}0.0000\\0.0006\\0.0000\\0.0004\\0.0256\\0.0000\\0.0001\\0.9761\end{bmatrix}$	$\begin{bmatrix}0.2216\\0.0924\\0.5199\\0.1600\\0.1007\\0.2262\\0.2585\\0.9780\\0.4668\\0.7097\\0.9053\\0.8353\\0.6998\end{bmatrix}$	$\begin{bmatrix}0.5754\\0.8850\\0.5136\\0.6945\\0.1099\\0.4552\\0.7095\\0.9782\\0.5892\\0.6749\\0.6672\\0.7165\\0.4854\end{bmatrix}$

Table 3: **Exact and Approximate Invertibility Checks for SW Model**

Note: Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage markup, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation. Number of measurements $\leq$ number of shocks. Imperfect information is not equivalent to perfect information for Cases 2 and 3 and this is verified by the rank conditions: EB and JB are not of full rank therefore both API and AII are not invertible. For approximate invertibility, there is no complete fundamentalness when both $\mathbb{F}^{PI}>0$ and $\mathbb{F}^{II}>0$. The fit of the innovations to the structural shocks is determined by the maximum eigenvalue of $\mathbb{F}$.

of the model and the number of shocks, and via simulation, is consistent with previous literature. For example, in the empirical exercise of Levine *et al.* (2012), the estimated NK model with the minimum amount of frictions produces the most notable differences between IRFs when assuming imperfect information for the agents.

In line with the empirical literature again, when we further add measurement errors to the measurement equations for the 4 real variables and the inflation (Case 3), the multivariate fit for fundamentalness or approximate invertibility of SW significantly declines for both the AII and API cases. It is very clear that, even with a medium-sized model like SW, it is the decreasing ratio of observables to shocks that drives a bigger wedge between API and AII, in the sense that the fundamentalness problem worsens for the performance of VARs, and the difference of empirical likelihood between perfect information and imperfect information models increases, with fewer observations by agents.

Finally, as expected, the overall fit also depends on the ratio of observables m to shocks $k \geq m$, in other words, the fewer the observations made by the agents compared to shocks the less well do VARs perform.

8 Conclusions

The description of invertibility as a 'missing information' problem on the part the econometrician is stressed in the econometrics literature on the subject, for example, Lippi and Reichlin (1994), Lutkepohl (2012) and Kilian and Lutkepohl (2017); but when they do refer informally to the underlying model that generates an MA process, they assume agents observe the shocks (our API case). The missing information of the econometrician is then *relative to the agents* in the model.[37]

Our paper looks at the problem where this information gap is closed and both econometrician and agents have the same imperfect information set. In our Theorem 2 we then have an extra condition over and above the PMIC for E-invertibility which demonstrates that considering the information of agents can make the invertibility problem worse. In this sense the appropriate choice of information assumption, consistent with market structure (complete vs incomplete markets) for the agents in the model can be seen as an important *additional* source of non-invertibility.

From Theorems 3 and 4, if the imperfect information solution for agents is not asymptotically equivalent to that under perfect information, then the impulse response functions of the former incorporate one or more Blaschke factors that cannot be picked up by an a-theoretical VAR econometrician. In the language of the time series literature, the econometrician is estimating a minimal spectral factorization of the data, whereas the data is actually generated by a response to structural shocks that corresponds to a non-minimal spectral factorization.

This paper lies within the tradition pioneered by Sims (1980) on the estimation-identification of SVARS. A more recent approach uses "external instruments" which are variables correlated with a particular shock of interest, but not with the other shocks.

[37]For instance, in Kilian and Lutkepohl (2017) they write on page 576: "The main argument in favor of nonfundamental shocks being important in economic analysis is that the econometrician may not have all the information that economic agents have."

External instruments can be used to directly estimate causal effects by direct IV regressions using the method of local projections (LP) of Jorda (2005). This method does not require invertibility. Stock and Watson (2018) compares the LP-IV approach with a more efficient SVAR-IV approach proposes a new test for invertibility which is applied to the study of Gertler and Karadi (2015).[38] It would be of interest to re-examine this method in the light of the information assumptions of agents in the assumed DSGE DGP.

As mentioned in Section 2.4 work in progress aims to show in the time domain how a variation of the AII solution (16)–(17) can be implemented that will match the results generated by Rondina and Walker (2017) for heterogeneous agents. Also our analysis can be generalized to allow for agents with different imperfect information observables m_t^A as studied in Lubik *et al.* (2018). These topics will be the subject for future research.[39]

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[38] Miranda-Agrippino and Ricco (2019) extends their result to the case when researcher only wants to partially identify the system, that is, to retrieve the dynamic effects of one or a subset of the structural shocks.

[39] Our imperfect information solution is currently available in Dynare. Bayesian estimation with II and fundamentalness tests are forthcoming. See Appendix J for details.

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Appendix

A Proofs of Theorems, Lemmas and Corollaries

A.1 Proof of Theorem 1: Transformation of System to PCL Form

A.1.1 The Problem Stated

An important feature of the RE solution procedure of the seminal paper Blanchard and Kahn (1980) is that it provided necessary and sufficient conditions for the existence and uniqueness of a solution for linearized model. The only general results on imperfect information solutions to rational expectations models date back to PCL, who utilize the Blanchard-Kahn set-up, and generalize this result.

Theorem 1 states that equation (1), re-expressed here

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t \tag{A.1}$$

with measurements

$$m_t = LY_t \tag{A.2}$$

can be written in the form (2) and (3) originally used by PCL, re-stated here as

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{A.3}$$

with agents' measurements given by

$$m_t = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{A.4}$$

To prove Theorem 1, the next section describes a completely novel algorithm for converting the state space (A.1), (A.2) under imperfect information to the form (A.3), (A.4). We assume that the system is 'proper', by which we mean the matrix A_1 is invertible; this precludes the possibility of a system that includes equations of the form $h^T Y_{t+1} = 0$, but it is fairly easy to take account of these as well.

A.1.2 An Iterative Algorithm

Although complicated, the basic stages for the conversion are fairly simple:

1. We first (Stages 1 to 3) find the singular value decomposition for the $n \times n$ matrix A_0 (which is typically of reduced rank $m < n$) which allows us to define a vector of m forward-looking variables that are linear combinations of the original Y_t.

2. We then introduce a novel iterative stage (Stage 4) which replaces any forward-looking expectations that use model-consistent updating equations. This reduces the number of equations with forward-looking expectations, while increasing the number of backward-looking equations one-for-one. But at the same time it introduces a dependence of the additional backward-looking equations on both state estimates $z_{t,t}$ $\left(\equiv E_t z_t | I_t^A\right)$ and estimates of forward-looking variables, $x_{t,t}$. This in turn implies that both (A.3) and (A.4) in general contain such terms.

3. A simple example may help to provide intuition for this iterative stage: Suppose

two of the equations in the system are of the form: $z_t = \rho z_t + \varepsilon_t$, $y_t = z_{t+1,t}$ (where both y_t and z_t are scalars) i.e., we have one backward-looking (BL) equation and one forward-looking (FL) equation. However using the first equation we can write $z_{t+1,t} = E_t z_{t+1} = \rho z_{t,t}$, hence substituting into the second equation, $y_t = \rho z_{t,t}$: i.e., we can use a model-consistent updating equation. Note, however, a crucial feature: since under II we cannot assume that z_t is directly observable, this updating equation is expressed in terms of the filtered state estimate $z_{t,t}$ rather than directly in terms of x_t We thus now have two BL equations, but one of these is expressed in term of a state estimate.

4. The iterative Stage 4 may need to be repeated a finite number of times. In the case of perfect information this is all that is needed, apart from defining what are the $t+1$ variables.

5. For imperfect information, we retain the same backward and forward looking variables as in the perfect information case, but the solution process is a little more intricate.

The detailed procedure for conversion of (A.1) and (A.2) to the form in (A.3) and (A.4) is as follows:

Stage 1: SVD and partitions of A_0. Obtain the singular value decomposition for the $n \times n$ matrix A_0: $A_0 = U_0 S_0 V_0^T$, where U_0, V_0 are unitary matrices. Assuming that only the first m values of the diagonal matrix S_0 are non-zero (rank$(A_0) = m < n$), we can rewrite this as $A_0 = U_1 S_1 V_1^T$, where U_1 are the first m columns of U_0, S_1 is the first $m \times m$ block of S_0 and V_1^T are the first m rows of V_0^T. In addition, U_2 are the remaining $n-m$ columns of U_0, and V_2^T are the remaining $n-m$ rows of V_0^T.

Stage 2: Extract FL subsystem from (A.3) using S_1 and U_1. Multiply (A.3) by $S_1^{-1}U_1^T$, which yields:

$$V_1^T Y_{t+1,t} + S_1^{-1}U_1^T A_1 Y_t = S_1^{-1}U_1^T A_2 Y_{t-1} + S_1^{-1}U_1^T \Psi \varepsilon_t \tag{A.5}$$

We can now define an initial subdivision of Y_t into an (initially) m-vector of forward-looking variables $x_t = V_1^T Y_t$, and and an $(n-m)$-vector of backward-looking variables $s_t = V_2^T Y_t$ (noting that $Y_t = V_1 x_t + V_2 s_t$), and use the fact that $I = VV^T = V_1 V_1^T + V_2 V_2^T$

to rewrite (A.3) as:

$$x_{t+1,t} + S_1^{-1}U_1^T A_1(V_1 x_t + V_2 s_t) = S_1^{-1}U_1^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + S_1^{-1}U_1^T \Psi\varepsilon_t \tag{A.6}$$

or simply:

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.7}$$

where $F_1 = S_1^{-1}U_1^T A_1 V_1,\ F_2 = S_1^{-1}U_1^T A_1 V_2,\ F_3 = S_1^{-1}U_1^T A_2 V_1,\ F_4 = S_1^{-1}U_1^T A_2 V_2$ and $F_5 = S_1^{-1}U_1^T\Psi$. This is a set of m forward-looking equations. Note that in the iterative Stage 4, the definition of x_t will usually change further, and thus at this stage x_t is not usually equal to its final form in (A.3).

Stage 3: Extract BL subsystem from (A.3) using U_2. Multiply A.3 by U_2^T which yields:

$$U_2^T A_1 Y_t = U_2^T A_2 Y_{t-1} + U_2^T \Psi\varepsilon_t \tag{A.8}$$

which can be rewritten as

$$U_2^T A_1(V_1 x_t + V_2 s_t) = U_2^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T \Psi\varepsilon_t \tag{A.9}$$

or more simply:

$$C_1 x_t + C_2 s_t = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.10}$$

where $C_1 = U_2^T A_1 V_1$, $C_2 = U_2^T A_1 V_2$, $C_3 = U_2^T A_2 V_1$, $C_4 = U_2^T A_2 V_2$ and $C_5 = U_2^T\Psi$. This is a set of $n-m$ backward-looking equations.

If C_2 is invertible then it is straightforward to multiply (A.3) by C_2^{-1}, and go straight to Stage 5. However if C_2 is not invertible we need to proceed to the next (iterative) stage.

Stage 4: Iterative transformation of FL equations using model-consistent updating. In this iterative stage we write (A.7) and (A.10) in the more general form:

$$\begin{aligned} x_{t+1,t} + F_1 x_t + F_2 s_t &= F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t && \text{(A.11)} \\ C_1 x_t + C_2 s_t + G_1 x_{t,t} + G_2 s_{t,t} &= C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t && \text{(A.12)} \end{aligned}$$

where by comparison of (A.12) with (A.10) we have introduced two new matrices, G_1

and G_2 that must be zero in the first stage of iteration. However, at the end of the first iteration of this stage we shall increase the dimension of s_t, and reduce the dimension of x_t one-for-one, which will require us to re-define all the matrices in (A.11) and (A.12), such that, from the second iteration onwards, G_1 and G_2 will be non-zero. The whole of Stage 4 may then need to be iterated a finite number of times.

First find, a matrix J_2 such that $J_2^T(C_2+G_2)$=0, by using the SVD of C_2+G_2 (noting that in the first iterative stage, $G_2 = 0$) Then take forward expectations of (A.12) and pre-multiply by J_2^T to yield:

$$J_2^T(C_1 + G_1)x_{t+1,t} = J_2^T C_3 x_{t,t} + J_2^T C_4 s_{t,t} \tag{A.13}$$

Then reduce the number of forward-looking variables by substituting for $x_{t+1,t}$ from (A.11). In addition find a matrix Q that has the same number of columns as $J_2^T(C_1 + G_1)$ and is made up of rows that are orthogonal to it. Then we define the following subdivision of x_t

$$\begin{bmatrix} \bar{x}_t \\ \hat{x}_t \end{bmatrix} = \begin{bmatrix} Q \\ J_2^T(C_1 + G_1) \end{bmatrix} x_t \qquad x_t = M_1\bar{x}_t + Q_2\hat{x}_t \tag{A.14}$$

where $[Q_1 \;\; Q_2] = \begin{bmatrix} Q \\ J_2^T(C_1 + G_1) \end{bmatrix}^{-1}$ From the substitution of $x_{t+1,t}$ into (A.13), we can then rewrite the system in terms of a new $\overline{m}$-vector of forward-looking variables $\bar{x}_t$, where $\overline{m}$ =rank$(C_2 + G_2) \leq m$, and $n - \overline{m}$ backward-looking variables $(s_t, \; \hat{x}_t)$:

$$\begin{aligned} & \bar{x}_{t+1,t} + QF_1Q_1\bar{x}_t + [QF_2 \;\; QF_1Q_2] \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ = \;\; & QF_3Q_1\bar{x}_{t-1} + [QF_4 \;\; QF_3Q_2] \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + QF_5\varepsilon_t \end{aligned} \tag{A.15}$$

$$\begin{aligned} & \begin{bmatrix} C_1Q_1 \\ J_2^T(C_1 + G_1)F_1Q_1 \end{bmatrix} \bar{x}_t + \begin{bmatrix} C_2 & C_1Q_2 \\ J_2^T(C_1 + G_1)F_2 & J_2^T(C_1 + G_1)F_1Q_2 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \\ + & \begin{bmatrix} G_1Q_1 \\ J_2^T C_3 Q_1 \end{bmatrix} \bar{x}_{t,t} + \begin{bmatrix} G_2 & G_1Q_2 \\ J_2^T C_4 & J_2^T C_3 Q_2 \end{bmatrix} \begin{bmatrix} s_{t,t} \\ \hat{x}_{t,t} \end{bmatrix} \end{aligned} \tag{A.16}$$

$$
= \begin{bmatrix} C_3Q_1 \\ J_2^T(C_1+G_1)F_3Q_1 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} C_4 & C_3Q_2 \\ J_2^T(C_1+G_1)F_4 & J_2^T(C_1+G_1)F_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix}
$$

$$
+ \begin{bmatrix} C_5 \\ J_2^T(C_1+G_1)F_5 \end{bmatrix} \varepsilon_t
$$

The number of forward-looking states has now usually decreased from m to $\overline{m} \leq m$; while the number of backward-looking states $\bar{s}_t = \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix}$ has increased by the same amount. In addition the relationship $Y_t = V_1 x_t + V_2 s_t$ has changed to

$$
Y_t = V_1 Q_1 \bar{x}_t + \begin{bmatrix} V_2 & V_1 Q_2 \end{bmatrix} \bar{s}_t \tag{A.17}
$$

Finally we redefine $x_t = \overline{x}_t, s_t = \overline{s}_t$. Having done so, the system in (A.15) and (A.16) is now of the form of (A.11) and (A.12), subject to an appropriate redefinition of matrices. Thus, from (A.16), for G_1, and G_2, for example, we have an iterative scheme whereby, in the $(i+1)$th iteration,

$$
G_1^{i+1} = \begin{bmatrix} G_1^i Q_1^i \\ \left(J_2^i\right)^T C_3^i Q_1^i \end{bmatrix}; \quad G_2^{i+1} \begin{bmatrix} G_2^i & G_1^i Q_2^i \\ \left(J_2^i\right)^T C_4^i & \left(J_2^i\right)^T C_3^i Q_2^i \end{bmatrix}
$$

where, eg G_1^i is the value of G_1 in the ith iteration, and $G_1^1 = 0, G_2^1 = 0$.

Repeat this stage until $C_2 + G_2$ is of full rank.

Proof of Theorem 1 for Perfect Information. In the perfect information case, the form (A.11), (A.12) with $s_t = s_{t,t}, x_t = x_{t,t}$ is generated after a finite number of iterations of Stage 3, where the number of iterations cannot exceed the number of variables. The forward looking variables are now x_t and the backward looking variables are s_t and x_{t-1}, and the system can be set up in Blanchard-Kahn form by defining $z_{t+1} = \begin{bmatrix} s_t \\ x_t \end{bmatrix}$. The only additional calculation is to invert $C_2 + G_2$ to obtain the equation for s_t, and to substitute into (A.11). □

Proof of Theorem 1 for Imperfect Information. From this point, we eschew the details of matrix manipulations, as these are much more straightforward to understand conceptually compared with those above.

Stage 5: C_2 non-singular after Stage 4. First form expectations of (A.12), and invert $C_2 + G_2$ to obtain $s_{t,t}$ in terms of $x_{t,t}, x_{t-1,t}, s_{t-1,t}, \varepsilon_{t,t}$. Then substitute this back into (A.12), and invert C_2 to yield an expression for s_t in terms of the above expected values and also $x_t, x_{t-1}, s_{t-1}, \varepsilon_t$. This can be further substituted into (A.11) to yield an expression for $x_{t+1,t}$ in terms of these variables and their expectations. Similarly the measurement equations $m_t = LY_t$ can now be expressed in terms of all these variables. It follows that if we define $z_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, then the system can now be described by (A.3). Note that, since $\dim(s_t) + \dim(x_t) = n$, in this final form $\dim(z_t) = n + rank\,(BB')$.

Stage 6: C_2 singular after Stage 4. We again start from (A.11) and (A.12), and regard x_t as the forward looking variable and (s_t, x_{t-1}) as the backward looking variables. Now advance these equations by changing t to $t+k:\ k = 1, 2, 3, ...$ and take expectations using information at time t, implying that $E_t s_{t+k} = E_t s_{t+k,t+k}$. Because $C_2 + G_2$ is invertible, we can rewrite these equations with just $x_{t+k+1,t}$ and $s_{t+k,t}$ on the LHS. Then the usual Blanchard-Kahn conditions for stable and unstable roots imply a saddlepath relationship of the form

$$x_{t+k+1,t} + N_1 s_{t+k,t} + N_2 x_{t+k,t} = 0 \tag{A.18}$$

where $[I\ N_1\ N_2]$ represents the eigenvectors of the unstable eigenvalues. In particular, this holds for $k = 0$, so if we substitute for $x_{t+1,t} = -N_1 s_{t,t} - N_2 x_{t,t}$ into (A.11), then together with (A.12) we obtain solutions for x_t, s_t in terms of $x_{t,t}, s_{t,t}, x_{t-1}, s_{t-1}, \varepsilon_t$. This is possible, because we have assumed the system is proper i.e., A_1 is invertible[40], and any manipulations of A_1 in the previous stages have been simple linear transformations of it to yield the matrices F_1, F_2, C_1, C_2. In addition, when we take expectations of (A.12) at time t, given that $C_2 + G_2$ is invertible, we obtain an equation for $s_{t,t}$ in terms of $x_{t,t}, s_{t-1,t}, x_{t-1,t}, \varepsilon_{t,t}$. It therefore follows that we can write s_t is terms of these latter variables as well as the variables above (excluding $s_{t,t}$). The same will be true of the the measurements $m_t = LY_t$.

At this point we have expressions for x_t and s_t, without any effect from $x_{t+1,t}$, so in principle we could solve the signal processing problem from this point onwards. However

[40]The algorithm can be reworked without too much much difficulty if for example some of the forward looking equations in (A.1) are of the form $S_0 E_t Y_{t+1} = 0$.

for consistency with the case of C_2 nonsingular, we can retrieve the representation of $x_{t+1,t}$ by substituting for s_t back into (A.11), and then the system has the same structure as that for the case C_2 nonsingular.

Finally, by defining $z_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, the converted form (A.3) becomes

$$\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ FF_4 & FF_3 & FF_2 & FF_1 \\ 0 & 0 & 0 & 0 \\ FF_8 & FF_7 & FF_6 & FF_5 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{A.19}$$

where $G_{13} = -C_2^{-1}C_1$, $G_{12} = C_2^{-1}C_3$, $G_{11} = C_2^{-1}C_4$, $P_1 = C_2^{-1}C_5$, $G_{33} = -F_2G_{13} - F_1$, $G_{32} = -F_2G_{12}+F_3$, $G_{31} = -F_2G_{11}+F_4$, $P_3 = -F_2P_1+F_5$, $FF_1 = -C_2^{-1}G_1+C_2^{-1}G_2(C_2+G_2)^{-1}(C_1+G_1)$, $FF_2 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_3$, $FF_3 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_4$, $FF_4 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_5$, $FF_5 = -F_2FF_1$, $FF_6 = -F_2FF_2$, $FF_7 = -F_2FF_3$ and $FF_8 = -F_2FF_4$. The C and F matrices are the reduction system matrices in (A.15) and (A.16) in the form of (A.11) and (A.12) (i.e., the iterative procedure that ensures invertibility to be achieved).

The measurements $m_t = LY_t$ can be written in terms of the states as $m_t = L(V_1x_t + V_2s_t)$, where V_1, V_2 have been updated by (A.17) through the same reduction procedure as above. Using (A.19), we show that m_t can be rewritten as

$$m_t = \begin{bmatrix} LV_2P_1 & LV_2G_{11} & LV_2G_{12} & LV_1 + LV_2G_{13} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix}$$

$$+ \begin{bmatrix} LV_2FF_4 & LV_2FF_3 & LV_2FF_2 & LV_2FF_1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} \quad \text{(A.20)}$$

So the observations (A.20) can now be cast into the form in (A.4)

$$m_t = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix}$$

where $M_1 = [LV_2P_1 \; LV_2G_{11} \; LV_2G_{12}]$ and $M_2 = LV_1 + LV_2G_{13}$. Similarly, $M_3 = [LV_2FF_4 \; LV_2FF_3 \; LV_2FF_2]$ and $M_4 = LV_2FF_1$. Thus the set-up is as required, with the vector of predetermined variables given by $[\varepsilon_t' \; s_{t-1}' \; x_{t-1}']'$, and the vector of jump variables given by x_t.

This completes the proof by construction for imperfect information. □

Example A.1 (Example of Stage 6 Being Needed for Imperfect Information)**.** Suppose that at the end of Stage 4, there is a system in scalar processes x_t and s_t,

$$x_{t+1,t} + \alpha x_t + s_t = \beta s_{t-1} + \varepsilon_t \qquad x_t - x_{t,t} + s_{t,t} = \gamma s_{t-1} \quad \text{(A.21)}$$

It is clear from examining these equations that they cannot be manipulated into BK form directly. However, if we now advance these equations by k periods and take expectations subject to I_t, one obtains two equations relating $x_{t+k+1,t}, s_{t+k,t}$ to $x_{t+k,t}, s_{t+k-1,t}$. Since this is true for all $k \geq 1$, and provided there is exactly one unstable eigenvalue corresponding to these dynamic relationships, it follows that there must be an expectational saddlepath relationship $x_{t+1,t} = -ns_{t,t}$. Substituting this into the first of the above equations allows us to solve in particular for s_t in terms of $x_t, s_{t,t}, s_{t-1}, \varepsilon_t$; from the second equation we can solve for $s_{t,t}$ in terms of $s_{t-1,t}$, so that we can replace the second equation by an equation for s_t in terms of $x_t, s_{t-1,t}, s_{t-1}, \varepsilon_t$. Redefining $z_{t+1} = s_t$, it is now straightforward to obtain the BK form for the first equation and the new second equation.

A.2 Proof of Lemma 3.5

Proof. Clearly $\tilde{A}$ must be stable, and the other PMIC condition discussed after (24) is that $\tilde{A} - \tilde{A}P^E\tilde{E}'(\tilde{E}P^E\tilde{E}')^{-1}\tilde{E}$ is stable. But if this latter condition does not hold then we have seen from (27) and the discussion following that P^E is not the appropriate solution of the Riccati equation. $\square$

A.3 Proof of Theorem 2

Proof. Using the expressions (30)–(29) for AII, and the invertibility requirement that $\tilde{A} - \tilde{A}\tilde{B}(\tilde{E}\tilde{B})^{-1}\tilde{E}$ has stable eigenvalues, we calculate the latter as the matrix

$$\begin{bmatrix} A - AP^AJ'(EP^AJ')^{-1}E & 0 \\ -F(I - P^AJ'(JP^AJ')^{-1}J)(JB)^{-1}JP^AJ'(EP^AJ')^{-1}E & F(I - B(JB)^{-1}J) \end{bmatrix} \tag{A.22}$$

If $F(I - B(JB)^{-1}J)$ has eigenvalues outside the unit circle, it immediately follows that AII is not E-invertible. If its the eigenvalues are inside the unit circle, it follows that the solution to (19) is $P^A = BB'$; this is because the Convergence Condition for P^A is that $F - FP^AJ'(JP^AJ')^{-1}J = F(I - B(JB)^{-1}J)$ is a stable matrix. Furthermore it follows that $A - AP^AJ'(EP^AJ')^{-1}E = A(I - B(EB)^{-1}E)$, so that (A.22) is a stable matrix as required for invertibility.

To show that invertibility implies that AII and API are equivalent, we note that (17) now implies that $\tilde{z}_t = B\varepsilon_t + (F(I - B(JB)^{-1}J))^t\tilde{z}_0$, which in dynamic equilibrium implies $\tilde{z}_t = B\varepsilon_t$. This implies that $z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t$, and hence that $z_{t+1} = \tilde{z}_{t+1} + z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t + B\varepsilon_{t+1} = Az_t + B\varepsilon_{t+1}$ as in the API case. In addition, from (18), $m_t = Ez_{t,t-1} + E\tilde{z}_t = Ez_t$, also as in the API case. If $F(I - B(JB)^{-1}J)$ is not a stable matrix, then $P^A \neq BB'$, and the overall dynamics of (16)-(17) are of a higher dimension than under API. $\square$

Proof. Writing (18) in terms of lagged state variables and shocks yields a coefficient matrix on the latter given by $EP^AJ'(JP^AJ')^{-1}JB$, and the rank of this is $\leq rank(JB) \leq rank(J)$. This immediately implies that the system is E-non-invertible. $\square$

Proof.

$$
\begin{aligned}
m_t^U &= \begin{bmatrix} UM_1 & UM_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \\
&= \begin{bmatrix} UM_2 G_{22}^{-1} G_{21} & UM_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \\
&= UM_2 G_{22}^{-1} \left(\mathbb{E}_t x_{t+1} - \begin{bmatrix} H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \right) + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix}
\end{aligned} \tag{A.23}
$$

where the last expression comes from substituting from (3). Noting that $\mathbb{E}_t x_{t+1} = -N\mathbb{E}_t z_{t+1}$, and that $\mathbb{E}_t z_{t+1}$ is dependent on $\mathbb{E}_t z_t$ and $\mathbb{E}_t x_t$, it follows that m_t^U is solely dependent on these too. In other words, m_t^U cannot be affected by current shocks ε_t, and is redundant information. □

A.4 Proof of Theorem 3

Proof. We first solve the steady state Riccati equation (27) corresponding to the matrices (30)-(32). It is easy to verify that $\tilde{P^E} = diag(M, P^A)$ where $M = Z - P^A J'(JP^A J')^{-1} JP^A$ and Z satisfies

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + P^A J'(JP^A J')^{-1} JP^A \tag{A.24}$$

For the innovations representation, we use the notation $s_t = [s_{1t}' \; s_{2t}']'$, rather than $s_t = [z_{t,t-1}' \; \tilde{z}_t']'$ as the notation for one-step ahead predictors of the latter will lead to confusion. We can then show that the steady state innovations representation corresponding to (26) is given by

$$\mathbb{E}_t s_{t+1} = \begin{bmatrix} A & AP^A J'(JP^A J')^{-1} J \\ 0 & F - FP^A J'(JP^A J')^{-1} J \end{bmatrix} \mathbb{E}_{t-1} s_t + \begin{bmatrix} AZE'(EZE')^{-1} \\ 0 \end{bmatrix} e_t \qquad e_t = m_t^E - \tilde{E}\mathbb{E}_{t-1} s_t \tag{A.25}$$

or more succinctly

$$\mathbb{E}_t s_{1,t+1} = A\mathbb{E}_{t-1} s_{1,t} + AZE'(EZE')^{-1} e_t \qquad e_t = m_t^E - E\mathbb{E}_{t-1} s_{1t} \tag{A.26}$$

The corresponding VARMA representation arises from defining $\xi_t = \mathbb{E}_{t-1}s_{1t} + ZE'(EZE')^{-1}e_t$ which yields

$$\xi_{t+1} = A\xi_t + ZE'(EZE')^{-1}e_{t+1} \qquad m_t^E = E\xi_t \qquad e_t \sim N(0, EZE') \tag{A.27}$$

The final step follows from comparing (A.27) with (16) and (17); clearly the dynamics of the RE saddle-path solution explained by the innovations process e_t are of smaller dimension that the dynamics yielding the impulse responses. □

A.5 Proof of Corollary 3.1

Proof. From the proof of Theorem 2 we have seen that the MA roots of the VARMA process include the eigenvalues of $F(I - B(JB)^{-1}J)$, while from (16)-(17), the AR roots include the eigenvalues of $F(I - P^A J'(JP^A J')^{-1}J)$. By Corollary 3.4, it follows that one or more of these are reciprocals of one another. Hence the transfer function from shocks to observables incorporates at least one Blaschke factor. It follows that IRFs of structural shocks from the latter cannot be linear combinations of IRFs from VAR residuals, which will only mimic the IRFs from the innovations process. □

A.6 Proof of Corollary 3.2

Proof. The state space equations describing the system, (16), (17), will be unchanged, as these depend on the measurements made by the agents. However if the information set of the econometrician is a subset of that of the agents, this means that in the notation of (2), we have $L^E = WL^A$ for some matrix W. It then follows that the measurement equation of the econometrician, following from (18), is given by $m_t = W(Ez_{t,t-1} + EPD'(DPD')^{-1}D\tilde{z}_t)$. Thus the innovations process and the VARMA as shown in the proof of Theorem 3 are changed merely by replacing E by WE, with the Riccati matrix Z also obtained with the same replacement of E. □

A.7 Proof of Theorem 5

Proof. Both of these results follow from finding the best fit of a linear combination of structural shocks and residuals, which can be expressed as

$$min_{a,b}\mathbb{E}(a'\varepsilon - b'e)^2 \;\; s.t. \;\; a'a = 1 \tag{A.28}$$

Given a, one obtains b via standard OLS techniques, and the problem reduces to minimizing $a'\mathbb{F}^{PI}a$ s.t. $a'a = 1$, with solution a equal to the eigenvector of the minimum eigenvalue of $\mathbb{F}^{PI}$. □

B Example 3: Simple NK Partial Equilibrium Model

Consider a New Keynesian Phillips curve dependent on the real marginal cost mc_t and a mark-up shock $\varepsilon_{1,t}$ *assumed exogenous*

$$\pi_t = \beta\pi_{t+1,t} + \lambda mc_t + \sigma_1\varepsilon_{1,t} \tag{B.1}$$

$$mc_{t+1} = \rho mc_t + \sigma_2\varepsilon_{2,t+1} \tag{B.2}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}$ and $(1-\theta)$ is the constant per period probability that the Calvo contract is reset and $\varepsilon_{i,t} \sim N(0,1)$. This of the Blanchard-Kahn state-space form:

$$\begin{bmatrix} \varepsilon_{1,t+1} \\ mc_{t+1} \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ mc_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ 0 \end{bmatrix}$$

B.1 API Solution

Consider first the solution under Agents' Perfect Information (API). To solve this we need to first go back (12) below from the paper and the saddle path satisfying

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix}(G+H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{B.3}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G+H)$ is the same as the dimension of x_t, then the system will be determinate.

To find N, consider the matrix of eigenvectors W satisfying

$$W(G+H) = \Lambda^U W \tag{B.4}$$

Then, as for G and H, partitioning W conformably with z_t and x_t, from PCL we have

$$N = -W_{22}^{-1} W_{21} \tag{B.5}$$

In our example

$$G+H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda/\beta & 1/\beta \end{bmatrix} \tag{B.6}$$

which has eigenvalues 0, ρ both less than unity and $\frac{1}{\beta} > 1$. Now write the ij element of W as w_{ij}, $i, j \in 1, 3$. Then corresponding to the eigenvalue $1/\beta$ we have the eigenvector

$$[w_{31}\, w_{32}\, w_{33}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1/\beta & -\lambda\beta & 1/\beta \end{bmatrix} = \frac{1}{\beta}[w_{31}\, w_{32}\, w_{33}] \tag{B.7}$$

leaving w_{31}, w_{32}, w_{33} to satisfy

$$\begin{aligned} -w_{33} &= w_{31} \\ \rho w_{32} - \frac{\lambda}{\beta} w_{33} &= \frac{1}{\beta} w_{32} \\ w_{33} \frac{1}{\beta} &= \frac{1}{\beta} w_{33} \end{aligned}$$

w.l.o.g. we can put $w_{33} = 1$. Hence $w_{31} = -1$ and $w_{32} = \frac{\lambda\beta}{\beta\rho - 1}$ giving $N = \begin{bmatrix} \beta & \frac{\lambda}{1-\beta\rho} \end{bmatrix}$

From our general solution procedure above, the following matrices are defined

$$A = F = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}; \quad E = -N = -\begin{bmatrix} \beta & \dfrac{\beta}{1-\beta\rho} \end{bmatrix}; \quad J = [\beta\ \beta]; \quad BB' = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

It follows that under API that

$$\pi_t = \beta\varepsilon_{1,t} + \frac{\lambda}{1-\beta\rho}mc_t \equiv \pi_t^{API} \tag{B.8}$$

Along with (B.2) we then have a **VAR(1)** process in $[\pi_t\ mc_t]'$ and $[\varepsilon_{1,t}\ \varepsilon_{2,t}]'$. In case of Nimark (2008) where $\varepsilon_{1,t} = 0$ this becomes

$$\pi_t = \frac{\lambda}{1-\beta\rho}mc_t \tag{B.9}$$

which is (11) in Nimark (2008).

B.2 Agents' Imperfect Information

We consider **agents' information sets**

1. Perfect Information (API) : $[\varepsilon_{1,t}\ mc_t\ \pi_t]'$

2. Imperfect Information (AII): π_t

3. Imperfect Information (AII): π_{t-1}

Case (1), API solution is above. Next consider Case (2) where agents have AII with π_t observed. Following our API solution in the main text we arrive at

$$\begin{aligned}
mc_t &= \rho mc_{t-1} + \varepsilon_{2,t} \\
\tilde{mc}_t &\equiv mc_t - mc_{t,t-1} = \frac{\rho}{\sigma_1^2 + p}(\sigma_1^2\tilde{mc}_{t-1} - p\varepsilon_{1,t-1}) + \varepsilon_{2,t} && \text{(B.10)} \\
\pi_t &= \beta\left(1 + \frac{\beta\rho p}{(1-\beta\rho)(\sigma_1^2+p)}\right)\varepsilon_{1,t} + \frac{\lambda}{1-\beta\rho}mc_t \\
&- \frac{\beta\rho\sigma_1^2}{(1-\beta\rho)(\sigma_1^2+p)}\tilde{mc}_t && \text{(B.11)}
\end{aligned}$$

where from the main text the agents' steady-state Ricatti equation is given by

$$P^A = FP^AF' - FP^AJ'(JP^AJ')^{-1}JP^AF' + BB' = Q^AP^A(Q^A)' + BB' \tag{B.12}$$

This has a solution

$$P^A = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & p \end{bmatrix} \qquad \text{where } p = \frac{\rho^2 p \sigma_1^2}{\sigma_1^2 + p} + \sigma_2^2$$

noting that $N - G_{22}^{-1} G_{21} = \begin{bmatrix} 0 & \frac{\beta\lambda\rho}{1-\beta\rho} \end{bmatrix}$, This is an **VARMA(1,1)** process in $[\pi_t \; mc_t \, \tilde{mc}_t]'$ and $[\varepsilon_{1,t} \; \varepsilon_{2,t}]'$.

Figure 4 shows the impulse response function following a negative marginal cost shock $\varepsilon_{2,t}$. The greater is σ_1^2, the greater is the difference between AII and API.

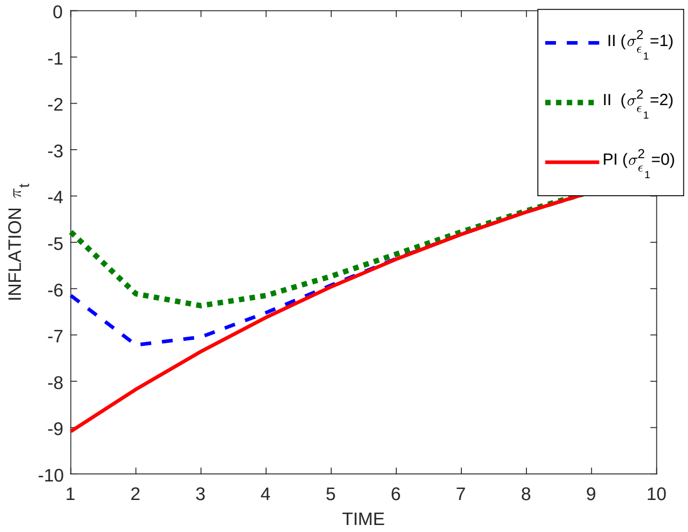

Figure 4: **Inflation Dynamics under Perfect (PI) and Imperfect Information (II)**

To obtain the innovations representation, we first solve for Z in (A.24); it is easy to verify that Z is given by

$$Z = P^E J'(JP^E J')^{-1} JP^E = \frac{1}{\sigma_1^2 + p} \begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix} \begin{bmatrix} \sigma_1^2 & p \end{bmatrix} \tag{B.13}$$

The innovations process that provides the VARMA for π_t, corresponding to (A.27) is then

$$\tilde{s}_{1,t} = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix} \tilde{s}_{1,t-1} + \frac{1}{\beta\sigma_1^2 + \frac{\beta}{1-\beta\rho} p} \begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix} \hat{\varepsilon}_t$$

$$\pi_t = \begin{bmatrix} \beta & \frac{\lambda}{1-\beta\rho} \end{bmatrix} \tilde{s}_{1,t}$$

from which it is readily seen that the system is back to a **VAR(1)** process as under PI. This illustrates Theorem 3 of our paper: even though II adds more persistence than under PI, the innovations process dynamics has the same dimensions in each case.

B.3 Nimark (2008)

Now consider the Nimark (2008) AII information case (3) and with only one shock $\varepsilon_{2,t}$. A fundamental difference is that he does not start with (B.1), which he argues in the NK standard model only under PI, but rather a forward-looking Phillips Curve with higher order expectations (6) derived from a model with idiosyncratic shocks. Does his solution in the limit as the latter dominate the aggregate component tend to our solution which is an II solution of (B.1)?

In the Nimark example the information set is $m_t^A = \pi_{t-1}$ and $\varepsilon_{1,t} = 0$. Then consistent with these information assumptions, the NK Phillips curve becomes

$$\pi_{t,t} = \beta\pi_{t+1,t} + \lambda mc_{t,t} \tag{B.14}$$

where we recall that $\lambda \equiv \frac{(1-\theta)(1-\beta\theta}{\theta}$.

Augmenting the state vector the state-space form is now:

$$\begin{bmatrix} mc_{t+1} \\ \pi_t \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} mc_t \\ \pi_{t-1} \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\lambda}{\beta} & 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} mc_{t,t} \\ \pi_{t-1,t} \\ \pi_{t,t} \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t+1} \\ 0 \\ 0 \end{bmatrix} \tag{B.15}$$

giving eigenvalues 0, ρ and $\frac{1}{\beta} > 1$. The eigenvector associated with the eigenvalue outside the unit circle

$$[w_{31}\, w_{32}\, w_{33}] \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 1 \\ -\lambda/\beta & 0 & 1/\beta \end{bmatrix} = \frac{1}{\beta}[w_{31}\, w_{32}\, w_{33}] \tag{B.16}$$

gives, w.l.o.g. $[w_{31}\, w_{32}\, w_{33}] = [\frac{\lambda}{\rho\beta-1}\, 0\, 1]$ and $N = [\frac{\lambda}{\rho\beta-1}\, 0]$. The agent's perfect information solution is therefore

$$\pi_t^{API} = \frac{\lambda}{1-\rho\beta} mc_t \tag{B.17}$$

For AII we need the matrices

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2G_{22}^{-1}G_{21} \tag{B.18}$$

$$A = G_{11} + H_{11} - (G_{12} + H_{12})N \qquad E = M_1 + M_3 - (M_2 + M_4)N \tag{B.19}$$

capturing intrinsic dynamics in the system. For our example these are

$$F = \begin{bmatrix} \rho & 0 \\ \lambda & 0 \end{bmatrix} \qquad E = J = [0\ 1] \qquad A = \begin{bmatrix} \rho & 0 \\ \frac{\lambda}{1-\beta\rho} & 0 \end{bmatrix} \tag{B.20}$$

Turning to the Riccati equation (B.12) it is easy to show a solution is

$$P^A = \begin{bmatrix} \frac{1}{1-\rho^2} & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^A = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \qquad P^AJ'(JP^AJ')^{-1}J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad AP^AJ'(JP^AJ')^{-1}J = 0 \tag{B.21}$$

It follows that $z_{t,t-1} = 0$, and the second element of $\tilde{z}_t = 0$. Hence

$$\begin{aligned} \pi_t &= -Nz_{t,t-1} - G_{22}^{-1}G_{21}\tilde{z}_t - (N - G_{22}^{-1}G_{21})P^AJ'(JP^AJ')^{-1}J\tilde{z}_t \\ &= 0 \end{aligned} \tag{B.22}$$

where $\tilde{z}_t' = [\tilde{mc}_t, \tilde{\pi}_{t-1}]'$.

Evidently this is different from Nimark's purported solution

$$\pi_t^{AII} = \pi_t^{API} + \lambda(\theta - ((1-\beta\rho))^{-1})\tilde{mc}_t \tag{B.23}$$

That the latter is an error is evident from his derived expression for inflation (his equation (6)), which is dependent only on expectations. Since in the limit of infinite variance idiosyncratic shocks, inflation cannot be driven by any shocks at all, it follows that any expectations based on observations of lagged inflation must be 0, and hence inflation is 0.[41]

[41]The solution in Nimark's equation (13) is exactly the solution of (B.15) when its final equation is

To elaborate, Nimark's representation of the solution is given by the hierarchy of higher order expectations:

$$\pi_t = (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k)} \tag{B.24}$$

where

$$x_{t|s}^{(k)} \equiv \int \mathcal{E}[x_{t|s}^{(k-1)}|I_s(j)]dj \tag{B.25}$$

and $I_t(j)$ is the Calvo price-setting firm's information at time t. When the only observation at time t is π_{t-1}, then there is a solution $\pi_t = 0$. This solution is completely consistent with $\pi_{t+1,t} = mc_{t,t} = 0$.

One can very easily check the case when $\rho = 0$. Nimark's solution in his equation (13) then asserts that π_t is proportional (in our notation) to $\varepsilon_{2,t}$ i.e. $\pi_t = \gamma\varepsilon_{2,t}$. In that case observation at time t of π_{t-1} yields information on $\varepsilon_{2,t-1}$, but this sheds no light on $mc_t = \varepsilon_{2,t}$ so that the best estimate of this at time t is therefore 0. Likewise, the best estimate of $\pi_{t+1} = \gamma\varepsilon_{2,t+1}$ at time t is 0, which implies by Nimark's equation (6) that $\pi_t = 0$.

C Extending the Sims Solution to the Imperfect Information Case

Sims (2002) sets up the model in the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi\eta_t \tag{C.1}$$

where y_t includes and forward-looking expectations, and η_t satisfy $E_t\eta_{t+1} = 0$; Γ_0 is in general singular. He then computes a QZ decomposition for Γ_0, Γ_1 such that the unstable part of the system is given by $Z_2 y_t$, which satisfies

$$\Lambda_{22} Z_2 y_t = \Omega_{22} Z_2 y_{t-1} + Q_2(\Psi\varepsilon_t + \Pi\eta_t) \tag{C.2}$$

where Λ_{22} is in general singular. $Z_2 y_{t-1}$ is solved forwards in time; perfect information

dependent on mc_t instead of $mc_{t,t}$.

then implies

$$E_{t-1}Z_2y_{t-1} = E_tZ_2y_{t-1}(= Z_2y_{t-1}) \tag{C.3}$$

which in turn implies that

$$Q_2(\Psi\varepsilon_t + \Pi\eta_t) = 0 \ \text{ and } \ Z_2y_{t-1} = 0 \tag{C.4}$$

Thus $Z_2y_t = 0$ represents the saddlepath relationship. Furthermore, assuming that the terms in η_t in the rest of the system defined by the QZ decomposition are linearly dependent on $Q_2\Pi$, it is then easy to solve for the remaining transformed states of the system as a vector autoregression in ε_t.

For the *imperfect information case*, (C.3) no longer holds. If we assume that η_t is known at time t, it follows that $E_{t-1}Z_2y_{t-1} = 0$, but

$$E_tZ_2y_{t-1} = -\Omega_{22}^{-1}Q_2(\Psi E_t\varepsilon_t + \Pi\eta_t) \tag{C.5}$$

It therefore follows that the remaining states Z_1y_t will be dependent on Z_1y_{t-1}, ε_t and in addition $E_tZ_2y_{t-1}$ and $Q_2\Psi E_t\varepsilon_t$, so that the overall solution will be as complicated as that derived by PCL. In particular, one has to define the updating equation for $E_tZ_2y_{t-1}$ in terms of $E_{t-1}Z_2y_{t-1}$ and the observations at time t, and solve for this in dynamic equilibrium.

D Equivalence of Various State Space Models

We show that all of the state-space models that are used in the statistics, control theory and econometrics literature can be represented by that used in the main text.

The usual model used in the statistics literature, Model 1, includes measurement error η_{1t}

$$s_{t+1} = A_1s_t + B_1\varepsilon_{1,t+1} \qquad m_t = C_1v_t + D_1\eta_{1t} \tag{D.1}$$

In the control theory literature, with possible correlation between ε_{2t} and measurement error η_{2t}, Model 2 is given by

$$w_{t+1} = A_2w_t + B_2\varepsilon_{2t} \qquad m_t = C_2w_t + D_2\eta_{2t} \tag{D.2}$$

In Fernandez-Villaverde *et al.* (2007) and much of the econometrics literature, Model 3 is given by

$$x_{t+1} = A_3 x_t + B_3 \varepsilon_{3,t+1} \qquad \text{(i.e., } x_t = A_3 x_{t-1} + B_3 \varepsilon_{3,t}\text{)} \qquad m_t = C_3 x_{t-1} + D_3 \varepsilon_{3t} \tag{D.3}$$

For Model 1, add η_{1t} to the state space, so that it can be rewritten as

$$\begin{bmatrix} \eta_{1,t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ v_t \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t+1} \\ \varepsilon_{1,t+1} \end{bmatrix} \qquad m_t = \begin{bmatrix} D_1 & C_1 \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ v_t \end{bmatrix} \tag{D.4}$$

For Model 2, if $D_2 = 0$, then the statistical properties of w_t are identical whether we date the shock as ε_{2t} or $\varepsilon_{2,t+1}$; thus Model 2 is equivalent to the main text model when $D_2 = 0$. Otherwise, include ε_{2t} and η_{2t} into the state space

$$\begin{bmatrix} \varepsilon_{2,t+1} \\ \eta_{2,t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_2 & 0 & A_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t} \\ \eta_{2,t} \\ w_t \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{2,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix} \qquad m_t = \begin{bmatrix} 0 & D_2 & C_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{2,t} \\ \eta_{2,t} \\ w_t \end{bmatrix} \tag{D.5}$$

Model 3 can be written in the form of the main text model by appending both ε_{3t} and x_{t-1} to the state space

$$\begin{bmatrix} \varepsilon_{3,t+1} \\ x_t \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{3,t} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ B_3 \end{bmatrix} \varepsilon_{3,t+1} \qquad m_t = \begin{bmatrix} D_3 & C_3 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{3,t} \\ x_{t-1} \\ w_t \end{bmatrix} \tag{D.6}$$

E Recoverability

A recent innovation in the economics literature by Chahrour and Jurado (2017) is the notion of recoverability, which they point out is a generalization of much earlier work by Kolmogorov (see Shiryayev (1992)), and which relates to situations for which the shocks are non-fundamental, so that the system of dynamic equations is non-invertible. We shall be calling on this notion subsequently because when the imperfect information solution differs from that of the perfect information solution, then the former will be characterized by non-invertibility (or non-fundamentalness of the shocks). The main point that they make is that if the VARMA is known, then it is possible (under mild conditions) to recover the values of all the shocks to have affected the VARMA process using the data, assuming observations over *all time*, as opposed to data only up to time t as available to economic agents in the model. In particular what this means is that for a finite set of data, one can obtain an accurate estimate of shocks that have taken place around the middle of the dataset.

To be more specific, suppose that the VARMA process is fully invertible, then the residuals as calculated above will converge to the true values of the shocks, so that the estimate of a shock at time t will be calculated using all past values of the observations. We illustrate with an example.

E.1 Example 4: Fundamental and non-Fundamental MA Processes

For example, if measurements $\{m_t^E : t \geq -\infty\}$ are generated by the MA(1) process

$$m_t^E = \varepsilon_t - \alpha\varepsilon_{t-1} = (1-\alpha L)\varepsilon_t, \quad -1 < \alpha < 1, \;\; \varepsilon_t \sim N(0, \sigma^2) \tag{E.1}$$

where L is the lag operator, then the root of $(1-\alpha L)$ lies outside the unit circle and the process is fundamental.[42] Then $\varepsilon_t = \sum_{s=0}^{\infty} \alpha^s m_{t-s}^E$. For a finite number of observations starting at $t = 0$, truncating this sum at $s = t$ will achieve a very close approximation (with probability 1) for values of t that are large enough to ensure that the variance of the untruncated terms, which equals $\alpha^{2t}\sigma^2/(1-\alpha^2)$ is below a certain threshold. However if $\alpha > 1$, then the above representation is non-fundamental and cannot converge. If instead we write the lag operator representation of ε_t as $\varepsilon_t = m_t^E/(1-\alpha L)$ as $\varepsilon_t = -\alpha^{-1}L^{-1}m_t^E/(1-\alpha^{-1}L^{-1})$, then we can rewrite the representation of the shocks as

$$\varepsilon_t = -\sum_{s=1}^{\infty} \alpha^{-s} m_{t+s}^E \tag{E.2}$$

[42] An MA process $m_t^E = \Phi(L)\varepsilon_t$ is a fundamental representation if the roots of $\Phi(L)$ lie outside the complex unit circle (see, for example, Lippi and Reichlin (1994) and Kilian and Lutkepohl (2017)).

Thus recovering the shocks requires summing over *future* values of the observations. Clearly for a finite sample of length T one cannot obtain an accurate approximation to the most recent shock ε_T, but one can obtain a good approximation to the earliest shocks provided that T is large enough.

One can readily extend this to the MA(2) case $m_t^E = (1-\alpha L)(1-\beta L)\varepsilon_t$ when $-1 < \beta < \alpha < 1$. Then the process is fundamental and we have

$$\varepsilon_t = \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha L} - \frac{\beta}{1-\beta L}\right) m_t^E = \frac{1}{\alpha-\beta}\left(\sum_{s=0}^{\infty}\alpha^{s+1}m_{t-s}^E - \sum_{s=0}^{\infty}\beta^{s+1}m_{t-s}^E\right) \tag{E.3}$$

When however $-1 < \alpha < 1 < \beta$, we can rewrite the expression for the shock as

$$\varepsilon_t = \frac{1}{\beta-\alpha}\left(-\frac{\alpha}{1-\alpha L} + \frac{L^{-1}}{1-\beta^{-1}L^{-1}}\right) m_t^E = \frac{1}{\beta-\alpha}\left(\sum_{s=0}^{\infty}\alpha^{s+1}m_{t-s}^E - \sum_{s=1}^{\infty}\beta^{-s+1}m_{t+s}^E\right) \tag{E.4}$$

so that recovering the shocks requires summing over *both past and future* values of the observations. For finite samples the approximating values of shocks at the beginning and end of the sample will be a poor fit to the true values.

Similarly when $-1 < \beta < 1 < \alpha$, we have

$$\varepsilon_t = \frac{1}{\alpha-\beta}\left(-\sum_{s=0}^{\infty}\beta^{s+1}m_{t-s}^E - \sum_{s=1}^{\infty}\alpha^{-s+1}m_{t+s}^E\right) \tag{E.5}$$

Finally when $-1 < \beta < \alpha < 1$, we can rewrite the expression for the shock as

$$\varepsilon_t = \frac{1}{\alpha-\beta}\left(-\frac{L^{-1}}{1-\alpha^{-1}L^{-1}} - \frac{L^{-1}}{1-\beta^{-1}L^{-1}}\right) m_t^E = \frac{1}{\alpha-\beta}\left(-\sum_{s=0}^{\infty}\alpha^{-s+1}m_{t+s}^E - \sum_{s=1}^{\infty}\beta^{-s+1}m_{t+s}^E\right) \tag{E.6}$$

so that recovering the shocks requires summing over *only future* values of the observations. Again for finite samples the approximating values of shocks at the beginning and end of the sample will be a poor fit to the true values.

E.2 Blaschke Factors and Spectral Factorization

If a square non-invertible system of n stationary measurements and n shocks in each period is estimated, then although the parameters of the system can be consistently estimated using maximum likelihood, the innovations process (i.e., the residuals) will nevertheless correspond to those of the statistically equivalent invertible system. They cannot therefore be matched to a linear transformation of the structural shocks, and the same will automatically hold true when a VAR approximation to the system is estimated, since by definition the latter is invertible. The literature, summarized by Kilian and Lutkepohl (2017) suggests using Blaschke factors on the lag operator representation of the VAR in order to 'flip' roots of the MA process from invertible to non-invertible.

To see how this works first consider the general MA process $m_t^E = \Phi(L)\varepsilon_t$ assumed to be fundamental and write

$$m_t^E = \Phi(L)\varepsilon_t = \Phi(L)B(L)B(L)^{-1}\varepsilon_t \equiv \Phi(L)^*\varepsilon_t^* \tag{E.7}$$

where $\varepsilon_t^* = B(L)^{-1}\varepsilon_t$ and $\Phi(L)^* = \Phi(L)B(L)$. Then Lippi and Reichlin (1994) show that Φ^* has roots inside the complex unit circle (so that $m_t^E = \Phi(L)^*\varepsilon_t^*$ is non-fundamental) if $B(L)$ is chosen to be a 'Blaschke matrix' which has two properties (i) all roots inside the complex unit circle and (ii) $B(L)^{-1} = B^*(L^{-1})$ where the asterik denotes the conjugate transpose. Then corresponding to our MA(2) fundamental example $\Phi(L) = (1-\alpha L)(1-\beta L)$ above with $-1 < \alpha,\ \beta < 1$ we have three non-fundamental representations $\Phi(L)B(L)$ corresponding to the Blaschke factors:

$$-1 < \alpha < 1 < \beta \quad : \quad B(L) = \frac{L-\alpha}{1-\alpha L} \tag{E.8}$$

$$-1 < \beta < 1 < \alpha \quad : \quad B(L) = \frac{L-\beta}{1-\beta L} \tag{E.9}$$

$$-1 < \alpha, \beta < 1 \quad : \quad B(L) = \left(\frac{L-\alpha}{1-\alpha L}\right)\left(\frac{L-\beta}{1-\beta L}\right) \tag{E.10}$$

For the four possible combinations of α and β one MA(2) representation will be fundamental and the other three non-fundamental. Only the fundamental one will be captured by the data VAR estimation. If the econometrician is estimating α, β she will be confronted with three non-fundamental and one fundamental processes with identical statistical properties (i.e., the same first and second moments). It therefore follows that one can only use recoverability to obtain the structural shock unambiguously if the four cases (E.3)–(E.6) can be separated by the econometrician by prior information on the location of α and β.

E.3 A Test of Fundamentalness

Lippi and Reichlin (1994), Fernandez-Villaverde *et al.* (2007), Kilian and Lutkepohl (2017) and others, have pointed out that non-invertibility is a missing information problem arising from econometricians not using the appropriate measurements. Choosing the right measurements may then alleviate the problem. Closely related to this idea and also to recoverability is a recent paper by Canova and Sahneh (2017), that shows how to test the residuals of a VAR model for fundamentalness. Suppose that a VARMA process m_t^E in shocks ε_t is estimated in the VAR form $\Phi(L)m_t^E = u_t$, where u_t are the residuals; then a linear transformation is applied to u_t in order to attempt to recover an approximation e_t to the structural shocks ε_t. However in principle there is no way that one can determine whether e_t is a linear transformation of the structural shocks ε_t using the VAR alone.

But suppose that there is an additional measurement m_{2t}^E available to the econometrician of the form $m_{2t}^E = \Theta_1(L)\varepsilon_t + \Theta_2(L)\varepsilon_{2t}$, which is dependent on the same shocks ε_t as the main variables m_t^E, and some additional shocks ε_{2t}. If there is no invertibility problem for m_t^E estimated as a VAR, then m_{2t}^E can be rewritten (as $t \to \infty$) as

$$m_{2t}^E = \Theta_1(L)e_t + \Theta_2(L)\varepsilon_{2t} \tag{E.11}$$

If there is an invertibility problem then (E.11) no longer applies, because at least one element of ε_t depends on future values of e_t via one or more Blaschke factors[43]. Thus conducting a standard Granger causality test of whether m_{2t}^E depends on future values of the recorded residuals e_t is sufficient to deduce whether the latter are fundamental or not.

F An Implication for Estimation

The innovations representation is closely connected to the use of the Kalman filter in the estimation of linear models. Suppose that the system is given by (20). Then the

[43]Suppose for example that $y_t = (1 - \alpha^{-1}L)\varepsilon_t$, where $\alpha < 1$, so that it is non-invertible. After this is estimated as a finite VAR, it can then be approximately written as $y_t = (1 - \alpha L)e_t$. It follows that $\varepsilon_t = \frac{(1-\alpha L)}{(1-\alpha^{-1}L)}e_t = \frac{-\alpha L^{-1}(1-\alpha L)}{(1-\alpha L^{-1})}e_t$, so that it is dependent on future values of e.

loglikelihood, lnL, for the system is given by

$$2lnL = -Trln(2\pi) - \sum_{t=1}^{T}[\ln\det(\text{cov}(e_t)) + e_t'(\text{cov}(e_t))^{-1}e_t] \tag{F.12}$$

where the innovations process $e_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$, T is the number of time periods and r is the dimension of m_t^E.

We use time varying version of (A.26) and (A.24) in order to evaluate the loglikelihood (F.12) for any given set of parameters, and define $\bar{v}_t = \bar{s}_{1t}$:

$$\bar{v}_{t+1} = A\bar{v}_t + AZ_tE'(EZ_tE')^{-1}e_t \qquad e_t \equiv m_t^E - E\bar{v}_t$$

$$Z_{t+1} = AZ_tA' - AZ_tE'(EZ_tE')^{-1}EZ_tA' + P^AJ'(JP^AJ')^{-1}JP^A \tag{F.13}$$

Initial values are $\bar{v}_1 = 0$, with Z_1 satisfying $Z_1 = AZ_1A' + P^AJ'(JP^AJ')^{-1}JP^A$. We note that it is inappropriate for the matrix P^A in (F.13) to be time-varying. This is because there is no guarantee that the matrix F has all its eigenvalues stable, which would mean that the conventional initial value, which assumes that the system is in a stochastic steady state, cannot be obtained. Instead we make the assumption that the overall system is in stochastic steady state, and the time-varying Riccati equation is only relevant for the innovations process e_t.

Recall what is meant by over-identification, or the singularity problem in estimation: if the number of observables exceeds the number of shocks, then the likelihood function will be singular[44]. We then obtain a further result:

Theorem 6. *If $rank(J) <$ the number of observables, then the RE saddle-path solution under imperfect information is singular.*

Proof of Theorem 6. If J is of full row rank, then it is easy to see that in general $P^AJ'(JP^AJ')^{-1}JP^A$ will have the same rank as J. If J is not of full row rank, then rank$(P^AJ'(JP^AJ')^{-1}JP^A) \leq$rank$(J)$ i.e., the 'effective' number of shocks is less than the number of observables. In such a case, we can solve for P^A by writing $J = UJ_1$, where J_1 has a smaller number of rows than J, and is of full row rank, and $U'U = I$. Then

[44] In the simplest case, for two regression equations that depend on the same single shock, the covariance matrix of the shocks cannot, as is required, be inverted.

an appropriate likelihood function is obtained by changing the observables from m_t^E to $U'm_t^E$. An alternative of course is to incorporate measurement error into the system, but then this would make the system non-square. □

Note that these results are only relevant when the measurements satisfy invertibility if agents were to have perfect information. If any of the measurements are lagged, then Theorem 6 does not apply.

G The RBC Model

We first consider the standard RBC model with a non-zero growth steady state. Then consider a simplified special case suitable for an analytical solution.

G.1 The Full Model

For the household:

$$\text{Utility}: U_t = U(C_t, L_t) \tag{G.14}$$

$$\text{Euler Consumption}: U_{C,t} = \beta R_t \mathbb{E}_t \left[U_{C,t+1} \right] \tag{G.15}$$

$$\text{Labour Supply}: \frac{U_{H,t}}{U_{C,t}} = -\frac{U_{L,t}}{U_{C,t}} = -W_t \tag{G.16}$$

$$\text{Leisure and Hours}: L_t \equiv 1 - H_t \tag{G.17}$$

where C_t is real consumption, L_t is leisure, R_t is the gross real interest rate set in period t to pay out interest in period $t+1$, H_t are hours worked and W_t is the real wage.

The Euler consumption equation, (G.15), where $U_{C,t} \equiv \frac{\partial U_t}{\partial C_t}$ is the marginal utility of consumption and $\mathbb{E}_t[\cdot]$ denotes rational expectations based on the agents' information set, describes the optimal consumption-savings decisions of the household. It equates the marginal utility from consuming one unit of income in period t with the discounted marginal utility from consuming the gross income acquired, R_t, by saving the income. For later use define $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the *real stochastic discount factor* over the interval $[t, t+1]$. (G.16) equates the real wage with the marginal rate of substitution between consumption and leisure.

Output and the firm behaviour is summarized by:

$$\begin{aligned}
\text{Output} : Y_t &= F(A_t, H_t, K_t) && \text{(G.18)} \\
\text{Labour Demand} : F_{H,t} &= W_t && \text{(G.19)} \\
\text{Capital Demand} : 0 &= \mathbb{E}_t\left[\Lambda_{t+1}(F_{K,t+1} - R_t + 1 - \delta)\right] && \text{(G.20)} \\
\text{Stochastic Discount Factor} : \Lambda_t &= \beta \frac{U_{C,t+1}}{U_{C,t}} && \text{(G.21)}
\end{aligned}$$

(G.18) is a production function where K_t is *beginning-of-period* t capital stock. Equation (G.19), where $F_{H,t} \equiv \frac{\partial F_t}{\partial H_t}$, equates the marginal product of labour with the real wage. (G.20), where $F_{K,t} \equiv \frac{\partial F_t}{\partial K_t}$, equates the marginal product of capital with the cost of capital. The model is completed with an output equilibrium, law of motion for capital and a balanced budget constraint with fixed lump-sum taxes.

$$\begin{aligned}
Y_t &= C_t + G_t + I_t && \text{(G.22)} \\
I_t &= K_{t+1} - (1-\delta)K_t && \text{(G.23)} \\
G_t &= T_t && \text{(G.24)}
\end{aligned}$$

We now generalize the model by adding the Smets and Wouters (2007) form of investment adjustment costs to the RBC model. The law of motion for capital becomes

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + (1 - S(X_t))I_t\,;\;\; S',\, S'' \geq 0\,;\; S(1) = S'(1) = 0 \\
X_t &\equiv \frac{I_t}{I_{t-1}}
\end{aligned}$$

We introduce capital producing firms that at time t convert I_t of output into $(1 - S(X_t))I_t$ of new capital sold at a real price Q_t and then maximize with respect to $\{I_t\}$ expected discounted profits. The first-order condition for the capital producers is

$$Q_t(1 - S(X_t) - X_t S'(X_t)) + E_t\left[\Lambda_{t,t+1}\, Q_{t+1} S'(X_{t+1}) X_{t+1}^2\right] = 1$$

Demand for capital by the wholesale firm owned by households is now given by

$$1 = R_t \mathbb{E}_t[\Lambda_{t,t+1}]$$

$$= \quad \frac{\mathbb{E}_t\left[\Lambda_{t,t+1}[(1-\alpha)\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)Q_{t+1}]\right]}{Q_t} \equiv \mathbb{E}_t[\Lambda_{t,t+1}R_{K,t+1}] \tag{G.25}$$

In (G.25) the right-hand-side is the discounted gross return to holding a unit of capital in from t to $t+1$. The left-hand-side is the discounted gross return from holding bonds, the opportunity cost of capital. Note that without investment costs, $S = 0$, $Q_t = 1$ (G.25) reduces to the standard Euler equation. We complete this set-up with the functional form for investment adjustment, $S(X) = \phi_X(X_t - 1)^2$, which completes the RBC model augmented with capital producers and monetary policy.

We now specify functional forms for production and utility and AR(1) processes for exogenous variables A_t and G_t. For production we assume a Cobb-Douglas function. The consumers' utility function is non-separable and consistent with a balanced growth path when the inter-temporal elasticity of substitution, $1/\sigma$ is not unitary. These functional forms, the associated marginal utilities and marginal products, and exogenous processes are given by

$$F(A_t, H_t, K_t) = (A_t H_t)^\alpha K_t^{1-\alpha} \tag{G.26}$$

$$F_H(A_t, H_t, K_t) = \frac{\alpha Y_t}{H_t} \tag{G.27}$$

$$F_K(A_t, H_t, K_t) = \frac{(1-\alpha)Y_t}{K_t} \tag{G.28}$$

$$\log A_t - \log \bar{A}_t = \rho_A(\log A_{t-1} - \log \bar{A}_{t-1}) + \varepsilon_{A,t} \tag{G.29}$$

$$\log G_t - \log \bar{G}_t = \rho_G(\log G_{t-1} - \log \bar{G}_{t-1}) + \varepsilon_{G,t} \tag{G.30}$$

$$U_t = \frac{(C_t^{(1-\varrho)} L_t^{\varrho})^{1-\sigma} - 1}{1-\sigma} \tag{G.31}$$

$$U_{C,t} = (1-\varrho)C_t^{(1-\varrho)(1-\sigma)-1}(1-H_t)^{\varrho(1-\sigma)} \tag{G.32}$$

$$U_{H,t} = -\varrho C_t^{(1-\varrho)(1-\sigma)}(1-H_t)^{\varrho(1-\sigma)-1} \tag{G.33}$$

(G.14) – (G.33) describe an equilibrium in U_t, C_t, W_t, Y_t, L_t, H_t, K_t, I_t, R_t, T_t, given A_t and G_t where for the latter we assume AR(1) processes about steady states $\bar{A}, \bar{G}$ driven by zero mean iid shocks $\varepsilon_{A,t}$ and $\varepsilon_{G,t}$.

Figures 5 and 6 show the deterministic IRFs in response to unanticipated shocks A_t and G_t.

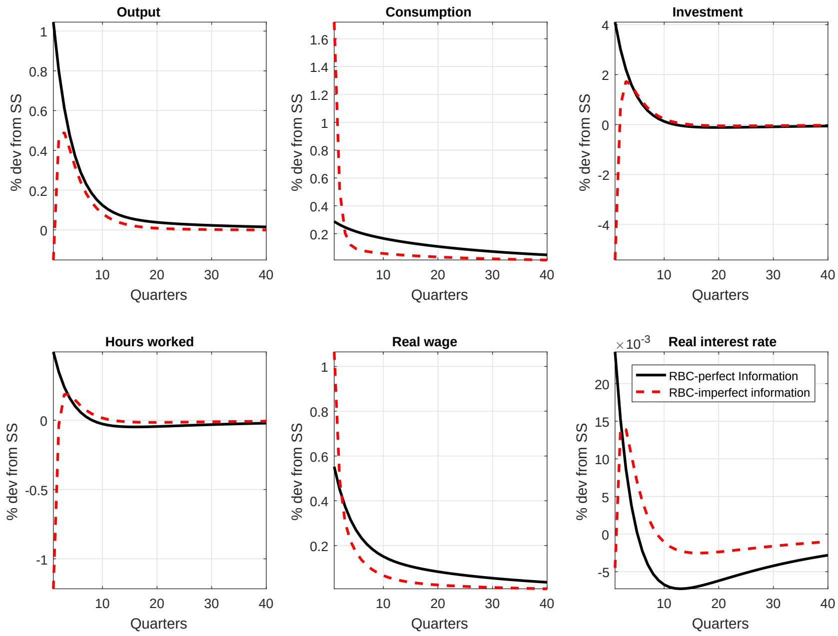

Figure 5: Model 1 Impulse Responses to a Technology Shock, A_t. Observables Y_t, R_t

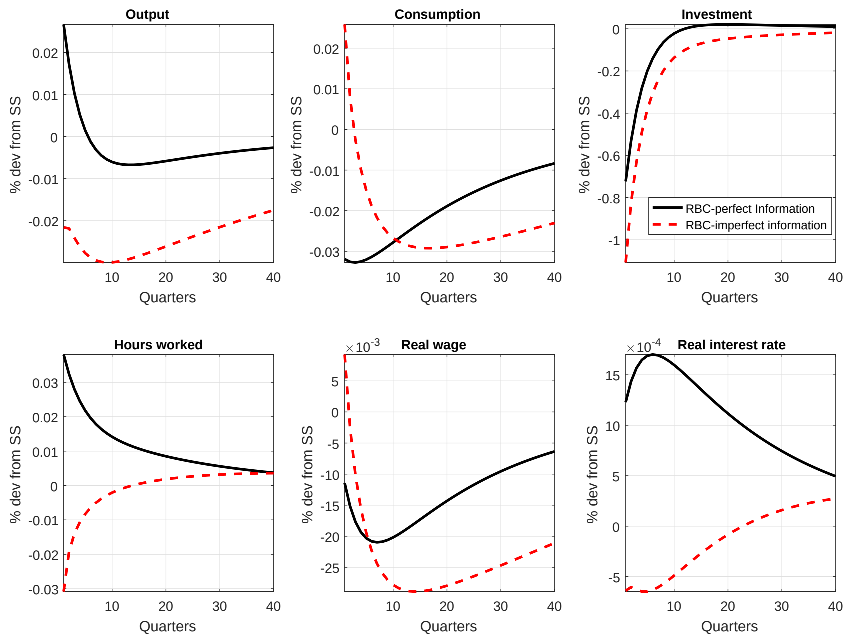

Figure 6: Model 1 Impulse Responses to a Government Spending Shock, G_t. Observables Y_t, R_t

Information Set	E-Invertibility under API?	A-Invertibility?	Notes	Eigenvalues of $\mathbb{F}^{PI}$ and $\mathbb{F}^{PI}$
RBC Case 1: $\sigma_c = 0.3$ and $\alpha = 0.6$				
$(C_t, I_t), (C_t, R_t), (C_t, R_{K,t})$ $(I_t, R_t), (I_t, H_t), (I_t, R_{K,t})$ $(H_t, R_t), (W_t, R_t), (C_t, R_{K,t})$	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0, 0]$
$(Y_t, C_t), (C_t, H_t)$ $(Y_t, H_t), (C_t, W_t)$ $(Y_t, I_t), (H_t, W_t)$ $(Y_t, W_t), (Y_t, R_t)$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) > 0$
$(Y_t, R_{K,t}), (H_t, R_{K,t})$ $(W_t, R_{K,t})$	NO	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) > 0$
RBC Case 2: $\sigma_c = 2$ and $\alpha = 0.6$				
$(C_t, I_t), (H_t, R_t)$ $(I_t, R_t), (I_t, W_t)$ $(I_t, R_{K,t})$	YES	YES	E, EB, J,JB are of full rank $A(I - B(EB)^{-1}E)$ is stable $F(I - B(JB)^{-1}J)$ is stable	$eig(\mathbb{F}^{PI}) \equiv eig(\mathbb{F}^{II}) = [0, 0]$
$(Y_t, C_t), (C_t, H_t), (I_t, H_t)$ $(Y_t, H_t), (C_t, W_t), (C_t, R_t)$ $(Y_t, I_t), (H_t, W_t), (W_t, R_t)$ $(Y_t, W_t), (Y_t, R_t)$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) > 0$
$(Y_t, R_{K,t}), (C_t, R_{K,t})$ $(H_t, R_{K,t}), (W_t, R_{K,t})$	NO	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is **not stable** J,JB are of full rank $F(I - B(JB)^{-1}J)$ is **not stable**	$eig(\mathbb{F}^{PI}) > 0$ $eig(\mathbb{F}^{II}) > 0$
$(R_t, R_{K,t})$	YES	NO	E, EB are of full rank $A(I - B(EB)^{-1}E)$ is stable J,JB are **rank deficient**	$eig(\mathbb{F}^{PI}) = [0, 0]$ $eig(\mathbb{F}^{II}) > 0$

Table 4: **Exact and Approximate Invertibility Checks for Full RBC Model**

Note: Check Conditions in Lemma 4.2 and Theorem 2. This is the full RBC model with investment adjustment costs and variable hours. We consider two cases for $(\sigma_c, \alpha) = (0.3, 0.6)$ and $(\sigma_c, \alpha) = (2, 0.6)$.

G.2 A Special Case in Linearized Form

The analytical example in Section 5, taken from Campbell (1994), is a linearized form of a special case of the full RBC model for which hours H_t are constant and normalized at unity, $G_t = 0$ leaving only one technology shock process and there are no investment adjustment costs so $S_t(X_t) = S_t'(X_t) = 0$ and $Q_t = 1$.

Then defining lower case variables $x_t \equiv \log(X_t/X) \approx \frac{X_t - X}{X}$ where X is the zero-growth steady state of X_t and linearising (G.25), (G.18), (G.15) and (G.25) we have

$$\begin{aligned} k_{t+1} &= (1-\delta)k_{t-1} + \frac{Y}{K}y_t - \frac{C}{K}c_t & \text{(G.34)} \\ y_t &= \alpha a_t + (1-\alpha)k_t & \text{(G.35)} \\ c_{t+1} &= c_t + \sigma r_t & \text{(G.36)} \\ r_t &= \mathbb{E}_t r^K_{t+1} = \frac{\alpha(1-\alpha)}{1+r}\left(\frac{A}{K}\right)^{\alpha} \mathbb{E}_t(a_{t+1} - k_{t+1}) & \text{(G.37)} \end{aligned}$$

where the steady state ratios are given by $\frac{Y}{K} = \frac{1-\alpha}{r+\delta}$, $\frac{C}{K} = \frac{r+\alpha\delta}{1-\alpha}$ and $\frac{A}{K} = \frac{r+\delta}{1-\alpha}$ where the steady state *net* real interest rate $r = R - 1$. Combining (G.34) – (G.37) gives

$$\begin{aligned} k_{t+1} &= \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2)c_t & \text{(G.38)} \\ c_{t+1} &= c_t + \kappa\mathbb{E}_t(a_{t+1} - k_{t+1}) & \text{(G.39)} \end{aligned}$$

where $\lambda_1 = 1 + r$, $\lambda_2 = \frac{\alpha(r+\delta)}{1-\alpha}$ and $\kappa \equiv \frac{\sigma\alpha(1-\alpha)}{1+r}\left(\frac{A}{K}\right)^{\alpha} = \frac{\sigma\alpha(1-\alpha)}{1+r}\left(\frac{r+\delta}{1-\alpha}\right)^{\alpha}$.

In the example $a_t = \varepsilon_{a,t}$ so (G.39) gives (36).

H Example 5: Fundamentalness Measures for RBC with News

We run our final exercise using a standard version of news shocks with one-period ahead shocks to A_t and G_t as in Blanchard *et al.* (2013) and Forni *et al.* (2017). The structural shocks of our full RBC model now follows

$$\begin{aligned} A_t &= A_{t-1} + \varepsilon_{a,t-1} & \text{(H.1)} \\ G_t &= G_{t-1} + \varepsilon_{g,t-1} & \text{(H.2)} \end{aligned}$$

The effects of anticipated changes in productivity and government policy are delayed with respect to the time at which agents get information about them.

As is standard, the news shocks are assumed to be observable by the agents under perfect information containing the past values of the innovations $\varepsilon_{a,t-1}$ and $\varepsilon_{g,t-1}$. They must also observe A_t and G_t with perfectly anticipated changes in the fundamentals to occur at future dates. Economic data, by reflecting the rational forward-looking behaviour of agents, can be used by the econometrician to estimate the shocks' volatilities. On the other hand, the agents and econometrician have the identical information sets under imperfect information and there is no longer information that provides inference on the news until the future period when it directly affects its fundamental. This clearly suggests structural non-fundamentalness with respect to agents' imperfect information set. In addition, when we add additional shocks to the PI system this also introduces non-fundamentalness into the model. This result will depend on the size of the model in general, and the horizon of anticipation periods in particular which introduces multiple latent state variables in model solutions.

The implications of our model embedded with news information have strong consequences for invertibility and empirical analysis (e.g., the validity of VAR methods). In other words, with our invertibility conditions and fundamentalness testing presented and discussed so far, we expect to find that the RBC's fundamentalness no longer holds under perfect information, where agents observe current shocks, and under imperfect information the structural shocks are non-fundamental too with respect to agents' information set, which, in this example, is assumed to be consistent with the combinations of observables in Table 5 when the system was found to be perfectly invertible.

Combinations of observables (where $m = k$)	Perfect information	Imperfect information
$(Y_t, C_t), (Y_t, H_t), (Y_t, I_t)$ $(Y_t, W_t), (C_t, H_t), (C_t, W_t)$ $(I_t, H_t), (I_t, W_t), (H_t, W_t)$	EB is rank deficient (=1) $A(I - B(EB)^{-1}E)$ is non-existent $eig(\mathbb{F}^{PI}) = [0, 1]$	JB is rank deficient (=1) $F(I - B(JB)^{-1}J)$ is non-existent $eig(\mathbb{F}^{PI}) = [1, 1]$

Table 5: **Nonfundamentalness Measures for RBC Model with One-period Anticipated News Shocks**

I A Note on the Forni *et al.* (2017) Model

This sets out the simple model in Section 2 of their paper. It illustrates that when there is no saddlepath involved, as there is in Theorem 4, then the size of the state space does not increase under imperfect information.

$$\begin{aligned}
\text{potential output} &: \quad y_t^* = y_{t-1}^* + \underbrace{\varepsilon_{t-1}}_{\text{news shock}} \quad ; \varepsilon_t \sim Ni.i.d(0, \sigma_\varepsilon^2) && \text{(I.1)} \\
\text{signal} &: \quad s_t = \varepsilon_t + \underbrace{v_t}_{\text{noise shock}} \quad ; v_t \sim Ni.i.d(0, \sigma_v^2) && \text{(I.2)} \\
\varepsilon_t, v_t \text{ uncorrelated} &: \quad \text{Hence } \sigma_s^2 = \sigma_\varepsilon^2 + \sigma_v^2 && \text{(I.3)} \\
\text{consumption} &: \quad c_t = \lim_{j \to \infty} \mathbb{E}[y_{t+j}^* | I_t] && \text{(I.4)} \\
\text{actual ouput in equilibrium} &: \quad y_t = c_t && \text{(I.5)} \\
\text{information set} &: \quad I_t = \{y_{t-k}^*, s_{t-k}, k \geq 0\} && \text{(I.6)}
\end{aligned}$$

Then from (I.1) and (I.4) we have

$$c_t = \mathbb{E}[y_{t+j}^* | I_t] = \mathbb{E}[y_{t+1}^* | I_t] = y_t^* + \mathbb{E}[\varepsilon_t | I_t] \tag{I.7}$$

From (I.2) the OLS projection of ε_t on s_t is given by

$$\mathbb{E}[\varepsilon_t | I_t] = \frac{\sigma_\varepsilon^2}{\sigma_s^2} s_t \equiv \gamma s_t = \gamma(\varepsilon_t + v_t) \tag{I.8}$$

Hence we have

$$\Delta c_t = \Delta y_t^* + \gamma \Delta(\varepsilon_t + v_t) = \gamma \varepsilon_t + (1 - \gamma)\varepsilon_{t-1} + \gamma(v_t - v_{t-1}) \tag{I.9}$$

The state-space form of the RE solution is then

$$\begin{bmatrix} \Delta y_t^* \\ \Delta c_t \\ s_t \end{bmatrix} = \begin{bmatrix} L & 0 \\ \gamma + (1-\gamma) & \gamma(1-L) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \tag{I.10}$$

In the absence of noise, $v_t = \sigma_v^2 = 0$, $\gamma = 1$ and agents observe the shock and we have

PI. Then

$$\Delta c_t = \varepsilon_t \tag{I.11}$$

and after a shock consumption jumps immediately to its new long-run level. But with II consumption jumps to $c_t = \gamma\varepsilon_t$ in the first period and reaches $c_{t+1} = c_t + (1-\gamma)\varepsilon_t = c_{t-1} + \varepsilon_t$

The spectrum of the two process $\Delta a, s$ is given by

$$E\left[\begin{bmatrix} L\varepsilon_t \\ \varepsilon_t + \nu_t \end{bmatrix}[L^{-1}\varepsilon_t \;\; \varepsilon_t + \nu_t]\right] = \begin{bmatrix} \sigma_\varepsilon^2 & L\sigma_\varepsilon^2 \\ L^{-1}\sigma_\varepsilon^2 & \sigma_\varepsilon^2 + \sigma_\nu^2 \end{bmatrix}$$

It is easy to show that an alternative spectral factorization of this joint process is

$$\begin{bmatrix} 1 & L\frac{\sigma_\varepsilon^2}{\sigma_s^2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ L^{-1}\frac{\sigma_\varepsilon^2}{\sigma_s^2} & 1 \end{bmatrix}$$

where $\sigma_u^2 = \sigma_\varepsilon^2\sigma_\nu^2/(\sigma_\varepsilon^2 + \sigma_\nu^2)$

This automatically yields equation (7) of Forni et al. Now Blaschke factors are defined as $(L-a)/(1-aL)$. In this particular case $a = 0$ so the Blaschke factor is merely L. So apply this just for the shock u_t i.e., change u_t to $L\bar{u}_t$. It now follows that ε_t, ν_t are just simple linear transformations of $\bar{u}_t, s_t$ because (9), when expressed in terms of the latter, requires a change of L^{-1} to 1.

J Dynare Implementation

Levine *et al.* (2019) describes the working and use of the Imperfect Information (Partial Information)[45] software that solves, simulates and estimates DSGE rational expectations (RE) models in Dynare under imperfect information. The software is a MATLAB based code and is now integrated into Dynare unstable version 4.6 (Link to the Unstable Versions). The solution techniques adopted are based on the work by Pearlman *et al.* (1986).

[45] Different terminologies are found in the literature. Most DSGE models are solved on the assumption that agents have perfect information of the current state as an endowment. This is the default option in Dynare. Under imperfect information this assumption is relaxed. The use of perfect/imperfect information corresponds to the standard use in dynamic game theory when describing the information of the history of play. See footnote 7.

In particular, the software:

1. Transforms Dynare's linearized model solutions into the Blanchard-Kahn form which is solved to yield a reduced-form system. See Theorem 1 of paper.
2. Provides the conditions for invertibility under which imperfect information is equivalent to perfect information. See Theorem 2 of paper.
3. Implements multivariate measures of goodness of fit of the innovation residuals to the fundamental shocks, and provides information as to how well VAR residuals correspond to the fundamentals in DSGE models. See Theorem 5 of paper.
4. Simulates the model and uses the resulting reduced-form solution to obtain theoretical moments and IRFs
5. Evaluates the reduced-form system via the Kalman filter to obtain the likelihood function for estimation purposes and results from an identified DSGE-VAR. See Appendix F of paper.