

# Gravitational Waves in a Codimension Two Braneworld

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## Abstract

We consider the propagation of gravitational waves in six dimensions induced by sources living on 3-branes in the context of a recent exact solution [1]. The brane geometries are de Sitter and the bulk is a warped geometry supported by a positive cosmological constant as well as a 2-form flux. We show that at low energies ordinary gravity is reproduced, and explicitly compute the leading corrections from six dimensional effects. After regulating the brane we find a logarithmic dependence on the cutoff scale of brane physics even for modes whose frequency is much less than this energy scale. We discuss the possibility that this dependence can be renormalized into bulk or brane counterterms in line with effective field theory expectations. We discuss the inclusion of Gauss-Bonnet terms that have been used elsewhere to regulate codimension two branes. We find that such terms do not regulate codimension two branes for compact extra dimensions.

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## I. INTRODUCTION

In this paper we discuss a simple six dimensional model, with two compact extra dimensions with matter living on two conical 3-branes. This model may well approximate an inflationary phase, or late-time dark energy dominated phase in scenarios with two large extra dimensions as proposed in the original scenario of Arkani-Hamed, Dimopolous and Dvali (ADD) [2, 3]. Our concern shall be how the matter on the brane backreacts on the bulk, and in particular the resulting gravitational waves that are generated. We find that at low energies, at leading order ordinary gravity is reproduced with the anticipated effective Newton constant. In addition there are a series of corrections suppressed by powers of  $R_c^2 E^2$  where  $R_c$  is the size of the extra dimensions. The magnitude of the leading order correction is sensitive to how we regulate the brane even at energy scales far below  $E \ll 1/l$  where  $l$  is the width of the brane. This suggests an apparent model dependence in dealing with codimension two branes. We discuss the possibility that this dependence can be renormalized, i.e. absorbed into brane and bulk counterterms. Recently similar ideas have been proposed as being useful for understanding black hole physics [4].

The proposal of ADD resolves the hierarchy problem between the Planck and electroweak scales by having the fundamental Planck mass of order the electroweak scale, and using the dilution effect of gravity in extra dimensions to give rise to an effectively weak four-dimensional gravity. In the context of string theory this typically introduces a new hierarchy between the size of several large extra dimensions and additional small extra dimensions and/or the fundamental Planck scale, something which may or may not be natural depending on the details of the moduli stabilization. In the supersymmetric large extra dimension scenario (SLED) it has been suggested that the same features of the large extra dimensions combined with the calming effects of SUSY can also be used to alleviate the cosmological constant hierarchy problem [5, 6], (see [7] for a recent review). A detailed understanding of whether this is true requires amongst other things an understanding of how changes in the brane tensions, of the type that may arise in phase transitions on the brane, influence the bulk dynamics. It has proven technically challenging [6] to answer this question for reasons that shall become apparent in the following. Here we shall focus on the simpler question of how the bulk dynamics is influenced by small matter perturbations on the brane.

Although great progress has been made in recent years in understanding the dynamics of codimension one branes, higher codimension branes remain something of an enigma. This arises because at the level of GR, distributional sources of arbitrary codimension are typically singular [8, 9]. As a result one needs to deal with ‘thick’ branes such as those described by regular defects from field

theory models [10, 11, 12]. An alternative approach is to modify Einstein gravity by the addition of higher derivative curvature terms which can potentially allow distributional sources to be well defined. In the case of codimension two branes one such approach that has been taken is to use Gauss-Bonnet terms [13, 14, 15, 16]. One of the main motivations for the use of these terms comes from string theory where they generically appear as a leading order quantum correction to gravity and guarantee a ghost-free action [17]. However, as will become apparent later this method of regularizing the branes with Gauss-Bonnet terms can only be applied for codimension two branes in a noncompact extra dimension, and is inconsistent for the more familiar case of compact extra dimensions. Some other inconsistencies have been suggested in the case of an axially symmetric bulk, where an isotropic braneworld cosmological ansatz seems to be incompatible with the model [15].

In string theory we are interested in the dynamics of D-branes of arbitrary dimension, and here we find that at the level of the supergravity, the geometry describing all but the D3-branes are singular. In string theory we are to understand that string  $\alpha'$  or  $g_s$  corrections will ‘regulate’ the singularity. However, at first sight there seems to be a contradiction with expectations from low energy effective field theory (EFT), since this suggests that we need to understand the string scale physics that regulates the brane in order to make predictions about low energy dynamics of relevance to cosmology. It is precisely this aspect that we would like to explore in the following by asking the simpler question, how does the production of gravitational waves induced by sources living on the brane depend on the detailed physics that resolves the brane?

We begin in section II by discussing approaches to regulating the brane geometries. Then in section III we introduce the background solution and calculate the tensor perturbations via a derivative expansion. This enables us to obtain order by order the modified equations for the gravitational waves on the brane in section IV. We discuss the presence of the logarithmic divergences that arise in the thin brane limit and whether these can be canceled by counterterms on the brane or in the bulk. After reviewing in detail the Kaluza-Klein limit of the general solution in section V, we compute in section VI, the effects of adding a Gauss-Bonnet term in the bulk and show that this cannot be used to regulate the divergences near the brane. Finally in section VII we conclude.

## II. DEALING WITH CODIMENSION TWO BRANES

In this section we discuss our approach to dealing with codimension two branes. We follow closely the approach of ref. [18].

### A. Thick Branes

One approach to dealing with codimension two objects is to regulate them by replacing them with a smooth stress energy source, i.e. a ‘thick brane’ [9, 19]. For instance one model of this would be as a codimension two topological defect arising for example from a six dimensional Abelian-Higgs model [10]. Figure 1 illustrates what the resulting geometry of the two extra dimensions would look like. The conical deficits at the poles are replaced with smooth geometries.

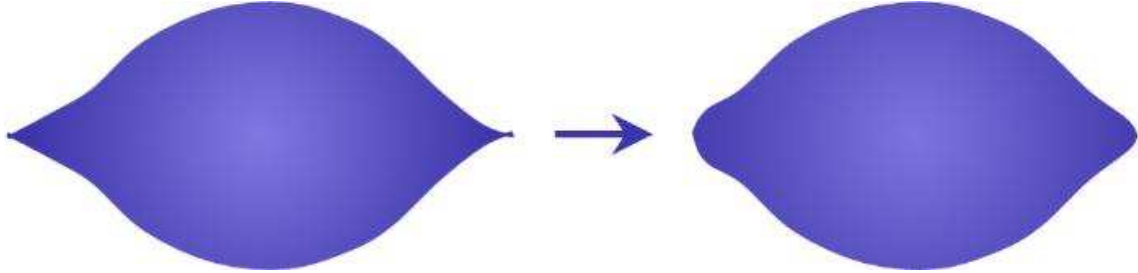


FIG. 1: Six dimensional compactified spacetime with regularized singularities.

Although the full behaviour of the solution in the polar regions may be quite complicated, provided the width  $l$  of the brane is much less than the scales of physical interest we anticipate that the bulk physics will only be sensitive to integrated effects over the brane, a perspective espoused in [18]. As in many works on this topic we shall restrict ourselves to axisymmetric solutions which is a consistent (if not entirely desirable) truncation. Given a six dimensional metric of the form

$$ds^2 = N(r)^2 dr^2 + \mathcal{L}(x, r)^2 d\varphi^2 + g_{\mu\nu}(x, r) dx^\mu dx^\nu, \quad (1)$$

with branes located at  $r_-$  and  $r_+ > r_-$ , we define the four dimensional stress energy of the brane by

$${}^{(4)}T_\nu^{\mu(\pm)} = \frac{1}{\sqrt{-g(r = \tilde{r}_\pm)}} \int_{r_\pm}^{\tilde{r}_\pm} dr d\varphi N(r) \mathcal{L}(r, x) \sqrt{-g} {}^{(6)}T_\nu^\mu, \quad (2)$$

with  $\tilde{r}_\pm = r_\pm \mp \epsilon$ . Here  ${}^{(6)}T_\nu^\mu$  is the regular stress energy source describing the thick brane which we assume vanishes outside the region  $|r - r_\pm| \leq \epsilon$ . Here  $\epsilon$  is related to the width of the brane by  $l = |\int_{r_\pm}^{\tilde{r}_\pm} N(r) dr|$ . The requirement that the metric is smooth as  $r \rightarrow r_\pm$  implies

$$\mathcal{L}(x, r_\pm) = 0, \quad (3)$$

$$N(r_\pm)^{-1} \partial_r \mathcal{L}(x, r_\pm) = \mp 1, \quad (4)$$

$$N(r_\pm)^{-1} \partial_r g_{\mu\nu}(x, r_\pm) = 0. \quad (5)$$

The key observation of [18] is that by integrating the Einstein equations over the brane thickness, one may infer an effective matching rule for the extrinsic curvature on the surface  $r = r_{\pm} \mp \epsilon$  in terms of the integrated brane stress energies. This matching rules play the role of the Israel junction conditions for codimension one branes.

In particular if one looks at the  ${}^{(6)}R_{\nu}^{\mu}$  equation we have [29]

$${}^{(6)}R_{\nu}^{\mu} = {}^{(4)}R_{\nu}^{\mu} - \mathcal{L}^{-1} \nabla^{\mu} \nabla_{\mu} \mathcal{L} - \frac{1}{N\sqrt{-g}} \partial_r (\sqrt{-g} \mathcal{L} K_{\nu}^{\mu}) = \kappa \left( {}^{(6)}T_{\nu}^{\mu} - \frac{1}{4} \delta_{\nu}^{\mu} {}^{(6)}T_M^M \right). \quad (6)$$

Here  $K_{\nu}^{\mu}$  is the extrinsic curvature defined by  $K_B^A = \frac{1}{2N} g^{AC} \partial_r g_{CB}$ . The dominant contribution is expected to come from the second order derivatives, and so on integrating around the brane we find

$$2\pi \mathcal{L}(\tilde{r}_{\pm}) K_{\nu}^{\mu}|_{r=\tilde{r}_{\pm}} = \pm \kappa \left( {}^{(4)}T_{\nu}^{\mu(\pm)} - \frac{1}{4} \delta_{\nu}^{\mu} {}^{(4)}T_M^M(\pm) \right) + \mathcal{O}(\epsilon). \quad (7)$$

Similarly we have

$$2\pi \mathcal{L}(\tilde{r}_{\pm}) K_{\varphi}^{\varphi}|_{r=\tilde{r}_{\pm}} \pm \pi \frac{\sqrt{-g}|_{r=r_{\pm}}}{\sqrt{-g}|_{r=\tilde{r}_{\pm}}} = \pm \kappa \left( {}^{(4)}T_{\varphi}^{\varphi(\pm)} - \frac{1}{4} {}^{(4)}T_M^M(\pm) \right) + \mathcal{O}(\epsilon). \quad (8)$$

Note that the  $T_{\varphi}^{\mu}$  terms all vanish due to the initial assumption of the form of the metric.

## B. Regularizing as a Codimension One Brane

The previous matching conditions amount to a statement about the extrinsic curvature defined on a codimension one surface in terms of the brane stress energy which is precisely the same physical information as if we really had a codimension one brane, localized for example at an orbifold fixed point as in the Randall-Sundrum model as shown in figure 2.

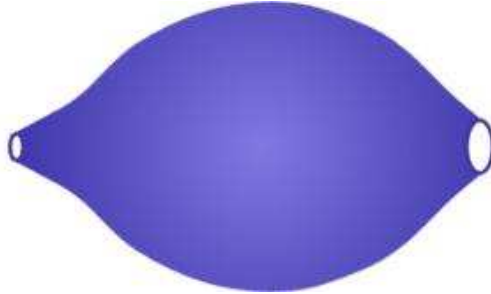


FIG. 2: Six dimensional compactified spacetime with codimension-one orbifold planes cutting off the singularity

Consequently we may regulate the codimension two brane as a codimension one brane located at an orbifold fixed point, where the Israel matching conditions would give

$$K_\nu^\mu|_{r=\tilde{r}_\pm} = \pm \frac{1}{2} \kappa \left( {}^{(5)}T_\nu^\mu(\pm) - \frac{1}{4} \delta_\nu^\mu {}^{(5)}T_M^M(\pm) \right) + \mathcal{O}(\epsilon) \quad (9)$$

$$K_\varphi^\varphi|_{r=\tilde{r}_\pm} = \pm \frac{1}{2} \kappa \left( {}^{(5)}T_\varphi^\varphi(\pm) - \frac{1}{4} {}^{(5)}T_M^M(\pm) \right) + \mathcal{O}(\epsilon). \quad (10)$$

Note that the factor of 1/2 arises because of the doubling of the extrinsic curvature at the fixed point. The five dimensional stress energy is related to the four dimensional stress energy by

$${}^{(5)}T_\nu^\mu(\pm) = \frac{1}{\pi \mathcal{L}(x, \tilde{r}_\pm)} \left( {}^{(4)}T_\nu^\mu(\pm) + \delta_\nu^\mu \left( {}^{(4)}T_r^r(\pm) + \frac{\pi \sqrt{-g}|_{r_\pm}}{\kappa \sqrt{-g}|\tilde{r}_\pm} \right) \right) \quad (11)$$

$${}^{(5)}T_\varphi^\varphi(\pm) = \frac{1}{\pi \mathcal{L}(x, \tilde{r}_\pm)} \left( {}^{(4)}T_\varphi^\varphi(\pm) + {}^{(4)}T_r^r(\pm) \right). \quad (12)$$

### C. Tensor perturbations

One of the problems with the above matching rules is that they only contain partial information without a specification of  ${}^{(4)}T_r^r$  and  ${}^{(4)}T_\varphi^\varphi$ . In the following we shall be interested in tensor perturbations about a fixed cosmological background (gravitational waves), where tensors are defined with respect to four dimensional observers on the brane. In this case we have  $\delta T_r^r = \delta T_\varphi^\varphi = 0$  and so

$$\delta {}^{(5)}T_\nu^\mu(\pm) = \frac{1}{\pi \mathcal{L}(x, \tilde{r}_\pm)} \delta {}^{(4)}T_\nu^\mu(\pm) \quad (13)$$

$$\delta {}^{(5)}T_\varphi^\varphi(\pm) = 0. \quad (14)$$

It is this simplification that will allow us to determine the bulk solution without further specification of the physics of the brane other than through the cutoff  $\epsilon$ .

## III. SIX DIMENSIONAL SOLUTIONS

We use the index conventions that Greek indices are four dimensional, labeling the transverse  $x^\mu$  directions, while small Roman indices are five dimensional, labeling the  $x^\mu$  and  $\varphi$  coordinates. The full six dimensional coordinates are represented by capital Roman indices:  $A = 0, \dots, 5$ . Our starting point is the action

$$S = \int d^6x \sqrt{-g} \frac{1}{2\kappa} \left( {}^{(6)}R - 2\Lambda - \frac{1}{2} F_{AB} F^{AB} \right), \quad (15)$$

for gravity and a bulk form field  $F_{AB} = \partial_A A_B - \partial_B A_A$ , ( $A_B$  being a  $U(1)$  gauge field) and cosmological constant  $\Lambda$ . For other six dimensional solutions see [20, 21] and for cosmology on codimension two branes, see for instance [22, 23].

### A. Background solution

We consider a special case of (1), where fluxes are present in the six dimensional bulk and  $\mathcal{L}(r, x)^2 = L^2 f(r)$ , leading to the compactification of [1], with a metric of the form

$$ds^2 = g_{AB} dx^A dx^B = f^{-1} dr^2 + L^2 f d\varphi^2 + g_{\mu\nu} dx^\mu dx^\nu \quad (16)$$

$$g_{\mu\nu} dx^\mu dx^\nu = H^2 r^2 q_{\mu\nu} dx^\mu dx^\nu, \quad (17)$$

where  $q_{\mu\nu}$  is the four-dimensional metric. For the background,  $q_{\mu\nu} = \gamma_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ ,  $a(\tau) = (-H\tau)^{-1}$  and  $f = 1 - \frac{\Lambda}{10}r^2 - \frac{\mu}{r^3} - \frac{b^2}{12r^6}$ , where  $\mu, b$  are some constants and  $\Lambda$  is the six dimensional cosmological constant. The gauge field is of the form  $A_M dx^M = \frac{b}{3r^3} L d\varphi$ .  $H$  is an arbitrary reference scale that we have included to keep dimension.  $L$  is included so that we can normalize  $\varphi \in [0, 2\pi]$ . The proper size of the  $\varphi$  direction is then  $2\pi\sqrt{f}L$ .

The properties of this solution are discussed more fully in [1], the main point is that provided  $\Lambda > 0$ , one can find solutions for which  $f$  vanishes linearly at two positive points on the real axis  $r = r_\pm$ , and for arbitrary values of the parameters these points will be locally Minkowski with a conical deficit angle signifying the positions of two conical 3-branes. To make this clear one may define  $\rho_\pm = \sqrt{\frac{\mp 2(r-r_\pm)}{\sigma_\pm}}$  and  $\sigma_\pm = \mp \frac{1}{2} f'(r_\pm)$ , for which the near brane metric becomes

$$ds^2 \approx d\rho_\pm^2 + L^2 \sigma_\pm^2 \rho_\pm^2 d\varphi^2 + H^2 r_\pm^2 q_{\mu\nu} dx^\mu dx^\nu. \quad (18)$$

The conical deficit angle is defined through  $2\pi(1 - \delta_\pm) = 2\pi L\sigma_\pm$  which corresponds to two branes of tension

$$T_\pm = \kappa^{-1}(1 - L\sigma_\pm), \quad (19)$$

where  $\kappa$  is the six-dimensional Newtonian constant. For fixed bulk cosmological constant  $\Lambda$ , we are free to choose  $\mu, b$  and  $L$  which gives us more than enough freedom to fit two branes of arbitrary tension. In fact we may set the flux to zero, i.e.  $b = 0$  and provided  $\Lambda > 0$  we can still find solutions with arbitrary brane tension by adjusting  $L$  and  $\mu$ . The geometry of each brane is de Sitter with Hubble constants  $H_\pm = 1/r_\pm$  which is only indirectly related to the brane tensions and the bulk flux and cosmological constant.

### Boundary conditions

Following the prescription of section II A, we now consider two codimension one branes located at  $r = \tilde{r}_\pm$ . The normal vector to the branes is  $N_A dx^A = f^{-1/2} dr$  and the extrinsic curvature is

$$K_\nu^\mu = \frac{\sqrt{f}}{r} \delta_\nu^\mu \quad K_\varphi^\varphi = \frac{f'}{2\sqrt{f}} \quad K_\mu^\varphi = 0 \quad K_\varphi^\mu = 0. \quad (20)$$

On the branes, the following junction condition should be satisfied:

$$\Delta [K \delta_b^a - K_b^a]_{\tilde{r}_\pm} = \kappa {}^{(5)}T_b^{a(\pm)}, \quad (21)$$

where  ${}^{(5)}T_b^{a(\pm)}$  is the stress-energy tensor for the gauge and matter fields introduced on the codimension one branes. The extrinsic curvature on the brane should hence satisfy

$$K_b^a(\tilde{r}_\pm) = \pm \frac{\kappa}{2} {}^{(5)}\tilde{T}_b^{a(\pm)} = \pm \frac{\kappa}{2} \left( {}^{(5)}T_b^{a(\pm)} - \frac{1}{4} {}^{(5)}T_c^{c(\pm)} \delta_b^a \right). \quad (22)$$

For the background, we should therefore have:

$${}^{(5)}\tilde{T}_\nu^{\mu(\pm)} = \pm \frac{2}{\kappa \tilde{r}_\pm} \sqrt{f(\tilde{r}_\pm)} \delta_\nu^\mu \quad (23)$$

$${}^{(5)}\tilde{T}_{\varphi\varphi}^{(\pm)} = \pm \frac{L^2}{\kappa} \sqrt{f(\tilde{r}_\pm)}. \quad (24)$$

### B. Tensor perturbations

We now study the cosmological perturbations around this background solution sourced by matter on the branes. To start with, we consider only four-dimensional tensor perturbations to the background solution. As elsewhere we only consider axisymmetric perturbations which is a consistent truncation. We therefore have  $q_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}(x^\mu, r)$  with  $\gamma^{\mu\nu} h_{\mu\nu} = 0$  and  $\gamma^{\mu\alpha} h_{\alpha\nu;\mu} = 0$ .

Working in this gauge, the only non-trivial contribution to the Ricci tensor is

$${}^{(6)}R_{\mu\nu}[g_{AB}] = {}^{(4)}R_{\mu\nu}[q_{\mu\nu}] - H^2 (3f + rf') q_{\mu\nu} - 2H^2 r f h_{\mu\nu,r} - \frac{1}{2} r^2 H^2 (f h_{\mu\nu,r})_{,r}, \quad (25)$$

the perturbed Einstein equation is therefore  $\delta^{(6)}\bar{R}_\nu^\mu = 0$

$$\partial_r (f \partial_r h_\nu^\mu) + \frac{4}{r} f \partial_r h_\nu^\mu = \frac{2}{H^2 r^2} \delta \bar{R}_\nu^\mu = -\frac{1}{H^2 r^2} \square h_\nu^\mu. \quad (26)$$

The operator  $\square$  may be expressed as

$$\square h_\nu^\mu = \left[ \square - 2H^2 \right] h_\nu^\mu, \quad (27)$$

$\square$  being the four-dimensional Laplacian:  $\square = \gamma^{\alpha\beta} D_\alpha D_\beta$ .



*Boundary conditions*

At the perturbed level, we consider some matter with stress-energy  $\delta^{(5)}T_b^{a(\pm)} = g^{ac}(\tilde{r}_\pm) \delta^{(5)}T_{cb}^{(\pm)}$  on each brane. Since we concentrate for now on four-dimensional tensor perturbations, we impose  $\delta^{(5)}T_b^{\varphi(\pm)} = 0$ ,  $\delta^{(5)}T^{(\pm)} = \delta^{(5)}T_a^{a(\pm)} = \delta^{(5)}T_\mu^{\mu(\pm)} = 0$  and  $\delta^{(5)}T_{\nu;\mu}^{\mu(\pm)} = 0$ .

From the junction conditions, we should have:  $\delta K_b^a(\tilde{r}_\pm) = \pm \frac{\kappa}{2} \delta^{(5)}T_b^{a(\pm)}$  with

$$\delta K_\nu^\mu = \frac{\sqrt{f}}{2} \partial_r h_\nu^\mu \quad (28)$$

$$\delta K_\varphi^\varphi = 0, \quad \delta K_\mu^\varphi = 0, \quad \delta K_\varphi^\mu = 0. \quad (29)$$

This relation implies:

$$\left. \frac{\sqrt{f}}{2} h_{\nu,r}^\mu \right|_{\tilde{r}_\pm} = \pm \frac{\kappa}{2} \delta^{(5)}T_\nu^{\mu(\pm)}, \quad (30)$$

we recall that  $h_\nu^\mu = \gamma^{\mu\alpha} h_{\alpha\nu}$ , whereas  $\delta^{(5)}T_b^{a(\pm)} = g^{ac}(\tilde{r}_\pm) \delta^{(5)}T_{cb}^{(\pm)}$ , hence  $\delta^{(5)}T_\nu^{\mu(\pm)} = \frac{1}{H^2 \tilde{r}_\pm^2} \gamma^{\mu\alpha} \delta^{(5)}T_{\alpha\nu}$ . From now on, we use the notation  ${}^{(5)}T_\nu^{\mu(\pm)} = \gamma^{\mu\alpha} \delta^{(5)}T_{\alpha\nu}$ , and hence the junction condition is:  $\partial_r h_\nu^\mu(\tilde{r}_\pm) = \pm \frac{\kappa}{H^2 \tilde{r}_\pm^2 \sqrt{f(\tilde{r}_\pm)}} {}^{(5)}T_\nu^{\mu(\pm)}$ . Furthermore, we may recall from the relation (13) between the five and the four-dimensional stress-energy tensor that  ${}^{(5)}T_\nu^{\mu(\pm)} = \frac{1}{\pi L \sqrt{f(\tilde{r}_\pm)}} {}^{(4)}T_\nu^{\mu(\pm)}$ , we hence have the boundary condition in terms of the four-dimensional stress-energy tensor:

$$\partial_r h_\nu^\mu(\tilde{r}_\pm) = \pm \frac{\kappa}{\pi L H^2 \tilde{r}_\pm^2 f(\tilde{r}_\pm)} {}^{(4)}T_\nu^{\mu(\pm)}. \quad (31)$$

#### IV. LOW-ENERGY EXPANSION

At low-energies, we may consider that the contribution from the four-dimensional derivatives to be small in comparison to the  $r$  derivatives. We can hence consider an expansion in the operator  $\square$ . The way of solving the Einstein equation (26), will be similar to the RS case. For that, we will express the solution of (26) as an expansion in  $\square$  and in what follows we omit any indices to lighten the notation and  $h$  will designate  $h_\nu^\mu$ , and similarly for  ${}^{(4)}T^{(\pm)}$ :

$$h(r, x^\mu) = \sum_{n \geq 0} \left( \frac{\square}{H^2} \right)^{n-1} h_n(r, x^\mu), \quad (32)$$

where we consider each  $h_n$  to be of order zero in  $\square/H^2$ . So each  $h_n$  have a similar weight in the expansion, but they are weighted by a factor  $(\square/H^2)^{n-1}$  which makes their effective contribution smaller and smaller in the low-energy regime. In particular we will consider the sum to be dominated by the zero mode  $h_0(r, x^\mu)$ .

We can now solve the modified Einstein equation for each  $h_n(r, x)$ ,  $n \geq 0$ :

$$\partial_r g_n(r, x) + \frac{4}{r} g_n(r, x) = -\frac{1}{r^2} h_{n-1}(r, x), \quad (33)$$

with the notation  $h_{-1}(r, x) = 0$  and  $g_n = f \partial_r h_n$ .

Each mode should satisfy as well the junction conditions. Using the constraint (30), we therefore have the boundary conditions for each mode:

$$\partial_r h_n(\tilde{r}_\pm, x^\mu) = 0 \quad \forall n \geq 0, n \neq 1 \quad (34)$$

$$\partial_r h_1(\tilde{r}_\pm, x^\mu) = \pm \frac{\kappa}{\pi L H^2 \tilde{r}_\pm^2} f(\tilde{r}_\pm). \quad (35)$$

At this point we should stress that in dealing with functions of  $\square$  (or  $\square$ ) we always have to be careful that there are implicitly homogeneous solutions of the equations. For instance in dealing with an equation of the form  $[\square + a\square^2]h = T$  we shall write its solution in the form  $h = \square^{-1}(1 - a\square + O(\square^2))T$ . Whilst this is a particular solution, we should also include the homogeneous solution satisfying  $[\square + a\square^2]h = 0$ . We shall take it as read in what follows that these homogeneous solutions should be included, and hence concentrate on the particular solution. We may point out that this method would give rise to precisely the same result as obtained in [24] if it was applied to a codimension one brane.

### A. Zeroth order

The Einstein equation for the zero mode (33) can be easily solved (in particular, it does not depend on  $f$ ), and the solution is simply

$$\partial_r h_0(r, x^\mu) = \frac{D_0(x^\mu)}{r^4 f}. \quad (36)$$

The constant  $D_0$  may be fixed using the junction condition (34) which fixes  $D_0 = 0$ , and so:

$$h_0(r, x^\mu) = C_0(x^\mu). \quad (37)$$

As we shall see in what follows,  $C_0$  will be fixed by the junction condition for the first mode since the zero mode acts as a source term for the first mode.

### B. First order

The first order mode can be found by solving

$$\partial_r g_1 + \frac{4}{r} g_1 = -\frac{1}{r^2} h_0 = -\frac{D_1(x^\mu)}{r^2}, \quad (38)$$

the solution of this equation satisfies

$$\partial_r h_1 = \frac{g_1}{f} = -\frac{C_0(x^\mu)}{3fr} + \frac{D_1(x^\mu)}{fr^4}. \quad (39)$$

We work for now in the region  $r_- < \tilde{r}_- < r < \tilde{r}_+ < r_+$ , and  ${}^{(1)}h(r, x^\mu)$  is hence regular. The constants  $C_0(x^\mu)$  and  $D_1(x^\mu)$  should be fixed from the junction conditions as follows. The junction condition for the first mode is given in (35). We therefore have:

$$C_0(x^\mu) = \frac{3\kappa}{\pi LH^2 (\tilde{r}_-^3 - \tilde{r}_+^3)} \left( \tilde{r}_-^2 {}^{(4)}\tau^{(-)} + \tilde{r}_+^2 {}^{(4)}\tau^{(+)} \right) \quad (40)$$

$$D_1(x^\mu) = \frac{\kappa \tilde{r}_-^2 \tilde{r}_+^2}{\pi LH^2 (\tilde{r}_-^3 - \tilde{r}_+^3)} \left( \tilde{r}_+ {}^{(4)}\tau^{(-)} + \tilde{r}_- {}^{(4)}\tau^{(+)} \right), \quad (41)$$

so that

$$\partial_r h_\nu^\mu = \frac{\kappa}{\pi LH^2 (\tilde{r}_-^3 - \tilde{r}_+^3) f(r)} \left[ \frac{\tilde{r}_+^2}{r^4} (\tilde{r}_-^3 - r^3) {}^{(4)}\tau_\nu^{\mu(+)} + \frac{\tilde{r}_-^2}{r^4} (\tilde{r}_+^3 - r^3) {}^{(4)}\tau_\nu^{\mu(-)} \right]. \quad (42)$$

In the low-energy limit, we are interested in the expression of the zero mode, which is *finite* in the limit where  $\epsilon \rightarrow 0$  and the same on both branes. Its contribution to the gravitational waves is given by:

$$\frac{H^2}{\square} h_{0\nu}^\mu(r_\pm, x^\mu) = -\frac{2\tilde{\kappa}_\pm}{\square} \left( {}^{(4)}\tau_\nu^{\mu(\pm)} + \frac{r_\pm^2}{r_\pm^2} {}^{(4)}\tau_\nu^{\mu(\mp)} \right) \quad (43)$$

$$\tilde{\kappa}_\pm = \frac{3r_\pm^2}{2\pi L (r_+^3 - r_-^3)} \kappa. \quad (44)$$

On the brane, the Hubble parameter is fixed to  $H_\pm = 1/r_\pm$ , by making the choice  $H = H_+$  or  $H = H_-$ , we can work in terms of the proper coordinates on either of the branes. By making this choice, we have:

$$\tilde{\kappa}_\pm = \tilde{\kappa} = \frac{3}{2\pi LH^2 (r_+^3 - r_-^3)} \kappa. \quad (45)$$

### C. Effective Newton constant

The effective gravitational coupling constant we have derived is exactly what one expects by naively integrated out the action according to the usual argument

$$\begin{aligned} \int d^6x \sqrt{-g} \frac{1}{2\kappa} {}^{(6)}R &= \int d^4x dr d\varphi \sqrt{-g} \frac{1}{2\kappa} g^{\mu\nu(4)} R_{\mu\nu} + \dots \\ &= 2\pi L \int d^4x dr r^2 H^2 \sqrt{-q} \frac{1}{2\kappa} q^{\mu\nu(4)} R_{\mu\nu} + \dots \\ &= \frac{2}{3} \pi LH^2 (r_+^3 - r_-^3) \int d^4x \sqrt{-q} \frac{1}{2\kappa} q^{\mu\nu(4)} R_{\mu\nu} + \dots \end{aligned}$$

and so

$$\tilde{\kappa}_{\pm} = \tilde{\kappa} = \frac{3}{2\pi LH^2 (r_+^3 - r_-^3)} \kappa. \quad (46)$$

This result was anticipated in ref. [1]. In the limit where  $r_+ \gg r_-$ , the zero mode couples uniquely to the matter on the brane at  $r = r_+$ :

$${}^{(0)}h_{\nu}^{\mu}(r_+, x^{\mu}) = -\frac{2\tilde{\kappa}}{\square} {}^{(4)}T_{\nu}^{\mu(+)}, \quad (47)$$

$${}^{(0)}h_{\nu}^{\mu}(r_-, x^{\mu}) = -\frac{2r_+^2 \tilde{\kappa}}{r_-^2 \square} {}^{(4)}T_{\nu}^{\mu(+)}. \quad (48)$$

This is similar to the single brane limit of the Randall-Sundrum scenario where conventional gravity is recovered on the positive tension brane only.

#### D. Second order

In order to understand the behaviour of the first mode  $h_1$ , we need first to constrain the remaining degree of freedom in  $h_1$  by imposing the boundary conditions on the second mode.

The first mode is given by the integration of (42). In particular we may use the relation:

$$\int \frac{1}{r^m f(r)} dr = \int \frac{r^{6-m}}{r^6 f(r)} dr = \sum_{i/f(r_i)=0} \frac{\log|r-r_i|}{r_i^m f'(r_i)}, \quad (49)$$

for any integer  $-1 \leq m \leq 6$ , where the integral has been performed by recognizing that the integrand is a ratio of two polynomials, with the numerator one being of lower order. Using this relation in the integral of (42), we hence have the expression for the first mode

$$h_1 = C_1 + \sum_{r_i} \mathcal{A}_i \log|r-r_i| \quad (50)$$

$$\mathcal{A}_i = \frac{\kappa}{\pi LH^2 f'(r_i) (\tilde{r}_-^3 - \tilde{r}_+^3)} \left[ \frac{\tilde{r}_+^2}{r_i^4} (\tilde{r}_-^3 - r_i^3) {}^{(4)}T_{\tau}^{(+)} + \frac{\tilde{r}_-^2}{r_i^4} (\tilde{r}_+^3 - r_i^3) {}^{(4)}T_{\tau}^{(-)} \right]. \quad (51)$$

We may now use this expression to derive the second mode, which using the equation (33), is of the form:

$$\begin{aligned} f\partial_r h_2 = g_2 &= \frac{1}{r^4} \left[ D_2 - \int r^2 h_1 dr \right] \\ &= \frac{1}{r^4} \left[ D_2 - \frac{1}{3} C_1 r^3 + z(r) \right] \end{aligned} \quad (52)$$

$$z(r) = -\frac{1}{3} \sum_{r_i} \mathcal{A}_i \left( -r \left( \frac{1}{3} r^2 + \frac{1}{2} r_i r + r_i^2 \right) + (r^3 - r_i^3) \log|r-r_i| \right). \quad (53)$$

From the boundary conditions,  $\partial_r h_2$  should vanish on both branes at  $\tilde{r}_\pm$ . This constraint fixes the constants  $C_1$  and  $D_2$  to

$$C_1 = \frac{3(z(\tilde{r}_-) - z(\tilde{r}_+))}{\tilde{r}_-^3 - \tilde{r}_+^3} \quad (54)$$

$$D_2 = \frac{\tilde{r}_+^3 z(\tilde{r}_-) - \tilde{r}_-^3 z(\tilde{r}_+)}{\tilde{r}_-^3 - \tilde{r}_+^3}, \quad (55)$$

leading to the following expression for the first mode:

$$h_1 = \frac{3(z(\tilde{r}_-) - z(\tilde{r}_+))}{\tilde{r}_-^3 - \tilde{r}_+^3} + \sum_{r_i} \mathcal{A}_i \log |r - r_i|. \quad (56)$$

We may check that in general, the first mode is not finite in the thin brane limit  $\epsilon \rightarrow 0$ . In particular, the dominant contribution to the first mode on the branes is logarithmically divergent

$$\begin{aligned} h_1(r_\pm) &= \mathcal{A}_\pm \log \epsilon + \mathcal{O}(\epsilon^0) \\ &= -\frac{\kappa \log \epsilon}{\pi L H^2 r_\pm^2 |f'(r_\pm)|} {}^{(4)}\mathcal{T}^{(\pm)} + \mathcal{O}(\epsilon^0). \end{aligned} \quad (57)$$

Putting all this together, we have the following effective four dimensional equations of motion on each brane

$$h_\pm = -\frac{2\tilde{\kappa}}{\square} \left( {}^{(4)}\mathcal{T}^{(\pm)} + \frac{r_\mp^2}{r_\pm^2} {}^{(4)}\mathcal{T}^{(\mp)} \right) - \frac{\kappa \log \epsilon}{\pi L H^2 r_\pm^2 |f'(r_\pm)|} {}^{(4)}\mathcal{T}^{(\pm)} + \mathcal{O}(\epsilon^0). \quad (58)$$

### E. Renormalization and EFT

The presence of the logarithmic dependence on the cutoff suggests an inherent model dependence in the form of the solutions. That is, it will be necessary to specify certain features of the brane physics in order to determine a unique solution in the bulk. At first sight one may not be so concerned about this, since the dependence is only logarithmic in the cutoff. However, this is a feature of the fact that we are doing linearized perturbations. The logarithmic divergence of the metric perturbations near the brane is in fact a signal of the onset of a generic anisotropic Kasner-like singularity which the conical singularity is unstable to. Consequently we only expect this dependence on  $\epsilon$  to get worse at higher orders. We stress that this logarithmic divergence is not the same as the more familiar infrared logarithmic divergence of massless scalar fields on two dimensional spacetimes, in fact we anticipate that for any codimension we will still find a logarithmic dependence for gravitational waves, where the infrared behaviour falls off as a power for higher codimension.

In EFT we are used to the idea that if we are interested in physics at a given energy scale, we can integrate out the modes whose masses are much larger than this energy scale, and the net effect is to just renormalize various counterterms in the effective action. It is tempting to apply the same philosophy here, the cutoff  $\epsilon$  is associated with an energy scale  $1/\epsilon$  at which modes which describe the brane itself will become excited. If we concentrate on physics at energy scales well below this, for instance at scales set by the size of the extra dimensions for which the higher derivative terms we have been discussing are still important, then it is natural to expect that all dependence on the scale  $1/\epsilon$  can be renormalized or canceled by various counterterms localized either on the brane or in the bulk.

Consider first the possibility that the  $\log \epsilon$  may be absorbed in brane counterterms that are local functions of the metric, curvature invariants and the matter degrees of freedom on the brane. This minimal possibility seems natural given that the high energy brane physics is localized at the brane itself. Such terms would correspond to redefinitions of the stress energy on each brane. The problem is that the metrics on each brane depend on the same combination of  ${}^{(4)}\tau^{(\pm)}$  at leading order and so any redefinition would give the same contribution on each brane. However in general the log divergence on each brane is different and so no simple renormalization of the stress energy on each brane will cancel the logarithmic dependence.

The next possibility is to include in the brane action functions of the extrinsic curvature. This is only consistent as a boundary condition if we also include higher order derivative terms in the bulk that increase the order of the Cauchy problem. Adding counterterms in the bulk as well as on the brane can allow one to cancel the  $\log \epsilon$  divergence on each brane, but closer inspection shows that this is typically always at the price of reintroducing it in the bulk. The key point is that as long as the fact that the perturbations diverge logarithmically near the brane is unaffected by the bulk counterterms, the logarithmic divergences will still show up at the brane. However, we have not as yet performed an exhaustive analysis of all the possible counterterms that could be used. In section VI we discuss an alternative approach based on using GB terms and show that also does not remove the  $\log \epsilon$  divergences.

#### *Position of the brane*

This apparent difficulty at reconciling these results with the intuition from effective field theory may be a consequence of the fact that the boundary condition approach we have used in section II A is not adequate. For instance, in this approach we are interpreting the metric evaluate at  $r = \tilde{r}_{\pm}$

to be the metric of the brane. However in practice the brane is a thick object smoothed over a region of width  $\epsilon$ . We have made the assumption that the variation of the metric across the brane thickness is suppressed by  $\epsilon$ . In fact this is not necessarily the case, suppose for example we assume that on the region  $|r - r_{\pm}| < \epsilon$ , the metric perturbation varies as  $h \approx A + B(r - r_{\pm})^2$ . Matching this form to the known form for  $|r - r_{\pm}| > \epsilon$  shows us that  $h(r = \tilde{r}_{\pm}) - h(r = r_{\pm}) \approx \frac{\kappa}{LH^2 r_{\pm}^2 |f'(r_{\pm})|} {}^{(4)}\tau^{(\pm)}$ . This variation is precisely of the same order as the first correction. The implication is that the coefficient of this correction will vary depending on where within the brane we choose to call the ‘brane position’. In some sense one can renormalize the  $\log \epsilon$  dependence into the ‘position’ of the brane. Allowing for more complicated evolution of the metric in the brane regime will allow for more possibilities to absorb the log dependence. However, we are left with the same conclusion, we need to specify in more detail the physics of the brane in the region  $|r - r_{\pm}| < \epsilon$  in order to make predictions for even the leading order six dimensional corrections to the gravitational wave propagation.

What seems to be lacking is a consistent way of separating the physics corresponding to different scales in the manner of effective field theory. It seems clear that something along these lines needs to be developed before a good understanding of the dynamics of codimension two branes can be achieved. Interesting work in this direction has been done in [4], and extending these ideas to arbitrary codimension branes seems to be crucial to capturing the essential physics of higher codimension braneworlds [25].

## F. Higher orders in the derivative expansion

This derivative expansion may be continued to arbitrary high order. It is important to check that no further divergences are introduced so that our approximations are self consistent. Actually we can show that the next orders are regular in the limit  $\epsilon \rightarrow 0$ . Using the expression (52) with the relations (54) and (55) for the integration constants, we see that  $\partial_r h_2$  is regular everywhere. Since  $\partial_r h_2$  vanishes on the branes,  $h_2$  must be regular at that point as well, and hence the second mode is regular everywhere. Furthermore, if for a given mode  $n \geq 2$ ,  $h_n$  is regular everywhere, then  $f(r)\partial_r h_{n+1}(r) = g_{n+1}(r) = \frac{1}{r^4} \int_r^{r^*} r' h_n(r') dr'$  is similarly finite everywhere. Since  $\partial_r h_{n+1}$  vanishes on the branes, the next mode  $h_{n+1} = \int g/f dr$  is therefore finite on the brane as well and regular everywhere. Therefore only the first mode has a logarithmic divergence, all further modes are finite in the  $\epsilon \rightarrow 0$  limit on the branes and this derivative expansion is hence well-defined and may be continued to higher orders.

## G. Compactification with one brane

It is possible to take a one-brane limit whereby the background constants are chosen so that the tension of one of the branes is zero and at the perturbed level the associated stress energy vanishes. We find that this is a well defined limit since the metric perturbations do not diverge at the smooth pole.

## V. KALUZA-KLEIN LIMIT

### A. Limit of the General solution

The general solutions we have considered so far which describe warped compactifications, contain the more familiar Kaluza-Klein compactifications as a special limit. In particular we may obtain solutions describing  $dS^4 \times S^2$  as follows: Redefine  $r = H^{-1} + \sigma\rho$  and specify the constants  $\mu$ ,  $b$  and  $L$  as

$$\mu = \frac{1}{15}\Lambda H^{-5} - \frac{1}{6}b^2 H^3, \quad \frac{1}{12}b^2 H^6 = \frac{1}{6}\Lambda H^{-2} + \sigma^2 - 1, \quad L = \frac{R_c}{\sigma}, \quad (59)$$

where

$$\frac{1}{R_c^2} = 2\Lambda - 9H^2. \quad (60)$$

Then on taking the limit  $\sigma \rightarrow 0$  we find the metric

$$ds^2 = ds_{dS^4}^2 + \tilde{f}(\rho)^{-1}d\rho^2 + R_c^2 \tilde{f}(\rho)d\varphi^2, \quad (61)$$

with  $\tilde{f}(\rho) = f(\rho)/\sigma^2 = 1 - \rho^2/R_c^2$ . This describes a direct product of four-dimensional de Sitter with Hubble constant  $H$  and a two-sphere with curvature radius  $R_c$ . After a suitable gauge transformation the gauge field is given by  $A = -(bH^4 R_c)\rho d\varphi$  corresponding to a constant flux. Solutions describing  $Minkowski^4 \times S^2$  and  $AdS^4 \times S^2$  are similarly obtained by taking  $H^2 = 0$  and  $H^2 < 0$  respectively.

### B. Kaluza-Klein solution

In the special KK limit, the exact behavior of the different modes may be computed exactly. This provides a useful test on the procedure we have presented, but allows us as well to understand



the behavior and the nature of the divergence beyond the zeroth order. The evolution equation in this limit  $Hr \rightarrow 1$  is simply

$$\partial_\rho \left( \tilde{f} \partial_\rho h \right) = -\square h \quad (62)$$

$$\partial_\rho h|_{\pm R_c \mp \epsilon} = \pm \frac{\kappa}{\pi R_c \tilde{f}(\pm R_c \mp \epsilon)} {}^{(4)}_{\mathcal{T}}(\pm), \quad (63)$$

where the four-dimensional indices  $\mu, \nu$  have been omitted to simplify the notation. The solution is therefore of the form:

$$h = C_1 \mathcal{P}_m(\rho/R_c) + C_2 \mathcal{Q}_m(\rho/R_c), \quad (64)$$

where  $\mathcal{P}_m$  and  $\mathcal{Q}_m$  are the Legendre polynomials and  $m = -1/2 + \sqrt{1/4 + R_c^2 \square}$ . The two integration functions  $C_{1,2}$  can be fixed using the boundary condition (63). Using these, the gravitational waves on each brane  $h^\pm$  are sourced by the matter perturbations in the following way:

$$h^\pm = -\frac{\kappa}{2\pi} \left[ \left( \log \frac{\epsilon}{2R_c} + H_m + H_{-m-1} \right) {}^{(4)}_{\mathcal{T}}(\pm) + \frac{\pi}{\sin m\pi} {}^{(4)}_{\mathcal{T}}(\mp) \right], \quad (65)$$

where  $H_m$  is the  $m^{\text{th}}$  Harmonic number:  $H_m = \sum_{k=1}^m k^{-1}$  which for noninteger  $m$  can be defined by  $H_m = \int_0^1 \frac{1-x^m}{1-x} dx$ . To leading order, we therefore have:

$$\begin{aligned} h^\pm = & -\frac{\kappa}{2\pi R_c^2 \square} \left[ {}^{(4)}_{\mathcal{T}}(+)+ {}^{(4)}_{\mathcal{T}}(-) \right] - \frac{\kappa}{2\pi} \left[ \left( 1 + \log \frac{\epsilon}{2R_c} \right) {}^{(4)}_{\mathcal{T}}(\pm) + {}^{(4)}_{\mathcal{T}}(\mp) \right] \\ & + \frac{\kappa R_c^2 \square}{2\pi} \left[ {}^{(4)}_{\mathcal{T}}(\pm) + \left( 1 - \frac{\pi^2}{6} \right) {}^{(4)}_{\mathcal{T}}(\mp) \right] + \dots \end{aligned} \quad (66)$$

The essential features are the same as those observed in the general case. First we may emphasize that only the first mode diverges and its divergence is very mild since it is only logarithmic in the cutoff parameter  $\epsilon$ . Another important feature is that this divergence only couples to the matter on the specific brane and *not* to the matter content of the other brane.

This represents a consistency check on the procedure used in the low-energy limit to derive each modes separately, since we recover the same result. We recover for both the leading and first order in the expansion, the results obtained in (43) and (57) corresponds precisely with the result in (66) in the limit  $\sigma \rightarrow 0$ .

## VI. EFFECT OF GAUSS-BONNET TERMS

Faced with the fact that higher codimension distributional sources are typically singular in GR, one popular approach to dealing with this is to include higher derivative terms in the bulk which allow distributional sources to be consistent with the equations of motion. In particular,

for codimension two branes the main focus has been on Gauss-Bonnet (GB) terms. The effect of introducing these terms in the bulk is to modify the boundary conditions in such a way that the singular solutions may be discarded [26]. For instance a simple analysis of a codimension two brane in uncompactified Minkowski space shows that at the perturbed level the bulk equations of motion are unaffected and hence the logarithmically diverging solution is still present, however the boundary conditions are modified so that we may ignore this solution. In this section we point out that as soon as we consider codimension two branes in a compactified space this procedure fails. The reason is simply that if we restrict ourself to the solution which is regular at one brane, this solution will inevitably diverge at the other brane (or pole if no brane is present).

Introducing a GB term in the six dimensional action:

$$S_{\text{GB}} = \frac{\alpha}{2\sqrt{\kappa}} \int d^6x \sqrt{-g} \left[ {}^{(6)}R^{ABCD} {}^{(6)}R_{ABCD} - 4 {}^{(6)}R^{AB} {}^{(6)}R_{AB} + {}^{(6)}R^2 \right], \quad (67)$$

where  $\alpha$  is a dimensionless parameter, the presence of this term modifies both the bulk Einstein equation and the junction condition. In particular, the modified Einstein equation is:

$${}^{(6)}G_{AB} + \alpha\sqrt{\kappa} \left( 2\mathcal{R}_{AB}^{\text{GB}} - \frac{1}{2}\mathcal{R}^{\text{GB}} g_{AB} \right) = -\Lambda g_{AB} + \left( F_A{}^C F_{BC} - \frac{1}{4}F^{CD}F_{CD} g_{AB} \right), \quad (68)$$

with

$$\mathcal{R}_{AB}^{\text{GB}} = {}^{(6)}R {}^{(6)}R_{AB} - 2 {}^{(6)}R_{AC} {}^{(6)}R^C{}_B - 2 {}^{(6)}R^{CD} {}^{(6)}R_{ACBD} + {}^{(6)}R_A{}^{DEF} {}^{(6)}R_{BDEF}. \quad (69)$$

In the Kaluza-Klein limit, we consider the bulk metric of the form

$$ds^2 = ds_{\text{dS}^4}^2 + \tilde{f}(\rho)^{-1} d\rho^2 + L^2 \tilde{f}(\rho) d\varphi^2. \quad (70)$$

The limit  $L = R_c$  corresponds to the previous situation where the brane tension vanishes, and the background geometry is smooth without any singularity. Taking  $L < R_c$  corresponds to the rugby ball geometry corresponding to two equal tension branes at each pole. The contribution of the GB term in this geometry is:

$$\left( 2\mathcal{R}^{\text{GB}A}{}_B - \frac{1}{2}\mathcal{R}^{\text{GB}} \delta^A{}_B \right) = \begin{pmatrix} -\frac{12H^2}{R_c^2} \delta^\mu{}_\nu & \\ & -12H^4 \delta^x{}_y \end{pmatrix}, \quad (71)$$

where the indices  $x, y$  run over the two extra dimensions:  $x, y = \varphi, \rho$ . These terms may hence be interpreted as a redefinition of the background parameters, and in particular, the metric (70) is a solution of the modified Einstein equation (68) if

$$\frac{1}{R_c^2} = \frac{1}{1 + 12\alpha\sqrt{\kappa}H^2} [2\Lambda - (9 + 12\alpha\sqrt{\kappa}H^2) H^2] \quad (72)$$

$$A_B dx^B = -\sqrt{1 - 3R_c^2 H^2 + 12\alpha\sqrt{\kappa}H^2 (R_c^{-2} - H^2)} \rho d\varphi. \quad (73)$$

In this limit, the form of the background solution is unaffected by the presence of the GB term, and in particular in the solution that describes  $Minkowski^4 \times S^2$ , ie. when  $H^2 = 0$ , the GB terms vanish at this order. If we concentrate on this specific solution, at the perturbed level, the contribution of these terms is simply:

$$\delta\mathcal{R}^{\text{GB}}{}^\mu{}_\nu = -\frac{1}{R_c^2} \square h^\mu{}_\nu, \quad (74)$$

and all other component vanish. The equation of motion of the gravitation waves is only very slightly modified:

$$\partial_\rho \left( \tilde{f} \partial_\rho h^\mu{}_\nu \right) = - \left( 1 - 2 \frac{\alpha \sqrt{\kappa}}{R_c^2} \right) \square h^\mu{}_\nu. \quad (75)$$

The gravitational waves hence behave identically as in (64), with only a modification of the parameter  $m$ :  $m = -\frac{1}{2} + \sqrt{\frac{1}{4} + R_c^2(1 - 2\alpha\sqrt{\kappa}/R_c^2)}$ . Although the boundary conditions might be modified we argue that the addition of this term can not remove the logarithmic divergence obtained in (65, 66). The key point is that the form of the solution (64) remains completely unaffected by the addition of the GB term. Both Legendre polynomials have a logarithmic divergence when  $\rho \rightarrow \pm R_c$  and the solution is therefore of the form:

$$h(\rho = R_c - \epsilon) \rightarrow -\frac{C_2}{2} \log \frac{\epsilon}{2R_c} \quad (76)$$

$$h(\rho = -R_c + \epsilon) \rightarrow \left( \frac{C_1 \sin m\pi}{\pi} + \frac{C_2 \cos m\pi}{2} \right) \log \frac{\epsilon}{2R_c}. \quad (77)$$

Requiring that the logarithmic divergence cancels on both branes, would fix both parameters  $C_1$  and  $C_2$  to zero, giving rise to the trivial solution. This is clearly an unphysical restriction, and so we conclude that the GB terms do not regulate the branes.

Although we have reached this conclusion for the case of linear perturbations around the Kaluza-Klein solution, it should be clear that the same result will occur in general. In fact it is a fundamental feature of the GB terms that they do not change the nature of the Cauchy problem (i.e. the differential equations remain second order) [27], as a result we are always dealing with ‘two solutions’ both of which will diverge at one or both branes even in the nonlinear case. Furthermore the GB terms will not significantly alter the asymptotic form of the solutions near the brane since they only become significant when the curvature becomes large, and for conical singularities the curvature remains finite and arbitrarily small up to the singularity itself. Demanding that the metric is conical at both branes is too restrictive a condition on the space of solutions that we will be left with essentially a trivial solution. This fact casts serious doubt on whether it makes sense to regulate codimension two branes with GB terms even in the uncompactified case, since if the

regulating physics is local then this case should be no different than the compactified case in the limit in which the extra dimensions are very large.

## VII. CONCLUSION

In the context of an explicit six dimensional braneworld model, we have considered the effective equations describing the propagation of bulk gravitational waves induced by matter sources living on the branes. We have shown that one obtains ordinary gravity at low energies, justifying the boundary condition approach we have used, and have explicitly determined the leading order modifications due to six dimensional effects. We have seen that there is an apparent model dependence in these corrections and have discussed the possibility of renormalizing this in brane or bulk counterterms, in accordance with expectations from effective field theory. Although it seems plausible that the cutoff dependence may be absorbed into counterterms localized near the brane, we find that it is technically difficult to do so. Our work highlights the need to develop a more consistent picture of how to infer bulk dynamics from the boundary conditions imposed by the brane physics using the ideas of effective field theory. We have shown that the use of Gauss-Bonnet terms to regulate codimension two branes is inconsistent when the extra dimensions are compact, suggesting in fact that even in the uncompactified case they represent an unphysical regularization. Since we focussed on tensor perturbations, much of the analysis was straightforward, but many questions remain such as how to deal with the boundary conditions in the scalar sector, and what happens to the logarithmic divergences in the nonlinear theory. Many of the features of codimension two branes discussed here will be present for higher codimension branes, although the codimension two is a specific case that can not be treated the same way as higher codimensional branes [28].

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