A Generalized Model of Advertised Sales

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May 4, 2020

Abstract

To better understand temporary price reductions or 'sales', this paper presents a generalized 'clearinghouse' framework of advertised sales and explores some applications. By viewing the firms as competing in utility and amending the conventional tie-break rule, we allow for multiple dimensions of firm heterogeneity in complex market environments. Moreover, we i) provide original insights into the number and types of firms that use sales, ii) offer new results on how firm heterogeneity affects market outcomes, iii) extend a common empirical 'cleaning' procedure, and iv) analyze a family of activities in sales markets, including persuasive advertising and obfuscation.

Keywords: Sales; Price Dispersion; Advertising; Clearinghouse; Heterogeneity

JEL Codes: L13; D43; M3

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1 Introduction

The evidence of price dispersion within markets is overwhelming, even when products are homogeneous (as reviewed by Baye et al. (2006)). Such price dispersion has a long-standing interest within the areas of industrial organization and marketing, but is also gaining increased attention from finance, development economics, and macroeconomics.[1] Empirical findings show that much price dispersion is due to temporary price reductions or 'sales' (e.g. Hitsch et al. (2019), Kaplan and Menzio (2015), Nakamura and Steinsson (2008), Hosken and Reiffen (2004)). One of the main theoretical explanations for such sales involves mixed strategies that arise from variation in consumers' search frictions and/or the existence of moderate advertising costs.[2] This literature continues to offer deep insights into sales, and is also receiving a renewed interest as an analytical foundation for many broader research areas, including price comparison platforms, advertising, obfuscation, choice complexity, and several issues within macroeconomics.[3]

However, while the predictions of such mixed strategy sales models are frequently consistent with empirical evidence, they often struggle to fully explain the observed differences in firms' pricing and advertising behaviors.[4] In particular, their restricted ability to allow for firm heterogeneity constrains the theoretical and empirical understanding of sales, and inhibits the wider related literatures. Indeed, as Baye and Morgan (2009, p.1151) state *"...little is known about asymmetric models within this class. Breakthroughs on this front would not only constitute a major theoretical advance, but permit a tighter fit between the underlying theory and empirics"*.

In response, this paper presents a substantially generalized 'clearinghouse' framework of advertised sales (e.g. Baye and Morgan (2001), Baye et al. (2004a)) and explores some example applications. Our main contribution is methodological - while its modeling assumptions sometimes differ to existing research, our framework can extend many of the past literature's sales predictions to more complex market settings while allowing for multiple dimensions of firm heterogeneity. Thus, we hope that our framework will

[1]E.g. Woodward and Hall (2012), Allen et al. (2014); Jensen (2007); Nakamura et al. (2018).

[2]E.g. Varian (1980), Burdett and Judd (1983), Stahl (1989), Janssen and Moraga-González (2004); Robert and Stahl (1993), Baye and Morgan (2001). See Baye et al. (2006) and Anderson and Renault (2018) for reviews.

[3]For reviews and recent examples, see Moraga-González and Wildenbeest (2012), Armstrong (2015), Spiegler (2015), and Ronayne (2019). For macroeconomic applications to nominal rigidities, output fluctuations and monetary policy, see Guimaraes and Sheedy (2011), Kaplan and Menzio (2016), and Burdett and Menzio (2017).

[4]E.g. Lach (2002), Baye et al. (2004a), Baye et al. (2004b), Lewis (2008), Chandra and Tappata (2011), Wildenbeest (2011), Giulietti et al. (2014), Lach and Moraga-González (2017), Zhang et al. (2018) and Pennerstorfer et al. (2019). Also see Potters and Suetens (2013) for experimental evidence.

help open up the future analysis of many remaining questions within the sales literature. As some examples, the paper then provides original insights into the number and type of firms that use sales, offers new predictions about the effects of firm asymmetries on market outcomes, and illustrates how the framework can be used to i) extend a 'cleaning' procedure that is commonly used in the empirical literature, and ii) analyze a family of activities in sales markets, including persuasive advertising and obfuscation.

As reviewed by Baye et al. (2006), the original clearinghouse framework considers a symmetric market with a single homogeneous good. Consumers are potentially split into 'non-shoppers' that are only willing to buy from a designated firm, and 'shoppers' that can buy from any firm. Firms choose their price, and whether to inform consumers of this price via some advertising channel. As consistent with observed sales behavior, the equilibrium involves each firm randomizing between selecting a high price without advertising, and advertising a lower price drawn from some support. The seminal model of sales by Varian (1980) obtains as the limiting case when advertising costs tend to zero.

We modify this clearinghouse framework in two important respects. First, we recast the firms as competing in (net) utility rather than prices. By drawing on the (symmetric, pure-strategy) model of competition in the utility space by Armstrong and Vickers (2001), we let each firm i select its utility, u_i, with a per-consumer profit function, $\pi_i(u_i)$, that depends upon the firm's underlying demand, products, costs, and pricing technology. With little increase in computation, this facilitates a high level of generality across complex market settings, involving downward-sloping demand, multiple products, or two-part tariffs, where the analysis of sales would often be otherwise impenetrable.

Second, we make a subtle change to the tie-break rule when the shoppers are indifferent over which firm to trade with. The existing literature assumes shoppers i) trade exclusively with advertising firm(s) in any tie between advertising and non-advertising firms, and ii) mix between the tied firms with equal probability in any other form of tie. Although consistent with an advertising channel involving a price-comparison platform where shoppers face additional visit costs to buy from non-listed firms (Baye and Morgan (2001)), it turns out that this 'traditional' tie-break rule impedes the analysis of sales under firm heterogeneity. In particular, it causes a substantial loss in tractability by i) making the exact form of equilibrium become dependent upon parameter sub-cases, and ii) prompting the existence of mass points in some firms' advertised price distributions.[5]

[5]For instance, even in a simple unit demand duopoly where firms only differ in their shares of non-shoppers, the equilibrium in Arnold et al. (2011) has two parameter sub-cases and one firm uses a mass point in advertised prices. Furthermore, the equilibrium does not converge to standard sales equilibria as advertising costs tend to zero (e.g. Narasimhan (1988)).

To resolve this problem, we take a different approach within a setting where shoppers receive all adverts before making their visit decisions. Here, we are free to select any tie-break rule because shoppers should be willing to buy from any advertising or non-advertising firm with the same expected utility. Moreover, rather than treating the tie-break rule as a modeling assumption, we are free to specify the tie-break rule *as part of equilibrium* in line with Simon and Zame's (1990) concept of 'endogenous sharing rules' (which they use to offer general results about equilibrium existence in n-player games). Under our assumptions, we show that the 'equilibrium tie-break rule' is uniquely defined for any firm that uses sales. This rule partially offsets any firm heterogeneities and ensures that all firms that use sales have the same incentive to employ a common upper bound in advertised utilities. In symmetric settings, it coincides with the traditional tie-break rule. However, in asymmetric settings, it offers a unique level of tractability by removing any parameter sub-cases and eliminating any mass points in advertised utility distributions. As such, it allows us to simultaneously permit i) any variation in firms' shares of non-shoppers, ii) any variation in firms' advertising costs, and iii) considerable variation in firms' profit functions. In equilibrium, firms may vary in advertising probabilities, utility distributions, and profits depending on the level and form of heterogeneity.[6]

Sections 2 and 3 introduce the framework and equilibrium analysis. We first present the equilibrium under duopoly, and show how it can generalize many predictions from the previous literature to more complex market settings with multiple forms of heterogeneity.[7] In addition, we offer further new insights by characterizing some common forms of sales that have remained unstudied within the clearinghouse literature, including cases where firms use two-part tariffs or non-price variables such as package size (e.g 'X% Free'). We then present the equilibrium for $n > 2$ firms. Here, the previous literature with heterogeneous firms is particularly scant - in a simple setting of unit demand and zero advertising costs, it suggests that only two firms can ever engage in sales behavior (Baye et al. (1992), Kocas and Kiyak (2006) and Shelegia (2012)). In contrast, and in better line with typical empirical findings (e.g. Lach (2002), Lewis (2008), Chandra and Tappata (2011)), our framework explains how any number of heterogeneous firms $k^* \in [2, n]$ can engage in equilibrium sales. In particular, we demonstrate how a rise in the cost of informative advertising can lead to an *increase* in the number of firms that use advertised

[6]The variation in profit functions is subject to a condition that is implicit within all of the existing literature - each firm would offer the same utility under monopoly, $u_i^m = u^m \geq 0 \; \forall i$. This does not restrict each firm's monopoly profits, and is innocuous in several market settings, including unit demand.

[7]Among many others, these include symmetric models such as Varian (1980), Baye et al. (2004a), Baye et al. (2006), and Simester (1997), and asymmetric models, such as Narasimhan (1988), Baye et al. (1992), Kocas and Kiyak (2006), and Wildenbeest (2011).

sales. Intuitively, despite the direct cost increase, higher advertising costs can prompt more firms to use sales by softening competition for the shoppers. Thus, if the costs of informative advertising fall in the digital era, we predict that fewer firms will engage in sales behavior. Finally, we provide a broad characterization of the types of firms that are most likely to use advertised sales. Ceteris paribus, these are firms with relatively low shares of loyal consumers, low advertising costs, and high profitability (under reasonable assumptions on market conditions). These results offer some clear empirical predictions, but currently remain untested within the literature.[8]

Section 4 explores some applications to illustrate how our framework can be used for future research. Section 4.1 uses the framework to assess and extend a common procedure within the large empirical literature on sales and price dispersion. This 'cleaning' procedure attempts to remove the effects of firm-level heterogeneities from raw price data by retrieving the residuals from a price regression involving firm-level fixed-effects.[9] Wildenbeest (2011) verifies the theoretical validity of the procedure in a setting of unit demand and zero advertising costs where the firms differ in quality and costs, but share the same value-cost margin. However, our more general framework shows how the procedure is invalid i) for downward-sloping demand (because the relationship between firms' offered prices and utilities becomes non-linear), and ii) under unit demand outside Wildenbeest's condition (because the firms no longer offer the same average utility). Moreover, we then offer the basis for modified methodologies that may be applied instead.

Section 4.2 uses the framework to study a family of games where each firm's share of non-shoppers is determined endogenously as a function of the firms' actions prior to sales competition. Among other examples, such actions are consistent with forms of persuasive advertising, sales-force methods, and obfuscation. Starting with Chioveanu (2008), Carlin (2009) and Wilson (2010), some related streams of literature have become popular in recent years.[10] However, almost all such models are based upon simple market settings with zero costs of informative advertising. An exception is Baye and Morgan (2009), but due to the consequent difficulties of analyzing asymmetries, they are unable to consider all possible subgames. Instead, they show the existence of a continuum of symmetric Nash equilibria

[8]Existing empirical studies often focus on different factors affecting firms' use of sales, such as market information, competition, or rivals' behavior (e.g. Lewis (2008), Chandra and Tappata (2011), Shankar and Bolton (2004), Ellickson and Misra (2008)).

[9]The residuals are then used in i) reduced-form studies of price dispersion, e.g. Sorensen (2000), Lach (2002), Brown and Goolsbee (2002), Barron et al. (2004), Lewis (2008), Chandra and Tappata (2011), Pennerstorfer et al. (2019), and Sherman and Weiss (2017), or ii) structural estimations, e.g. Wildenbeest (2011), Moraga-González et al. (2013), Giulietti et al. (2014), Allen et al. (2014), and An et al. (2017).

[10]For a review, and for some wider related models, see Grubb (2015) and Spiegler (2015).

and a unique symmetric equilibrium in *secure strategies*. In contrast, our framework can characterize the unique symmetric *subgame-perfect* Nash equilibrium across a general market setting. Moreover, while their equilibria imply that an increase in the cost of informative advertising can have a positive, negative or zero effect on the prior actions, our framework offers a unique empirically testable prediction. As the costs of informative advertising decrease, sales markets should experience i) a reduction in loyalty-enhancing actions, such as persuasive advertising, and ii) an increase in loyalty-reducing actions, such as some forms of obfuscation. This offers a first theoretical connection between the costs of informative advertising and equilibrium levels of persuasive advertising, and gives a new advertising-costs-based explanation for why firms increase their obfuscation tactics in response to advances in digital technology (e.g. Ellison and Ellison (2009)).

Finally, Section 4.3 provides some new comparative static results that could be utilized for future research regarding the effects of *firm-level* characteristics on sales and market performance. For instance, standard results show that an industry-wide increase in advertising costs deters the use of sales and raises firms' profits. However, we can isolate the effects of an increase in a single firm's advertising costs - we show that firms still reduce their use of sales, but that it is rival rather than own advertising costs that matter in determining profits. Similarly, we isolate the effects of an increase in an individual firm's share of non-shoppers. In contrast to results under the traditional tie-break rule (e.g. Arnold et al. (2011)), this induces the firm to set lower average utility offers as more consistent with standard results under zero advertising costs (e.g. Narasimhan (1988)). Lastly, we study changes in firms' profit functions or 'profitability'. These results are new even in a symmetric industry-wide setting - an industry-wide increase in profitability, such as a reduction in costs or an increase in per-consumer demand, will always increase firms' use of sales. Further, an increase in a single firm's profitability will increase its sale probability and prompt it to use higher average offers in most common market settings.

Related Literature: Armstrong and Vickers (2001) introduced competition in utility to study price discrimination in a symmetric, pure-strategy setting. In contrast, we transfer their utility approach into an asymmetric (clearinghouse) model to study mixed strategy sales. Some past sales papers have referred to competition in utility (Simester (1997), Hosken and Reiffen (2007), Wildenbeest (2011), Dubovik and Janssen (2012), Anderson et al. (2015)). However, they only use it to compute equilibria in specific settings, and do not use the associated profit function, $\pi(u)$, to explore any general results or implications. In a recent paper, Armstrong and Vickers (2019) take a different line. They use a dual approach by writing an individual consumer's surplus as a function of the associated per-consumer profit to analyze the effects of price discrimination within a duopoly sales model with asymmetric shares of non-shoppers. Some other work also exists in non-clearinghouse settings. First, Anderson et al. (2015) allow for firm heterogeneity in a model where firms must advertise to earn positive profits, and where all consumers are shoppers. Contrary to us, they find that only two firms can ever use advertised sales when firms are heterogeneous. As such, they cannot analyze how market factors affect the number and type of firms that use sales, or connect to the larger theoretical or empirical clearinghouse literature. Instead, they focus on some interesting results regarding equilibrium selection and welfare. Second, some papers consider clearinghouse-style frameworks but under an assumption of horizontally differentiated products (e.g. Galeotti and Moraga-González (2009), Moraga-González and Wildenbeest (2012)). These papers exhibit pure-strategy price equilibria without price dispersion, and therefore do not share our focus on sales.

2 Model

Let there be $n \geq 2$ firms, $i \in \{1, ..., n\}$. Also suppose there is a unit mass of risk-neutral consumers that have a zero outside option. Each firm i competes by choosing a utility offer (net of any associated payments), $u_i \in \mathbb{R}_{\geq 0}$. All consumers have identical preferences and so each consumer values firm i's offer at precisely u_i.[11]

The maximum possible profit that firm i can extract per consumer when providing an offer, u_i, is defined as $\pi_i(u_i)$. Following Armstrong and Vickers (2001), the exact source of utility and form of profit function can depend upon a rich set of demand, product, and cost conditions.[12] We assume that $\pi_i(u_i)$ is independent of the number of consumers served.[13] Further, to ensure that any sales equilibrium is well-behaved, we make some mild technical assumptions: i) $\pi_i(u_i)$ is strictly quasi-concave in u_i with a unique maximizer at firm i's 'monopoly utility' level, $u_i^m \in [0, \infty)$, ii) $\pi_i(u_i^m) \equiv \pi_i^m > 0$, iii) $\pi_i(u_i)$ is twice

[11]This assumption of identical preferences is standard within the mixed strategy sales literature. If, in contrast, consumers' preferences were sufficiently heterogeneous then any mixed strategy sales equilibrium would be replaced by a pure strategy non-sales equilibrium.

[12]As two simple examples, consider the following where firm i sells a single good at price p_i with marginal cost c_i. First, suppose each consumer has a unit demand and values firm i's good at V_i. Firm i's utility offer is the associated consumer surplus, $u_i = V_i - p_i$, while its profits per consumer equal $\pi_i(u_i) = V_i - c_i - u_i$. Second, let each consumer have a linear demand for firm i's good, $q_i(p_i) = a_i - b_i p_i$. Firm i's utility offer equals $u_i = (a_i - b_i p_i)^2/2b_i$, and by using $p_i = (a_i - \sqrt{2b_i u_i})/b_i$, one can write $\pi_i(u_i) = \frac{1}{b_i}[a_i - b_i c_i - \sqrt{2b_i u_i}][\sqrt{2b_i u_i}]$. Further examples including more general downward-sloping demand, multi-product firms, two-part tariffs, and non-price sales are later detailed in Appendix A.

[13]Similar to Armstrong and Vickers (2001), this is needed for the profit function to remain well-defined. However, it rules out some empirically relevant features such as scale-economies or capacity constraints.

continuously differentiable for all $u_i > u_i^m$, and iv) there exists a finite break-even utility $\hat{u}_i > u_i^m$ where $\pi_i(\hat{u}_i) = 0$.

Consumers are initially uninformed about firms' utility offers. However, each firm can choose whether or not to advertise in order to inform consumers of its offer, $\eta_i \in \{0, 1\}$. In line with some previous versions of the clearinghouse model (e.g. Baye et al. (2004a)), we assume i) all advertising must be truthful, ii) any advert is observed by all relevant consumers, and iii) firms' advertising costs are exogenous. However, in contrast to the previous literature, we also assume that iv) advertising costs can differ across firms, as consistent with different advertising capabilities or channels, and v) each firm's advertising cost is strictly positive, $A_i > 0\ \forall i$.[14]

There are two types of consumers, 'non-shoppers' and 'shoppers', in respective proportions, $\theta \in (0, 1)$ and $(1 - \theta)$. Non-shoppers ignore all adverts. They simply visit their designated 'local' firm and buy according to their underlying demand function, or exit. Our framework allows the firms to have asymmetric shares of non-shoppers, $\theta_i > 0$, with $\sum_{i=1}^{n} \theta_i = \theta$. In contrast, the remaining 'shopper' consumers pay attention to adverts and can buy from any firm. However, to simplify exposition, we assume that shoppers can only visit one firm. Hence, shoppers choose between i) visiting an advertising firm to buy from its known utility offer, ii) visiting a non-advertising firm to discover its utility offer and potentially buy, or iii) exiting the market immediately.[15]

We analyze the following game. In Stage 1, each firm chooses its utility offer, $u_i \in \mathbb{R}_{\geq 0}$, and its advertising decision, $\eta_i \in \{0, 1\}$. To allow for mixed strategies, define i) $\alpha_i \in [0, 1]$ as firm i's advertising probability, ii) $F_i^A(u)$ as firm i's utility distribution when advertising, and iii) $F_i^N(u)$ as firm i's utility distribution when not advertising, both on support $\mathbb{R}_{\geq 0}$. In Stage 2, consumers observe any adverts, form beliefs about the (expected) utility provided by any non-advertising firm, u_i^e, and then make their visit and purchase decisions in accordance with the strategies outlined above.

We define a 'tie' as any situation where the shoppers are indifferent over visiting a set of firms, T, where $|T| \in [2, n]$. Any firm within the tied set, $i \in T$, must have advertised (or be expected to offer) a common level of utility, u, while any firm outside the tie, $j \neq T$, must have advertised (or be expected to offer) a utility strictly lower than u. A 'tie-break rule' then assigns the probability (or proportion) with which the shoppers visit each tied

[14]Strictly positive advertising costs help ensure that each firm refrains from advertising with positive probability in our later equilibrium. This is needed for our tie-break rule to be effective in providing tractability. See footnote 22 for more.

[15]These assumptions can be substantially generalized by allowing shoppers to visit firms sequentially provided that i) the cost of any first visit is not too large, and ii) each shopper may only purchase from a single firm ('one-stop shopping'). For technical details see Appendix C2.

firm. In particular, for any $i \in T$, let firm i's 'tie-break probability', $x_i(\eta, u, T) \in [0, 1]$, depend upon the tied firms' advertising decisions, $\eta = \{\eta_i\}_{i \in T}$, the tied utility level, u, and the set of tied firms, T, where $\sum_{i \in T} x_i(\eta, u, T) = 1$.

The existing literature assumes that shoppers i) trade exclusively and symmetrically with advertised firm(s) in any tie between advertised and non-advertised firms, and ii) trade symmetrically with all tied firms in any other form of tie. This can be summarized as follows for $n = 2$: $x_1((1, 0), u, \{1, 2\}) = 1$, $x_1((0, 1), u, \{1, 2\}) = 0$, and $x_1((1, 1), u, \{1, 2\}) = x_1((0, 0), u, \{1, 2\}) = 0.5$ for all u. In contrast, while we also assume that $x_i(\cdot)$ is independent of u, we depart from the literature's approach in two ways.

First, we assume that the tie-break probabilities are independent of firms' advertising decisions. In particular, Assumption X lets the tie-break probabilities depend only on the set of tied firms, T. While this assumption may be restrictive in some situations, it permits sufficient flexibility for us to manipulate the tie-break rule. Furthermore, Assumption X remains consistent with our context where shoppers receive all adverts before making their visit decisions and so have no reason to favor advertising firms.[16]

$$x_i(\eta, u, T) = x_i(T) \quad \forall \eta, u \qquad \text{(Assumption X)}$$

Second, instead of assuming that the tie-break probabilities are otherwise symmetric, $x_i(T) = |T|^{-1}$, we build on the concept of endogenous sharing rules (Simon and Zame (1990)) to specify them endogenously as part of the game equilibrium. Hence, we focus on perfect Bayesian equilibria (PBE), where in addition to specifying the players' equilibrium strategies and beliefs, we also specify the profile of equilibrium tie-break probabilities, $x^*(T)$= $\{x_1^*(T), ..., x_n^*(T)\}$, for all possible T. As detailed below, this approach, together with Assumption X, will allow us to manipulate the tie-break probabilities to help improve equilibrium tractability.

Finally, we discuss two remaining assumptions. First, while our framework offers a significant increase in generality, it cannot avoid an assumption that is implicit across the *entire* previous literature. We are the first to state it:

$$u_i^m = u^m \quad \forall i \qquad \text{(Assumption U)}$$

This does not require firms to have the same monopoly price or the same monopoly profits, only the same level of monopoly utility. However, Assumption U is not innocuous. Although it is trivially satisfied under unit demand or two-part tariffs because u_i^m is

[16]Footnote 19 also later explains how Assumption X can be partially relaxed.

then always zero (given our assumption that all consumers have identical preferences), it is restrictive under downward-sloping demand and linear prices. Here, one must either i) restrict attention to symmetric profit functions $\pi_i(u) = \pi(u)$, or ii) introduce some binding lower bound on firms' utility offers, $u^{min} \geq \min\{u_i^m, ..., u_n^m\}$, as consistent with an unmodeled competitive fringe, or a price cap policy when firms' profit functions differ only in costs. Outside Assumption U, the power of our tie-break approach is lost.[17]

Second, we let all firms have some basic potential to use advertised sales. Specifically, we let each firm i's profits from not advertising with $u_i = u^m$ and selling only to its non-shoppers, $\theta_i \pi_i^m$, be less than its profits from advertising an offer just above u^m and gaining the shoppers, $[\theta_i + (1-\theta)]\pi_i^m - A_i$. The resulting Assumption A is relatively mild and just ensures that each firm's advertising cost is not prohibitively large.

$$A_i \leq (1-\theta)\pi_i^m \quad \forall i \qquad \text{(Assumption A)}$$

3 Equilibrium Analysis

Section 3.1 considers some preliminary findings before Section 3.2 provides some results on the equilibrium tie-break probabilities. Section 3.3 then completes the equilibrium analysis for duopoly ($n = 2$), before Section 3.4 tackles the more complex case of a broader oligopoly ($n > 2$). Any formal proofs are listed in Appendix B.

3.1 Preliminary Results

Lemma 1. *In any equilibrium, each firm i must set $u_i = u^m$ if it does not advertise, $\eta_i = 0$, and set $u_i > u^m$ if it advertises, $\eta_i = 1$.*

Any firm that does not advertise cannot use its unobserved utility offer to attract more consumers. Instead, it will find it optimal to set the monopoly utility level, u^m, because its non-shoppers and any visiting shoppers cannot visit elsewhere. In addition, no firm will ever wish to advertise an offer of u^m. Specifically, any firm i that advertises u^m (with positive probability) could profitably deviate by not advertising. By doing so, it would reduce its advertising costs, $A_i > 0$, while having no impact on its units sold since $x_i(T)$ is independent of advertising decisions via Assumption X.

[17] Suppose $u_i^m > u^m$ for some i. In the event where all firms set their monopoly utility, the shoppers will then strictly prefer to visit firm i. As later explained, this implies there will be no ties in equilibrium. Thus, the tie-break probabilities are redundant and cannot be used to ensure a tractable equilibrium.

Lemma 1 offers several implications. First, in equilibrium, if no advert is observed from firm i, then shoppers must correctly believe $u_i^e = u^m$. Moreover, if no adverts are observed from any firm, then shoppers must believe that all the firms are tied with $u_i^e = u^m$ for all i via Assumption U. Second, if firm i advertises, firm i's lowest advertised utility will always be strictly larger than its non-advertised utility, u^m. Therefore, from this point forward, we will simply refer to firm i's utility distribution *unconditional on advertising*, $F_i(u)$, where firm i sets u^m without advertising with probability $1 - \alpha_i = F_i(u^m) \in [0, 1]$, and uses advertised sales on $u > u^m$ with total probability α_i.

Firm i will then be said to use 'sales' if it advertises an offer above u^m with positive probability, $\alpha_i > 0$. In any given equilibrium, we will refer to k^* as the number of firms that use sales and K^* as the set of firms that use sales. A 'sales equilibrium' will be defined as any equilibrium where $k^* \geq 1$. In any given sales equilibrium, we will denote $\bar{u} > u^m$ as the minimum level of u for which $F_i(u) = 1$ for all i. By adapting standard arguments, we can state:

Lemma 2. *In any sales equilibrium, at least two firms use sales, $k^* \geq 2$, and for at least two firms i and j, u is a point of increase of $F_i(u)$ and $F_j(u)$ at any $u \in (u^m, \bar{u}]$. Any firm which uses sales, $i \in K^*$, has no point masses in $F_i(u)$ for $u > u^m$ and advertises with an interior probability, $\alpha_i = 1 - F_i(u^m) \in (0, 1)$. When $n = 2$, any sales equilibrium has both firms advertising on $(u^m, \bar{u}]$ without gaps.*

Thus, any firm that uses sales will set an unadvertised 'regular' offer of u^m with probability $(1 - \alpha_i) \in (0, 1)$, together with randomized discounted offers $u \in (u^m, \bar{u}]$ with probability α_i.[18] When $n = 2$, Lemma 2 demonstrates that any sales equilibrium will involve both firms using sales on the same full support $(u^m, \bar{u}]$. However, when $n > 2$, similar to the insights of Baye et al. (1992) for zero advertising costs, it implies that there may be multiple forms of sales equilibria with firms using different supports. Indeed, provided at least two firms mix on any given interval within $(u^m, \bar{u}]$, other advertising firms need not be active on the same interval. Hence, to avoid these significant complications and potential multiplicities when $n > 2$, we follow Chioveanu (2008) by focusing only on sales equilibria where all advertising firms use the full convex support, $(u^m, \bar{u}]$. Hence, for all i with $\alpha_i > 0$, we assume u is a point of increase of $F_i(u)$ for all $u \in (u^m, \bar{u}]$.

[18] This sales behavior is consistent with the empirical evidence cited in the introduction. While Hitsch et al. (2019) find that firms use different regular prices, this need not be inconsistent with all firms using regular offers of u^m provided the firms differ in (perceived) quality or service levels.

3.2 Equilibrium Tie-Break Probabilities

Lemma 3. *Ties can only occur with positive probability in equilibrium when all firms refrain from advertising,* $\eta_i = 0 \ \forall i$.

In contrast to the existing literature where ties are possible between non-advertising and advertising firms, Lemma 3 shows that ties are only possible in our model when *all* firms choose not to advertise. Hence, the only tie-break probability that can be relevant in equilibrium is $x_i(N)$ where N denotes the set of all firms, and so from this point forward, we simply denote $x_i(N) \equiv x_i$, and $x^* = \{x_1^*, ..., x_n^*\}$ as a set of equilibrium tie-break probabilities. This difference to the literature arises from our Assumption X which ensures that the tie-break rule is independent of firms' advertising decisions.[19]

For a given x_i^*, firm i's expected profits from not advertising with $u_i = u^m$ equal

$$\pi_i^m[\theta_i + (1-\theta)x_i^*\Pi_{j\neq i}(1-\alpha_j)]. \tag{1}$$

Firm i will always trade with its θ_i non-shoppers, but it will also trade with the $(1-\theta)$ shoppers if i) all other firms also choose not to advertise, which occurs with probability $\Pi_{j\neq i}(1-\alpha_j)$, and ii) the shoppers visit i in the subsequent tie, which occurs with tie-break probability x_i^*.

If firm i uses sales in equilibrium, then we know from Lemma 2 that it must use an interior probability, $\alpha_i \in (0,1)$. Hence, under the requirements of a mixed-strategy equilibrium and our assumption that all advertising firms use the full support $(u^m, \bar{u}]$, firm i must expect to earn the same level of equilibrium profits, $\bar{\Pi}_i$, from i) setting $u_i = u^m$ and not advertising, and ii) advertising any $u_i \in (u^m, \bar{u}]$. We can then state the following.

Lemma 4. *Consider any sales equilibrium with a given set of tie-break probabilities,* x^*. *Then, if firm* i *uses sales, its equilibrium profits are uniquely defined as*

$$\bar{\Pi}_i = \theta_i \pi_i^m + \frac{x_i^*}{1-x_i^*}A_i. \tag{2}$$

Hence, the equilibrium profits of any firm i that uses sales will derive from its share of non-shoppers, θ_i, its advertising costs, A_i, and its equilibrium tie-break probability, x_i^*.

[19] Assumption X can be partially relaxed. If, instead, advertising firms were assigned (slightly) lower tie-break probabilities than non-advertising firms, then our results would not change - advertising u^m would still be dominated and ties could still only occur when all firms refrain from advertising. However, if advertising firms were assigned significantly higher tie-break probabilities than non-advertising firms, we would move closer to the existing literature. Specifically, advertising u^m would not be dominated and so ties could also exist between advertising and non-advertising firms at u^m in ways that would generate the literature's associated loss in equilibrium tractability.

To begin to understand more about the equilibrium tie-break probabilities, it is useful to now define the following expressions, where $\theta_{-i} = \theta - \theta_i$ refers to the total share of non-shoppers that are not designated to firm i:

$$\chi_i(u) \equiv 1 - \frac{A_i}{\pi_i(u)(1-\theta_{-i}) - \theta_i \pi_i^m} \tag{3}$$

$$\tilde{u}_i \equiv \pi_i^{-1}\left(\frac{\theta_i \pi_i^m + A_i}{1-\theta_{-i}}\right). \tag{4}$$

Intuitively, $\chi_i(u)$ is the level of x_i^* at which firm i's equilibrium profits, (2), are equal to the maximum profits that firm i could obtain from advertising an offer, u, and successfully attracting all the shoppers, $\pi_i(u)(1-\theta_{-i}) - A_i$. Further, $\tilde{u}_i$ can then be understood as the level of utility at which $\chi_i(u) = 0$; where firm i's maximum profits from advertising are equal to its lowest possible profits from not advertising, $\theta_i \pi_i^m > 0$. Hence, firm i will never advertise $u_i > \tilde{u}_i$. Formally, we define $\chi_i(u)$ on $[u^m, \tilde{u}_i]$ with $\chi_i'(u) < 0$. Using Assumption A, we then know that $\chi_i(u^m) = 1 - \frac{A_i}{(1-\theta)\pi_i^m} \in [0, 1)$ is weakly larger than $\chi_i(\tilde{u}_i) = 0$, or equivalently, $\tilde{u}_i \geq u^m$ for all i.

Lemma 5. *Consider any sales equilibrium with a given upper utility bound $\bar{u} > u^m$. Then, if firm i uses sales, i) the upper bound must satisfy $\bar{u} \leq \tilde{u}_i$, and ii) firm i's equilibrium tie-break probability is uniquely defined as:*

$$x_i^* = \chi_i(\bar{u}). \tag{5}$$

If firm i uses sales on $u_i \in (u^m, \bar{u}]$, then we know from above that $\tilde{u}_i$ must be weakly larger than $\bar{u}$ in order for firm i to be willing to set offers up to $\bar{u}$. Moreover, if firm i uses sales, then its equilibrium profits, $\bar{\Pi}_i$, must equal its expected profits from advertising any $u_i \in (u^m, \bar{u}]$. Hence, by setting $\bar{\Pi}_i$ equal to its expected profits from advertising $\bar{u}$, where it would attract the shoppers for sure, $(1-\theta_{-i})\pi_i(\bar{u}) - A_i$, we know that firm i's equilibrium tie-break probability, x_i^*, must equal $\chi_i(\bar{u})$; any other $x_i \neq \chi_i(\bar{u})$ is incompatible with a sales equilibrium under our assumptions. Thus, the equilibrium tie-break probabilities for any firms using sales must ensure that all such firms have exactly the same incentive to employ the common upper utility bound, $\bar{u}$.

To help understand this further, first consider a fully symmetric setting. Here, the firms already have identical incentives and so (5) implies that any firms that engage in sales will share a common equilibrium tie-break probability. More importantly, now consider an example asymmetric setting where only firms 1 and 2 engage in sales, and where firm 1 is relatively more willing to advertise higher utilities, $\tilde{u}_1 > \tilde{u}_2$.[20] In equilibrium, to ensure that the firms have the same incentive to adopt a common upper bound, (5) implies that firm 1 must be assigned a larger equilibrium tie-break probability, $x_1^* > x_2^*$. This acts to make firm 1 (firm 2) relatively less (more) aggressive by enhancing (reducing) its expected payoffs from not advertising. However, as later shown, it is not the case that x_1^* and x_2^* prompt the firms to play symmetric strategies. Thus, the equilibrium tie-break probabilities do not fully neutralize the firms' heterogeneities, they just partially offset them.[21]

3.3 Duopoly ($n = 2$)

As now formalized, the equilibrium under duopoly is unique and takes one of two forms. When advertising costs are sufficiently low, there is a sales equilibrium where both firms engage in sales, otherwise, there is a non-sales equilibrium.

Lemma 6. *When $n = 2$, any sales equilibrium has a unique upper utility bound, $\bar{u}$, (implicitly) defined by (6), and each firm's advertising probability and offer distribution are uniquely defined by (7) and (8).*

$$\chi_1(\bar{u}) + \chi_2(\bar{u}) = 1 \tag{6}$$

$$\alpha_i = 1 - \frac{A_j}{x_i^*(1-\theta)\pi_j^m} \tag{7}$$

$$F_i(u) = \frac{\theta_j(\pi_j^m - \pi_j(u)) + (A_j/x_i^*)}{(1-\theta)\pi_j(u)} \tag{8}$$

When $n = 2$, previous results have shown that any sales equilibrium must involve both firms and that $x_i^* = \chi_i(\bar{u})$. Hence, as the tie-break probabilities must sum to one, the unique equilibrium upper bound must satisfy (6). One can then derive each firm's advertising probability and offer distribution from the fact that each firm must earn its equilibrium profits over $u = u^m$ and $u \in (u^m, \bar{u}]$.[22]

[20] From inspection of (4), this could arise because firm 1 has a relatively lower advertising cost, A_1, a relatively lower share of non-shoppers, θ_1, a relatively lower level of per-consumer profits at u^m, π_1^m, and/or a relatively higher level of per-consumer profits at $\bar{u}$, $\pi_1(\bar{u})$.

[21] One may ask why the shoppers should behave in accordance with (5). Similar questions arise within the wider concept of endogenous sharing rules. As noted by Simon and Zame (1990, p.863), "The answer is, as always, that equilibrium theory never explains why any agents would act in any particular way. Equilibrium theory is intended to explain how agents behave, not why."

[22] In an extreme case where the firms are asymmetric but $A_i = A_j \to 0$, the only way for the firms to

Proposition 1 now shows that the characterized sales equilibrium exists uniquely if advertising costs are sufficiently low, $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$. In contrast, if advertising costs are higher, the firms are deterred from competing against each other - instead, there exists a unique non-sales equilibrium where both firms select u^m and refrain from advertising.

Proposition 1. *Given Assumptions X, U and A, the game has the following unique equilibrium (where consumers always expect non-advertising firms to offer u^m):*

a) If advertising costs are low, $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$, each firm i offers $u_i = u^m$ and does not advertise with probability $(1 - \alpha_i) \in (0, 1)$ according to (7), and advertises a sale offer $u_i \in (u^m, \bar{u}]$ according to (8) with probability α_i, where the upper bound, $\bar{u}$, solves (6), and firm i's equilibrium tie-break probability, $x_i^* = 1 - x_j^* \in (0, 1)$, is given by (5).

b) If advertising costs are high, $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} \geq 1 - \theta$, both firms offer $u_i = u^m$ and never advertise, $\alpha_i = 0$, and firm i's equilibrium tie-break probability equals $x_i^* = 1 - x_j^* \in [\chi_i(u^m), 1 - \chi_j(u^m)]$.

One can demonstrate the implications of Proposition 1 by specifying the exact source of utility and profit function. As a very simple example, consider a symmetric setting with unit demand. Following footnote 12, let $u_i = V - p_i$ and $\pi(u_i) = V - c - u_i$, where $u^m = 0$ and $\pi^m = V - c$. Proposition 1 then implies a clearinghouse equilibrium with $x_i^* = 0.5$, $\bar{\Pi}_i = \frac{\theta(V-c)}{2} + A$, $\alpha_i = 1 - \frac{2A}{(1-\theta)(V-c)}$, and $\bar{u} = \frac{2(1-\theta)(V-c)-4A}{2-\theta}$. By using $F_i(p) = 1 - F_i(u)$, one can then further derive $F_i(p) = 1 - \frac{\theta(V-p)+4A}{2(1-\theta)(p-c)}$, with $p^m = V - u^m = V$ and $\underline{p} = V - \bar{u} = c + \frac{\theta(V-c)+4A}{2-\theta}$. Further details and additional example settings involving downward-sloping demand, and multi-product firms are included in Appendix A.

Appendix A also shows how Proposition 1 can be used to characterize two common forms of sales that have remained unstudied within the clearinghouse literature. First, it characterizes sales behavior when firms use two-part tariffs. Existing theoretical work is very limited on this - we only know of Hendel et al. (2014) which shows how sales with non-linear prices can emerge in a dynamic context with storable goods. In contrast, our framework considers a simpler clearinghouse setting while allowing for full asymmetry. We show that equilibrium sales will involve marginal cost pricing and firms mixing between not advertising a high fixed fee, and advertising a stochastic lower fixed fee. While there is little empirical analysis available, our predictions are consistent with several anecdotal examples and some wider forms of evidence.[23]

share a common upper utility bound is for $x_i^* \to 1$ and $x_j^* \to 0$. This limit equilibrium converges to the equilibrium of a model that allows for $A = 0$ explicitly without our tie-break rule. See Appendix C1 for full details.

[23] In practice, the use of such tariffs may also be driven by heterogeneous consumer preferences. However,

Second, it characterizes sales when firms hold prices constant but compete with some non-price variable. This setting covers a broad set of commonly observed marketing practices, including i) temporary extensions to package size or quantity, such as 'X% Free' offers and 'bonus packs', ii) temporary increases in product quality or content, such as the inclusion of free items or 'premiums', and iii) other temporary increases in product value, such as the use of consumer finance deals, prize draws, or charity donations (see the discussions in Chen et al. (2012), Palazon and Delgado-Ballester (2009)). As consistent with these phenomena, we show that equilibrium sales will involve firms mixing between not advertising a minimum 'regular' package size/product value, and advertising a sale with an increased package size/product value.

3.4 $n > 2$ Firms

We now extend the analysis beyond the simpler case of duopoly to offer new results about the number and type of firms that use sales in markets with $n > 2$ firms. Here, the sales literature with heterogeneous firms is scant because existing models quickly become intractable. Most notably, as part of their analysis, Baye et al. (1992, Lemmas 7'-14') establish that only two firms can ever engage in sales behavior in a unit-demand clearinghouse model with zero advertising costs when firms differ in their shares of non-shoppers. The remaining firms with relatively larger shares of non-shoppers are less willing to compete and prefer to always set high (non-sale) prices to their non-shoppers. This finding has been extended to allow firms to vary in their product values (Kocas and Kiyak (2006)) or costs (Shelegia (2012)).

However, this 'two-firm' prediction contrasts to common empirical findings where multiple heterogeneous sellers exhibit sales behavior (e.g. Lach (2002), Lewis (2008), Chandra and Tappata (2011)). Instead, within our more general framework, we now demonstrate how *any* number of heterogeneous firms, $k^* \in [2, n]$, can engage in equilibrium sales. In particular, we explain the factors that determine the number of firms that use sales in equilibrium, and provide a broad characterization of the types of firms that are likely to use sales, depending on their advertising costs, non-shopper shares, and profit functions.

To proceed, we use (4) to index the firms in (weakly) decreasing order of $\tilde{u}_i$ from 1 to n, such that firm n is the least willing to advertise high utilities. We then focus on

consistent with our predictions, most UK suppliers of broadband, land-line and TV packages, as well as many gym facilities and sports clubs offer sales with reduced monthly fees but unchanged prices for charged services. Our predictions are also consistent with a finding in Giulietti et al. (2014) which suggests that firms play mixed strategies with the implied 'final bill' for an average consumer in the British electricity market where suppliers often employ two-part tariffs.

characterizing sales equilibria in two settings: i) a quasi-symmetric setting where $u^m < \tilde{u}_i = \tilde{u}$ for all i, and ii) a strict asymmetric setting where $u^m < \tilde{u}_n < ... < \tilde{u}_1$.[24]

Unlike the duopoly case, the tie-break probabilities can now generate potential sources of sales equilibrium multiplicity. This can arise for two reasons. First, sales equilibria are now possible where at least one firm does not advertise. Here, Lemma 5 is insufficient to pin down a unique equilibrium value of x_i^* for firms that never advertise, $\alpha_i = 0$. Second, equilibrium multiplicity can also exist at knife-edge cases where $\tilde{u}_i = \bar{u}$ for some firm i. Here, Lemma 5 implies that $x_i^* = 0$ such that firm i is indifferent between using sales or not. To avoid both of these ambiguities which are largely uninteresting from an economic perspective, we focus on sales equilibria where i) firms that *never* advertise receive a zero equilibrium tie-break probability, $x_i^* = 0$ if $\alpha_i = 0$, and ii) advertising firms receive a positive equilibrium tie-break probability, $x_i^* > 0$ if $\alpha_i > 0$. Lemma 7 now provides a preliminary step, before Proposition 2 summarizes our main equilibrium result.

Lemma 7. *Consider any sales equilibrium that satisfies our restrictions with a given upper utility bound, $\bar{u} > u^m$. Firm i uses sales if and only if $\tilde{u}_i > \bar{u}$. Hence, i) if $k^* = n$ then $\bar{u} \in (u^m, \tilde{u}_n)$, and ii) if $k^* \in [2, n)$ then $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$ and $K^* = \{1, ..., k^*\}$.*

The basic intuition is straightforward - firm i will only be willing to engage in sales within a given sales equilibrium if the upper bound, $\bar{u}$, is lower than the maximum utility that firm i could possibly wish to advertise, $\tilde{u}_i$, from (4). From this logic, Lemma 7 then goes on to make two immediate statements about the number and identity of firms that will use sales for a given $\bar{u}$. If all firms use sales, $k^* = n$, then it must be that the upper bound is sufficiently low such that firm n is willing to use sales, $\tilde{u}_n > \bar{u}$. Alternatively, if only $k^* \in [2, n)$ firms use sales, then the firms using sales must be those with the highest values of $\tilde{u}_i$, $K^* = \{1, ..., k^*\}$. In particular, it must be that $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$ such that $\bar{u} < \tilde{u}_i$ for $i \in K^* = \{1, ..., k^*\}$, but $\bar{u} \geq \tilde{u}_j$ for the remaining firms $j = \{k^* + 1, ..., n\}$.

By using this together with an approach similar to Section 3.3, we now characterize a unique sales equilibrium under the assumption that a sales equilibrium exists.[25] To avoid undue repetition of technical details, Proposition 2 jumps to the main result (see the proof for full details).

[24] A third setting where a subset of firms have the same $\tilde{u}$ but where some remaining firms differ in $\tilde{u}$ can also be analyzed but is omitted for brevity due to its unnecessary complications.

[25] While Proposition 2 demonstrates equilibrium uniqueness, we are unable to prove existence for the general case when $n > 2$ as it is difficult to verify that $F_i'(u) > 0$ over the relevant u for all $i \in K^*$. However, existence can be guaranteed by further specifying the model, e.g. if the firms i) are sufficiently symmetric, ii) differ only in their advertising costs, or iii) differ only in their profit functions when $\pi_i(u) = t_i\pi(u)$ where $t_i > 0$. An explicit asymmetric example is provided in Section 3.4.1.

Proposition 2. *When a sales equilibrium exists under our restrictions, it is unique. In such an equilibrium, firms $i \leq k^*$ engage in sales with interior probabilities, $\alpha_i \in (0,1)$, while any remaining firms, $j > k^*$, never advertise $\alpha_j = 0$; k^*, x^*, and $\bar{u}$ are uniquely defined as follows:*

$$k^* = \begin{cases} n & \text{if} \quad \sum_{i=1}^{n} \chi_i(\tilde{u}_n) < 1 < \sum_{i=1}^{n} \chi_i(u^m) \\ k \in [2, n) & \text{if} \quad \sum_{i=1}^{k} \chi_i(\tilde{u}_k) < 1 \leq \sum_{i=1}^{k} \chi_i(\tilde{u}_{k+1}) \end{cases} \tag{9}$$

$$x_i^* = \begin{cases} \chi_i(\bar{u}) \in (0,1) & \text{if} \quad i \leq k^* \\ 0 & \text{if} \quad i > k^* \end{cases} \tag{10}$$

$$\sum_{i=1}^{k^*} \chi_i(\bar{u}) = 1 \tag{11}$$

Before discussing the economic intuition and implications of Proposition 2, it is useful to provide a sketch of the technical proof. First, for a given $k^* \in [2, n]$, we derive the unique set of equilibrium tie-breaking probabilities, x^* in (10), and the unique upper bound, $\bar{u}$ in (11). As the non-advertising firms have $x_i^* = 0$ by assumption, the values of $x_i^* \in (0,1)$ for the advertising firms follow from Lemma 5 and must sum to one. Second, using Lemma 7, we then specify the conditions for the equilibrium upper bound, $\bar{u}$, to be consistent with the stated number of advertising firms, k^*. In particular, for a given k^*, we require (9) to ensure that $\bar{u} \in (u^m, \tilde{u}_n)$ if $k^* = n$, and $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$ if $k^* \in [2, n)$. Third, we show that (9) always specifies a unique equilibrium value of $k^* \in [2, n]$ provided advertising costs are sufficiently low: $\sum_{i=1}^{n} \chi_i(u^m) > 1$ or equivalently, $\sum_{i=1}^{n} \frac{A_i}{\pi_i^m} < (n-1)(1-\theta)$.[26] Finally, given k^*, the proof derives the unique advertising probabilities, $\alpha_i \in (0,1)$, utility distributions, $F_i(u)$, and profits, $\bar{\Pi}_i$.

To examine the economic intuition of Proposition 2, Sections 3.4.1 and 3.4.2 now further discuss the number and type of firms that use equilibrium sales, in turn.

3.4.1 The Number of Firms that Use Sales

First, consider the quasi-symmetric case, where $\tilde{u}_i = \tilde{u} > u^m$ for all i. Here, from (4), $\chi_i(\tilde{u}) = 0$ for all i, and so the conditions in (9) can never be satisfied for $k^* \in [2, n)$.

[26] In contrast, if $\sum_{i=1}^{n} \chi_i(u^m) \leq 1$, then there is no solution for $k^* \in [2, n]$. Instead, in parallel to the duopoly case, there is a non-sales equilibrium with $k^* = 0$ for appropriate values of x^*.

Instead, any sales equilibrium must have $k^* = n$. If two firms wish to use sales, then they all wish to use sales. This equilibrium then resembles that under duopoly; all firms engage in sales on $(u^m, \bar{u}]$, and the unique set of equilibrium tie-breaking probabilities, x^*, ensures that each firm has an identical incentive to adopt the common upper bound. However, the resulting tie-break probabilities, utility distributions, and advertising probabilities, need not be symmetric in equilibrium - they are only symmetric if the firms also have identical advertising costs, non-shopper shares, and profit functions. Equilibrium existence can be further demonstrated with $F_i'(u) > 0 \; \forall i$ over the relevant range if the firms are sufficiently symmetric.

Now consider the strict asymmetric setting, where $u^m < \tilde{u}_n < ... < \tilde{u}_1$. Here, the intuition regarding k^* is more complex. However, some broad insights can be gained by simplifying to symmetric advertising costs, $A_i = A \; \forall i$, where any changes in A do not affect the ranking of firms in terms of $\tilde{u}_i$.

Corollary 1. *Suppose firms are strictly asymmetric, $u^m < \tilde{u}_n < ... < \tilde{u}_1$, but advertising costs are symmetric, $A_i = A \; \forall i$. When a sales equilibrium exists, i) only two firms use sales when advertising costs are sufficiently small, $k^* = 2$ when $A \to 0$, but ii) all firms use sales when advertising costs are moderate, $k^* = n$ when $A \to \frac{(n-1)(1-\theta)}{\sum_i 1/\pi_i^m}$.*

When $A \to 0$, our findings are in line with the existing literature's two-firm result and generalize it to a broad range of market settings. When $A \to 0$, competition for the shoppers is fierce. Hence, the only way for any firms to have exactly the same incentives to employ a common upper bound is to give the firm with the highest incentive to advertise, firm 1, almost all the shoppers in a tie, $x_1^* \to 1$. In equilibrium, only firms 1 and 2 then use sales with $\bar{u} = \tilde{u}_2$.

However, once we allow for higher advertising costs, the two-firm result becomes a special case of a new and more general relationship. Indeed, from (9), any number of heterogeneous firms $k^* \in [2, n]$ may now use sales in equilibrium. At the extreme, Corollary 1 states that *all* firms can use sales provided advertising costs are moderate. This appears paradoxical at first because an increase in A reduces the direct incentives for each firm to use sales, as evidenced by the associated reduction in $\tilde{u}_i$. However, the increase in A also softens the competition for the shoppers and reduces $\bar{u}$ in a way that prompts firms with lower $\tilde{u}_i$ to start using sales. Indeed, for moderate A, $\bar{u}$ can fall below $\tilde{u}_n$ such that *all* firms use sales. Thus, if the costs of informative advertising fall in the digital era, then Corollary 1 suggests that fewer firms might opt to use sales.

To conclude this subsection, we provide the following example. Assume unit demand such that $\pi_i(u) = \Psi_i - u$ and $\pi_i^m = \Psi_i > 0$, where $\Psi_i = V_i - c_i$ denotes the value-cost markup. Suppose $n = 3$ and let the firms be symmetric aside from $\Psi_1 > \Psi_2 > \Psi_3 > 0$ such that $u^m < \tilde{u}_3 < \tilde{u}_2 < \tilde{u}_1$. The conditions in (9) then imply that $k^* = 2$ if $A \leq \underline{A}$, and $k^* = 3$ if $A \in (\underline{A}, \bar{A})$.[27] Firm 1 has the largest equilibrium tie-breaking probability, but it still advertises with the highest probability and offers the highest average utility offers. Equilibrium existence can be demonstrated with $F_i'(u) > 0$ over the relevant range $\forall i \leq k^*$ provided the firms are not too asymmetric.[28]

3.4.2 The Types of Firms that Use Sales

To further explore the intuition of Proposition 2, we now consider its implications for the types of firms that are most likely to use sales. The existing literature only considers some specific dimensions under unit demand and zero advertising costs (e.g. Baye et al. (1992), Kocas and Kiyak (2006), Shelegia (2012)). However, in our general setting, we can offer a broad characterization. In particular, when $k^* < n$, Proposition 2 implies that the firms using sales will be the firms with the highest values of $\tilde{u}_i$ in (4). Corollary 2 then follows immediately and generally because $\tilde{u}_i$ is strictly decreasing in A_i and θ_i.

Corollary 2. *Suppose a sales equilibrium exists with $k^* < n$. Ceteris paribus, the firms with the lowest advertising costs, A_i, and shares of non-shoppers, θ_i, will use sales.*

However, understanding how a firm's profit function, $\pi_i(u)$, will impact its use of sales is more difficult because variations in profit functions may affect firms' profits differently at different utility levels. To proceed, we focus on the following functional form, $\pi_i(u) = \pi(u, \rho_i)$, where $\pi(\cdot)$ is common across firms, and $\rho_i > 0$ is a parameter representing firm i's profitability. We assume that $\pi(u, \rho_i)$ is twice continuously differentiable and increasing in ρ_i for all $u \geq u^m$, where u^m maximizes $\pi(u, \rho_i)$ for any ρ_i.

Corollary 3. *Suppose a sales equilibrium exists with $k^* < n$, and that firms have symmetric shares of non-shoppers, $\theta_i = (\theta/n)\ \forall i$. Ceteris paribus, the firms that use sales will be those with the highest profitability, ρ_i, if increases in ρ_i raise per-consumer profits relatively more at higher rather than lower utility levels, $\pi_{\rho u}(u, \rho) \geq 0\ \forall u > u^m$.*

Corollary 3 suggests that the effects of a firm's profitability, ρ_i, on its use of sales are ambiguous. However, it predicts that more profitable firms are more likely to use sales

[27] In particular, $\underline{A} \equiv (1-\theta)\sqrt{(\Psi_1 - \Psi_3)(\Psi_2 - \Psi_3)}$ and $\bar{A} = 2(1-\theta)\left[\frac{1}{\Psi_1} + \frac{1}{\Psi_2} + \frac{1}{\Psi_3}\right]^{-1}$.

[28] Specifically, existence can be demonstrated for $\frac{1}{\Psi_1} + \frac{1}{\Psi_2} > \frac{1}{\Psi_3}$. This condition also ensures that $\underline{A} < \bar{A}$ and that Assumption A is satisfied for all $A < \bar{A}$.

if $\pi_{\rho u}(u,\rho) \geq 0 \; \forall u > u^m$. This condition is satisfied for several common situations. For instance, it applies under unit demand or two-part tariffs, where ρ_i captures an increase in per-consumer demand or a reduction in marginal cost, and where $\pi_{\rho u}(\cdot) = 0$ as $\pi_\rho(u,\rho)$ is independent of u. Thus, in these instances, firms with larger per-consumer demand or lower marginal costs should be more likely to use sales. Alternatively, it also applies under downward sloping demand.[29]

4 Applications

Although our main contribution is methodological, this section provides three applications to illustrate how our framework can be used to help future research. Section 4.1 examines a common procedure used in empirical work on sales and price dispersion. Section 4.2 considers a family of games where firms engage in a prior activity to influence their share of non-shoppers, and Section 4.3 presents a number of comparative statics to further analyze the effects of firm heterogeneities on sales.

4.1 Implications for Empirical Procedures

Within a given market, price dispersion is broadly divided into two forms. The first 'temporal' form involves price differences that vary over time, such as those generated by sales. The second 'spatial' form arises from persistent inter-firm heterogeneities related to firms' characteristics. As listed in the introduction, many studies within the large empirical literature on sales attempt to focus on the temporal form by using a 'cleaning' procedure. This procedure retrieves a set of price residuals from raw price data by using a regression involving observable firm characteristics or firm-level fixed effects. The price residuals are then interpreted as resulting from a homogeneous symmetric market and used to perform a reduced-form analysis or structural estimation. We now use our framework to better understand when this procedure is valid, and to suggest some modifications for it to be applied more widely.

Wildenbeest (2011) provides the only formal justification for the cleaning procedure under a specific set of market conditions. A version of his arguments can be derived in our framework, where in contrast, we generalize to positive advertising costs. Under unit demand and single products, suppose that firms vary in product quality and costs

[29]Under downward-sloping demand with a minimum utility constraint, $u^{min} \geq \min\{u_i^m, ..., u_n^m\}$, ρ_i is best interpreted as a (sufficiently small) reduction in firm i's marginal cost. There, $\pi_{\rho u}(\cdot) > 0$, because a reduction in marginal cost profitably extends over a larger number of units for higher u.

subject to a common value-cost markup, $V_i - c_i = \Psi \; \forall i$. This implies symmetric profit functions, $\pi_i(u) = \Psi - u \; \forall i$. Hence, if firms also have symmetric shares of non-shoppers and advertising costs, then any sales equilibrium will involve symmetric utility distributions, $F_i(u) = F(u) \; \forall i$. Importantly, firms' subsequent *price* distributions are then simple translations of each other as $p_i(u_i) = V_i - u_i$ under unit demand. Therefore, after observing a panel of price observations, one can obtain a measure of firms' utility offers (under the assumption of a stationary, finitely repeated game). Specifically, one can regress the raw price data on a set of firm-level fixed effects, $p_{it} = \alpha + \delta_i + \varepsilon_{it}$, to soak up the effects of the firm heterogeneities and return a set of 'cleaned' residuals that correctly proxy the utilities up to a positive constant.[30]

Downward-Sloping Demand: For the procedure to be valid under downward-sloping demand, one first needs a revised condition to ensure that firms' profit functions are symmetric. While a more general condition can be provided, it is sufficient for our purposes to focus on the following simple case where each firm i has a marginal cost $c_i \geq 0$, and a linear per-consumer demand function that varies only in its intercept, $q_i(p_i) = a_i - bp_i$ where $a_i \geq 0$ and $b > 0$. From footnote 12, we know $u_i = (a_i - bp_i)^2/2b$ and $\pi_i(u) = \frac{1}{b}[a_i - bc_i - \sqrt{2bu}][\sqrt{2bu}]$. Hence, profit functions are symmetric if $a_i - bc_i = \Psi \; \forall i$. This restriction maintains some sense of Wildenbeest's constant value-cost assumption. Under this new condition, one would then aim to recover the firms' utility draws from the raw price data. However, unlike unit demand, the relationship between prices and utilities is non-linear, $u_i = (a_i - bp_i)^2/2b$. Therefore, the cleaning procedure no longer provides correct estimates of utility up to a positive constant.[31] Instead, to recover the utility draws, one would have to use a more complex, data-intensive procedure to estimate some of the demand parameters.

Asymmetric Utility Distributions: We now return to unit demand but depart from Wildenbeest's constant value-cost condition. Here, firms' utility distributions will be asymmetric, $F_i(u) \neq F(u)$, with different mean utilities, u_i^{ave}, and so firms' price distributions are no longer simple translations of each other. Hence, any fixed-effects regression cannot correctly proxy utilities up to a positive constant. Instead, one could use the following modification. Rather than using fixed-effects, each firm's utilities, $u_i = V_i - p_i$, could be estimated more directly from the price data by inferring V_i. As firms will set their

[30] The estimated residuals equal $\hat{\varepsilon}_{it} \equiv p_{it} - p_i^{ave}$ where p_i^{ave} is the average price of firm i. We also know that $p_{it} = V_i - u_{it}$ and $p_i^{ave} = V_i - u^{ave}$, where u^{ave} is the average utility from the symmetric utility distribution. Hence, $\hat{\varepsilon}_{it} \equiv -(u_{it} - u^{ave})$. For many applications, such as the estimation of search costs, this is sufficient as only the difference in utilities matters.

[31] Specifically, $p_{it} = (a_i/b) - \sqrt{(2u_{it}/b)}$ with residuals, $\hat{\varepsilon}_{it} = p_{it} - p_i^{ave} \equiv -[\sqrt{(2u_{it}/b)} - \sqrt{(2u^{ave}/b)}]$.

highest price equal to V_i with probability $(1 - \alpha_i) > 0$, V_i can be inferred from firm i's maximum observed price, or by establishing its 'regular' price using a statistical procedure (e.g. Hosken and Reiffen (2004)). Having recovered the utilities, one could then use our framework to analyze the observed price dispersion or estimate a structural model. For instance, by using our theoretical predictions, one could use data on prices and advertising frequencies to estimate each firm's share of non-shoppers, θ_i, marginal cost, c_i, and/or advertising cost, A_i.

4.2 Investment and Obfuscation Games

We now illustrate how our framework's capacity to allow for asymmetries can also help develop wider theoretical results. Specifically, we provide some results for a family of games where each firm's share of non-shoppers is determined endogenously as a function of the firms' actions prior to sales competition. Such actions can be interpreted as any (long-run) marketing activity that influences consumer loyalty, such as i) investments in persuasive advertising or sales-force methods, or ii) some forms of obfuscation that influence the level of complexity or search costs within the market.

Starting with Chioveanu (2008), Carlin (2009) and Wilson (2010) some related streams of literature have become popular in recent years (see Grubb (2015) and Spiegler (2015) for a review, and some wider related models). However, as explained in the introduction, only Baye and Morgan (2009) allow for positive costs of informative advertising, $A > 0$; yet, due to the consequent difficulties of analyzing asymmetries, they only show the existence of a continuum of symmetric Nash equilibria and a unique symmetric equilibrium in *secure strategies*. In contrast, by utilizing the flexibility of our framework, we can characterize the unique symmetric *subgame-perfect* Nash equilibrium (SPNE) across a general market setting (when it exists). Moreover, while their equilibria imply that an increase in the cost of informative advertising can have a positive, negative or zero effect on the prior actions, we offer a unique, empirically testable prediction.

To begin, consider a market with $n \geq 2$ otherwise symmetric firms. In Stage 1, each firm i chooses its level of action, $e_i \geq 0$, with an associated unit cost equal to $\tau \geq 0$. Each firm i's share of non-shoppers, $\theta_i(e_i, e_{-i}) \in (0, 1)$, is determined by its own action, e_i, and the actions of its rivals, given by the vector e_{-i}. Furthermore, we assume $\theta_i(e, e_{-i}) = \theta_j(e, e_{-j})$, $\forall i, j$, whenever e_{-j} is obtained from e_{-i} by permutation. Then, in Stage 2, having observed the resulting shares, $\{\theta_1(\cdot), ..., \theta_n(\cdot)\}$, the firms compete in utility with informative advertising costs, $A > 0$, in line with our main model and under our previously stated conditions.

The exact form of the function, $\theta_i(e_i, e_{-i})$, is allowed to depend upon the type of action. For instance, under '(own) loyalty-increasing actions', $\theta_i(e_i, e_{-i})$ is strictly increasing in firm i's action, e_i, and (weakly) decreasing in a rival's action, $e_{j\neq i}$, as consistent with persuasive advertising. Alternatively, under '(own) loyalty-decreasing actions', $\theta_i(e_i, e_{-i})$ is strictly decreasing in e_i and (weakly) increasing in $e_{j\neq i}$, as consistent with some forms of obfuscation. However, in either case, we assume that the total proportion of non-shoppers, $\theta(\cdot) = \sum_{i=1}^{n} \theta_i(\cdot)$, is increasing and concave in any firm's action, e_i, with $\theta(\cdot) \to 1$ as $\sum_{i=1}^{n} e_i(\cdot) \to \infty$.[32]

We now seek to characterize the SPNE of the two-stage game. In particular, we wish to characterize a symmetric SPNE where the firms select positive actions before actively competing in a sales equilibrium (with $e_i = e > 0$ and $\alpha_i = \alpha > 0\ \forall i$). To proceed, note that firm i will select its action, e_i, to maximize its equilibrium profits (net of any direct action costs), $\bar{\Pi}_i = \theta_i(\cdot)\pi^m + \frac{x_i^*}{1-x_i^*}A - \tau e_i$. However, for our current purposes, it is better to use an alternative expression based on the expected profits from advertising the upper utility bound and attracting all the shoppers, $\bar{\Pi}_i = \pi(\bar{u}(\cdot))[1-\sum_{j\neq i}\theta_j(\cdot)] - A - \tau e_i$, where $\bar{u}(\cdot)$ is now a function of the firms' actions. We can then state the following.[33]

Proposition 3. *i) When a symmetric SPNE exists with $e_i = e^* > 0$ and $\alpha_i = \alpha^* > 0\ \forall i$, the unique level of action, e^*, satisfies the following FOC:*

$$-\pi(\bar{u}(\cdot))(n-1)\frac{\partial\theta_j(\cdot)}{\partial e_i} + \Big(1 - \frac{(n-1)}{n}\theta(\cdot)\Big)\pi'(\bar{u}(\cdot))\frac{\partial\bar{u}(\cdot)}{\partial e_i} - \tau = 0 \tag{12}$$

ii) When actions are own loyalty-increasing (or decreasing), each firm's equilibrium action, e^*, is strictly increasing (decreasing) in the costs of informative advertising, A.

Aside from increasing firm i's total action costs by τ, the FOC in (12) suggests that a marginal increase in e_i generates two effects on the profits of firm i. The first effect changes the sum of available consumers that firm i can attract with a given offer of $\bar{u}$ by influencing its $(n-1)$ rivals' shares of non-shoppers. Under own loyalty-increasing

[32]Hence, under a persuasive advertising interpretation, an increase in e_i expands the total proportion of persuaded (non-shopper) consumers, $\theta(\cdot)$, as well as increasing firm i's proportion of non-shoppers, $\theta_i(\cdot)$. Under the obfuscation interpretation, an increase in e_i reduces firm i's proportion of non-shoppers, $\theta_i(\cdot)$. However, the increase in e_i also makes the firms harder to compare and so it also increases the total proportion of non-shoppers, $\theta(\cdot)$.

[33]While Proposition 3 demonstrates uniqueness, equilibrium existence remains difficult to verify in the general case. However, it is clear that the costs of informative advertising, A, must be moderate. If A is sufficiently small, no symmetric equilibrium exists because firm i's stage 2 profits exhibit a saddle point in e_i. This mirrors the existing literature with $A = 0$, e.g. Chioveanu (2008). However, if A is too large, then no sales equilibrium will exist.

actions, this effect is (weakly) positive by reducing rivals' non-shoppers, but under own loyalty-decreasing actions, the effect is (weakly) negative. In contrast, the second effect is always positive for either form of action. For a given sum of available consumers, it increases firm i's profits per-consumer. Specifically, an increase in e_i softens competition by raising the total proportion of non-shoppers, $\theta(\cdot)$, which prompts a reduction in the equilibrium value of $\bar{u}$, such that $\pi'(\bar{u})\frac{\partial \bar{u}}{\partial e_i} > 0$.

Now consider the comparative statics. An increase in the costs of informative advertising, A, enhances both of the main effects in (12). Within the first effect, an increase in A lowers $\bar{u}$ by reducing the firms' incentives to use advertised sales. This enhances firm i's reward from attracting available consumers, $\pi(\bar{u})$. Thus, to reduce rival non-shopper shares, firm i will wish to select a higher action under own loyalty-increasing investment, and a lower action under own loyalty-decreasing investment. Within the second effect, an increase in A always enhances the incentives for firm i to select a higher action by increasing the ability of actions to soften competition; a higher A makes $\bar{u}$ more sensitive to e_i.

Thus, under own loyalty-increasing actions both positive effects become larger, and so increases in the costs of informative advertising prompt higher equilibrium actions, $\partial e/\partial A > 0$. Hence, reductions in the costs of informative advertising may be beneficial to markets for two reasons: a direct effect in increasing sales competition, but also an indirect effect in reducing brand loyalty and thereby further reducing prices. To our knowledge, this result offers the first theoretical prediction about how the costs of informative advertising affect equilibrium levels of loyalty-enhancing marketing activities such as persuasive advertising. It remains untested empirically.

Under own loyalty-decreasing actions, increases in A prompt the first effect to become more negative and the second effect to become more positive. However, the first effect dominates such that increases in A lower equilibrium actions, $\partial e/\partial A < 0$. Hence, under an obfuscation interpretation, our model predicts that obfuscation should increase in response to reductions in informative advertising costs. This complements several theories that explain how firms' obfuscation levels should rise after a fall in search costs due to advances in search technologies (e.g. Ellison and Ellison (2009), Ellison and Wolitzky (2012)).

4.3 Comparative Statics

This sub-section provides some new comparative static results. For symmetric market cases, the findings extend standard clearinghouse results to a generalized market setting.

More substantially, for asymmetric market cases where the existing literature has offered limited results, we offer several new insights by isolating the effects of individual firm characteristics on sales and market performance. In line with footnote 8, these predictions remain untested empirically, as the empirical literature has focused on other issues. For simplicity, we derive the statics under duopoly. However, related results can also be derived for $n > 2$.

4.3.1 Changes in a Firm's Share of Non-Shoppers

Under symmetry, our framework produces a generalized form of the standard clearinghouse result - an increase in the proportion of non-shoppers, θ, (and associated reduction in shoppers, $1-\theta$) leads to a lower sales probability, α, higher equilibrium profits, $\bar{\Pi}$, and lower average offers, $E(u)$. More interestingly, we can analyze a change in an individual firm's share of non-shoppers, θ_i. As these effects are difficult to characterize, we focus on evaluating a small increase in θ_i at the point of symmetry. To proceed, one must also stipulate whether the increase is associated with a reduction in shoppers, $1-(\theta_i+\theta_j)$, or rival non-shoppers, $\theta_j = \theta - \theta_i$. We first consider the latter:

Proposition 4. *In an otherwise symmetric market, consider a small increase in firm i's non-shoppers θ_i (and associated reduction in θ_j). Starting from a point of symmetry, $\theta_i = \theta_j$, this increases firm i's equilibrium profits, $\bar{\Pi}_i$, decreases firm i's sales probability, α_i, and average offer, $E(u_i)$, and generates the opposite effects on firm j.*

Ceteris paribus, an increase in θ_i reduces $\tilde{u}_i$ and makes firm i less willing to offer higher utilities. However, to maintain the firms' incentives to employ a common $\bar{u}$ in equilibrium, this is partially offset by a reduction in firm i's equilibrium tie-break probability, x_i^* (and an associated increase in x_j^*). Hence, when combined, these effects lead firm i (firm j) to gain higher (lower) equilibrium profits, use sales with a lower (higher) probability, and set lower (higher) average utility offers. While intuitive, the last result about average utility offers differs from Arnold et al. (2011) which considers asymmetric θ_i with unit demand and $A > 0$ under the past literature's tie-break rule. Instead, they suggest an increase in θ_i leads firm i to become *more* aggressive in its advertised prices and so offer *higher* average utility offers. Unlike our results, this finding conflicts with standard results under $A = 0$ such as Narasimhan (1988).[34]

[34] With two exceptions, our findings remain robust in the alternative case where the increase in θ_i comes from a reduction in shoppers. First, an increase in θ_i now raises $\bar{\Pi}_j$ because there is no reduction in θ_j. Second, an increase in θ_i can provide reversed effects on α_j and $E(u_j)$ if advertising costs are relatively

4.3.2 Changes in a Firm's Advertising Costs

Under symmetry, one can verify a generalized form of the standard result - an increase in advertising costs, A, leads to a lower sales probability, α, higher equilibrium profits, $\bar{\Pi}$, and lower average offers, $E(u)$. More substantially, our framework can now isolate the effects from a change in an individual firm's advertising cost, A_i.

Proposition 5. *In an otherwise symmetric market, a small increase in firm i's advertising cost, A_i, leads to no change in firm i's equilibrium profits, $\bar{\Pi}_i$, an increase in firm j's equilibrium profits, $\bar{\Pi}_j$, and a decrease in both firms' sales probabilities and average offers, α_k and $E(u_k)$ for $k = i, j$.*

Ceteris paribus, an increase in A_i reduces $\tilde{u}_i$ and makes firm i less willing to offer higher utilities. However, to maintain a common $\bar{u}$, this is also accompanied by a reduction in firm i's equilibrium tie-break probability, x_i^*, (and associated increase in x_j^*). This leads to no aggregate effect on $\bar{\Pi}_i$ because the direct effect from A_i is exactly offset by the indirect effect from x_i^*. However, an increase in A_i raises firm j's profits, $\bar{\Pi}_j$, because the indirect effect raises x_j^*. Hence, given our assumptions, the standard profit result under symmetry is driven by the rise in rival, rather than own, advertising costs. Finally, the increase in A_i reduces both firms' use of sales, and prompts a subsequent reduction in their average offers.

4.3.3 Changes in a Firm's Profit Function

As previously explained, studying variations in firms' profit functions is difficult at a general level. However, we now provide some comparative statics by using the form of profit function introduced in Section 3.4.2, $\pi(u, \rho)$. In particular, we focus on situations where an increase in firm i's profitability, ρ_i, raises firm i's per-consumer profits relatively more at higher rather than lower utility levels, $\pi_{\rho u}(u, \rho) \geq 0 \ \forall u > u^m$, which we argued was consistent with many market environments.

Under symmetry, one can derive a new result; an industry-wide increase in profitability, ρ, increases firms' equilibrium profits, while also inducing an increase in the probability of sales, α, and average utility offers, $E(u)$. In addition, we can also isolate the effects from a change in an individual firm's profitability, ρ_i. The existing literature has only been able to consider a few specific cases involving individual changes in marginal costs or product

high. This arises due to the opposing effects of a decrease in shoppers, and an increase in x_j^* (which varies in A). However, firm i still offers a lower average utility than firm j.

values under unit demand and zero advertising costs (e.g. Kocas and Kiyak (2006)). Related technical difficulties also remain in our framework. However, by evaluating a small change at the point of symmetry, we can substantially improve upon past results:

Proposition 6. *In an otherwise symmetric market, consider a small increase in firm i's profitability, ρ_i. Starting from a point of symmetry, $\rho_i = \rho_j$, this increases firm i's equilibrium profits, $\bar{\Pi}_i$, sales probability, α_i, and average offer, $E(u_i)$, but decreases firm j's equilibrium profits, $\bar{\Pi}_j$.*

An increase in ρ_i unambiguously increases firm i's equilibrium profits and overall industry profits. Further, in the common cases where $\pi_{\rho u}(u, \rho) \geq 0$, an increase in ρ_i also raises firm i's incentive to advertise higher utilities. This prompts an increase in firm i's equilibrium tie-break probability, x_i^*, to maintain a common $\bar{u}$. Nevertheless, firm i increases it sales probability, α_i, and its average offer, $E(u_i)$, while firm j receives lower equilibrium profits, $\bar{\Pi}_j$.[35]

5 Conclusions

Due to the associated technical complexities, existing clearinghouse sales models are unable to fully consider the effects of firm heterogeneity. This restricts theoretical understanding, empirical analysis, and policy guidance with regards to sales and price dispersion, and many other topics in related literatures. The current paper has tried to fill this gap by providing a generalized clearinghouse sales framework. In addition, the paper has i) provided original insights into the number and types of firms that use sales, ii) offered new results on how firm heterogeneity affects market outcomes, iii) extended a 'cleaning' procedure that is commonly used within the empirical literature, and iv) analyzed a family of games to study persuasive advertising and obfuscation in sales markets. By opening up the analysis of sales with firm heterogeneity, we hope that our framework can enable future research to further address many theoretical and empirical issues that remain under-explored.

[35] In general, the effects on α_j and $E(u_j)$ are more nuanced. However, for $\bar{u}$ sufficiently close to u^m, an increase in ρ_i leads both firms to increase their sales probabilities and average offers.

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Appendix A - Example Equilibrium Specifications

This appendix contains a number of equilibrium specifications by further detailing the exact source of utility and profit function. To help exposition, we illustrate the main steps within a context of full symmetry. Here, using the results from Sections 3.3 and 3.4, this also permits us to easily include the equilibrium details for $n \geq 2$ firms rather than just duopoly. Section A1 summarizes the basic n-firm equilibrium under full symmetry. Sections A2-A4 then detail the equilibrium under unit demand, downward-sloping demand, and multiple products, while discussing how our framework can reproduce and substantially extend many of the past literature's key predictions for pricing, advertising, and purchasing behavior. Sections A5-A6 demonstrate how Proposition 1 can be used to characterize some common forms of sales that have remained unstudied within the clearinghouse literature, including cases where firms use two-part tariffs or non-price variables such as package size (e.g 'X% Free').

A1. Equilibrium under Full Symmetry

Using the results from Sections 3.3 and 3.4, we can now summarize the basic n-firm equilibrium under full symmetry. Here, all firms will use sales in equilibrium and the equilibrium tie-break rule will be fully symmetric, $x_i^* = \frac{1}{n}$. In addition, for all i, we know $\bar{\Pi}_i = \frac{\theta\pi^m}{n} + \frac{A}{n-1}$, $\alpha_i = 1 - \left(\frac{\frac{n}{n-1}A}{(1-\theta)\pi^m}\right)^{\frac{1}{n-1}}$, $\bar{u} = \pi^{-1}\left(\frac{\theta\pi^m + \frac{n^2}{n-1}A}{\theta + n(1-\theta)}\right)$ and $F_i(u) = \left(\frac{\theta[\pi^m - \pi(u)] + \frac{n^2}{n-1}A}{n(1-\theta)\pi(u)}\right)^{\frac{1}{n-1}}$.

A2. Equilibrium with Unit Demand

Building on footnote 12, suppose $u_i = V - p_i$ and $\pi(u_i) = V - c - u_i$, where $u_i^m = 0$ and $\pi^m = V - c$. Under full symmetry, this produces a simple clearinghouse equilibrium with $x_i^* = \frac{1}{n}$, $\bar{\Pi}_i = \frac{\theta(V-c)}{n} + \frac{A}{n-1}$, $\alpha_i = 1 - \left(\frac{\frac{n}{n-1}A}{(1-\theta)(V-c)}\right)^{\frac{1}{n-1}}$, and $\bar{u} = \frac{n(1-\theta)(V-c)-\frac{n^2}{n-1}A}{\theta+n(1-\theta)}$. By using $F_i(p) = 1 - F_i(u)$, one can further derive $F_i(p) = 1 - \left(\frac{\theta(V-p)+\frac{n^2}{n-1}A}{n(1-\theta)(p-c)}\right)^{\frac{1}{n-1}}$ with $p^m = V - u^m = V$ and $\underline{p} = V - \bar{u}$. This collapses to the (popularized) equilibrium of Varian (1980) when $A \to 0$. Under firm heterogeneity, the past literature has considered various asymmetries in non-shopper shares, product values and/or costs in the duopoly setting under the restriction, $A_i = A_j = 0$. As detailed in Appendix C1, our framework can obtain these equilibria in the limit when $A_i = A_j \to 0$ while allowing for any θ_i, c_i, and V_i. Moreover, our framework can also extend them to allow for positive and asymmetric advertising costs.

A3. Equilibrium with Downward-Sloping Demand

Suppose each consumer has a downward-sloping demand function for firm i's good, $q_i(p_i)$, and that firm i has a constant marginal cost, $c_i \geq 0$. Firm i then has a per-consumer profit function equal to $\pi_i(p_i) = (p_i - c_i)q_i(p_i)$, and the utility at firm i can be given by its associated consumer surplus, $u_i = S(p_i, q_i(p_i))$. Under our sales equilibrium, each firm i then chooses its price to maximize its profits subject to supplying its required utility draw, u_i, with $p_i^*(u_i) = \text{argmax}_{p_i} \ \pi_i(p_i)$ subject to $S(p_i, q_i(p_i)) = u_i$. It also follows that $p_i^m = \text{argmax}_{p_i} \ \pi_i(p_i)$, with $u_i^m = S(p_i^m, q_i(p_i^m))$ and $\pi_i^m \equiv \pi_i(u_i^m) = \pi_i(p_i^m)$.

Under full symmetry, this produces a standard clearinghouse equilibrium (e.g. Baye et al. (2004a)). Specifically, it follows that $x_i^* = \frac{1}{n}$, $\bar{\Pi}_i = \frac{\theta\pi(p^m)}{n} + \frac{A}{n-1}$, $\alpha_i = 1 - \left(\frac{\frac{n}{n-1}A}{(1-\theta)\pi(p^m)}\right)^{\frac{1}{n-1}}$, and $\bar{u} = \pi^{-1}\left(\frac{\theta\pi(p^m)+\frac{n^2}{n-1}A}{\theta+n(1-\theta)}\right)$. Using $F_i(p) = 1 - F_i(u)$, one can then find $F_i(p) = 1 - \left(\frac{\theta[\pi(p^m)-\pi(p)]+\frac{n^2}{n-1}A}{n(1-\theta)\pi(p)}\right)^{\frac{1}{n-1}}$ with $\underline{p} = p^*(\bar{u})$ and $\bar{p} = p^*(u^m) = p^m$.[36] Our framework shows how this clearinghouse equilibrium can be generalized to allow for asymmetric advertising costs, non-shopper shares, and profit functions (provided there is some binding minimum utility constraint if needed).

[36] For comparison, standard clearinghouse models express the price distribution conditional on advertising. In our model, this equates to $F_i^A(p) = \frac{1-F_i(u)}{\alpha_i}$.

A4. Equilibrium with Downward-Sloping Demand and Multiple Products

An equilibrium with downward-sloping demand where firms sell multiple products can be derived as an extension of A3 above. In particular, now suppose firm i has $K_i \geq 1$ products, where $\mathbf{c_i} = \{c_{i1}, ...c_{iK_i}\}$, $\mathbf{p_i} = \{p_{i1}, ...p_{iK_i}\}$ and $\mathbf{q_i}(\mathbf{p_i}) = \{q_{i1}(\mathbf{p_i}), ...q_{iK_i}(\mathbf{p_i})\}$ denote the associated vectors of (constant) marginal costs, prices, and product demand functions per consumer. Many of the steps from section A3, then apply immediately. In particular, under suitable demand assumptions, there exists a unique price vector that maximizes a firm's profits subject to supplying a given utility draw u across its products, such that $\mathbf{p_i^*}(u) = \text{argmax}_{\mathbf{p_i}} \quad \pi_i(\mathbf{p_i}) = \mathbf{q_i^*}(\mathbf{p_i})'(\mathbf{p_i} - \mathbf{c_i})$ subject to $S(\mathbf{p_i}, \mathbf{q_i^*}(\mathbf{p})) = u$, with resulting profits per-consumer, $\pi_i(u) \equiv \pi_i(\mathbf{p_i^*}(\mathbf{u}))$.[37] Under full symmetry, this reproduces versions the equilibrium of Simester (1997) when marginal costs are zero, $K \geq 1$, and $A \to 0$. More substantially, for any marginal costs and any K, we can permit positive asymmetric advertising costs, and asymmetric shares of non-shoppers.

A5. Equilibrium with Two-Part Tariffs

In line with the associated discussion in Section 3.3, suppose the firms employ two-part tariffs to better extract consumer surplus as consistent with a form of oligopolistic first-degree price discrimination. In particular, reconsider the analysis in A3 and A4 above where firm i has $K_i \geq 1$ products, downward-sloping demand functions, $\mathbf{q_i}(\cdot)$, and marginal costs, $\mathbf{c_i}$. However, now let each firm i set a K_i-dimensional vector of marginal prices (per unit of consumption), $\mathbf{p_i}$, and a single fixed fee, $f_i \geq 0$. It then follows that $\pi_i(\mathbf{p_i}, f_i) = \mathbf{q_i}(\mathbf{p_i})'(\mathbf{p_i} - \mathbf{c_i}) + f_i$ and $u_i = S(\mathbf{p_i}, \mathbf{q_i}(\mathbf{p_i})) - f_i$ where $S(\mathbf{p_i}, \mathbf{q_i}(\mathbf{p_i}))$ denotes a consumer's surplus at i gross of i's fixed fee. To generate any utility, u', i will choose $\mathbf{p_i}$ and f_i to maximize $\pi_i(\mathbf{p_i}, f_i)$ subject to $S(\mathbf{p_i}, \mathbf{q_i}(\mathbf{p_i})) - f_i = u'$. This implies marginal cost pricing across each product, $\mathbf{p_i} = \mathbf{c_i}$, together with a suitably adjusted fixed fee, $f_i = S(\mathbf{c_i}, \mathbf{q_i}(\mathbf{c_i})) - u'$. The full equilibrium can then be derived using $\pi_i(u) = S(\mathbf{c_i}, \mathbf{q_i}(\mathbf{c_i})) - u$, $u^m = 0$ and $\pi_i(u^m) = S(\mathbf{c_i}, \mathbf{q_i}(\mathbf{c_i}))$.

[37] This constrained pricing decision can be thought of as a Ramsey problem. Individual prices are hard to fully characterize, but with additional restrictions, firms can be shown to optimally use lower prices on products that are more price-elastic and complementary to other products. See Armstrong and Vickers (2001) and Simester (1997) for more discussion.

A6. Equilibrium with Non-Price Sales

For ease of exposition, consider a fully symmetric market with single products and unit demand. In line with the associated discussion in Section 3.3, suppose each firm's price is fixed at $p > 0$, and that each firm employs some other sales variable $z_i \in [\underline{z}, \overline{z}]$. To avoid any unnecessary complications, we make two assumptions to ensure unique correspondences between z_i, u_i, and $\pi(u_i)$. First, let both the consumers' willingness to pay for i's product, $V(z_i)$, and i's marginal (per unit) cost, $c(z_i)$, be strictly increasing in z_i, such that $u(z_i) = V(z_i) - p$ is strictly increasing in z_i, and $\pi(z_i) = p - c(z_i)$ is strictly decreasing in z_i. Second, let $u(\underline{z}) = V(\underline{z}) - p \geq 0$ and $\pi(\overline{z}) = p - c(\overline{z}) > 0$. Because profits and utilities are monotone in z, we have $z = V^{-1}(p + u)$. We can then derive $\pi(u) = p - c(V^{-1}(p + u))$ and $u^m = V(\underline{z}) - p$. To ensure the equilibrium exists, we can verify that $\pi(u^m) = p - c(\underline{z}) > 0$ and $\pi'(u) < 0$. The full equilibrium can then be explicitly derived and shown to exhibit the features listed in Section 3.3.

Appendices B and C can be found in the online supplementary appendix.

Online Supplementary Appendix:

Appendix B - Main Proofs

Proof of Lemma 1. First, note that $u_i < u^m$ is always strictly dominated by $u_i = u^m$ for any $\eta_i \in \{0,1\}$. Increasing u_i to u^m would i) raise firm i's profits per consumer as $\pi_i'(u) > 0$ for $u_i < u^m$, and yet ii) never reduce the number of consumers that it trades with. Second, $u_i > u^m$ is strictly dominated by $u_i = u^m$ when $\eta_i = 0$. Reducing u_i to u^m would i) strictly increase firm i's profits per-consumer as $\pi_i'(u) < 0$ for $u_i > u^m$, but ii) never reduce the number of consumers that it trades with, since non-advertised offers are unobserved to consumers and consumers can only visit one firm. Third, for any tie-break probability, $x_i(T) \in [0,1]$, setting $u_i = u^m$ and $\eta_i = 1$ with positive probability is strictly dominated by setting $u_i = u^m$ and $\eta_i = 0$. Given $u_i = u^m$, moving from advertising to not advertising would i) strictly reduce firm i's advertising costs, $A_i > 0$, and ii) never reduce the number of consumers that it trades with since $x_i(T)$ is independent of advertising decisions via Assumption X. □

Proof of Lemma 2. First, any sales equilibrium must have $k^* \geq 2$ because there can be no sales equilibrium with $k^* = 1$. If so, firm i would win the shoppers with probability one whenever advertising as then $u_i > u^m$ and $u_j = u^m$ $\forall j \neq i$. Hence, in such instances, i's strategy cannot be defined as it would always want to relocate its probability mass closer to u^m. Second, given this, one can then adapt standard arguments (e.g. Baye et al. (1992)), to show that for at least two firms i and j, u must be a point of increase of $F_i(u)$ and $F_j(u)$ at any $u \in (u^m, \bar{u}]$. Third, by adapting standard arguments (e.g. Narasimhan (1988), Baye et al. (1992), Arnold et al. (2011)) firms cannot use point masses on any $u > u^m$. Fourth, any firm with $\alpha_i > 0$ must have $\alpha_i \in (0,1)$ in equilibrium. To see this, suppose $\alpha_i = 1$ for some i and note from above that at least two firms must randomize just above u^m. If so, the expected profits from advertising just above u^m must equal $\theta_j \pi_j^m - A_j$ for at least one such firm $j \neq i$ as there can be no mass points at $u > u^m$. However, firm j could earn $\theta_j \pi_j^m > \theta_j \pi_j^m - A_j$ from not advertising; a contradiction. Finally, suppose $n = 2$. As a consequence of previous arguments, in any sales equilibrium both firms must share a common advertised utility support, $(u^m, \bar{u}]$, with no gaps. □

Proof of Lemma 3. Assume the opposite and consider the following exhaustive cases. First, consider a potential tie involving at least one advertising firm and at least one non-advertising firm. If so, any advertising firms in T must set $u > u^m$, and any non-advertising firms in T must set u^m in equilibrium; a contradiction. Second, consider a potential tie involving only advertising firms. For such a tie to arise, at least two firms must put positive probability mass on some utility level, $u > u^m$. However, such mass points cannot exist in equilibrium via Lemma 2. Third, consider a potential tie involving only non-advertising firms, but where $|T| < n$. If so, the firms in T must set u^m, and any remaining firm, $j \notin T$, must set $u_j > u^m$ in equilibrium, a contradiction. □

Proof of Lemma 4. Firm i's expected profits from advertising just above u^m must equal $\pi_i^m[\theta_i + (1-\theta)\Pi_{j\neq i}(1-\alpha_j)] - A_i$, where for a cost of A_i it can win the shoppers outright with the probability that its rivals set u^m and do not advertise, $\Pi_{j\neq i}(1-\alpha_j)$. If firm i uses sales, we know from the text that its expected profits from advertising an offer just above u^m must equal its expected profits from not advertising, (1). Hence, by equating these two expressions one can solve for

$$\Pi_{j\neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}. \tag{13}$$

The expression in (2) can then be derived by plugging this back in to (1). □

Proof of Lemma 5. Suppose firm i uses sales in equilibrium and $\bar{u} > u^m$. i) For this to be optimal, it must be that $\bar{u} \leq \tilde{u}_i$. Suppose not. Then from the derivation of (4), we know $\pi_i(\bar{u})(1-\theta_{-i}) - A_i < \theta_i \pi_i^m$ such that firm i would strictly prefer to deviate from $u_i = \bar{u}$. ii) To derive (5), note that (1) expresses $\bar{\Pi}_i$ for a given x_i^*, and that i must expect to earn $\bar{\Pi}_i$ for $u_i = u^m$ and for all $u_i \in (u^m, \bar{u}]$. If i set $u_i = \bar{u}$ it would attract the shoppers with probability one because there are no mass points on $u \in (u^m, \bar{u}]$. Hence, it must be that $\bar{\Pi}_i = (1-\theta_{-i})\pi_i(\bar{u}) - A_i$. Solving this implies $x_i^* = \chi_i(\bar{u})$. □

Proof of Lemma 6. First, given $x_1^* + x_2^* = 1$ and $x_i^* = \chi_i(\bar{u})$, it must be that $\chi_1(\bar{u}) + \chi_2(\bar{u}) = 1$. $\chi_1(\bar{u}) + \chi_2(\bar{u})$ is defined on $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$ and is strictly decreasing. Hence, we know the solution for $\bar{u}$ will be unique, if it exists. Second, the expression for α_i can be calculated using (13) from the proof of Lemma 4. There, we found $\Pi_{j\neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}$, and so the unique expression (7) follows for $n = 2$. Third, to derive $F_i(u)$, we require firm i's equilibrium profits, $\bar{\Pi}_i$, to equal its expected profits for all $u_i \in (u^m, \bar{u}]$,

$\pi_i(u)[\theta_i + (1-\theta)F_j(u)] - A_i$. Using (2) and rearranging for $F_j(u)$ implies the unique expression (8). □

Proof of Proposition 1. Part a). If a sales equilibrium exists, Lemmas 1-6 have characterized its unique properties. We now demonstrate that this sales equilibrium exists and that no other equilibrium can exist when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$.

First, we show that no other equilibrium can exist. The only other candidate is a non-sales equilibrium where $\alpha_1 = \alpha_2 = 0$ and $u_1 = u_2 = u^m$. For this to be an equilibrium, we require that no firm i can profitably deviate to advertising a utility slightly above u^m to attract all the shoppers. For a given x_i^*, this requires $\pi_i^m[\theta_i + x_i^*(1-\theta)] \geq \pi_i^m[\theta_i + (1-\theta)] - A_i$ or $\frac{A_i}{\pi_i^m} \geq (1-\theta)(1-x_i^*)$. The same condition for j yields $\frac{A_j}{\pi_j^m} \geq (1-\theta)x_i^*$, and so for both to hold we need $1 - \frac{A_i}{(1-\theta)\pi_i^m} \leq x_i^* \leq \frac{A_j}{(1-\theta)\pi_j^m}$. However, no such x_i^* can exist when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$.

Second, we demonstrate the unique sales equilibrium exists. For this, it is sufficient to show that $\chi_1(\bar{u}) + \chi_2(\bar{u}) = 1$ implies a solution $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$. This follows as $\chi_1(\bar{u}) + \chi_2(\bar{u})$ is i) strictly decreasing in $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$, ii) below 1 for $\bar{u}$ sufficiently close to $\min\{\tilde{u}_1, \tilde{u}_2\}$ and iii) above 1 for $\bar{u}$ sufficiently close to u^m when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$. It then follows that $x_i^* = \chi_i(\bar{u}) \in (0, 1)$ for $i = \{1, 2\}$. One can then verify that $\alpha_i^* = 1 - \frac{A_j}{x_i^*(1-\theta)\pi_j^m} \in (0, 1)$, $F_i(\cdot)$ is increasing over $(u^m, \bar{u}]$, and $F_i(\bar{u}) = 1$ for both firms.

Part b). As demonstrated in Part a), a sales equilibrium only exists when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1 - \theta$. However, we now demonstrate that a non-sales equilibrium exists when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} \geq 1 - \theta$. From above, a non-sales equilibrium requires $1 - \frac{A_i}{(1-\theta)\pi_i^m} \leq x_i^* \leq \frac{A_j}{(1-\theta)\pi_j^m}$ for each i, or equivalently, $x_i^* = 1 - x_j^* \in [\chi_i(u^m), 1 - \chi_j(u^m)]$. This interval is non-empty when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} \geq 1 - \theta$. □

Proof of Lemma 7. First, let $\tilde{u}_i > \bar{u}$. To show why $\alpha_i > 0$ in equilibrium, suppose not, with $\alpha_i = 0$. From our restrictions, firm i would then have $x_i^* = 0$. Thus, by the definition of $\tilde{u}_i$, i would be indifferent between never advertising, and advertising $\tilde{u}_i$ provided it attracted all the shoppers. Given $\tilde{u}_i > \bar{u}$, i must then strictly prefer to deviate to set $\eta_i = 1$ with $u_i = \bar{u}$ where it could win the shoppers with probability one; a contradiction. Second, let $\tilde{u}_i \leq \bar{u}$. To show why $\alpha_i = 0$ in equilibrium, suppose not, with $\alpha_i > 0$. From our restrictions, firm i would then have $x_i^* > 0$. Thus, using the definition of $\tilde{u}_i$, i would be unwilling to advertise over the whole required support $u \in (u^m, \bar{u}]$, and would strictly prefer to deviate to $\alpha_i = 0$. Finally, statements i) and ii) in the Lemma then follow immediately given our two settings where $u^m < \tilde{u}_i = \tilde{u}$ for all i, or $u^m < \tilde{u}_n < ... < \tilde{u}_1$. □

Proof of Proposition 2. In line with the sketch of the proof under the proposition, we proceed by proving a number of claims.

Claim 1: In any sales equilibrium under our restrictions, a) the equilibrium tie-break probabilities, x^*, and upper bound, $\bar{u}$, are uniquely (implicitly) defined by (10) and (11), and b) these solutions must satisfy (9) for k^* to be consistent with equilibrium.

Proof of 1a: We know from (5), that any advertising firm, $i \leq k^*$, must have $x_i^* = \chi_i(\bar{u})$. From Lemma 7, an advertising firm must have $\tilde{u}_i > \bar{u}$ such that $x_i^* = \chi_i(\bar{u}) > 0$ as required. In addition, from our restrictions, $x_i^* = 0$ for all non-advertising firms, $i > k^*$. Hence, (10) applies. As $\sum_{i=1}^{n} x_i^*$ must sum to one, it then also follows that $\bar{u}$ is implicitly defined by (11). Note $\sum_{i=1}^{k^*} \chi_i(\bar{u})$ is strictly decreasing on $\bar{u} \in (u^m, \tilde{u}_{k^*})$. Hence, the solution for $\bar{u}$ will be unique.

Proof of 1b: First, suppose $k^* = n$. Then from Lemma 7, we require the solution to (11) to lie within $\bar{u} \in (u^m, \tilde{u}_n)$. Thus, we require $\sum_{i=1}^{n} \chi_i(u^m) > 1$ and $\sum_{i=1}^{n} \chi_i(\tilde{u}_n) < 1$ as consistent with (9). Note that $\bar{u} \in (u^m, \tilde{u}_n)$ also guarantees a unique interior value for $x_i^* \in (0,1)$ $\forall i \leq k^*$. Second, suppose $k^* \in [2, n)$. Then from Lemma 7, we require the solution to (11) to lie within $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$. Thus, we require $\sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*+1}) \geq 1$ and $\sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*}) < 1$ as consistent with (9). Note that $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$ also guarantees a unique interior value for $x_i^* \in (0,1)$ $\forall i \leq k^*$ under our restrictions.

Claim 2: Whenever a sales equilibrium exists under our restrictions, $k^* \in [2, n]$ is uniquely defined by (9) provided $1 < \sum_{i=1}^{n} \chi_i(u^m)$.

Proof: Using Claim 1, it is useful to summarize and re-notate the following results. First, for any $k^* \in [2, n]$, $\sum_{i=1}^{k^*} \chi_i(\bar{u})$ is strictly decreasing on $\bar{u} \in (u^m, \tilde{u}_{k^*})$. Second, using (9), if $k^* = n$, then we require $\underline{I}_n \equiv \sum_{i=1}^{n} \chi_i(\tilde{u}_n) < 1 < \sum_{i=1}^{n} \chi_i(u^m) \equiv \bar{I}_n$. Third, if $k^* = k \in (2, n]$, then we require $\underline{I}_k \equiv \sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*}) < 1 \leq \sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*+1}) \equiv \bar{I}_k$. Hence, for k^* to be uniquely defined, there must exist exactly one value of k^* for which either $1 \in (\underline{I}_n, \bar{I}_n)$ or $1 \in (\underline{I}_k, \bar{I}_k]$. Provided $\sum_{i=1}^{n} \chi_i(u^m) \equiv \bar{I}_n > 1$, this then follows because i) $\underline{I}_{z+1} = \bar{I}_z$ for any $z \in (2, n]$ (as $\sum_{i=1}^{z+1} \chi_i(\tilde{u}_{z+1}) = \sum_{i=1}^{z} \chi_i(\tilde{u}_{z+1})$ given $\chi_{z+1}(\tilde{u}_{z+1}) = 0$ from (4)), and ii) $\underline{I}_2 \equiv \sum_{i=1}^{2} \chi_i(\tilde{u}_2) < 1$ (as $\sum_{i=1}^{2} \chi_i(\tilde{u}_2) = \chi_1(\tilde{u}_2) \in (0,1)$).

Claim 3: Whenever a sales equilibrium exists under our restrictions, the firms' advertising probabilities and offer distributions are uniquely defined. Firms $i > k^*$ have $\alpha_i = 0$ and $u_i = u^m$, and firms $i \leq k^*$ have:

$$\alpha_i = 1 - \frac{\left[\Pi_{j=1}^{k^*} \gamma_j(u^m))\right]^{\frac{1}{k^*-1}}}{\gamma_i(u^m)} \in (0,1) \tag{14}$$

$$F_i(u) = \frac{\left[\Pi_{j=1}^{k^*}\gamma_j(u)\right]^{\frac{1}{k^*-1}}}{\gamma_i(u)} \tag{15}$$

$$where \qquad \gamma_i(u) = \frac{\pi_i(\bar{u})(1-\theta_{-i}) - \theta_i\pi_i(u)}{(1-\theta)\pi_i(u)} \tag{16}$$

In addition, $\forall i$, each firm i's equilibrium profits remain equal to (2).

Proof: The behavior of firms $i > k^*$ follows immediately from Lemma 1. To derive (14), first recall (13) from the proof of Lemma 4, $\Pi_{j\neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}$. As $\alpha_i = 0$ for all $i > k^*$, this also equals $\Pi_{j\neq i\in K^*}(1-\alpha_j)$. After plugging in $x_i^* = \chi_i(\bar{u})$, $\Pi_{j\neq i\in K^*}(1-\alpha_j) = \gamma_i(u^m)$, where $\gamma_i(u)$ is given by (16). By then multiplying this equation across the k^* firms, we get $\Pi_{i=1}^{k^*}[\Pi_{j\neq i\in K^*}(1-\alpha_j)] \equiv \Pi_{i=1}^{k^*}(1-\alpha_i)^{k^*-1} = \Pi_{i=1}^{k^*}\gamma_i(u^m)$, such that $\Pi_{i=1}^{k^*}(1-\alpha_i) = \left[\Pi_{i=1}^{k^*}\gamma_i(u^m)\right]^{\frac{1}{(k^*-1)}}$. Then, by returning to $\Pi_{j\neq i\in K^*}(1-\alpha_j) = \gamma_i(u^m)$ and multiplying both sides by $1-\alpha_i$ we get $\Pi_{j=1}^{k^*}(1-\alpha_j) = (1-\alpha_i)\gamma_i(u^m)$, which after substitution provides (14). Similar steps can be then used to derive the unique utility distributions, (15). One can verify that $\alpha_i \in (0,1)$ and $F_i(\bar{u}) = 1$ $\forall i \leq k^*$ as required given $\bar{u} \in (\tilde{u}_{k^*+1}, \tilde{u}_{k^*}]$. Finally, to verify each firm's equilibrium profits, remember that each firm must earn (1) for a given set of tie-break probabilities. After substituting out for $\Pi_{j\neq i}(1-\alpha_j)$ from above, this equals (2). Note that (2) applies not only to firms that use sales, but also to those that do not because they have $x_i^* = 0$ under our assumptions such that $\bar{\Pi}_i = \theta_i\pi_i^m$ as consistent with them pricing only to their non-shoppers. □

Proof of Corollary 1. i) Let $A \to 0$. Using (3) and past results, $\sum_{i=1}^{k^*}\chi_i(\tilde{u}_{k^*}) = \sum_{i=1}^{k^*-1}\chi_i(\tilde{u}_{k^*}) \to (k^*-1)$ for any $k^* \in [2,n]$. Hence, the conditions in (9) can only be satisfied when $k^* = 2$. ii) Let $A \to \frac{(n-1)(1-\theta)}{\sum_{i=1}^n \frac{1}{\pi_i^m}}$. Using (3), $\sum_{i=1}^n \chi_i(u^m) = n - \frac{nA}{(1-\theta)\sum_{i=1}^n \pi_i^m} \to 1$ such that the solution to $\bar{u}$ in (11) converges to $u^m < \tilde{u}_n$ from above. Hence, it must be that $\bar{u} \in (u^m, \tilde{u}_n)$ as only consistent with $k^* = n$. □

Proof of Corollary 3. From above, firms with a higher $\tilde{u}_i$ are more likely to use sales. Hence, we require $\frac{\partial\tilde{u}_i}{\partial\rho_i} > 0$. Rewrite (4) as $(1-\theta_{-i})\pi(\tilde{u}_i,\rho_i) - A_i = \theta_i\pi(u^m,\rho_i)$. Then note that $\frac{\partial\tilde{u}_i}{\partial\rho_i} = \frac{\theta_i\pi_\rho(u^m,\rho_i)-(1-\theta_{-i})\pi_\rho(\tilde{u}_i,\rho_i)}{(1-\theta_{-i})\pi_u(\tilde{u}_i,\rho_i)}$. As $\pi_u(\tilde{u}_i,\rho_i) < 0$ given $\tilde{u}_i > u^m$, then $\frac{\partial\tilde{u}_i}{\partial\rho_i}$ is positive whenever $\frac{1-\theta_{-i}}{\theta_i} > \frac{\pi_\rho(u^m,\rho_i)}{\pi_\rho(\tilde{u}_i,\rho_i)}$. This is satisfied when $\theta_i = (\theta/n)$ $\forall i$ and $\pi_{\rho u} \geq 0$ for $u > u^m$ because i) $\frac{1-\theta_{-i}}{\theta_i} = \frac{n-(n-1)\theta}{\theta} > 1$ given $\theta \in (0,1)$, and ii) $\frac{\pi_\rho(u^m,\rho_i)}{\pi_\rho(\tilde{u}_i,\rho_i)} \leq 1$ given $\tilde{u}_i > u^m$. □

Proof of Proposition 3. i) Given $\bar{\Pi}_i = \pi(\bar{u}(\cdot))(1-\sum_{j\neq i}\theta_j(\cdot)) - A - \tau e_i$, firm i's first-order condition wrt e_i can be expressed by (12) when evaluated at symmetry with

$\theta_j(\cdot) = \theta(\cdot)/n\ \forall j$. ii) For the comparative statics, we first re-write the FOC in terms of model primitives by using (11) to derive $\frac{\partial \bar{u}(\cdot)}{\partial e_i}$. When evaluated at symmetry, this equals $\frac{[\pi^m+\pi(\bar{u}(\cdot))(n-1)]}{\pi'(\bar{u}(\cdot))[n-(n-1)\theta(\cdot)]}\left(\frac{\partial\theta_i(\cdot)}{\partial e_i} - (n-1)\frac{\partial\theta_j(\cdot)}{\partial e_i}\right)$ where $\pi(\bar{u}(\cdot)) = \frac{\theta(\cdot)\pi^m + \frac{An^2}{(n-1)}}{n-(n-1)\theta(\cdot)}$. By substituting these in and rearranging, one can rewrite the FOC as: $\frac{\partial\theta_i(\cdot)}{\partial e_i}(\pi^m + An) + \frac{\partial\theta_j(\cdot)}{\partial e_i}[\pi^m(1-\theta(\cdot))(n-1) - An] - \tau[n - \theta(\cdot)(n-1)] = 0$. We now denote the LHS of this equation as $H(\cdot)$ and apply the implicit function theorem. At any symmetric equilibrium, the associated second-order condition must be negative, such that $\frac{\partial H(\cdot)}{\partial e_i} \equiv \frac{\partial^2\bar{\Pi}_i}{\partial e_i^2} < 0$. Hence, it follows that $\frac{\partial e}{\partial A} \gtreqless 0$ if $\frac{\partial H(\cdot)}{\partial A} = n\big(\frac{\partial\theta_i(\cdot)}{\partial e_i} - \frac{\partial\theta_j(\cdot)}{\partial e_i}\big) \gtreqless 0$. Hence, given our assumptions about the form of $\theta_i(\cdot)$, the statics follow as $\frac{\partial H(\cdot)}{\partial A} > 0$ under own loyalty-increasing actions, but $\frac{\partial H(\cdot)}{\partial A} < 0$ under own loyalty-decreasing actions. □

Proof of Proposition 4. Let $\pi_i(u) = \pi(u)$, $A_i = A$ and $\theta_j = \theta - \theta_i$. From (6), $\frac{\partial\bar{u}}{\partial\theta_i} = 0$ after we impose symmetry ex post with $\theta_i = \theta_j = \theta/2$. By using this with the derivative of (5), we gain $\frac{\partial x_i^*}{\partial\theta_i} = -\frac{A[\pi^m - \pi(\bar{u})]}{[\pi(\bar{u})(1-(\theta/2)) - (\theta/2)\pi^m]^2} < 0$. These two results also help us find the remaining derivatives. Using (2) or $\bar{\Pi}_i = (1-\theta_j)\pi_i(\bar{u}) - A_i$ gives $\frac{\partial\bar{\Pi}_i}{\partial\theta_i} = \pi(\bar{u}) > 0$ and $\frac{\partial\bar{\Pi}_j}{\partial\theta_i} = -\pi(\bar{u}) < 0$, and using (7) gives $\frac{\partial\alpha_i}{\partial\theta_i} = -\frac{[\pi^m - \pi(\bar{u})]}{(1-\theta)\pi^m} < 0$, and $\frac{\partial\alpha_j}{\partial\theta_i} = \frac{\pi^m - \pi(\bar{u})}{(1-\theta)\pi^m} > 0$. Further, from (8), $\frac{\partial F_i}{\partial\theta_i} = \frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} > 0$ and $\frac{\partial F_j}{\partial\theta_i} = -\frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} < 0$ for all relevant u, such that $E(u_i)$ decreases and $E(u_j)$ increases. □

Proof of Proposition 5. Given $\pi_i(u) = \pi(u)$ and $\theta_i = \theta/2$, note from (5) and (6) that $A_i + A_j = \pi(\bar{u})(1-\frac{\theta}{2}) - \frac{\theta}{2}\pi^m = \frac{A_j}{x_i}$, such that $x_i^* = \frac{A_j}{A_i + A_j}$. For the profit results, substitute x_i^* into (2) to give $\bar{\Pi}_i = \frac{\theta}{2}\pi^m + A_j$. For the remaining results, substitute x_i^* into (7) to give $\alpha_i = 1 - \frac{A_i + A_j}{(1-\theta)\pi^m}$, and into (8) to obtain $F_i(u) = \frac{(\theta/2)[\pi^m - \pi(u)] + [A_i + A_j]}{(1-\theta)\pi(u)}$. An increase in A_i then decreases α_i and α_j, and increases $F_i(u)$ and $F_j(u)$ for all relevant u. □

Proof of Proposition 6. Given $A_i = A$ and $\theta_i = \theta/2$, note from (6) that $\frac{\partial\bar{u}}{\partial\rho_i}|_{\rho_i=\rho_j=\rho} = \frac{(1-(\theta/2))\pi_\rho(\bar{u},\rho) - (\theta/2)\pi_\rho(u^m,\rho)}{-(2-\theta)\pi_u(\bar{u},\rho)}$. This is positive as both the denominator and numerator are positive given $\theta \in (0,1)$, $\pi_{\rho u}(\cdot) \geq 0$ and $\bar{u} > u^m$. Then, using (5) and the above, $\frac{\partial x_i^*}{\partial\rho_i} = \frac{A[(2-\theta)\pi_\rho(\bar{u},\rho) - \theta\pi_\rho(u^m,\rho)]}{[(2-\theta)\pi(\bar{u},\rho) - \theta\pi(u^m,\rho)]^2}$, which has the same sign as $\frac{\partial\bar{u}}{\partial\rho_i}|_{\rho_i=\rho_j=\rho}$. Note $\bar{\Pi}_i = (1-\frac{\theta}{2})\pi(\bar{u},\rho_i) - A$. At the point of symmetry, it then follows that $\frac{\partial\bar{\Pi}_i}{\partial\rho_i} = (1-\frac{\theta}{2})\left(\pi_\rho(\bar{u},\rho) + \frac{\partial\bar{u}}{\partial\rho_i}\pi_u(\bar{u},\rho)\right)$ which equals $\frac{1}{2}[(1-(\theta/2))\pi_\rho(\bar{u},\rho) + (\theta/2)\pi_\rho(u^m,\rho)] > 0$. Similarly, note $\bar{\Pi}_j = (1-\frac{\theta}{2})\pi(\bar{u},\rho_j) - A$. Then $\frac{\partial\bar{\Pi}_j}{\partial\rho_i} = \frac{1}{2}\theta\pi_\rho(u^m,\rho)$ which has the opposite sign of $\frac{\partial\bar{u}}{\partial\rho_i}|_{\rho_i=\rho_j=\rho}$. Using (7), one can then prove $\frac{\partial\alpha_i}{\partial\rho_i}$ has the same sign as $\frac{\partial\bar{u}}{\partial\rho_i}|_{\rho_i=\rho_j=\rho}$. Using (8) one can show that $\frac{\partial F_i(u)}{\partial\rho_i}$ has the opposite sign to $\frac{\partial\bar{u}}{\partial\rho_i}|_{\rho_i=\rho_j=\rho}$ for all relevant u. □

Appendix C - Supplementary Equilibrium Details

Sections C1 and C2 provide extra information about the equilibrium when i) advertising costs tend to zero, and ii) the single visit assumption is relaxed.

C1. Equilibrium when Advertising Costs Tend to Zero

To ease exposition and to best connect to the existing literature, we illustrate the case of near-zero advertising costs for the duopoly equilibrium. Suppose the firms are asymmetric, but $A_1 = A_2 = A \to 0$. The equilibrium depends upon $\tilde{u}_1 \gtrless \tilde{u}_2$. Without loss of generality, suppose $\tilde{u}_i < \tilde{u}_j$ such that $\pi_i(u)(1-\theta_j)-A-\theta_i\pi_i^m < \pi_j(u)(1-\theta_i)-A-\theta_j\pi_j^m$ at $u \in (u^m, \tilde{u}_i]$. Using (5) and (6), for $\bar{u}$ to exist within $(u^m, \tilde{u}_i]$ and for x_i^* and x_j^* to be well defined, it must be that $\bar{u} \to \tilde{u}_i$ such that $x_i^* \to 0$ and $x_j^* \to 1$. Given this, we know $\lim_{A\to 0}\bar{\Pi}_i = \theta_i\pi_i^m$ and $\lim_{A\to 0}\bar{\Pi}_j = \lim_{A\to 0}(1-\theta_i)\pi_j(\bar{u}) = (1-\theta_i)\pi_j\left(\pi_i^{-1}\left(\frac{\theta_i\pi_i^m}{1-\theta_j}\right)\right) > \theta_j\pi_j^m$. Further, from (8), we know $\lim_{A\to 0}F_i(u) = \lim_{A\to 0}\frac{\bar{\Pi}_j-\theta_j\pi_j(u)}{(1-\theta)\pi_j(u)}$ and $\lim_{A\to 0}F_j(u) = \lim_{A\to 0}\frac{\bar{\Pi}_i-\theta_i\pi_i(u)}{(1-\theta)\pi_i(u)}$. Finally, from (7), $\alpha_j \to 1$, while firm i advertises with probability $\lim_{A\to 0}\alpha_i = 1-\frac{\bar{\Pi}_j-\theta_j\pi_j^m}{(1-\theta)\pi_j^m} \in (0,1)$. This limit equilibrium converges to the equilibrium of a model that allows for $A = 0$ explicitly without our tie-break rule. There, both firms advertise with probability one and use equivalent utility distributions except that firm i advertises u^m with a probability mass equivalent to $\lim_{A\to 0}(1-\alpha_i)$.

To show how this connects to much of the past literature which has considered various asymmetries in non-shopper shares, product values and/or costs under unit demand and the restriction, $A_i = A_j = 0$, consider the following example. Suppose consumers have unit demands. From above, the equilibrium then depends upon $\tilde{u}_1 \gtrless \tilde{u}_2$, or $(1-\theta_1)(V_1 - c_1) - (1-\theta_2)(V_2-c_2) \lesseqgtr 0$. For instance, when this is negative, $x_1^* \to 0$ and $x_2^* \to 1$, such that $\bar{\Pi}_1 \to \theta_1(V_1-c_1)$, and $\bar{\Pi}_2 \to (1-\theta_1)[(V_2-c_2)-\bar{u}]$, where $\bar{u} \to \left(\frac{(1-\theta)(V_1-c_1)}{1-\theta_2}\right)$. By then denoting $\Delta V = V_1 - V_2$, and noting that $F_1(u_2) = Pr(u_1 \leq u_2) = 1-F_1(p_2+\Delta V)$ and $F_2(u_1) = 1-F_2(p_1-\Delta V)$, it follows that $F_1(p) = 1-\left[\frac{\bar{\Pi}_2-\theta_2(p-\Delta V-c_2)}{(1-\theta)(p-\Delta V-c_2)}\right] = 1 + \frac{\theta_2}{1-\theta} - \frac{(1-\theta_1)(\theta_1(V_1-c_1)+(1-\theta_2)(c_1-c_2-\Delta V))}{(1-\theta_2)(1-\theta)(p-\Delta V-c_2)}$ on $[V_1-\bar{u}, V_1)$ and $F_2(p) = 1-\left[\frac{\bar{\Pi}_1-\theta_1(p+\Delta V-c_1)}{(1-\theta)(p+\Delta V-c_1)}\right] = 1-\left[\frac{\theta_1(V_2-p)}{(1-\theta)(p+\Delta V-c_1)}\right]$ on $[V_2-\bar{u}, V_2)$, where $\alpha_2 \to 1$ but where firm 1 refrains from advertising with probability $1-\alpha_1 = 1-F_1(V_1) \in (0,1)$.

C2: Relaxing the Single Visit Assumption

Here, we explain how the model can be generalized to allow the shoppers to sequentially visit multiple firms. We focus on duopoly - similar (more lengthy) arguments can also be made for $n > 2$ firms. Suppose the cost of visiting any first firm is $s(1)$ and the cost of visiting any second firm is $s(2)$. The main model implicitly assumes $s(1) = 0$ and $s(2) = \infty$. However, we now use some arguments related to the Diamond paradox (Diamond, 1971) to show that our equilibrium remains under sequential visits for any $s(2) > 0$ provided that i) the costs of any first visit are not too large, $s(1) \in [0, u^m)$, and ii) shoppers can only purchase from a single firm. The latter 'one-stop shopping' assumption is frequently assumed in consumer search models and the wider literature on price discrimination.

First, suppose $s(1) \in [0, u^m)$ but maintain $s(2) = \infty$. Beyond $s(1) = 0$, this now permits cases where $s(1) \in (0, u^m)$ provided $u^m > 0$ as consistent with downward-sloping demand and linear prices. In this case, shoppers will still be willing to make a first visit and the equilibrium will remain unchanged.

Second, suppose $s(1) \in [0, u^m)$ but allow for any $s(2) > 0$ subject to a persistent 'one-stop shopping' assumption such that shoppers cannot buy from more than one firm. By assumption, the behavior of the non-shoppers will remain unchanged. Therefore, to demonstrate that our equilibrium remains robust, we need to show that shoppers will endogenously refrain from making a second visit. Initially suppose that the firms keep playing their original equilibrium strategies and that a given shopper receives $h \in \{0, 1, 2\}$ adverts. Given $s(2) > 0$ and one-stop shopping, the gains from any second visit will always be strictly negative for all h. In particular, if $h = 0$, then any second visit would be sub-optimal as both firms will offer u^m. Alternatively, if $h \geq 1$, then a shopper will first visit the firm with the highest advertised utility, $u^* > u^m$, and any second visit will be be sub-optimal as it will necessarily offer $u < u^*$. Now suppose that the firms can deviate from their original equilibrium strategies. To see that the logic still holds, note that only the behavior of any non-advertising firms is relevant and that such firms are unable to influence any second visit decisions due to their inability to communicate or commit to any $u < u^m$. Hence, firms' advertising and utility incentives remain unchanged and the original equilibrium still applies.