APPROXIMATE MODULARITY: KALTON'S CONSTANT IS NOT SMALLER THAN 3

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Abstract. Kalton and Roberts [*Trans. Amer. Math. Soc.*, 278 (1983), 803–816] proved that there exists a universal constant $K \leqslant 44.5$ such that for every set algebra $\mathcal{F}$ and every 1-additive function $f\colon \mathcal{F} \to \mathbb{R}$ there exists a finitely-additive signed measure μ defined on $\mathcal{F}$ such that $|f(A) - \mu(A)| \leqslant K$ for any $A \in \mathcal{F}$. The only known lower bound for the optimal value of K was found by Pawlik [*Colloq. Math.*, 54 (1987), 163–164], who proved that this constant is not smaller than 1.5; we improve this bound to 3 already on a non-negative 1-additive function.

1. Introduction

Let Ω be a set, $\mathcal{F}$ be a set algebra over Ω, and $\Delta \geqslant 0$. A function $f\colon \mathcal{F} \to \mathbb{R}$ is Δ-*additive*, whenever $f(\varnothing) = 0$ and

$$|f(A) + f(B) - f(A \cup B)| \leqslant \Delta \quad (A, B \in \mathcal{F}, A \cap B = \varnothing).$$

Quite clearly, 0-additive maps are nothing but signed, finitely-additive measures on $\mathcal{F}$. For brevity, we refer to 0-additive functions as additive. Kalton and Roberts proved in [4] a rather surprising stability theorem for Δ-additive maps, which asserts that there exists a universal constant (we follow Pawlik's convention [8] and refer to it as *Kalton's constant*) $K \leqslant 44.5$ (independent of the choice of $\mathcal{F}$) such that for every Δ-additive function $f\colon \mathcal{F} \to \mathbb{R}$ there exists a (signed, finitely-additive) measure $\mu\colon \mathcal{F} \to \mathbb{R}$ such that

$$\sup_{A \in \mathcal{F}} |f(A) - \mu(A)| \leqslant K \cdot \Delta. \tag{1.1}$$

In 2014, Bondarenko, Prymak, and Radchenko decreased the upper bound for K from 44.5 to 38.8 (see [2, Proof of Corollary 1.2]).

Results of this kind (that is, including ours) are of importance in Functional Analysis, for example, in the theory of twisted sums of (quasi-)Banach spaces and certain stability problems of vector measures [4, 5]. Improving Kalton's constant may likely fine-tune various optimisation algorithms in machine learning and algorithmic game theory (see [3, Section 1.2] and references therein for more details). Moreover, recently there have been efforts to extend the validity of the Kalton–Roberts theorem to lattices [1].

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An analogous and closely related stability problem for ε-modular ($\varepsilon > 0$) set functions was recently studied by Feige, Felfman, and Talgam-Cohen [3]. In order to present it, we require a piece of notation.

Let $m \in \mathbb{N}$ (we are reserving n for a different purpose), assume that $\Omega_m = \{1, 2, \ldots, m\}$ (to fit the setting in [3]), set $\mathcal{F} := 2^{\Omega_m}$ and let $f\colon \mathcal{F} \to \mathbb{R}$ be a function such that $f(\varnothing) = 0$, where 2^{Ω_m} denotes the power set of Ω_m. Then, f is additive (that is, it is a finitely-additive signed measure), if and only if, it satisfies the *modular identity*:

$$f(A) + f(B) = f(A \cup B) + f(A \cap B) \quad (A, B \in \mathcal{F}).$$

Functions that assume a possibly non-zero value at the empty set and that satisfy the modular identity are for this reason called *modular*. For $\varepsilon > 0$, a function $f\colon \mathcal{F} \to \mathbb{R}$ is then termed *ε-modular*, whenever

$$|f(A) + f(B) - f(A \cup B) - f(A \cap B)| \leqslant \varepsilon \quad (A, B \in \mathcal{F}). \tag{1.2}$$

Also, f is said to be *weakly-ε-modular*, whenever (1.2) is satisfied for every sets A, B so that $A \cap B = \varnothing$, in particular, if $f(\varnothing) = 0$, then the properties of being weakly-ε-modular and ε-additive are equivalent. Moreover, every weakly-ε-modular function is 2ε-modular (see [3, Proposition 2.1]).

The main results in [3] state that there are universal constants $K_s < 12.62$ (*the strong Kalton constant*) and $K_w < 24$ (*the weak Kalton constant*) so that for every ε-modular function f there is a modular function ν_1 with

$$\sup_{A \in \mathcal{F}} |f(A) - \nu_1(A)| \leqslant \varepsilon K_s, \tag{1.3}$$

and for every weakly-ε-modular function h there is a modular function ν_2 with

$$\sup_{A \in \mathcal{F}} |h(A) - \nu_2(A)| \leqslant \varepsilon K_w. \tag{1.4}$$

It is also worth emphasising the inequalities between K_s and K_w [3, Corollary 2.7]), namely

$$\tfrac{1}{2} K_w \leqslant K_s \leqslant K_w. \tag{1.5}$$

Here, the constants K_w and K_s are depending on m, and so we also write $K_w(m) \equiv K_w$ and $K_s(m) \equiv K_s$.

Remark 1 (Inequality between Kalton's constants). Here, we draw a clear picture of inequalities between all Kalton's constants K, K_w and K_s. Denote by $K(m)$ the optimal Kalton constant for 1-additive functions defined on 2^{Ω_m}, to emphasize m dependence. Clearly, if f is ε-additive, then it is weakly-ε-modular and the converse is not true as $f(\varnothing)$ may be non-zero. However if f is weakly-ε-modular and $f(\varnothing) = a \neq 0$, then by shifting, we get $g = f - a \cdot \mathbb{1}_{\mathcal{F}}$, which is ε-additive, as for any $A, B \in \mathcal{F}$ we have

$$\begin{aligned} |g(A) + g(B) - g(A \cup B)| &= |f(A) - a + f(B) - a - f(A \cup B) + a| \\ &= |f(A) + f(B) - f(A \cup B) - f(A \cap B)| \\ &\leqslant \varepsilon. \end{aligned}$$

Similarly, for any modular function ν with $\nu(\varnothing) = b \neq 0$ one can construct an additive function by setting $\mu = \nu - b \cdot \mathbb{1}_{\mathcal{F}}$.

Let g be weakly-ε-modular so that $g(\varnothing) = a \neq 0$. Set $f = g - a\mathbb{1}_{\mathcal{F}}$ and note that it is ε-additive. There is an additive function μ so that

$$\sup_{A\in\mathcal{F}} |f(A) - \mu(A)| \leqslant \varepsilon K(m).$$

Let $\nu_a = \mu + a\mathbb{1}_{\mathcal{F}}$ then ν_a is modular and $\nu_a(\varnothing) = a$ also

$$\sup_{A} |g(A) - \nu_a(A)| = \sup_{A} |f(A) + a - \mu(A) - a| = \sup_{A} |f(A) - \mu(A)| \leqslant K(m)\varepsilon.$$

Thus, $K_w(m) \leqslant K(m)$. Now, let f be ε-additive, so it is weakly-ε-modular. Consequently, there is a modular function ν so that $\sup_{A\in\mathcal{F}} |f(A) - \nu(A)| \leqslant \varepsilon K_w(m)$. Assume that $\nu(\varnothing) = c \neq 0$. Then $\mu = \nu - c\mathbb{1}_{\mathcal{F}}$ is additive and

$$|f(A) - \mu(A)| \leqslant |f(A) - \nu(A) + c| \leqslant |f(A) - \nu(A)| + |f(\varnothing) - c| \leqslant 2K_w(m).$$

Thus, $K(m) \leqslant 2K_w(m)$. Hence, we have the following inequalities between Kalton's constants

$$\tfrac{1}{2}K_w(m) \leqslant K_s(m) \leqslant K_w(m) \leqslant K(m) \leqslant 2K_w(m). \tag{1.6}$$

Lower bounds. The results concerning estimating K, K_w and K_s from below have been so far rather scarce. In 1987, Pawlik published a paper [8], where Kalton's constant K was estimated from below by 3/2. Recently, his result has been reviewed in [3, Appendix A, Appendix C]. Moreover, Feige *et al.* have proved that $K_s \geqslant 1$ [3, Theorem 1.2].

2. Main results

The aim of the present paper is to improve known lower bounds on Kalton's constant by obtaining the following inequality.

Main Theorem. $K \geqslant 3$.

In order to prove the Main Theorem, we require the following fact. Let $\mathcal{F}_m := 2^{\Omega_m}$, that is, the power set of an m-element set (so that $\mathcal{F}_m$ has 2^m elements) and denote by $K(m)$ the optimal Kalton constant for 1-additive functions defined on $\mathcal{F}_m$ only. Then the sequence $(K(m))_{m=1}^{\infty}$ is increasing and

$$K = \lim_{m\to\infty} K(m) = \sup_{m\in\mathbb{N}} K(m).$$

(This follows from a standard compactness argument; see the first paragraph of the proof of [4, Theorem 4.1] for details.) In other words, it is sufficient to work with finite set algebras.

Remark 2. It is clear that both sequences $(K_w(m))_{m=1}^{\infty}$, $(K_s(m))_{m=1}^{\infty}$ are increasing. As K is the limit of $K(m)$ as $m \to \infty$, we have

$$K_w(m) \leqslant K(m) \leqslant \sup_{m\in\mathbb{N}} K(m) = K, \tag{2.1}$$

then, by monotone convergence theorem, the sequence $(K(m))_{m\in\mathbb{N}}$ is convergent, and similarly $K_w := \lim_{m\to\infty} K_w(m) = \sup_{m\in\mathbb{N}} K_w(m)$ similarly, as $K_s(m) \leqslant K_w(m)$. Then, $K_s := \lim_{m\to\infty} K_s(m) = \sup_{m\in\mathbb{N}} K_s(m)$. This leads to the estimates

$$\tfrac{1}{2}K_w \leqslant K_s \leqslant K_w \leqslant K \leqslant 2K_w, \tag{2.2}$$

which show that indeed there is no dependence on m.

As an immediate corollary to our Main Theorem and (2.2), we obtain the lower bound for $K_w = \lim_{m\to\infty} K_w(m)$ (see also [3, Theorem 1.3], where it is proved that already $K_w(20) \geqslant \frac{3}{2}$).

Corollary. $K_w \geqslant \frac{3}{2}$.

3. Proof of the main result

Let $\Omega_{k,n}$ be a set of cardinality $n\cdot k$; we write $\Omega_{k,n}$ as the disjoint union of sets $X_1,\ldots,X_n$ each set having cardinality k, where $n,k \geqslant 2$. We define $f_{k,n}$ by setting

- $f_{k,n}(\varnothing) = 0$;
- $f_{k,n}(A) = 3$ for every set A with $A\cap X_j \neq \varnothing$ for all $j \leqslant n$ and $A \cap X_j = X_j$ for at least one j;
- $f_{k,n}(B) = 1$ for all other sets B.

In particular, $f_{k,n}(\Omega_{k,n}) = 3$. It is a matter of direct verification that each function $f_{k,n}$ is 1-additive and so weakly-1-modular, which yields also that $f_{k,n}$ is 2-modular.

Proof of the Main Theorem. Let $\mu_{k,n}$ be a measure that minimises the distance from $f_{k,n}$ to the space of measures on $\Omega_{k,n}$. Choose indices $i_1,\ldots,i_n$ that realise $\gamma^1_{k,n},\ldots,\gamma^n_{k,n}$, where

$$\gamma^j_{k,n} = \min_{i\in X_j} |\mu_{k,n}(\{i\})| \quad (j = 1,\ldots,n).$$

We *claim* that for all j and n we have $\gamma^j_{k,n} \to 0$ as $k\to\infty$. Assume not. Then $\gamma^j_{k,n} \geqslant \gamma$ for some $\gamma > 0$ and infinitely many k.

Let

$$M = \sup_{k,n} \sup_{A\subseteq\Omega_{k,n}} |\mu_{k,n}(A)|.$$

If $M = \infty$, the theorem would have been proved, so we may assume that M is finite. (Of course, it follows from the Kalton–Roberts theorem that $M \leqslant 44.5 + 3$, but there is no need to invoke such a deep result here.) As k increases over the chosen infinite set, the number of those $i \in X_j$ for which $\mu_{k,n}(\{i\})$ are either all positive or all negative increases to infinity; let A_j denote the subset of X_j comprising such elements of the same sign. In particular,

$$|\mu_{k,n}(A_k)| \geqslant \gamma \cdot |A_k| \to \infty$$

as $k\to\infty$; a contradiction.

Let us note that

$$\begin{aligned} n\cdot K(k\cdot n) &\geqslant n\cdot \sup_{A\subseteq \Omega_{k,n}} |f_{k,n}(A)-\mu_{k,n}(A)| \\ &\geqslant \sum_{j=1}^{n} |f_{k,n}(X_j\cup\{i_\ell\colon \ell\neq j\})-\mu_{k,n}(X_j)-\mu_{k,n}(\{i_\ell\colon \ell\neq j\})| \\ &= \sum_{j=1}^{n} |3-\mu_{k,n}(X_j)-\mu_{k,n}(\{i_\ell\colon \ell\neq j\})| \\ &\geqslant \sum_{j=1}^{n} (3-\mu_{k,n}(X_j)-\mu_{k,n}(\{i_\ell\colon \ell\neq j\})) \\ &= 3n-\mu_{k,n}(\Omega_{k,n})-\sum_{j=1}^{n}\mu_{k,n}(\{i_\ell\colon \ell\neq j\}). \end{aligned}$$

We have

$$K(k\cdot n)\geqslant 3-\frac{1}{n}\mu_{k,n}(\Omega_{k,n})-\frac{1}{n}\sum_{j=1}^{n}\mu_{k,n}(\{i_\ell\colon \ell\neq j\})),$$

which shows that

$$\begin{aligned} K &\geqslant 3-\frac{1}{n}\limsup_{k\to\infty}\mu_{k,n}(\Omega_{k,n})-\frac{1}{n}\limsup_{k\to\infty}\sum_{j=1}^{n}\mu_{k,n}(\{i_\ell\colon \ell\neq j\}) \\ &= 3-\frac{1}{n}\limsup_{k\to\infty}\mu_{k,n}(\Omega_{k,n})-\frac{1}{n}\limsup_{k\to\infty}\sum_{j=1}^{n}\sum_{\ell\neq j}\gamma^{\ell}_{k,n} \\ &= 3-\frac{1}{n}\limsup_{k\to\infty}\mu_{k,n}(\Omega_{k,n}) \\ &\geqslant 3-\frac{M}{n}, \end{aligned}$$

because

$$\sum_{j=1}^{n}\sum_{\ell\neq j}\gamma^{\ell}_{k,n}\to 0$$

as $k\to\infty$ (and n is fixed).

□

4. The constants $K(m)$

The proof of the Main Theorem has an asymptotic nature as it involves all constants $K(m)$ at once. Given the value of m, it would be thus desirable to find lower (and upper) estimates for $K(m)$ as well. This can be achieved by estimating the distances of the functions appearing in the proof of the Main Theorem to the space of measures.

We start with the following lemma, which asserts that it is always possible to find a measure minimising the distance to $f_{k,n}$ that is constant on singletons from the respective partitions. (As the supremum norm that we consider here is not strictly convex, there is no guarantee for the uniqueness of the element that minimises a distance to a subspace.)

For $n, k \in \mathbb{N}$, denote by $\mathcal{F}_{k,n}$ the power-set of $\Omega_{k,n}$ and let $\mathcal{S}_{k,n}$ be the set of all self-bijections of $\Omega_{k,n}$ that leave each set X_j invariant ($j \leqslant n$). Then $\mathcal{S}_{k,n}$ has exactly $(k!)^n$ elements.

Lemma 1. *Let $n, k \in \mathbb{N}$. Then there exists a measure ν that minimises the distance from $f_{k,n}$ to the space of measures on $\Omega_{k,n}$ with the property for every $j \leqslant n$ the function $x \mapsto \nu(\{x\})$ is constant on X_j.*

Proof. Let μ be any measure that minimises the distance from $f_{k,n}$ to the space of measures. For any self-bijection σ of $\Omega_{k,n}$, the composition $\mu \circ \sigma$ defines a measure again. Let us observe that the measure

$$\nu = \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}_{k,n}} \mu \circ \sigma$$

has the desired properties. Indeed, it is clear that the function $x \mapsto \nu(\{x\})$ is constant on the respective sets X_j ($j \leqslant n$) as we consider only bijections that leave each set X_j invariant. Let then prove that ν also minimises the distance to the space of measures. Indeed, by convexity of balls (here, in $\ell_\infty(\mathcal{F}_{k,n})$), we have

$$\sup_{A \in \mathcal{F}_{k,n}} \left| f_{k,n}(A) - \frac{1}{(k!)^n} \sum_{\sigma \in \mathcal{S}} (\mu \circ \sigma)(A) \right| \leqslant \sup\nolimits_{A \in \mathcal{F}_{k,n}} \left| f_{k,n}(A) - \mu(A) \right|.$$

As μ was chosen to minimise the distance, the proof is complete. □

4.1. **The case $n = 2$.** Let $n = 2$, $k \in \mathbb{N}$, and let ν be a measure as in the statement of Lemma 1. In that case, $\Omega_{k,2} = X_1 \cup X_2$. Denote $x = \nu(X_1)$ and $y = \nu(X_2)$. In this case, we have essentially three types of sets to consider:

- $X_1 \cup \{\omega_2\}$, where $\omega_2 \in X_2$,
- $X_2 \cup \{\omega_1\}$, where $\omega_1 \in X_1$,
- $\Omega_{k,2} \setminus \{\omega_1, \omega_2\}$, where $\omega_1 \in X_1$ and $\omega_2 \in X_2$.

Thus, we seek to minimise the following expressions simultaneously:

$$\left| x + \frac{y}{k} - 3 \right|, \left| y + \frac{x}{k} - 3 \right|, \left| x + y - \frac{x+y}{k} - 1 \right|$$

with respect to (x, y). We then arrive at the following system of equations:

$$\begin{cases} \frac{k-1}{k}(x+y) - 1 = 3 - x - \frac{y}{k} \\ \frac{k-1}{k}(x+y) - 1 = 3 - y - \frac{x}{k}, \end{cases}$$

which has the unique solution:

$$\begin{cases} x = \frac{4k}{3k-1} \\ y = \frac{4k}{3k-1}. \end{cases}$$

In that case, the lower estimates for $K(m)$ are suboptimal as asymptotically they yield the inequality $K \geqslant 5/3$.

4.2. **The case** $n \geqslant 3$**.** Analogously to the case $n = 2$, let us denote $x_i = \nu(X_i)$ for $i = 1, \ldots, n$. For every $j \leqslant n$, let us pick $\omega_j \in X_j$ $(j \leqslant n)$. By Lemma 1, we may restrict our attention to measures that assume equal values on singletons from the respective sets X_j. In other words, it is enough to consider the following sets:

- $\Omega_{k,n} \setminus \{\omega_\ell \colon \ell \leqslant n\}$;
- $X_j \cup \{\omega_\ell \colon \ell \leqslant n, \ell \neq j\}$ $\quad (j \leqslant n)$.

Thus, this time, we seek to minimise the following expressions simultaneously:

$$\left|\sum_{\ell \leqslant n} x_\ell - \frac{\sum_{\ell \leqslant n} x_\ell}{k} - 1\right|, \; \left|x_j + \sum_{\ell \neq j} \frac{x_\ell}{k} - 3\right| \; (j \leqslant n)$$

with respect to $(x_1, x_2, \ldots, x_n)$. In particular, for $j \leqslant n$, we have

$$\frac{k-1}{k} \sum_{\ell \leqslant n} x_\ell - 1 = 3 - x_j - \sum_{\ell \neq j} \frac{x_\ell}{k}. \tag{4.1}$$

The sum $t = \sum_{\ell \leqslant n} x_\ell$ may be then computed by adding these equations together. More specifically, $t = \frac{4nk}{(n+1)k-1}$. It follows from (4.1) that for any $j \leqslant n$ we have

$$\frac{k-1}{k} t - 1 = 3 - \frac{k-1}{k} x_j - \frac{t}{k}.$$

Finally, for every $j \leqslant n$, we have

$$x_j = \frac{k}{k-1}\Big(4 - \frac{4nk}{(n+1)k-1}\Big) = \frac{4k}{(n+1)k-1}.$$

Since the double sequence

$$a_{k,n} = 3 - \frac{4k}{(n+1)k-1} - \frac{n-1}{k} \frac{4k}{(n+1)k-1}$$

converges to 3 as $k, n \to \infty$, we may estimate $K(m)$ from below by $a_{k,n}$, where k, n are such that $m = k \cdot n$.

In particular, we could restrict, for example, to $n = k$. In this case, for $n = k = 10$, we get approximately 2.305, for $n = k = 20$, it is approximately 2.628, and for $n = k = 200$, we arrive at 2.96.

5. Closing remarks

Feige, Feldman, and Talgam-Cohen remarked that *obtaining good lower bounds on* K_s *is also not easy. Part of the difficulty is that even if one comes up with a function* f *that is a candidate to yield the lower bound, verifying that it is* ε*-modular involves checking roughly* 2^{2^n} *approximate modularity equations* [3, p. 69].

Motivated by the above statement, we have found a suitable candidate for the function(s) $f_{k,2}$ using a Python script, which gave us a lower estimate of $5/3$ for K. Subsequently, we added more degrees of freedom (by defining $f_{k,n}$) in an analogous manner. Let us briefly explain our approach, which would probably make the proof of the main result less *ad hoc*.

We consider the set $\Omega_{k,2}$ for $k \in \mathbb{N}$ and $k \geqslant 2$ so that $|\Omega_{k,2}| \geqslant 4$. Let $A = [a_{ij}]_{i,j=1}^3$ be a real matrix. We then define a function $f\colon \mathcal{F}_{k,2} \to \mathbb{R}$ by asserting that

$$\begin{array}{lll} f_k(\varnothing) = 0, & f_k(Y') = a_{12}, & f_k(X_2) = a_{13}, \\ f_k(X') = a_{21}, & f_k(X' \cup Y') = a_{22}, & f_k(X' \cup X_2) = a_{23}, \\ f_k(X_1) = a_{31}, & f_k(X_1 \cup Y') = a_{32}, & f_k(X_1 \cup X_2) = a_{33} \end{array}$$

as long as and X', Y' are proper, non-empty subsets of X_1 and X_2, respectively.

Lemma 2. *The function f is 1-additive (weakly-1-modular) if and only if the following conditions are satisfied:*

(i) $|a| \leqslant 1$ *for* $a \in \{a_{12}, a_{21}, a_{22}\}$.
(ii) $|2a_{12} - a_{13}| \leqslant 1$,
(iii) $|2a_{21} - a_{31}| \leqslant 1$,
(iv) $|a_{13} + a_{31} - a_{33}| \leqslant 1$,
(v) $|a_{12} + a_{21} - a_{22}| \leqslant 1$,
(vi) $|2a_{22} - b| \leqslant 1$ *for* $b \in \{a_{23}, a_{32}, a_{33}\}$,
(vii) $|a_{12} + a_{22} - a_{23}| \leqslant 1$,
(viii) $|a_{21} + a_{22} - a_{32}| \leqslant 1$,
(ix) $|a_{33} - c| \leqslant 1$ *for* $c \in \{a_{32} + a_{12}, a_{23} + a_{21}\}$.

In particular, $a_{13}, a_{31}, a_{23}, a_{32}, a_{33} \in [-3, 3]$.

Proof. Straightforward verification. ∎

Effectively, Pawlik's construction corresponds to the matrix

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}.$$

Having implemented the conditions from Lemma 2 in Python, we run a simple script that listed for us *all* 1-additive functions of that form that take values from the list $(-3, -2.5, -2, \ldots, 2, 2.5, 3)$. (By Lemma 2, the numbers -3 and 3 are extremal values for the range of such functions.) Overall, we found in total 38,034 such functions that are non-zero (excluding those that differ only by the sign, we had only 19,017 functions to investigate after all). Using a convex optimisation solver SCS ([6, 7]), we filtered out those functions whose distance to the space of measures is at least 1.4 in the case $k = 4$ (that is, functions on an 8-element set), having found only two:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Obviously, the former one corresponds to functions $f_{k,2}$ that we consider in the present paper.

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