Arc-disjoint strong spanning subdigraphs in compositions and products of digraphs

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Abstract

A digraph $D = (V, A)$ has a *good decomposition* if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong. Let T be a digraph with vertices $u_1, \dots, u_t$ $(t \geq 2)$ and let $H_1, \dots H_t$ be digraphs such that H_i has vertices u_{i,j_i}, $1 \leq j_i \leq n_i$. Then the *composition* $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\{u_{i,j_i} \colon 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$A(Q) = \cup_{i=1}^{t} A(H_i) \cup \{u_{ij_i}u_{pq_p} \colon u_iu_p \in A(T), 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}.$$

For digraph compositions $Q = T[H_1, \dots H_t]$, we obtain sufficient conditions for Q to have a good decomposition and a characterization of Q with a good decomposition when T is a strong semicomplete digraph and each H_i is an arbitrary digraph with at least two vertices.

For digraph products, we prove the following: (a) if $k \geq 2$ is an integer and G is a strong digraph which has a collection of arc-disjoint cycles covering all vertices, then the Cartesian product digraph $G^{\square k}$ (the kth power of G with respect to Cartesian product) has a good decomposition; (b) for any strong digraphs G, H, the strong product $G \boxtimes H$ has a good decomposition.

Keywords: strong spanning subdigraph; decomposition into strong spanning subdigraphs; semicomplete digraph; digraph composition; Cartesian product; strong product.

AMS subject classification (2010): 05C20, 05C70, 05C76, 05C85.

1 Introduction

We refer the readers to [1, 2, 6] for graph-theoretical notation and terminology not given here. A digraph $D = (V, A)$ is *strongly connected* (or

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strong) if there exists a path from x to y and a path from y to x in D for every pair of distinct vertices x, y of D. A digraph D is *k-arc-strong* if $D - X$ is strong for every subset $X \subseteq A$ of size at most $k-1$.

A digraph D is *semicomplete* if for every pair x, y of distinct vertices of D, there is at least one arc between x and y. In particular, a tournament is a semicomplete digraph, where there is exactly one arc between x, y for every pair x, y of distinct vertices. A digraph D is *locally semicomplete* if the out-neighborhood and in-neighborhood of every vertex of D induce semicomplete digraphs.

An *out-branching* B_s^+ (respectively, *in-branching* B_s^-) in a digraph $D = (V, A)$ is a connected spanning subdigraph of D in which each vertex $x \neq s$ has precisely one arc entering (leaving) it and s has no arcs entering (leaving) it. The vertex s is the *root* of B_s^+ (respectively, B_s^-).

Edmonds [9] characterized digraphs having k arc-disjoint out-branchings rooted at a specified root s. Furthermore, there exists a polynomial algorithm for finding k arc-disjoint out-branchings from a given root s if they exist (see p. 346 of [1]). However, if we ask for the existence of a pair of arc-disjoint branchings B_s^+, B_s^- such that the first is an out-branching rooted at s and the latter is an in-branching rooted at s, then the problem becomes NP-complete (see Section 9.6 of [1]). In connection with this problem, Thomassen [12] posed the following conjecture: There exists an integer N so that every N-arc-strong digraph D contains a pair of arc-disjoint in- and out-branchings.

Bang-Jensen and Yeo generalized the above conjecture as follows.[1] A digraph $D = (V, A)$ has a *good decomposition* if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong [4].

Conjecture 1.1 *[5] There exists an integer N so that every N-arc-strong digraph D has a good decomposition.*

For a general digraph D, it is a hard problem to decide whether D has a decomposition into two strong spanning subdigraphs.

Theorem 1.1 *[5] It is NP-complete to decide whether a digraph has a good decomposition.*

Clearly, every digraph with a good decomposition is 2-arc-strong. Bang-Jensen and Yeo characterized semicomplete digraphs with a good decomposition.

Theorem 1.2 *[5] A 2-arc-strong semicomplete digraph D has a good decomposition if and only if D is not isomorphic to S_4, where S_4 is obtained from the complete digraph with four vertices by deleting a cycle of length 4 (see Figure 1). Furthermore, a good decomposition of D can be obtained in polynomial time when it exists.*

The following result extends Theorem 1.2 to locally semicomplete digraphs.

[1] Every strong digraph D has an out- and in-branching rooted at any vertex of D.

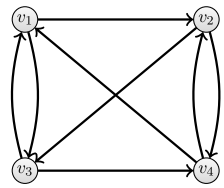

Figure 1: Digraph S_4

Theorem 1.3 *[4] A 2-arc-strong locally semicomplete digraph D has a good decomposition if and only if D is not the second power of an even cycle.*

Let T be a digraph vertices $u_1, \dots, u_t$ $(t \geq 2)$ and let $H_1, \dots H_t$ be digraphs such that H_i has vertices $u_{i,j_i},\ 1 \leq j_i \leq n_i$. Then the *composition* $Q = T[H_1, \dots, H_t]$ is a digraph with vertex set $\{u_{i,j_i} \colon 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$A(Q) = \cup_{i=1}^{t} A(H_i) \cup \{u_{ij_i}u_{pq_p} \colon u_i u_p \in A(T), 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}.$$

In this paper, we continue research on good decompositions in classes of digraphs and consider digraph compositions and products.

In Section 2, for a digraph composition $Q = T[H_1, \dots H_t]$, we obtain sufficient conditions for Q to have a good decomposition (Theorem 2.2) and a characterization of Q with a good decomposition when T is a strong semicomplete digraph and each H_i is an arbitrary digraph with at least two vertices (Theorem 2.3). Remarkably, in Theorem 2.3 as in Theorem 1.2, there are only a finite number of exceptional digraphs, which for Theorem 2.3 is three. Thus, as Theorems 1.2 and 1.3, Theorem 2.3 confirms Conjecture 1.1 for a special class of digraphs.

In Section 3, for digraph products, we prove the following: (a) if $k \geq 2$ is an integer and G is a strong digraph which arcs can be partitioned into cycles, then the Cartesian product digraph $G^{\Box k}$ (the kth power of G with respect to Cartesian product) has a good decomposition (Theorem 3.4); (b) for any strong digraphs G, H, the strong product $G \boxtimes H$ has a good decomposition (Theorem 3.7). Necessary definitions of the digraph products are given in Section 3.

Simple examinations of our constructive proofs show that all our decompositions can be found in polynomial time.

We conclude the paper in Section 4, where we pose a number of open problems.

2 Compositions of digraphs

The composition of digraphs is a useful concept in digraph theory, see e.g., [1]. In particular, they are used in the Bang-Jensen-Huang characterization of quasi-transitive digraphs and its structural and algorithmic applications for quasi-transitive digraphs and their extensions; see e.g., [1, 2, 8].

Let us start from a simple observation, which will be useful in the proofs of the theorems of this section.

Lemma 2.1 *Let D be a digraph on t vertices ($t \geq 2$) and let $H'_1, \dots, H'_t$ be digraphs with no arcs. If an induced subdigraph Q^* of $Q' = D[H'_1, \dots, H'_t]$ with at least one vertex in each H'_i, $i \in [t]$ has a good decomposition, then so have Q'.*

Proof: Let $\{u_{i,1}, \dots, u_{i,n_i}\}$ be the set of vertices of H'_i for every $i \in [t]$. For every $i \in [t]$, let $H_i^{(m_i)}$ be the subdigraph of H'_i induced by $\{u_{i,1}, \dots, u_{i,m_i}\}$, where $1 \leq m_i \leq n_i$. Without loss of generality, let $Q^* = D[H_1^{(m_1)}, \dots, H_t^{(m_t)}]$ and let Q^* have a decomposition into arc-disjoint strong spanning subdigraphs Q_1^*, Q_2^*. To extend this decomposition to Q', for every i, j, where $i \in [t]$ and $j \in \{1, 2\}$, add to Q_j^* the vertices $u_{i,m_i+1}, \dots, u_{i,n_i}$ and let them have the same in- and out-neighbors as $u_{i,1}$. (This way the inserted vertices will keep Q_1^* and Q_2^* strongly connected.) □

The following theorem gives sufficient conditions for a digraph composition to have a good decomposition. As in Theorem 1.2, S_4 will denote the digraph obtained from the complete digraph with four vertices by deleting a cycle of length 4.

Theorem 2.2 *Let T be a digraph with vertices $u_1, \dots, u_t$ ($t \geq 2$) and let $H_1, \dots, H_t$ be digraphs. Let the vertex set of H_i be $\{u_{i,j_i} \colon 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ for every $i \in [t]$. Then $Q = T[H_1, \dots, H_t]$ has a good decomposition if at least one of the following conditions holds:*

(a) *T is a 2-arc-strong semicomplete digraph and $H_1, \dots, H_t$ are arbitrary digraphs, but Q is not isomorphic to S_4;*

(b) *T has a Hamiltonian cycle and one of the following conditions holds:*

- *t is even and $n_i \geq 2$ for every $i = 1, \dots, t$;*
- *t is odd, $n_i \geq 2$ for every $i = 1, \dots, t$ and at least two distinct subdigraphs H_i have arcs;*
- *t is odd and $n_i \geq 3$ for every $i = 1, \dots, t$ apart from one i for which $n_i \geq 2$.*

(c) *T and all H_i are strong digraphs with at least two vertices.*

Proof: For every $i \in [t]$, let H'_i be the digraph obtained from H_i by deleting all arcs. Let $Q' = T[H'_1, \dots, H'_t]$. We will prove parts of the theorem one by one.

Part (a) If T is not isomorphic to S_4 then we are done by Theorem 1.2 and Lemma 2.1. Now assume that T is isomorphic to S_4, but Q is not isomorphic to S_4. Let the vertices of T be u_1, u_2, u_3, u_4 and its arcs

$$u_1u_2, u_2u_1, u_3u_4, u_4u_3, u_1u_4, u_2u_3, u_4u_2, u_3u_1.$$

Since Q is not isomorphic to S_4, at least one of H_1, H_2, H_3, H_4 has at least two vertices. Without loss of generality, let H_1 have at least two vertices.

Consider the subdigraph Q^* of Q' induced by $\{u_{1,1}, u_{1,2}, u_{2,1}, u_{3,1}, u_{4,1}\}$. Then Q^* has two arc-disjoint strong spanning subdigraphs: Q_1^* with arcs

$$\{u_{1,1}u_{2,1}, u_{2,1}u_{1,2}, u_{1,2}u_{4,1}, u_{4,1}u_{3,1}, u_{3,1}u_{1,1}\}$$

and Q_2^* with arcs

$$\{u_{2,1}u_{1,1}, u_{1,1}u_{4,1}, u_{4,1}u_{2,1}, u_{2,1}u_{3,1}, u_{3,1}u_{1,2}, u_{1,2}u_{2,1}\}.$$

It remains to apply Lemma 2.1 to obtain a good decomposition of Q' and thus of Q.

Part (b) Without loss of generality, assume that $u_1u_2\dots u_tu_1$ is a Hamiltonian cycle of T. Let $U = \bigcup_{i=1}^{t}\{u_{i,1}, u_{i,2}\}$.

Case 1: t is even and $n_i \geq 2$ for every $i = 1, \dots, t$. The following arc sets induce arc-disjoint strong spanning subdigraphs Q_1^*, Q_2^* of $Q'[U]$:

$$\{u_{i,j}u_{i+1,j}\colon\ 1 \leq i \leq t-1, 1 \leq j \leq 2\} \cup \{u_{t,1}u_{1,2}, u_{t,2}u_{1,1}\}; \qquad (1)$$

$$\{u_{i,j}u_{i+1,j')}\colon\ 1 \leq i \leq t-1, 1 \leq j \leq 2\} \cup \{u_{t,1}u_{1,1}, u_{t,2}u_{1,2}\}, \qquad (2)$$

where $j' = j + 1 \pmod 2$.

It remains to apply Lemma 2.1 to obtain a good decomposition of Q' and thus of Q.

Case 2: t is odd, $n_i \geq 2$ for every $i = 1, \dots, t$ and at least two distinct subdigraphs H_i have arcs. Let e_p, e_q be arcs in two distinct subdigraphs H_p and H_q. We may assume that both end-vertices of e_p and e_q are in U. Observe that while Q_1^* (with arcs listed in (1)) is strong, Q_2^* (with arcs listed in (2)) forms two arc-disjoint cycles C and Z. We may assume that the tail (head) of e_p (e_q) is in C and the head (tail) of e_p (e_q) is in Z (otherwise, relabel vertices in $\{u_{p,1}, u_{p,2}\}$ and/or $\{u_{q,1}u_{q2}\}$). Thus, adding e_p and e_q to Q_2^* makes it strong. To obtain two arc-disjoint strong spanning subdigraphs of Q from Q_1^*, Q_2^*, let every vertex $u_{i,j}$ for $j \geq 3$ and $1 \leq i \leq t$ have the same out- and in-neighbors as $u_{i,1}$ in Q'.

Case 3: t is odd and $n_i \geq 3$ for every $i \in [t]$ apart from one i for which $n_i \geq 2$. Without loss of generality, assume that $n_1 \geq 2$ and $n_i \geq 3$ for all $i \in \{2, 3, \dots, t\}$.

First we consider the subcase in which $t = 3$, $n_1 = 2$, and $n_2 = n_3 = 3$. Then Q' has two arc-disjoint spanning subdigraphs Q_1^* and Q_2^* with arc sets

$$\{u_{1,1}u_{2,1}, u_{3,1}u_{1,1}, u_{1,2}u_{2,2}, u_{1,2}u_{2,3}, u_{3,2}u_{1,2}, u_{3,3}u_{1,2}, u_{2,1}u_{3,2}, u_{2,2}u_{3,1}, u_{2,3}u_{3,3}\},$$

$$\{u_{1,1}u_{2,2}, u_{1,1}u_{2,3}, u_{3,2}u_{1,1}, u_{3,3}u_{1,1}, u_{1,2}u_{2,1}, u_{3,1}u_{1,2}, u_{2,1}u_{3,3}, u_{2,2}u_{3,2}, u_{2,3}u_{3,1}\},$$

respectively. Observe that Q_1^* and Q_2^* are strong since they contain the closed walks through all vertices, respectively:

$$u_{1,2}u_{2,2}u_{3,1}u_{1,1}u_{2,1}u_{3,2}u_{1,2}u_{2,3}u_{3,3}u_{1,2};\ u_{1,1}u_{2,2}u_{3,2}u_{1,1}u_{2,3}u_{3,1}u_{1,2}u_{2,1}u_{3,3}u_{1,1}.$$

Now we extend the previous subcase to that in which $n_1 = 2$ and $n_i = 3$ for all $i \in \{2, 3, \ldots, t\}$. First replace index 3 in every vertex of the form $u_{3,i}$ by t in the two arc sets of the previous subcase. Then replace every arc of the form $u_{2,i}u_{t,j}$ in Q_1^* by the path $u_{2,i}u_{3,i} \ldots u_{t-1,i}u_{t,j}$. In Q_2^*, we replace $u_{2,1}u_{t,3}$ by the path $u_{2,1}u_{3,2}u_{4,1}u_{5,2} \ldots u_{t-1,1}u_{t,3}$, replace $u_{2,2}u_{t,2}$ by the path $u_{2,2}u_{3,1}u_{4,2}u_{5,1} \ldots u_{t-1,2}u_{t,2}$, replace $u_{2,3}u_{t,1}$ by the path $u_{2,3}u_{3,2}u_{4,3}u_{5,2} \ldots u_{t-1,3}u_{t,1}$, and finally add the path $u_{2,2}u_{3,3}u_{4,2}u_{5,3} \ldots u_{t-1,2}$.

Finally, we extend the previous subcase to the general one using Lemma 2.1.

Part (c) For $j \in \{1, 2\}$, let T_j be the subdigraph of Q induced by vertex set $\{u_{i,j} \colon 1 \le i \le t\}$. Clearly, $T_1 \cong T_2 \cong T$ and T_1 and T_2 are strong.

Let Q_1 be the spanning subdigraph of Q with arc set $A(Q_1) = A(T_1) \cup (\bigcup_{i=1}^{t} A(H_i))$. Observe that Q_1 is strong since T_1 and each H_i are strong, and T_1 has a common vertex with each H_i, where $1 \le i \le t$.

Let Q_2 be the spanning subdigraph of Q with arc set $A(Q_2) = A(Q) \setminus A(Q_1)$. To see that Q_2 is strong, we only need to find a strong subdigraph in Q_2 which contains x and y for each pair of distinct vertices x and y in Q_2. We will consider two cases.

Case 1: $x \in V(T_1)$. Without loss of generality, we assume that $x = u_{1,1}$ and $y \in \{u_{1,2}, u_{2,1}, u_{2,2}\}$. We first consider the subcase that $y = u_{2,1}$. Observe that there is at least one arc entering and one arc leaving $u_{1,2}$ ($u_{2,2}$) in T_2, and so there are two arcs, say a and b (c and d), with opposite directions between x (y) and T_2 in Q_2. Then by adding the arcs a, b, c, d, and the vertices x, y to T_2, we obtain a strong subdigraph T_2' of Q_2 which contains both x and y, as desired. For the case that $y \in \{u_{1,2}, u_{2,2}\}$, we just add the arcs a, b, and the vertex x to T_2, and then obtain a strong subdigraph T_2'' of Q_2 which contains both x and y.

Case 2: $x \notin V(T_1)$. Without loss of generality, we assume that $x = u_{1,2}$ and $y \in \{u_{1,1}, u_{2,1}, u_{1,3}, u_{2,2}, u_{2,3}\}$ (if $u_{1,3}$ and $u_{2,3}$ exist). By Case 1 and the fact that $T_2 \cong T$ is strong, we are done if $y \in \{u_{1,1}, u_{2,2}\}$. For the case that $y = u_{2,1}$, by adding the arcs c, d and the vertex y to T_2, we can obtain a strong subdigraph T_2''' of Q_2 which contains both x and y. With a similar argument, we can get the desired strong subdigraph for the case that $y \in \{u_{1,3}, u_{2,3}\}$.

Hence, we complete the argument and conclude that Q has a good decomposition. $\square$

We will use Theorem 2.2 to prove the following characterization for certain compositions $T[H_1, \ldots, H_t]$, where T is a strong semicomplete digraph. In the characterization, $\overline{K_p}$ will stand for the digraph of order p with no arcs. Also, $\overrightarrow{C}_k$ and $\overrightarrow{P}_k$ will denote the cycle and path with k vertices, respectively.

Theorem 2.3 *Let T be a strong semicomplete digraph on $t \ge 2$ vertices and let $H_1, \ldots, H_t$ be arbitrary digraphs, each with at least two vertices. Then*

$Q = T[H_1, \ldots, H_t]$ *has a good decomposition if and only if* Q *is not isomorphic to one of the following three digraphs:* $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$. $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$.

Proof: Let us first prove the 'only if' part of the theorem, i.e. $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ and $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ do not have good decompositions. By Lemma 2.1, it suffices to show that neither $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ nor $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ has a good decomposition. The proof is by reductio ad absurdum.

Suppose that $Q = \overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K_2}, \overline{K_2}]$ has a decomposition into two strong spanning subdigraphs D_1, D_2. Since Q has 13 arcs, without loss of generality, we may assume that D_1 is a Hamiltonian cycle of Q. Since the arc of H_1 cannot be in a Hamiltonian cycle of Q, without loss of generality, let $D_1 = u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$. Then the remaining arcs of Q form two disjoint cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and a single arc between them, a contradiction to the assumption that D_2 is strong.

Suppose that $Q = \overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$ has a decomposition into two strong spanning subdigraphs D_1, D_2. Since Q has 16 arcs and has no Hamiltonian cycle, each of D_1, D_2 has 8 arcs. Since Q has only cycles of lengths 3 and 6 and D_1 is strong, without loss of generality, we may assume that D_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and a path $u_{2,1}u_{3,3}u_{1,1}$. Then D_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and a path $u_{2,2}u_{3,3}u_{1,2}$. Observe that D_2 is not strong, a contradiction.

Now we will show the 'if' part of the theorem by reductio ad absurdum as well. Assume that Q is not isomorphic to either of the three digraphs, but has no good decomposition.

By Camion's Theorem [7], T has a Hamiltonian cycle $C = u_1u_2 \ldots u_tu_1$. Thus, Conditions (b) of Theorem 2.2 are applicable. By the conditions, t must be odd and for at least two distinct indexes $p, q \in \{1, 2, \ldots, t\}$, we have $n_p = n_q = 2$.

Suppose $t \geq 5$. Then there will be arcs between H_i and H_{i+2} in Q for every $i = 1, 2, \ldots, t-2$. Recall Case 2 of Part (b) of the proof of Theorem 2.2. The arcs between H_i and H_{i+2} arcs can be used to make D_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 2.2. Thus, Q has a good decomposition, a contradiction. Hence, $t = 3$ and, without loss of generality, $n_1 = n_2 = 2$ and $n_3 \geq 2$.

Suppose that T has opposite arcs. One of these arcs will not be on the Hamiltonian cycle C of T and will correspond to four or more arcs in Q. Now recall Case 2 of Part (b) of the proof of Theorem 2.2. Two of the above-mentioned arcs can be used to make D_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 2.2. Thus, Q has a good decomposition, a contradiction. Hence, $T = \overrightarrow{C}_3$.

Suppose that $n_3 \geq 4$. To get a contradiction, by Lemma 2.1 it suffices to show that $Q = \overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_4}]$ has a decomposition into two strong spanning subdigraphs D_1, D_2, where D_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and two paths $u_{2,1}u_{3,4}u_{1,1}$ and $u_{2,2}u_{3,3}u_{1,2}$ and D_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$ and two paths $u_{2,1}u_{3,3}u_{1,1}$ and $u_{2,2}u_{3,4}u_{1,2}$. Thus, $n_3 \leq 3$.

Now consider the case of $n_1 = n_2 = 2$ and $n_3 = 3$. Since Q is not isomorphic to $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_3}]$, it has an arc in either H_1 or H_2 or H_3, and by Conditions (b) of Theorem 2.2, only one of H_1, H_2, H_3 has an arc a. Without loss of generality, assume that if H_1 has an arc then $a = u_{1,2}u_{1,1}$, if H_2 has an arc then $a = u_{2,1}u_{2,2}$ and if H_3 has an arc then $a = u_{3,2}u_{3,1}$. Then Q has a decomposition into two spanning subdigraphs D_1, D_2, where D_1 consists of a cycle $u_{1,1}u_{2,1}u_{3,1}u_{1,2}u_{2,2}u_{3,2}u_{1,1}$ and a path $u_{2,1}u_{3,3}u_{1,1}$ and D_2 consists of two cycles $u_{1,1}u_{2,2}u_{3,1}u_{1,1}$ and $u_{1,2}u_{2,1}u_{3,2}u_{1,2}$, a path $u_{2,2}u_{3,3}u_{1,2}$ and arc a. Observe that both D_1 and D_2 are strong, a contradiction.

It remains to consider the case of $n_1 = n_2 = n_3 = 2$. Since Q is not isomorphic to $\overrightarrow{C}_3[\overline{K_2}, \overline{K_2}, \overline{K_2}]$, at least one of H_1, H_2 and H_3 has an arc. By Conditions (b) of Theorem 2.2, only one of H_1, H_2 and H_3 has an arc. Without loss of generality, assume that H_1 has an arc. Suppose that H_1 has two arcs. Then $H_1 = \overrightarrow{C}_2$. Then we can use the arcs of H_1 to make D_2 strong instead of arcs e_p and e_q used in Case 2 of Part (b) of the proof of Theorem 2.2. Thus, Q has a good decomposition, a contradiction. Hence, if H_1 has an arc, it must have just one arc. This concludes our proof. □

3 Products of digraphs

The *Cartesian product* $G \Box H$ of two digraphs G and H is a digraph with vertex set $V(G \Box H) = V(G) \times V(H) = \{(x, x') \colon x \in V(G), x' \in V(H)\}$ and arc set $A(G \Box H) = \{(x, x')(y, y') \colon xy \in A(G), x' = y', \text{ or } x = y, x'y' \in A(H)\}$. By definition, the Cartesian product is associative and commutative (up to isomorphism), and $G \Box H$ is strongly connected if and only if both G and H are strongly connected [10]. We define the kth powers with respect to Cartesian product as $D^{\Box k} = \underbrace{D \Box D \Box \cdots \Box D}_{k \text{ times}}$.

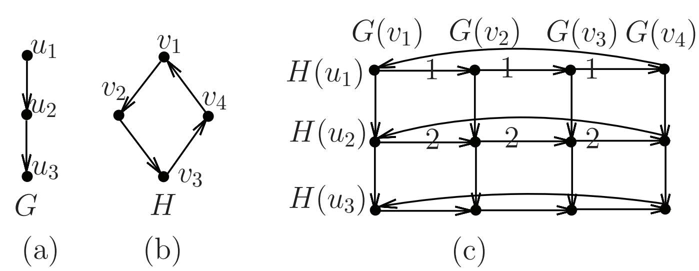

Figure 2: Two digraphs G, H and their Cartesian product.

In the arguments of this section, we will use the following terminology and notation. Let G and H be two digraphs with $V(G) = \{u_i \colon 1 \le i \le n\}$ and $V(H) = \{v_j \colon 1 \le j \le m\}$. For simplicity, we let $u_{i,j} = (u_i, v_j)$ for $1 \le i \le n, 1 \le j \le m$. We use $G(v_j)$ to denote the subdigraph of $G \Box H$ induced by vertex set $\{u_{i,j} \colon 1 \le i \le n\}$, where $1 \le j \le m$, and use $H(u_i)$ to denote the subdigraph of $G \Box H$ induced by vertex set $\{u_{i,j} \colon 1 \le j \le m\}$, where $1 \le i \le n$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. (For example, as shown in Figure 2, $G(v_j) \cong G$ for $1 \le j \le 4$ and $H(u_i) \cong H$ for $1 \le i \le 3$.) For $1 \le j_1 \ne j_2 \le m$, u_{i,j_1} and u_{i,j_2} belong to the same digraph $H(u_i)$, where $u_i \in V(G)$; we call u_{i,j_2} the *vertex corresponding to* u_{i,j_1} in $G(v_{j_2})$; for $1 \le i_1 \ne i_2 \le n$, we call $u_{i_2,j}$ the vertex corresponding to $u_{i_1,j}$ in $H(u_{i_2})$. Similarly, we can define the subdigraph *corresponding* to some other subdigraph. For example, in Fig. 2(c), let P_1 (P_2) be the path labelled 1 (2) in $H(u_1)$ ($H(u_2)$), then P_2 is called the path *corresponding* to P_1 in $H(u_2)$.

Lemma 3.1 *For any integer $n \ge 2$, the product digraph $D = \overrightarrow{C}_n \Box \overrightarrow{C}_n$ can be decomposed into two arc-disjoint Hamiltonian cycles.*

Proof: Let $G = H \cong \overrightarrow{C}_n$; moreover $G = u_1u_2 \dots u_nu_1$ and $H = v_1v_2 \dots v_nv_1$. Let $P_i = G(v_i) - u_{n-i,i}u_{n+1-i,i}$ for $1 \le i \le n-1$ and $P_n = G(v_n) - u_{n,n}u_{1,n}$. Let $Q_i = H(u_i) - u_{i,n-i}u_{i,n+1-i}$ for $1 \le i \le n-1$ and $Q_n = H(u_n) - u_{n,n}u_{n,1}$. Furthermore, let

$$D' = \left(\bigcup_{i=1}^{n-1} (P_i \cup \{u_{n-i,i}u_{n-i,i+1}\}) \right) \cup (P_n \cup \{u_{n,n}u_{n,1}\})$$

and

$$D'' = \left(\bigcup_{i=1}^{n-1} (Q_i \cup \{u_{i,n-i}u_{i+1,n-i}\}) \right) \cup (Q_n \cup \{u_{n,n}u_{1,n}\})$$

By the construction, the subdigraphs D' and D'' are Hamiltonian cycles of D. For example, see Figure 3 for the case that $n = 5$ (the Hamiltonian cycle D' consists of five "vertical" paths P_i of order five and five "horizontal" arcs, D'' consists of five "horizontal" paths Q_i of order five and five "vertical" arcs, furthermore, these two cycles are symmetric about the diagonal.) □

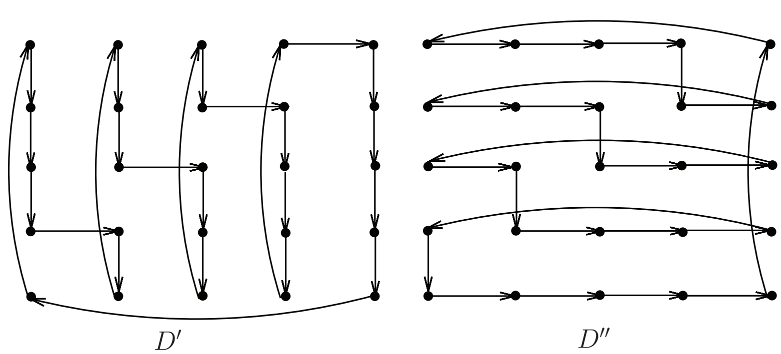

Figure 3: Two arc-disjoint Hamiltonian cycles for the case $n = 5$.

Note that deciding whether a digraph D has a collection of arc-disjoint cycles covering all vertices of D can be done in polynomial time using network flows. Indeed, assign lower bound 1 and upper bound $\min\{d^-(x), d^+(x)\}$ to every vertex x in D and lower bound 0 and upper bound 1 to every arc of D. Observe that the resulting network has a feasible flow if and only if D has a collection of arc-disjoint cycles covering all vertices of D. Observe that the existence of a flow in a network with lower and upper bounds on vertices and arcs can be decided in polynomial time, see e.g., Chapter 4 in [1]. Moreover, we can compute such a flow in polynomial time (if it exists) and obtain the corresponding collection of cycles in D. The following lemma may be of independent interest.

Lemma 3.2 *Let G be a strong digraph with at least two vertices which has a collection of arc-disjoint cycles covering all its vertices. Then the product digraph $D = G\Box G$ has a good decomposition. Moreover, such a good decomposition can be found in polynomial time.*

Proof: By the arguments in the paragraph before this lemma, we may assume that we are given a collection $(P_0, P_1, P_2, \cdots, P_p)$ of arc-disjoint cycles covering all vertices of G. For each $h \in \{0, 1, 2, \cdots, p\}$, let G_h denote the digraph with vertices $\bigcup_{i=0}^{h} V(P_i)$ and arcs $\bigcup_{i=0}^{h} A(P_i)$. Now we will prove the lemma by induction on the number of cycles in the collection.

For the base step, by Lemma 3.1, we have that $G_0\Box G_0 = P_0\Box P_0$ can be decomposed into two arc-disjoint strong spanning subdigraphs.

For the inductive step, we assume that $G_h\Box G_h$ $(0 \leq h \leq p-1)$ can be decomposed into two arc-disjoint strong spanning subdigraphs D'_h and D''_h. We will construct two arc-disjoint strong spanning subdigraphs in $G_{h+1}\Box G_{h+1}$.

If $V(G_h) \subseteq V(P_{h+1})$, then P_{h+1} is a Hamiltonian cycle of G_{h+1}, and we are done by Lemma 3.1. If $V(P_{h+1}) \subseteq V(G_h)$, then G_h is a strong spanning subdigraph of G_{h+1}, and we are also done by the induction hypothesis.

In the following argument, we assume that $V(G_h) \setminus V(P_{h+1}) \neq \emptyset$ and $V(P_{h+1}) \setminus V(G_h) \neq \emptyset$. Without loss of generality, for the first copies of G_h and P_{h+1} in $G_h\Box G_h$ and $P_{h+1}\Box P_{h+1}$, let $V(G_h) = \{u_i\colon 1 \leq i \leq t\}$, $V(P_{h+1}) = \{u_i\colon s \leq i \leq \ell\}$. We have $1 < s \leq t < \ell$. For the second copies of G_h and P_{h+1} in $G_h\Box G_h$ and $P_{h+1}\Box P_{h+1}$, we will use v_i's rather than u_i's.

By Lemma 3.1, in $G_{h+1}\Box G_{h+1}$, the subdigraph $P_{h+1}\Box P_{h+1}$ can be decomposed into two arc-disjoint strong spanning subdigraphs $\overline{D}'_h$ and $\overline{D}''_h$. Observe that

$$V(G_h\Box G_h)\cap V(P_{h+1}\Box P_{h+1}) \supseteq \{u_{t,t}\} \text{ and } A(G_h\Box G_h)\cap A(P_{h+1}\Box P_{h+1}) = \emptyset.$$

For $1 \leq j \leq s-1$, let $G_{h,j}$ be the subdigraph of $G(v_j)$ corresponding to P_{h+1}. For $t+1 \leq j \leq \ell$, let $G_{h,j}$ be the subdigraph of $G(v_j)$ corresponding to G_h. For $1 \leq i \leq s-1$, let $H_{h,i}$ be the subdigraph of $H(u_i)$ corresponding to P_{h+1}. For $t+1 \leq i \leq \ell$, let $H_{h,i}$ be the subdigraph of $H(u_i)$ corresponding to G_h.

Now let D'_{h+1} be a union of the following strong digraphs: D'_h, $\overline{D}'_h$, $H_{h,i}$ and $G_{h,j}$ for all $t+1 \leq i, j \leq \ell$. Observe that D'_{h+1} is a strong spanning subdigraph of $G_{h+1}\Box G_{h+1}$ since $\overline{D}'_h$ has at least one common vertex with each of D'_h, $H_{h,i}$ and $G_{h,j}$ for all $t+1 \leq i, j \leq \ell$. Let D''_{h+1} be a spanning subdigraph of $G_{h+1}\Box G_{h+1}$ with $A(D''_{h+1}) = A(G_{h+1}\Box G_{h+1})\setminus A(D'_{h+1})$. Observe that D''_{h+1} is the union of D''_h, $\overline{D}''_h$, $H_{h,i}$ and $G_{h,j}$ for all $1 \leq i, j \leq s-1$.

And D''_h has at least one common vertex with each of $\overline{D}''_h$, $H_{h,i}$ and $G_{h,j}$ for all $1 \le i, j \le s-1$, thus D''_{h+1} is strong.

Hence, we complete the inductive step and conclude that $D = G\Box G$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Moreover, by the above argument, these subdigraphs can be found in polynomial time. □

Lemma 3.3 *For any two strong digraphs G and H, if G has a good decomposition, then the product digraph $D = G\Box H$ has a good decomposition.*

Proof: Let $V(G) = \{u_i \colon 1 \le i \le n\}, V(H) = \{v_j \colon 1 \le j \le m\}$, and G contain two arc-disjoint strong spanning subdigraphs G_1 and G_2. For $1 \le i \le 2$ and $1 \le j \le m$, let $G_{i,j}$ be the subdigraph of $G(v_j)$ corresponding to G_i. As shown in Figure 4, let D' be the union of $H(u_1)$ and $G_{1,j}$ for all $1 \le j \le m$, and let D'' be a subdigraph of D with $V(D'') = V(D)$ and $A(D'') = A(D) \setminus A(D')$. Note that D'' is the union of $H(u_i)$ and $G_{2,j}$ for all $2 \le i \le n, 1 \le j \le m$. Since $H(u_i)$, $G_{1,j}$ and $G_{2,j}$ $(1 \le i \le n, 1 \le j \le m)$ are strong, both D' and D'' are strong spanning subdigraphs of D. This completes the proof. □

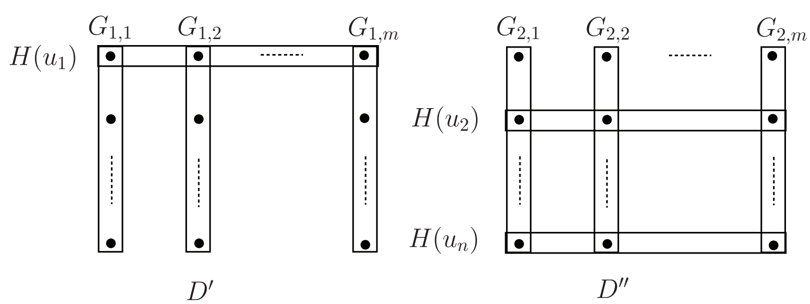

Figure 4: Two arc-disjoint strong spanning subdigraphs in Lemma 3.3.

By the definition of $D^{\Box k}$, associativity of the Cartesian product (up to isomorphism), and Lemmas 3.2 and 3.3, we can obtain the following result on $G^{\Box k}$ for any integer $k \ge 2$.

Theorem 3.4 *Let G be a strong digraph of order at least two which has a collection of arc-disjoint cycles covering all its vertices and let $k \ge 2$ be an integer. Then the product digraph $D = G^{\Box k}$ has a good decomposition. Moreover, for any fixed integer k, such a good decomposition can be found in polynomial time.*

The *strong product* $G \boxtimes H$ of two digraphs G and H is a digraph with vertex set $V(G \boxtimes H) = V(G) \times V(H) = \{(x, x') \colon x \in V(G), x' \in V(H)\}$ and arc set $A(G \boxtimes H) = \{(x, x')(y, y') \colon xy \in A(G), x' = y',\ or\ x = y, x'y' \in A(H),\ or\ xy \in A(G),\ x'y' \in A(H)\}$. By definition, $G\Box H$ is a spanning subdigraph of $G \boxtimes H$, and $G \boxtimes H$ is strongly connected if and only if both G and H are strongly connected [10]. In the following argument, we will still use the terminology and notation introduced earlier in this section, since $G\Box H$ is a spanning subdigraph of $G \boxtimes H$.

Lemma 3.5 *For any two integers $n, m \geq 2$, the product digraph $D = \overrightarrow{C}_n \boxtimes \overrightarrow{C}_m$ has a good decomposition.*

Proof: Let $\overrightarrow{C}_n = u_1u_2\dots u_nu_1$ and $\overrightarrow{C}_m = v_1v_2\dots v_m$. Let D' be the spanning subdigraph of D which is the union of $G(v_j)$ for $1 \leq j \leq m$ and the following additional m arcs: $\{u_{n,j}u_{1,j+1}\colon 1 \leq j \leq m-1\} \cup \{u_{1,m}u_{2,1}\}$. Observe that D' is strong. Let D'' be a spanning subdigraph of D with $A(D'') = A(D) \setminus A(D')$. To see that D'' is strong, observe that it contains $H(u_i)$ for $1 \leq i \leq n$ and arcs $\{u_{i,1}u_{i+1,2}\colon 1 \leq i \leq n-1\} \cup \{u_{n,m}u_{1,1}\}$. $\square$

We will use the following decomposition of strong digraphs.

An *ear decomposition* of a digraph D is a sequence $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_t)$, where P_0 is a cycle or a vertex and each P_i is a path, or a cycle with the following properties:
(a) P_i and P_j are arc-disjoint when $i \neq j$.
(b) For each $i = 0, 1, 2, \cdots, t$: let D_i denote the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$. If P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to $V(D_{i-1})$.
(c) $\bigcup_{j=0}^{t} A(P_j) = A(D)$.

The following result is well-known, see e.g., [1].

Theorem 3.6 *Let D be a digraph with at least two vertices. Then D is strong if and only if it has an ear decomposition. Furthermore, if D is strong, every cycle can be used as a starting cycle P_0 for an ear decomposition of D, and there is a linear-time algorithm to find such an ear decomposition.*

Theorem 3.7 *For any strong digraphs G and H with at least two vertices, the product digraph $D = G \boxtimes H$ has a good decomposition. Moreover, such a decomposition can be found in polynomial time.*

Proof: By Theorem 3.6, G has an ear decomposition $\mathcal{P} = (P_0, P_1, P_2, \cdots, P_p)$ and H has an ear decomposition $\mathcal{Q} = (Q_0, Q_1, Q_2, \cdots, Q_q)$, such that P_0 is a cycle of G and Q_0 is a cycle of H. Let G_i denote the subdigraph of G with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$ and let H_i denote the subdigraph of H with vertices $\bigcup_{j=0}^{i} V(Q_j)$ and arcs $\bigcup_{j=0}^{i} A(Q_j)$.

We will prove the theorem by induction on $r \in \{0, 1, \dots, p+q\}$. For the base step, by Lemma 3.5, we have that $P_0 \boxtimes Q_0$ can be decomposed into two arc-disjoint strong spanning subdigraphs. For the inductive step, we assume that $r = h + g < p + q$ $(h \leq p, g \leq q)$ and $G_h \boxtimes H_g$ can be decomposed into two arc-disjoint strong spanning subdigraphs D' and D''.

Since strong product is a commutative operation, without loss of generality it suffices to prove that $G_{h+1} \boxtimes H_g$ $(h < p)$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Let $V(G_h) = \{u_1, u_2, \dots, u_\ell\}$, $V(H_g) = \{v_1, v_2, \dots, v_m\}$ and $v_1v_s \in A(H_g)$. Let $P_{h+1,j}$ be the subdigraph of $G(v_j)$ corresponding to P_{h+1} for $1 \leq j \leq m$. We will consider two cases.

Case 1: P_{h+1} is a cycle. Let $P_{h+1} = u_\ell u_{\ell+1} \dots u_n u_\ell$. Observe that every $P_{h+1,j}$ for $1 \le j \le m$ shares vertex $u_{\ell,j}$ with D'. Thus, the union U_1 of D' and $P_{h+1,j}$ for $1 \le j \le m$ is a strong spanning subdigraph of $G_{h+1} \boxtimes H_g$. Let $V(U_2) = V(G_{h+1} \boxtimes H_g)$ and $A(U_2) = A(G_{h+1} \boxtimes H_g) \setminus A(U_1)$.

Observe that $A(U_2)$ contains $A(D'')$, $A(H(u_i))$ for $\ell+1 \le i \le n$ and $\{u_{i,1}u_{i+1,s}\colon\ \ell \le i \le n-1\} \cup \{u_{n,1}u_{\ell,s}\}$. Thus, U_2 is strong.

Case 2: P_{h+1} is a path. Let $P_{h+1} = u_\ell u_{\ell+1} \dots u_{n-1} u_t$, where $t < \ell$. Let U_1 be the union of D' and $P_{h+1,j}$ for $1 \le j \le m$. Observe that U_1 is a spanning subdigraph of $G_{h+1} \boxtimes H_g$ and strong since every $P_{h+1,j}$ for $1 \le j \le m$ shares its end-vertices with D'. Let $V(U_2) = V(G_{h+1} \boxtimes H_g)$ and $A(U_2) = A(G_{h+1} \boxtimes H_g) \setminus A(U_1)$. Observe that $A(U_2)$ contains $A(D'')$, $A(H(u_i))$ for $\ell+1 \le i \le n-1$ and $\{u_{i,1}u_{i+1,s}\colon\ \ell \le i \le n-2\} \cup \{u_{n-1,1}u_{t,s}\}$. Thus, U_2 is strong.

Hence, we complete the inductive step and conclude that $D = G \boxtimes H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Furthermore, by Theorem 3.6, the proof of Lemma 3.5, and the argument of this theorem, we can conclude that these two strong spanning subdigraphs can be found in polynomial time. □

The *lexicographic product* $G \circ H$ of two digraphs G and H is a digraph with vertex set $V(G \circ H) = V(G) \times V(H) = \{(x, x')\colon\ x \in V(G), x' \in V(H)\}$ and arc set $A(G \circ H) = \{(x, x')(y, y')\colon\ xy \in A(G),\ \textit{or}\ x = y\ \textit{and}\ x'y' \in A(H)\}$ [10]. By definition, $G \boxtimes H$ is a spanning subdigraph of $G \circ H$, so the following result holds by Theorem 3.7: For any strong connected digraphs G and H with orders at least 2, the product digraph $D = G \circ H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. Moreover, these two arc-disjoint strong spanning subdigraphs can be found in polynomial time. In fact, we can get a more general result.

A digraph is *Hamiltonian decomposable* if it has a family of Hamiltonian dicycles such that every arc of the digraph belongs to exactly one of the dicycles. Ng [11] gives the most complete result among digraph products.

Theorem 3.8 *[11] If G and H are Hamiltonian decomposable digraphs, and $|V(G)|$ is odd, then $G \circ H$ is Hamiltonian decomposable.*

Theorem 3.8 implies that if G and H are Hamiltonian decomposable digraphs, and $|V(G)|$ is odd, then $G \circ H$ can be decomposed into two arc-disjoint strong spanning subdigraphs. It is not hard to extend this result as follows: for any strong digraphs G and H of orders at least 2, if H contains $\ell \ge 1$ arc-disjoint strong spanning subdigraphs, then the product digraph $D = G \circ H$ can be decomposed into $\ell + 1$ arc-disjoint strong spanning subdigraphs.

4 Open Problems

We have characterized digraphs $T[H_1, \dots, H_t]$, where T is strong semicomplete and every H_i is arbitrary with at least two vertices, which have a good decomposition. It is a natural open problem to extend the characterization to all such digraphs, where some H_i's can have just one vertex. Of course, the extended characterization would generalize also Theorem 1.2.

A digraph Q is *quasi-transitive*, if for any triple x, y, z of distinct vertices of Q, if xy and yz are arcs of Q then either xz or zx or both are arcs of Q. For a recent survey on quasi-transitive digraphs and their generalizations, see a chapter [8] by Galeana-Sánchez and Hernández-Cruz. Bang-Jensen and Huang [3] proved that a quasi-transitive digraph is strong if and only if $Q = T[H_1, \ldots, H_t]$, where T is a strong semicomplete digraph and each H_i is a non-strong quasi-transitive digraph or has just one vertex. Thus, a special case of the above problem is to characterize strong quasi-transitive digraphs with a good decomposition. This would generalize Theorem 1.2 as well.

We believe that these characterizations will confirm Conjecture 1.1 for the classes of quasi-transitive digraphs and digraphs $T[H_1, \ldots, H_t]$, where T is strong semicomplete. In the absence of the characterizations, it would still be interesting to confirm the conjecture at least for quasi-transitive digraphs.

In Lemma 3.2, we show that $G\Box H$ contains a pair of arc-disjoint strong spanning subdigraphs when $G \cong H$. However, the following result implies Lemma 3.2 cannot be extended to the case that $G \ncong H$, since it is not hard to show that the Cartesian product digraph of any two cycles has a pair of arc-disjoint strong spanning subdigraphs if and only if it has a pair of arc-disjoint Hamiltonian cycles.

Theorem 4.1 *[13] The Cartesian product $\overrightarrow{C_p}\Box\overrightarrow{C_q}$ is Hamiltonian if and only if there are non-negative integers d_1, d_2 for which $d_1+d_2 = \gcd(p, q) \geq 2$ and $\gcd(p, d_1) = \gcd(q, d_2) = 1$.*

However, Lemma 3.2 could hold for the case that $G \ncong H$ if we add other conditions. As shown in Lemma 3.3, we know $G\Box H$ contains a pair of arc-disjoint strong spanning subdigraphs when one of G and H contains a pair of arc-disjoint strong spanning subdigraphs. So the following open question is interesting: for any two strong digraphs G and H, neither of which contain a pair of arc-disjoint strong spanning subdigraphs, under what condition the product digraph $G\Box H$ contains a pair of arc-disjoint strong spanning subdigraphs?

Furthermore, we may also consider the following more challenging question: under what conditions the product digraph $G\Box H$ ($G \boxtimes H$) has more (than two) arc-disjoint strong spanning subdigraphs?

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