The horofunction boundary and Denjoy-Wolff type theorems

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Abstract

In this thesis we will study the horofunction boundary of metric spaces, in particular the Funk, reverse-Funk and Hilbert's metrics, and one of its applications, Denjoy-Wolff type theorems. In a Denjoy-Wolff type setting we will show that Beardon points are star points of the union of the ω-limit sets. We will also show that Beardon and Karlsson points are not unique in $\mathbb{R}^2$. In fact, we will show one can have a continuum of Karlsson points. We will establish two Denjoy-Wolff type theorem that confirm the Karlsson-Nussbaum conjecture for classes of non-expanding maps on Hilbert' metric spaces. For unital Euclidean Jordan algebras we will give a description of the intersection of closed horoballs with the boundary of the cone as the radius tends to minus infinity.

We will expand on results by Walsh by establishing a general form for the Funk and reverse Funk horofunction boundaries of order-unit spaces. We will also give a full classification of the horofunctions of JH-algebras and the horofunctions and Busemann points of the spin factors for the Funk, reverse Funk and Hilbert metrics. Finally we will show that there exists a reverse-Funk non-Busemann horofunction for the cone of positive bounded self-adjoint operators on an infinite dimensional Hilbert space, the infinite dimensional spin factors and a space in which the pure states are weak* closed, answering a question raised by Walsh in [66].

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Contents

Chapter 1

Introduction and overview

The horofunction boundary was first introduced by Gromov in [27]. It provides a natural way to add a "boundary at infinity" to a metric space. One of the strengths of the horofunction boundary is that we can define one for every metric space and it does not require the space to have special properties like the Gromov boundary of a δ-hyperbolic space, see [9]. The union of the original metric space with the horofunction boundary is also a compact set, unlike other boundaries such as the bordification, see [9]. The horofunction boundary has applications in the study of isometry groups of metric spaces, see [45, 64, 65], the analysis of Denjoy-Wolff type problems in metric spaces, see [6, 25, 35, 41, 42], and establishing multiplicative ergodic theorems [26]. In general the horofunction boundary is hard to determine, though it has been found for a variety of metric spaces, including special cases of, normed spaces [29, 31, 30, 62], Hilbert metric spaces [41, 65] and Teichmüller spaces [36, 67], see [37] for an overview.

In this thesis we will be particularly interested in Hilbert metric spaces. Hilbert's (cross-ratio) metric d_h was first defined by Hilbert in [32] on an open bounded convex set C of a finite dimensional real vector space in the following way:

$$d_h(x, y) = \log\left(\frac{\|x' - y\|\|y' - x\|}{\|x' - x\|\|y' - y\|}\right) \qquad (x, y \in C)$$

where x' and y' are the intersections of the line through x and y and the boundary of C such that x lies between x' and y and y lies between x and y'. Hilbert metric spaces are a generalisation of Klein's model of the real hyperbolic space and have been studied intensively, see [57] for an overview. In [8] Birkhoff noted a connection between Hilbert's cross-ratio metric and the order structure of the cone. This gave rise to Birkhoff's version of Hilbert's metric, which gives an alternative definition of Hilbert's metric between the rays of the cone solely using the partial order. This version was popularised by Bushell in

[11]. Birkhoff's version of Hilbert's metric also gives rise to two pseudo-metrics, the Funk and reverse-Funk metrics, whose sum forms Hilbert's metric.

In this thesis we will be studying the geometry and applications of the horofunction boundary for order-unit spaces. For the applications specifically we will look at Denjoy-Wolff type theorems. Below we will give an overview of the main results of this thesis.

First we will recall the necessary background in Chapter 2, introducing partially ordered vector spaces, order-unit spaces, the Funk and reverse-Funk metric and Hilbert's metric, and the horofunction boundary.

In Chapter 3 we study Denjoy-Wolff type theorems. The classical Denjoy-Wolff theorem describes the accumulation points of the orbits of fixed point free holomorphic maps on the open unit disc of $\mathbb{C}$. Beardon expanded this result to metric spaces, and for finite dimensional Hilbert metric spaces Karlsson and Nussbaum independently conjectured that the accumulation points of the orbits of fixed point free non-expansive maps should satisfy similar behaviour. For numerous special cases the conjecture has been shown to hold, see [6, 35, 46].

We will in particular focus on Denjoy-Wolff type theorems by Beardon [6] and Denjoy-Wolff type theorems by Karlsson [35]. The methods in their proofs give rise to a special class of points; which we call Beardon and Karlsson points. Karlsson and Noskov have shown in [40] that Karlsson points are star points of the accumulation set of the orbit and, in Theorem 3.1.11, we will show that Beardon points are star points of the accumulation set of the orbit as well. Though Karlsson and Beardon points are unique for an open bounded strictly convex set of a vector space equipped with Hilbert's metric, we will show this is not the case in general. In Example 3.3.6 we will also show that for $\mathbb{R}^2$ equipped with the Euclidean norm the Karlsson and Beardon points in general are not unique. In fact, we will show that one can find a continuum of Karlsson points.

Finally, we will prove two special cases of the Karlsson-Nussbaum conjecture. In Theorem 3.4.1 we will show that the conjecture holds if the map is an isometry instead of non-expansive. In Theorem 3.4.8 we show that, if f is a fixed point free non-expansive map on a finite dimensional Hilbert metric space such that the images of the limit maps of (f^n) are closed, then the Karlsson-Nussbaum conjecture holds. This result is analogous to a result by Chu and Rigby in [14] for bounded symmetric domains.

In Chapter 4 we will provide an alternate version of another result by Chu and Rigby. In Theorem 5.10 in [14] Chu and Rigby give an explicit description of the closed horoballs on bounded symmetric domains, using the Pierce decomposition of Jordan algebras. We will give a description of the intersection of closed horoballs of Euclidean Jordan algebras as the radius tends to minus infinity.

In the remainder of the thesis we focus on the horofunction boundary of infinite dimensional order-unit spaces. A lot of our results are based on work in [66], by Walsh, which gives a classification of the Busemann points of order-unit spaces. Busemann points are horofunctions that are the limits of almost geodesics which were introduced by Rieffel in [59]. Walsh in particular has shown that in $C(K)$, where K is a compact Hausdorff space, all horofunctions of the Funk and reverse-Funk geometry are Busemann points. Walsh raised the question if for general order-unit spaces this remains the case [66, Question 6.6]. In [62] Walsh shows that for finite dimensional order-unit spaces the horofunctions of the reverse-Funk geometry are always Busemann points, and the horofunctions of the Funk geometry are all Busemann points if and only if the pure states are weak* closed. For infinite dimensional order-unit spaces there are no known necessary and sufficient conditions for which all horofunctions are Busemann points.

In Chapter 5 we will recall Kuratowski-Painlevé convergence. Kuratowski-Painlevé convergence is used to define a limit of a net of subsets of a Hausdorff space. The convergence is not always topological; if a Hausdorff space X is not locally compact, then there is no topology on the power set of X for which the limits coincide with the Kuratowski-Painlevé limit. One of the advantages of Kuratowski-Painlevé convergence is that the convergence is "compact", i.e. for every net there exists a convergent subnet. One can define the limit of a net of maps from a Hausdorff space to $\mathbb{R}$ by taking the Kuratowski-Painlevé limit of the epigraphs or hypographs. These modes of convergence play an important role in Walsh's classification of the Busemann point in $C(K)$ and in Chapter 7.

We will study and expand Walsh's results from [66] in Chapter 6. In the first two sections we will provide the proofs of the results in more detail. In the final section we use the classification of the Busemann point in $C(K)$ to describe the horofunction boundary of order-unit spaces. In Theorem 6.3.7 and Theorem 6.3.8 we will show that the horofunctions on order-unit spaces of the reverse-Funk and Funk geometry respectively are always of a specific form. This form is determined by the respective epigraph or hypograph limit of the evaluation maps of the net defining the horofunction. To obtain this result we use work by Kalauch, Lemmens and van Gaans [34] which shows the existence of an order dense embedding from an order-unit space into the continuous functions on the weak* closure of the pure states. We can show that any horofunction on the original order-unit space corresponds to a unique horofunction of the continuous functions on the weak* closure of the pure states, whose form is already known due to Walsh. Unfortunately the reverse is not true. Therefore, though we have a description of the horofunction boundary of an order-unit space, we do not have a classification.

In Chapter 7 we will provide an answer to the first part of Question 6.6 in [66]. We will show that there exist order-unit spaces for which not all horofunctions of the reverse-Funk geometry are Busemann points. Theorem 7.1.20 shows that there exists a reverse-Funk non-Busemann horofunction on $B(H)_{sa}$, the space of self-adjoint bounded operators on a complex Hilbert space. For this we will use that the pure states of $B(H)_{sa}$ are not weak* closed. As this was an important condition in [62] for all horofunctions of the Funk geometry to be Busemann point, one might think that it will be sufficient to require the pure states to be weak* closed, but in Theorem 7.1.22 we will show that, even for the reverse-Funk geometry, this is not a sufficient condition.

Finally, in Theorem 7.2.1 and Theorem 7.3.14 we will give a classification of the horofunctions and Busemann points of the Hilbert geometry of the spin factors and JH-algebras. Spin factors are well known and important Jordan algebras. They are one of the main building blocks of the Euclidean Jordan algebras, which were classified by Jordan, von Neumann and Wigner in [56]. Spin factors are also used as a model for real hyperbolic spaces. JH-algebras are Jordan algebras which are also a Hilbert space, where the multiplication map is self-adjoint with respect to the inner product. Roelands and Wortel have classified the unital JH-algebras as a finite direct sum of Euclidean Jordan Algebras and spin factors. In Theorem 7.3.5 and Theorem 7.3.8 respectively we will classify the horofunction of the reverse-Funk and Funk geometry of a finite direct sum of order-unit spaces in terms of the horofunction boundaries of the terms of the direct sum. As the horofunction boundary of Euclidean Jordan algebras has been fully classified by Lemmens, Lins, Nussbaum and Wortel in [41], this allows us to give the full classification of the horofunction boundary of the Hilbert geometry of JH-algebras. For both the infinite dimensional spin factors and infinite dimensional JH-algebras we find non-Busemann horofunctions for the Funk, reverse-Funk and Hilbert geometry.

Chapter 2

Preliminaries

2.1 Partially ordered vector spaces

Recall that a *partial order* on a set X is a binary relation which for all $x, y, z \in X$ satisfies the properties

1. (reflexive) $x \leq x$,
2. (anti-symmetry) $x \leq y$ and $y \leq x$ implies that $x = y$ and
3. (transitive) $x \leq y$ and $y \leq z$ implies that $x \leq z$.

A real vector space $(X, \leq)$ is a *partially ordered vector space* if it is equipped with a partial order $\leq$ which for all $\lambda \geq 0$ and $x, y, z \in X$ satisfies the properties

1. $x \leq y$ implies that $x + z \leq y + z$ and
2. $x \leq y$ implies that $\lambda x \leq \lambda y$.

We call a subset $X_+ \subset X$ of a real vector space a *cone* if X_+ is convex, for all $\lambda \geq 0$ we have $\lambda X_+ = X_+$ and $X_+ \cap -X_+ = \{0\}$. Cones and partially ordered vector spaces are linked. For a partially ordered vector space $(X, \leq)$ the set $X_+ = \{x \in X : x \geq 0\}$ is a cone, and for a cone $(X_+ \cup \{0\}) \subset X$ of a real vector space the relation $x \leq y$ if and only if $y - x \in X_+$ is a partial order. An element $u \in X_+$ is called an *order-unit* of X if for all $x \in X$ there is a $\lambda > 0$ such that $-\lambda u \leq x \leq \lambda u$. If X is *Archimedean*, i.e. for all $x \in X_+$ we have that $nx \leq u$ for all $n \in \mathbb{N}$ if and only if $x \in -X_+$, then X can be equipped with the *order-unit norm*

$$\|x\|_u = \inf\{\lambda > 0 : -\lambda u \leq x \leq \lambda u\}.$$

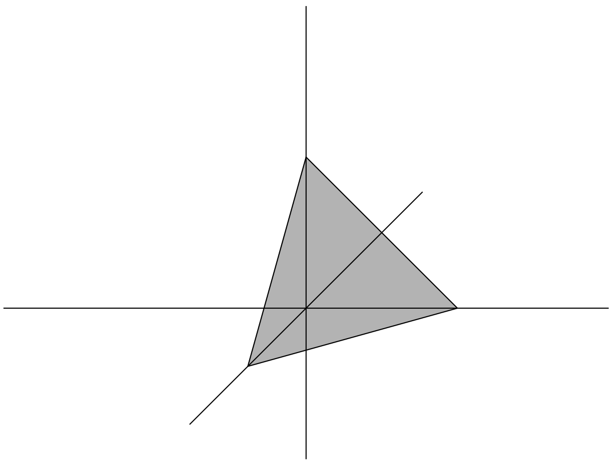
The triple (X, X_+, u), where X is an Archimedean partially ordered vector space with cone X_+ and order-unit u is called an *order-unit space*. One can show that in an order-unit space every order-unit lies in the interior of the cone with respect to the order-unit norm. Also every element in the interior of the cone is an order-unit.

Let $(X, \leq), (Y, \leq)$ be a partially ordered vector spaces. We call a map $f : X \to Y$ *order-preserving* if for all $x, y \in X$ with $x \leq y$ we have that $f(x) \leq f(y)$. If f is linear and order-preserving, we call f *positive*. It is easy to verify that f is positive if and only if it maps the cone of X into the cone of Y. We call f *strictly positive* if $f(x) > 0$ for all $x \in X_+^\circ$. Furthermore, if X is a normed partially ordered vector space we denote the set of all continuous linear functionals by X^* and the set of all positive continuous linear functionals by X_+^*.

2.1.1 Example. Consider $\mathbb{R}^n$ with the standard positive cone

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}.$$

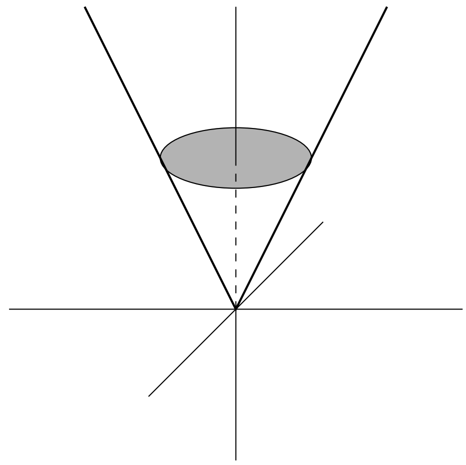
This cone defines an order on the coordinates; for all $x, y \in \mathbb{R}^n$ we have $x \leq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$.



positive cone

2.1.2 Example. Consider $\mathbb{R}^{n+1}$ with the *Lorentz cone*

$$\Lambda_n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sqrt{x_2^2 + \ldots + x_{n+1}^2} < x_1\}.$$



Lorentz cone

One can find a general version of the Lorentz cone for any normed vector space X by considering the vector space $V = \mathbb{R} \times X$ with cone

$$V_+ = \{(\lambda, x) \in V : \|x\| \leq \lambda\}.$$

We will call such a cone a *base cone*. If H is a Hilbert space, then $V = \mathbb{R} \times H$ admits a natural Jordan algebra structure, whose cone of squares is V_+, see Chapter 4. These spaces are known as spin-factors and will be discussed in more detail in Section 7.2.

2.1.3 Example. Let $X = M_{n\times n}(\mathbb{R})_{sa}$ be the space of real, self-adjoint, $n \times n$ matrices. Recall that a matrix A is positive semidefinite if for all $x \in \mathbb{R}^n$ we have $x^T Ax \geq 0$. The set

$$X_+ = \{A \in M_{n\times n}(\mathbb{R})_{sa} : A \text{ positive semidefinite}\}$$

is a cone of $M_{n\times n}(\mathbb{R})_{sa}$. Recall that a matrix A is positive semidefinite if and only if its eigenvalues are non-negative. Note that the identity matrix I is an order-unit. The associated order-unit norm is given by its *spectral radius* $\|A\|_I = \rho(A) = \max\{|r| : r \in \sigma(A)\}$.

2.1.4 Definition. Let X be a vector space with cone X_+. Let $x, y \in X_+$, we call x *comparable* to y if there exists an $\lambda \in \mathbb{R}_{>0}$ such that $\frac{1}{\lambda}x \leq y \leq \lambda x$. We call the set

$$P_x := \{y \in X_+ : x \text{ comparable to } y\}.$$

a *part* of X_+.

Note that comparability is an equivalence relation and the parts of X_+ are equivalence classes.

2.1.5 Proposition. *If X is a vector space with cone X_+, then the parts of X_+ are convex.*

Proof. Let $x \in X_+$ and let P_x be a part of X_+. Let $y, z \in P_x$ and let $\lambda_1, \lambda_2 > 0$ such that $\frac{1}{\lambda_1}x \leq y \leq \lambda_1 x$ and $\frac{1}{\lambda_2}x \leq z \leq \lambda_2 x$. For all $c \in (0,1)$ we find that

$$\frac{\lambda_1(1-c)+\lambda_2 c}{\lambda_1\lambda_2}x \leq cy + (1-c)z \leq (\lambda_1 c + \lambda_2(1-c))x.$$

Picking

$$\lambda = \max\left(\lambda_1 c + \lambda_2(1-c), \frac{\lambda_1\lambda_2}{\lambda_1(1-c)+\lambda_2 c}\right)$$

gives that $cy + (1-c)z \in P_x$. □

2.1.6 Definition. Let X be a topological vector space and let X_+ be a closed cone. Then $F \subset X_+$ is called a *face* of X_+ if F is non-empty, convex, and if for all $x, y \in X_+$ for which there exists a $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y \in F$, we have $x, y \in F$.

Faces and parts are closely related.

2.1.7 Proposition. *Let X be an order-unit space with closed cone X_+. Every face is a union of parts.*

Proof. Suppose there is a face F which is not the union of parts. As parts are equivalence classes on the closed cone this means that there exists a part P such that there are $x, y \in P$ for which $x \in F$ and $y \notin F$. Now let $0 < \lambda < 1$ be such that $\lambda y \leq x$. Then $\frac{1}{1-\lambda}x - \frac{\lambda}{1-\lambda}y \in P$, since

$$x = \frac{1-\lambda}{1-\lambda}x \leq \frac{1}{1-\lambda}x - \frac{\lambda}{1-\lambda}y \leq \frac{1}{1-\lambda}x.$$

However we find that

$$x = \lambda y + (1-\lambda)\left(\frac{1}{1-\lambda}x - \frac{\lambda}{1-\lambda}y\right),$$

so by definition $y \in F$ which is a contradiction. □

In fact, the parts of X_+ are precisely the relative interiors of the faces of X_+, see [42, Lemma 1.2.2]. Since X_+ is a face of X_+ it follows that X_+ is the disjoint union of parts. Finally we will show that every convex set in the boundary of the cone is contained in a face.

2.1.8 Proposition. *Let X be an order-unit space with closed cone X_+. If $C \subset \partial X_+$ is a convex set, then C is contained in a face of X_+.*

Proof. We will first construct a face containing C and then show it is contained in the boundary of the cone. Consider

$$F_1 = \{x \in X_+ : \exists y \in X_+ \text{ such that there is a } 0 < \lambda < 1 \text{ such that } \lambda x + (1-\lambda)y \in C\}$$

and for all $n > 1$ we define inductively

$$F_n = \{x \in X_+ : \exists y \in X_+ \text{ such that there is a } 0 < \lambda < 1 \text{ such that } \lambda x + (1-\lambda)y \in F_{n-1}\}.$$

Note that $F = \bigcup_{n>0} F_n$ is a face, as for all x, y and $\lambda \in (0,1)$ there exists and n such that $\lambda x + (1-\lambda)y \in F_n$ and thus $x, y \in F_{n+1} \subset F$. Furthermore we claim that for all n the set F_n is contained in the boundary. To see this, let $x \in X_+^\circ$ and a $y \in X_+$. For all $0 < \lambda < 1$ we have that $\lambda x + (1-\lambda)y \in X_+^\circ$ by convexity, hence, as $C \subset \partial X_+$, we have that $F_1 \subset \partial X_+$. By induction it follows this holds for all n. □

2.2 Hemi-metric spaces

Let M be a set we call a map $d : M \times M \to \mathbb{R}$ a *hemi-metric* if for all $x, y, z \in M$ it satifies the properties

- (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ and
- $d(x,y) = d(y,x) = 0$ if and only if $x = y$.

We call (M,d) a *hemi-metric space*. Note that a hemi-metric need not be positive. If additionally d satisfies the properties

- (Non-negativity) $d(x,y) \geq 0$ and
- (Symmetry) $d(x,y) = d(y,x)$,

then we call d a *metric* and (M,d) a *metric space*. Note that every metric is a hemi-metric.

2.2.1 Proposition. *Let (M,d) be a hemi-metric space. Then $\delta : M \times M \to \mathbb{R}$ given by*

$$\delta(x,y) = \max(d(x,y), d(y,x)) \qquad (x, y \in M)$$

is a metric.

Proof. One can easily verify that δ is symmetric and satisfies the triangle inequality. Since for all $x, y \in M$ we have that $0 = d(x,x) \leq d(x,y) + d(y,x)$ we have that δ is non-negative. If $\delta(x,y) = 0$, then $d(x,y)$ and $d(y,x)$ are both non-positive hence, by the inequality above, $d(x,y) = d(y,x) = 0$, so $x = y$. $\square$

We call a hemi-metric space (M,d) *complete* if its associated metric space (M,δ) is complete.

2.2.2 Lemma. *Let (M,d) be a hemi-metric space. For all $y \in M$ the functions $x \mapsto d(x,y)$ and $x \mapsto d(y,x)$ are Lipschitz continuous with constant 1 with respect to δ.*

Proof. Let $x, x', y \in M$. Using the triangle inequality we find $d(x,y) - d(x',y) \leq d(x,x')$ and $d(x',y) - d(x,y) \leq d(x',x)$, so

$$|d(x,y) - d(x',y)| \leq \delta(x,x').$$

The proof for $x \mapsto d(y,x)$ is similar. $\square$

2.2.3 Definition. Let (M,d) be a hemi-metric space, let $x \in M$ and let $r \in \mathbb{R}$. We call $B_r(x) = \{y \in M : d(x,y) < r\}$ the *open (hemi-metric) ball* of radius r around x.

Note that if $y \in B_r(x)$, it does not necessarily imply that $x \in B_r(y)$. Let (M,d) be a hemi-metric space. If $x, y \in M$ and $r, s \in \mathbb{R}$, $z \in B_r(x) \cap B_s(y)$ and $t = \min(r - d(x,z), s - d(y,z))$, then $B_t(z) \subset B_r(x) \cap B_s(y)$, since for all $u \in B_t(z)$ we have

$$\begin{aligned} 0 &> d(z,u) - \min(r - d(x,z), s - d(y,z)) \\ &= \max(d(x,z) + d(z,u) - r, d(y,z) + d(z,u) - s) \\ &\geq \max(d(x,u) - r, d(y,u) - s). \end{aligned}$$

So the open hemi-metric balls form a basis for a topology.

Let (M_1, d_1) and (M_2, d_2) two hemi-metric spaces. We call a map $f : M_1 \to M_2$ an isometry if $d_1(x,y) = d_2(f(x), f(y))$ for all $x, y \in M_1$. We call f *non-expansive* if $d_1(x,y) \geq d_2(f(x), f(y))$ for all $x, y \in M_1$. We call f *strictly non-expansive* if $d_1(x,y) > d_2(f(x), f(y))$ for all $x, y \in M_1$, $x \neq y$. We call f a *contraction* if there exists an $r \in (0,1)$ such that $rd_1(x,y) > d_2(f(x), f(y))$ for all $x, y \in M_1$, $x \neq y$.

It is well known that every contraction map in a complete metric space has a unique fixed point.

2.2.4 Theorem (Contraction mapping theorem, [3]). *Let (M, d) be a complete metric space, then every contraction has a unique fixed point.*

This theorem does not hold for strictly non-expansive maps. Consider for example the map $f : [1, \infty) \to [1, \infty)$ given by $f(x) = x + \frac{1}{x}$ which is strictly non-expansive with respect to the Euclidean distance, but has no fixed point. One can easily show that if a strictly non-expansive map has a fixed point, it is unique. For non-expansive maps one cannot guarantee existence or uniqueness of fixed point. This can be easily verified as every translation on a vector space is norm non-expansive.

2.3 The Funk, reverse-Funk and Hilbert metric

Natural hemi-metrics appear in the study of cones in order-unit space. One can define Birkhoff's version of Hilbert's metric using the *gauge function*,

$$M(x/y) = \inf\{\lambda > 0 : x \leq \lambda y\} \quad (x \in X_+, y \in X_+^\circ).$$

Note that, as y is an order-unit, $M(x/y)$ is finite. We can now define the *Funk metric* as

$$d_F(x, y) = \log M(x/y) \qquad (x, y \in X_+^\circ)$$

the *reverse-Funk metric* as

$$d_R(x, y) = \log M(y/x) \qquad (x, y \in X_+^\circ)$$

and *Hilbert's metric* as

$$d_H(x, y) = \log(M(x/y)M(y/x)) \qquad (x, y \in X_+^\circ).$$

Note that Hilbert's metric is the sum of the Funk metric and the reverse-Funk metric. One can prove that Hilbert's metric is a pseudo metric on X_+°, i.e. it is non-negative, symmetric, it satisfies $d_H(x, x) = 0$ for all x and the triangle inequality. It is well known [42] that Hilbert's metric is a metric on the rays of the interior cone. This is a direct consequence of the fact that d_H is invariant under scaling; for all $x, y \in X_+^\circ$ and $\lambda, \mu > 0$ we have $M(\lambda x/\mu y) = \frac{\lambda}{\mu} M(x/y)$, so $M(\lambda x/y)M(y/\lambda x) = M(x/y)M(y/x)$.

One can also prove that the Funk and the reverse-Funk metrics are hemi metrics on X_+°. By Proposition 2.2.1 it follows that *Thompson's metric* d_T given by

$$d_T(x, y) = \log(\max(M(x/y), M(y/x))) = \max(d_F(x, y), d_R(x, y)) \qquad (x, y \in X_+^\circ).$$

is a metric on the interior of the cone. Note that by Lemma 2.2.2 we have that d_H is Lipschitz continuous with constant 2 with respect to d_T.

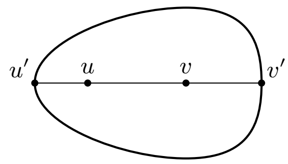
2.3.1 Example. Consider $\mathbb{R}^{n+1}$ with the standard positive cone

$$\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0\}.$$

For $x, y \in (\mathbb{R}^{n+1}_+)^\circ$ we find

$$\begin{aligned} d_F(x,y) &= \log\left(\max_{1\leq i\leq n+1} \frac{x_i}{y_i}\right), \\ d_R(x,y) &= \log\left(\max_{1\leq i\leq n+1} \frac{y_i}{x_i}\right), \\ d_H(x,y) &= \log\left(\max_{1\leq i,j\leq n+1} \frac{x_i}{x_j}\frac{y_j}{y_i}\right), \\ d_T(x,y) &= \log\left(\max_{1\leq i\leq n+1} \max\left(\frac{x_i}{y_i}, \frac{y_i}{x_i}\right)\right). \end{aligned}$$

Hilbert's metric can also be defined using a cross-ratio product. Let X be a real normed space and let $C \subset X$ be an open bounded convex set. Hilbert's cross-ratio metric is defined as follows. Let u and v be different elements of C and let $l_{u,v}$ be the line through u and v. Let u' and v' be the intersection of $l_{x,y}$ and the boundary of C such that u is between u' and v and v is between u and v'.



Hilbert's cross-ratio metric is given by

$$d_h(u,v) = \log\left(\frac{\|u'-v\|\|v'-u\|}{\|u'-u\|\|v'-v\|}\right) \qquad (u, v \in C).$$

Note that $\overline{C}$ can always be viewed as a slice of a cone in $Y = \mathbb{R} \times X$, by taking

$$Y_+ = \{\lambda(1,x) \in Y : \lambda \geq 0, x \in \overline{C}\}.$$

Then $(1, \overline{C})$ is the slice of the cone at height 1. It is well-known that Hilbert's cross-ratio metric and Hilbert's metric coincide on the interior of the cone, see [42].

2.3.2 Example. Consider $\mathbb{R}^{n+1}$ with the Lorentz cone

$$\Lambda_n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sqrt{x_2^2 + \ldots + x_{n+1}^2} \leq x_1\}.$$

The disc

$$\mathbf{D} = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sqrt{x_2^2 + \ldots + x_{n+1}^2} < x_1 = 1\}$$

equipped with the metric $\frac{1}{2}d_H$ is known as Klein's model, which is a model of the real n-dimensional hyperbolic space.

If in a Hilbert metric space the elements of two converging sequences stay close to each other, the limits will be in the same part of the cone.

2.3.3 Lemma. *Let X be an order-unit space with closed cone X_+ equipped with Hilbert's metric d_H and let $x, y \in \partial X_+$. If (x_n) and (y_n) are sequences in X_+° converging in norm to x and y respectively and there exists an $M > 0$ such that $d_H(x_n, y_n) < M$ for all $n \in \mathbb{N}$, then x and y are comparable.*

Proof. Let $u \in X_+$ be a unit. By continuity of the norm, and since d_H is invariant under scaling, we may assume that $\|x_n\|_u = \|y_n\|_u = 1$. Note that, as x_n and y_n are positive, we have that

$$\|x_n\|_u = \inf\{\lambda > 0 : x_n \leq \lambda u\} = 1 \text{ and } \|y_n\|_u = \inf\{\lambda > 0 : y_n \leq \lambda u\} = 1.$$

Let $\alpha_n = M(x_n/y_n)$ and $\beta_n = M(y_n/x_n)$. Note that $\alpha_n, \beta_n \geq 1$ as $x_n \leq \alpha_n y_n \leq \alpha_n u$ and $y_n \leq \beta_n x_n \leq \beta_n u$. Therefore, since $\log(\alpha_n \beta_n) < M$, we find that α_n and β_n are bounded. So by taking a further subsequence we may assume that (α_n) and (β_n) converge to some $\alpha, \beta \in [1, e^M]$ respectively. We find that

$$x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \alpha_n y_n = \alpha y$$

and

$$y = \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} \beta_n x_n = \beta x.$$

□

2.4 Horofunctions

Let (M, d) be a hemi-metric space. Fix $b \in M$ as a base point and consider the embedding $i : M \to C(M)$ given by

$$i(x)(y) = d(y, x) - d(b, x) \quad (x, y \in M) \tag{2.4.1}$$

where $C(M)$ is equipped with the topology of compact convergence; see [53, §46]. Recall that by Lemma 2.2.2 i is continuous with respect to the metric δ given by

$$\delta(x,y) = \max(d(x,y), d(y,x)) \qquad (x, y \in M).$$

Now consider the family $i(M)$. Note that by Lemma 2.2.2 we have

$$|i(x)(y) - i(x)(y')| = |d(y,x) - d(y',x)| \leq \delta(y,y'),$$

hence $i(M)$ is equicontinuous. Furthermore, note that for all $x, y \in M$ we have that $|i(x)(y)| \leq \delta(y,b)$, so $\{i(x)(y) : x \in M\}$ has compact closure in $\mathbb{R}$. By Ascoli's Theorem; see [53, Theorem 47.1] we find that $i(M)$ has compact closure in $C(M)$. The closure $\overline{i(M)}$ is called the *horofunction compactification* of (M,d). The set $\overline{i(M)} \setminus i(M)$ is called the *horofunction boundary* of (M,d) and its elements are called *horofunctions*.

Horofunctions are a fundamental tool in metric geometry and have found applications in numerous fields including, geometric group theory [21], ergodic theory [26, 38], nonlinear operator theory [42, 25] and metric and non-commutative geometry [59]. In this thesis we will focus on their properties and their applications in proving Denjoy-Wolff type theorems.

In many applications of horofunctions one uses horoballs. Let $\xi \in \overline{i(M)} \setminus i(M)$ be a horofunction, then

$$H_\xi(r) = \{x \in M : \xi(x) \leq r\} \qquad (r \in \mathbb{R})$$

is a *horoball* of ξ. Horoballs have a number of useful properties. First we recall the definition of a net and a subnet.

2.4.1 Definition. We call a set $(J, \leq)$ a directed set if $\leq$ is a partial order and every two elements have an upper bound, i.e. for every $x, y \in J$ there exists a $z \in J$ such that $z \geq x$ and $z \geq y$. A net in X is a map $f : J \to X$ from a directed set J to X. We will denote a net by $(x_\alpha)_{\alpha \in J}$ or (x_α) if there is no ambiguity.

2.4.2 Definition. Let X be a set and let $(x_\alpha)_{\alpha \in A}$ and $(x_\beta)_{\beta \in B}$ be nets in X. We call $(x_\beta)_{\beta \in B}$ a subnet of $(x_\alpha)_{\alpha \in A}$ if there exists a map $f : B \to A$ such that $x_\beta = x_{f(\beta)}$ for all $\beta \in B$, f is order-preserving, i.e., for all $\beta, \gamma \in B$ with $\beta \leq \gamma$ we have $f(\beta) \leq f(\gamma)$, and f is final, i.e., for all $\alpha \in A$ there exist a $\beta \in B$ such that $\alpha \leq f(\beta)$.

2.4.3 Proposition. *Let X be a vector space and let $M \subset X$ be equipped with a hemi-metric. If all hemi-metric balls in M are convex, then all horoballs are convex.*

Proof. Let b be a base point and let $\xi \in \overline{i(M)} \setminus i(M)$ be a horofunction. Let (x_α) be a net in X such that $\lim_\alpha i(x_\alpha) = \xi$. Let $r \in \mathbb{R}$ and let $x, y \in H_\xi(r)$. As the metric balls are convex we find for all $0 \leq \lambda \leq 1$

$$\begin{aligned}\xi(\lambda x + (1-\lambda)y) &= \lim_\alpha d(\lambda x + (1-\lambda)y, x_\alpha) - d(b, x_\alpha) \\ &\leq \lim_\alpha \lambda d(x, x_\alpha) - \lambda d(b, x_\alpha) + (1-\lambda)d(y, x_\alpha) - (1-\lambda)d(b, x_\alpha) \\ &= \lambda\xi(x) + (1-\lambda)\xi(y).\end{aligned}$$

□

We can show that the Funk, reverse-Funk, Hilbert and Thompson metrics all have convex hemi-metric balls.

2.4.4 Lemma. *If $(X, \|\cdot\|_u)$ is an order-unit space with closed cone X_+ equipped with the reverse-Funk metric, then every closed ball $B_r[x] = \{y \in X_+^\circ : d_R(x, y) \leq r\}$ is a convex subset of X.*

Proof. Let $y_1, y_2 \in B_r[x]$ and let $0 < \alpha_1, \alpha_2 \leq e^r$ such that $y_1 \leq \alpha_1 x$ and $y_2 \leq \alpha_2 x$. For $t \in [0, 1]$ we have

$$ty_1 + (1-t)y_2 \leq (t\alpha_1 + (1-t)\alpha_2)x \leq e^r x$$

hence $ty_1 + (1-t)y_2 \in B_r[x]$. □

The proof for the Funk metric is similar.

2.4.5 Lemma. *If $(X, \|\cdot\|_u)$ is an order-unit space with closed cone X_+ equipped with the Funk metric, then every closed ball $B_r[x] = \{y \in X_+^\circ : d_F(x, y) \leq r\}$ is a convex subset of X.*

Proof. Let $y_1, y_2 \in B_r[x]$ and let $0 < \alpha_1, \alpha_2 \leq e^r$ such that $x \leq \alpha_1 y_1$ and $x \leq \alpha_2 y_2$. For $t \in [0, 1]$ we have

$$e^r(ty_1 + (1-t)y_2) \geq t\alpha_1 y_1 + (1-t)\alpha_2 y_2 \geq x$$

hence $ty_1 + (1-t)y_2 \in B_r[x]$. □

The proofs of Hilbert's metric and Thompson's metric use similar methods and can be found in [42, Lemma 2.6.1, Lemma 2.6.2].

We call a metric space *proper* if all closed balls are compact. For proper metric spaces the horofunction boundary can be viewed as a boundary at infinity.

2.4.6 Proposition. *Let (M,d) be proper a metric space. If $x \in M$ and (x_α) is a net in M converging to x, then $\lim_\alpha i(x_\alpha) = i(x)$.*

Proof. Fix $b \in M$. For all $y \in M$ we find

$$\begin{aligned}\lim_\alpha i(x_\alpha)(y) &= \lim_\alpha d(y,x_\alpha) - d(b,x_\alpha) \\ &\leq \lim_\alpha d(y,x) + 2d(x,x_\alpha) - d(b,x) = i(x) \\ &= d(y,x) - d(b,x) = \lim_\alpha d(y,x) - 2d(x_\alpha,x) - d(b,x) \\ &\leq \lim_\alpha d(y,x_\alpha) - d(b,x_\alpha) = \lim_\alpha i(x_\alpha)(y).\end{aligned}$$

□

It is an easy consequence to see that if M is proper all horofunction are generated by unbounded nets.

2.4.7 Corollary. *Let (M,d) be a proper metric space. If (x_α) is a net such that $\xi = \lim_\alpha i(x_\alpha) \in \overline{i(M)} \setminus i(M)$ is a horofunction, then (x_α) is unbounded.*

In general there can exist horofunctions generated by bounded nets. We will discuss these in more detail in Chapter 6.

Recall that a topological space is metrizable if and only if there exists a metric such that the topologies coincide.

2.4.8 Proposition. *If (M,d) is a proper metric space, then $\overline{i(M)}$ equipped with the topology of compact convergence is metrizable.*

Proof. To see this note that on $\overline{i(M)}$ the topology of compact convergence is equivalent to the topology of pointwise convergence, see Step 3 on page 291 of [53]. Also note that for $x_0 \in M$ we can write $M = \bigcup_{n=1}^\infty B_n[x_0]$, where $B_n[x_0]$ is the closed ball around x_0 of radius n. As M is proper $B_n[x_0]$ is a compact metric space and hence separable. So M is separable as the countable union of separable sets, hence we can find a countable set $\{x_n \in M : n \in \mathbb{N}\}$ which lies dense in M. It is easy to verify that $\hat{d} : C(M) \times C(M) \to \mathbb{R}$ given by

$$\hat{d}(f,g) = \sum_{j=1}^\infty \frac{\min(|f(x_j) - g(x_j)|, 1)}{2^j} \qquad (f,g \in C(M))$$

is a metric.

We will show that the metric topology and the topology of pointwise convergence coincide on $\overline{i(M)}$. Recall that a basis of the topology of pointwise convergence on $\overline{i(M)}$

consists of sets of the form $U = \{f \in \overline{i(M)} : |f(w) - g(w)| < \varepsilon\}$ with $\varepsilon > 0$, $w \in M$ and $g \in \overline{i(M)}$. We want to show there is an open metric ball in $\overline{i(M)}$ contained in U. Let $n \in \mathbb{N}$ such that $d(x_n, w) < \frac{\varepsilon}{3}$ and consider the metric ball $B = \{f \in \overline{i(M)} : \hat{d}(f,g) < \frac{\varepsilon}{3 \cdot 2^n}\}$. Let $f \in B$ and let $(i(x_\alpha))$ and $(i(z_\alpha))$ be nets in $i(M)$ converging to f and g respectively. We find

$$\begin{aligned}|f(w) - g(w)| &\leq \lim_\alpha |i(x_\alpha)(w) - i(x_\alpha)(x_n)| + |f(x_n) - g(x_n)| + |i(z_\alpha)(x_n) - i(z_\alpha)(w)| \\ &\leq \lim_\alpha |d(x_\alpha, w) - d(x_\alpha, x_n)| + |f(x_n) - g(x_n)| + |d(z_\alpha, w) - d(z_\alpha, x_n)| \\ &\leq 2d(x_n, w) + 2^n \hat{d}(f,g) < \varepsilon\end{aligned}$$

so $B \subset U$.

Now let $\varepsilon > 0$, $g \in \overline{i(M)}$ and let $B = \{f \in i(M) : \hat{d}(f,g) < \varepsilon\}$. We want to show there is an open set in the topology of pointwise convergence contained in B. Let $n \in \mathbb{N}$ be such that $\sum_{j=n+1}^{\infty} 2^{-j} < \frac{\varepsilon}{2}$ and let

$$U = \{f \in i(M) : |f(x_j) - g(x_j)| < \frac{\varepsilon}{2} \text{ for all } 1 \leq j \leq n\}.$$

Then for all $f \in U$ we have

$$\hat{d}(f,g) = \sum_{j=1}^{n} \frac{\min(|f(x_j) - g(x_j)|, 1)}{2^j} + \sum_{j=n+1}^{\infty} \frac{\min(|f(x_j) - g(x_j)|, 1)}{2^j} < \varepsilon$$

so $f \in B$. □

One of the consequences of Proposition 2.4.8 is that for (M,d) a proper metric space $\overline{i(M)}$ is first countable, so we can use sequences instead of nets.

Chapter 3

Denjoy-Wolff type theorem

One famous application of horofunctions are Denjoy-Wolff type theorems. The Denjoy-Wolff theorem is a theorem in the field of complex analysis proven by Denjoy [19] and Wolff [69] in 1926. The theorem states that any holomorphic self-map of the open unit disk without a fixed point has a unique accumulation point on the boundary of the unit disc.

3.0.1 Theorem. *If D is the open unit disk of $\mathbb{C}$ and $f : D \to D$ is a fixed point free holomorphic map, then there is a unique point $z_0 \in \partial D$ such that for all $z \in D$ it holds that*

$$\lim_{n\to\infty} f^k(z) = z_0.$$

Beardon noted that this result can be viewed in a geometric context and gave a proof of the Denjoy-Wolff theorem using only geometrical methods, see [5]. In [6] Beardon expanded this result to a general geometric setting for metric spaces with boundary that is similar to the boundary of a hyperbolic space. Let us recall some basic terminology.

3.0.2 Definition. Let (X, τ) be a Hausdorff topological space. Let $f : X \to X$ be a map. For $x \in X$ we call

$$O(x, f) = \{f^n(x) : n \in \mathbb{N}\}$$

the *orbit* of x, and

$$\omega(x, f) = \{y \in A : \exists (n_k) \text{ sequence in } \mathbb{N} \text{ such that } \lim_{k\to\infty} f^{n_k}(x) = y \text{ with respect to } \tau\}$$

the *ω-limit set* of x.

Karlsson and Nussbaum independently conjectured the following generalization of the Denjoy-Wolff theorem;

3.0.3 Conjecture. *Let $f : \Sigma \to \Sigma$ be a fixed point free non-expansive map on a finite dimensional Hilbert metric space (Σ, d_H). Then there exists a convex set $\Omega \subset \partial\Sigma$ such that for each $x \in \Sigma$ the ω-limit set $\omega(x, f)$ lies in Ω.*

The conjecture is still an open problem, though it has been solved for a number of special cases like strictly convex sets by Beardon [6] and polyhedral domains by Lins [46]. There are also cases with special maps such as maps with a strictly positive translation number which was solved by Karlsson [35] and Nussbaum [55]. In this chapter we will study Denjoy-Wolff type theorems by Beardon and Karlsson. Their results are a generalisation of the following observation due to Wolff [70].

3.0.4 Theorem. *If D is the open unit disk of $\mathbb{C}$ and $f : D \to D$ is a fixed point free holomorphic map, then there is a horofunction $\xi \in \overline{i(D)}\backslash i(D)$ with respect to the hyperbolic metric on D such that f leaves the horoballs of ξ invariant.*

The results from Beardon and Karlsson provide good examples of applications of horofunctions and give rise to so called Beardon and Karlsson points which we will study in more detail. At the end of this chapter we will prove two special cases of the Karlsson-Nussbaum Conjecture 3.0.3.

3.1 Beardon's Theorem

We will make two minor alterations to Beardon's original proof. First, Beardon's proof uses horoballs which are defined using a different definition. Though the two definitions of horoballs can be shown to be equivalent for the specific conditions of Beardon's theorem, we will instead follow the proof in [42], which uses our definition of a horoball. Second, Beardon's original Theorem only considers the case where the map is strictly non-expansive. One can show, using Całka's theorem [12, Theorem 5.6], a more general case for maps with a certain fixed point property.

3.1.1 Theorem (Całka's theorem,)**.** *Let (M, d) be a proper metric space. If $f : M \to M$ is non-expansive and there exists a $y \in M$ such that the orbit $O(y)$ has a bounded subsequence, then $O(x)$ is bounded for all $x \in M$.*

3.1.2 Definition. Let (M, d) be a metric space and let $f : M \to M$ be a map. We say f has the *fixed point property* if f has a fixed point in M whenever there exists a $x \in M$ such that $O(x, f)$ is bounded.

We can show that strictly non-expansive maps have the fixed point property.

3.1.3 Example. Let (M,d) be a metric space, let $f : M \to M$ be a strictly non-expansive map and suppose there exists an $x \in M$ such that $O(x,f)$ is bounded. Then $\omega(x,f)$ is bounded and, since M is first countable we find that $\omega(x,f) = \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x) : k \geq n\}}$, hence $\omega(x,f)$ is closed and bounded and therefore compact. It follows that, as $f(\omega(x,f)) = \omega(x,f)$, we can find $y, z \in \omega(x,f)$ such that

$$d(f(y), f(z)) = \operatorname{diam}(\omega(x,f)) = \sup\{d(u,v) : u, v \in \omega(x,f)\}.$$

Since f is strictly non-expansive we have

$$\operatorname{diam}(\omega(x,f)) = d(f(y), f(z)) \leq d(y,z) \leq \operatorname{diam}(\omega(x,f))$$

with equality if and only if $y = z$. So $\omega(x,f) = \{y\}$ and, since $f(\omega(x,f)) = \omega(x,f)$ we find $f(y) = y$.

Let X be a finite dimensional vector space and let $C \subset X$ be a convex bounded set equipped with Hilbert's metric. By Corollary 3.6' in [55] any non-expansive map $f : X \to X$ has the fixed point property.

Beardon's proof [6] consists of two parts. First Beardon proves a generalisation of Wolff's theorem 3.0.4.

3.1.4 Theorem. *Let (M,d) be a proper metric space such that $M \subset A$ is a precompact open subset of a first-countable Hausdorff space (A,τ) and the topology of M coincides with the topology τ of A. If*

(i) for all sequences (x_n) and (y_n) in M, converging to distinct points $x, y \in \partial M$ respectively, we have

$$\lim_{n\to\infty} d(x_n, z) = \infty \text{ and } \lim_{n\to\infty} d(x_n, y_n) - \max(d(x_n, z), d(y_n, z)) = \infty \qquad (z \in M)$$

(ii) and $f : M \to M$ is a fixed point free non-expansive map such that there exists a sequence of contractions (f_n) converging pointwise to f,

then there exists a horofunction ξ such that f leaves the horoballs of ξ invariant.

Proof. By Theorem 2.2.4 we can find (x_n), the sequence of the unique fixed points of (f_n) in M. As M is precompact and A is first countable we may assume, by taking a subsequence if necessary, that (x_n) converges to some $x \in \overline{M}$. Note that $x \in \partial M$, as otherwise it is a fixed point of f. Indeed, as f is the pointwise limit of contractions f_n, if $x \in M$ we have

$$\begin{aligned} d(x, f(x)) &\leq \lim_{n\to\infty} d(x, x_n) + d(f_n(x_n), f_n(x)) + d(f_n(x), f(x)) \\ &\leq \lim_{n\to\infty} d(x, x_n) + d(x, x_n) + d(f_n(x), f(x)) = 0, \end{aligned}$$

which implies that x is a fixed point of f. Hence we may also assume, by taking a further subsequence if necessary, that $(i(x_n))$ converges to some horofunction ξ. Let $y \in M$. We find

$$\begin{aligned} \xi(f(y)) &= \lim_{n\to\infty} d(f(y), x_n) - d(b, x_n) \\ &\leq \liminf_{n\to\infty} d(f(y), f_n(y)) + d(f_n(y), f_n(x_n)) - d(b, x_n) \\ &\leq \liminf_{n\to\infty} d(f(y), f_n(y)) + d(y, x_n) - d(b, x_n) \\ &= \lim_{n\to\infty} d(y, x_n) - d(b, x_n) = \xi(y). \end{aligned}$$

Hence f leaves the horoballs of ξ invariant. □

Theorem 3.1.4 can be used to obtain a Denjoy-Wolff type theorem.

3.1.5 Theorem (Beardon's Theorem [6]). *Let (M, d) be a proper metric space such that $M \subset A$ is a precompact open subset of a first-countable Hausdorff space (A, τ) and the topology of M coincides with the topology τ of A. If*

(i) for all sequences (x_n) and (y_n) in M, converging to distinct points $x, y \in \partial M$ respectively, we have

$$\lim_{n\to\infty} d(x_n, z) = \infty \text{ and } \lim_{n\to\infty} d(x_n, y_n) - \max(d(x_n, z), d(y_n, z)) = \infty \qquad (z \in M)$$

(ii) and $f\colon M \to M$ is a fixed point free non-expansive map with the fixed point property such that there exists a sequence of contractions (f_n) converging pointwise to f,

then there exists a point $x \in \partial M$ such that for all y it holds that

$$\lim_{n\to\infty} f^n(y) = x.$$

Proof. Let ξ, (x_n) and x be as in the proof of Theorem 3.1.4. Suppose the orbit $(f^n(y))$ has a bounded subsequence. By Theorem 3.1.1 we have that the entire orbit is bounded. Thus

f has a fixed point, as f has the fixed point property. Since f is fixed point free it follows that all subsequences of the orbit of y are unbounded. So $\omega(y, f) \subset \partial M$. Furthermore, taking $r = \xi(y)$ we get that $y \in H_\xi(r)$, where $H_\xi(r)$ is the horoball of ξ with radius r. By Theorem 3.1.4 f leaves the horoballs invariant, so the accumulation points of $(f^n(y))$ all lie in $\partial M \cap \overline{H_\xi(r)}$. Let $z \in \partial M \cap \overline{H_\xi(r)}$ and let (z_n) be a sequence in $H_\xi(r)$ converging to z with respect to τ. Fix $\varepsilon > 0$ and note that since (z_n) is in $H_\xi(r)$, for every $n \in \mathbb{N}$ we can find a k_n such that $k_n > k_{n-1}$ and for all $k \geq k_n$ it holds that

$$d(z_n, x_k) - d(b, x_k) < r + \varepsilon.$$

From this it follows that

$$d(z_n, x_{k_n}) - d(b, x_{k_n}) < r + \varepsilon$$

for all $n \in \mathbb{N}$, and thus

$$\lim_{n\to\infty} d(z_n, x_{k_n}) - \max\left(d(b, x_{k_n}), d(b, z_n)\right) \leq r + \varepsilon.$$

As

$$\lim_{n\to\infty} d(z_n, x_{k_n}) - \max\left(d(b, x_{k_n}), d(b, z_n)\right) = \infty$$

if $x \neq z$, we find that $\partial M \cap \overline{H_\xi(r)} \subset \{x\}$. By compactness of $\overline{M}$ every orbit has at least one accumulation point, hence for all $y \in M$ we have

$$\lim_{n\to\infty} f^n(y) = x.$$

□

Note that if X is a finite dimensional vector space with a strictly convex closed cone X_+ equipped with Hilbert's metric, then by Lemma 2.3.3 if we identify X with A and a slice of X_+ with M, then they satisfy the conditions of Theorem 3.1.5.

The proof of Theorem 3.1.5 depends mainly on the properties of the horofunction ξ and the geometry of the cone. The point $x \in \partial M$ however is an interesting point in its own right.

3.1.6 Definition. Let X be a finite dimensional vector space, let $C \subset X$ be a convex bounded open set equipped with Hilbert's metric and let $f : C \to C$ be a fixed point free non-expansive map. We call $x \in \partial C$ a *Beardon point* if there exist a sequence of contractions $f_n : C \to C$ converging pointwise to f with unique fixed points (x_n) such that a subsequence (x_{n_k}) converges to x.

Due to a result by Walsh [63] we find that the horofunction ξ in the proof of Theorem 3.1.5 has a uniquely associated Beardon point.

3.1.7 Theorem. *[63, Theorem 1.3] Let X be a finite dimensional vector space and $C \subset X$ be a bounded open convex subset equipped with Hilbert's metric. If a sequence (x_n) is such that $(i(x_n))$ converges to a horofunction, then the sequence converges to some point in ∂C, i.e. every horofunction has a unique associated point in the boundary.*

In general the converse of this result is not true, i.e. not every point in the boundary has a unique associated horofunction.

3.1.8 Example. Consider the 2-simplex

$$\Delta_n = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}.$$

Recall that for all $x, y \in \Delta_n^\circ$ Hilbert's distance between x and y is given by

$$d_H(x, y) = \log\left(\max_{1 \le i,j \le 3} \frac{x_i}{x_j} \frac{y_j}{y_i}\right)$$

Consider the sequences $(x_n) = ((\frac{n+1}{n+2}, \frac{1}{2n+4}, \frac{1}{2n+4}))$ and $(y_n) = ((\frac{n+1}{n+2}, \frac{2}{3n+6}, \frac{1}{3n+6}))$. We may assume that for the sequences $(i(x_n))$ and $(i(y_n))$ there exist subsequence that converges to horofunctions ξ and γ respectively. Let $b = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and let $z = (\frac{1}{2}, \frac{3}{10}, \frac{1}{5})$ and consider

$$\begin{aligned}\lim_{n\to\infty} i(x_n)(z) &= \lim_{n\to\infty} \log\left(\frac{2n+2}{n+2} \cdot \frac{3n+6}{5}\right) - \log\left(\frac{3n+3}{n+2} \cdot \frac{2n+4}{3}\right) \\ &= \log\left(\frac{3}{5}\right) = \xi(z)\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} i(y_n)(z) &= \lim_{n\to\infty} \log\left(\frac{2n+2}{n+2} \cdot \frac{3n+6}{5}\right) - \log\left(\frac{3n+3}{n+2} \cdot (n+2)\right) \\ &= \log\left(\frac{2}{5}\right) = \gamma(z)\end{aligned}$$

so $\xi \neq \gamma$. Note however that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = (1, 0, 0)$.

Beardon points have some useful properties.

3.1.9 Definition. Let $A \subset X$ be a subset of a vector space X and let $x \in A$. We call x a *star point* of A if for all $y \in A$ the (straight) line segment $\overline{xy}$ between x and y is in A.

We will show that every Beardon point is a star point of $\bigcup_{x \in X} \omega(x, f)$. For this we need the following result.

3.1.10 Lemma. *Let X be a finite dimensional vector space and let C be a bounded open convex set equipped with Hilbert's metric d_H. Let (x_n) and (y_n) be sequences in C converging to $x, y \in \partial C$. If the line segment $\overline{xy}$ between x and y is not contained in ∂C, then for all $z \in C$ we have*

$$\lim_{n\to\infty} d_H(x_n, y_n) - \max(d_H(x_n, z), d_H(y_n, z)) = \infty.$$

Proof. For all $n \in \mathbb{N}$ and $0 \leq \lambda \leq 1$, let $u_{n,\lambda} = \lambda x_n + (1-\lambda)y_n$. Suppose

$$u = \lim_{n\to\infty} u_{n,\lambda} = \lambda x + (1-\lambda)y \notin \partial C.$$

for some $0 < \lambda < 1$. Let $z \in C$, consider

$$\begin{aligned} d_H(x_n, y_n) - \max(d_H(x_n, z), d_H(y_n, z)) &= d_H(x_n, u_{n,\lambda}) + d_H(y_n, u_{n,\lambda}) \\ &\quad - \max(d_H(x_n, z), d_H(y_n, z)) \\ &\geq d_H(x_n, z) + d_H(y_n, z) - 2d_H(u_{n,\lambda}, z) \\ &\quad - \max(d_H(x_n, z), d_H(y_n, z)) \\ &= \min(d_H(x_n, z), d_H(y_n, z)) - 2d_H(u_{n,\lambda}, z). \end{aligned}$$

Since $\lim_{n\to\infty} \min(d_H(x_n, z), d_H(y_n, z)) = \infty$ and $\lim_{n\to\infty} d_H(u_{n,\lambda}, z) = d_H(u, z) < \infty$, the result follows. $\square$

Recall that the *convex hull* of a subset A of a vector space X is the smallest convex set C containing A.

3.1.11 Theorem. *Let X be a finite dimensional vector space and let $C \subset X$ be a bounded open convex set equipped with Hilbert's metric d_H. Let $f : C \to C$ be a non-expansive fixed point free map and let $\xi \in \overline{i(C)} \setminus i(C)$ be a horofunction such that f leaves the horoballs of ξ invariant. Let $x \in \partial C$ be such that there is a sequence (x_n) in C such that (x_n) converges to x in norm topology and $\lim_{n\to\infty} i(x_n) = \xi$. If Ω is the convex hull of $\bigcup_{y\in C} \omega(y; f)$, then x is a star point of $\partial C \cap \Omega$.*

Proof. Let (x_n) be a sequence in C such that $i(x_n)$ converges to ξ. Note that all we have to prove is that for all $y_0 \in C$ and all $y \in \omega(y_0, f)$ the line segment between x and y is in ∂C. Let $r \in \mathbb{R}$ be such that $y_0 \in H_\xi(r)$. Since f leaves the horoballs of ξ invariant, we find that $\xi(f^n(y)) \leq r$. Now note that, as $(i(x_n))$ converges pointwise to ξ, we can find a subsequence (x_{n_k}) such that

$$|\xi(f^k(y)) - d_H(f^k(y), x_{n_k}) + d_H(b, x_{n_k})| < \frac{1}{k},$$

thus

$$\lim_{k\to\infty} |\xi(f^k(y)) - d_H(f^k(y), x_{n_k}) + d_H(b, x_{n_k})| = 0.$$

Now suppose that the line segment between x and y is not contained in ∂C. Then by Lemma 3.1.10 we have

$$\begin{aligned}\lim_{k\to\infty} \xi(f^k(y)) &= \lim_{k\to\infty} d_H(f^k(y), x_{n_k}) - d_H(b, x_{n_k}) \\ &\geq \lim_{k\to\infty} d_H(f^k(y), x_{n_k}) - \max(d_H(b, x_{n_k}), d_H(b, f^k(y))) = \infty\end{aligned}$$

This however is in contradiction with $\xi(f^n(y)) \leq r$. □

This result is very similar to the Karlsson-Noskov theorem [40] which we will discuss in the next section. Note that the theorem not only concerns Beardon points, but any point in the boundary that is associated with some horofunction which horoballs are left invariant by the map f. The following Denjoy-Wolff type theorem is an easy consequence of Theorem 3.1.11.

3.1.12 Corollary. *Let X be a finite dimensional normed vector space and let $C \subset X$ be a bounded open convex set equipped with Hilbert's metric d_H. Let $f : C \to C$ be a non-expansive fixed point free map and let $\xi \in \overline{i(C)} \setminus i(C)$ be a horofunction such that f leaves the horoballs of ξ invariant. Let $x \in \partial C$ be such that there is a sequence (x_n) in C such that (x_n) converges to x in norm topology and $\lim_{n\to\infty} i(x_n) = \xi$. If the set*

$$\{y \in \partial C : \overline{xy} \subset \partial C\}$$

is convex, then there exists a convex set $\Omega \subset \partial C$ such that for each $y \in C$ we have $\omega(y, f) \subset \Omega$.

3.2 Karlsson's theorem

Karlsson's theorem is another variation on Wolff's theorem 3.0.4. Instead of showing all horoballs of a horofunction are invariant it shows a single orbit will remain in a horoball of radius 0. The advantages of Karlsson's theorem over Beardon's theorem 3.1.5 are that there is no need to find contractions approximating f. Moreover, for specific f it can be shown that this particular orbit will be contained in horoballs with radius tending to $-\infty$. For these kind of maps one can show a Denjoy-Wolff type theorem.

3.2.1 Definition. Let (X,d) be a metric space, let $f : X \to X$ be a non-expansive map and let $x \in X$. Then we call

$$\tau_f = \lim_{n\to\infty} \frac{1}{n} d(x, f^n(x))$$

the *translation number* of f.

It is easy to show, using the non-expansiveness of f, that the translation number exists and is independent of x. Indeed note that for all $x, y \in X$ and all $n \in \mathbb{N}$ we have that

$$\begin{aligned} d(x,f^n(x)) - d(y,f^n(y)) &\le d(x,y) + d(y,f^n(x)) - d(y,f^n(y)) \\ &\le d(x,y) + d(f^n(x),f^n(y)) \le 2d(x,y) \end{aligned}$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} |d(x,f^n(x)) - d(y,f^n(y))| \le \lim_{n\to\infty} \frac{2d(x,y)}{n} = 0.$$

The existence of the translation numbers is a direct consequence of Fekete's subadditive lemma.

3.2.2 Lemma (Fekete's subadditive lemma)**.** *Let (a_n) be a subadditive sequence of real numbers, i.e. $a_{n+m} \le a_n + a_m$, then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n>0} \frac{a_n}{n} < \infty.$$

Proof. Let $\varepsilon > 0$. We can find an $N \in \mathbb{N}$ such that $\frac{a_N}{N} < \inf_{n>0} \frac{a_n}{n} + \varepsilon$. For all $n > N$ we can find a $k_n \in \mathbb{N}$ and an $0 \le r_n < N$ such that $n = k_n N + r_n$. Now for $n > \frac{N}{\varepsilon}$ we find

$$\inf_{k>0} \frac{a_k}{k} \le \frac{a_n}{n} = \frac{a_{k_n N + r_n}}{n} \le \frac{k_n a_N + a_{r_n}}{n} \le \frac{a_N}{N} + \frac{r_n a_1}{n} \le \frac{a_N}{N} + \frac{N|a_1|}{n} \le \inf_{k>0} \frac{a_k}{k} + (1+|a_1|)\varepsilon.$$

Letting ε tend to 0 gives the required result. □

Karlsson uses the translation number to bound the values of a special horofunction of a single orbit.

3.2.3 Theorem ([35])**.** *Let (M,d) be a proper metric space and let $f : M \to M$ be a non-expansive map with translation number τ_f. Then for all $x \in X$ there exists a function $\xi \in \overline{i(X)}$ such that for all m*

$$\xi(f^m(x)) \le -\tau_f m.$$

Proof. Let $x \in X$ and consider $O(x,f)$. Note that for all $\varepsilon > 0$ the sequence $d(x, f^n(x)) - n(\tau_f - \varepsilon)$ is unbounded. Let (ε_k) be a strictly positive sequence converging to 0. By

Lemma 3.2.2 we can find $(f^{n_k}(x))$, a subsequence of $(f^n(x))$ such that $(i(f^{n_k}(x)))$ converges to some horofunction ξ and for all $m < n_k$ we have that $d(x, f^{n_k}(x)) - n_k(\tau_f - \varepsilon_k) > d(x, f^m(x)) - m(\tau_f - \varepsilon_k)$.
Then we find that for all $m \in \mathbb{N}$ we have

$$\begin{aligned}
\xi(f^m(x)) &= \lim_{k\to\infty} d(f^m(x), f^{n_k}(x)) - d(x, f^{n_k}(x)) \\
&\leq \liminf_{k\to\infty} d(x, f^{n_k - m}(x)) - d(x, f^{n_k}(x)) \\
&= \liminf_{k\to\infty} d(x, f^{n_k - m}(x)) - (n_k - m)(\tau_f - \varepsilon_k) \\
&\qquad - d(x, f^{n_k}(x)) + n_k(\tau_f - \varepsilon_k) - m(\tau_f - \varepsilon_k) \\
&\leq \liminf_{k\to\infty} -m(\tau_f - \varepsilon_k) = -m\tau_f.
\end{aligned}$$

□

3.2.4 Remark. Note that the function ξ in Karlsson's theorem is not necessarily a horofunction. If f is fixed point free, then the orbits of f will be unbounded and ξ will be a horofunction, as (M, d) is assumed to be proper.

A consequence of Karlson's theorem is, that for any finite dimensional Hilbert metric space, if the translation number is strictly positive, then Conjecture 3.0.3 holds as proven by Karlsson, Metz and Noskov [39, Theorem 5.2] and Nussbaum [55, Theorem 4.25]. Just like with Beardon's Theorem, Karlsson's Theorem gives rise to a special set of points.

3.2.5 Definition. Let X be a finite dimensional normed vector space and let $C \subset X$ be an open bounded convex set equipped with Hilbert's metric. Let $f : C \to C$ be a fixed-point free non-expansive map and for $x \in C$ let $(f^{n_k}(x))$ be a subsequence of $(f^n(x))$ such that for all $m < n_k$ we have that $d(x, f^{n_k}(x)) > d(x, f^m(x))$ and $\lim_{k\to\infty} f^{n_k}(x) = z \in \partial C$. We call z a *Karlsson point.*

Note that if f is fixed point free, then the horofunctions found in Theorem 3.2.3 have associated Karlsson points. It is not known if the converse is true, i.e., if we can find a horofunction with the properties described in Theorem 3.2.3 for every Karlsson point. Karlsson and Noskov have shown that every Karlsson point is a star point, see [40].

3.2.6 Theorem. *Let X be a finite dimensional vector space and let $C \subset X$ be an open bounded convex set equipped with the Hilbert metric. If $f : C \to C$ is a fixed-point free non-expansive map and Ω is the convex hull of $\bigcup_{y\in C} \omega(y; f)$, then every Karlsson point is a star point of $\partial C \cap \Omega$.*

Proof. Let $x \in C$ and let $a_n = d_H(f^n(x), x)$ for all $n \geq 1$. Let $(f^{n_k}(x))$ be a subsequence such that $(i(f^{n_k}(x)))$ converges to some horofunction ξ and for all $m < n_k$ we have $a_m < a_{n_k}$. Now let $u \in \omega(x, f)$. Then there exists a subsequence (m_k) and a $y \in X$ such that $\lim_{n\to\infty} f^{m_n}(y) = u$, by taking a further subsequence of n_k we may assume that $m_k < n_k$. Now consider the Gromov product

$$\begin{aligned}
-\infty &\leq \limsup_{k\to\infty} d_H(f^{n_k}(x), f^{m_k}(y)) - \max(d_H(f^{n_k}(x), x), d_H(f^{m_k}(y), x) \\
&\leq \limsup_{k\to\infty} d_H(f^{n_k}(x), f^{m_k}(y)) - d_H(f^{n_k}(x), x) \\
&\leq \limsup_{k\to\infty} d_H(f^{n_k}(x), f^{m_k}(x)) + d_H(f^{m_k}(x), f^{m_k}(y)) - a_{n_k} \\
&\leq \limsup_{k\to\infty} a_{n_k - m_k} - a_{n_k} + d_H(x, y) \leq d_H(x, y) < \infty.
\end{aligned}$$

The result follows by Lemma 3.1.10. □

3.3 Uniqueness of Karlsson and Beardon points

In Chapter 9 of [57] Karlsson shows that the Karlsson-Nussbaum conjecture 3.0.3 holds for bounded convex sets in $\mathbb{R}^2$ equipped with Hilbert's metric. In his proof he makes use of the properties of both the Karlsson and Beardon points. In the same chapter Karlsson claims that if the Beardon point and the Karlsson point coincide then the Karlsson-Nussbaum conjecture would hold. In a bounded strictly convex subset of $\mathbb{R}^n$ equipped with Hilbert's metric it is easy to see that, as Beardon and Karlsson points are star points, the Beardon and Karlsson points are unique and coincide. This leads us to ask the question if in general the Karlsson and the Beardon point are unique. In this section we will show that in general this is not the case. The example we will construct comes from a special class of Hilbert's metric non-expansive maps. To introduce them, let us recall some basic terminology.

3.3.1 Definition. Let X, Y be real vector spaces. We call a map $f : X \to Y$ *homogeneous (of degree one)* if $f(\lambda x) = \lambda f(x)$.

3.3.2 Proposition. *Let X be a topological vector space with closed cone X_+. If $f : X_+^\circ \to X_+^\circ$ is homogeneous and order-preserving, then f is non-expansive with respect to Hilbert's metric.*

Proof. Let $x, y \in X_+^\circ$. We have to show that $M(f(x)/f(y))M(f(y)/f(x)) \leq M(x/y)M(y/x)$. Note that

$$f(x) \leq f(M(x/y)y) = M(x/y)f(y),$$

so $M(f(x)/f(y)) \leq M(x/y)$. Similarly we find that $M(f(y)/f(x)) \leq M(y/x)$. □

3.3.3 Corollary. *Let X be a vector space with closed cone X_+. If a map $f : X_+^\circ \to X_+^\circ$ is a positive linear operator, then f is non-expansive with respect to Hilbert's metric.*

3.3.4 Example. In this example we will show that in general a Karlsson point is not unique.

Consider the 2-simplex

$$\Delta_2 = \{x \in \mathbb{R}_+^3 : x_1 + x_2 + x_3 = 1\}$$

equipped with Hilbert's metric d_H. Consider the positive operator

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Note that, as it is linear and leaves $\mathbb{R}_+^3$ invariant, it is order-preserving and homogeneous. Hence $B : \Delta_2^\circ \to \Delta_2^\circ$, defined by $Bx = Ax/\|Ax\|_1$ is non-expansive with respect to Hilbert's metric. Now let $x = (1,1,1)^T$ and let $\hat{x} = x/\|x\|_1$. We find the following

$$A^k x = \begin{cases} \left(2^{\frac{1}{2}k+\frac{1}{2}}, 2^{\frac{1}{2}k-\frac{1}{2}}, 2^{-k}\right)^T & \text{if } k \text{ is odd.} \\ \left(2^{\frac{1}{2}k}, 2^{\frac{1}{2}k}, 2^{-k}\right)^T & \text{if } k \text{ is even.} \end{cases}$$

Note that since $B^k\hat{x} = A^k x/\|A^k x\|_1$ we find that

$$d_H(B^k\hat{x}, \hat{x}) = d_H(A^k x, x) = \begin{cases} \log\left(2^k \cdot 2^{\frac{1}{2}k+\frac{1}{2}}\right) = \left(\frac{3}{2}k + \frac{1}{2}\right)\log(2) & \text{if } k \text{ is odd.} \\ \log\left(2^k \cdot 2^{\frac{1}{2}k}\right) = \frac{3}{2}k\log(2) & \text{if } k \text{ is even.} \end{cases}$$

We can show that every accumulation point of $O(\hat{x}, B)$ is a Karlsson point. Indeed, for all even $k \in \mathbb{N}$ we find that

$$\begin{aligned} d_H(B^{k-1}\hat{x}, \hat{x}) &= \left(\frac{3}{2}k - 1\right)\log(2) < \frac{3}{2}k\log(2) = d_H(B^k\hat{x}, \hat{x}) \\ &< \left(\frac{3}{2}k + 2\right)\log(2) = d_H(B^{k+1}\hat{x}, \hat{x}), \end{aligned}$$

so for all $k \in \mathbb{N}$ and all $m < k$ we have that $d_H(B^k\hat{x}, \hat{x}) > d_H(B^m\hat{x}, \hat{x})$. Note that

$$\lim_{k\to\infty} \frac{A^{2k}x}{\|A^{2k}x\|_1} = \lim_{k\to\infty} \frac{(2^k, 2^k, 2^{-2k})^T}{\|(2^k, 2^k, 2^{-2k})^T\|_1} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)^T$$

and

$$\lim_{k\to\infty} \frac{A^{2k+1}x}{\|A^{2k+1}x\|_1} = \lim_{k\to\infty} \frac{(2^{k+1}, 2^k, 2^{-2k-1})^T}{\|(2^{k+1}, 2^k, 2^{-2k-1})^T\|_1} = \left(\frac{2}{3}, \frac{1}{3}, 0\right)^T.$$

As both points are accumulation points we have shown there can be more than one Karlsson point.

In the following example we will show the same for Beardon points.

3.3.5 Example. Consider the 2-simplex

$$\Delta_2 = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$$

equipped with Hilbert's metric d_H. Consider the positive operator

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Note that, as it is linear and leaves $\mathbb{R}^3_+$ invariant, it is order-preserving and homogeneous. Hence $B : \Delta_2^\circ \to \Delta_2^\circ$, defined by $Bx = Ax/\|Ax\|_1$ is non-expansive with respect to Hilbert's metric. Consider the perturbations

$$A_n x = Ax + \frac{\|x\|_1}{n}(1,1,1)^T.$$

A_n is order-preserving and homogeneous, hence $B_n : \Delta_2^\circ \to \Delta_2^\circ$, defined by $B_n x = A_n x/\|A_n x\|_1$ is non-expansive with respect to d_H, in fact B_n is a contraction, see the proof of Theorem 4.3 in [41]. One can show that for

$$x_n = \begin{pmatrix} \frac{4}{6-n+\sqrt{n^2+4n+36}} \\ \frac{4}{6-n+\sqrt{n^2+4n+36}} \\ \frac{4}{2n+12+\sqrt{n^2+4n+36}} \end{pmatrix} \text{ and } \lambda_n = \frac{3n+6+\sqrt{n^2+4n+36}}{4n}$$

we have $A_n x_n = \lambda_n x_n$, hence x_n is a fixed point of B_n. So

$$\lim_{n\to\infty} \begin{pmatrix} \frac{4}{6-n+\sqrt{n^2+4n+36}} \\ \frac{4}{6-n+\sqrt{n^2+4n+36}} \\ \frac{4}{2n+12+\sqrt{n^2+4n+36}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

is a Beardon point. Now consider another sequence of perturbations

$$\hat{A}_n x = Ax + \frac{\|x\|_1}{n}(2,1,1)^T.$$

$\hat{A}_n$ is order-preserving and homogeneous, hence $\hat{B}_n : \Delta_2^\circ \to \Delta_2^\circ$, defined by $\hat{B}_n x = \hat{A}_n x/\|\hat{A}_n x\|_1$ is non-expansive with respect to d_H, in fact $\hat{B}_n$ is a contraction, see the proof of Theorem 4.3 in [41]. One can show that for

$$\hat{x}_n = \begin{pmatrix} \frac{8}{8-n+\sqrt{n^2+8n+64}} \\ \frac{4}{8-n+\sqrt{n^2+8n+64}} \\ \frac{4}{2n+8+\sqrt{n^2+8n+64}} \end{pmatrix} \text{ and } \hat{\lambda}_n = \frac{3n+8+\sqrt{n^2+8n+64}}{4n}$$

we have $\hat{A}_n\hat{x}_n = \hat{\lambda}_n\hat{x}_n$, hence $\hat{x}_n$ is a fixed point of $\hat{B}_n$. So

$$\lim_{n\to\infty} \begin{pmatrix} \frac{8}{8-n+\sqrt{n^2+8n+64}} \\ \frac{4}{8-n+\sqrt{n^2+8n+64}} \\ \frac{4}{2n+8+\sqrt{n^2+8n+64}} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

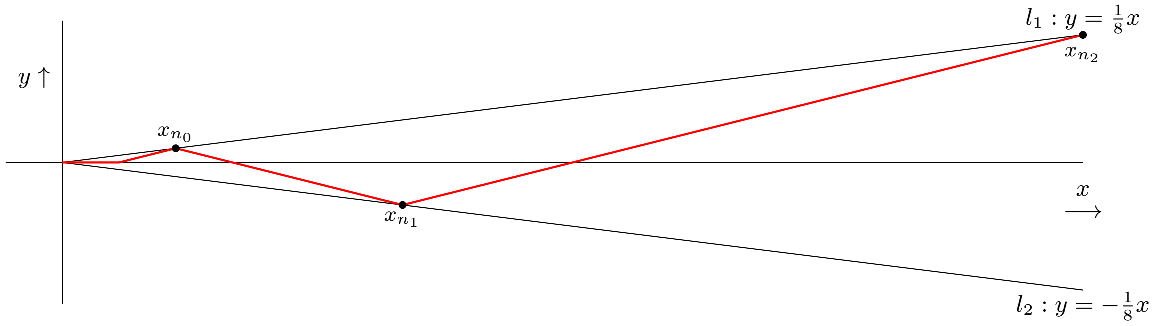
is also a Beardon point.

We can also introduce the concept of Karlsson and Beardon points in different kinds of spaces. Consider $\mathbb{R}^n$ equipped with the Euclidean distance $\|x-y\|_2$ for all $x, y \in \mathbb{R}^n$. One can show that the horofunction boundary can be identified with the Euclidean unit sphere. Take $b = 0$ as base point and let (x_n) be a sequence in $\mathbb{R}^n$ such that $(i(x_n))$ converges to some horofunction $\xi \in \overline{i(\mathbb{R}^n)} \setminus i(\mathbb{R}^n)$. Note that, as $\mathbb{R}^n$ is proper, by Proposition 2.4.7 $\|x_n\|_2$ tends to infinity. By taking a subsequence we may assume that $\left(\frac{x_n}{\|x_n\|_2}\right)$ converges to some $x \in S^{n-1}$ where S^{n-1} denotes the unit sphere. Note that this subsequence gives rise to the same horofunction. Indeed,

$$\begin{aligned} \xi(y) &= \lim_{n\to\infty} (\|y-x_n\|_2 - \|x_n\|_2) \\ &= \lim_{n\to\infty} \left(\sqrt{\|y\|_2^2 + \|x_n\|_2^2 - 2\langle x_n, y\rangle} - \|x_n\|_2\right) \\ &= \lim_{n\to\infty} \frac{\|y\|_2^2 + \|x_n\|_2^2 - 2\langle x_n, y\rangle - \|x_n\|_2^2}{\sqrt{\|y\|_2^2 + \|x_n\|_2^2 - 2\langle x_n, y\rangle} + \|x_n\|_2} \\ &= \lim_{n\to\infty} \frac{\frac{\|y\|_2^2}{\|x_n\|_2} - 2\langle \frac{x_n}{\|x_n\|_2}, y\rangle}{\sqrt{\frac{\|y\|_2^2+1}{\|x_n\|_2^2} - 2\langle \frac{x_n}{\|x_n\|_2}, \frac{y}{\|x_n\|_2}\rangle} + 1} = -\langle x, y\rangle. \end{aligned}$$

So we can identify the horofunction boundary with the Euclidean unit sphere, which, like in Beardon's theorem, is strictly convex. Also note that, like Hilbert's metric, the Euclidean metric balls are strictly convex. One might expect that in this situation Beardon and Karlsson points will be unique again, this however is not the case.

3.3.6 Example. Consider $\mathbb{R}^2$ with Euclidean norm. We use Theorem 11.3 in [68] which states that if we find some non-expansive sequence (x_n), then there exists a non-expansive map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x_n) = x_{n+1}$. Using a strategy similar to Lins in [46] we will show that the Karlsson point is not unique, indeed, we will show there is a continuum of Karlsson points. We do this by going back and forth between the lines $l_1 : y = -\frac{1}{8}x$ and $l_2 : y = \frac{1}{8}x$ following lines parallel to $l_3 : y = \frac{1}{4}x$ and $l_4 : y = -\frac{1}{4}x$. To ensure the sequence created is non-expansive we need to make sure that the angle between l_3 and l_4 is not too large, and every time we switch between a line parallel to l_3 and a line parallel to l_4 or the other way around, we need to decrease the step size sufficiently. In this example we will halve the step sizes. This will give us the construction in the figure below.



Here n_i are the indices of the sequence elements for which x_{n_i} lies on l_1 for i even, and x_{n_i} lies on l_2 for i odd.

We can calculate the n_i, for this first consider the point $a = (n, \frac{n}{8}) \in l_1$ and let $l_5 : y = -\frac{1}{4}x + \frac{3n}{8}$ be the line parallel to l_4 passing through a. l_5 intersects l_2 in $b = (3n, -\frac{3n}{8})$, let $l_6 : y = \frac{1}{4}x - \frac{9n}{8}$ be the line parallel to l_3 passing through b. l_6 intersects l_1 in $c = (9n, \frac{9n}{8})$. One can easily see that $c_1 - b_1 = 3(b_1 - a_1)$, so if it takes m steps to go between a and b, then, as we use half the step length between b and c, it takes $6m$ steps between b and c.

Using this we define $x_0 = (0, 0)$, $x_1 = (1, 0)$ and $x_2 = (2, \frac{1}{4})$, which gives us $n_0 = 2$, it then takes 8 steps to get from $(2, \frac{1}{4})$ to $(6, -\frac{3}{4})$ giving us $n_1 = 10$ and subsequently it takes 3 times the distance to travel between the two lines. Combined with the fact that the step length gets halved we find $n_i = 2 + \sum_{j=0}^{i-1} 8 \cdot 6^j$. We can then define for $n > 2$

$$x_n = \begin{cases} x_{n-1} + 2^{-i}(1, \frac{1}{4}) & \text{if } n_{i-1} < n \leq n_i \text{ and } i \text{ is even.} \\ x_{n-1} + 2^{-i}(1, -\frac{1}{4}) & \text{if } n_{i-1} < n \leq n_i \text{ and } i \text{ is odd.} \end{cases}$$

We first need to show that the sequence (x_n) is non-expansive. Let $m, n \in \mathbb{N}$. It is easy to see that

$$\|x_{n+1} - x_{m+1}\|_2 \leq \|x_n - x_m\|_2$$

if $n = m + 1$, so we may assume $m + 1 < n$. Let $x_n = (y_1, y_2)$ and let $x_m = (z_1, z_2)$ and let $u_1 = y_1 - z_1 > 0$ and let $u_2 = y_2 - z_2$. Let $i, j \in \mathbb{N}$ such that $n_i < m + 1 \leq n_{i+1}$ and $n_j < n + 1 \leq n_{j+1}$. If $i = j$ then clearly $x_n - x_m = x_{n+1} - x_{m+1}$ so we may assume that $j > i$, from which it follows that $2^{-i} \geq 2^{-j+1}$. Note that $u_2 \leq \frac{1}{4}u_1$ and also note that since $m + 1 < n$ we have that $u_1 \geq (2^{-i} + 2^{-i-1}) = \frac{3}{2} \cdot 2^{-i}$. We now find

$$\begin{aligned}
\|x_{n+1} - x_{m+1}\|_2^2 &\leq (u_1 - (2^{-i} - 2^{-j}))^2 + (u_2 + \frac{1}{4}(2^{-i} + 2^{-j}))^2 \\
&= u_1^2 - 2(2^{-i} - 2^{-j})u_1 + 2^{-2i} - 2^{-(i+j)+1} + 2^{-2j} \\
&\quad + u_2^2 + \frac{1}{2}(2^{-i} + 2^{-j})u_2 + 2^{-2i-4} + 2^{-(i+j)-3} + 2^{-2j-4} \\
&= \|x_m - x_n\|_2^2 + \frac{17}{16} \cdot 2^{-2i} + \frac{17}{16} \cdot 2^{-2j} \\
&\quad + (\frac{1}{2}u_2 - 2u_1)2^{-i} + (\frac{1}{2}u_2 + 2u_1)2^{-j} - 15 \cdot 2^{-(i+j)-3} \\
&\leq \|x_m - x_n\|_2^2 + \frac{17}{16} \cdot 2^{-2i} + \frac{17}{16} \cdot 2^{-2j} - \frac{15}{8}u_1 2^{-i} + \frac{17}{8}u_1 2^{-j} - 15 \cdot 2^{-(i+j)-3} \\
&\leq \|x_m - x_n\|_2^2 + \frac{17}{16} \cdot 2^{-2i} + \frac{17}{16} \cdot 2^{-2j} - \frac{30}{16}u_1 2^{-i} + \frac{17}{16}u_1 2^{-i} - 15 \cdot 2^{-(i+j)-3} \\
&\leq \|x_m - x_n\|_2^2 + \frac{17}{16} \cdot 2^{-2i} + \frac{17}{16} \cdot 2^{-2j} - \frac{13}{16}u_1 2^{-i} - 15 \cdot 2^{-(i+j)-3} \\
&\leq \|x_m - x_n\|_2^2 + \frac{17}{16} \cdot 2^{-2i} + \frac{17}{16} \cdot 2^{-2j} - \frac{39}{32}2^{-2i} - 15 \cdot 2^{-2j-2} \\
&= \|x_m - x_n\|_2^2 - \frac{5}{32}2^{-2i} - \frac{43}{16} \cdot 2^{-2j} < \|x_m - x_n\|_2^2
\end{aligned}$$

This proves that (x_n) is non-expansive. Hence by Theorem 11.3 in [68] there exists a non-expansive map f such that $f(x_n) = x_{n+1}$. Furthermore, let $0 < m < n$ and let $x_n = (y_1, y_2)$ and $x_m = (z_1, z_2)$. Note that $|y_2 - z_2| \leq \frac{1}{4}(y_1 - z_1)$, $|z_1| < y_1$, $|y_2| \leq \frac{1}{4}y_1$ and $z_2 \leq \frac{1}{4}z_1$. Using this we find

$$\begin{aligned}
\|x_m\|_2^2 = z_1^2 + z_2^2 = z_1^2 + (z_2 - y_2)(z_2 + y_2) + y_2^2 &= \\
\leq z_1^2 + \frac{1}{16}(y_1 - z_1)(y_1 + z_1) + y_2^2& \\
\leq \frac{15}{16}z_1^2 + \frac{1}{16}y_1^2 + y_2^2 \leq y_1^2 + y_2^2 = \|x_n\|_2^2&
\end{aligned}$$

Hence for any subsequence of (x_{k_n}) satisfying that $(i(x_{k_n}))$ converges to some horofunction ξ, it follows that ξ is a Karlsson point. Recall that the horofunctions can be represented by considering the projections on the unit sphere. Since for i large enough

$$X_i = \left\{ \frac{x_n}{\|x_n\|_2} : n_{i-1} \leq n \leq n_i \right\}$$

gives an arbitrarily fine partition of the arc between $(\frac{8}{\sqrt{65}}, \frac{1}{\sqrt{65}})$ and $(\frac{8}{\sqrt{65}}, -\frac{1}{\sqrt{65}})$ we see that the Karlsson points of f form a continuum on the unit circle.

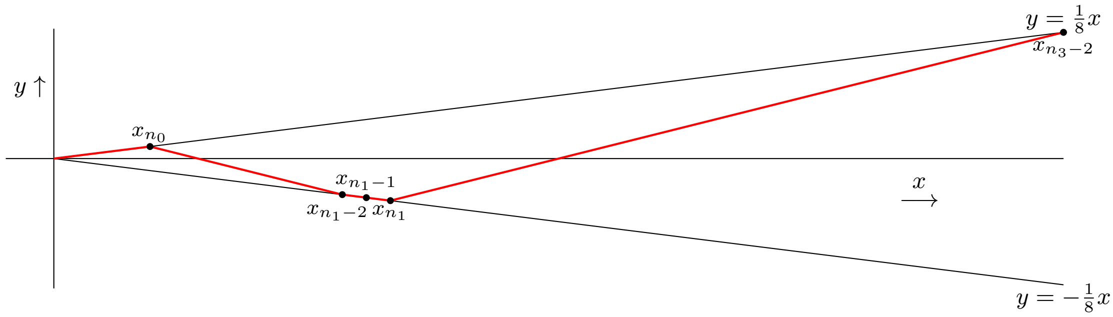
With a slight modification to the above example we can also ensure that there are multiple Beardon points. This is done by going back and forth between l_1 and l_2 as above, but every time upon reaching either l_1 or l_2 we take two steps along l_1 or l_2 respectively.

Using similar calculations as above we can again calculate the n_i. Let $a = (n, \frac{n}{8})$ and let l_5 be a line parallel to l_4 passing through a. By the above l_5 intersects l_2 in $b' = (3n, -\frac{3n}{8})$. Let s be the step length between a and b'. Taking two more steps along l_2 gives us $b = (3n+2s, -\frac{3n+2s}{8})$. With a similar procedure we find $c = (9n+7s, \frac{9n+7s}{8})$. One can easily see that $c_1 - b_1 = 3(b_1 - a_1) - s$, so if it takes m steps between a and b, then, as we use half the step length between b and c, it takes $6m - 2$ steps between b and c.

Using this we define $x_0 = (0,0), x_1 = (1, \frac{1}{8})$, and $x_2 = (2, \frac{1}{4})$, and $n_0 = 2$, $n_1 = 12$ and for $i \geq 2$ we find $n_i = 2 + \sum_{j=0}^{i-1} 10 \cdot 6^j - 2j$. We can then define for $n > 2$

$$x_n = \begin{cases} x_{n-1} + 2^i(1, \frac{1}{4}) & \text{if } n_{i-1} < n \leq n_i - 2 \text{ and } i \text{ is even.} \\ x_{n-1} + 2^i(1, \frac{1}{8}) & \text{if } n_i - 2 < n \leq n_i \text{ and } i \text{ is even.} \\ x_{n-1} + 2^i(1, -\frac{1}{4}) & \text{if } n_{i-1} < n \leq n_i - 2 \text{ and } i \text{ is odd.} \\ x_{n-1} + 2^i(1, -\frac{1}{8}) & \text{if } n_i - 2 < n \leq n_i \text{ and } i \text{ is odd.} \end{cases}$$

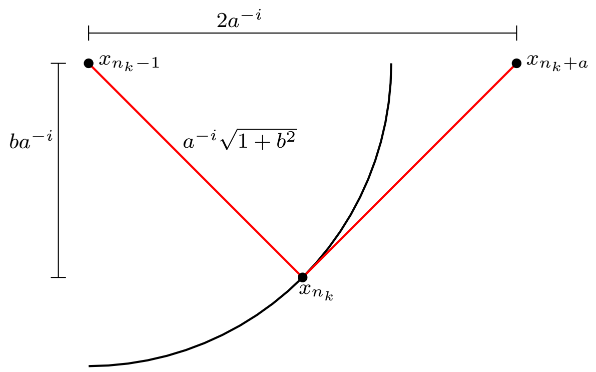
This gives us a construction like the figure below



We can use the same calculation as above to show that (x_n) is a non-expansive sequence and therefore by Theorem 11.3 in [68] there exists a non-expansive map f such that $f(x_n) = x_{n+1}$. Now for $0 < r < 1$ define $f_r = rf$ and note that f_r converges to f pointwise if $r \to 1$. One can easily see that for all i the point x_{n_i-2} is a fixed point of f_{r_i} for some $r_i \in (0,1)$. Furthermore as the step size decreases and the distance from the origin increases we find that $r_0 < r_1 < \ldots < 1$ and $\lim_{i\to\infty} r_i = 1$. It follows that $\xi_+(\cdot) = -\frac{8}{\sqrt{65}}\langle(1, \frac{1}{8}), \cdot\rangle$ and $\xi_-(\cdot) = -\frac{8}{\sqrt{65}}\langle(1, -\frac{1}{8}), \cdot\rangle$ are Beardon points with respect to f.

Finally we will show that with some adaptations we can use this method to get a continuum of Karlsson points with an angle which is arbitrarily close to 90 degrees. For this let $a \in \mathbb{N}$ and $0 < b < 1$. For all $0 < \delta < b$ we can go back and forth between the lines $l_1 : y = (b-\delta)x$ and $l_2 : y = -(b-\delta)x$ following lines parallel to $l_3 : y = bx$ and $l_4 : y = -bx$. Every time we switch between a line parallel with l_3 and a line parallel with l_4, or the other way around, we multiply the step size by a^{-1}. We can define n_i and x_i similar to our first example.

Let $m, n \in \mathbb{N}$ and let $x_n = (y_1, y_2)$, $x_m = (z_1, z_2)$, $u_1 = y_1 - z_1 > 0$ and let $u_2 = y_2 - z_2$. Let $i, j \in \mathbb{N}$ such that $n_i < m+1 \leq n_{i+1}$ and $n_j < n+1 \leq n_{j+1}$. As before we may assume that $n > m+1$ and $j > i$. Suppose $u_1 < 2a^{-i}$, then there must exists a k such that $x_m = x_{n_k-1}$ and an $l < a$ such that $x_n = x_{n_k+l}$. But in this case $\|x_{n+1} - x_{m+1}\|_2 \leq a^{-i}\sqrt{1+b^2} \leq \|x_n - x_m\|_2$, see the figure below, so we may assume that $u_1 \geq 2a^{-i}$.



From this figure, and the fact that $u_1 \geq 2a^{-i}$ and $j > i$ we may also deduce that $u_2 \leq b(u_1 - 2a^{-i})$. Using this we find:

$$\begin{aligned}
\|x_{n+1} - x_{m+1}\|_2^2 &\leq (u_1 - (a^{-i} - a^{-j}))^2 + (u_2 + b(a^{-i} + a^{-j}))^2 \\
&= \|x_n - x_m\|_2^2 + 2bu_2(a^{-i} + a^{-j}) - 2u_1(a^{-i} - a^{-j}) \\
&\quad\quad + (1+b^2)a^{-2i} + (1+b^2)a^{-2j} + 2(b^2-1)a^{-i-j} \\
&\leq \|x_n - x_m\|_2^2 + 2b^2(u_1 - 2a^{-i})(a^{-i} + a^{-j}) - 2u_1(a^{-i} - a^{-j}) \\
&\quad\quad + (1+b^2)a^{-2i} + (1+b^2)a^{-2j} + 2(b^2-1)a^{-i-j} \\
&= \|x_n - x_m\|_2^2 + 2u_1(b^2-1)a^{-i} + 2u_1(b^2+1)a^{-j} \\
&\quad\quad + (1-3b^2)a^{-2i} + (1+b^2)a^{-2j} - 2(b^2+1)a^{-i-j}
\end{aligned}$$

Now if $b^2 \leq \frac{a-1}{a+1}$ we find that

$$\begin{aligned}
(b^2-1)a^{-i} + (b^2+1)a^{-j} &\leq (b^2-1)a^{-i} + (b^2+1)a^{-i+1} \\
&\leq -\frac{2}{a+1}a^{-i} + (\frac{2}{a+1})a^{-i} \leq 0.
\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1}-x_{m+1}\|_2^2 \le \|x_n-x_m\|_2^2 &+ 4(b^2-1)a^{-2i} + 4(b^2+1)a^{-i-j} \\ &+ (1-3b^2)a^{-2i} + (1+b^2)a^{-2j} - 2(b^2+1)a^{-i-j} \\ &= \|x_n-x_m\|_2^2 + (b^2-3)a^{-2i} + 2(b^2+1)a^{-i-j} + (1+b^2)a^{-2j} \\ &\le \|x_n-x_m\|_2^2 + (b^2-3)a^{-2i} + 2(b^2+1)a^{-2i-1} + (1+b^2)a^{-2i-2}\end{aligned}$$

Now let $k \in \mathbb{R}$ such that $0 < \frac{1}{k} \le 1-b^2 < 1$. Then it holds that

$$\|x_{n+1}-x_{m+1}\|_2^2 \le \|x_n-x_m\|_2^2 - \frac{2k+1}{k}a^{-2i} + \frac{4k-2}{k}a^{-2i-1} + \frac{2k-1}{k}a^{-2i-2} \le \|x_n-x_m\|_2^2$$

if and only if

$$-(2k+1)a^2 + (4k-2)a + 2k - 1 \le 0.$$

Solving this and letting k tend to infinity gives us that for $0 < b < 1$ and $a \ge \max(1 + \sqrt{2}, \frac{1+b^2}{1-b^2}$ we have that (x_n) is a non-expansive sequence. By Theorem 11.3 in [68] there exists a non-expansive map f such that $f(x_n) = x_{n+1}$. Furthermore, let $0 < m < n$ and let $x_n = (y_1, y_2)$ and $x_m = (z_1, z_2)$. Note that $|y_2 - z_2| \le b(y_1 - z_1)$, $z_1 < y_1$, $|y_2| \le by_1$ and $|z_2| \le bz_1$. Using this we find

$$\begin{aligned}\|x_m\|_2^2 = z_1^2 + z_2^2 &= z_1^2 + (z_2-y_2)(z_2+y_2) + y_2^2 = \\ &\le z_1^2 + \frac{1}{b^2}(y_1-z_1)(y_1+z_1) + y_2^2 \\ &\le \frac{b^2-1}{b^2}z_1^2 + \frac{1}{b^2}y_1^2 + y_2^2 \le y_1^2 + y_2^2 = \|x_n\|_2^2\end{aligned}$$

Hence for any subsequence of (x_{k_n}) satisfying that $(i(x_{k_n}))$ converges to some horofunction ξ, it follows that ξ is a Karlsson point. Therefore we can find a continuum of Karlsson points with an angle which is arbitrarily close to 90 degrees.

Ernest Ryu independently proved that Karlsson points are not necessarily unique in [60].

3.4 Two Denjoy-Wolff type Theorems

In this section we will prove two special cases of the Karlsson-Nussbaum conjecture 3.0.3, where we have special conditions on the map f.

3.4.1 Theorem. *Let X be a finite dimensional vector space and let C be a bounded open convex set equipped with the Hilbert metric d_H. Let $f : C \to C$ be a fixed point free isometry. Then there exists a convex set $\Omega \subset \partial C$ such that for each $x \in C$ we have $\omega(x, f) \subset \Omega$.*

We will prove this theorem by utilizing a theorem from Walsh [65, Theorem 1.3] classifying the isometries on cones equipped with Hilbert's metric.
Recall that, if C is a bounded open convex set in a vector space X, then we can identify C in $Y = \mathbb{R} \times X$ with a slice of the open cone

$$Y_+^\circ = \{(\lambda, x) \in Y : \lambda > 0 \text{ and } \frac{x}{\lambda} \in C\}.$$

It is easy to verify that if we equip Y_+° and C with Hilbert's metric, the natural bijection from C to $\Sigma = \{(1, x) \in Y_+^\circ\}$ sending x to $(1, x)$ is an isometry. So for the proof of Theorem 3.4.1 we only have to consider order-unit spaces. Therefore we can use Birkhoff's version of Hilbert's metric using the gauge function.

3.4.2 Definition. Let X be an order-unit space with closed cone X_+. Recall the gauge function

$$M(x/y) = \inf\{\beta > 0 : x \leq \beta y\} \qquad (x, y \in X_+^\circ).$$

We call a map $f : X_+^\circ \to X_+^\circ$ *gauge-preserving* if $M(x/y) = M(f(x)/f(y))$ for all $x, y \in X_+^\circ$, we call it *gauge-reversing* if $M(x/y) = M(f(y)/f(x))$ for all $x, y \in X_+^\circ$.

Walsh has classified the isometries in [65].

3.4.3 Theorem. *Let X be a finite dimensional order-unit space with closed cone X_+ equipped with Hilbert's metric. Every isometry on X_+° arises as the projection of either a gauge-preserving or gauge-reserving map.*

We can now use a result from Noll and Schäfer in [54] which shows that every gauge-preserving map is the restriction of an isomorphism, and a result by Lins and Nussbaum which shows that if f is a (linear) isomorphism the Karlsson-Nussbaum conjecture holds. This shows that the conjecture holds for gauge-preserving maps, we can then use the fact that the square of a gauge-reversing map is gauge-preserving to prove Theorem 3.4.1.

3.4.4 Theorem ([54])**.** *Let X be a finite dimensional order-unit space with closed cone X_+. Every gauge-preserving map is the restriction of an isomorphism to X_+°.*

3.4.5 Theorem (Theorem 2, [47])**.** *Let X be a finite dimensional vector space with closed cone X_+. Let $f : X \to X$ be a positive linear map, let $\varphi \in X_+^*$ be a strictly positive linear functional and let*

$$\Sigma_\varphi = \{x \in X_+^\circ : \varphi(x) = 1\}.$$

If $T : \Sigma_\varphi \to \Sigma_\varphi$ given by $T(x) = f(x)/\varphi(f(x))$ has no fixed points, then there exists some $u \in \overline{\Sigma_\varphi} \setminus \Sigma_\varphi$ such that $\omega(x, T) \subset P_u \cap (\overline{\Sigma_\varphi} \setminus \Sigma_\varphi)$ for all $x \in \Sigma_\varphi$. Recall that P_u is the part of u.

Recall that by Corollary 2.1.5 parts are convex sets. We can now prove Theorem 3.4.1.

Proof 3.4.1. Let $Y = \mathbb{R} \times X$ be the order-unit space with open cone

$$Y_+^\circ = \{(\lambda, x) \in Y : \lambda > 0 \text{ and } \frac{x}{\lambda} \in C\}.$$

We can view f as an isometry on $(1, C)$. By Theorem 3.4.3 f is the projection of a gauge-preserving or gauge-reversing map $\tilde{f}$ on Y_+°. If $\tilde{f}$ is gauge-preserving, then by Theorem 3.4.4 it follows that $\tilde{f}$ is the restriction of an isomorphism on Y and by Theorem 3.4.5 we find that there exists some $u \in \partial Y_+$ such that for all $x \in X$ the set of accumulation points $\omega((1, x), f) \in P_u \cap (1, \partial C)$.

If f is gauge-reversing, then f^2 is a gauge-preserving map and by the same reasoning there exists some $u \in \partial Y_+$ such that for all $x \in X$ we have $\omega((1, x), f^2) \in P_u \cap (1, \partial C)$. In particular, for all $x \in X$ it holds that $\omega(f(x), f^2) \subset P_u \cap \partial C$. As $\omega(x, f) = \omega(x, f^2) \cup \omega(f(x), f^2)$ we find that $\omega(x, f) \subset P_u \cap \partial C$. □

Our second Denjoy-Wolff type theorem is analogous to Theorem 6.3 [14] by Chu and Rigby.

3.4.6 Definition. Let X be a normed vector space, let $A \subset X$ be some subset of X and let $f : A \to A$ be a map. We call $l : A \to \overline{A}$ a limit function of f if l is the pointwise limit of some subsequence (f^{n_k}).

For "well-behaved" spaces and maps these limit functions always exist, as shown in the following proposition.

3.4.7 Proposition. *Let X be a normed vector space and let (M, d) be a proper metric space such that $M \subset X$ is a precompact open subset and the topologies defined by the metric and the norm coincide on M. If $f : M \to M$ is a non-expansive map, then for every subsequence of (f^n) there exists a further subsequence which converges pointwise to a limit map.*

Proof. We will prove this using Ascoli's theorem [53, 47.1]. Note that for all $x, y \in M$ and $n \in \mathbb{N}$ we have

$$d(f^n(x), f^n(y)) \leq d(x, y),$$

from which it follows that $\{f^n(x) : n \in \mathbb{N}\}$ is equicontinuous with respect to the metric topology, and, since the topologies coincide, with respect to the norm topology. Furthermore, for all $x \in M$ the closure of

$$\{f^n(x) : n \in \mathbb{N}\}$$

is compact with respect to the norm, as it is contained in $\overline{M}$ which is compact. By Ascoli's theorem the set $\{f^n : n \in \mathbb{N}\}$ is contained in a compact set of $C(M, \overline{M})$, the set of norm continuous maps from M to $\overline{M}$ equipped with the topology of compact convergence. Since singletons of M are compact in the norm topology, it follows that the topology of pointwise convergence is weaker than the topology of compact convergence. Hence every subsequence of (f^n) has a further subsequence which converges pointwise to a limit map. □

3.4.8 Theorem. *Let X be a finite dimensional Banach space with closed cone X_+ equipped with the Hilbert metric d_H and let $f : X_+^\circ \to X_+^\circ$ be an order-preserving homogeneous map with no eigenvectors. Let φ be a positive linear functional and let*

$$\Sigma = \{x \in X_+^\circ : \varphi(x) = 1\}.$$

If $T : \Sigma \to \Sigma$, given by $T(x) = \frac{f(x)}{\varphi(f(x))}$ for all $x \in \Sigma$, is such that for every limit map l of T the image $l(\Sigma)$ is closed, then there exists a convex set $\Omega \subset \partial X_+$ such that for all $x \in \Sigma$ we have $\omega(x, T) \subset \Omega$.

Proof. By Theorem 4.3 from [41] we can find a sequence (T_n) of contractions converging pointwise to T. By Theorem 3.1.4 there exists a horofunction ξ such that T leaves the horoballs of ξ invariant. Furthermore by Proposition 3.1.10 we know that horoballs of ξ are convex. Hence

$$\tilde{\Omega} := \bigcap_{r>0} \overline{H_\xi(-r)} \subset \partial X_+$$

is convex. By Proposition 2.1.8 we have that there is a face $F \subset \partial X_+$ containing $\tilde{\Omega}$. Now let (x_n) be a sequence in X_+° such that $x_n \in H_\xi(-n)$. Since the horoball of a Beardon point is T-invariant we find that for every limit map l of T we have $l(x_n) \in \overline{H_\xi(-n)}$. Thus, since $\overline{\Sigma}$ is compact, by taking a further subsequence if required, we may assume that

$$\lim_{n\to\infty} l(x_n) \in \tilde{\Omega}.$$

Since $l(\Sigma)$ is closed there exists an $x \in \Sigma$ such that $l(x) = \lim_{n\to\infty} l(x_n)$. Note that, since f is order preserving, for all $y \in X_+^\circ$ we have

$$M(l(x)/l(y)) = \lim_{n\to\infty} M(T^{k_n}(x)/T^{k_n}(y)) = \lim_{n\to\infty} M(f^{k_n}(x)/f^{k_n}(y)) \leq M(x/y) < \infty$$

and in a similar way we have $M(l(y)/l(x)) \leq M(y/x)$. So for all $x, y \in \Sigma$ we find that $l(x)$ and $l(y)$ are comparable. It follows that there is a part $P \subset \partial(\Sigma)$ such that $l(\Sigma) \subset P$. Furthermore, since $\tilde{\Omega} \cap P \subset F$ is non-empty and by Proposition 2.1.7 F is a union of parts, we have that $l(\Sigma) \subset P \subset F$. As F is independent of l, we find that all accumulation points lie in the convex set $F \subset \partial X_+$. □

Chapter 4

Jordan Algebras

Jordan algebras can be viewed as order-unit spaces with additional algebraic structure. In this chapter we will provide an alternate version of a result by Chu and Rigby. In Theorem 5.10 in [14] Chu and Rigby give an explicit description of closed horoballs on bounded symmetric domains, using the Pierce decomposition of Jordan Algebras. We will give a description of the intersection of the boundary of the cone of a Euclidean Jordan algebra with closed horoballs whose radius tends to minus infinity. Before we do this we will briefly recall the basic concepts of the theory of Jordan algebras required in this chapter. One can find a thorough introduction in [22] by Faraut and Korányi and in [48] by McCrimmon.

4.0.1 Definition. Let X be a vector space over $\mathbb{R}$ equipped with a bilinear map $\bullet$: $X \times X \to X, (x, y) \mapsto xy$. We call X a *Jordan algebra* and $\bullet$ a *Jordan product* if for all $x, y \in X$ we have

$$xy = yx$$
$$x(x^2y) = x^2(xy)$$

and for all $x \in X$ the map $L(x) : X \to X, y \mapsto xy$ is linear. If there is an $e \in X$ such that $ex = x$ for all $x \in X$ we call e the *unit* of X and X a *unital Jordan algebra.* Furthermore, if X is finite dimensional and there exists an inner product $\langle \cdot, \cdot \rangle$ on X which is associative with respect to the Jordan product, i.e. for all $x, y, z \in X$ holds

$$\langle xy, z\rangle = \langle y, xz\rangle,$$

then we call X a *Euclidean Jordan algebra.*

If the Jordan product can be confused with a different product, e.g. the matrix product or an operator product, we will denote the Jordan product as $x \bullet y$. We will simply write xy otherwise.

Unital Euclidean Jordan algebras are all *formally real*, i.e. for all $x, y \in X$ it holds that $x^2 + y^2 = 0$ if and only if $x = y = 0$. This can be easily verified as

$$\|x\|^2 + \|y\|^2 = \langle x, x\rangle + \langle y, y\rangle = \langle e, x^2\rangle + \langle e, y^2\rangle = \langle e, x^2 + y^2\rangle = \langle e, 0\rangle = 0$$

if and only if $x = y = 0$. Conversely, it is known that all unital finite dimensional formally real Jordan algebras are unital Euclidean Jordan algebras, see [22, Proposition VIII 4.2].

4.0.2 Example. We will give some examples of well known Euclidean Jordan algebras.

1. The space $\mathbb{R}^n$ with the pointwise product and standard inner product.

2. The space of bounded self-adjoint operators of a finite dimensional real inner product space H, $B(H)_{sa}$ equipped with the following Jordan product $A \bullet B = \frac{1}{2}(AB + BA)$ for $A, B \in B(H)_{sa}$ with as inner product trace(AB).

3. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a finite dimensional real inner product space, the space $X = \mathbb{R} \times H$ with inner product

$$\langle(\lambda, x), (\mu, y)\rangle = \lambda\mu + \langle x, y\rangle_H \qquad ((\lambda, x), (\mu, y) \in X)$$

and product

$$(\lambda, x) \bullet (\mu, y) = (\lambda\mu + \langle x, y\rangle_H, \mu x + \lambda y) \qquad ((\lambda, x), (\mu, y) \in X).$$

This family of Euclidean Jordan algebras is known as *spin factors*.

For the remainder of this chapter we will only consider finite dimensional unital Jordan algebras with unit element e.

Let $(X, \bullet)$ be a Euclidean Jordan algebra. We call two elements $x, y \in X$ *orthogonal* if $x \bullet y = 0$. Recall that an element $x \in X$ is called an *idempotent* if $x^2 = x$. We call an idempotent *primitive* if it is non-zero and can not be written as the sum of two different non-zero idempotents. We call a set of pairwise orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ a *Jordan frame* if

$$\sum_{i=1}^{n} e_i = e.$$

4.0.3 Theorem (Spectral theorem, Theorem III.1.2 [22])**.** *Let $(X, \bullet)$ be a Jordan algebra. Then for all $x \in X$ there exists a Jordan frame $\{e_1, \ldots, e_n\}$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$x = \sum_{i=1}^{n} \lambda_i e_i.$$

The numbers $\lambda_1, \ldots, \lambda_n$ are uniquely determined by x.

The set $\{\lambda_1, \ldots \lambda_n\}$ is called the *spectrum* of x. Using the spectrum we can equip X with a natural cone X_+, the set of all elements with positive spectrum. It is an easy consequence of the Spectral theorem 4.0.3 that this set is also the set of squares

$$X_+ = \{x \in X : \exists y \in X \text{ such that } y^2 = x\}.$$

For the remainder of this chapter X_+ refers to the cone of squares. Using the Spectral theorem 4.0.3 it is easy to see that the interior of X_+ with respect to the order-unit norm $\|\cdot\|_e$ is the set of squares of invertible elements.

4.0.4 Definition. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space with cone X_+. We call X_+ *self-dual* if

$$X_+^* = \{x \in X : \langle x, y\rangle \geq 0, \forall y \in X_+\} = X_+.$$

We call X_+ *homogeneous* if the automorphism group

$$G(X_+) = \{A \in GL(X) : AX_+ = X_+\}$$

acts transitively on X_+°, i.e. for all $x, y \in X_+^\circ$ there exists an $A \in G(X_+)$ such that $Ax = y$. We call X_+ symmetric if it is both self-dual and homogeneous.

If X is a Euclidean Jordan algebra with cone of squares X_+, then, by the Koecher-Vinberg theorem [22, Theorem III.2.1], X_+° is a symmetric cone.

We call the linear map $U(x) = 2L(x)^2 - L(x^2)$ the *quadratic representation* of x. Using the spectral theorem it is easy to verify that $x \in X_+^\circ$ is invertible and that $U(x^{-\frac{1}{2}})x = e$. Furthermore, for all invertible $x \in X$ we have that $U(x)$ is invertible and $U(x)^{-1} = U(x^{-1})$. It follows that $U(x)$ is in the automorphism group $G(X_+)$ if x is invertible, see Proposition III.2.2 [22]. Thus for all $x, y \in X_+^\circ$ we have

$$M(x/y) = \inf\{\beta > 0 : U(y^{-\frac{1}{2}})x \leq \beta e\} = \max\{\lambda \in \sigma\left(U(y^{-\frac{1}{2}})x\right)\}.$$

From this it follows that

$$d_H(x, y) = \log\left(\max\{\lambda \in \sigma\left(U(y^{-\frac{1}{2}})x\right)\}\max\{\lambda \in \sigma\left(U(x^{-\frac{1}{2}})y\right)\}\right)$$

4.1 Pierce decomposition

Let $(X, \bullet)$ be a Euclidean Jordan algebra and let $x \in X$ be an idempotent one can show (Proposition III.1.3) that the eigenvalues of $L(x)$ are 0, $\frac{1}{2}$ and 1. The decomposition of X in the eigenspaces $X = V(x; 0) + V(x; \frac{1}{2}) + V(x; 1)$ is called the *Pierce decomposition.*

4.1.1 Proposition (Proposition IV.1.1 [22]). *If $(X, \bullet)$ is a Jordan algebra and x an idempotent, then $V(x; 1)$ and $V(x; 0)$ are Jordan subalgebras. Furthermore*

$$\begin{aligned} &V(x;1) \bullet V(x;0) = \{0\}, \\ &(V(x;1) + V(x;0)) \bullet V(x;\frac{1}{2}) \subset V(x;\frac{1}{2}) \textit{ and} \\ &V(x;\frac{1}{2}) \bullet V(x;\frac{1}{2}) \subset V(x;1) + V(x;0) \end{aligned}$$

One can find projections on $V(x; 1)$, $V(x; \frac{1}{2})$ and $V(x; 0)$. It is easy to verify that $L(x)(2L(x) - I)$ is a projection on $V(x; 1)$, $4L(x)(I - L(x))$ is a projection on $V(x; \frac{1}{2})$ and $(I - L(x))(2L(x) - I)$ is a projection on $V(x; 0)$.

A consequence of Proposition 4.1.1, is that if $x \in X$ is a primitive idempotent, then $V(x; 1) = \mathbb{R}x$. One can verify this by applying the spectral theorem 4.0.3 to the Jordan algebra $V(x; 1)$.

4.1.2 Theorem (Theorem IV.2.1 [22]). *Let $(X, \bullet)$ be a Euclidean Jordan algebra. Let $\{e_1, \ldots, e_n\}$ be a Jordan frame and let $V_{ii} = V(e_i; 1) = \mathbb{R}e_i$ and $V_{ij} = V(e_i; \frac{1}{2}) \cap V(e_j; \frac{1}{2})$. Then*

(i) *X decomposes in the direct sum*

$$X = \bigoplus_{i \le j} V_{ij}.$$

(ii) *If we define P_{ij} to be the orthogonal projection on V_{ij}, then $P_{ii} = U(e_i)$ and $P_{ij} = 4L(e_i)L(e_j)$.*

(iii) *And furthermore*

$$\begin{aligned} &V_{ij} \bullet V_{ij} \subset V_{ii} + V_{jj}, \\ &V_{ij} \bullet V_{jk} \subset V_{ik} \textit{ if } i \neq k, \\ &V_{ij} \bullet V_{kl} = \{0\} \textit{ if } \{i,j\} \cap \{k,l\} = \emptyset. \end{aligned}$$

The decomposition $X = \bigoplus_{i \leq j} V_{ij}$ as defined in Theorem 4.1.2 is call the *joint Pierce decomposition.*

4.1.3 Example. Consider the spin factor $X = \mathbb{R} \times \mathbb{R}^n$ with Jordan product

$$(\lambda, x) \bullet (\mu, y) = (\lambda\mu + \langle x, y\rangle, \lambda y + \mu x\rangle \qquad ((\lambda, x), (\mu, y) \in X).$$

This has unit element $e = (1, 0)$ and note that if $(\lambda, x) \in X$ is an idempotent, then

$$(\lambda, x) = (\lambda, x)^2 = (\lambda^2 + \|x\|^2, 2\lambda x)$$

from which it follows that $\lambda = \frac{1}{2}$ and $\|x\| = \frac{1}{2}$. It follows that all non-unit, non-zero idempotents are primitive. For a unit vector $x \in \mathbb{R}^n$ one can easily verify that the only idempotent orthogonal to $e_1 = (\frac{1}{2}, \frac{1}{2}x)$ is $e_2 = (\frac{1}{2}, -\frac{1}{2}x)$. As $e_1 + e_2 = e$ we find that $\{e_1, e_2\}$ is a Jordan frame and

$$\begin{aligned} V_{11} &= \mathbb{R}e_1, \\ V_{22} &= \mathbb{R}e_2, \\ V_{12} &= \{(0, y) \in X : \langle x, y\rangle = 0\}. \end{aligned}$$

4.2 Horoballs of the symmetric cone

Let $(X, \bullet)$ be a Euclidean Jordan algebra with unit e. Recall that the cone of squares X_+ is the set of all elements with positive spectrum. By applying the Spectral theorem 4.0.3 one can verify that the inverse map $(x \mapsto x^{-1})$ is well-defined on X_+° and leaves X_+° invariant. It is well known that the inverse map is gauge-reversing on the interior of the cone, i.e.

$$M(x/y) = M(y^{-1}/x^{-1}),$$

see [44, page 8]. This was utilised by Lemmens, Lins, Nussbaum and Wortel [41] to classify the Hilbert horofunction boundary for Euclidean Jordan algebras.

4.2.1 Theorem (Theorem 5.6 [41])**.** *If X_+° is a symmetric cone in a Euclidean Jordan Algebra $(X, \bullet)$ with as base point the unit e, then the horofunctions of the Hilbert geometry are precisely the functions of the following form:*

$$\xi(x) = \log(M(y/x)) + \log(M(z/x^{-1})) \qquad (x \in X_+^\circ),$$

where $y, z \in \partial X_+$ such that $\|y\|_e = \|z\|_e = 1$ and $y \bullet z = 0$.

We can use this to gain more information on the geometry of the horoballs. In the case of bounded symmetric domains there is actually a full description of the horoballs known by Chu and Rigby [14, Theorem 5.11]. We will do something similar for Euclidean Jordan algebras for the intersection of the closure of the horoballs with the boundary of the cone. For this we will need the following two lemmas.

4.2.2 Lemma. *Let $(X, \bullet)$ be a finite dimensional Euclidean Jordan algebra, with closed symmetric cone X_+ and unit e. Let $\{e_1, ..., e_n\}$ be a Jordan frame of X. For all $1 \leq i \leq n$, if $w \in V(e_i; \frac{1}{2})$, then $e_i w^2 = \frac{1}{2}\|w\|^2 e_i$. Furthermore, for all $1 \leq i < j \leq n$, if $w \in V_{ij}$ then $w^2 = \frac{1}{2}\|w\|^2(e_i + e_j)$.*

Proof. Let $1 \leq i \leq n$ and let $w \in V(e_i; \frac{1}{2})$. Note that by Proposition 4.1.1 we have that $w^2 \in V(e_i; 1) + V(e_i; 0)$. As e_i is a primitive idempotent by Theorem 4.1.2 we have that $e_i w^2 \in V(e_i; 1) = \mathbb{R}e_i$. Consider

$$\begin{aligned}\langle e_i w^2, e_i\rangle &= \langle L(e_i)w^2, e_i\rangle = \langle w^2, L(e_i)e_i\rangle \\ &= \langle w^2, e_i\rangle = \frac{1}{2}\langle w, w\rangle = \frac{1}{2}\|w\|^2,\end{aligned}$$

hence $e_i w^2 = \frac{1}{2}\|w\|^2 e_i$.

Now let $1 \leq i < j \leq n$ and let $w \in V_{ij}$. By Theorem 4.1.2 we have that $w^2 \in V_{ii} + V_{jj}$ and $w \in V(e_i; \frac{1}{2}) \cap V(e_j; \frac{1}{2})$. From the above it follows that $w^2 = (e_i + e_j)w^2 = \frac{1}{2}\|w\|^2(e_i + e_j)$. ☐

4.2.3 Lemma. *Let $(X, \bullet)$ be a finite dimensional Euclidean Jordan algebra, with closed symmetric cone X_+ and unit e. Let $\{e_1, ..., e_n\}$ be a Jordan frame of X. For all $1 \leq i \leq n$, if (w_m) is a sequence in $V(e_i; \frac{1}{2})$, then $\lim_{k\to\infty} w_m^2 - e_i w_m^2 = 0$ if and only if (w_m) converges to 0.*

Proof. First, note that $V(e_i; \frac{1}{2}) = \sum_{j\neq i} V_{ij}$. Indeed, since $4L(e_i)(I - L(e_i))$ is a projection on $V(e_i; \frac{1}{2})$ by Theorem 4.1.2 we find

$$\begin{aligned}V(e_i; \frac{1}{2}) &= 4L(e_i)(I - L(e_i))X = 4L(e_i)(I - L(e_i))\left(\sum_{j=1}^{n} V_{jj} + \sum_{1\leq j<k\leq n} V_{jk}\right) \\ &= 4L(e_i)\left(\sum_{j\neq i} V_{jj} + \sum_{j\neq i} \frac{1}{2}V_{ij} + \sum_{j\neq i\neq k} V_{jk}\right) = \sum_{j\neq i} V_{ij}.\end{aligned}$$

So for all $m \in \mathbb{N}$ we can write $w_m = \sum_{j \neq i} w_{mij}$ with $w_{mij} \in V_{ij}$. By Theorem 4.1.2 we find

$$w_m^2 = \sum_{j \neq i} w_{mij}^2 + 2 \sum_{\substack{0 \leq j < k \leq n \\ j \neq i \neq k}} w_{mij} w_{mik} \in \sum_{j \neq i} w_{mij}^2 + \sum_{\substack{1 \leq j < k \leq n \\ j \neq i \neq k}} V_{jk}.$$

By Lemma 4.2.2 we find

$$\sum_{j \neq i} w_{mij}^2 = \sum_{j \neq i} \frac{1}{2} \|w_{mij}\|^2 (e_i + e_j)$$

hence by Theorem 4.1.2 it follows that $e_i w_m^2 = \sum_{j \neq i} \frac{1}{2} \|w_{mij}\|^2 e_i$. As by Lemma 4.2.2 we have that $e_i w_m^2 = \frac{1}{2} \|w_m\|^2 e_i$, we conclude that $\|w_m\|^2 = \sum_{j \neq i} \|w_{mij}\|^2$. So

$$w_m^2 - e_i w_m^2 = w_m^2 - \frac{1}{2} \|w_m\|^2 e_i \in \sum_{j \neq i} \frac{1}{2} \|w_{mij}\|^2 e_j + \sum_{\substack{1 \leq j < k \leq n \\ j \neq i \neq k}} V_{jk}.$$

By Theorem 4.1.2 we have that $X = \bigoplus_{1 \leq j \leq k \leq n} V_{jk}$, so $\lim_{m \to \infty} w_m^2 - e_i w_m^2 = 0$ if and only if $(\|w_{mij}\|)$ tends to 0 for all $j \neq i$. Since $(\|w_m\|^2) = (\sum_{j \neq i} \|w_{mij}\|^2)$ it follows that $\lim_{m \to \infty} w_m^2 - w_m^2 \bullet e_i = 0$ if and only if w_m converges to 0. ☐

We can now prove the main theorem of this chapter.

4.2.4 Theorem. *Let $(X, \bullet)$ be a finite dimensional Euclidean Jordan algebra, with closed symmetric cone X_+ and unit e. If ξ is a horofunction of the Hilbert geometry given by*

$$\xi(x) = \log(M(y/x)) + \log(M(z/x^{-1})) \qquad (x \in X_+^\circ)$$

where $y, z \in \partial C$ such that $y \bullet z = 0$ and $\|y\|_e = \|z\|_e = 1$, then

$$\bigcap_{r > 0} \overline{H_{-r}(\xi)} = V(z; 0) \cap X_+$$

where $\overline{H_{-r}(\xi)}$ is the norm closure of the horoball of ξ of radius $-r$.

Proof. Let $J = \{e_1, \ldots, e_n\}$ be a Jordan frame such that $z = \sum_{i=1}^k \alpha_i e_i$ and $y = \sum_{i=k+1}^l \beta_i e_i$. We will first prove that $V(z; 0) \cap X_+ \subset \bigcap_{r > 0} \overline{H_{-r}(\xi)}$. Let $x \in V(z; 0) \cap X_+$. Since $V(z; 0) = V(\sum_{i=1}^k e_i; 0) = \bigoplus_{k+1 \leq i \leq j \leq n} V_{ij}$ we can find $\mu_i \in \mathbb{R}$ and $x_{ij} \in V_{ij}$ such that

$$x = \sum_{i=k+1}^n \mu_i e_i + \sum_{k+1 \leq i < j \leq n} x_{i,j}.$$

For all $m \in \mathbb{N}$ we define

$$x_m = \sum_{i=1}^k \frac{1}{m^2} e_i + \sum_{i=k+1}^n (\frac{1}{m} + \mu_i) e_i + \sum_{k+1 \leq i < j \leq n} x_{ij}.$$

Clearly x_m converges to x in norm as $m \to \infty$ and $x_m \in X_+^\circ$ as $x_m > x + \frac{1}{m^2}e \geq \frac{1}{m^2}e$ as $x \in X_+$.

As $\sum_{i=1}^k e_i$ is an idempotent by Proposition 4.1.1 $V(z;0) = V(\sum_{i=1}^k e_i;0)$ is a Jordan subalgebra. As the cones of Euclidean Jordan algebras are the set of squares we have $V(z;0)_+ = V(z;0) \cap X_+$. It is easy to verify that $u = \sum_{i=k+1}^n e_i$ is the unit of $V(z;0)$ and therefore an order-unit of $V(z;0)$. Since $x \in V(z;0) \cap X_+ = V(z;0)_+$ we can find a $\beta > 0$ such that $x \leq \beta u$. Then $x_m \leq \frac{1}{m^2}\sum_{i=1}^k e_i + (\frac{1}{m} + \beta)u$. Thus

$$x_m^{-1} \geq m^2 \sum_{i=1}^k e_i + (\frac{m}{\beta m + 1})u \geq m^2 z \qquad \text{and}$$
$$x_m \geq x + \frac{1}{m}\sum_{i=k+1}^n e_i \geq \frac{1}{m}\sum_{i=k+1}^n e_i \geq \frac{1}{m}y$$

we have $M(y/x_m) \leq m$ and $M(z/x_m^{-1}) \leq m^{-2}$. Hence

$$\xi(x_m) = \log(M(y/x_m)) + \log(M(z/x_m^{-1})) \leq \log(m) - 2\log(m).$$

So we find that $\xi(x_m)$ tends to $-\infty$ if $m \to \infty$ and hence $V(z;0) \cap X_+ \subset \bigcap_{r>0} \overline{H_{-r}(\xi)}$.

For the opposite inclusion let (x_m) be a sequence in X_+° such that $\|x_m\|_e = 1$, $\lim_{m\to\infty} x_m = x \in \partial X_+$ and $\lim_{m\to\infty} \xi(x_m) = -\infty$. Note that, as $x_m \leq e$ implies that

$$y \leq M(y/x_m)x_m \leq M(y/x_m)e,$$

we have

$$M(y/x_m) \geq M(y/e) = \|y\|_e = 1.$$

In particular this means that

$$-\infty = \liminf_{m\to\infty} \log\left(M(y/x_m)M(z/x_m^{-1})\right) \geq \liminf_{m\to\infty} \log\left(M(z/x_m^{-1})\right).$$

So we have to prove that, if $\lim_{m\to\infty} M(z/x_m^{-1}) = 0$, then $x \in V(z;0)$.

To understand the limit of $(M(z/x_m^{-1}))$ we will first examine the limit of $(M(e_i/x_m^{-1}))$. Recall that

$$M(e_i/x_m^{-1}) = M(U(x_m^{\frac{1}{2}})e_i/e) = M((2L(x_m^{\frac{1}{2}})^2 e_i - L(x_m)e_i)/e).$$

Let $\lambda_m \in \mathbb{R}$, $u_{m,\frac{1}{2}} \in V(e_i;\frac{1}{2})$ and $u_{m,0} \in V(e_i;0)$ such that $x_m^{\frac{1}{2}} = \lambda_m e_i + u_{m,\frac{1}{2}} + u_{m,0}$. Recall that by Lemma 4.2.2 we have $e_i u_{m,\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i$. Using this and Proposition 4.1.1 we find

$$x_m = (\lambda_m^2 + \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2)e_i + 2u_{m,0} \bullet u_{m,\frac{1}{2}} + \lambda_m u_{m,\frac{1}{2}} + (u_{m,0}^2 + u_{m,\frac{1}{2}}^2 - \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i)$$

and

$$L(x_m)e_i = (\lambda_m^2 + \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2)e_i + u_{m,0} \bullet u_{m,\frac{1}{2}} + \frac{\lambda_m}{2}u_{m,\frac{1}{2}}.$$

Furthermore

$$\begin{aligned}
2L(x_m^{\frac{1}{2}})^2 e_i &= 2(\lambda_m e_i + u_{m,\frac{1}{2}} + u_{m,0}) \bullet (\lambda_m e_i + \frac{1}{2}u_{m,\frac{1}{2}}) \\
&= 2(\lambda_m^2 e_i + \frac{\lambda_m}{4}u_{m,\frac{1}{2}} + \frac{\lambda_m}{2}u_{m,\frac{1}{2}} + \frac{1}{2}u_{m,\frac{1}{2}}^2 + u_{m,0} \bullet u_{m,\frac{1}{2}}) \\
&= (2\lambda_m^2 + \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2)e_i + \frac{3\lambda_m}{2}u_{m,\frac{1}{2}} + u_{m,0} \bullet u_{m,\frac{1}{2}} + (u_{m,\frac{1}{2}}^2 - \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i)
\end{aligned}$$

This gives us

$$2L(x_m^{\frac{1}{2}})^2 e_i - L(x_m)e_i = \lambda_m^2 e_i + \lambda_m u_{m,\frac{1}{2}} + (u_{m,\frac{1}{2}}^2 - \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i)$$

Now suppose that $\lim_{m\to 0} U(x_m^{\frac{1}{2}})e_i = 0$. As $U(x_m^{\frac{1}{2}})e_i$ is a direct sum of $V(e_i;1), V(e_i;\frac{1}{2})$ and $V(e_i;0)$ we find that $\lim_{m\to\infty} U(x_m^{\frac{1}{2}})e_i = 0$ if and only if

$$\begin{aligned}
\lim_{m\to\infty} u_{m,\frac{1}{2}}^2 - \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i = 0, \\
\lim_{m\to\infty} \lambda_m u_{m,\frac{1}{2}} = 0 \text{ and} \\
\lim_{m\to\infty} \lambda_m^2 e_i = 0.
\end{aligned}$$

Note that $\lim_{m\to\infty} \lambda_m^2 e_i = 0$ if and only if $\lim_{m\to\infty} \lambda_m = 0$. Furthermore, by Lemma 4.2.3 we find that $\lim_{m\to\infty} u_{m,\frac{1}{2}}^2 - \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 e_i = 0$ if and only if $\lim_{m\to\infty} u_{m,\frac{1}{2}} = 0$. Note that $\lim_{m\to\infty} \lambda_m u_{m,\frac{1}{2}}$ if $\lim_{m\to\infty} \lambda_m = 0$, hence $\lim_{m\to 0} U(x_m^{\frac{1}{2}})e_i = 0$ if and only if $\lim_{m\to\infty} u_{m,\frac{1}{2}} = 0$ and $\lim_{m\to\infty} \lambda_m = 0$.
Now we consider the components of x. Note that if $\lim_{m\to\infty} u_{m,\frac{1}{2}} = 0$ and $\lim_{m\to\infty} \lambda_m = 0$, then $\lim_{m\to\infty} \lambda_m^2 + \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 = 0$ and $\lim_{m\to\infty} 2u_{m,0} \bullet u_{m,\frac{1}{2}} + \lambda_m u_{m,\frac{1}{2}} = 0$. Also note that if $\lim_{m\to\infty} \lambda_m^2 + \frac{1}{2}\|u_{m,\frac{1}{2}}\|^2 = 0$ then clearly $\lim_{m\to\infty} u_{m,\frac{1}{2}} = 0$ and $\lim_{m\to\infty} \lambda_m = 0$. Thus we conclude that $\lim_{m\to\infty} U(x_m^{\frac{1}{2}})e_i = 0$ if and only if $\lim_{m\to\infty} x_m \in V(e_i;0)$.

Now consider

$$U\left(x_m^{\frac{1}{2}}\right)z = \sum_{i=1}^{k} \alpha_i U\left(x_m^{\frac{1}{2}}\right)e_i.$$

We know from Proposition IV.3.2 in [22] that the primitive idempotents are exactly the extreme rays of the cone. Furthermore recall that as $x_m^{\frac{1}{2}}$ is invertible, $U(x_m^{\frac{1}{2}})$ is invertible, so $U(x_m^{\frac{1}{2}})$ is a linear isomorphism which leaves X_+ invariant. Consequently it sends extreme rays to extreme rays, i.e. $U(x_m^{\frac{1}{2}})e_i$ is the multiple of a primitive idempotent. As $\alpha_i > 0$ we find that

$$\begin{aligned}\lim_{m\to\infty} M(U(x_m^{\frac{1}{2}})z/e) &= \lim_{m\to\infty} M(\sum_{i=1}^{k} \alpha_i U(x_m^{\frac{1}{2}})e_i/e)\\ &= \lim_{m\to\infty} \|\sum_{i=1}^{k} \alpha_i U(x_m^{\frac{1}{2}})e_i\|_e = 0\end{aligned}$$

if and only if $\lim_{m\to\infty} M(U(x_m^{\frac{1}{2}})e_i/e) = 0$ for all $1 \leq i \leq k$, i.e. $\lim_{m\to\infty} x_m \in V(e_i; 0)$. $\square$

Chapter 5

Kuratowski-Painlevé convergence

In the remainder of this thesis we will focus on the horofunction boundary itself. We will give descriptions and classifications of the horofunctions of the Funk, reverse-Funk and Hilbert geometry for the cone of, possibly infinite dimensional, order-unit spaces. For general order-unit spaces we will give only a description. This result is based on the work of Walsh in [66]. In this paper Walsh gives a classification of the Busemann points of the Funk, reverse-Funk and Hilbert geometry for the cone of an order-unit spaces. The Busemann points are a special class of horofunctions which we will introduce in Chapter 6. In particular Walsh shows that the horofunctions of the Funk and reverse-Funk geometry of the cone of positive continuous functions on a compact set are all Busemann points. For this result we need Kuratowski-Painlevé convergence which we will introduce in this chapter. In Chapter 6 we will give the proof of Walsh's results in [66] and extend it to a description of the horofunction boundary of general order-unit spaces. Finally we will give a partial answer to the following question which was raised by Walsh in [66]. Is it possible for reverse-Funk geometries to have non-Busemann horofunctions? In Section 7 we will show that this is possible even for some "well-behaved" spaces. As part of this effort we will also classify the horofunctions and Busemann points of the Funk, reverse-Funk, and Hilbert geometry of the cone of spin-factors and JH-algebras.

5.1 Kuratowski-Painlevé convergence

Every function $f : X \to Y$ is just a representation of the graph of the function

$$\{(x, y) \in X \times Y : f(x) = y\}.$$

Kuratowski-Painlevé convergence deals with the convergence of nets of sets and thus can be used to define convergence on nets of functions.

5.1.1 Definition. Let J' be a subset of a directed set J. J' is called *residual* if there exists an $\alpha \in J$ such that $\alpha' \in J'$ for all $\alpha' \geq \alpha$. J' is called *cofinal* if for all $\alpha \in J$ there exist an $\alpha' \geq \alpha$ such that $\alpha' \in J'$.

5.1.2 Definition. Let (X, τ) be a Hausdorff space and let $(A_\alpha)_{\alpha \in J}$ be a net of subsets of X. We call x_0 a *limit point* of $(A_\alpha)_{\alpha \in J}$ if for every neighbourhood U of x_0 there exist a residual subset $J' \subset J$ such that for all $\alpha \in J'$ we have $A_\alpha \cap U \neq \emptyset$.
We call x_0 a *cluster point* of $(A_\alpha)_{\alpha \in J}$ if for every neighbourhood U of x_0 there exist a cofinal subset $J' \subset J$ such that for all $\alpha \in J'$ we have $A_\alpha \cap U \neq \emptyset$.
We denote the set of limits point of (A_α) as Li A_α and the set of cluster points as Ls A_α.

Note that Li $A_\alpha \subset$ Ls A_α. We will call Li A_α and Ls A_α the lower closed limit and the upper closed limit. The fact that both are closed is a direct consequence of a result by Choquet in [13].

5.1.3 Proposition. *Let (X, τ) be a Hausdorff space and let $(A_\alpha)_{\alpha \in J}$ be a net of subsets of X. Then*

$$\text{Li } A_\alpha = \bigcap \left\{ \overline{\bigcup_{\alpha \in J'} A_\alpha} : J' \textit{ is a cofinal subset of } J \right\}$$

and

$$\text{Ls } A_\alpha = \bigcap \left\{ \overline{\bigcup_{\alpha \in J'} A_\alpha} : J' \textit{ is a residual subset of } J \right\}.$$

Proof. The proof for the formula of Li A_α and Ls A_α are similar so we will only provide the proof for Ls A_α. Let $x \in$ Ls A_α and let J' be a residual subset of J and let $\alpha \in J$ be such that for all $\alpha \leq \alpha' \in J$ we have $\alpha' \in J'$. As $x \in$ Ls A_α, we know that for all neighbourhoods U of x there is an $\alpha' \geq \alpha$ such that $A_{\alpha'} \cap U \neq \emptyset$, so

$$x \in \overline{\bigcup_{\alpha' \geq \alpha} A_{\alpha'}} \subset \overline{\bigcup_{\alpha' \in J'} A_{\alpha'}}.$$

Now let $x \in X \backslash \text{Ls } A_\alpha$, i.e., there is a neighbourhood V of x such that $\{\alpha \in J : V \cap A_\alpha \neq \emptyset\}$ is not cofinal. Then $J' = \{\alpha \in J : V \cap A_\alpha = \emptyset\}$ is residual and $x \notin \overline{\bigcup_{\alpha \in J'} A_\alpha}$. □

Note that an immediate consequence of Proposition 5.1.3 is that the set of limits points and the set of cluster point are both closed.

5.1.4 Definition. Let (X, τ) be a Hausdorff space let A be a subset of X and let (A_α) be a net of subsets of X. We call (A_α) Kuratowski-Painlevé convergent to A if Li $A_\alpha =$ Ls $A_\alpha = A$. We write $A = K - \lim A_\alpha$.

As Li $A_\alpha \subset$ Ls A_α, there are two easy ways to verify if a net of sets is Kuratowski-Painlevé convergent.

5.1.5 Lemma (Lemma 5.2.4, [7]). *Let X be a Hausdorff space and let (A_α) be a net of subsets of X and let $A \subset X$. (A_α) is Kuratowski-Painlevé convergent if* Ls $A_\alpha \subset$ Li A_α. *Moreover, (A_α) is Kuratowski-Painlevé convergent to A if* Ls $A_\alpha \subset A \subset$ Li A_α.

Note that for any open set $U \subset X$ and subset $A \subset X$ we have $U \cap A = \emptyset$ if and only if $U \cap \overline{A} = \emptyset$, Thus $K - \lim A_\alpha = K - \lim \overline{A_\alpha}$. This allows us to compare Kuratowski-Painlevé convergence to the convergence in the Fell topology, a topology on the closed sets of X introduced by Fell in [23].

5.1.6 Definition. Let X be a Hausdorff space and let $\mathrm{CL}(X)$ be the set of closed subsets of X. The Fell topology τ_F on $\mathrm{CL}(X)$ is the topology generated by the base which consists of all sets of the following form, for all open sets $V \subset X$ the sets $V^- = \{C \in \mathrm{CL}(X) : C \cap V \neq \emptyset\}$ and for all open set $W \subset X$ with a compact complement the sets $W^+ = \{C \in \mathrm{CL}(X) : C \subset W\}$.

One can also define the Fell topology in non-Hausdorff spaces, which has applications in functional analysis. However this does make the Fell topology harder to work with due to lack of unique limits. As we are mainly interested in metric spaces, we will only consider the Fell topology in Hausdorff spaces. For Hausdorff spaces Kuratowski-Painlevé convergence always implies convergence with respect to the Fell topology [20], as the following proposition shows.

5.1.7 Proposition. *Let X be a Hausdorff space, let $(A_\alpha)_{\alpha \in J}$ be a net of subsets of X and let $\emptyset \neq A \subset X$ be closed. If (A_α) is Kuratowski-Painlevé convergent to A, then $(\overline{A_\alpha})$ converges to A in the Fell topology.*

Proof. First let $V \subset X$ be an open set such that $A \in V^-$, i.e. $A \cap V \neq \emptyset$. Let $x \in A \cap V$ and note that V is a neighbourhood of X. As Li $A_\alpha = A$ there exists an α such that for all $\alpha' \geq \alpha$ we have $A_{\alpha'} \cap V \neq \emptyset$ and in particular $\overline{A_{\alpha'}} \in V^-$. Finally let $W \subset X$ be an open set such that $X \setminus W$ is compact and $A \subset W$. Now suppose there is some cofinal set $J' \subset J$ such that for all $\alpha \in J'$ we have $A_\alpha \cap (X \setminus W) \neq \emptyset$. As $X \setminus W$ is compact this implies there is a cluster point of (A_α) in $X \setminus W$ which contradicts $K - \lim A_\alpha = A$. □

The converse of this statement only is true if X is locally compact. In fact, by a result by Mrówka in [51], if X is not locally compact then there exist no topology that coincides with

Kuratowski-Painlevé convergence. In particular one can see that convergence in the Fell topology and Kuratowski-Painlevé convergence do not coincide if X is not locally compact as the Fell topology is Hausdorff if and only if X is locally compact [7, Proposition 5.1.2], while Kuratowski-Painlevé convergence is always unique if X is Hausdorff.

5.1.8 Proposition. *Let X be a locally compact Hausdorff space, let $(A_\alpha)_{\alpha\in J}$ be a net of subsets of X and let $A \subset X$ be closed. If $(\overline{A_\alpha})$ converges to A in the Fell topology, then (A_α) is Kuratowski-Painlevé convergent to A.*

Proof. By Lemma 5.1.5 we only have to prove Ls $A_\alpha \subset A \subset$ Li A_α. So let $x \in$ Ls A_α be a cluster point and suppose $x \notin A$. As X is a locally compact Hausdorff space there is a compact neighbourhood C of x which is disjoint from A. As C has non-empty interior C° and x is a cluster point there exists a cofinal set $J' \subset J$ such that for all $\alpha \in J'$ we have $A_\alpha \cap C^\circ \neq \emptyset$. So for all $\alpha \in J'$ it holds that $A_\alpha \notin (X \setminus C)^+$. This contradicts that $(\overline{A_\alpha})$ converges to A in the Fell topology.
Now let $x \in A$ and suppose A is not a limit point. Then there is some open neighbourhood U of x and a cofinal set $J' \subset J$ such that for all $\alpha \in J'$ we have $A_\alpha \cap U = \emptyset$. In particular this means $A_\alpha \notin U^-$ which contradicts that $(\overline{A_\alpha})$ converges to A in the Fell topology. □

If X is not locally compact then the result does not hold, as illustrated by the following example.

5.1.9 Example. Let $B_1[0] \subset \ell_2$ be the closed unit ball, let $x \in \ell_2$ with $\|x\| = 2$ and let

$$K(x) = \{K \subset \ell_2 : x \in K, K \text{ compact and } K \cap B_1[0] = \emptyset\}$$

be the sets of all compact sets containing x which do not intersect the closed unit ball. Note that $K(x)$ is a directed set when equipped with the partial order $U \leq V$ if and only if $U \subset V$ for all $U, V \in K(x)$. Now for every $U \in K(x)$ we define $r_U = \max_{y\in U} \|y\|$, note that $2 \leq r_U < \infty$. Since $B_{r_U^{-1}}[x]$ is not contained in U we can find some $x_U \in B_{r_U^{-1}}[x] \setminus U$. As $\{x, nx\} \in K(x)$ for all $n > 1$, we can easily see that the net $(r_u)_{U\in K(x)}$ tends to infinity and therefore $\lim_{U\in K(x)} x_U = x$.

Now let $e_n \in \ell_2$ be the standard unit vectors in ℓ_2. Note that since every $U \in K(x)$ is compact, the set $\{2e_n \in U : n \in \mathbb{N}\}$ is finite. Hence we can define the net $(A_U)_{U\in K(x)}$ of subsets of ℓ_2 given by

$$A_U = \begin{cases} B_1[0] \cup \{x_U\} & \text{if } |\{2e_n \in U : n \in \mathbb{N}\}| \text{ is odd.} \\ B_1[0] & \text{if } |\{2e_n \in U : n \in \mathbb{N}\}| \text{ is even.} \end{cases}$$

Clearly x is a cluster point but not a limit point for $(A_U)_{U\in K(x)}$, hence (A_U) is not Kuratowski-Painlevé convergent. However one can show that (A_U) converges to $B_1[0]$ in the Fell topology. To see this let $V \subset \ell_2$ be an open set such that $B_1[0] \in V^-$. We have that $A_U \in V^-$ for all $U \in K(x)$. Let $W \subset \ell_2$ be an open set containing $B_1[0]$ for which $K = \ell_2 \setminus W$ is compact. Note that if $K \notin K(x)$ then $\inf_{y\in K} \|x - y\| > 0$. If we pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \inf_{y\in K} \|x - y\|$ then for all $U \geq \{x, ne_1\}$ we have that $\|x - x_U\| \leq \frac{1}{n}$. Hence $x_U \in W$ and therefore $A_U \subset W$. If $K \in K(x)$, then for all $U \geq K$ we have that $A_U \subset W$. It follows that (A_U) converges to $B_1[0]$ in the Fell topology.

As in general infinite dimensional order-unit spaces are not locally compact we will only look at Kuratowski-Painlevé convergence and not at a topology. Finally one important reason for using Kuratowski-Painlevé convergence is that it is "compact" for any Hausdorff space, i.e. every net of subsets has a Kuratowski-Painlevé convergent subnet. Mrowka gave a proof of this result in [52] using Tychonoff's theorem [53, Theorem 37.3].

5.1.10 Theorem (Mrowka's theorem)**.** *Let X be a Hausdorff space. If $(A_\alpha)_{\alpha\in J}$ is a net of subsets of X, then (A_α) has a Kuratowski-Painlevé convergent subnet.*

Proof. Let $O(X)$ be the set of open sets of X. Let $\{0, 1\}$ be equipped with the discrete topology and consider $\{0, 1\}^{O(X)}$ equipped with the product topology. For all $\alpha \in J$ we define $f_\alpha \in \{0, 1\}^{O(X)}$ by

$$f_\alpha(U) = \begin{cases} 1 & \text{if } A_\alpha \cap U \neq \emptyset. \\ 0 & \text{if } A_\alpha \cap U = \emptyset. \end{cases}$$

Since by Tychonoff's theorem $\{0, 1\}^{O(X)}$ is compact $(f_\alpha)_{\alpha\in J}$ has a convergent subnet $(f_\alpha)_{\alpha\in\hat{J}}$.

We will now show that for the subnet $(A_\alpha)_{\alpha\in\hat{J}}$ it holds that Ls $A_\alpha \subset$ Li A_α. Applying Lemma 5.1.5 will then give us that $(A_\alpha)_{\alpha\in\hat{J}}$ is Kuratowski-Painlevé convergent.

Let $x \in X$ be a cluster point of $(A_\alpha)_{\alpha\in\hat{J}}$ and let U be a neighbourhood of x. For some cofinal $J' \subset \hat{J}$ it holds that for all $\alpha \in J'$ we have $f_\alpha(U) = 1$. But since $(f_\alpha)_{\alpha\in\hat{J}}$ converges this implies that $\lim_\alpha f_\alpha(U) = 1$ and thus there is a residual $J^\dagger \subset \hat{J}$ such that $f_\alpha(U) = 1$ for all $\alpha \in J^\dagger$, so x is a limit point of $(A_\alpha)_{\alpha\in\hat{J}}$. □

5.2 Epiconvergence and hypoconvergence

Let X be a topological space and let $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ be a map. We define the *epigraph* of f as $\text{epi}(f) = \{(x, r) \in X \times \overline{\mathbb{R}} : r \geq f(x)\}$ and we define the *hypograph* of f as $\text{hypo}(f) = \{(x, r) \in X \times \overline{\mathbb{R}} : r \leq f(x)\}$.

5.2.1 Definition. Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is called *upper-semicontinuous* if for all $x \in X$ either for all $\varepsilon > 0$ there exists an open neighbourhood $U \subset X$ of x such that for all $y \in U$ we have $f(x) + \varepsilon > f(y)$ if $f(x) > -\infty$, or, for all $N \in \mathbb{R}$ there exists an open neighbourhood $U \subset X$ of x such that $f(y) < N$ for all $y \in U$ if $f(x) = -\infty$. We call f *lower-semicontinuous* if $-f$ is upper-semicontinuous.

If X is Hausdorff there is a useful equivalent definition for upper and lower semicontinuity.

5.2.2 Proposition. *Let $f : X \to \overline{\mathbb{R}}$ be a map on a Hausdorff space X. Then f is upper-semicontinuous at $x \in X$ if and only if for all nets (x_α) converging to x we have $\limsup_\alpha f(x_\alpha) \leq f(x)$.*

Proof. Note that the result is evident if $f(x) = -\infty$, so we may assume $f(x) > -\infty$.
Suppose f is upper-semicontinuous at $x \in X$, but there exists a net (x_α) in X such that $\limsup_\alpha f(x_\alpha) > f(x)$, by taking a subnet we may assume $\lim_\alpha f(x_\alpha) > f(x)$. Let $\varepsilon = \frac{1}{2}(\lim_\alpha f(x_\alpha) - f(x))$. There exists a neighbourhood U of x such that for all $y \in U$ we have $f(x) \geq f(y) - \varepsilon$. As $x_\alpha \in U$ for α large enough, this is a contradiction.
Now assume for all nets (x_α) in X converging to x we have $\limsup_\alpha f(x_\alpha) \leq f(x)$, but there is an $\varepsilon > 0$ such that for all neighbourhoods U of x there is a $y_U \in U$ such that $f(y_U) - \varepsilon \geq f(x)$. Note that the set of neighbourhoods of x is a directed set when equipped with the partial order $U \leq V$ if and only if $U \supset V$. Then (y_U) is a net converging to x with $\limsup_U f(y_U) \geq f(x) + \varepsilon > f(x)$ which is a contradiction. □

A similar result can be obtained for lower-semicontinuous functions.

5.2.3 Proposition. *Let $f : X \to \overline{\mathbb{R}}$ be a map on a Hausdorff space X. Then f is lower-semicontinuous at $x \in X$ if and only if for all nets (x_α) converging to x we have $\liminf_\alpha f(x_\alpha) \geq f(x)$.*

From the results above we find the following easy consequence.

5.2.4 Corollary. *Let $f : X \to \overline{\mathbb{R}}$ be a map on a compact Hausdorff space X. If f is upper-semicontinuous, then there is a $y \in X$ such that $f(y) = \sup_{x\in X} f(x)$. If f is lower-semicontinuous, then there is a $y \in X$ such that $f(y) = \inf_{x\in X} f(x)$.*

For reasons of convenience for the rest of the thesis we will denote "$\sup_{x\in X} f(x)$" by "$\sup f$" and "$\inf_{x\in X} f(x)$" by "$\inf f$" if there is no ambiguity.

Proof 5.2.4. Let f be upper-semicontinuous and let (x_α) be a net such that $\lim_\alpha f(x_\alpha) = \sup f$. Since X is compact, by taking a further subsequence, we may assume that x_α converges to some $y \in X$. By Proposition 5.2.2 it follows that

$$f(y) \geq \limsup_\alpha f(x_\alpha) = \sup f.$$

The proof for the case in which f is lower-semicontinuous is similar. □

If X is Hausdorff there is another equivalent definition of upper or lower semi-continuity using the epigraph and the hypograph.

5.2.5 Proposition. *Let X be a Hausdorff space and let $f : X \to \overline{\mathbb{R}}$ be an extended real function. Then f is upper-semicontinuous if and only if its hypograph is closed with respect to the product topology of $X \times \overline{\mathbb{R}}$. Likewise f is lower-semicontinuous if and only if its epigraph is closed with respect to the product topology of $X \times \overline{\mathbb{R}}$.*

Proof. Suppose f is upper-semicontinuous and let $(x, r) \in X \times \mathbb{R}$ such that $r > f(x)$. If $f(x) > -\infty$ let $\varepsilon = \frac{1}{4}(r - f(x))$. Then there exists a neighbourhood U of x such that for all $y \in U$ we have $r - \varepsilon > f(x) + \varepsilon > f(y)$, so $(U \times (r - \varepsilon, r + \varepsilon)) \cap \text{hypo}(f) = \emptyset$. If $f(x) = -\infty$, then there exists an open neighbourhood $U \subset X$ of x such that $f(y) < r - 1$ for all $y \in U$, so $(U \times (r - \frac{1}{2}, r + \frac{1}{2})) \cap \text{hypo}(f) = \emptyset$, hence $\text{hypo}(f)$ is closed.
Now suppose $\text{hypo}(f)$ is closed and let $x \in X$. If $f(x) = -\infty$ then for all $N \in \mathbb{N}$ there exists a neighbourhood U of x and a $\varepsilon > 0$ such that $(U \times (N - \varepsilon, N + \varepsilon)) \cap \text{hypo}(f) = \emptyset$, as $\text{hypo}(f)$ is closed. In particular we find that for all $y \in U$ we have $f(y) < N$. If $f(x) > -\infty$, then for all $\varepsilon > 0$ there exists a $\mu > 0$ and a neighbourhood U of x such that $(U \times (f(x) + \varepsilon - \mu, f(x) + \varepsilon + \mu)) \cap \text{hypo}(f) = \emptyset$, so $f(y) < f(x) + \varepsilon$ for all $y \in U$, hence f is upper-semicontinuous.
The proof for the second equivalence relation is an easy consequence of the above and the fact that $\text{epi}(-f) = -\text{hypo}(f)$. □

5.2.6 Corollary. *The Kuratowski-Painlevé limit of a net of hypographs is always upper-semicontinuous and the Kuratowski-Painlevé limit of a net of epigraphs is always lower-semicontinuous.*

Proof. This is a direct consequence of the fact that the Kuratowski-Painlevé limit of a net of sets is always closed by Proposition 5.1.3. The result then follows from Proposition 5.2.5. □

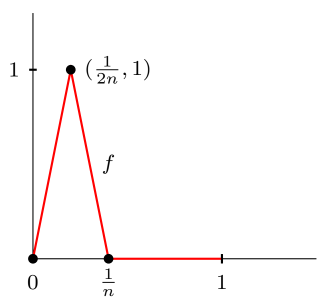
This motivates the idea to call a net of functions (f_α) convergent to some f if and only if the epigraphs or hypographs of f_α are Kuratowski-Painlevé convergent to epigraph or hypograph of f.

5.2.7 Definition. Let X be a Hausdorff space and let (f_α) be a net of functions from X to $\overline{\mathbb{R}}$. We say (f_α) is *Kuratowski-Painlevé epi-convergent* to a function f if $\text{epi}(f) = K - \lim \text{epi}(f_\alpha)$.
Similarly we say (f_α) is *Kuratowski-Painlevé hypo-convergent* to a function f if $\text{hypo}(f) = K - \lim \text{hypo}(f_\alpha)$.

As we will only consider the limits of epigraphs and hypographs with respect to Kuratowski-Painlevé convergence we will shorten Kuratowski-Painlevé epi-convergent and Kuratowski-Painlevé hypo-convergent to *epi-convergent* and *hypo-convergent* respectively. A function f is continuous if it is both upper and lower-semicontinuous. Yet it is easy to find a net of continuous functions which are epi or hypo-convergent to a function that is not continuous.

5.2.8 Example. Consider the sequence of functions (f_n) where $f_n : [0,1] \to \mathbb{R}$ is given by

$$f_n(x) = \begin{cases} 2nx \text{ if } 0 \leq x < \frac{1}{2n}. \\ 2 - 2nx \text{ if } \frac{1}{2n} \leq x < \frac{1}{n}. \\ 0 \text{ if } \frac{1}{n} \leq x \leq 1. \end{cases}$$



One can see that the sequence (f_n) is hypo-convergent to the function f for which $f(0) = 1$ and $f(x) = 0$ for all $0 < x \leq 1$ which is not a continuous function. Also note that evidently (f_n) converges pointwise to $f \equiv 0$, it follows that hypo-convergence is not stronger or weaker than pointwise convergence. One can construct a similar example for epi-convergence.

For first countable Hausdorff spaces there is a more intuitive way to verify hypo-convergence of a sequence of functions. Recall that a space is *first countable* if each point has a countable neighbourhood basis.

5.2.9 Theorem. *Let X be a first-countable Hausdorff space and let (f_n) be a sequence of upper-semicontinuous functions in X and let $f : X \to \overline{\mathbb{R}}$ be an upper-semicontinuous function. Then (f_n) is hypo-convergent to f if and only if for all $x \in X$ the following hold:*

(i) There exist (x_n) converging to x such that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

(ii) If (x_n) converges to x, then $\limsup_{n\to\infty} f_n(x_n) \leq f(x)$.

Proof. Suppose (f_n) is hypo-convergent to f and let $x \in X$. We will first show that (ii) holds.

Let (x_n) be a sequence converging to x. Suppose that $l = \limsup_{n\to\infty} f_n(x_n) > f(x) \geq -\infty$. If $f(x) = -\infty$, then let (x_{n_k}) be a subsequence such that $f_{n_k}(x_{n_k}) > l - 1 > -\infty$. It follows that $(x_{n_k}, l-1) \in \text{hypo}(f_{n_k})$ for all $k \in \mathbb{N}$, so $(x, l-1)$ is a cluster point of $(\text{hypo}(f_n))$. Since $(\text{hypo}(f_n))$ is Kuratowski-Painlevé convergent to $\text{hypo}(f)$ we have $(x, l-1) \in \text{hypo}(f)$ which is a contradiction. If $f(x) > -\infty$, then we can find some $\varepsilon > 0$ and a subsequence (f_{n_k}) such that for all $k \in \mathbb{N}$ we have $f_{n_k}(x_{n_k}) > f(x)+\varepsilon$. In particular this means that $(x_{n_k}, f(x)+\varepsilon) \in \text{hypo}(f_{n_k})$ for all $k \in \mathbb{N}$, so $(x, f(x)+\epsilon)$ is a cluster point of $(\text{hypo}(f_n))$. Since $(\text{hypo}(f_n))$ is Kuratowski-Painlevé convergent to $\text{hypo}(f)$ we have $(x, f(x)+\epsilon) \in \text{hypo}(f)$ which is a contradiction proving (ii).
To prove (i) let $\{U_n : n \in \mathbb{N}\}$ be a neighbourhood basis of $(x, f(x))$ with $U_n \supset U_{n+1}$ for all n. Since $(x, f(x))$ is a limit point of $(\text{hypo}(f_n))$ we can find a monotone increasing sequence (N_n) such that for all $m \geq N_n$ we have that $\text{hypo}(f_m) \cap U_n \neq \emptyset$. We can now find a sequence $((x_n, c_n))$ such that $(x_n, c_n) \in \text{hypo}(f_n) \cap U_m$ for all $N_m \leq n < N_{m+1}$. By construction $((x_n, c_n))$ converges to $(x, f(x))$. This proves (i) since it follows from (ii) that

$$f(x) = \lim_{n\to\infty} c_n = \liminf_{n\to\infty} c_n \leq \liminf_{n\to\infty} f_n(x_n) \leq \limsup_{n\to\infty} f_n(x_n) \leq f(x).$$

Now suppose that (i) and (ii) hold. By (i) we have for all x that there exists a sequence $((x_n, f(x_n))$ converging to $(x, f(x))$, so $(x, f(x)) \in \mathrm{Li}\ \mathrm{hypo}(f_n)$. Then for all $c < f(x)$ we find that $(x_n, f(x_n) - f(x) + c)$ converges to (x, c) hence $\mathrm{hypo}(f) \subset \mathrm{Li}\ \mathrm{hypo}(f_n)$.

Finally let $(x, c) \in X \times \mathbb{R}$ be a cluster point of $(\mathrm{hypo}(f_n))$ and suppose that $c > f(x)$. As X is first-countable there exists a subsequence $(\mathrm{hypo}(f_{n_k}))$ for which there exist $(x_{n_k}, c_{n_k}) \in \mathrm{hypo}(f_{n_k})$ such that (x_{n_k}) converges to x and (c_{n_k}) converges to c. Now we can define a sequence $((x_n, c_n))$ by picking $x_n = x$ and $c_n = f_n(x)$ for all $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$. Then by (ii) we find

$$\limsup_{n\to\infty} f_n(x_n) \leq f(x) < c = \lim_{k\to\infty} c_{n_k} \leq \limsup_{n\to\infty} c_n \leq \limsup_{n\to\infty} f_n(x_n)$$

which is a contradiction with (ii). □

A similar result can be obtained for convergence in the epigraph topology, see [7, Theorem 5.3.5]. Finally we will show the following result which is similar to [7, Theorem 5.3.6].

5.2.10 Proposition. *Let X be a compact Hausdorff space. If (f_α) is a hypo-convergent net of upper-semicontinuous functions on X to some function f, then* $\lim_\alpha \sup f_\alpha = \sup f$. *If (f_α) is an epi-convergent net of lower-semicontinuous function on X to some function f, then* $\lim_\alpha \inf f_\alpha = \inf f$.

Proof. First suppose that $\sup f > \liminf_\alpha \sup f_\alpha$ and let $x \in X$ be such that $f(x) > \liminf_\alpha \sup f_\alpha$. Let $\varepsilon = \frac{1}{2}(f(x) - \liminf_\alpha \sup f_\alpha)$ and consider the open set $U = X \times (f(x) - \varepsilon, f(x) + \varepsilon)$. We find that for all α' there is a $\beta \geq \alpha'$ such that $\sup f_\beta < \liminf_\alpha \sup f_\alpha + \varepsilon$. So $U \cap \mathrm{hypo}(f_\beta) = \emptyset$, hence $(x, f(x))$ is not a limit point of $(\mathrm{hypo}(f_\alpha))$, which is a contradiction as (f_α) is hypo-convergent to f.

Note that, as X is compact, we can find a net $(x_\alpha)_{\alpha \in I}$ such that $f_\alpha(x_\alpha) = \sup f_\alpha$. Also by compactness we can find a subnet $(x_\alpha)_{\alpha \in J}$ of $(x_\alpha)_{\alpha \in I}$ such that $(x_\alpha)_{\alpha \in J}$ converges to some $x \in X$ and $\lim_{\alpha \in J} \sup f_\alpha = \limsup_{\alpha \in I} f_\alpha$. Then for every open neighbourhood U of x in X and every open subset V of $\limsup_{\alpha \in I} \sup f_\alpha$ in $\overline{\mathbb{R}}$ we have that for every index $\alpha \in I$ there exists an $\beta \in J$ with $\beta \geq \alpha$ such that

$$\mathrm{hypo}(f_\beta) \cap (U \times V) \neq \emptyset.$$

Hence $(x, \lim_\alpha \sup f_\alpha)$ is a cluster point of $(\mathrm{hypo}(f_\alpha))$, and thus $\sup f \geq f(x) \geq \lim_\alpha \sup f_\alpha$. The proof for the second part of the proposition is similar. □

5.2.11 Lemma. *Let X be a Hausdorff space and let $(g_\alpha)_{\alpha\in J} : X \to \overline{\mathbb{R}}$ be a net of maps converging to $g : X \to \overline{\mathbb{R}}$ in the hypograph topology. If $h : X \to [a,b] \subset \mathbb{R}_{>0}$ is a continuous function, then $(\frac{g_\alpha}{h})_{\alpha\in J}$ converges to $\frac{g}{h}$ in the hypograph topology.*

Proof. Note that, since h is continuous and $0 < a \leq h(x) \leq b$ for all $x \in X$ we have that the map $\phi : X \times \overline{\mathbb{R}} \to X \times \overline{\mathbb{R}}$ given by $\phi((x,r)) = (x, \frac{r}{h(x)})$ and its inverse ϕ^{-1} given by $\phi^{-1}((x,r)) = (x, rh(x))$ are continuous. We can use this to show that (x,r) is a cluster or limit point of hypo(g_α) if and only if $(x, \frac{r}{h(x)})$ is a cluster or limit point of hypo$(\frac{g_\alpha}{h})$ respectively. Indeed let (x,r) be a cluster point of hypo(g_α). Let U be a neighbourhood of $(x, \frac{r}{h(x)})$, then $\phi^{-1}(U)$ is a neighbourhood of (x,r) and there exists a cofinal index set J' of J such that for all $\alpha \in J'$ we have hypo$(g_\alpha) \cap \phi^{-1}(U) \neq \emptyset$. Then for all $\alpha \in J'$ we have

$$\phi(\text{hypo}(g_\alpha) \cap \phi^{-1}(U)) = \text{hypo}(\frac{g_\alpha}{h}) \cap U \neq \emptyset,$$

so if (x,r) is a cluster point of hypo(g_α), then $(x, \frac{r}{h(x)})$ is a cluster point of hypo$(\frac{g_\alpha}{h})$. The rest of the assertion follows using similar arguments. □

Chapter 6

The horofunction boundary of infinite dimensional order-unit spaces

6.1 Busemann points

In this chapter we will examine the horofunction boundary of the Funk, reverse-Funk and Hilbert metric of infinite dimensional order-unit spaces. While there has been a lot of interest and research on the horofunction boundary of finite dimensional spaces, see for example [41, 63], the horofunction boundary of infinite dimensional order-unit spaces is still relatively unknown. Only recently Walsh classified the Busemann points of infinite dimensional order-unit spaces in [66]. In this section we will introduce these results following Walsh's proof and then expand on them to give a description of all horofunctions of infinite dimensional order-unit spaces. To start we will introduce Busemann points.
Busemann points were first introduced by Rieffel in [59]. These special horofunctions are known to be particularly useful in the study of isometric problems in metric spaces, see for instance [45, 66, 65].

6.1.1 Definition. A net (x_α) in a hemi-metric space (M,d) with base point $b \in M$ is *almost geodesic* if, for all $\varepsilon > 0$ there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have

$$d(b, x_{\alpha'}) \geq d(b, x_\alpha) + d(x_\alpha, x_{\alpha'}) - \varepsilon.$$

A horofunction $\xi \in \overline{i(M)} \setminus i(M)$ is called a *Busemann point* if there exists an almost geodesic net (x_α) in X such that $\xi = \lim_\alpha i(x_\alpha)$. The Busemann points can be viewed as a more "well-behaved" subclass of the horofunctions. Recall that if an order-unit space (X, X_+, u) is finite dimensional by Corollary 2.4.7 horofunctions are only generated by unbounded nets. This need not be the case for infinite dimensional order-unit spaces.

6.1.2 Example. Consider $X = \mathbb{R}\times\ell_2$ with closed cone $X_+ = \{(\lambda, x) \in \mathbb{R}\times\ell_2 : \lambda \geq \|x\|_2\}$. Note that X is an order-unit space with order-unit $e = (1, 0)$. Let $e_n \in \ell_2$ be the standard unit vectors, i.e., $e_n(n) = 1$ and $e_n(m) = 0$ for all $m \neq n$. Note that the sequence $((1, \frac{1}{2}e_n))$ is bounded with respect to Hilbert's metric, but one can show that $(i_H((1, \frac{1}{2}e_n)))$ converges to a horofunction. We will show this in detail in Section 7.2.

Restricting to Busemann point is a good way to exclude horofunctions generated by finite nets. Walsh showed in [66] that all Busemann points can be derived from unbounded almost geodesic nets. Recall that by Proposition 2.4.6 a net (x_α) in a metric space M does not converge to a horofunction if (x_α) converges to some $x \in M$.

6.1.3 Proposition. *Let (x_α) be a net in a complete metric space M. If (x_α) is almost geodesic and bounded, then (x_α) converges to some $x \in M$.*

Proof. Let b be a base point. The first step is to prove that $d(b, x_\alpha)$ converges to some $r \in \mathbb{R}$. To see this we define for an index A the supremum $r_A = \sup_{\alpha\geq A} d(b, x_\alpha)$ which exists as the net is bounded. Let $\varepsilon > 0$ and let A be an index such that for all $\alpha' \geq \alpha \geq A$ we have

$$d(b, x_{\alpha'}) \geq d(b, x_\alpha) + d(x_\alpha, x_{\alpha'}) - \varepsilon.$$

Let $\alpha_A \geq A$ be such that $0 \leq r_A - d(b, x_{\alpha_A}) < \varepsilon$. Then for all $\alpha' \geq \alpha_A$ we have

$$r_A \geq d(b, x_{\alpha'}) \geq d(b, x_{\alpha_A}) + d(x_{\alpha_A}, x_{\alpha'}) - \varepsilon \geq d(b, x_{\alpha_A}) - \varepsilon \geq r_A - 2\varepsilon.$$

So for all $\alpha', \alpha \geq \alpha_A$ we find that $|d(b, x_{\alpha'}) - d(b, x_\alpha)| \leq 2\varepsilon$. Hence $(d(b, x_\alpha))$ is a Cauchy net from which it follows that $\lim_\alpha d(b, x_\alpha) = r$ for some $r \in \mathbb{R}$.

Now let $\varepsilon > 0$ and let A be an index such that for all $\alpha' \geq \alpha \geq A$ we have $|r - d(b, x_\alpha)| < \varepsilon$ and

$$d(b, x_{\alpha'}) \geq d(b, x_\alpha) + d(x_\alpha, x_{\alpha'}) - \varepsilon.$$

It follows that

$$d(x_\alpha, x_{\alpha'}) \leq d(b, x_{\alpha'}) - d(b, x_\alpha) + \varepsilon < 3\varepsilon$$

Hence (x_α) is a Cauchy net. The proposition follows by completeness. □

6.1.4 Definition. A net of real-valued functions (f_α) on a set B is called *almost non-increasing* if, for any $\varepsilon > 0$, there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have $f_\alpha \geq f_{\alpha'} - \varepsilon$.

An almost geodesic net can be represented by a net of almost non-increasing functions using the natural embedding $i : M \to C(M)$, given by $i(x) = d(\cdot, x) - d(b, x)$.

6.1.5 Proposition. *Let M be a hemi-metric space and let $b \in M$ be a base point. A net (x_α) in M is almost geodesic if and only if $(i(x_\alpha)) = (d(\cdot, x_\alpha) - d(b, x_\alpha))$ is a net of almost non-increasing functions.*

Proof. Suppose (x_α) is an almost geodesic net in M and let $\varepsilon > 0$ be given. Then there exists some index A such that for all $\alpha' \geq \alpha \geq A$ we have

$$d(x_{\alpha'}, b) \geq d(b, x_\alpha) + d(x_\alpha, x_{\alpha'}) - \varepsilon.$$

Now for every $x \in M$ we find

$$d(x, x_\alpha) - d(b, x_\alpha) \geq d(x, x_\alpha) + d(x_\alpha, x_{\alpha'}) - d(b, x_{\alpha'}) - \varepsilon \geq d(x, x_{\alpha'}) - d(b, x_{\alpha'}) - \varepsilon$$

from which follows that $(i(x_\alpha))$ is an almost non-increasing net.
Now suppose $(i(x_\alpha))$ is an almost non-increasing net and let $\varepsilon > 0$ be given. Then there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have $i(x_\alpha) \geq i(x_{\alpha'}) - \varepsilon$, in particular we find

$$-d(b, x_\alpha) = i(x_\alpha)(x_\alpha) \geq i(x_{\alpha'})(x_\alpha) - \varepsilon = d(x_\alpha, x_{\alpha'}) - d(b, x_{\alpha'}) - \varepsilon$$

from which follows that (x_α) is an almost geodesic net. □

This representation can be useful to gain a better understanding on Busemann points. In our case we will use an extension of Dini's theorem, see [24].

6.1.6 Lemma. *Let (f_α) be a net of almost non-increasing functions on a Hausdorff space Y. Then (f_α) converges pointwise to some $f : Y \to \overline{\mathbb{R}}$. If for all α, f_α is upper-semicontinuous, then f is upper-semicontinuous. If furthermore Y is compact, then*

$$\limsup_\alpha \sup f_\alpha = \sup f.$$

Proof. Let $\varepsilon > 0$ and $x \in Y$. As (f_α) is almost non-increasing there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have $f_\alpha(x) \geq f_{\alpha'}(x) - \varepsilon$. From this it follows that

$$\liminf_\alpha f_\alpha(x) \geq \limsup_\alpha f_\alpha(x) - \varepsilon.$$

Letting ε tend to 0 gives us that f_α converges pointwise to some $f : Y \to \overline{\mathbb{R}}$.
Now assume that f_α is upper-semicontinuous for all α. Let $x \in Y$, let (x_β) be a net in

Y converging to x and let $\varepsilon > 0$. By Proposition 5.2.2 we have $f_\alpha(x) \geq \limsup_\beta f_\alpha(x_\beta)$ for all α. Furthermore as f_α converges pointwise to f we find that $f(x) + \varepsilon \geq f_\alpha(x)$ for α large enough and, as (f_α) is almost non-increasing, we have $f_\alpha \geq f - \varepsilon$ for α large enough. Combining this gives

$$f(x) \geq f_\alpha(x) - \varepsilon \geq \limsup_\beta f_\alpha(x_\beta) - \varepsilon \geq \limsup_\beta f(x_\beta) - 2\varepsilon.$$

By Proposition 5.2.2 f is upper-semicontinuous.
Finally suppose moreover that Y is compact. Then, as Y is compact, by Corollary 5.2.4 we have for all α that there exists an $x_\alpha \in Y$ such that $f_\alpha(x_\alpha) = \sup f_\alpha$. Furthermore as Y is compact there is a subnet such that x_α converges to some $x \in Y$ and taking a further subnet we may also assume that $\lim_\alpha f_\alpha(x_\alpha)$ exists as $\overline{\mathbb{R}}$ is compact. Let $\varepsilon > 0$, since f is the pointwise limit of a non-increasing net of functions (f_α) we can find an α such that $f(x) \geq f_\alpha(x) - \varepsilon$ and for all $\alpha' \geq \alpha$ we have that $f_\alpha \geq f_{\alpha'} - \varepsilon$. Finally, since f_α is upper-semicontinuous, by Proposition 5.2.2 we find

$$\begin{aligned}\sup f \geq f(x) \geq f_\alpha(x) - \varepsilon &\geq \limsup_{\alpha'} f_\alpha(x_{\alpha'}) - \varepsilon \geq \limsup_{\alpha'} f_{\alpha'}(x_{\alpha'}) - 2\varepsilon \\ &= \lim_{\alpha'} f_{\alpha'}(x_{\alpha'}) - 2\varepsilon = \lim_{\alpha'} (\sup f_{\alpha'}) - 2\varepsilon \geq \sup \lim_{\alpha'} f_{\alpha'} - 2\varepsilon = \sup f - 2\varepsilon.\end{aligned}$$

Letting ε tend to 0 gives that $\lim_\alpha \sup f_\alpha = \sup f$. □

6.2 A classification of Busemann points

To classify the Busemann points of order-unit spaces we recall some basic terminology.

6.2.1 Definition. Let (X, X_+, u) be an order-unit space and let X' be its dual space. We call the set of positive functionals $X'_+ = \{\varphi \in X' : \varphi(x) \geq 0 \text{ for all } x \in X_+\}$ the *dual cone* and we call $S(X) = \{\varphi \in X' : \varphi(x) \geq 0 \text{ for all } x \in X_+ \text{ and } \varphi(u) = 1\}$ the *state space* of X. We call the elements of $S(X)$ *states*. We call the extreme points of $S(X)$ the *pure states* and denote the set of pure states by $E(X)$.

Note that the state space $S(X)$ is convex, closed with respect to the weak*-topology and contained in the dual unit ball. The last follows from the fact that for all $\varphi \in S(X)$ we have that $\|\varphi\| = \sup\{|\varphi(x)| : \|x\|_u \leq 1\} = \varphi(u) = 1$, as φ is a positive functional and if $\|x\|_u \leq 1$, then $-u \leq x \leq u$. Therefore by the Banach-Alaoglu Theorem $S(X)$ is weak*-compact. Also note that for every positive non-zero functional $\varphi \in X'$ it holds that $\varphi/\varphi(u) \in S(X)$.

6.2.2 Definition. Let X be a vector space and let $C \subset X$ be a convex subset. We call a function $f : C \to \overline{\mathbb{R}}$ *affine* if for all $x, y \in C$ and all $\lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y).$$

We will denote the set of all restriction of continuous affine functions on X to C by $A(C, X)$.

In this section we will show the following result due to Walsh in [66]. Recall that for a vector space X and $x \in X$ the map $f_x : X' \to \mathbb{R}$ given by $f_x(\varphi) = \varphi(x)$ for all $\varphi \in X'$ is called the *evaluation map* of x.

6.2.3 Theorem. *[66, Proposition 8.8] Let (X, X_+, u) be a complete order-unit space. The Busemann points of Hilbert's geometry on X_+ are precisely the functions of the following form:*

$$\xi_H(x) = \log \sup_{\varphi \in E(X)} \frac{\varphi(x)}{g(\varphi)} + \log \sup_{\varphi \in E(X)} \frac{h(\varphi)}{\varphi(x)} \qquad (x \in X_+)$$

where g is a weak lower-semicontinuous non-negative affine function on $S(X)$ with infimum 1 and h is a weak* upper-semicontinuous non-negative affine function on $S(X)$ with supremum 1, g and h are not evaluation maps of some $x \in X_+$, and for all $\varphi \in S(X)$, either $h(\varphi) = 0$ or $g(\varphi) = \infty$.*

The proof of this Theorem is long and technically involved. It might be insightful to consider the result through a simple example.

6.2.4 Example. Consider the order-unit space $(\mathbb{R}^n, \mathbb{R}^n_+, u)$ where $\mathbb{R}^n_+ = \mathbb{R}^n_{\geq 0}$ and u is the constant one vector. Note that $\|\cdot\|_u = \|\cdot\|_\infty$. The dual space of $\mathbb{R}^n$ can be identified with $\mathbb{R}^n$ itself, where a linear functional $y \in \mathbb{R}^n$ is given by coordinatewise multiplication, i.e.

$$y(x) = \sum_{i=1}^n y(i)x(i) \qquad (x \in \mathbb{R}^n).$$

The state space is given by

$$S(\mathbb{R}^n) = \{x \in \mathbb{R}^n : x(i) \geq 0 \text{ for all } 1 \leq i \leq n \text{ and } \sum_{i=1}^n x(i) = 1\}$$

and the pure states are the standard unit vectors e_i. Consider that for all $x, y \in (\mathbb{R}^n_+)^\circ$ we have

$$\begin{aligned} M(x/y) &= \inf\{\beta > 0 : x \leq \beta y\} \\ &= \inf\{\beta > 0 : x(i) \leq \beta y(i) \text{ for all } 1 \leq i \leq n\} \\ &= \max_i \frac{x(i)}{y(i)} = \sup_{e_i \in E(\mathbb{R}^n)} \frac{xe_i}{ye_i}. \end{aligned}$$

In [41] Lemmens, Lins, Nussbaum and Wortel show in Remark 5.7 that the horofunction of the Funk, reverse-Funk and Hilbert geometry are given by the functions of the form

$$\xi_F(x) = \log \max_i x(i)z(i),$$

$$\xi_R(x) = \log \max_i x(i)^{-1}y(i),$$

$$\xi_H(x) = \log \max_i x(i)z(i) + \log \max_i x(i)^{-1}y(i),$$

where $y, z \in \partial\mathbb{R}^n_+$ such that $\|y\|_\infty = \|z\|_\infty = 1$ and $y(i)z(i) = 0$ for all i.

We will show how we can view the horofunctions above in the form described in Theorem 6.2.3. Let $y, z \in \partial\mathbb{R}^n_+$ such that $\|y\|_\infty = \|z\|_\infty = 1$ and $y(i)z(i) = 0$ for all i. Consider the sequence (h_n) given by

$$h_n(i) = \begin{cases} y(i) & \text{if } y(i) > 0. \\ \frac{1}{z(i)n^2} & \text{if } z(i) > 0. \\ \frac{1}{n} & \text{else.} \end{cases}$$

For all n we define $g_n = 1/h_n$. Note that $\lim_{n\to\infty} h_n = y$ and $\lim_{n\to\infty} g_n/\|g_n\|_\infty = z$. Furthermore recall from section 4.2 that the inverse map $(x \mapsto x^{-1})$ is gauge-reversing. Hence for all $x \in (\mathbb{R}^n_+)^\circ$ we have

$$\begin{aligned} i_F(h_n)(x) &= d_F(x, h_n) - d_F(u, h_n) = \log(M(x/h_n)) - \log(M(u/h_n)) \\ &= \log(M(h_n^{-1}/x^{-1})) - \log(M(h_n^{-1}/u^{-1})) = \log(M(g_n/x^{-1})) - \log(M(g_n/u)) \\ &= \log\left(\sup_{e_i \in E(\mathbb{R}^n)} \frac{g_n(e_i)}{x^{-1}(e_i)}\right) - \log(\|g_n\|_u) = \log\left(\sup_{e_i \in E(\mathbb{R}^n)} \frac{x(e_i)g_n(e_i)}{\|g_n\|_u}\right) \end{aligned}$$

Define $z^{-1} \in \overline{\mathbb{R}}^n$ by $z^{-1}(i) = z(i)^{-1}$ if $z(i) > 0$ and $z^{-1}(i) = \infty$ if $z(i) = 0$. Note that $z^{-1} : S(X) \to \overline{\mathbb{R}}$ given by coordinatewise multiplication can be viewed as a weak* lower-semicontinuous affine function on $S(\mathbb{R}^n)$, with $\inf z^{-1} = 1$. Furthermore note that $\lim_{n\to\infty} \|g_n\|_\infty/g_n = z^{-1}$. By Lemma 6.1.6 we find

$$\begin{aligned} \lim_{n\to\infty} i_F(h_n)(x) &= \log\left(\lim_{n\to\infty} \sup_{e_i \in E(\mathbb{R}^n)} \frac{g_n(e_i)x(e_i)}{\|g_n\|_u}\right) \\ &= \log\left(\sup_{e_i \in E(\mathbb{R}^n)} \frac{x(e_i)}{z^{-1}(e_i)}\right) = \log\left(\max_i x(i)z(i)\right) \end{aligned}$$

Similarly for all $x \in (\mathbb{R}^n_+)^\circ$ we have

$$\begin{aligned} i_R(h_n)(x) &= d_R(x, h_n) - d_R(u, h_n) = \log(M(h_n/x)) - \log(M(h_n/u)) \\ &= \log(M(h_n/x)) - \log(M(h_n/u)) = \log\left(\sup_{e_i \in E(\mathbb{R}^n)} \frac{h_n(e_i)}{x(e_i)}\right). \end{aligned}$$

By Lemma 6.1.6 we find

$$\lim_{n\to\infty} i_R(h_n)(x) = \log\left(\lim_{n\to\infty}\sup_{e_i\in E(\mathbb{R}^n)}\frac{h_n(e_i)}{x(e_i)}\right)$$
$$= \log\left(\sup_{e_i\in E(\mathbb{R}^n)}\frac{y(e_i)}{x(e_i)}\right) = \log\left(\max_i x^{-1}(i)y(i)\right).$$

Note that y, when viewed as a function on $S(\mathbb{R}^n)$, is continuous, affine and $\sup y = 1$.

Since $y(i)z(i) = 0$ for all i we have that $z(i) = 0$ if $y(i) > 0$, it follows that for all $x \in S(\mathbb{R}^n)$ either $y(x) = \sum_{i=1}^n x(i)y(i) = 0$, i.e. for all i we have that if $y(i) > 0$, then $x_i = 0$, or $z^{-1}(x) = \sum_{i=1}^n x(i)z^{-1}(i) = \infty$, i.e. for some i we have $x_i > 0$ and $z(i) = 0$.

As in the example, we will prove Theorem 6.2.3 by first proving the result for the Busemann points of the Funk and reverse-Funk geometry. To do this we will need some preliminary results.

6.2.5 Proposition. *If (X, X_+, u) is an order-unit space, then for $x \in X_+$ and $y \in X_+^\circ$ we have*

$$M(x/y) = \{\inf \beta \geq 0 : x \leq \beta y\} = \sup_{\varphi\in S(X)}\frac{\varphi(x)}{\varphi(y)} = \sup_{\varphi\in E(X)}\frac{\varphi(x)}{\varphi(y)}.$$

Proof. We will first prove that for all $x, y \in X$ we have that $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in X'_+$. As X'_+ is the set of positive functionals the implication from left to right is trivial. Now let $x, y \in X$ such that $y - x \notin X_+$, then by Hahn-Banach separation theorem there is a linear functional $\varphi \in X'$ and $s \in \mathbb{R}$ such that for all $z \in X_+$ we have $\varphi(y - x) < s \leq \varphi(z)$. Note that $s \leq 0$ as $0 \in X_+$. Now suppose there is a $z \in X_+$ such that $\varphi(z) < 0$. As $\frac{2sz}{\varphi(z)} \in X_+$ we have $s < \varphi(\frac{2sz}{\varphi(z)}) = 2s < s$ which is a contradiction. So φ is a positive functional such that $\varphi(y - x) < 0$ proving our claim.

As non-zero positive linear functionals are strictly positive on the interior of the cone, we can normalise any $\varphi \in X'_+ \setminus \{0\}$ by $\varphi/\varphi(u) \in S(X)$. From this we easily find that for all $x, y \in X$ we have that $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in S(X)$.
Finally we will show that for all $x, y \in X$ we have that $\varphi(x) \leq \varphi(y)$ for all $\varphi \in S(X)$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in E(X)$. Note that the implication from left to right is trivial. Let $x, y \in X$, let $\psi \in S(X)$ and suppose that $\varphi(x) \leq \varphi(y)$ for all $\varphi \in E(X)$. By the Krein-Milman theorem the convex hull of the pure states is weak* dense in $S(X)$ so we can find $\varphi_{\alpha,j} \in E(X)$ and $\lambda_{\alpha,j} \in [0, 1]$ such that $\sum_j \lambda_{\alpha,j} = 1$ for all α and $(\sum_j \lambda_{\alpha,j}\varphi_{\alpha,j})$ is a net weak* convergent to ψ. We find that

$$\psi(x) = \lim_\alpha \sum_j \lambda_{\alpha,j}\varphi_{\alpha,j}(x) \leq \lim_\alpha \sum_j \lambda_{\alpha,j}\varphi_{\alpha,j}(y) = \psi(y).$$

It follows that

$$M(x/y) = \{\inf \beta \geq 0 : x \leq \beta y\} = \inf\{\beta \geq 0 : \frac{\varphi(x)}{\varphi(y)} \leq \beta \text{ for all } \varphi \in S(X)\}$$
$$= \sup_{\varphi \in S(X)} \frac{\varphi(x)}{\varphi(y)} = \sup_{\varphi \in E(X)} \frac{\varphi(x)}{\varphi(y)}.$$

□

Let (X, X_+, u) be an order-unit space. If we take $z \in X_+^\circ$ and normalise to $x = z/M(u/z)$, then by Proposition 6.2.5 we find

$$i_F(z) = i_F(x) = d_F(\cdot, x) - d_F(u, x) = \log(M(\cdot, x)) = \sup_{\varphi \in S(X)} \frac{\varphi(\cdot)}{\varphi(x)} = \sup_{\varphi \in S(X)} \frac{\varphi(\cdot)}{f_x(\varphi)}$$

where $f_x : X' \to \mathbb{R}, \varphi \mapsto \varphi(x)$ is the evaluation map of x on X'. And similarly for the reverse-Funk metric if we take $x = z/M(z/u)$ we find

$$i_R(z) = i_R(x) = \sup_{\varphi \in S(X)} \frac{f_x(\varphi)}{\varphi(\cdot)}.$$

In order to prove Theorem 6.2.3 we now need to prove two things for the Funk part. First that

$$\lim_\alpha \sup_{\varphi \in S(X)} \frac{\varphi(\cdot)}{f_{x_\alpha}(\varphi)} = \sup_{\varphi \in S(X)} \frac{\varphi(\cdot)}{f(\varphi)}$$

where f is the limit of (f_α) with respect to hypoconvergence. Second we need that for every f which is weak* lower-semicontinuous non-negative affine function of $S(X)$ with infimum 1 there exists an almost-geodesic net (x_α) such that f_{x_α} converges to f.
We need to prove similar properties for the reverse-Funk part, but as the proofs are very similar we will only show the proof for the Funk part. One can find a detailed proof for the reverse-Funk part in [66]. To prove the Funk part we will use the following results from [1].

6.2.6 Proposition. *[1, Proposition I.1.2] Let $K \subset X$ be a compact convex set of a locally convex Hausdorff space X. If $f : K \to (-\infty, \infty]$ is a lower-semicontinuous affine function, then for all $x \in X$*

$$f(x) = \sup\{a(x) : a \in A(K, X), a < f\}.$$

The proof of the following result can be found in the proof of Corollary I.1.4 in [1]

6.2.7 Lemma. *Let $K \subset X$ be a compact convex set of a locally convex Hausdorff space X. If $f : K \to (-\infty, \infty]$ is a lower-semicontinuous affine function, then the set*

$$\{a \in A(K, X) : a < f\}$$

is a directed set with respect to $\leq$.

We can now prove a slightly altered version of Corollary I.1.4 in [1].

6.2.8 Corollary. *Let $K \subset X$ be a compact convex set of a locally convex Hausdorff space X. If $f : K \to (-\infty, \infty]$ is a lower-semicontinuous affine function, then there exists an increasing net in $A(K, X)$ converging pointwise to f. Furthermore, if $f > 0$, then there exists an increasing net of strictly positive functions in $A(K, X)$.*

Proof. For the first part note that by Lemma 6.2.7 $(a)_{a<f}$ is a net in $A(K, X)$. Note that $(a)_{a<f}$ is increasing, and by Proposition 6.2.6 it converges pointwise to f. If $f > 0$, note that $\inf_K f > 0$, as K is compact. Hence the subnet $(a)_{\frac{1}{2}\inf f \leq a < f}$ consists entirely of positive functions. □

We will also use the following result, which is part of Theorem II.1.8 in [1], due to Kadison.

6.2.9 Proposition. *If (X, X_+, u) is a complete order-unit space, then $\Phi : X \to C(S(X))$, given by*

$$\Phi(x)(\varphi) = \varphi(x) \qquad (x \in X, \varphi \in S(X)),$$

satisifies $\Phi(X) = A(S(X), X')$.

This result shows that for every $f \in A(S(X), X')$ there is a unique $x \in X$ such that f is the evaluation map of x. With some effort we can make Kadison's result more precise.

6.2.10 Corollary. *If (X, X_+, u) be a complete order-unit space and $f \in A(S(X), X')$, then f is strictly positive on the state space $S(X)$ if and only if f is the evaluation map of some $x \in X_+^\circ$.*

Proof. We will first show the implication from right to left. Let $u \in X$ be an order-unit such that $\varphi(u) = 1$ for all $\varphi \in S(X)$. For all $x \in X_+^\circ$ we can find a $\lambda_x > 0$ such that $x \geq \lambda_x u$. For the evaluation map f_x of x we find for all $\varphi \in S(X)$ that

$$f_x(\varphi) = \varphi(x) \geq \lambda_x \varphi(u) = \lambda_x > 0.$$

To show the implication from left to right, suppose that f is strictly positive on $S(X)$. By Proposition 6.2.9 there exists a unique $x \in X$ such that $f = f_x$, the evaluation map of x. Suppose that $x \in X \setminus X_+^\circ$, note that there exists some $\lambda > 0$ such that $z = x + \lambda u \in \partial X_+$ as u is an order-unit. We will construct a linear functional $\psi \in S(X)$ such that $\psi(z) = 0$. Note that the line $\{\alpha z : \alpha \in \mathbb{R}\}$ is a closed convex subset of X and X_+° an open convex subset of X. By the Hahn-Banach separation theorem there exists a linear functional $\varphi \in X'$ and some $s \in \mathbb{R}$ such that for all $\alpha \in \mathbb{R}$ and all $y \in X_+^\circ$ we have

$$\varphi(\alpha z) \leq s < \varphi(y).$$

As $\alpha\varphi(z) \leq s$ for all $\alpha \in \mathbb{R}$ we find that $\varphi(z) = 0$. Furthermore, since $\varphi(\alpha y) > s$ for all $\alpha > 0$ and all $y \in X_+^\circ$ we find that $s = 0$. Hence $\psi = \varphi/\varphi(u) \in S(X)$ and

$$\psi(x) = \psi(z) - \lambda\psi(u) = -\lambda < 0.$$

□

Finally we need the following two results due to Walsh in [66], for the first we will follow the proof of Walsh, for the second result we will use a variation on the proof of the first result.

6.2.11 Lemma. *Let $K \subset X$ be a compact convex set of a locally convex Hausdorff space X and let $f_1, f_2 : K \to (0, \infty]$ be upper-semicontinuous affine functions. If for all positive $g \in A(K, X)$ we have $\sup_K \frac{f_1}{g} \leq \sup_K \frac{f_2}{g}$, then $f_1 \leq f_2$.*

Proof. Let y be an extreme point of K and let $\mathbf{1}_{\{y\}}$ be the indicator function of $\{y\}$, so $f(x) = 1$ if $x = y$ and $f(x) = 0$ for all $x \in K \setminus \{y\}$. We also define $\frac{1}{\mathbf{1}_{\{y\}}}$ as the function for which $\frac{1}{\mathbf{1}_{\{y\}}}(y) = 1$ and $\frac{1}{\mathbf{1}_{\{y\}}}(x) = \infty$ otherwise. This is an affine lower-semicontinuous function on K, so by Corollary 6.2.8 we can find an increasing net (g_α) of positive affine continuous functions converging to $\frac{1}{\mathbf{1}_{\{y\}}}$. We can now apply Lemma 6.1.6 to find

$$f_1(y) = \sup_K f_1\mathbf{1}_{\{y\}} = \lim_\alpha \sup_K \frac{f_1}{g_\alpha} \leq \lim_\alpha \sup_K \frac{f_2}{g_\alpha} = \sup_K f_2\mathbf{1}_{\{y\}} = f_2(y).$$

So $f_1(y) \leq f_2(y)$ for any extreme point y of K. By Choquet's Theorem [58, Page 14] we have that $f_1 \leq f_2$. □

6.2.12 Lemma. *Let $K \subset X$ be a compact convex set of a locally convex Hausdorff space X and let $f_1, f_2 : K \to (0, \infty]$ be lower-semicontinuous affine functions. If for all positive $g \in A(K, X)$ we have $\sup_K \frac{g}{f_1} \leq \sup_K \frac{g}{f_2}$, then $f_1 \geq f_2$.*

Proof. As f_2 is a lower-semicontinuous affine function, by Corollary 6.2.8 we can find an increasing net (g_α) of positive continuous affine functions converging pointwise to f_2. Applying Lemma 6.1.6 gives us

$$\sup_K \frac{f_2}{f_1} = \lim_\alpha \sup_K \frac{g_\alpha}{f_1} \leq \lim_\alpha \sup_K \frac{g_\alpha}{f_2} = 1,$$

and thus $f_2 \leq f_1$. □

We can now classify the Busemann points of the Funk and reverse-Funk geometry.

6.2.13 Theorem. *[66, Theorem 7.1] Let (X, X_+, u) be a complete order-unit space. The Busemann points of the Funk geometry on X_+ are precisely the functions of the following form:*

$$\xi_F(x) = \log \sup_{\varphi \in E(X)} \frac{\varphi(x)}{g(\varphi)} \qquad (x \in X_+)$$

where g is a weak lower-semicontinuous non-negative affine function on $S(X)$ with infimum 1 which is not the evaluation map of some $x \in X_+$.*

Proof. Let ξ_F be of the above form. By Corollary 6.2.8 there exists an increasing net (f_α) of strictly positive continuous affine functions on $S(X)$ converging pointwise to g. Using Corollary 6.2.10 we can find a corresponding net (x_α) in X_+° such that $f_\alpha = f_{x_\alpha}|_{S(X)}$ for all α, where $f_{x_\alpha}|_{S(X)}$ is the restriction of the evaluation map of x_α to $S(X)$. Note that for all $x \in X_+^\circ$ the net $(\frac{f_x}{f_\alpha})$ is decreasing and converges pointwise to $(\frac{f_x}{g})$. By Lemma 6.1.6 and Proposition 6.2.5 we find

$$\lim_\alpha d_F(x, x_\alpha) = \lim_\alpha \log \sup_{\varphi \in S(X)} \frac{\varphi(x)}{f_\alpha(\varphi)} = \log \sup_{\varphi \in S(X)} \frac{\varphi(x)}{g(\varphi)} = \log \sup_{\varphi \in E(X)} \frac{\varphi(x)}{g(\varphi)}.$$

In particular $\lim_\alpha d_F(u, x_\alpha) = \log \sup_{\varphi \in S(X)} \frac{1}{g(\varphi)} = 0$, as $\inf g = 1$, So $i_F(x_\alpha)$ converges to ξ and is almost non-increasing, which by Proposition 6.1.5 shows that ξ is a Busemann point.

Now let (x_α) be an almost geodesic net in X_+° such that $i_F(x_\alpha)$ converges to a Busemann point, by scaling if necessary we may assume that for all α we have $d_F(u, x_\alpha) = 0$. For all α we denote $f_\alpha = f_{x_\alpha}$. By Proposition 6.1.5 $(i_F(x_\alpha))$ is a non-increasing net, so for all $\varepsilon > 0$ there is an index A such that for all $\alpha' \geq \alpha \geq A$ we have for all $x \in X_+^\circ$ that

$$\sup_{\varphi \in S(X)} \frac{\varphi(x)}{f_{\alpha'}(\varphi)} \leq e^\varepsilon \sup_{\varphi \in S(X)} \frac{\varphi(x)}{f_\alpha(\varphi)}.$$

By Lemma 6.2.12 it follows that $f_{\alpha'} \geq e^{-\varepsilon} f_\alpha$. Hence $(-\log(f_\alpha))$ is a almost non-increasing net of weak* continuous functions, so by applying Lemma 6.1.6 we find $(-\log(f_\alpha))$ converges pointwise to some weak* upper-semicontinuous function $-\log(g)$. It follows that g is weak* lower-semicontinuous. Furthermore by applying Lemma 6.1.6 again we find for all $x \in X_+^\circ$ that

$$\xi(x) = \lim_\alpha \log \sup_{\varphi \in E(X)} \frac{\varphi(x)}{f_\alpha(\varphi)} = \log \sup_{\varphi \in E(X)} \frac{\varphi(x)}{g(\varphi)},$$

which is the required form. Finally note that $0 = \xi(u) = \log \sup_{\varphi \in S(X)} \frac{1}{g(\varphi)}$ gives that $\inf g = 1$ and, since ξ is a horofunction, g cannot be the restriction of the evaluation map of some $x \in X_+^\circ$ to $S(X)$. ∎

6.2.14 Theorem. *[66, Theorem 6.3] Let (X, X_+, u) be a complete order-unit space with state space $S(X)$. The Busemann points of the reverse-Funk geometry on X_+ are precisely the functions of the following form:*

$$\xi_R(x) = \log \sup_{\varphi \in E(X)} \frac{h(\varphi)}{\varphi(x)} \qquad (x \in X_+)$$

where h is a weak upper-semicontinuous non-negative affine function on $S(X)$ with supremum 1 which is not the evaluation map of some $x \in X_+$.*

The proof of this result is similar to that of Theorem 6.2.13 and can be found in full detail in [66].

Finally, to prove Theorem 6.2.3 we will first show that a Busemann point of the Hilbert geometry is the sum of a Busemann point of the Funk geometry and a Busemann point of the reverse-Funk geometry. Then we will show which particular combinations of Busemann points of the Funk and reverse-Funk geometry give rise to a Busemann point of the Hilbert geometry.

6.2.15 Proposition. *Let (X, X_+, u) be a complete order-unit space. A net (x_α) in (X_+°) is almost geodesic in the Hilbert geometry if and only if it is almost geodesic in the Funk and reverse-Funk geometry*

Proof. For all α, α' indices we define

$$R(\alpha, \alpha') = d_R(u, x_\alpha) + d_R(x_\alpha, x_{\alpha'}) - d_R(u, x_{\alpha'})$$
$$F(\alpha, \alpha') = d_F(u, x_\alpha) + d_F(x_\alpha, x_{\alpha'}) - d_F(u, x_{\alpha'})$$
$$H(\alpha, \alpha') = d_H(u, x_\alpha) + d_H(x_\alpha, x_{\alpha'}) - d_H(u, x_{\alpha'})$$

By the triangle inequality R, F and H are non-negative and clearly $H = F+R$. Now let $\varepsilon > 0$ and suppose there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have $H(\alpha, \alpha') < \varepsilon$. Since R and F are non-negative, it follows that $R(\alpha, \alpha') < \varepsilon$ and $F(\alpha, \alpha') < \varepsilon$. In the same way if there exists an index A such that for all $\alpha' \geq \alpha \geq A$ we have $R(\alpha, \alpha') < \frac{\varepsilon}{2}$ and $F(\alpha, \alpha') < \frac{\varepsilon}{2}$, then $H(\alpha, \alpha') < \varepsilon$. The result follows by Proposition 6.1.5. $\square$

So a horofunction of the Hilbert geometry ξ_H is a Busemann point if and only if there is a net (x_α) which is almost geodesic and converges to some ξ_F and ξ_R in the Funk and reverse-Funk geometry respectively such that $\xi_H = \xi_F + \xi_R$. One can show such a net exists for any g a non-negative affine weak* lower-semicontinuous function and h a non-negative affine weak* upper-semicontinuous function on $S(X)$ with $\sup g = \inf h = 1$ and

$g(\varphi) = 0$ if $h(\varphi)$ is finite. To see this, for g and h satisfying the conditions above consider the following set

$$\mathcal{C} = \{(f, f') \in A(S(X), X') \times A(S(X), X') : f = \lambda f' \text{ for some } \lambda \in (0, 1] \text{ and } g < f \leq f' < h\}. \tag{6.2.1}$$

Recall that by Proposition 6.2.9 f and f' are evaluation maps for some $x \in X$ and as f and f' are strictly positive by Corollary 6.2.10 we have $x \in X_+^\circ$.

6.2.16 Example. Again it might be useful to view $\mathcal{C}$ in the context of Example 6.2.4. Consider the order-unit space $(\mathbb{R}^n, \mathbb{R}^n_+, u)$ where $\mathbb{R}^n_+ = \mathbb{R}^n_{\geq 0}$ and u is the constant one vector. Let $y, z \in \partial\mathbb{R}^n_+$ with $\|y\|_\infty = \|z\|_\infty = 1$. Taking $g = y$ and $h = z^{-1}$ as defined in Example 6.2.4 we find that

$$\mathcal{C} = \{(\lambda x, x) \in X' \times X' : \lambda \in (0, 1] \text{ and for all } i \text{ we have } y(i) < \lambda x(i) \leq x(i) < z^{-1}(i)\}.$$

We can define a partial order on $\mathcal{C}$ the following way: $(f_1, f_1') \preceq (f_2, f_2')$ if and only if $f_1 \geq f_2$ and $f_1' \leq f_2'$. We will show that with respect to this partial order $\mathcal{C}$ is a directed set and the net $((f, f'))_{f \in \mathcal{C}}$ converges to (g, h).

6.2.17 Lemma. *Let (X, X_+, u) be a complete order-unit space. If g is a non-negative affine weak* lower-semicontinuous function and h is a non-negative affine weak* upper-semicontinuous function on $S(X)$ with $\sup g = \inf h = 1$ and $g(\varphi) = 0$ if $h(\varphi)$ is finite and $\mathcal{C}$ is as in equation (6.2.1), then $g = \inf\{f : (f, f') \in \mathcal{C}\}$ and $h = \sup\{f' : (f, f') \in \mathcal{C}\}$.*

Proof. Consider $X' \times \mathbb{R}$ equipped with the product topology where X' is equipped with the weak*-topology and $\mathbb{R}$ is equipped with the Euclidean topology. Note that by Proposition 5.2.5 we have that

$$\operatorname{hypo}(g)_{\geq 0} = \{(\varphi, \lambda) \in S(X) \times \mathbb{R} : g(\varphi) \geq \lambda \geq 0\}$$

and

$$\operatorname{epi}(h) = \{(\varphi, \lambda) \in S(X) \times \mathbb{R} : h(\varphi) \leq \lambda\}$$

are closed convex sets. Furthermore $\operatorname{hypo}(g)_{\geq 0}$ is compact, as $\operatorname{hypo}(g)_{\geq 0} \subset S(X) \times [0, 1]$ and $S(X) \times [0, 1]$ is compact. Recall that $S(X)$ is weak*-compact.

Let $\psi \in S(X)$ and let $\lambda < h(\psi)$. The convex hull $H = \operatorname{co}(\operatorname{hypo}(g)_{\geq 0} \cup \{(\psi, \lambda)\})$ is a convex compact set which is disjoint with $\operatorname{epi}(h)$, as h is affine. So by the Hahn-Banach separation theorem [33, Theorem 1.2.10] there exists a continuous linear functional σ on

$X' \times \mathbb{R}$, strongly separating H and epi(h), i.e. we can find $c, s, t \in \mathbb{R}$ such that for all $x \in H$ and $y \in \text{epi}(h)$ we have $\sigma(x) < s < c < t < \sigma(y)$. As for all $\varphi \in S(X)$ it holds that $\sigma(\varphi, g(\varphi)) < \sigma(\varphi, h(\varphi))$ we have $\sigma((0,1)) > 0$. Furthermore, the hyperplane $\sigma(\varphi, r) = c$ strongly separates H and epi(h). Then $\hat{\sigma}$, given by

$$\hat{\sigma}(\varphi) = \frac{c - \sigma((\varphi, 0))}{\sigma((0,1))}$$

for all $\varphi \in A$ where A is the affine hull of $S(X)$, is an affine weak*-continuous function. As for every θ in the affine hull of $S(X)$ we have $\theta(u) = 1$, it is easy to check that we can extend $\hat{\sigma}$ to a linear functional σ' on the linear span of A. Using the Hahn-Banach extension theorem [33, Theorem 1.2.14] we can extend σ' to a linear functional $f \in X''$. Note that $(f|_{S(X)}, f|_{S(X)}) \in \mathcal{C}$. Since $\sigma(\varphi, f(\varphi)) = c$ for all $\varphi \in A$, we find that the hyperplane $\{(\varphi, f(\varphi)) : \varphi \in X'\}$ strongly separates H and epi(h). Thus $f(\psi) > \lambda$ and as λ can be chosen arbitrarily close to, but not larger than $h(\psi)$, we find that $\sup\{f'(x) : (f, f') \in \mathcal{C}\} = h(x)$.

The proof for g goes in the same way, using that hypo$(g)_{\geq 0}$ is a compact convex set and for all (ψ, λ) with $g(\psi) < \lambda$ the convex hull $H = \text{co}\,(\text{epi}(h)_{\geq 0} \cup \{(\varphi, \lambda)\})$ is convex and closed and hypo$(g)_{\geq 0}$ and H are disjoint as g is affine. ☐

6.2.18 Lemma. *Let h be a weak* lower-semicontinuous affine function on $S(X)$ bounded from below. If $\{f_1, \ldots, f_n\}$ is a finite set of weak* upper-semicontinuous affine functions on $S(X)$ such that $f_i < h$ for all i, then there exists an $f \in A(S(X), X')$ such that for all i we have $f_i < f < h$.*

Proof. The proof of this lemma is similar to that of Lemma 6.2.17, but this time we may not assume that the f_i are bounded below. For all $1 \leq i \leq n$ we define

$$\text{hypo}(f_i)_h = \{(\varphi, \lambda) \in S(X) \times \mathbb{R} : f_i(\varphi) \geq \lambda \geq \inf(h) - 1\} \cup \{(\varphi, \inf(h) - 1) : \varphi \in S(X)\}$$

Note that hypo$(f_i)_h$ is a weak* compact set disjoint with epi(h). Furthermore, as h is affine it follows that the convex hull

$$H = \text{co}(\bigcup_{i=1}^{n} \text{hypo}(f_i)_{\inf(h)-1})$$

is convex, compact and disjoint with epi(h). We now find an $f \in A(S(X); X')$ strongly separating H and epi(h) in the same way as Lemma 6.2.17. ☐

6.2.19 Corollary. *Let g be a weak* upper-semicontinuous affine function on $S(X)$ bounded from below. If $\{f_1, \dots, f_n\}$ be a finite set of weak* lower-semicontinuous affine functions on $S(X)$ such that $f_i > g$ for all i, then there exists an $f \in A(S(X); X')$ such that for all i we have $f_i > f > g$.*

Proof. Apply Lemma 6.2.18 to $-g$. □

6.2.20 Lemma. *The set $\mathcal{C}$, as defined in equation (6.2.1), is a directed set with respect to $\preceq$.*

Proof. Let $(f_1, f_1'), (f_2, f_2') \in \mathcal{C}$. By Lemma 6.2.18 there exists an $f' \in A(S(X), X')$ such that $f_1', f_2' < f' < h$. Now consider the weak* lower-semicontinuous function $\min(f_1, f_2) - g$ which due to compactness attains it infimum on $S(X)$. As $f_1, f_2 > g$ this gives that $\inf \min(f_1, f_2) - g > 0$ and so we can find an $\varepsilon \in (0, 1)$ such that $\inf \min(f_1, f_2) - g > \varepsilon > 0$. Let $\delta > 0$ such that $0 < \delta f'(\varphi) < \varepsilon$ for all $\varphi \in S(X)$. It follows that $a = g + \delta f'$ is a non-negative weak* upper-semicontinuous function on $S(X)$ such that

$$g, \delta f_1', \delta f_2' < a < f_1, f_2.$$

As for all $\varphi \in S(X)$ for which $h(\varphi)$ is finite we have that $g(\varphi) = 0$, it follows that $a < \delta h$. Using Corollary 6.2.19 we find a real-valued continuous linear functional b such that $g < a < b < f_1, f_2, \delta h$. Furthermore note that $f_1', f_2' < \frac{a}{\delta} < \frac{b}{\delta} < h$ hence

$$(f_1, f_1'), (f_2, f_2') \preceq (b, \frac{b}{\delta})$$

from which follows that $\mathcal{C}$ is directed. □

Now we can finally prove Theorem 6.2.3. Recall that we already have classified the Busemann points of the Funk and reverse-Funk geometry in Theorem 6.2.13 and Theorem 6.2.14. Moreover we have shown that a horofunction ξ_H of the Hilbert geometry is a Busemann point if it is the sum of ξ_F and ξ_R, Busemann points of the Funk and reverse-Funk geometry respectively, such that there exists a net (x_α) in X_+° which is almost geodesic with respect to the Funk and reverse-Funk metric and $\lim_\alpha i_F = \xi_F$ and $\lim_\alpha i_R = \xi_R$ by Proposition 6.2.15.

Proof Theorem 6.2.3. Let g, h satisfy the conditions in the Theorem. We can now define the set $\mathcal{C}$ as in equation (6.2.1). We can now find a net $((g_\alpha, h_\alpha))_{\alpha \in \mathcal{C}}$ in $\mathcal{C}$ by taking

$(g_\alpha, h_\alpha) = \alpha$. By Corollary 6.2.10 there is a net (x_α) in X_+° and $\lambda_\alpha \in (0,1]$ such that g_α and h_α are the evaluation maps of $\lambda_\alpha x_\alpha$ and x_α respectively.

Fix $x \in X_+^\circ$ and let f_x be the evaluation map of x. By definition $(\frac{g_\alpha}{f_x})$ and $(\frac{f_x}{h_\alpha})$ are decreasing nets of functions on $S(X)$, so by Lemma 6.1.6 both $(\frac{g_\alpha}{f_x})$ and $(\frac{f_x}{h_\alpha})$ converge pointwise to some function and by Lemma 6.2.17 it then follows that $(\frac{g_\alpha}{f_x})$ and $(\frac{f_x}{h_\alpha})$ converge pointwise to $(\frac{g}{f_x})$ and $(\frac{f_x}{h})$ respectively.
Furthermore by Lemma 6.2.17 it follows that

$$\lim_\alpha \sup_{\varphi \in S(X)} \frac{g_\alpha}{f_x} = \sup_{\varphi \in S(X)} \frac{g}{f_x} \text{ and } \lim_\alpha \sup_{\varphi \in S(X)} \frac{f_x}{h_\alpha} = \sup_{\varphi \in S(X)} \frac{f_x}{h}$$

If $x = u$, then, as for all $\varphi \in S(X)$ it holds that $\varphi(u) = \inf h = \sup g = 1$, we have

$$\lim_\alpha d_F(u, \lambda_\alpha x_\alpha) = \lim_\alpha d_R(u, x_\alpha) = 0.$$

So for all $x \in X_+^\circ$ we have

$$\lim_\alpha i_F(x_\alpha)(x) = \lim_\alpha d_F(x, x_\alpha) - d_F(u, x_\alpha) = \log \sup_{\varphi \in S(X)} \frac{g}{f_x}$$

and

$$\lim_\alpha i_R(x_\alpha)(x) = \lim_\alpha d_R(x, x_\alpha) - d_R(u, x_\alpha) = \log \sup_{\varphi \in S(X)} \frac{f_x}{h}.$$

Note that as $(d_F(u, \lambda_\alpha x_\alpha))$ and $(d_R(u, x_\alpha))$ converge to 0, and $(d_F(x, \lambda_\alpha x_\alpha))$ and $(d_R(x, x_\alpha))$ are decreasing, $(i_F(x_\alpha)(x))$ and $(i_R(x_\alpha)(x))$ are almost non-increasing nets. Therefore, by Proposition 6.1.5, (x_α) is almost geodesic with respect to the Funk and reverse-Funk geometry. By Proposition 6.2.15 ξ_H is a Busemann point.

Finally suppose ξ_H is a Hilbert Busemann point. Then there exists an almost geodesic net (x_α) in X_+° with respect to Hilbert's metric. By Theorem 6.2.13 and Theorem 6.2.14 there exists a weak* lower-semicontinuous non-negative affine function g on $S(X)$ with infimum 1 and a weak* upper-semicontinuous non-negative affine function h on $S(X)$ with supremum 1 such that g and h are not evaluation map of some $x \in X_+$ and

$$\lim_\alpha i_F(x_\alpha)(x) = \log \sup_{\varphi \in S(X)} \frac{\varphi(x)}{g(\varphi)} \qquad (x \in X_+^\circ)$$

and

$$\lim_\alpha i_R(x_\alpha)(x) = \log \sup_{\varphi \in S(X)} \frac{h(\varphi)}{\varphi(x)} \qquad (x \in X_+^\circ).$$

Recall that for all $\varphi \in S(X)$ we have

$$g(\varphi) = \lim_\alpha \varphi(M(u/x_\alpha)x_\alpha) \text{ and } h(\varphi) = \lim_\alpha \varphi\left(\frac{x_\alpha}{M(x_\alpha/u)}\right).$$

so it follows that

$$\log\frac{g(\varphi)}{h(\varphi)} = \lim_\alpha \log\frac{\varphi(x_\alpha)M(u/x_\alpha)}{\varphi(x_\alpha)/M(x_\alpha/u)} = \lim_\alpha \log(M(u/x_\alpha)M(x_\alpha/u)) = \lim_\alpha d_H(u,x_\alpha).$$

By Proposition 6.1.3 we know that (x_α) is not bounded, and thus $\lim_\alpha d_H(u,x_\alpha)$ diverges to infinity. Hence for all $\varphi\in S(X)$ either $g(\varphi)=\infty$ or $h(\varphi)=0$. □

6.3 A description of horofunctions

We will now show a similar result for general Funk and reverse-Funk horofunctions of order-unit spaces. For this we will first consider the horofunction boundary of $C(K)_+$, the positive continuous functions on a compact Hausdorff space K, which was described by Walsh in [66]. It is well-known that $C(K)$ equipped with this cone is an order-unit space with order-unit u the constant 1 function. The dual space of $C(K)$ is known to be $\mathrm{rca}(K)$, the space of all regular finite real-valued Borel measures on K, see Example 1.10.6 in [49]. One can easily see that the positive linear functionals $\mathrm{rca}(K)_+$ are all regular finite positive real-valued Borel measures on K. Then the state space is $S(C(K))=\{\mu\in \mathrm{rca}(K)_+ : \mu(K)=1\}$, the probability measures, and the pure states are the Dirac masses, i.e. measures δ_x for some $x\in K$ such that for all $f\in C(K)$ we have $\delta_x(f)=\int_K f d\delta_x = f(x)$, hence $E(X)=\{\delta_x\in\mathrm{rca}(K) : x\in K\}$. Using this we can state the following result.

6.3.1 Theorem. *[66, Proposition 9.1] Let K be a compact Haussdorf space. The horofunctions of the reverse-Funk Geometry of $C(K)$ are precisely the functions of the following form:*

$$\xi_R(f) = \log\sup_{x\in K}\frac{g(x)}{f(x)} \qquad (f\in C(K)_+^\circ)$$

where $g: K\to[0,1]$ is an upper-semicontinuous function with supremum 1 which is not both positive and continuous. Furthermore every horofunction is a Busemann point.

Before we can prove this theorem we will need a few preliminary results. First we need a variation of the Lesbegue's monotone convergence theorem by Baranov and Woracek, see [4, Proposition 2.13].

6.3.2 Proposition. *Let K be a compact Hausdorff space and let μ be a regular Borel measure such that μ is positive, complete and $\mu(K)<\infty$. If I is a directed set and $(f_i)_{i\in I}$ is a monotone non-increasing net of upper-semicontinuous function, where $f_i: K\to[0,\infty]$ for all $i\in I$, then if we set $f(x)=\inf_{i\in I} f_i(x)$ for all $x\in K$ we find*

$$\int_K f\ d\mu = \inf_{i\in I}\int_K f_i\ d\mu.$$

6.3.3 Lemma. *If K is a compact Hausdorff space and $g : K \to \mathbb{R}$ is an upper-semicontinuous function, then $g = \inf\{f \in C(K) : f \geq g\}$.*

Proof. Let $f \in C(K)$ be such that $f \geq g$ and let $x \in K$ such that $\varepsilon = f(x) - g(x) > 0$. As f is continuous and g is upper-semicontinuous there exists an open neighbourhood U of x such that for all $y \in U$ we have $|f(x) - f(y)| < \frac{\varepsilon}{4}$ and $g(y) \leq g(x) + \frac{\varepsilon}{4}$. As K is normal, Urysohn's lemma gives us there exists a continuous function $h : K \to [0,1]$, such that $h(x) = 1$ and $h(y) = 0$ for all $y \in K \setminus U$. Define $\hat{f} : K \to \mathbb{R}$ by

$$\hat{f}(y) = f(y) - \frac{\varepsilon}{2}h(y) \qquad (y \in K).$$

Note that for all $y \in K \setminus U$ we have $\hat{f}(y) = f(y) \geq g(y)$ and for all $y \in U$ we have

$$\hat{f}(y) = f(y) - \frac{\varepsilon}{2}h(y) \geq f(x) - \frac{\varepsilon}{4} - \frac{\varepsilon}{2}h(y) \geq \varepsilon + g(x) - \frac{3\varepsilon}{4} \geq g(y).$$

So $f \geq \hat{f} \geq g$. Since $\hat{f}$ is continuous we find that $g = \inf\{f \in C(K) : f \geq g\}$. □

Finally we need the following result by Walsh which was provided in private communication with the author.

6.3.4 Lemma. *Let K be a compact Hausdorff space. If $g : K \to \mathbb{R}$ is an upper-semicontinuous function, then $\hat{g} : S(C(K)) \to \mathbb{R}$ given by*

$$\hat{g}(\mu) = \int_K g \, d\mu \qquad \mu \in S(C(K))$$

is a weak upper-semicontinuous function.*

Proof. Let (μ_α) be a net in $S(C(K))$ converging to μ in the weak* topology and let $f \in C(K)$ be such that $f \geq g$. Then

$$\limsup_\alpha \hat{g}(\mu_\alpha) \leq \limsup_\alpha \mu_\alpha(f) = \mu(f).$$

Note that the set $I = \{f \in C(K) : f \geq g\}$ is a downward directed set. By Lemma 6.3.3 $(f)_{f \in I}$ is a net converging pointwise to g. Applying Proposition 6.3.2 gives us

$$\inf_{f \in I} \mu(f) = \mu(g).$$

Combining these two results we find

$$\limsup_\alpha \hat{g}(\mu_\alpha) \leq \inf_{f \in I} \mu(f) = \mu(g).$$

By Proposition 5.2.2 $\hat{g}$ is weak* upper-semicontinuous. □

We can now prove Theorem 6.3.1

Proof Theorem 6.3.1. Let $g : K \to [0,1]$ be an upper-semicontinuous function with supremum 1 which is not both positive and continuous. We define

$$\hat{g}(\mu) = \int_K g d\mu \qquad \mu \in S(C(K)).$$

Note that $\hat{g}$ is an affine function on $S(C(K))$ with supremum 1. By Lemma 6.3.4 $\hat{g}$ is weak* upper-semicontinuous. Suppose that $\hat{g}$ is both positive and continuous. Note that if $\hat{g}$ is positive, then for all $x \in K$ we have $0 < \hat{g}(\delta_x) = g(x)$, hence g is positive. It follows by the definition of g that g is not continuous. In particular there is an $x \in K$ and an $\varepsilon > 0$ such that for all open neighbourhoods U of x there exists a $y \in U$ such that $|g(y) - g(x)| > \varepsilon$. Consider δ_x and recall that a weak* neighbourhood basis of δ_x is given by sets of the form

$$V = \{\mu \in C(K)' : |\delta_x(g_i) - \mu(g_i)| < \hat{\varepsilon} \text{ for all } 1 \leq i \leq n\}$$

where $n \in \mathbb{N}$, $\hat{\varepsilon} > 0$ and $g_1, ..., g_n \in C(K)$. Note that, as $g_1, ..., g_n$ are continuous functions for all $\hat{\varepsilon}$ there is an open neighbourhood U of x such that for all $y \in U$ we have $|g_i(x) - g_i(y)| < \hat{\varepsilon}$ for all $1 \leq i \leq n$. Hence for every set V in the neighbourhood base of δ_x we can find a neighbourhood U of x such that $\delta_y \in V$ for all $y \in U$. Since there exist a $y \in U$ such that $|g(y) - g(x)| > \varepsilon$ we find that $|\hat{g}(\delta_x) - \hat{g}(\delta_y)| > \varepsilon$. Thus $\hat{g}$ is not weak* continuous, which is a contradiction.
So $\hat{g}$ is a weak* upper-semicontinuous function on $S(C(K))$ with supremum 1 which is not both positive and continuous.

By Proposition 6.2.5 and the fact the the pure states are the Dirac masses we find for all $f \in C(K)_+^\circ$ that

$$\log \sup_{x \in K} \frac{g(x)}{f(x)} = \log \sup_{\delta_x \in E(C(K)))} \frac{\hat{g}(\delta_x)}{\delta_x(f)} = \log \sup_{\mu \in S(C(K))} \frac{\hat{g}(\mu)}{\mu(f)}$$

which by Theorem 6.2.14 is a Busemann point with respect to the reverse-Funk geometry.

Now let (g_α) be a net in $C(K)_+^\circ$ converging to a horofunction ξ_R with respect to the reverse-Funk geometry. By scaling we may assume that $\sup g_\alpha = 1$. Note that by Proposition 5.2.5 $(\text{hypo}(g_\alpha))$ is a net of closed subsets of $K \times \mathbb{R}$, so by Mrowka's Theorem 5.1.10 it has a Kuratowski-Painlevé convergent subsequence. So (g_α) has a hypo-convergent subsequence converging to a upper-semicontinuous function g. By Proposition 5.2.10 we have that $\sup g = \lim_\alpha \sup g_\alpha = 1$. Thus for all $f \in C(K)_+$ we have that $\frac{g_\alpha}{f}$ and $\frac{g}{f}$ are upper-semicontinuous functions and by Lemma 5.2.11 $(\frac{g_\alpha}{f})$ is hypo-convergent to $\frac{g}{f}$. Therefore by Proposition 5.2.10 we find

$$\sup_{x\in K}\frac{g(x)}{f(x)}=\lim_\alpha\sup_{x\in K}\frac{g_\alpha(x)}{f(x)}.$$

Hence

$$\xi_R=\log\sup_{x\in K}\frac{g(x)}{f(x)}\qquad (f\in C(K)_+^\circ).$$

As ξ_R is a horofunction it follows that g is not both strictly positive and continuous as required. □

6.3.5 Theorem. *[66, Proposition 9.2] Let K be a compact Haussdorf space. The horofunctions of the Funk geometry of $C(K)$ are precisely the functions of the following form:*

$$\xi_F(f)=\log\sup_{x\in K}\frac{f(x)}{h(x)}\qquad (f\in C(K)_+^\circ)$$

where $h:K\to[1,\infty]$ is a lower-semicontinuous function with infimum 1 which is not both finite and continuous. Furthermore every horofunction is a Busemann point.

Proof. Let ξ_F be a horofunction of the Funk geometry and let (h_α) be a net in $C(K)_+^\circ$ such that $\lim_\alpha i_F(h_\alpha)=\xi_F$ and $\inf_{x\in K}h_\alpha(x)=1$. Note that if $1/h_\alpha$ converges to some $g\in C(K)_+^\circ$ with respect to the reverse-Funk metric, then h_α converges to $1/g\in C(K)_+^\circ$ in the Funk metric, so by taking a further subnet if required we may assume that $\lim_\alpha i_R(1/h_\alpha)$ converges to some horofunction ξ_R in the reverse-Funk geometry. By Theorem 6.3.1 there is an upper-semicontinuous function g on K with supremum 1 which is not both strictly positive and continuous such that

$$\xi_R(f)=\log\sup_{x\in K}\frac{g(x)}{f(x)}\qquad (f\in C(K)_+^\circ).$$

Note that $h=1/g$ is a lower-semicontinuous function on K with infimum 1 which is not both finite and continuous. Furthermore for all $f\in C_+^\circ$ we have

$$\lim_\alpha i_F(h_\alpha)(f)=\lim_\alpha i_R(1/h_\alpha)(1/f)=\log\sup_{x\in K}\frac{g(x)}{f(x)^{-1}}=\log\sup_{x\in K}\frac{f(x)}{h(x)}.$$

One can use a similar method to show that for every lower-semicontinuous function h on K with infimum 1 which is not both finite and continuous

$$\xi_F(f)=\log\sup_{x\in K}\frac{f(x)}{h(x)}\qquad (f\in C(K)_+^\circ)$$

is a Funk horofunction. □

Finite dimensional order-unit spaces for which all Funk and reverse-Funk horofunctions are Busemann points are not rare. In fact, Walsh has shown in [63] that for finite dimensional order-unit space all horofunctions of the reverse-Funk geometry are Busemann points and all horofunctions of the Funk geometry are Busemann points if and only if the pure states are weak*-closed. In Chapter 7 we will show that this is no longer the case in infinite dimension where even "well-behaved" spaces contain non-Busemann horofunctions. In the remainder of this section we will give a description of the horofunctions of a general order-unit space using a result due to Kalauch, Lemmens and van Gaans in [34]. Recall that in a partially order vector space X a linear subspace $Y \subset X$ is called *order dense* if for all $x \in X$ we have that $x = \inf\{y \in Y : y \geq x\}$.

6.3.6 Theorem. *[34, Theorem 10] If (X, X_+, u) is an order-unit space and $\psi : X \to C(\overline{E(X)}^*)$ is a map from X to the weak* closure of $E(X)$ given by $\psi(x) = f_x$ for $x \in X$, then $\psi(X)$ is order dense in $C(\overline{E(X)}^*)$.*

Note that ψ is injective and gauge-preserving. We will use this to show that horofunctions in X are horofunctions in $\psi(X)$.

6.3.7 Theorem. *Let (X, X_+, u) be an order-unit space. If ξ_R is a horofunction of the reverse-Funk geometry, then there exists an upper-semicontinuous function $g : \overline{E(X)}^* \to [0, 1]$ with supremum 1 which is not both positive and continuous such that*

$$\xi_R(x) := \log \sup_{y \in E(X)} \frac{g(y)}{y(x)}, \qquad (x \in X_+),$$

Furthermore g has an affine extension to $S(X)$ if and only if $\xi(x)$ is a reverse-Funk Busemann point.

Proof. Let (x_α) be a net in X_+° such that $(i_R(x_\alpha))$ converges to a horofunction ξ_R and let ψ be as in Theorem 6.3.6. By scaling if necessary we may assume that $d_R(u, x_\alpha) = 0$. We will first prove that $(i_R(\psi(x_\alpha)))$ converges to a horofunction. For this we need to show that

$$d_R(f, \psi(x_\alpha)) = \log(M(\psi(x_\alpha)/f))$$

converges for all $f \in C(\overline{E(X)}^*)_+^\circ$.

Let $f \in C(\overline{E(X)}^*)_+^\circ$, then by Theorem 6.3.6 we have $f = \inf\{\psi(b) : b \in X \text{ and } \psi(b) \geq f\}$. Note that $\psi(b) \geq f$ for some $b \in X$ implies $\varphi(b) > 0$ for all $\varphi \in \overline{E(X)}^*$. By the proof of Proposition 6.2.5 this implies that $b \in X_+^\circ$. Since ψ is Gauge preserving we know that

$$\beta_{b,\alpha} := M(x_\alpha/b) = M(\psi(x_\alpha)/\psi(b)).$$

We define $\beta_\alpha^* = \sup_{\psi(b)\geq f} \beta_{b,\alpha}$ and claim that $\beta_\alpha^* = M(\psi(x_\alpha)/f)$. Clearly

$$\psi(x_\alpha) \leq \inf_{\psi(b)\geq f} \beta_{b,\alpha}\psi(b) \leq \beta_\alpha^* \inf_{\psi(b)\geq f} \psi(b) = \beta_\alpha^* f,$$

so $M(\psi(x_\alpha)/f) \leq \beta_\alpha^*$. Now suppose there is some $\hat{\beta}_\alpha < \beta_\alpha^*$ such that $M(\psi(x_\alpha)/f) \leq \hat{\beta}_\alpha$.Let (b_n) be a sequence in X_+° such that $\psi(b_n) \geq f$ and

$$\lim_{n\to\infty} \beta_{b_n,\alpha} = \lim_{n\to\infty} M(x_\alpha/b_n) = \beta_\alpha^*.$$

As $\overline{E(X)}^*$ is compact, by Proposition 6.2.5 we have that for every n there exists a $\phi_n \in \overline{E(X)}^*$ such that

$$\beta_{b_n,\alpha} = \sup_{\varphi\in\overline{E(X)}^*} \frac{\varphi(x_\alpha)}{\varphi(b_n)} = \frac{\phi_n(x_\alpha)}{\phi_n(b_n)}$$

Let $\varepsilon = \beta_\alpha^* - \hat{\beta}_\alpha$. For n large enough we find

$$\psi(x_\alpha)(\phi_n) = \beta_{b_n,\alpha}\psi(b_n)(\phi_n) > (\beta_\alpha^* - \varepsilon)\psi(b_n)(\phi_n) = \hat{\beta}_\alpha\psi(b_n)(\phi_n) \geq \hat{\beta}_\alpha f(\phi_n).$$

Hence $\psi(x_\alpha) \not\leq \hat{\beta}_\alpha f$, which is a contradiction. So we conclude that $M(\psi(x_\alpha)/f) = \beta_\alpha^*$.

Now all we have to prove is that

$$\lim_\alpha \exp(d_R(f, \psi(x_\alpha))) = \lim_\alpha M(\psi(x_\alpha)/f) = \lim_\alpha \beta_\alpha^* = \lim_\alpha \sup_{\psi(b)\geq f} \beta_{b,\alpha}$$

exists. Let $(y_\alpha)_{\alpha\in A}$ be a net in $\overline{E(X)}^*$ such that $\sup_{y\in\overline{E(X)}^*} \frac{\psi(x_\alpha)(y)}{f(y)} = \frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)}$. Such a net exist as $\overline{E(X)}^*$ is compact. Also by compactness there exists a subnet $(y_\alpha)_{\alpha\in B}$ of $(y_\alpha)_{\alpha\in A}$ such that $\lim_{\alpha\in B} y_\alpha = \overline{y} \in \overline{E(X)}^*$ and

$$\lim_{\alpha\in B} \frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)} = \limsup_{\alpha\in A} \frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)}.$$

Finally let $(b_\beta)_{\beta\in C}$ be a net in X_+° such that $\psi(b_\beta) \geq f$ and $\lim_{\beta\in C} \psi(b_\beta)(\overline{y}) = f(\overline{y})$. As $\psi(x_\alpha)$ is continuous and bounded and f is continuous and $\inf f > 0$, we find that for all $\varepsilon > 0$ that for $\alpha \in B$ large enough we have $\frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)} \leq \frac{\psi(x_\alpha)(\overline{y})}{f(\overline{y})} + \varepsilon$. It follows that for all $\varepsilon > 0$ we have

$$\begin{aligned}
\sup_{\psi(b)\geq f} \exp \xi_R(b) &= \sup_{\psi(b)\geq f} \lim_{\alpha\in A} \beta_{b,\alpha} = \sup_{\psi(b)\geq f} \liminf_{\alpha\in A} \beta_{b,\alpha} \\
&\leq \liminf_{\alpha\in A} \sup_{\psi(b)\geq f} \beta_{b,\alpha} = \liminf_{\alpha\in A} M(\psi(x_\alpha)/f) \leq \limsup_{\alpha\in A} M(\psi(x_\alpha)/f) \\
&= \limsup_{\alpha\in A} \sup_{y\in\overline{E(X)}^*} \frac{\psi(x_\alpha)(y)}{f(y)} = \limsup_{\alpha\in A} \frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)} \\
&= \lim_{\alpha\in B} \frac{\psi(x_\alpha)(y_\alpha)}{f(y_\alpha)} \leq \lim_{\alpha\in B} \frac{\psi(x_\alpha)(\overline{y})}{f(\overline{y})} + \varepsilon \\
&= \lim_{\beta\in C} \lim_{\alpha\in B} \frac{\psi(x_\alpha)(\overline{y})}{\psi(b_\beta)(\overline{y})} \frac{\psi(b_\beta)(\overline{y})}{f(\overline{y})} + \varepsilon = \lim_{\beta\in C} \lim_{\alpha\in B} \frac{\psi(x_\alpha)(\overline{y})}{\psi(b_\beta)(\overline{y})} + \varepsilon \\
&\leq \sup_{\psi(b)\geq f} \lim_{\alpha\in B} \sup_{y\in\overline{E(X)}^*} \frac{\psi(x_\alpha)(y)}{\psi(b)(y)} + \varepsilon = \sup_{\psi(b)\geq f} \lim_{\alpha\in B} \beta_{b,\alpha} + \varepsilon \\
&= \sup_{\psi(b)\geq f} \exp \xi_R(b) + \varepsilon
\end{aligned}$$

By letting ε tend to 0 we find that $\lim_{\alpha\in A} \sup_{y\in\overline{E(X)}^*} \frac{\psi(x_\alpha)(y)}{f(y)} = \sup_{\psi(b)\geq f} \xi_R(b)$, hence $(i_R(\psi(x_\alpha)))$ converges to a horofunction of the reverse-Funk geometry.

By Theorem 6.3.1, for some subnet of $(\psi(x_\alpha))$ and for all $a \in X_+$ we know that

$$\begin{aligned}
\lim_\alpha d(a, x_\alpha) - d(u, x_\alpha) &= \lim_\alpha d(\psi(a), \psi(x_\alpha)) - d(u, \psi(x_\alpha)) \\
&= \log \sup_{y\in\overline{E(X)}^*} \frac{g(y)}{\psi(a)(y)} = \log \sup_{y\in E(X)} \frac{g(y)}{y(a)}
\end{aligned}$$

where $g : \overline{E(X)}^* \to [0,1]$ is an upper-semicontinuous non-negative function with supremum 1. Since we know the entire net converges we find that

$$\xi_R(a) = \log \sup_{y\in\overline{E(X)}^*} \frac{g(y)}{y(a)}.$$

Finally by Theorem 6.2.14 we find that $\xi_R(a)$ is Busemann if and only if g is affine on $S(X)$. □

We can achieve a similar result for the Funk geometry.

6.3.8 Theorem. *Let (X, X_+, u) be an order-unit space. If ξ_F is a horofunction of the reverse-Funk geometry, then there exists an lower-semicontinuous function $h : \overline{E(X)}^* \to [1\infty]$ with infimum 1 which is not both bounded and continuous such that*

$$\xi(x) := \log \sup_{y\in E(X)} \frac{\langle y, x\rangle}{h(y)}, \qquad (x \in X_+),$$

Furthermore h has an affine extension to $S(X)$ if and only if $\xi(x)$ is a Funk Busemann point.

Proof. Let ψ be as in Theorem 6.3.6. The proof of this theorem is similar to the proof of Theorem 6.3.7, using the fact that for any $f \in \psi(X)$ we have

$$f = -(-f) = -\inf\{\psi(b) \in \psi(X) : \psi(b) \geq -f\} = \sup\{\psi(b) \in \psi(X) : \psi(b) \leq f\}.$$

□

It should be noted this is not a classification for the horofunctions of the Funk and reverse-Funk Geometry, but only a description, i.e. not every function of the form given in Theorem 6.3.7 or Theorem 6.3.8 is a horofunction in the reverse-Funk or Funk geometry respectively. This also makes it harder to give a meaningful description for horofunctions in the Hilbert geometry for general order-unit spaces.

For the remainder of this thesis we will work on two things, first we will give an answer to the question posed in [66] by Cormac Walsh whether there are order-unit spaces with non-Busemann horofunctions in the reverse-Funk geometry, then we will classify the horofunctions of the Funk, reverse-Funk and Hilbert geometry of spin-factors and more generally JH-algebras.

Chapter 7

Non-Busemann horofunctions

7.1 Non-Busemann horofunctions in the reverse-Funk geometry

We will now give a number of examples of spaces with non-Busemann horofunctions in the reverse-Funk geometry. The first space we will consider is $B(H)_{sa}$, the self-adjoint bounded linear maps on a Hilbert space H. Recall that a *Banach algebra* X is a complete normed vector space which is an associative algebra such that for all $x, y \in X$ we have $\|xy\| \leq \|x\|\|y\|$.

7.1.1 Definition. Let X be a complex Banach algebra, we call a map $* : X \to X, x \mapsto x^*$ an *involution* if for all $a, b \in \mathbb{C}$ and all $x, y \in X$ we have

(i) $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$,

(ii) $(xy)^* = y^*x^*$,

(iii) and $(x^*)^* = x$.

We call a complex Banach Algebra X with unit element e and involution map $*$ a C^**-algebra* if for all $x \in X$ we have $\|x^*x\| = \|x\|^2$. We call an element $x \in X$ *self-adjoint* if $x = x^*$ and we denote $X_{sa} = \{x \in X : x^* = x\}$ to be the set of self-adjoint elements. We call a set $A \subset X$ *self-adjoint* if for all $x \in A$ we have $x^* \in A$. For a C^*-algebra X with unit e we call the set $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ has no multiplicative inverse}\}$ the spectrum of x. X has a natural cone, consisting of all self adjoint element with positive spectrum

$$X_+ = \{x \in X : x^* = x \text{ and } \sigma(x) \subset \mathbb{R}_{\geq 0}\}.$$

The following result for the cone of C^*-algebras is well known.

7.1.2 Proposition (Theorem 4.2.6, [33]). *Let X be a C^*-algebra with unit and let $x \in X$. Then the following are equivalent* *(i)* $x \in X_+$.

(ii) *There exists a* $y \in X_+$ *such that* $x = y^2$.

(iii) *There exists a* $z \in X$ *such that* $x = zz^*$.

7.1.3 Example. Let $\ell^\infty(\mathbb{C})$ be the set of bounded complex sequences with coordinatewise multiplication and involution map $*$ sending a sequence $(x_n) \in \ell^\infty(\mathbb{C})$ to its complex conjugate $(\overline{x_n})$. Note that the multiplicative identity is the constant 1 sequence and it is easy to see that the standard positive cone

$$\begin{aligned}\ell^\infty_+ &= \{(x_n) \in \ell^\infty(\mathbb{C}) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\} \\ &= \{(x_n) \in \ell^\infty(\mathbb{C}) : (x_n)^* = (x_n) \text{ and } \sigma((x_n)) \subset \mathbb{R}_+\}.\end{aligned}$$

7.1.4 Example. Let H be a Hilbert space and let $B(H)$ be the set of bounded linear operators on X. $B(H)$ is a C^*-algebra when equipped with the composition as multiplication and the adjoint as the involution map $*$. Note that the identity operator I is the multiplicative identity and the cone is given by the self-adjoint operators with positive spectrum, note that the concepts of the spectrum of an operator and the spectrum of an element of a C^*-algebra coincide. By Proposition 7.1.2 we have

$$B(H)_+ = \{A \in B(H)_{sa} : \sigma(A) \subset [0, \infty)\} = \{A \in B(H)_{sa} : \langle Ax, x\rangle \geq 0 \text{ for all } x \in H\}.$$

Let X be a C^*-algebra with unit e. Any element $x \in X$ can be written as $x = a + ib$ where a and b are self-adjoint by taking $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$. Note that for all $0 \neq x \in X_+$ we can define $z = x + ix$. For all $\lambda > 0$ we have that $\lambda x - z = (\lambda - 1)x + ix$ which is not self-adjoint, so $\lambda x \not\geq z$. It follows that a C^*-algebra is not an order-unit space, though one can easily check that (X_{sa}, X_+, e) is a real order-unit space.

Though X is not an order-unit space one can still introduce the concept of the state space. We call a linear functional $\varphi \in X'$ a *state* if φ is positive and $\varphi(e) = 1$, as usual we denote $S(X)$ to be the set of all states of X and we call $S(X)$ the *state space* of X. We call the extreme points of $S(X)$ the *pure states* and we denote $E(X)$ to be the set of all pure states of X.

This definition may appear confusing with our former definition of states and the state space on order-unit spaces, but the state space of the C^*-algebra X can be identified with the state space of the order-unit space (X_{sa}, X_+, e). To see this note that any linear functional $\varphi_0 \in (X_{sa})'$ can be uniquely extended to a linear functional $\varphi \in X'$ by

$$\varphi(x) = \frac{1}{2}(\varphi_0(x + x^*) - i\varphi_0(ix - ix^*)) \qquad (x \in X),$$

so the map $\Phi : X_{sa} \to X'$ given by $\Phi(\varphi(0)) = \varphi$ is an isomorphism. As X and X_{sa} have the same cone we find that φ_0 is positive if and only if φ is positive, and since $\varphi_0(e) = \varphi(e)$ we find that φ_0 is a state if and only if φ is a state. It follows that $\Phi(S(X_{sa})) = S(X)$. Furthermore, as Φ is an isomorphism, we find that for any $\varphi, \psi_1, \psi_2 \in X_{sa}$ and $\lambda \in (0, 1)$ that $\varphi = \lambda\psi_1 + (1 - \lambda)\psi_2$ if and only if $\Phi(\varphi) = \lambda\Phi(\psi_1) + (1 - \lambda)\Phi(\psi_2)$, so φ is a pure state if and only if $\Phi(\varphi)$ is a pure state. For more details see [2, Page 51].

7.1.5 Example. Consider $\ell^\infty(\mathbb{C})$. It is known that $\ell^\infty(\mathbb{C})$ is isomorphic isometric to $C(\beta\mathbb{N})$, the continuous functions on the Stone-Čech compactification of $\mathbb{N}$. As $\beta\mathbb{N}$ is a compact Hausdorff space, we find that the state space and pure states of $\ell^\infty(\mathbb{C})$ is identical to the state space and pure states $\ell^\infty(\mathbb{C})_{sa} = \ell^\infty(\mathbb{R})$ which by the previous section we know to be the set of probability measures on $\beta\mathbb{N}$ and the Dirac masses. Since ℓ_∞ is an commutative C^*-algebra we can find a different way of classifying the pure states.

7.1.6 Proposition (Proposition 4.4.1, [33])**.** *Let X be a commutative C^*-algebra, then a linear fuctional $\varphi \in X$ is a pure state if and only if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in X$.*

From this it follows that the pure states of ℓ^∞ are precisely the multiplicative linear functionals. The state space and pure states of $B(H)$ are harder to find, we will study the geometry of the state space and particular the pure states of $B(H)$ later in this section.

The following results can mostly be found as results or exercises in [33] by Kadison and Ringrose. We have included proofs for the readers convenience. We will first show a Cauchy-Schwartz type inequality for C^*-algebras.

7.1.7 Lemma. *Let X be a $C*$-algebra and let φ be a positive linear functional on X. Then for all $x, y \in X$ we have*

(i) $\varphi(x^*y) = \overline{\varphi(y^*x)}$ *and*

(ii) $|\varphi(xy)|^2 \leq \varphi(x^*x)\varphi(y^*y)$.

Proof. By Proposition 7.1.2 it holds that for all $\lambda \in \mathbb{C}$ and all $x, y \in X$ we have

$$\begin{aligned} 0 &\leq \varphi((\lambda x + y)^*(\lambda x + y)) \\ &= |\lambda|^2\varphi(x^*x) + \overline{\lambda}\varphi(x^*y) + \lambda\varphi(y^*x) + \varphi(y^*y) \\ &= |\lambda|^2\varphi(x^*x) + \operatorname{Re}\left(\overline{\lambda}\varphi(x^*y) + \lambda\varphi(y^*x)\right) + \varphi(y^*y). \end{aligned}$$

From the above we find $\operatorname{Im}(\lambda\varphi(y^*x)) = -\operatorname{Im}(\overline{\lambda}\varphi(x^*y))$. Now pick $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\operatorname{Re}(\lambda\varphi(y^*x)) = 0$. We find

$$\begin{aligned} |\varphi(y^*x)| &= \left(\operatorname{Re}(\lambda\varphi(y^*x))^2 + \operatorname{Im}(\lambda\varphi(y^*x))^2\right)^{\frac{1}{2}} = \left(\operatorname{Im}(\overline{\lambda}\varphi(x^*y))^2\right)^{\frac{1}{2}} \\ &\leq \left(\operatorname{Re}(\overline{\lambda}\varphi(x^*y))^2 + \operatorname{Im}(\overline{\lambda}\varphi(x^*y))^2\right)^{\frac{1}{2}} = |\varphi(x^*y)|. \end{aligned}$$

In a similar way we can show that $|\varphi(x^*y)| \leq |\varphi(y^*x)|$ which yields (i), as $\operatorname{Im}(\varphi(y^*x)) = -\operatorname{Im}(\varphi(x^*y))$.

For (ii) let λ be such that $\lambda\varphi(y^*x) = |\lambda||\varphi(y^*x)|$. Then

$$0 \leq \varphi((\lambda x + y)^*(\lambda x + y)) = |\lambda|^2\varphi(x^*x) + |2\lambda||\varphi(y^*x))| + \varphi(y^*y)$$

hence the quadratic-formula gives us

$$4|\varphi(y^*x))|^2 \leq 4\varphi(x^*x)\varphi(y^*y)$$

which gives us (ii). □

Let H be a Hilbert space and let $x \in H$ be a unit vector. We call $\varphi_x \in B(H)'$ given by

$$\varphi_x(A) = \langle Ax, x\rangle \qquad (A \in B(H))$$

a *vector state*. Note that φ_x is linear, $\varphi_x(I) = 1$ and by Proposition 7.1.2 we have that for all $A \in B(H)_+$ there exists a $B \in B(H)$ such that $B^*B = A$, so $\varphi(A) = \langle Ax, x\rangle = \langle Bx, Bx\rangle \geq 0$. So φ_x is a state.

7.1.8 Proposition. *If H be a Hilbert space, then all vector states of $B(H)$ are pure.*

Proof. Let $x \in H$ be a unit vector and let φ_x be a vector state. Let P_x be the projection on $\mathbb{C}x$ and let $P_{x^\perp}$ be the projection on $(\mathbb{C}x)^\perp$. Clearly $\varphi_x(P_x) = 1$ and $\varphi_x(P_{x^\perp}) = 0$. Let $\psi_1, \psi_2 \in S(B(H))$ and $\lambda \in (0, 1)$ such that $\lambda\psi_1 + (1-\lambda)\psi_2 = \varphi_x$. Note that $\psi_i(P_x) \leq$

$\psi_i(I) = 1$ hence $\psi_i(P_x) = 1$. Also note that $\psi_i(P_{x^\perp}) = 0$. By Lemma 7.1.7 for all $A \in B(H)$ we have

$$|\psi_i(AP_{x^\perp})|^2 \leq \psi_i(AA^*)\psi_i(P_{x^\perp}) = 0.$$

Hence for all $A \in B(H)$ we have

$$\psi_i(A) = \psi_i(AP_x) + \psi_i(AP_{x^\perp}) = \psi_i(AP_x).$$

Using this and Lemma 7.1.7 we also find for all $A \in B(H)$

$$\psi_i(A) = \overline{\psi_i(A^*)} = \overline{\psi_i(A^*P_x)} = \psi_i(P_xA).$$

Finally note that for all $y \in H$ we have

$$P_xAP_xy = \langle y, x\rangle P_xAx = \langle Ax, x\rangle\langle y, x\rangle x = \langle Ax, x\rangle P_xy.$$

Thus

$$\psi_i(A) = \psi_i(P_xAP_x) = \langle Ax, x\rangle\psi_i(P_x) = \varphi_x(A).$$

□

Though not every pure state of $B(H)$ is a vector state one can show that the vector states lie dense in the weak* closure of the pure states.

7.1.9 Theorem (Theorem 4.3.9, [33])**.** *Let X be a C^*-algebra with unit e and let $A \subset X$ be a self-adjoint subspace containing e. If $S_0 \subset S(X)$ is a subset of the space state of A, then the following are equivalent:*

(i) If $x \in A$ and $\varphi(x) \geq 0$ for all $\varphi \in S_0$, then $x \in A_+$.

(ii) For all self-adjoint $x \in A$ we have $\|x\| = \sup\{|\varphi(x)| : \varphi \in S_0\}$.

(iii) $\overline{\mathrm{co}(S_0)}^* = S(A)$.

(iv) $E(A) \subset \overline{S_0}^*$.

7.1.10 Corollary. *Let H be a Hilbert space, then for every self-adjoint operator $A \in B(H)_{sa}$ we have*

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Furthermore if S_0 is the set of the vector states then $\overline{S_0}^* = E(B(H))$ and $\overline{\mathrm{co}(S_0)}^* = S(B(H))$.

Proof. Since

$$B(H)_+ = \{A \in B(H) : \langle Ax, x\rangle \geq 0 \text{ for all } x \in H\}$$

we know that $A \in B(H)_+$ if and only if $\varphi_x(A) \geq 0$ for all $x \in H$ where φ_x is the vector state of x. The result now follows from Theorem 7.1.9. □

7.1.11 Definition. Let A, B be C*-algebras. We call $\varphi : A \to B$ a *-homomorphism if it is linear, multiplicative, carries the unit to the unit and respects the involution. i.e. for $a, b \in A$, $\lambda, \mu \in \mathbb{C}$ and e_A and e_B the respective unit vectors of A and B we have

(i) $\varphi(\lambda a + \mu b) = \lambda\varphi(a) + \mu\varphi(b)$.

(ii) $\varphi(ab) = \varphi(a)\varphi(b)$.

(iii) $\varphi(e_A) = e_B$.

(iv) $\varphi(a^*) = \varphi(a)^*$.

One of the advantages of a *-homomorphism is that it send states to states and pure states to pure states.

7.1.12 Proposition (Exercise 4.6.22, [33])**.** *Let $\varphi : X \to Y$ be a surjective *-homomorphism between two C*-algebras and let ρ be a linear functional on Y. Then*

(i) $\rho \circ \varphi$ is a state if and only if ρ is a state.

(ii) $\rho \circ \varphi$ is a pure state if and only if ρ is a pure state.

Proof. We start by proving (i). Suppose ρ is a state. Clearly $\rho(\varphi(e_X)) = \rho(e_Y) = 1$ and $\rho \circ \varphi$ is linear. Also by Theorem 7.1.2 we know that for any positive $x \in X$, there exists a $y \in X$ such that $x = y^*y$. Hence

$$\rho(\varphi(x)) = \rho(\varphi(y^*y)) = \rho(\varphi(y)^*\varphi(y)) \geq 0$$

which proves that $\rho \circ \varphi$ is a state. Now suppose $\rho \circ \varphi$ is a state. Clearly since φ is a *-homomorphism we have $1 = \rho(\varphi(e_X)) = \rho(e_Y)$. Let $x \in Y$ be positive, again there exists a $y \in Y$ such that $x = y^*y$ is positive. As φ is surjective there exists a $z \in X$ such that $\varphi(z) = y$, we find

$$\rho(x) = \rho(y^*y) = \rho(\varphi(z^*z)) \geq 0$$

which completes the proof of (i).

Now for (ii) suppose that $\rho \circ \varphi$ is a pure state and let $\lambda \in (0,1)$ and ψ_1, ψ_2 states on Y such that $\rho = \lambda\psi_1 + (1-\lambda)\psi_2$. Then, as $\rho \circ \varphi = \lambda\psi_1 \circ \varphi + (1-\lambda)\psi_2 \circ \varphi$ and $\rho \circ \varphi$ is a pure state, we have $\psi_1 \circ \varphi = \psi_2 \circ \varphi = \rho \circ \varphi$. Since φ is surjective we have $\psi_1 = \psi_2 = \rho$, so ρ is a pure state.

Finally suppose ρ is a pure state. Let $\lambda \in (0,1)$ and let ψ_1, ψ_2 be states on X such that $\rho \circ \varphi = \lambda\psi_1 + (1-\lambda)\psi_2$. Let $x \in \ker(\varphi)$, note that $\ker(\varphi)$ is a self adjoint subalgebra as φ is a *homomorphism. In particular we find that $x^*x \in \ker(\varphi)$. By Proposition 7.1.2 we have that $x^*x \in X_+$, so $\psi_1(x^*x), \psi_2(x^*x) \geq 0 = \rho(\varphi(x^*x))$. It follows that for $i = 1, 2$ we have $\psi_i(x^*x) = 0$ and by Proposition 7.1.7 we find

$$0 \leq |\psi_i(x)| = |\psi_i(e^*x)| \leq \psi_i(e^*e)\psi_i(x^*x) = \psi_i(x^*x) = 0$$

So ψ_1 and ψ_2 are 0 on $\ker(\varphi)$. Therefore, for $i = 1, 2$, we can define a linear functional $\hat{\psi}_i$ of Y by $\hat{\psi}_i(y) = \psi_i(x)$ where $y \in Y$ and $x \in \varphi^{-1}(y)$. It is easy to verify that for $i = 1, 2$ we have $\hat{\psi}_i \circ \varphi = \psi_i$. By (i) it follows that $\hat{\psi}_i$ is a state and, as

$$\rho \circ \varphi = \lambda\psi_1 + (1-\lambda)\psi_2 = \lambda\hat{\psi}_1 \circ \varphi + (1-\lambda)\hat{\psi}_2 \circ \varphi$$

and ρ is a pure state, we find that $\rho = \hat{\psi}_1 = \hat{\psi}_2$. Hence $\rho \circ \varphi = \psi_1 = \psi_2$, so $\rho \circ \varphi$ is a pure state. $\square$

Finally before we can show that the pure states of $B(H)$ are not weak*-closed we need the following results.

7.1.13 Theorem (Theorem 4.3.13(iv), [33])**.** *Let X be a C^*-algebra with unit e and let $A \subset X$ be a self-adjoint subspace of X containing e. If ρ is a pure state of A, then ρ extends to a pure state of X.*

7.1.14 Lemma. *Let H be a Hilbert space. If (x_n) is a sequence in H converging weakly to 0, then (x_n) is bounded.*

Proof. Suppose (x_n) is not bounded. Let (x_{n_k}) be a subsequence of (x_n) such that $4\|x_{n_k}\| \leq \|x_{n_{k+1}}\|$ for all k and $\frac{1}{\|x_{n_i}\|^2}\langle x_{n_k}, x_{n_i}\rangle < \frac{1}{2k-2}$ for all $i < k$. Note that we can find a subsequence satisfying the second condition as $\lim_{n\to\infty}\langle x_n, x_{n_i}\rangle = 0$. Also note that, since

$$\|\sum_{k=1}^{\infty} \frac{x_{n_k}}{\|x_{n_k}\|^2}\| \leq \sum_{k=1}^{\infty} \frac{1}{\|x_{n_k}\|} \leq \frac{1}{\|x_{n_1}\|}\sum_{k=1}^{\infty}\frac{1}{4^k} < \infty,$$

we have that $(\sum_{k=1}^{n} \frac{x_{n_k}}{\|x_{n_k}\|^2}$ is a Cauchy sequence. Since H is a Hilbert space it converges to some $x \in H$. Then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\langle x, x_{n_k}\rangle &= \sum_{i=1}^{k-1} \frac{\langle x_{n_k}, x_{n_i}\rangle}{\|x_{n_i}\|^2} + 1 + \sum_{i=k+1}^{\infty} \frac{\langle x_{n_k}, x_{n_i}\rangle}{\|x_{n_i}\|^2} \\ &\geq -\frac{1}{2} + 1 - \sum_{i=k+1}^{\infty} \frac{\|x_{n_k}\|}{\|x_{n_i}\|} \\ &\geq \frac{1}{2} - \sum_{i=k+1}^{\infty} \frac{1}{4^{i-k}} = \frac{1}{6}\end{aligned}$$

Hence $\lim_{n\to\infty}\langle x, x_{n_k}\rangle$ either does not exist or is non-zero. It follows that (x_n) does not converge weakly to 0. □

We will now show that there exists a pure state of $B(H)$ which is 0 on the compact operators. This pure state can be used to create a state which is not pure, but lies in the weak* closure of the pure states.

7.1.15 Lemma (Exercise 4.6.57, [33]). *If H is an infinite dimensional Hilbert space, then there exists a pure state ρ of $B(H)$ which is 0 on $\mathcal{K} \subset B(H)$, the set of compact operators.*

Proof. Consider the C*-algebra ℓ_∞ with involution $(x_n)^* = (\overline{x_n})$ and unit $e = (1)_{\mathbb{N}}$. Note that $c \subset \ell_\infty$, the set of converging sequences, is a commutative C*-algebra. Let $\rho_0 : c \to \mathbb{C}$ be a linear functional given by

$$\rho_0((x_n)) = \lim_{n\to\infty} x_n. \qquad ((x_n) \in c)$$

By Proposition 7.1.6 we have that ρ_0 is a pure state and by Theorem 7.1.13 we can extend ρ_0 to a pure state ρ on ℓ_∞.

Now consider $\mathcal{L}_0$ the set of all sequences (u_n) in H which are weak convergent to 0. Note that $\mathcal{L}_0$ is a vector sequence space. Let $(u_n), (v_n) \in \mathcal{L}_0$. As (u_n) and (v_n) weakly converge to 0, by Lemma 7.1.14 we have $(\|u_n\|), (\|v_n\|) \in \ell_\infty$. By the Cauchy-Schwarz inequality we have

$$|\langle u_n, v_n\rangle| \leq \|u_n\|\|v_n\|$$

so $(\langle u_n, v_n\rangle) \in \ell_\infty$. We define $\langle\cdot,\cdot\rangle_0 : \mathcal{L}_0 \times \mathcal{L}_0 \to \mathbb{C}$ by

$$\langle (u_n), (v_n)\rangle_0 = \rho((\langle u_n, v_n\rangle)_{n\in\mathbb{N}}) \qquad ((u_n), (v_n) \in \mathcal{L}_0).$$

Note that, since ρ respects conjugacy, $\langle\cdot,\cdot\rangle_0$ is conjugate symmetric. Clearly $\langle\cdot,\cdot\rangle_0$ is linear and, as ρ is positive, $\langle\cdot,\cdot\rangle_0$ is semi-positive definite. Thus $\langle\cdot,\cdot\rangle_0$ is a semi-inner product and the corresponding $\|\cdot\|_0$ is a semi-norm.

Now let $\mathcal{N}_0 = \{(u_n) \in \mathcal{L}_0 : \|(u_n)\|_0 = 0\}$ and define $\mathcal{L}_1 = \mathcal{L}_0/\mathcal{N}_0$. Then $\langle\cdot,\cdot\rangle_1$, given by $\langle x+\mathcal{N}_0, y+\mathcal{N}_0\rangle_1 = \langle x,y\rangle_0$ for all $x,y \in \mathcal{L}_0$, is an inner product. We denote by $\|\cdot\|_1$ its associated norm. Let $\mathcal{L}$ be the completion of $\mathcal{L}_1$, then $(\mathcal{L},\|\cdot\|_1)$ is a Hilbert space. Consider the linear operator $\pi_0 : B(H) \to B(\mathcal{L}_1)$ given by

$$\pi_0(T)((u_n)+\mathcal{N}_0) = (Tu_n)+\mathcal{N}_0 \qquad (T \in B(H), (u_n)+\mathcal{N}_0 \in \mathcal{L}_1).$$

Claim: π_0 is well-defined and continuous.

Note that for all $(u_n) \in \mathcal{L}_0$ we find, as ρ is positive, that

$$\|(Tu_n)\|_0 = \rho((\|Tu_n\|)) \leq \rho(\|T\|(\|u_n\|)) = \|T\|\rho((\|u_n\|)) = \|T\|\|(u_n)\|_0.$$

In particular this means that if $(u_n) \in \mathcal{N}_0$, then $(Tu_n) \in \mathcal{N}_0$, so π_0 is well-defined and

$$\|\pi_0(T)(u_n+\mathcal{N}_0)\|_1 \leq \|T\|\|(u_n)+\mathcal{N}_0\|_1.$$

As π_0 is continuous we can extend it uniquely to a continuous linear operator $\pi : B(H) \to B(\mathcal{L})$.

Claim: π is a *-homomorphism with kernel $\mathcal{K}$.

Clearly π is linear, multiplicative and $\pi(I) = I_{\mathcal{L}}$. Also note that for $(u_n),(v_n) \in \mathcal{L}_0$ we have

$$\begin{aligned}\langle\pi(T)(u_n)+\mathcal{N}_0,(v_n)+\mathcal{N}_0\rangle_1 &= \rho((\langle Tu_n,v_n\rangle)) = \rho((\langle u_n,T^*v_n\rangle))\\ &= \langle(u_n)+\mathcal{N}_0,\pi(T^*)(v_n)+\mathcal{N}_0\rangle_1\end{aligned}$$

So $\pi(T)^* = \pi(T^*)$ and therefore a π is a *-homomorphism. Now let $T \in B(H)$ such that $\pi(T) = 0$. Note that this is the case if and only if for all $(u_n) \in \mathcal{L}_0$ we have that $(Tu_n) \in \mathcal{N}_0$ i.e. $\lim_{n\to\infty}\|Tu_n\| = 0$ for all $(u_n) \in \mathcal{L}_0$. As T is compact if and only if for all sequences (u_n) weakly converging to some u we have $\lim_{n\to\infty}\|T(u-u_n)\| = 0$, see [16, Proposition VI.3.3], we find that $\ker(\pi) = \mathcal{K}$.

Then by Lemma 7.1.12 we find that $\rho\circ\pi$ is a pure state on $B(H)$ which is 0 on $\mathcal{K}$. □

7.1.16 Theorem (Exercise 4.6.69, [33])**.** *If H is infinite dimensional, then the pure states of $B(H)$ are not weak* closed.*

Proof. Let $x \in H$ with $\|x\| = 1$, let φ_x be its corresponding vector state and let ρ be a pure state of $B(H)$ which is 0 on $\mathcal{K}$, the set of compact operators of $B(H)$. Such state exists by Lemma 7.1.15. Consider the state $\omega = \lambda\varphi_x + (1-\lambda)\rho$ for some $\lambda \in (0,1)$. We will prove that ω is the weak* limit of vector states. We know by Corollary 7.1.10 that the pure states of $B(H)$ are contained in the weak* closure of the set of vector states. Hence there exists a net of unit vectors (y'_α) in H such that the sequence of vector states $(\varphi_{y'_\alpha})$ weak* converges to ρ. Now we know that the sets

$$\{\theta \in B(H)' : \|(\omega - \theta)A_i\| < \varepsilon \text{ for all } i = 0, \ldots, m\} \quad (\varepsilon > 0, m \in \mathbb{N}, A_1, \ldots, A_m \in B(H))$$

form a neighbourhood basis around ω for the weak* topology. Now let $\varepsilon > 0$, $m \in \mathbb{N}$ and let $A_1, \ldots, A_m \in B(H)$. Let P be the projection on $\text{span}(x, A_1x, \ldots, A_mx, A_1^*x, \ldots, A_m^*x)$ and define $y_\alpha = (I-P)y'_\alpha / \|(I-P)y'_\alpha\|$. Note that

$$y_\alpha \perp \text{span}(x, A_1x, \ldots, A_mx, A_1^*x, \ldots, A_m^*x))$$

for all α. Furthermore, note that P, A_iP, A_i^*P and PA_iP are compact operator for all $i = 1, \ldots, m$. Also note that

$$\begin{aligned}\lim_\alpha \|(I-P)y'_\alpha\|^2 &= \lim_\alpha \langle (I-P)y'_\alpha, (I-P)y'_\alpha\rangle = \lim_\alpha \langle (I-P)y'_\alpha, y'_\alpha\rangle \\ &= \lim_\alpha \varphi_{y'_\alpha}(I-P) = \rho(I-P) = 1.\end{aligned}$$

Let $z_\alpha = \lambda^{\frac{1}{2}}x + (1-\lambda)^{\frac{1}{2}}y_\alpha$ and consider

$$\begin{aligned}\lim_\alpha |(\omega - \varphi_{z_\alpha})A_i| &= \lim_\alpha |\lambda\langle A_ix, x\rangle + (1-\lambda)\rho(A_i) - \lambda\langle A_ix, x\rangle - (1-\lambda)\langle A_iy_\alpha, y_\alpha\rangle \\ &\quad - \lambda^{\frac{1}{2}}(1-\lambda)^{\frac{1}{2}}\langle y_\alpha, A_i^*x\rangle - \lambda^{\frac{1}{2}}(1-\lambda)^{\frac{1}{2}}\langle A_ix, y_\alpha\rangle| \\ &= \lim_\alpha |(1-\lambda)\rho(A_i) - (1-\lambda)\langle A_iy_\alpha, y_\alpha\rangle| \\ &= \lim_\alpha |(1-\lambda)\rho(A_i) - \frac{1-\lambda}{\|(I-P)y'_\alpha\|^2}(\langle A_iy'_\alpha, y'_\alpha\rangle - \langle A_iPy'_\alpha, y'_\alpha\rangle \\ &\quad - \langle y'_\alpha, A_i^*Py'_\alpha\rangle - \langle PA_iPy'_\alpha, y'_\alpha\rangle)| \\ &= |(1-\lambda)(\rho(A_i) - \rho(A_i) - \rho(A_iP) - \overline{\rho(A_i^*P)} - \rho(PA_iP))| = 0.\end{aligned}$$

Hence for α large enough we find

$$\varphi_{z_\alpha} \in \{\theta \in B(H)' : \|(\omega - \theta)A_i\| < \epsilon \text{ for all } i = 0, \ldots, m\}.$$

So ω is contained in the weak* closure of the pure states. $\square$

Note that we need H to be infinite dimensional, as for finite dimensional spaces the pure states of $B(H)$ are weak* closed. To see this recall the following result.

7.1.17 Proposition. *Let H be a Hilbert space. If (x_α) is a net of unit vectors in H converging to a unit vector x in norm, then the vector states (φ_{x_α}) converge in operator norm to φ_x.*

Proof. Let (x_α) be an net of unit vectors in H converging to some unit vector x. Let $A \in B(H)$. Consider

$$\begin{aligned} |\langle Ax_\alpha, x_\alpha\rangle - \langle Ax, x\rangle| &\leq |\langle Ax_\alpha, x_\alpha - x\rangle| + |\langle A(x_\alpha - x), x\rangle| \\ &\leq \|A\|\|x_\alpha\|\|x_\alpha - x\| + \|A\|\|x_\alpha - x\|\|x\| \end{aligned}$$

□

7.1.18 Corollary. *Let H be a finite dimensional Hilbert space. Then the set of pure states of $B(H)$ is weak* closed.*

Proof. Let (x_α) be a net of unit vectors in H such that φ_{x_α} converges in the weak* topology. As H is finite dimensional we can find a norm convergent subnet (x_β) converging to some unit vector x. By Proposition 7.1.17 it follows that $\lim_\alpha \varphi_{x_\alpha} = \varphi_x$. As by Corollary 7.1.10 the vector states lie dense in the weak* closure of the pure states, it follows that all pure states are vector states and that they are weak* closed. □

For H infinite dimensional we can now show there exists a reverse-Funk non-Busemann horofunction.

7.1.19 Lemma. *Let H a Hilbert space and let $y, z \in H$ unit vectors. If P_y and P_z are projections on $\mathbb{C}y$ and $\mathbb{C}z$ respectively, then*

$$P_y + P_z \leq (1 + |\langle z, y\rangle|)I.$$

Proof. We know that

$$P_y + P_z \leq r(P_y + P_z)I$$

where $r(P_y + P_z)$ is the spectral radius of $P_y + P_z$. We know that

$$r(P_y + P_z) = \sup_{x, \|x\|=1} \langle (P_y + P_z)x, x\rangle$$

Let $x = x_1 + x_2 \in H$ with $x_1 \in \operatorname{span}(y,z)$ and $x_2 \in \operatorname{span}(y,z)^\perp$. Consider that

$$\langle (P_y + P_z)x, x\rangle = \langle (P_y + P_z)x_1, x_1\rangle.$$

So the supremum is attained on $\operatorname{span}(y,z)$. Note that on $\operatorname{span}(y,z)$ we can rewrite $P_y + P_z$ by

$$(P_y + P_z)(ay + bz) = \begin{pmatrix} 1 & \langle z, y\rangle \\ \langle y, z\rangle & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix}$$

Hence the spectral radius is the largest eigenvalue of A which is given by the equation

$$(1-\lambda)^2 - |\langle y, z\rangle|^2 = \lambda^2 - 2\lambda - |\langle y, z\rangle|^2 + 1.$$

This equation has solutions $\lambda = 1 \pm |\langle y, z\rangle|$. Hence

$$r(P_y + P_z) = 1 + |\langle y, z\rangle|.$$

□

7.1.20 Theorem. *Let H be an infinite dimensional Hilbert space. Then there exists a net of affine functions on $S(B(H))$ which has non-affine limit in the hypograph topology.*

Proof. Let x, ρ and $\omega = \lambda\varphi_x + (1-\lambda)\rho$ be as in the proof of Theorem 7.1.16. Let (y_α) be a net in H such that the net of vector states (φ_{y_α}) converges to ω in the weak* topology and let P_{y_α} be the orthogonal projections on $\mathbb{F}y_\alpha$. We define $\theta_\alpha : S(B(H)) \to \mathbb{C}$ by

$$\theta_\alpha(\psi) = \psi(P_{y_\alpha}).$$

Note that, since the hypographs of θ_α are subsets of a Hausdorff space, by Mrowka's theorem 5.1.10 we may assume that, by taking a subnet if necessary, (θ_α) is hypograph convergent to some $\theta : S(B(H)) \to \mathbb{C}$. We will now show that θ is not affine.

First note that $(\omega, \lim_\alpha \theta_\alpha(\varphi_{y_\alpha}))$ is a limit point of $(\text{hypo}(\theta_\alpha))$ as $(\varphi_{y_\alpha}, \theta_\alpha(\varphi_{y_\alpha})) \in (\text{hypo}(\theta_\alpha))$. So

$$\theta(\omega) \geq \lim_\alpha \theta_\alpha(\varphi_{y_\alpha}) = 1.$$

Also note that for all states $\psi \in S(B(H))$ we have $\theta_\alpha(\psi) = \psi(P_{y_\alpha}) \leq \psi(I) = 1$ we have that $\theta(\psi) \leq 1$. In particular we find that $\theta(\omega) = 1$ and $\theta(\rho) \leq 1$.

Let z be some unit vector with corresponding vector state φ_z. We will show that $\theta(\varphi_z) \leq \lambda$. As θ is the hypograph limit of θ_α we have that $(\varphi_z, \theta(\varphi_z))$ is a limit point of $(\text{hypo}(\theta_\alpha))$. Since the set

$$X_z = \{U_\varepsilon = U \times (\theta(\varphi_z) - \varepsilon, \theta(\varphi_z) + \varepsilon) \subset S(B(H)) \times \mathbb{R} : U \text{ is a neighbourhood of } \varphi_z\}$$

is a neighbourhood base of $(\varphi_z, \theta(\varphi_z))$ we can define for every $U_\varepsilon \in X_z$ an α_{U_ε} such that for all $\alpha \geq \alpha_{U_\varepsilon}$ we have $\mathrm{hypo}(\theta_\alpha) \cap U_\varepsilon \neq \emptyset$ and $|\varphi_{y_\alpha}(P_z) - \omega(P(z))| < \varepsilon$. Hence for every $U_\varepsilon \in X_z$ we can fix a $\psi_{U_\varepsilon} \in U$ such that $\theta_{\alpha_{u_\varepsilon}}(\psi_{U_\varepsilon}) \geq \theta(\varphi_z) - \varepsilon$. Note that X_z equipped with the partial order $\leq$ given by $U_\varepsilon \leq V_\delta$ if and only if $U_\varepsilon \supseteq V_\delta$ is a directed set. As X_z is a neighbourhood base of $(\varphi_z, \theta(\varphi_z))$ it follows that the net (ψ_{U_ε}) converges to φ_z in the weak* topology. Hence for all $\varepsilon > 0$ and U_δ big enough we have

$$\varphi_z(P_z) - \psi_{U_\delta}(P_z) = 1 - \psi_{U_\delta}(P_z) < \varepsilon \Leftrightarrow 1 - \varepsilon < \psi_{U_\delta}(P_z) \leq 1.$$

Furthermore, by Lemma 7.1.19 we have $\psi_{U_\delta}(P_z + P_{y_{\alpha_{U_\delta}}}) \leq 1 + |\langle z, y_{\alpha_{U_\delta}}\rangle|$ as $P_z + P_{y_{\alpha_{U_\delta}}} \leq (1 + |\langle z, y_{\alpha_{U_\delta}}\rangle|)I$ and ψ_{U_δ} is positive. Hence, for U_δ big enough, it holds that

$$\begin{aligned}\theta_{\alpha_{U_\delta}}(\psi_{U_\delta}) = \psi_{U_\delta}(P_{y_{\alpha_{U_\delta}}}) &= \psi_{U_\delta}(P_z + P_{y_{\alpha_{U_\delta}}}) - \psi_{U_\delta}(P_z) \\ &\leq 1 + |\langle z, y_{\alpha_{U_\delta}}\rangle| - \psi_{U_\delta}(P_z) < \epsilon + |\langle z, y_{\alpha_{U_\delta}}\rangle|.\end{aligned}$$

Finally note that $\limsup_{U_\delta} |\varphi_{y_{\alpha_{U_\delta}}}(P_z) - \omega(P_z)| \leq \limsup_{U_\delta} \delta = 0$. So we find

$$\begin{aligned}\theta(\varphi_z) &\leq \limsup_{U_\delta} \theta_{\alpha_{U_\delta}}(\psi_{U_\delta}) + \delta \leq \limsup_{U_\delta} |\langle z, y_{\alpha_{U_\delta}}\rangle| + \delta \\ &= \limsup_{U_\delta} (|\varphi_{y_{\alpha_{U_\delta}}}(P_z)|)^{\frac{1}{2}} + \delta = |\omega(P_z)|^{\frac{1}{2}} = \lambda|\langle z, x\rangle| \leq \lambda < 1.\end{aligned}$$

Thus

$$\theta(\omega) = 1 = \lambda + (1 - \lambda) > \lambda\theta(\varphi_x) + (1 - \lambda)\theta(\rho).$$

Since $\lambda\varphi_x + (1 - \lambda)\rho = \omega$ it follows that θ is not affine. □

It now follows immediately from Theorem 6.3.7 and Theorem 7.1.20 that $B(H)$ has a non-Busemann reverse-Funk horofunction.

One thing to note from our example of a reverse-Funk non-Busemann horofunction in $B(H)$ is that the proof depended on the fact that the pure states are not weak* closed. We have already seen that the pure states being weak* closed is a necessary and sufficient condition for all Funk horofunctions to be non-Busemann for finite dimensional spaces in [63], so it is not unreasonable to hypothesize this might be a sufficient condition for reverse-Funk horofunctions in infinite dimensional spaces. This turns out not to be the case.

Let K be a compact Hausdorff space and let $C(K)$ be the space of continuous functions on K. Consider $X = \mathbb{R} \times C(K)$ with closed cone

$$X_+ = \{(\lambda, f) \in X : \|f\|_\infty \leq \lambda\}$$

and order-unit $e = (1,0)$. This is a similar construction to that of the spin-factors earlier. Recall that the dual space of $C(K)$ is $\mathrm{rca}(K)$, the set of regular finite real-valued Borel measures on K. It follows that the dual space of X is given by

$$X' = \{\varphi_{\alpha,\mu} : \alpha \in \mathbb{R}, \mu \in \mathrm{rca}(K) \text{ and } \varphi_{\alpha,\mu}((\lambda, f)) = \alpha\lambda + \int_K f \, d\mu \text{ for all } (\lambda, f) \in X\}.$$

For convenience we will denote $\varphi_{\alpha,\mu}$ as (α, μ).

Recall that for every regular finite real-valued Borel measure μ there exist two unique positive regular finite real-valued Borel measures μ_+ and μ_- with disjoint support such that $\mu = \mu_+ - \mu_-$. Using this notation we can see that the positive functionals are of the form (α, μ) where $\alpha \geq 0$ and $\int_K d\mu_+ + \int_K d\mu_- \leq \alpha$. So the state space is given by

$$S(X) = \{(1, \mu) : \int_K d\mu_+ + \int_K d\mu_- \leq 1$$

and the pure states, $E(X)$, are precisely those elements of the form $(1, \delta_x)$ and $(1, -\delta_x)$, where δ_x is Dirac mass of x, i.e. $\int_K f(y)\, d\delta_x = f(x)$ for $x \in K$ and all $f \in C(K)$. As with $C(K)$, the pure states of X are weak* closed.

7.1.21 Proposition. *Let K be a compact Hausdorff space and let $C(K)$ be the space of continuous functions on K. If $X = \mathbb{R} \times C(K)$ is an order-unit space with cone $X_+ = \{(\lambda, f) \in X : \|f\|_\infty < \lambda\}$ and order-unit $e = (1, 0)$, then the pure states of X are weak* closed.*

Proof. Let $((1, \mu_\alpha))$ be a net of pure states converging in the weak* topology to $(1, \mu) \in S(X)$. Note that we can find a net (x_α) in X such that $\mu_\alpha = \pm\delta_{x_\alpha}$. As K is compact, by taking a subnet if required, we may assume that (x_α) converges to some $x \in K$. Furthermore, by taking a subnet if required, we may assume that $\mu_\alpha = \delta_{x_\alpha}$ for all α or $\mu_\alpha = -\delta_{x_\alpha}$ for all α. As the proofs are similar we will assume that $\mu_\alpha = \delta_{x_\alpha}$ for all α.

Let $\lambda \in \mathbb{R}$ and let $f \in C(K)$. As f is continuous we find

$$(1, \mu)((\lambda, f)) = \lim_\alpha (1, \mu_\alpha)((\lambda, f)) = \lim_\alpha \lambda + f(x_\alpha) = \lambda + f(x) = (1, \delta_x)((\lambda, f)).$$

It follows that $\mu = \delta_x$ which proves the result. □

We will now use the results obtained in Chapter 6 to show that $\mathbb{R} \times C(K)$ has a non-Busemann horofunction. Recall that a point x in a subset A of a topological space X is called *isolated* if it has a neighbourhood in X containing no other point of A.

7.1.22 Theorem. *Let K be a compact Hausdorff space and let $C(K)$ be the space of continuous functions on K. Let $X = \mathbb{R} \times C(K)$ be an order-unit space with cone $X_+ = \{(\lambda, f) \in X | \|f\|_\infty < \lambda\}$ and order-unit $e = (1,0)$. If K has a non-isolated point, then there exists a non-Busemann horofunction of X in the reverse-Funk geometry.*

Proof. Let y be a non-isolated point of K and consider the function $g : K \to [0,1]$ given by

$$\tilde{g}(x) = \begin{cases} 1 \text{ if } x = y. \\ 0 \text{ else.} \end{cases}$$

Note that g is an upper-semicontinuous function with supremum 1 which is not positive and not continuous, as y is a non-isolated point. Let N_y be the set of neighbourhoods of y. Since K is normal for every neighbourhood $U \in N_y$ we can use Urysohn's lemma to find a continuous $\tilde{g}_U : K \to [0,1]$ for which $\tilde{g}_U(y) = 1$ and $\tilde{g}_U(x) = 0$ for all $x \in K \setminus U$. By equipping N_y with the order $U \leq V$ if and only if $V \subset U$ we find, since y is a non-isolated point, that $(\tilde{g}_U)_{U \in N_y}$ is a net. We will now show that $(\tilde{g}_U)_{U \in N_y}$ is hypo-convergent to $\tilde{g}$.

Let (x, r) be a cluster point of $(\text{hypo}(\tilde{g}_U))_{U \in N_y}$, note that $r \leq 1$ as $\|\tilde{g}_U\|_\infty \leq 1$. If $r \leq 0$, then for all $U \in N_Y$ we have $(x, r) \in \text{hypo}(\tilde{g}_U)$, so (x, r) is a limit point. If $1 \geq r > 0$ and $x \neq y$, then, since K is Hausdorff, we can find disjoint neighbourhoods U of y and W of x. For all $V \in N_y$ with $V \geq U$ we have that $U \times (\frac{1}{2}r, \frac{3}{2}r) \cap \text{hypo}(\tilde{g}_V) = \emptyset$. This implies that (x, r) is not a cluster point for $r > 0$. If $x = y$, then $(y, r) \in (x, r) \in \text{hypo}(\tilde{g}_U)$ for all $U \in N_Y$, so we have that (y, r) is a limit point. Combining these facts shows that $(\text{hypo}(\tilde{g}_U))_{U \in N_y}$ is hypo-convergent to $\tilde{g}$.

Now for all $U \in N_y$ we define $g_U : E(X) \to [-1,1]$ given by $g_U(\pm\delta_x) = \int_K \tilde{g}_U \, d(\pm\delta_x) = \pm\tilde{g}_U(x)$ for all $x \in K$.

Claim: g_U is hypo-convergent to g where $g(\delta_y) = 1$ and for all $\mu \in E(X) \setminus \{\delta_y\}$ we have $g(\mu) = 0$.

To see this first let $x \in K$ and let $(\pm\delta_x, r)$ be a cluster point of $(\text{hypo}(g_U))_{U \in N_y}$, note that $r \leq 1$, as $g_U \leq 1$. If $r \leq 0$, then for all $U \in N_Y$ we have $(\pm\delta_x, r) \in \text{hypo}(g_U)$, so $(\pm\delta_x, r)$ is a limit point. If $1 \geq r > 0$ and $x \neq y$ then we can find disjoint neighbourhoods U of x and W of y. By Urysohn's lemma we can find a continuous function $f : K \to [0,1]$ such that $f(x) = 1$ and $f(z) = 0$ for all $z \in K \setminus U$. We know that for all $\varepsilon > 0$ the set

$$U_{\pm x, \varepsilon} = \{\mu \in E(X) : |\int_K f \, d(\pm\delta_x) - \int_K f \, d\mu| = | \pm 1 - \int_K f \, d\mu| < \varepsilon\}$$

is a neighbourhood of $\pm\delta_x$. For all $V > W$ we have $\text{hypo}(g_V) \cap (U_{\pm x,r} \times (\frac{1}{2}r, \frac{3}{2}r)) = \emptyset$. So $(\pm\delta_x, r)$ is not a cluster point of $(\text{hypo}(g_U))_{U\in N_y}$ for $r > 0$ and $x \neq y$. If $1 \geq r > 0$ and $x = y$, then note that, since $(\delta_y, r) \in \text{hypo}(g_U)$ for all $U \in N_y$ and $r \leq 1$, we have that (δ_y, r) is a limit point for all $r \leq 1$. Finally note that for all $\varepsilon > 0$ the set

$$U_{-y,\varepsilon} = \{\mu \in E(X) : |\int_K 1 \; d(-\delta_y) - \int_K 1 \; d\mu| = |1 + \int_K 1 \; d\mu| < \varepsilon\}$$

is a neighbourhood of $-\delta_y$. For all $V > K$ we have that $\text{hypo}(g_V) \cap (U_{-y,r} \times (\frac{1}{2}r, \frac{3}{2}r)) = \emptyset$ hence $(-\delta_y, r)$ is not a cluster point of $(\text{hypo}(g_U))_{U\in N_y}$ for $r > 0$. Combining these facts proves our claim.

By Proposition 5.2.10 and Lemma 5.2.11 we can define $\xi_R : X_+ \to \mathbb{R}$ which for all $(\lambda, f) \in X$ is given by

$$\begin{aligned}
\xi(R)((\lambda, f)) &= \lim_{U\in N_y} i_R((1, \tilde{g}_U))((\lambda, f)) \\
&= \lim_{U\in N_y} \log \sup_{(1,\mu)\in E(X)} \frac{(1, g_u)((1, \mu))}{(\lambda, f)((1, \mu))} \\
&= \log \sup_{(1,\mu)\in E(X)} \frac{(1, g)((1, \mu))}{(\lambda, f)((1, \mu))} \\
&= \log \sup_{(1,\mu)\in E(X)} \frac{1 + \int_K g \; d\mu}{\lambda + \int_K f \; d\mu}.
\end{aligned}$$

Note that ξ_R is a reverse-Funk horofunction, as g is not continuous and thus $\xi_R \notin \overline{i_R(X)}$. Moreover as for $x \neq y$ we have

$$\frac{1}{2}(1, g)((1, \delta_{+x})) + \frac{1}{2}(1, g)((1, \delta_{-x})) = 0 \neq \frac{1}{2} = \frac{1}{2}(1, g)((1, \delta_{+y})) + \frac{1}{2}(1, g)((1, \delta_{-y}))$$

and $\frac{1}{2}(1, \delta_{+x}) + \frac{1}{2}(1, \delta_{-x}) = \frac{1}{2}(1, \delta_{+y}) + \frac{1}{2}(1, \delta_{-y})$ we find that g has no affine extension to the state space of X, hence by Theorem 6.3.7 ξ_R is a non-Busemann horofunction of X. $\square$

Finally one can easily find that K has a non-isolated point if $C(K)$ is infinite dimensional.

7.1.23 Lemma. *Let K be a compact Hausdorff set. If $C(K)$ is infinite dimensional, then K has an non-isolated point.*

Proof. Suppose K only has isolated points. Then every $x \in K$ has a neighbourhood $U_x = \{x\}$. As $\bigcup_{x\in K} U_x$ covers K and K is compact we find that K is finite. But then we have that $C(K) = \mathbb{R}^{|K|}$, which is a finite dimensional vector space. $\square$

7.2 The horofunction boundary of spin factors

As promised we will now give a classification of the horofunction boundary of spin factors, see also [15]. Spin factors have multiple incarnations. Besides being order-unit spaces and Euclidean Jordan algebras, we can view spin factors as hyperbolic spaces.

The study of the geometry of the infinite dimensional real hyperbolic space $\mathbb{H}^\infty$ has gained significant momentum since it was popularised by Gromov in [28], see [10], [17], [18], [43] and [50]. We will work with the following model of hyperbolic space.

Let $(H, \langle \cdot, \cdot \rangle)$ be an infinite dimensional Hilbert space and let $X = \mathbb{R} \times H$ be a spin factor. Let $Q : X \to \mathbb{R}$ be the quadratic form,

$$Q((\lambda, x)) = \lambda^2 - \langle x, x \rangle \qquad ((\lambda, x) \in X)$$

Recall that the spin factor X has cone

$$X_+ = \{(\lambda, x) \in X : \|x\| < \lambda\}.$$

Let $B : X \times X \to \mathbb{R}$ be the bilinear form associated with Q,

$$B((\lambda, x), (\mu, y)) = \lambda\mu - \langle x, y \rangle \qquad ((\lambda, x), (\mu, y) \in X)$$

Consider the hyperboloid

$$\mathbf{H} = \{u \in X_+ : Q(u) = 1\}.$$

We can define a metric, called the *hyperboloid metric*, δ_h on $\mathbf{H}$ by

$$\cosh(\delta_h(u, v)) = B(u, v) \qquad (u, v \in \mathbf{H}).$$

We call $\mathbb{H}^\infty = (\mathbf{H}, \delta_h)$ the *hyperboloid model* of the infinite dimensional real hyperbolic space. It is known that on the hyperboloid $\mathbf{H}$ the metric δ_h coincides with a scaled version of Hilbert metric $\frac{1}{2}d_H$, see [42, Chapter 2.3]. The disc

$$\mathbf{D} = \{(\lambda, x) \in X : \lambda = 1 \text{ and } \|x\| < 1\}$$

equipped with $\frac{1}{2}d_H$ is known as *Klein's model* and is a different model of the hyperbolic space. We have already seen the finite dimensional version of Klein's Model in Example 2.1.2. We will be using Klein's model to study the spin factors for the rest of this chapter.

For finite dimensional real hyperbolic spaces $\mathbb{H}^n$ it is well known that $\partial\mathbb{H}^n$ coincides with the horofunction boundary. In infinite dimensions this is no longer the case. Indeed we will show that in $\mathbb{H}^\infty$ the Busemann points correspond to $\partial\mathbb{H}^\infty$, and that there are many horofunctions that are not Busemann points. This phenomenon is caused by the fact $\mathbb{H}^\infty$ is not proper. In fact, we will prove the following theorem.

7.2.1 Theorem. *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional Hilbert space and let $X = \mathbb{R}\times H$. The horofunctions of X with respect to the Hilbert Geometry are the functions of the following form:*

$$\xi((\gamma, y)) = 2\log\left(\frac{\gamma - \langle\hat{x}, y\rangle + \sqrt{(\gamma - \langle\hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1-r^2)}}{(1+r)\sqrt{\gamma^2 - \|y\|^2}}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

where either $\|\hat{x}\| < 1$ and $\|\hat{x}\| < r \leq 1$ or $\|\hat{x}\| = r = 1$. Furthermore, ξ is a Busemann if and only if $r = 1$ and $\|\hat{x}\| = 1$, in which case

$$\xi((\gamma, y)) = \log\left(\frac{(\gamma - \langle\hat{x}, y\rangle)^2}{\gamma^2 - \|y\|^2}\right) \qquad ((\gamma, y) \in X_+^\circ).$$

In terms of δ_h on the hyperboloid model we get the following reformulation of Theorem 7.2.1.

7.2.2 Corollary. *The horofunctions of $\mathbb{H}^\infty = (\boldsymbol{H}, \delta_h)$ are precisely the functions of the form*

$$\xi(v) = \log\left(\frac{B(\hat{u}, v) + \sqrt{(B(\hat{u}, v))^2 - (1-r^2)}}{(1+r)}\right) \qquad (v \in \boldsymbol{H})$$

where $0 < r \leq 1$ and $\hat{u} \in \boldsymbol{D}$ such that $0 \leq 1 - r^2 < Q(\hat{u})$, or, $r = 1$ and $\hat{u} \in \partial\boldsymbol{D}$. Furthermore, ξ is a Busemann point if and only if $r = 1$ and $\hat{u} \in \partial\boldsymbol{D}$, in which case

$$\xi(v) = \log\left(B(\hat{u}, v)\right) \qquad (v \in \boldsymbol{H}).$$

To prove Theorem 7.2.1, we will first calculate the gauge functions. We can use these to classify the horofunctions of the Funk, reverse Funk and Hilbert geometry. The final results will then further classify the Busemann points.

7.2.3 Proposition. *Let H be a Hilbert space and let $X = \mathbb{R}\times H$. For all $(\mu, x), (\gamma, y) \in X_+^\circ$ we have*

$$M((\mu, x)/(\gamma, y)) = \frac{\gamma\mu - \langle x, y\rangle + \sqrt{(\gamma\mu - \langle x, y\rangle)^2 - (\mu^2 - \|x\|^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2}.$$

Proof. We know that

$$\begin{aligned} M((\mu,x)/(\gamma,y)) &= \inf\{\beta>0 : (\mu,x)\le \beta(\gamma,y)\} \\ &= \inf\{\beta>0 : (\gamma\beta-\mu)^2 \ge \|\beta y - x\|^2 \text{ and } \gamma\beta-\mu\ge 0\}. \end{aligned}$$

So we have to solve

$$(\gamma\beta-\mu)^2 - \|\beta y - x\|^2 = (\gamma^2-\|y\|^2)\beta^2 - 2(\gamma\mu - \langle x,y\rangle)\beta + (\mu^2-\|x\|^2) = 0,$$

which has solutions

$$\beta_\pm = \frac{\gamma\mu - \langle x,y\rangle \pm \sqrt{(\gamma\mu-\langle x,y\rangle)^2 - (\mu^2-x^2)(\gamma^2-\|y\|^2)}}{\gamma^2-\|y\|^2}.$$

Note though, that

$$\begin{aligned} \gamma\beta_- - \mu &= \gamma\frac{\gamma\mu - \langle x,y\rangle - \sqrt{(\gamma\mu-\langle x,y\rangle)^2 - (\mu^2-\|x\|^2)(\gamma^2-\|y\|^2)}}{\gamma^2-\|y\|^2} - \mu \\ &= \gamma\frac{(\gamma\mu-\langle x,y\rangle)^2 - (\gamma\mu-\langle x,y\rangle)^2 + (\mu^2-\|x\|^2)(\gamma^2-\|y\|^2)}{(\gamma\mu-\langle x,y\rangle + \sqrt{(\gamma\mu-\langle x,y\rangle)^2 - (\mu^2-\|x\|^2)(\gamma^2-\|y\|^2)})(\gamma^2-\|y\|^2)} - \mu \\ &= \frac{\gamma(\mu^2-\|x\|^2)}{\gamma\mu-\langle x,y\rangle + \sqrt{(\gamma\mu-\langle x,y\rangle)^2 - (\mu^2-\|x\|^2)(\gamma^2-\|y\|^2)}} - \mu \\ &\le \frac{\gamma(\mu^2-\|x\|^2)}{\gamma\mu-\|x\|\|y\| + \sqrt{(\gamma\mu-\|x\|\|y\|)^2 - (\mu^2-\|x\|^2)(\gamma^2-\|y\|^2)}} - \mu \\ &= \frac{\gamma(\mu^2-\|x\|^2)}{\gamma\mu-\|x\|\|y\| + |\mu\|y\|-\gamma\|x\||} - \mu \\ &= \frac{\mu\|x\|\|y\| - \gamma\|x\|^2 - |\mu^2\|y\| - \mu\gamma\|x\||}{\gamma\mu-\|x\|\|y\| + |\mu\|y\|-\gamma\|x\||}. \end{aligned}$$

We find that if $\mu\|y\| < \gamma\|x\|$, then clearly $\gamma\beta_- - \mu < 0$. If $\mu\|y\|\ge\gamma\|x\|$, then consider

$$\begin{aligned} \gamma\beta_- - \mu &\le \frac{\mu\|x\|\|y\| - \gamma\|x\|^2 - \mu^2\|y\| + \mu\gamma\|x\|}{\gamma\mu - \|x\|\|y\| + \mu\|y\| - \gamma\|x\|} \\ &= \frac{(\mu\|y\|-\gamma\|x\|)(\|x\|-\mu)}{(\gamma+\|y\|)(\mu-\|x\|)} = -\frac{\mu\|y\|-\gamma\|x\|}{\gamma+\|y\|} \le 0. \end{aligned}$$

Hence we find that $M((\mu,x)/(\gamma,y)) = \beta_+$. ☐

For all $u,v\in X_+$ we can rewrite this result using the quadratic and bilinear forms as

$$M(u/v) = \frac{B(u,v) + \sqrt{B(u,v)^2 - Q(u)Q(v)}}{Q(v)}.$$

Note that if $u,v\in \mathbf{H}$, then using Proposition 7.2.3 we find

$$\begin{aligned} \frac{1}{2}d_H(u,v) &= \frac{1}{2}\log\left(M(u/v)M(v/u)\right) \\ &= \log(B(u,v) + \sqrt{B(u,v)^2-1}) = \cosh^{-1}(B(u,v)), \end{aligned}$$

which shows that indeed on X_+ the hyperbolic metric δ_h coincides with Birkhoff's version of Hilbert's metric $\frac{1}{2}d_H$. We also need the following basic result from functional analysis.

7.2.4 Lemma. *Let (x_α) be a net in a Hilbert space H such that x_α converges in the weak topology to some $x \in H$ and $(\|x_\alpha\|)$ converges to some $r \geq 0$. Then $r \geq \|x\|$. Moreover, if $r = \|x\|$, then (x_α) converges to x in the norm topology.*

Proof. Note that

$$\|x\|^2 = \lim_\alpha |\langle x, x_\alpha\rangle| \leq \lim_\alpha \|x\|\|x_\alpha\| = r\|x\|.$$

Now suppose that $r = \|x\|$. Then

$$\lim_\alpha \|x - x_\alpha\|^2 = \lim_\alpha \|x\|^2 + \|x_\alpha\|^2 - 2\langle x, x_\alpha\rangle = 0.$$

□

Using this we can now characterise the horofunctions of the Hilbert geometry of the spin factor as follows.

7.2.5 Theorem. *Let H be an infinite dimensional Hilbert space and let $X = \mathbb{R} \times H$. The horofunctions of X with respect to the Hilbert geometry are precisely the functions of the following form:*

$$\xi((\gamma, y)) = 2\log\left(\frac{\gamma - \langle \hat{x}, y\rangle + \sqrt{(\gamma - \langle \hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1 + r)\sqrt{\gamma^2 - \|y\|^2}}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

where either $\|\hat{x}\| < 1$ and $\|\hat{x}\| < r \leq 1$ or $\|\hat{x}\| = r = 1$.

Proof. Let $((1, x_\alpha))$ be a net in X_+° such that $d_H(\cdot, (1, x_\alpha)) - d_H((1, 0), (1, x_\alpha))$ converges to a horofunction. By taking a subnet we may assume that (x_α) weakly converges to some $\hat{x} \in H$ as the unit ball is weakly compact and $(\|x_\alpha\|)$ converges to some $r \leq 1$. Note that by Lemma 7.2.4, $r \geq \|\hat{x}\|$. Let $(\gamma, y) \in X_+^\circ$. Using Proposition 7.2.3 we find

$$\begin{aligned}
M((1, x_\alpha)/(1, 0)) &= 1 + \|x_\alpha\| \\
M((1, 0))/(1, x_\alpha)) &= \frac{1 + \|x_\alpha\|}{1 - \|x_\alpha\|^2} \\
M((1, x_\alpha)/(\gamma, y)) &= \frac{\gamma - \langle x_\alpha, y\rangle + \sqrt{(\gamma - \langle x_\alpha, y\rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{\gamma^2 - \|y\|^2} \\
M((\gamma, y))/(1, x_\alpha)) &= \frac{\gamma - \langle x_\alpha, y\rangle + \sqrt{(\gamma - \langle x_\alpha, y\rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{1 - \|x_\alpha\|^2}.
\end{aligned}$$

Hence

$$\begin{aligned} i((1,x_\alpha))((\gamma,y)) =& \log(M((\gamma,y)/(1,x_\alpha))M((1,x_\alpha)/(\gamma,y))) \\ & - \log(M((1,0)/(1,x_\alpha))M((1,x_\alpha)/(1,0))) \\ =& 2\log\left(\frac{\gamma-\langle x_\alpha,y\rangle+\sqrt{(\gamma-\langle x_\alpha,y\rangle)^2-(1-\|x_\alpha\|^2)(\gamma^2-\|y\|^2)}}{\sqrt{\gamma^2-\|y\|^2}\sqrt{1-\|x_\alpha\|^2}}\right) \\ & - 2\log\left(\frac{1+\|x_\alpha\|}{\sqrt{1-\|x_\alpha\|^2}}\right) \\ =& 2\log\left(\frac{\gamma-\langle x_\alpha,y\rangle+\sqrt{(\gamma-\langle x_\alpha,y\rangle)^2-(1-\|x_\alpha\|^2)(\gamma^2-\|y\|^2)}}{(1+\|x_\alpha\|)\sqrt{\gamma^2-\|y\|^2}}\right). \end{aligned}$$

Taking the limit gives us

$$\xi((\gamma,y)) = 2\log\left(\frac{\gamma-\langle\hat{x},y\rangle+\sqrt{(\gamma-\langle\hat{x},y\rangle)^2-(\gamma^2-\|y\|^2)(1-r^2)}}{(1+r)\sqrt{\gamma^2-\|y\|^2}}\right).$$

Note that if $r=\|\hat{x}\|<1$, then $\xi=i(1,\hat{x})$. So $r>\|\hat{x}\|$, if $\|\hat{x}\|<1$.

Now suppose that a function is of the form as described above. Note that all we need to do is find a net $((1,x_\alpha))$ in X_+° such that (x_α) converges weakly to $\hat{x}$ and $(\|x_\alpha\|)$ converges to r. Then it will give rise to the desired horofunction by the above. If $\|\hat{x}\|=1$, consider the sequence $((1,(1-\frac{1}{n})\hat{x}))$, clearly this sequence converges strongly to $(1,\hat{x})$ and gives rise to a horofunction by the above. If $\|\hat{x}\|<1$, then let (e_n) be an orthonormal sequence in H, which exists as $\dim(H)=\infty$, and consider the sequence $((1,\hat{x}+\sqrt{r^2-\|\hat{x}\|^2}e_n))$. Note that $(\hat{x}+\sqrt{r^2-\|\hat{x}\|^2}e_n)$ converges weakly to $\hat{x}$, since (e_n) converges weakly to 0. Also note that

$$\lim_{n\to\infty}\|\hat{x}+\sqrt{r^2-\|\hat{x}\|^2}e_n\|^2 = \lim_{n\to\infty} r^2+2\sqrt{r^2-\|\hat{x}\|^2}\langle\hat{x},e_n\rangle = r^2.$$

□

In a similar way we find the horofunctions of the reverse-Funk and Funk geometry.

7.2.6 Theorem. *Let H be an infinite dimensional Hilbert space and let $X=\mathbb{R}\times H$. The horofunctions of X with respect to the reverse Funk geometry are precisely the functions of the following form:*

$$\xi((\gamma,y)) = \log\left(\frac{\gamma-\langle\hat{x},y\rangle+\sqrt{(\gamma-\langle\hat{x},y\rangle)^2-(\gamma^2-\|y\|^2)(1-r^2)}}{(1+r)(\gamma^2-\|y\|^2)}\right) \qquad ((\gamma,y)\in X_+^\circ)$$

where either $\|\hat{x}\|<1$ and $\|\hat{x}\|<r\leq 1$ or $\|\hat{x}\|=r=1$.

7.2.7 Theorem. *Let H be an infinite dimensional Hilbert space and let $X = \mathbb{R} \times H$. The horofunctions of X with respect to the Funk geometry are precisely the functions of the following form:*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y\rangle + \sqrt{(\gamma - \langle \hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1+r)}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

where either $\|\hat{x}\| < 1$ and $\|\hat{x}\| < r \leq 1$ or $\|\hat{x}\| = r = 1$.

Note that the proof of Theorem 7.2.5 also shows that ξ is a Busemann point if $\|\hat{x}\| = 1$. We can show that these are the only horofunctions that are Busemann points.

7.2.8 Theorem. *Let H be an infinite dimensional Hilbert space and let $X = \mathbb{R} \oplus H$, let $\hat{x} \in H$ and $\|\hat{x}\| \leq r \leq 1$ and let*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y\rangle + \sqrt{(\gamma - \langle \hat{x}, y\rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1+r)\sqrt{\gamma^2 - \|y\|^2}}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

be a horofunction. Then ξ is a Busemann point of the Hilbert geometry if and only if $\|\hat{x}\| = r = 1$.

Proof. In Theorem 7.2.5 we already proved that if $\|\hat{x}\| = r = 1$, then ξ is a Busemann point. Now suppose that ξ is a Busemann point and let $((1, x_\alpha))$ be an almost geodesic net such that $i((1, x_\alpha))$ converges to ξ. Combining Proposition 6.1.3 and Theorem 7.2.5 gives us that $d_H((1,0),(1,x_\alpha))$ is not bounded, so $\lim_\alpha \|x_\alpha\| = r = 1$. Note that we can rewrite the horofunction as

$$\xi((1, y)) = \log\left(\frac{(1 - \langle \hat{x}, y\rangle)^2}{1 - \|y\|^2}\right). \qquad ((1, y) \in X_+^\circ)$$

Now suppose $\|\hat{x}\| < 1$. Let $\epsilon > 0$ and let A be such that for all $\alpha' \geq \alpha \geq A$ we have

$$\varepsilon + d_H((1,0),(1,x_{\alpha'})) \geq d_H((1,0),(1,x_\alpha)) + d_H((1,x_\alpha),(1,x_{\alpha'}))$$

As in the proof of Theorem 7.2.5, using Proposition 7.2.3 we find

$$\begin{aligned}\varepsilon \geq &\log\left(\frac{(1 + \|x_\alpha\|)^2}{1 - \|x_\alpha\|^2}\right) - \log\left(\frac{(1 + \|x_{\alpha'}\|)^2}{1 - \|x_{\alpha'}\|^2}\right) \\ &+ 2\log\left(\frac{1 - \langle x_\alpha, x_{\alpha'}\rangle + \sqrt{(1 - \langle x_\alpha, x_{\alpha'}\rangle)^2 - (1 - \|x_\alpha\|^2)(1 - \|x_{\alpha'}\|^2)}}{\sqrt{1 - \|x_{\alpha'}\|^2}\sqrt{1 - \|x_\alpha\|^2}}\right).\end{aligned}$$

Taking the exponential we find

$$e^{\frac{1}{2}\varepsilon} \geq \frac{1 - \langle x_\alpha, x_{\alpha'}\rangle + \sqrt{(1 - \langle x_\alpha, x_{\alpha'}\rangle)^2 - (1 - \|x_\alpha\|^2)(1 - \|x_{\alpha'}\|^2)}}{(1 - \|x_\alpha\|)(1 + \|x_{\alpha'}\|)}.$$

As this holds for all $\alpha' \geq \alpha$, we can take the limit with respect to α' to get

$$\begin{aligned} e^{\frac{1}{2}\varepsilon} &\geq \lim_{\alpha'} \frac{1 - \langle x_\alpha, x_{\alpha'} \rangle + \sqrt{(1 - \langle x_\alpha, x_{\alpha'} \rangle)^2 - (1 - \|x_\alpha\|^2)(1 - \|x_{\alpha'}\|^2)}}{(1 - \|x_\alpha\|)(1 + \|x_{\alpha'}\|)} \\ &= \frac{1 - \langle x_\alpha, \hat{x} \rangle}{1 - \|x_\alpha\|}. \end{aligned}$$

Finally, as this holds for all $\alpha \geq A$, we can take the limit with respect to α to find

$$e^{\frac{1}{2}\varepsilon} \geq \lim_{\alpha} \frac{1 - \langle x_\alpha, \hat{x} \rangle}{1 - \|x_\alpha\|} = \infty,$$

which is a contradiction. □

Using similar arguments we can obtain the same result for the Busemann points of the reverse-Funk and Funk geometry.

7.2.9 Theorem. *Let H be an infinite dimensional Hilbert space and let $X = \mathbb{R} \oplus H$, let $\hat{x} \in H$ and $\|\hat{x}\| \leq r \leq 1$ and let*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y \rangle + \sqrt{(\gamma - \langle \hat{x}, y \rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1 + r)(\gamma^2 - \|y\|^2)}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

be a horofunction. Then ξ is a Busemann point of the reverse Funk geometry if and only if $\|\hat{x}\| = r = 1$.

7.2.10 Theorem. *Let H be an infinite dimensional Hilbert space and let $X = \mathbb{R} \oplus H$, let $\hat{x} \in H$ and $\|\hat{x}\| \leq r \leq 1$ and let*

$$\xi((\gamma, y)) = \log\left(\frac{\gamma - \langle \hat{x}, y \rangle + \sqrt{(\gamma - \langle \hat{x}, y \rangle)^2 - (\gamma^2 - \|y\|^2)(1 - r^2)}}{(1 + r)}\right) \qquad ((\gamma, y) \in X_+^\circ)$$

be a horofunction. Then ξ is a Busemann point of the Funk geometry if and only if $\|\hat{x}\| = r = 1$.

Theorem 7.2.1 follows from Theorem 7.2.5 and Theorem 7.2.8.

7.3 The horofunction boundary of JH-algebras

Finally we will classify the horofunction boundary of unital JH-algebras which are a special class of Jordan algebras.

7.3.1 Definition. Let $(X, \langle \cdot, \cdot \rangle)$ be a Jordan Algebra with an inner product which is associative with the Jordan product, i.e.

$$\langle xy, z \rangle = \langle y, xz \rangle.$$

If X is a Hilbert space we call X a JH-algebra. If furthermore X has a unit we call X a unital JH-algebra.

Note that a finite dimensional JH-algebra is a Euclidean Jordan Algebra and recall that a Jordan algebra can be equipped with the cone of squares $X_+ = \{x^2 : x \in X\}$. It was observed by Roelands and Wortel that unital JH-algebras can be characterised in the following way.

7.3.2 Theorem. *A unital JH-algebra is a finite direct sum of finite dimensional unital formally real Jordan Algebras and spin factors.*

This result and its proof can be found in [61, Lemma 7.15]. As mentioned in Chapter 4 the finite dimensional unital formally real Jordan algebras are exactly the unital Euclidean Jordan algebras. It is known by the Jordan-von Neumann-Wigner theorem [56] that the unital Euclidean Jordan algebras can be classified as finite direct sums of the spaces of self-adjoint matrices over $\mathbb{R}$, $\mathbb{C}$ or the quarternions $\mathbb{H}$, the 3×3-matrices over the octonions $\mathbb{O}$, or finite dimensional spin factors. We will not require this classification for our purposes as the horofunction boundary of the unital Euclidean Jordan algebras is fully classified by Lemmens, Lins, Nussbaum and Wortel.

7.3.3 Theorem (Theorem 5.6 [41])**.** *Let X be a unital Euclidean Jordan algebra with unit e and cone X_+. If we use e as base point, then the following assertions hold:*

(i) The horofunctions of the Funk geometry are precisely the functions of the following form

$$\xi_F(x) = d_R(x^{-1}, z) \qquad (x \in X_+^\circ),$$

where $z \in \partial X_+$ with $\|z\|_e = 1$.

(ii) The horofunctions of the reverse-Funk geometry are precisely the functions of the following form

$$\xi_R(x) = d_R(x, y) \qquad (x \in X_+^\circ),$$

where $y \in \partial X_+$ with $\|y\|_e = 1$.

(iii) The horofunctions of the Hilbert geometry are precisely the functions of the following form

$$\xi_H(x) = d_R(x^{-1}, z) + d_R(x, y) \qquad (x \in X_+^\circ),$$

where $y, z \in \partial X_+$ with $\|y\|_e = \|z\|_e = 1$ and $yz = 0$.

Recall that we have referred to a part of this theorem before in Section 4.2. Combined with our own classification of the horofunction boundary of the spin factors in Section 7.2 we can find the horofunction boundary of general unital JH-algebras. For this we will first study the horofunction boundary of finite direct sums of order-unit spaces.

Let $\{(X_k, (X_k)_+, e_k) : 1 \leq k \leq n\}$ be a set of order-unit spaces and consider

$$X = \bigoplus_{k=1}^{n} X_k$$

the direct sum. First note that

$$X_+ = \bigotimes_{k=1}^{n} (X_k)_+$$

is a cone of X. It follows that $(X, X_+, (e_1, \ldots, e_k))$ is an order-unit space. Indeed, for any $x = (x_1, \ldots, x_n) \in X$ we can take $M = \max\{M_k : x_k \leq M_k e_k\}$ and we find that $Me - x \in X_+$. We denote $\pi_k : X \to X_k$, given by $\pi_k((x_1, \ldots, x_n)) = x_k$ for $(x_1, \ldots x_n) \in X$, to be the linear projections maps onto X_k. Note that if $\varphi \in S(X)$ is a state, then as φ is linear and positive on the subspace $\{x \in X : \pi_i(x) = 0 \text{ for all } i \neq k\}$ we can find $\varphi_k \in S(X_k)$ and $\lambda_k \in [0, 1]$ such that $\sum_{k=1}^{n} \lambda_k = 1$ and

$$\varphi(x) = \sum_{k=1}^{n} \lambda_k \varphi_k(\pi_k(x))$$

i.e., a state of X is the convex combination of states of X_k. From this it follows that the pure states are given by

$$E(X) = \{\varphi_k \circ \pi_k \in X' : k \in \{1, \ldots, n\}, \varphi_k \in E(X_k)\}$$

i.e., the pure states of X are the composition of a linear projection π_k and a pure state of X_k. We can use this to find the horofunctions of the Funk and reverse-Funk geometry of X in terms of the horofunctions of the Funk and reverse-Funk geometry of X_k. First though we will show the following "Pigeonhole principle" for nets.

7.3.4 Proposition. *Let X be a set, let $(x_\alpha)_{\alpha \in J}$ be a net in X and let $f : X \to \{1, \ldots, n\}$ be a map. We define $J_i = \{\alpha \in J : f(x_\alpha) = i\}$. Then for some $1 \leq i \leq n$ we have that $(x_\alpha)_{\alpha \in J_i}$ is a subnet of $(x_\alpha)_{\alpha \in J}$.*

Proof. To prove this all we need to show is that there is an $i \in \{1, \ldots, n\}$ such that for all $\alpha, \beta \in J$ there is a $\gamma \in J_i$ such that $\alpha, \beta \leq \gamma$. Suppose this is not true. Then we can find $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in J$ such that there is no $\gamma \in J_i$ such that $\alpha_i, \beta_i \leq \gamma$. However as J is directed there is a $\gamma \in J$ such that $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \leq \gamma$. Since $J = \bigcup_{i=1}^{n} J_i$ this is a contradiction. □

We will now classify the horofunctions of the direct sum space. We will do this in two steps. We will first classify the set $\overline{i_R(X_+^\circ)}$. Then we further classify which of these functions are horofunctions.

7.3.5 Theorem. *Let $\{(X_k, (X_k)_+, e_k) : 1 \leq k \leq n\}$ be a set of order-unit spaces and let $\bigoplus_{k=1}^n X_k$ be an order-unit space with cone $X_+ = \bigotimes (X_k)_+$. The horofunctions of the reverse Funk geometry are precisely the functions of the form*

$$\xi_R(y) = \max_k \xi_k(\pi_k(y)) + \log(r_k) \qquad (y \in X_+^\circ)$$

where for all $k \in \{1, \dots, n\}$ we have $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ and $r_k \in [0,1]$ with $\max_k r_k = 1$ such that for some k we have that $r_k = 0$ or ξ_k is a horofunction.

Proof. Let $e = (e_1, \dots, e_n)$ be an order-unit of X. Let $S(X_k)$ be the state space of X_k with respect to e_k, and let $S(X)$ be the state space of X with respect to e. Let $x, y \in X_+$ and recall that by Proposition 6.2.5 we have

$$M(x/y) = \sup_{\varphi \in E(X)} \frac{\varphi(x)}{\varphi(y)}.$$

Since the pure states of X are the composition of a projection on X_k for some k and a pure state of X_k we find

$$M(x/y) = \max_k \sup_{\varphi \in E(X_k)} \frac{\varphi(\pi_k(x))}{\varphi(\pi_k(y))}.$$

Now let $\xi_R \in \overline{i_R(X_+^\circ)}$ and let (x_α) be a net in X_+° such that $(i_R(x_\alpha))$ converges to ξ_R. By taking a further subnet if necessary, we may assume that $(i_R(\pi_k(x_\alpha)))$ converges to some $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ for all k. Fix $y \in X_+$ and define $z_\alpha = M(x_\alpha/e)^{-1}x_\alpha$. Consider

$$\begin{aligned} i_R(x_\alpha)(y) &= d_R(y, x_\alpha) - d_R(e, x_\alpha) \\ &= \log(M(z_\alpha/y)) - \log(M(z_\alpha/e)) \\ &= \log\left(\max_k \sup_{\varphi \in E(X_k)} \frac{\varphi(\pi_k(z_\alpha))}{\varphi(\pi_k(y))}\right). \end{aligned}$$

So for each α and y we can find k such that

$$i_R(x_\alpha)(y) = \log\left(\sup_{\varphi \in E(X_{k_{\alpha,y}})} \frac{\varphi(\pi_{k_{\alpha,y}}(z_\alpha))}{\varphi(\pi_{k_{\alpha,y}}(y))}\right).$$

By Proposition 7.3.4 there exists a $k_y \in \{1, \dots, n\}$ such that there is a subnet for which

$$\lim_\alpha i_R(x_\alpha)(y) = \lim_\alpha \log\left(\sup_{\varphi \in E(X_{k_y})} \frac{\varphi(\pi_{k_y}(z_\alpha))}{\varphi(\pi_{k_y}(y))}\right) = \lim_\alpha \log(M(\pi_{k_y}(z_\alpha)/\pi_{k_y}(y))).$$

Note that, since $z_\alpha = M(x_\alpha/e)^{-1}x_\alpha \in X_+^\circ$, for all k we find that $0 < M(\pi_k(z_\alpha)/\pi_k(e)) \leq 1$. So we may assume that, by taking a further subnet if necessary, that $(M(\pi_k(z_\alpha)/\pi_k(e)))$ converges to some $r_k \in [0,1]$ for all k. Furthermore note that, as $y \in X_+^\circ$, we have $d_R(\pi_{k_y}(y), e_{k_y}) < \infty$. So if $\lim_\alpha M(\pi_{k_y}(z_\alpha)/\pi_{k_y}(e)) = 0$, then

$$\begin{aligned}\lim_\alpha i_R(z_\alpha)(y) &= \lim_\alpha d_R(\pi_{k_y}(y), \pi_{k_y}(z_\alpha)) \\ &\leq \lim_\alpha d_R(\pi_{k_y}(y), e_{k_y}) + d_R(e_{k_y}, \pi_{k_y}(z_\alpha)) = -\infty,\end{aligned}$$

which contradicts $\xi_R(y) \in \mathbb{R}$. It follows that $r_{k_y} \in (0,1]$. Hence

$$\begin{aligned}\xi_R(y) = \lim_\alpha i_R(x_\alpha)(y) &= \lim_\alpha \log\left(M(\pi_{k_y}(z_\alpha)/\pi_{k_y}(y))\right) \\ &\quad - \log(M(\pi_{k_y}(z_\alpha)/\pi_{k_y}(e))) + \log(M(\pi_{k_y}(z_\alpha)/\pi_{k_y}(e))) \\ &= \xi_{k_y}(y) + \log(r_{k_y})\end{aligned}$$

Now finally note that for the created subnet and for all k we have

$$\begin{aligned}\xi_k(y) + \log(r_k) &= \lim_\alpha \log\left(M(\pi_k(z_\alpha)/\pi_k(y))\right) \\ &= \log\left(\sup_{\varphi \in E(X_k)} \frac{\varphi(\pi_k(z_\alpha))}{\varphi(\pi_k(y))}\right) \\ &\leq \log\left(\sup_{\varphi \in E(X_{k_y})} \frac{\varphi(\pi_{k_y}(z_\alpha))}{\varphi(\pi_{k_y}(y))}\right) \\ &= \xi_{k_y}(\pi_{k_y}(y)) + \log(r_{k_y}) = \xi_R(y).\end{aligned}$$

So we find that

$$\xi_R(y) = \max_k \xi_{k_y}(\pi_k(y)) + \log(r_k)$$

where for all $k \in \{1, \ldots, n\}$ we have $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ and $r_k \in [0,1]$ with $\max_k r_k = 1$.

Now let ξ be of the form

$$\xi_R(y) = \max_k \xi_k(\pi_k(y)) + \log(r_k) \qquad (y \in X_+^\circ)$$

where for all $k \in \{1, \ldots, n\}$ we have $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ and $r_k \in [0,1]$ with $\max_k r_k = 1$. Let $(x_{i,\alpha})_{\alpha \in A_i}$ be nets in $(X_i)_+^\circ$ such that $\lim_{\alpha \in A_i} i_R(x_{i,\alpha}) = \xi_i$ and $d_R(e, x_\alpha) = 0$. Let $A_0 = \mathbb{N}$, we define the index set

$$A = A_0 \times A_1 \times \ldots \times A_n$$

with lexicographical partial order, i.e., for all $\alpha, \alpha' \in A$ we have $\alpha \leq \alpha'$ if and only if $\alpha(i) \leq \alpha'(i)$ for all $0 \leq i \leq n$. Note that A is a directed set. Now consider the net $(x_\alpha)_{\alpha \in A}$ in X_+°, where $\pi_i(x_\alpha) = r_i x_{i,\alpha}$ if $r_i > 0$ and $\pi_i(x_\alpha) = \frac{1}{\alpha(0)} x_{i,\alpha}$ if $r_i = 0$.

Note that $(x_{k,\alpha(k)})_{\alpha \in A}$ is a subnet of $(x_{k,\alpha})_{\alpha \in A_k}$ for all $1 \leq k \leq n$. Hence for all $1 \leq k \leq n$, if $r_k > 0$, then

$$\lim_{\alpha \in A} i_R(\pi_k(x_\alpha)) = \lim_{\alpha \in A} i_R(r_k x_{k,\alpha(k)}) = \xi_k$$

and

$$\lim_{\alpha \in A} M(\pi_k(x_\alpha)/e_k) = \lim_{\alpha \in A} M(r_k x_{k,\alpha}/e_k) = r_k.$$

If $r_k = 0$, then

$$\lim_{\alpha \in A} i_R(\pi_k(x_\alpha)) = \lim_{\alpha \in A} i_R\left(\frac{1}{\alpha(0)} x_{k,\alpha(k)}\right) = \xi_k.$$

and

$$\lim_{\alpha \in A} M(\pi_k(x_\alpha)/e_k) = \lim_{\alpha \in A} M\left(\frac{1}{\alpha_0} x_{k,\alpha}/e_k\right) = 0.$$

We will show that a subnet of $(i_R(x_\alpha))_{\alpha \in A}$ converges to ξ_R.

As $\overline{i_R(X_+^\circ)}$ is compact we know there is a subnet $(i_R(x_\alpha))_{\alpha \in A'}$ which converges to some $\gamma \in \overline{i_R(X_+^\circ)}$. Fix $y \in X_+^\circ$. As before, by Proposition 7.3.4 we can find a $1 \leq k_y \leq n$ and a subnet $(x_\alpha)_{\alpha \in \hat{A}}$ such that

$$\begin{aligned}
\lim_{\alpha \in A'} i_R(x_\alpha)(y) = \lim_{\alpha \in \hat{A}} i_R(x_\alpha)(y) &= \lim_{\alpha \in \hat{A}} i_R(x_\alpha)(y) \max_k \log\left(\sup_{\varphi \in E(X_k} \frac{\varphi(x_{k,\alpha_k})}{\varphi(\pi_k(y))}\right) \\
&= \lim_{\alpha \in \hat{A}} i_R(x_\alpha)(y) \log\left(\sup_{\varphi \in E(X_k} \frac{\varphi(x_{k_y,\alpha_{k_y}})}{\varphi(\pi_{k_y}(y))}\right) \\
&= \lim_{\alpha \in \hat{A}} i_R(x_{k_y,\alpha_{k_y}}) + \log(M(x_{k_y,\alpha_{k_y}}/e_k)) \\
&= \xi_{k_y} + \log(r_{k_y})
\end{aligned}$$

By similar arguments as above we find that

$$\lim_{\alpha \in A'} i_R(x_\alpha)(y) = \max_k \xi_k + \log(r_k).$$

Finally, note that $x \in X_+^\circ$ if and only if for all $k \in \{1, \ldots, n\}$ there exist $x_k \in (X_k)_+^\circ$ such that $M(x_k/e_k) = 1$ and $r_1, ..., r_n > 0$ such that $x = (r_1 x_1, ..., r_n x_n)$. Let $z = M(x/e)^{-1} x = x / \max_k r_k$, then, using the above, for all y we find

$$i_R(x)(y) = i_R(z)(y) = \max_k i_R(z_k)(\pi_k(y)) + \log(r_k) - (\max_i \log(r_i)).$$

So for all functions ξ of the form

$$\xi_R(y) = \max_k \xi_k(\pi_k(y)) + \log(r_k) \qquad (y \in X_+^\circ)$$

where for all $k \in \{1, \dots, n\}$ we have $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ and $r_k \in [0,1]$ with $\max_k r_k = 1$, we find that $\xi \in i_R(X_+^\circ)$ if and only if for all $k \in \{1, \dots n\}$ we have $\xi_k = i_R(z_k)$ for some $z_k \in (X_k)_+^\circ$ and $r_k > 0$. □

Theorem 7.3.5 gives rise to some interesting horofunctions. First of all it should be noted that the functions ξ_k in Theorem 7.3.5 do not have to be horofunctions for ξ to be a horofunction, as can be seen from the following example.

7.3.6 Example. Let $Y = \mathbb{R} \times l_2$ be a spin factor and let $X = Y \times Y$. Let (x_n) be a sequence in X given by

$$x_n = ((1, \frac{1}{2}e_n), (1, \frac{1}{2}e_1)).$$

By Theorem 7.3.5 $(i_R(x_n))$ converges to some function ξ of the form

$$\xi_R(y) = \max(\xi_1(\pi_1(y)) + \log(r_1), \xi_2(\pi_2(y)) + \log(r_2)) \qquad (y \in X_+^\circ$$

where for $k = 1, 2$ we have $\xi_k \in \overline{i_R((X_k)_+^\circ)}$ and $r_k \in [0,1]$. Note that in this case $r_1 = r_2 = 1$. Using Proposition 7.2.3 we find that for all $(1, y) \in Y_+^\circ$ we have

$$\begin{aligned}
\xi_1((1,y)) &= \lim_{n\to\infty} \log\left(\frac{1 - \frac{1}{2}\langle y, e_n\rangle + \sqrt{(1 - \frac{1}{2}\langle y, e_n\rangle)^2 - (1 - \frac{1}{4}\|e_n\|^2)(1 + \|y\|^2)}}{(1 + \frac{1}{2}\|e_n\|)(1 - \|y\|^2)}\right) \\
&= \log\left(\frac{1 + \sqrt{1 - \frac{3}{4}(1 + \|y\|^2)}}{\frac{3}{2}(1 - \|y\|^2)}\right) \\
&= \log\left(\frac{2 + \sqrt{1 + 3\|y\|^2}}{3(1 - \|y\|^2)}\right)
\end{aligned}$$

By Theorem 7.2.6 we find that ξ_1 is a horofunction. Again by using Proposition 7.2.3 we find that

$$\begin{aligned}
\xi_2((1,y)) &= \lim_{n\to\infty} \log\left(\frac{1 - \frac{1}{2}\langle y, e_1\rangle + \sqrt{(1 - \frac{1}{2}\langle y, e_1\rangle)^2 - (1 - \frac{1}{4}\|e_1\|^2)(1 + \|y\|^2)}}{(1 + \frac{1}{2}\|e_1\|)(1 - \|y\|^2)}\right) \\
&= \log\left(\frac{2 - y(1) + \sqrt{(2 - y(1))^2 - 3 + 3\|y\|^2}}{3(1 - \|y\|^2)}\right) = i_R((1, \frac{1}{2}e_1))((1,y)).
\end{aligned}$$

Note that

$$\xi_1((1,0)) = \log(1) \leq \log(2) = \xi_2((1, -\frac{1}{2}e_1)),$$

hence

$$\xi(((1,0),(1,-\frac{1}{2}e_1))) = \xi_2((1,-\frac{1}{2}e_1)) = \log(2).$$

So for certain elements the maximum might appear at a non-horofunction.

In fact, if one of the scalars r_k in Theorem 7.3.5 equals zero, then ξ_k can be a non-horofunction for all k.

7.3.7 Example. Let X be as in Example 7.3.6 and let (x_n) be a sequence in X given by

$$x_n = ((1, \frac{1}{2}e_1), (n, \frac{n}{2}e_1)).$$

Then $r_1 = 0$ and $r_2 = 1$ and $\lim_{n\to\infty} i_R(x_n)(y) = \xi(y) = i_R((1, \frac{1}{2}e_1))(\pi_2(y))$.

We can find similar results for the horofunctions of the Funk geometry. The proofs for these are similar to those for the reverse-Funk geometry, hence will be omitted.

7.3.8 Theorem. *Let $\{X_k : 1 \leq k \leq n\}$ be a set of order-unit spaces and let $X = \bigoplus_{k=1}^n X_k$ be an order-unit space with cone $X_+ = \bigoplus (X_k)_+$. The horofunctions of the Funk geometry on X_+ are of the form*

$$\xi_F(y) = \max_k \xi_k(\pi_k(y)) + \log(r_k) \qquad (y \in X_+^\circ)$$

where for all $k \in \{1, \ldots, n\}$ we have $\xi_k \in \overline{i_F((X_k)_+^\circ)}$ and $r_k \in [0,1]$ with $\max_k r_k = 1$ such that for some k we have that $r_k = 0$ or ξ_k is a horofunction.

Though it is hard to give a classification of the horofunctions of the Hilbert geometry for general direct sums of order-unit spaces, it is possible for the JH-algebras. For this we will use the classification of the spin factors in Section 7.2 and the aforementioned classification of the unital Euclidean Jordan algebras in [41].

Combining these results with the classification of spin factors in Section 7.2 and Theorem 7.3.5 and Theorem 7.3.8, we can classify the horofunctions of the Hilbert geometry of JH-algebras. To help with the formulation of the result, we will use the following definition.

7.3.9 Definition. Let (X, X_+, e) be an order-unit space where the cone X_+ is equipped with Hilbert's metric. We call a function $\xi \in \overline{i_H(X_+^\circ)}$ *finite* if for all nets (x_α) in X_+ with $\lim_\alpha i(x_\alpha) = \xi$ we have that $d_H(e, x_\alpha)$ is bounded. If ξ is not finite, we call ξ *infinite*.

Note that in a finite dimensional order-unit space (X, X_+, e) all horofunctions are infinite. One can verify that all $\xi \in i_H(X_+^\circ)$ are finite.

7.3.10 Lemma. *Let (M, d) be a proper geodesic metric space. If (x_α) is a net in M such that $\lim_\alpha i(x_\alpha) = i(x)$ for some $x \in M$, then (x_α) converges to x.*

Proof. Fix $b \in M$ as a base point and let (x_α) be a net in M such that $\lim_\alpha i(x_\alpha) = i(x)$ for some $x \in M$. Suppose that (x_α) has a subnet $(x_\beta)_{\beta \in B}$ such that $(d(b, x_\beta))_{\beta \in B}$ diverges to infinity. We define $\hat{x}_\beta$ to be on the geodesic between x_β and b such that $d(\hat{x}_\beta, b) = 2d(x, b)$, note that, since $(d(b, x_\beta))_{\beta \in B}$ diverges to infinity, $\hat{x}_\beta$ exists for β large enough. Also note that, since X is proper, a subnet of $(\hat{x}_\beta)$ converges to some $\hat{x} \in M$. We find that

$$-d(b,x) < d(\hat{x},x) - d(b,x) = i(x)(\hat{x}) = \lim_\alpha i(x_\alpha)(\hat{x}) = \lim_{\beta \in B} d(\hat{x}, x_\beta) - d(b, x_\beta)$$
$$\leq \lim_{\beta \in B} d(\hat{x}, \hat{x}_\beta) + d(\hat{x}_\beta, x_\beta) - d(b, x_\beta) = \lim_{\beta \in B} -d(b, \hat{x}_\beta) = -2d(b,x)$$

which is a contradiction. So (x_α) is bounded. Since M is proper we may assume, by taking a further subnet if required, that (x_α) converges to some $\hat{x} \in M$. By Proposition 2.4.6 we have that $i(x) = \lim_\alpha i(x_\alpha) = i(\hat{x})$.

Finally suppose that $\hat{x} \neq x$. Let x' be on the geodesic between x and $\hat{x}$ such that $d(x, x') = d(x', \hat{x}) = \frac{1}{2} d(x, \hat{x})$. Consider

$$0 = i(x)(x') - i(\hat{x})(x') = d(x', x) - d(b, x) - d(x', \hat{x}) + d(b, \hat{x}) = d(b, \hat{x}) - d(b, x).$$

So $d(b, \hat{x}) - d(b, x) = 0$. Consider

$$0 = i(x)(x) - i(\hat{x})(x) = d(b, \hat{x}) - d(b, x) - d(x, \hat{x}) = -d(x, \hat{x}) < 0.$$

Hence (x_α) converges to x. □

If M is not a proper metric space the result does not need to hold.

7.3.11 Example. Consider $X = \{0\} \times [0, \infty) \cup \mathbb{N}_{>0} \times (0, \infty)$ equipped with the metric d given by

$$d((n,r),(m,s)) = \begin{cases} |r-s| & \text{if } n = m. \\ r+s & \text{if } n \neq m. \end{cases} \qquad ((n,r),(m,s) \in X)$$

X is known as an $\mathbb{R}$-tree. If we take $(0,0)$ as a base point then for all $(m,r) \in X$ we have

$$\lim_{n\to\infty} i((n,n))((m,r)) = \lim_{n\to\infty} d((m,r),(n,n)) - d((0,0),(n,n))$$
$$= \lim_{n\to\infty} n + r - n = r = i((0,0))((m,r)).$$

So $((n,n))$ is unbounded, but $(i((n,n)))$ converges to $i((0,0))$.

As a Euclidean Jordan algebra is finite dimensional, the rays of the interior of the cone equipped with Hilbert's metric is a proper geodesic metric space. Therefore we have the following corollary.

7.3.12 Corollary. *Let (X, X_+, e) be a Euclidean Jordan algebra with closed cone X_+ equipped with Hilbert's metric. If (x_α) is a net in X_+° such that $\lim_\alpha i_H(x_\alpha) = i_H(x)$ for some $x \in X_+^\circ$ with $\|x\| = 1$, then $(\frac{x_\alpha}{\|x_\alpha\|})$ converges to x. Furthermore, $\lim_\alpha d_H(e, x_\alpha) = d_H(e, x)$.*

A similar result can be obtained for spin factors.

7.3.13 Lemma. *Let $X = \mathbb{R} \times \mathbb{H}$ be a spin factor. If $\xi \in \overline{i(X_+^\circ)}$ is finite, then there exists an $0 \leq r < 1 \in \mathbb{R}$ such that for any net $((1, x_\alpha))$ in X_+° such that $\lim_\alpha i_H((1, x_\alpha)) = \xi$ we have $\lim_\alpha d_H((1,0),(1,x_\alpha)) = \frac{1+r}{1-r}$.*

Proof. Let $\xi \in \overline{i(X_+^\circ)}$ be finite and let $((1, x_\alpha))$ in X_+° be a net such that $\lim_\alpha i_H((1, x_\alpha)) = \xi$. By taking a subsequence if necessary we may assume that (x_α) converges weakly to some $x \in H$ and $(\|x_\alpha\|)$ converges to some $0 \leq r \leq 1$. Note that, if $\lim_\alpha \|x_\alpha\| = 1$, then by Proposition 7.2.3 we have

$$\lim_\alpha d_H((1,0),(1,x_\alpha)) = \lim_\alpha \frac{1 + \|x_\alpha\|}{1 - \|x_\alpha\|} = \infty,$$

so $r < 1$. Furthermore, by Proposition 7.2.3 for all $(\gamma, y) \in X_+^\circ$ we have

$$\begin{aligned}
\xi((\gamma, y)) &= \lim_\alpha i_H(x_\alpha)((\gamma, y)) \\
&= \lim_\alpha 2 \log \left(\frac{\gamma - \langle x_\alpha, y \rangle + \sqrt{(\gamma - \langle x_\alpha, y \rangle)^2 - (1 - \|x_\alpha\|^2)(\gamma^2 - \|y\|^2)}}{(1 + \|x_\alpha\|)\sqrt{\gamma^2 - \|y\|^2}} \right) \\
&= 2 \log \left(\frac{\gamma - \langle x, y \rangle + \sqrt{(\gamma - \langle x, y \rangle)^2 - (1 - r^2)(\gamma^2 - \|y\|^2)}}{(1 + r)\sqrt{\gamma^2 - \|y\|^2}} \right)
\end{aligned}$$

so ξ is uniquely determined by x and r, hence for every net $((1, y_\alpha))$ in X_+° such that $\lim_\alpha i_H((1, y_\alpha))$ we have that (y_α) weakly converges to x and $(\|y_\alpha\|)$ converges to r. It follows that

$$\lim_\alpha d_H((1,0),(1,y_\alpha)) = \lim_\alpha \frac{1 + \|y_\alpha\|}{1 - \|y_\alpha\|} = \frac{1+r}{1-r}.$$

□

We can now classify the Hilbert horofunction boundary of JH-algebras.

7.3.14 Theorem. *Let $X = \oplus_{i=1}^{n} X_i$ be a JH-algebra. The horofunctions of the Hilbert geometry are precisely the functions of the form*

$$\xi_H(z) = \max_i [\xi_{R,i}(\pi_i(z)) + \log(r_i)] + \max_j [\xi_{F,j}(\pi_j(z)) + \log(s_j)],$$

where $\xi_{R,i} \in \overline{i_R((X_i)_+^\circ)}$ and $\xi_{F,i} \in \overline{i_F((X_i)_+^\circ)}$ such that $\xi_{R,i} + \xi_{F,i} = \xi_{H,i} \in \overline{i_H((X_i)_+^\circ)}$, $r_i, s_i \in [0,1]$ such that $\max_i r_i = \max_i s_i = 1$ and precisely one of the following conditions is satisfied:

(i) For all $i \in \{1, \ldots, n\}$, if $\xi_{H,i}$ is finite, then we have $\min(r_i, s_i) = 0$.

(ii) For all $i \in \{1, \ldots, n\}$ we have that $\xi_{H,i}$ is finite, $r_i > 0$,

$$s_i = \lim_\alpha \exp(d_H(e_i, x_{\alpha,i}) - d_H(e_I, x_{\alpha,I}))\frac{r_I}{r_i}$$

where $I = argmax_j \{\lim_\alpha d_H(e_j, x_{\alpha,j}) - \log(r_i)\}$ and $\xi_{H,k}$ is a horofunction of X_i for some k.

Proof. We will first show that the horofunctions of the Hilbert geometry are always of this form. Let ξ_H be a horofunction and let $(x_\alpha) = ((x_{\alpha,1}, \ldots, x_{\alpha,n}))$ be a net in X_+° such that $\lim_\alpha i_H(x_\alpha) = \xi_H$. First note, as $i_H(x_\alpha) = i_R(x_\alpha) + i_F(x_\alpha)$, that horofunctions of the Hilbert geometry are the sum of a horofunction of the Funk geometry and the reverse-Funk geometry. Since Theorem 7.3.5 and Theorem 7.3.8 give a description of the horofunctions of the Funk and reverse-Funk geometry we find that

$$\xi_H(z) = \max_i \xi_{R,i}(\pi_i(z)) + \log(r_i) + \max_j \xi_{F,j}(\pi_j(z)) + \log(s_j),$$

where $\xi_{R,i} \in \overline{i_R((X_i)_+^\circ)}$, $\xi_{F,i} \in \overline{i_F((X_i)_+^\circ)}$ and $r_i, s_i \in [0,1]$ such that $\max_i r_i = \max_i s_i = 1$. Recall from the proof of Theorem 7.3.5 that $\xi_{R,i} = \lim_\alpha i_R(\pi_i(x_\alpha))$ and $\xi_{F,i} = \lim_\alpha i_F(\pi_i(x_\alpha))$, hence $\xi_{R,i} + \xi_{F,i} = \xi_{H,i} \in \overline{i_H((X_i)_+^\circ)}$. Now all we have to do is show that ξ_H satisfies condition (i) or condition (ii). Suppose ξ_H does not satisfy condition (i), i.e., there exists some j such that $\xi_{H,j}$ is a finite horofunction and $r_j, s_j > 0$. We will show that ξ_H satisfies condition (ii). Recall that by the proof of Theorem 7.3.5 and Theorem 7.3.8, and by Lemma 7.3.4 we have

$$r_j = \lim_\alpha \frac{\|x_{\alpha,j}\|_{e_j}}{\|x_\alpha\|_e} = \lim_\alpha \min_i \frac{M(x_{\alpha,j}/e_j)}{M(x_{\alpha,i}/e_i)} = \min_i \lim_\alpha \frac{M(x_{\alpha,j}/e_j)}{M(x_{\alpha,i}/e_i)}$$

and

$$s_j = \lim_\alpha \frac{M(e_j/x_{\alpha,j})}{M(e/x_\alpha)} = \lim_\alpha \min_i \frac{M(e_j/x_{\alpha,j})}{M(e_i/x_{\alpha,i})} = \min_i \lim_\alpha \frac{M(e_j/x_{\alpha,j})}{M(e_i/x_{\alpha,i})}.$$

Note that, as $\xi_{H,j}$ is finite, we have that

$$\lim_\alpha \log(M(x_{\alpha,j}/e_j)M(e_j/x_{\alpha,j})) = \lim_\alpha d_H(x_{\alpha,j}, e_j) < \infty.$$

Suppose there exists some k such that $\xi_{H,k}$ is not finite, then, since

$$\lim_\alpha \log(M(x_{\alpha,k}/e_k)M(e_k/x_{\alpha,k})) = \lim_\alpha d_H(x_{\alpha,k}, e_k) = \infty$$

we have that $M(x_{\alpha,k}/e_k)M(e_k/x_{\alpha,k})$ diverges to infinity. Since by assumption $r_j, s_j > 0$, we find

$$0 < r_j s_j = \min_i \lim_\alpha \frac{M(x_{\alpha,j}/e_j)}{M(x_{\alpha,i}/e_i)} \cdot \min_i \lim_\alpha \frac{M(e_j/x_{\alpha,j})}{M(e_i/x_{\alpha,i})} \leq \lim_\alpha \frac{M(x_{\alpha,j}/e_j)M(e_j/x_{\alpha,j})}{M(x_{\alpha,k}/e_k)M(e_k/x_{\alpha,k})} = 0.$$

So $\xi_{H,i}$ is finite for all $i \in \{1, \ldots, n\}$.

Suppose there exists some k such that $r_k = 0$ or $s_k = 0$. Then for some I and J we have that

$$0 = r_k s_k = \lim_\alpha \frac{M(x_{\alpha,k}/e_k)M(e_k/x_{\alpha,k})}{M(x_{\alpha,I}/e_I)M(e_J/x_{\alpha,J})} = \lim_\alpha \frac{\exp(d_H(x_{\alpha,k}, e_k))}{M(x_{\alpha,I}/e_I)M(e_J/x_{\alpha,J})}.$$

As $\exp(d_H(x_{\alpha,k}, e_k)) \geq 1$ we find that $M(x_{\alpha,I}/e_I)M(e_J/x_{\alpha,J})$ tends to infinity. Hence, since $M(x_{\alpha_j}/e_j)M(e_j/x_{\alpha,j})$ is finite we find

$$0 < r_j s_j = \min_i \lim_\alpha \frac{M(x_{\alpha,j}/e_j)}{M(x_{\alpha,i}/e_i)} \cdot \min_i \lim_\alpha \frac{M(e_j/x_{\alpha,j})}{M(e_i/x_{\alpha,i})} \leq \lim_\alpha \frac{M(x_{\alpha_j}/e_j)M(e_j/x_{\alpha,j})}{M(x_{\alpha,I}/e_I)M(e_J/x_{\alpha,J})} = 0.$$

So $r_i, s_i > 0$ for all $i \in \{1, \ldots, n\}$.

Finally, as $r_k > 0$ for all $k \in \{1, \ldots, n\}$, by Lemma 7.3.4 there is an I such that for all $k \in \{1, \ldots, n\}$ we have

$$\begin{aligned} s_k &= \min_i \lim_\alpha \frac{M(e_k/x_{\alpha,k})}{M(e_i/x_{\alpha,i})} = \lim_\alpha \frac{M(e_k/x_{\alpha,k})}{M(e_I/x_{\alpha,I})} \\ &= \lim_\alpha \frac{\exp(d_H(x_{\alpha,k}, e_k))(r_k \|x_\alpha\|_e)^{-1}}{\exp(d_H(x_{\alpha,I}, e_I))(r_I \|x_\alpha\|_e)^{-1}} \\ &= \lim_\alpha \frac{r_I}{r_k} \exp(d_H(x_{\alpha,k}, e_k) - d_H(x_{\alpha,I}, e_I)). \end{aligned}$$

So ξ_H satisifies either condition (i) or condition (ii).

For the second part of the proof, we will show that any function of this form is a horofunction of the Hilbert geometry. Let ξ_H be of the form above. For every $i \in \{1, \ldots, n\}$ we can be in one of the following three cases:

- Case 1, there is a $x_i \in X_i$ such that $i_H(x_i) = \xi_{H,i}$.

- Case 2, X_i is finite dimensional and $\xi_{H,i}$ is a horofunction in which case we can use Theorem 7.3.3.
- Case 3, X_i is infinite dimensional and $\xi_{H,i}$ is a horofunction in which case X_i is a spin-factor and we can use the results of Section 7.2.

For each case we will define a sequence $(y_{i,m})$ in $(X_i)_+^\circ$ such that $\lim_{n\to\infty} M(y_{i,m}/e_i) = r_i$ and $\lim_{m\to\infty} \frac{1}{m^2} M(e_i/y_{i,m}) = s_i$, and if $r_i > 0$ we have $\lim_{m\to\infty} i_R(y_{i,m}) = \xi_{R,i}$ and if $s_i > 0$ we have $\lim_{m\to\infty} i_F(y_{i,m}) = \xi_{F,i}$. Then we can define the sequence (y_m), given by $y_m = (y_{1,m}, \dots, y_{n,m})$ for all $m \in \mathbb{N}$. Using the same arguments as in the proof of Theorem 7.3.5 we find

$$\lim_{m\to\infty} i_H(y_m) = i_F(y_m) + i_R(y_m) = \max_i \xi_{R,i} + \log(r_i) + \max_j \xi_{F,j} + \log(s_j).$$

For each case we have to consider 4 scenarios; $r_i, s_i > 0$, $r_i = s_i = 0$, $r_i > 0 = s_i$ and $s_i > 0 = r_i$. Note that if $\xi_{H,i}$ is finite and $r_i, s_i > 0$, then ξ_H satisfies condition (ii), hence

$$s_i = \lim_\alpha \exp(d_H(e_i, x_{\alpha,i}) - d_H(e_I, x_{\alpha,I}))\frac{r_I}{r_i}$$

where $I = \operatorname{argmax}_j\{\lim_\alpha d_H(e_j, x_{\alpha,j}) - \log(r_i)\}$. By Lemma 7.3.12 and Lemma 7.3.13 we find that s_i is uniquely determined by r_i, r_I, $\xi_{H,i}$ and $\xi_{H,I}$, so we only have to consider the scenarios $r_i > 0$, $s_i > 0 = r_i$ and $r_i = s_i = 0$.

For case 1, let $x_i \in X_i$ such that $i_H(x_i) = \xi_{H,i}$. Note that by Lemma 7.3.12 and Lemma 7.3.13 $\xi_{H,i}$ is finite. If $r_i > 0$ we define

$$y_{i,m} = \frac{r_i x_i}{\|x_i\|_{e_i}}.$$

If $s_i > r_i = 0$ we define

$$y_{i,m} = M(e_i/x_i)\frac{x_i}{s_i m^2}.$$

Finally, if $s_i = r_i = 0$ we define

$$y_{i,m} = \frac{1}{m} x_i.$$

For case 2, note that if X_i is finite dimensional, then it is a finite dimensional Euclidean Jordan algebra with unit $e_i \in (X_i)_+^\circ$, and thus by Theorem 7.3.3 we know that there are $y, z \in \partial(X_i)_+$ with $\|y\|_{e_i} = \|z\|_{e_i} = 1$ and $y \bullet z = 0$ such that for all $x \in (X_i)_+$ we have $\xi_{F,i}(x) = d_R(x^{-1}, z)$ and $\xi_{R,i}(x) = d_R(x, y)$. Let $\{c_1, \dots, c_k\}$ be a Jordan frame such that $y = \sum_{j=1}^{p} \lambda_j c_j$ and $z = \sum_{j=p+1}^{q} \mu_j c_j$ with $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$ and $1 = \mu_{p+1} \geq \mu_2 \geq \ldots \geq \mu_q > 0$. If $r_i > 0 = s_i$ we define

$$y_{i,m} = \sum_{j=1}^{p} r_i \lambda_j c_j + \sum_{j=p+1}^{k} \frac{1}{m} c_j \in (X_i)_+^\circ.$$

If $s_i > 0 = r_i$ we define

$$y_{i,m} = \sum_{j=1}^{p} \frac{1}{m} c_j + \sum_{j=p+1}^{q} \frac{1}{m^2 s_i \mu_j} c_j + \sum_{j=q+1}^{k} \frac{1}{m} c_j \in (X_i)_+^\circ.$$

If $r_i, s_i > 0$ we define

$$y_{i,m} = \sum_{j=1}^{p} r_i \lambda_j c_j + \sum_{j=p+1}^{q} \frac{1}{m^2 s_i \mu_j} c_j + \sum_{j=q+1}^{k} \frac{1}{m} c_j \in (X_i)_+^\circ.$$

Finally if $r_i = s_i = 0$ we define

$$y_{i,m} = \sum_{j=1}^{k} \frac{1}{n} c_j.$$

For case 3, we know that X_i is a spin-factor $\mathbb{R} \times H$. By Theorem 7.2.6 and Theorem 7.2.7 we know that there exist $0 < r \leq 1$ and a $\hat{y} \in H$ with $\|\hat{y}\| \leq r$ such that

$$\xi_{R,i}((1,x)) = \log\left(\frac{1 - \langle \hat{y}, x\rangle + \sqrt{(1 - \langle \hat{y}, x\rangle)^2 - (1 - r^2)(1 - \|x\|^2)}}{(1+r)(1 - \|x\|^2)}\right)$$

and

$$\xi_{F,i}((1,x)) = \log\left(\frac{1 - \langle \hat{y}, x\rangle + \sqrt{(1 - \langle \hat{y}, x\rangle)^2 - (1 - r^2)(1 - \|x\|^2)}}{1+r}\right).$$

Let (d_m) be an orthonormal sequence of H, note that $(\hat{y} + \sqrt{r - \|\hat{y}\|^2} d_n)$ converges weakly to $\hat{y}$ and $(\|\hat{y} + \sqrt{r - \|\hat{y}\|^2} d_n\|)$ converges to r. We can now define the $y_{i,m}$ for finite and infinite $\xi_{H,i}$, as $\lim_{m\to\infty} M(y_{i,m}/e_i)$ and $\lim_{m\to\infty} \frac{1}{m^2} M(e_i/y_{i,m})$ are harder to compute for the spin factors we will show the calculation for $r_i, s_i > 0$ where $\xi_{H,i}$ is infinite. The other limits can be solved in similar fashion. If $\xi_{H,i}$ is finite we define $y_{i,m}$ as follows:
If $r_i > 0$ we define

$$y_{i,m} = \frac{r_i}{1+r}(1, \hat{y} + \sqrt{r - \|\hat{y}\|^2} d_m).$$

If $s_i > 0 = r_i$ we define

$$y_{i,m} = \frac{1}{m^2 s_i}(1, \hat{y} + \sqrt{r - \|\hat{y}\|^2} d_m).$$

Finally if $r_i = s_i = 0$ we define

$$y_{i,m} = \frac{1}{n}(1, 0).$$

If $\xi_{H,i}$ is infinite, then $r = 1$ and we define $y_{i,m}$ as follows:

If $r_i > 0 = s_i$ we define

$$y_{i,m} = \frac{(1+m)r_i}{2m}(1, (1 - \frac{1}{1+m})(\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m)).$$

If $s_i > 0 = r_i$ we define

$$y_{i,m} = \frac{1}{ms_i}(1, (1 - \frac{1}{1+m})(\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m)).$$

If $r_i = s_i = 0$ we define

$$y_{i,m} = \frac{1}{m}(1, 0).$$

Finally if $r_i, s_i > 0$ we define

$$y_{i,m} = \frac{1 + r_i s_i m^2}{2s_i m^2}(1, (1 - \frac{2}{1 + r_i s_i m^2})(\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m)).$$

We will now show that for this case $\lim_{m\to\infty} M(y_{i,m}/e_i) = r_i$ and $\lim_{m\to\infty} \frac{1}{m^2} M(e_i/y_{i,m}) = s_i$. By Proposition 7.2.3 and, since $(\|\hat{y} + \sqrt{r - \|\hat{y}\|^2} d_n\|)$ converges to 1, we have

$$\begin{aligned}
\lim_{m\to\infty} M(y_{i,m}/e_i) &= \lim_{m\to\infty} \frac{1 + r_i s_i m^2}{2s_i m^2} \left(1 + \|(1 - \frac{2}{1 + r_i s_i m^2})(\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m)\|\right) \\
&= \lim_{m\to\infty} \frac{1 + r_i s_i m^2}{2s_i m^2} \left(1 + \frac{r_i s_i m^2 - 1}{1 + r_i s_i m^2} \|\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m\|\right) \\
&= \lim_{m\to\infty} \frac{1 + r_i s_i m^2}{2s_i m^2} \left(1 + \frac{r_i s_i m^2 - 1}{1 + r_i s_i m^2}\right) \\
&= \lim_{m\to\infty} \frac{1 + r_i s_i m^2}{2s_i m^2} \frac{2r_i s_i m^2}{1 + r_i s_i m^2} = r_i
\end{aligned}$$

and

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m^2} M(e_i/y_{i,m}) &= \lim_{m\to\infty} \frac{1}{m^2} \frac{2s_i m^2}{1 + r_i s_i m^2} \left(\frac{1}{1 - \|(1 - \frac{2}{1 + r_i s_i m^2})(\hat{y} + \sqrt{1 - \|\hat{y}\|^2} d_m)\|}\right) \\
&= \lim_{m\to\infty} \frac{2s_i}{1 + r_i s_i m^2} \frac{1}{1 - \frac{r_i s_i m^2 - 1}{1 + r_i s_i m^2}} \\
&= \lim_{m\to\infty} \frac{2s_i}{1 + r_i s_i m^2} \frac{1 + r_i s_i m^2}{2} = s_i.
\end{aligned}$$

For (y_m) defined as above we find

$$\lim_{m\to\infty} i_H(y_m) = \max_i \xi_{R,i} + \log(r_i) + \max_j \xi_{F,j} + \log(s_j).$$

□

Note that the proof of Theorem 7.3.14 depends on the fact we can approach horofunctions in spin factors and unital Euclidean Jordan algebras with sequences instead of nets.

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