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# Angular momentum of free variable mass systems is partially conserved

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## Abstract

Variable mass systems are a classic example of open systems in classical mechanics with rockets being a standard practical example. Due to the changing mass, the angular momentum of these systems is not generally conserved. Here, we show that the angular momentum vector of a free variable mass system is fixed in inertial space and, thus, is a partially conserved quantity. It is well known that such conservation rules allow simpler approaches to solving the equations of motion. This is demonstrated by using a graphical technique to obtain an analytic solution for the second Euler angle that characterizes nutation in spinning bodies.

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## 1. Introduction

This brief note presents developments central to attitude motion analysis of free variable mass systems such as space rockets. Whereas the rigid-body motion of constant mass systems [1] has been scrutinized since the times of Euler, similar studies on variable mass systems are more recent. Contemporary researchers [2, 3] consider the mid-20th century document of Rosser et al [4] on the rotational behavior of rockets to be the first to address the topic of rigid-body motion of systems with changing mass. Rosser's study inspired work on rocket flight dynamics [5, 6] and general variable mass systems [7] which

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10 utilized discrete models for mass loss. A control volume approach [8, 9] to  
account for continuous mass variation subsequently emerged which has since  
become the modelling standard amongst the community of researchers on rocket  
flight dynamics. Recent work using the control volume formulation has focused  
on equation of motion formulation for general variable mass systems [10, 11],  
15 modeling and analysis of rocket-type systems [12, 13, 14, 15] and an abstraction  
of the rocket problem [16], and stability of transverse rotational motion in solid  
rocket motors [17]. The developments presented here on angular momentum  
also utilize this control volume formulation.

Since the Explorer-1 anomaly [18], the angular momentum property of spin-  
20 ning bodies has received significant attention in the field of spacecraft dynamics  
and controls as it provides a platform for attitude stability analysis [19, 20] and  
informs innovative attitude control strategies, such as dual-spin satellites [21].  
Angular momentum conservation of freely spinning bodies is the backbone that  
permits explicit solutions to the second Euler angle, or the nutation angle, of  
25 torque-free axisymmetric systems [22]. However, this is only true in the case  
of constant mass systems. In this paper, we show that the angular momentum  
vector of variable mass systems possesses a similar useful property.

We begin by showing that the angular momentum vector of any torque-free  
variable mass system has a fixed direction in space and, thus, is a partially  
30 conserved quantity. In comparison to previous analytical studies, the presenta-  
tion here does not assume axial symmetry in the internal mass flow or system  
geometry. In other words, the developments here are kept completely general  
so as to be applicable to a broad set of systems including, but not limited to,  
rockets. Following this conservation result, we demonstrate its utility in graphi-  
35 cally determining the second Euler angle of mass-varying systems. We conclude  
by briefly discussing the availability of analytic and closed-form solutions in the  
case of axisymmetric rockets. Thus, our work provides a basis to analyze the  
rotational motion of variable mass systems.

The work presented here creates avenues for future work. The conserva-  
40 tion result provides a foundation for developing criteria for motion stability for

a variety of mass-varying systems. For example, one can imagine parallels to Poincot's geometric interpretations of motion stability in constant mass systems. Such stability analyses will naturally lead to informing attitude control system design and development for a variety of vehicles. Further, mass-varying systems transcend aerospace applications such as rockets; they are also found in marine engineering where vehicles utilize mass variation for propulsion. More broadly, the conservation property will interest researchers concerned with the identification of conservation laws for mass-varying systems [23]. Thus, this result will impact a wide community of researchers.

## 2. Angular Momentum of a Variable Mass System

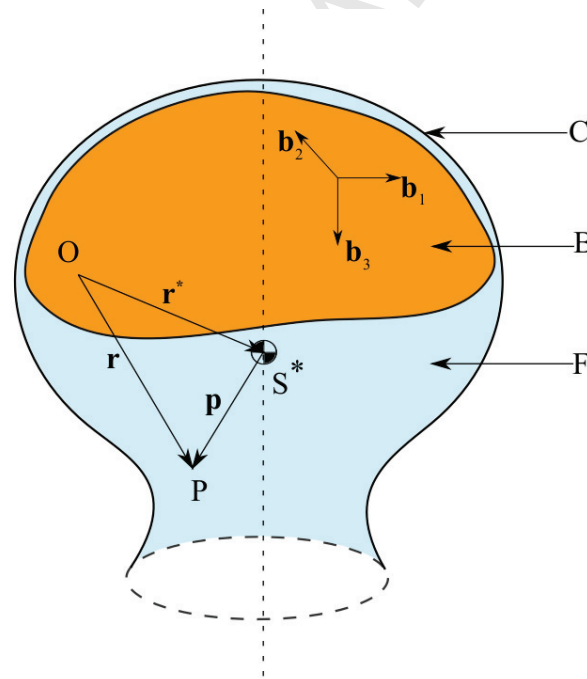


Figure 1: General Variable Mass System

Figure 1 is that of a system with mass variation comprising a consumable rigid base  $B$  and a fluid phase  $F$ . A massless shell  $C$  of volume  $V_0$  and surface

area  $S_0$  is attached to  $B$ . It is assumed that mass can enter or exit  $C$  through the region represented by a dashed ellipse. The shell and everything within it is considered to be of interest, while any matter outside of it is not. At any instant, there is a definite set of matter within the region  $C$  which obeys the laws of mechanics. At another instant,  $C$  may contain a different set of matter but it too must obey the laws of mechanics at that instant. Thus, the angular momentum principle can be applied to  $C$  and its contents to derive the vector equation of attitude motion that are valid at each instant of time.

At any general instant of time, there is a definite set of matter within  $C$ . At that instant, the angular momentum of this constant mass system about its mass center  $S^*$ , denoted  $\mathbf{H}^*$ , is given by

$$\mathbf{H}^* = \int_V \rho \mathbf{p} \times \mathbf{v} dV, \quad (1)$$

where  $V$  is the volume occupied by the contents of the constant mass system at the instant of interest,  $\rho$  is the mass density,  $\mathbf{p}$  is a position vector from  $S^*$  to an arbitrary particle  $P$  within  $C$ , and  $\mathbf{v}$  is the inertial velocity of  $P$ . It is easier to visualize the motion of particles such as  $P$  from  $B$ , as opposed to the inertial frame, and thus the angular momentum expression is reformulated as

$$\mathbf{H}^* = \int_V \rho \mathbf{p} \times [\mathbf{v}^o + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}] dV, \quad (2)$$

where  $\mathbf{r}$  is a position vector from  $O$  to  $P$ ,  $\mathbf{v}^o$  is the inertial velocity of  $O$ ,  $\mathbf{v}_r$  is the velocity of  $P$  relative to  $B$ , and  $\boldsymbol{\omega}$  is the inertial angular velocity of  $B$ . For spacecraft, the components of the vector terms in Equation (2) are usually available in the body-fixed frame  $\mathbf{b}_i$  for  $i = (1, 2, 3)$ ; this is assumed to be the case in the developments presented here. Equation (2) is then expanded as

$$\begin{aligned} \mathbf{H}^* &= \int_V \rho \mathbf{p} dV \times \mathbf{v}^o + \int_V \rho \mathbf{p} \times \mathbf{v}_r dV \\ &+ \int_V \rho \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{r}) dV. \end{aligned} \quad (3)$$

The first integral on the right-hand side of Equation (3) evaluates to zero by virtue of the definition of a mass center. Further, from Figure 1, it is evident

that  $\mathbf{r} = \mathbf{r}^* + \mathbf{p}$  where  $\mathbf{r}^*$  is the position vector from  $O$  to  $S^*$  so Equation (3) can be rewritten as

$$\begin{aligned} \mathbf{H}^* &= \int_V \rho \mathbf{p} \times \mathbf{v}_r \, dV + \int_V \rho \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{r}^*) \, dV \\ &\quad + \int_V \rho \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) \, dV. \end{aligned} \quad (4)$$

The second volume integral on the right-hand side of Equation (4) evaluates to zero, again, by the definition of a mass center. Equation (4) is now written in  
80 a compact form as

$$\mathbf{H}^* = \int_V \rho \mathbf{h} \, dV, \quad (5)$$

where  $\mathbf{h}$  is

$$\mathbf{h} = \mathbf{p} \times [\mathbf{v}_r + (\boldsymbol{\omega} \times \mathbf{p})]. \quad (6)$$

Equation (5) now gives the instantaneous angular momentum of a constant mass system. The angular momentum principle applied to this constant mass system about its mass center is

$$\mathbf{M}^* = \frac{{}^N D \mathbf{H}^*}{Dt}, \quad (7)$$

85 where  $\mathbf{M}^*$  is the sum of all moments due to external forces on the constant mass system, and  $\frac{{}^N D}{Dt}$  is the material derivative observed from an inertial frame  $N$ . In the case of torque-free motion,  $\mathbf{M}^* = \mathbf{0}$  which when used in Equation (7) gives

$$\mathbf{0} = \frac{{}^N D}{Dt} \int_V \rho \mathbf{h} \, dV. \quad (8)$$

Note that, in Equation (8),  $\mathbf{H}^*$  has been expressed in its integral form, given  
90 by Equation (5). The above equation tells us that the angular momentum of the constant mass system is invariant. If we choose to switch from the inertial reference frame to a reference frame attached to  $B$  then Equation (8) can be rewritten as

$$\mathbf{0} = \frac{{}^B D}{Dt} \int_V \rho \mathbf{h} \, dV + \boldsymbol{\omega} \times \int_V \rho \mathbf{h} \, dV. \quad (9)$$

In the above form, the two terms on the right hand side of Equation (9) focus on  
95 the constant mass system. Attention can be transferred to the control volume with fluxing matter via two operations. Firstly, Reynolds Transport Theorem

is invoked on the first term on the right-hand side of Equation (9). Secondly, noticing that, at the instant for which the above equation is derived,  $V = V_0$ . As a result, Equation (9) becomes

$$\begin{aligned} \mathbf{0} = & \frac{{}^B d}{dt} \int_{V_0} \rho \mathbf{h} dV + \int_{S_0} \rho \mathbf{h} (\mathbf{v}_r \cdot \mathbf{n}) dS \\ & + \boldsymbol{\omega} \times \int_{V_0} \rho \mathbf{h} dV. \end{aligned} \quad (10)$$

100 In the above equation,  ${}^B d/dt$  is a time derivative taken in a reference frame attached to  $B$ , and  $\mathbf{n}$  is an outwardly directed unit normal from a surface of  $C$  through which mass enters and/or exits; note that the orientation of  $\mathbf{n}$  is fixed relative to  $C$ . If  $\mathbf{v}_r \cdot \mathbf{n} = u$ , where  $u$  is a general scalar variable, Equation (10) can be rewritten as

$$\mathbf{0} = \frac{{}^B d\mathbf{H}_0^*}{dt} + \int_{S_0} \rho \mathbf{h} u dS + \boldsymbol{\omega} \times \mathbf{H}_0^*, \quad (11)$$

105 where  $\mathbf{H}_0^*$  is the angular momentum of the variable mass system and is

$$\mathbf{H}_0^* = \int_{V_0} \rho \mathbf{h} dV. \quad (12)$$

Since  $V = V_0$  at a particular instant,  $\mathbf{H}^*$  and  $\mathbf{H}_0^*$  are identical but their time derivatives are generally not identical since their evolution in time is associated with changing sets of matter. Since our interest is in understanding the behaviour of the variable mass system's angular momentum from an inertial  
110 frame, we revert the time derivative in Equation (11) to  $N$

$$\mathbf{0} = \frac{{}^N d\mathbf{H}_0^*}{dt} + \int_{S_0} \rho \mathbf{h} u dS. \quad (13)$$

In the above equation,  ${}^N d/dt$  is a time derivative taken in the inertial reference frame  $N$ . Any vector can be expressed as a combination of a scalar and a unit vector directed along the vector itself. So,  $\mathbf{h}$  is rewritten as  $\mathbf{h} = h\mathbf{n}_h$ , where  $\mathbf{n}_h$  is a unit vector directed along  $\mathbf{h}$  whose magnitude is  $h$ . As a result, Equation  
115 (12) can be written as

$$\mathbf{H}_0^* = \left( \int_{V_0} \rho h dV \right) \mathbf{n}_h \quad (14)$$

and Equation (13) as

$$\frac{N d\mathbf{H}_o^*}{dt} = \left( - \int_{S_o} \rho h u dS \right) \mathbf{n}_h \quad (15)$$

Equation (14) asserts that  $\mathbf{H}_o^*$  is generally not of constant magnitude while Equations (14) and (15) assert that it is always directed along the  $\mathbf{n}_h$  vector, which it will now be proved is an inertially fixed vector.

120 Let  $\mathbf{n}_f$  and  $\mathbf{n}_g$  be two unit vectors which form a dextral set with  $\mathbf{n}_h$  such that  $\mathbf{n}_f \times \mathbf{n}_g = \mathbf{n}_h$  and so on. This dextral set of unit vectors are attached to an imaginary reference frame  $Q$  whose inertial angular velocity is expressed as

$$\boldsymbol{\omega}^Q = \Omega_1 \mathbf{n}_f + \Omega_2 \mathbf{n}_g + \Omega_3 \mathbf{n}_h. \quad (16)$$

The time rate of change of  $\mathbf{n}_h$  in the inertial frame is

$$\frac{N d\mathbf{n}_h}{dt} = \frac{Q d\mathbf{n}_h}{dt} + \boldsymbol{\omega}^Q \times \mathbf{n}_h \quad (17)$$

where the first term on the right-hand side of Equation (17) evaluates to zero since  $\mathbf{n}_h$  is fixed in  $Q$ . Then, substituting for  $\boldsymbol{\omega}^Q$  from Equation (16) in Equation 125 (17) gives

$$\frac{N d\mathbf{n}_h}{dt} = \Omega_2 \mathbf{n}_f - \Omega_1 \mathbf{n}_g. \quad (18)$$

Further, Equation (15) is rewritten as

$$\frac{N d}{dt} \int_{V_o} \rho h dV \mathbf{n}_h = \left( - \int_{S_o} \rho u h dS \right) \mathbf{n}_h \quad (19)$$

or

$$\begin{aligned} \frac{d}{dt} \left( \int_{V_o} \rho h dV \right) \mathbf{n}_h + \left( \int_{V_o} \rho h dV \right) \frac{N d\mathbf{n}_h}{dt} \\ = \left( - \int_{S_o} \rho u h dS \right) \mathbf{n}_h. \end{aligned} \quad (20)$$

The result from Equation (18) is substituted in Equation (20) to give

$$\begin{aligned} \frac{d}{dt} \left( \int_{V_o} \rho h dV \right) \mathbf{n}_h + \left( \int_{V_o} \rho h dV \right) (\Omega_2 \mathbf{n}_f - \Omega_1 \mathbf{n}_g) \\ = \left( - \int_{S_o} \rho u h dS \right) \mathbf{n}_h. \end{aligned} \quad (21)$$

130 The above equation, when rewritten in component form, leads to  $\Omega_1 = \Omega_2 = 0$ . Using these values for  $\Omega_1$  and  $\Omega_2$  in Equation (18) gives  $\frac{N d\mathbf{n}_h}{dt} = \mathbf{0}$ , which



explains that  $\mathbf{n}_h$  is an inertially fixed unit vector thus, also making  $Q$  an inertial frame. By extension, it is also evident that the angular momentum of a variable mass system is also an inertially fixed vector as it is directed along  $\mathbf{n}_h$ .

### 135 3. Discussion

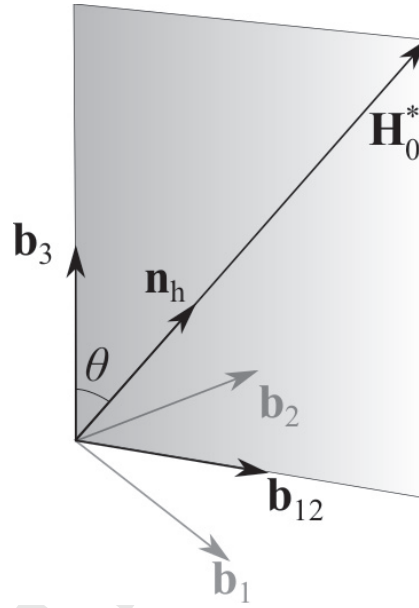


Figure 2: Graphical Evaluation of Nutation Angle

As mentioned in the introductory section, this directional conservation of the angular momentum vector is extremely useful in attitude determination. Figure 2 shows the setup of the angular momentum vector  $\mathbf{H}_0^*$  relative to the body-fixed principal directions  $\mathbf{b}_i (i = 1, 2, 3)$ . The angular momentum vector is seen to lie in the plane made by  $\mathbf{b}_3$  and  $\mathbf{b}_{12}$ , the latter is a unit vector in the  $\mathbf{b}_1$ - $\mathbf{b}_2$  plane. The angular momentum vector can then be expressed as a linear combination of two vectors in the  $\mathbf{b}_{12} - \mathbf{b}_3$  plane as

$$\mathbf{H}_0^* = \mathbf{H}_{12} + \mathbf{H}_3 = H_{12}\mathbf{b}_{12} + H_3\mathbf{b}_3 \quad (22)$$

where

$$\mathbf{b}_{12} = \frac{\mathbf{H}_{12}}{H_{12}}. \quad (23)$$

Our interest here is in evaluating  $\theta$ , the angle between  $\mathbf{b}_3$  and  $\mathbf{n}_h$ . Since  $\mathbf{n}_h$  is  
 145 an inertially fixed vector,  $\theta$  gives attitude information about the system from  
 an inertial reference frame and, for the 3-1-3 Euler sequence of rotations, it is  
 the second Euler angle.

From fig. 2, it is clear that

$$\theta = \tan^{-1}\left(\frac{H_{12}}{H_3}\right). \quad (24)$$

In the case of variable mass systems, the difficulty lies in evaluating  $H_{12}$  and  
 150  $H_3$  because the internal flow pattern of the fluid phase is not known everywhere  
 inside the control volume. However, for systems such as space-rockets, it is rea-  
 sonable to assume that the fluid flow relative to the rigid base is axisymmetric.  
 Consequently, the expression to the angular momentum in eq. (12) simplifies to  
 $\mathbf{H}_0^* = \int_{V_0} \rho \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p}) dV$ . Further, it is reasonable to assume that the rocket  
 155 is axisymmetric about  $\mathbf{b}_3$ . Then, the angular momentum can be expressed as

$$\mathbf{H}_0^* = \mathbf{I}_0^* \cdot \boldsymbol{\omega} = I\omega_{12}\mathbf{b}_{12} + J\omega_3\mathbf{b}_3, \quad (25)$$

where  $\omega_{12}$  is the angular speed in the  $\mathbf{b}_{12}$  direction,  $\omega_3$  is the spin rate in the  
 $\mathbf{b}_3$  direction, and  $I$  and  $J$  are moment of inertia scalars. The angular speeds  
 and moments of inertia are known parameters; in the case of the axisymmetric  
 cylinder [24], these parameters are known explicitly for a variety of idealized  
 160 models of mass loss. Thus,  $\theta$  is also known explicitly and is given by

$$\theta = \tan^{-1}\left(\frac{I\omega_{12}}{J\omega_3}\right). \quad (26)$$

The expression for  $\theta$  in eq. (26) is identical to that for an axisymmetric constant-  
 mass system [22], but the properties of the parameters are inherently different  
 due to mass variation; in the constant mass case all these parameters are con-  
 stant and, thus,  $\theta$  is constant.

165 In the classical mechanics literature,  $\theta$  is referred to as the nutation angle [1]  
 and  $\mathbf{b}_3$  is the spin axis. In the case of spacecraft, growths in this angle have an

undesirable effect on its heading direction. A control strategy for this nutation instability, suggested by eq. (26), is to impart a high spin rate to an axisymmetric rocket, thus, increasing the angular momentum in the spin direction. Such  
170 nutation instabilities have been observed in solid rocket motors [25] but the cause for the instability remains an open problem.

In summary, it has been shown that the angular momentum of a free variable mass system is inertially fixed and is, thus, a partially conserved quantity. This result can serve as the foundation for analytical and geometric examinations  
175 of the rotational motions of variable mass systems. Further, the utility of this result has been demonstrated with a brief discussion on graphically evaluating the second Euler angle without integrating the differential equation of motion. This analytical result provides footing for investigating nutation stability and developing control algorithms for a variety of systems with mass variation.

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