Correlation for permutations

J. Robert Johnson* Imre Leader† Eoin Long‡

April 2, 2020

Abstract

In this note we investigate correlation inequalities for 'up-sets' of permutations, in the spirit of the Harris–Kleitman inequality. We focus on two well-studied partial orders on S_n, giving rise to differing notions of up-sets. Our first result shows that, under the strong Bruhat order on S_n, up-sets are positively correlated (in the Harris–Kleitman sense). Thus, for example, for a (uniformly) random permutation π, the event that no point is displaced by more than a fixed distance d and the event that π is the product of at most k adjacent transpositions are positively correlated. In contrast, under the weak Bruhat order we show that this completely fails: surprisingly, there are two up-sets each of measure $1/2$ whose intersection has arbitrarily small measure.

We also prove analogous correlation results for a class of non-uniform measures, which includes the Mallows measures. Some applications and open problems are discussed.

1 Introduction

Let $X = \{1, 2, \ldots, n\} = [n]$. A family $\mathcal{F} \subset \mathcal{P}(X) = \{A : A \subset X\}$ is an *up-set* if given $F \in \mathcal{F}$ and $F \subset G \subset X$ then $G \in \mathcal{F}$. The well-known

*School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK. E-mail: r.johnson@qmul.ac.uk.

†DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK. E-mail: i.leader@dpmms.cam.ac.uk.

‡School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK. E-mail: e.long@bham.ac.uk.

and very useful Harris–Kleitman inequality [7, 9] guarantees that any two up-sets from $\mathcal{P}(X)$ are positively correlated. In other words, if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ are both up-sets then

$$\frac{|\mathcal{A} \cap \mathcal{B}|}{2^n} \geq \frac{|\mathcal{A}|}{2^n} \times \frac{|\mathcal{B}|}{2^n}.$$

The result has been very influential, and was extended several times to cover more general contexts [6, 8, 1]. However, for the most part, these results tend to focus on distributive lattices (such as $\mathcal{P}(X)$) and it is natural to wonder whether correlation persists outside of this setting.

In this note we aim to explore analogues of the Harris–Kleitman inequality for sets of permutations. There are two particularly natural notions for what it means for a family of permutations to be an up-set here, and the level of correlation that can be guaranteed in these settings turns out to differ greatly.

We write S_n for the set of all permutations of $[n]$, which throughout the paper we regard as ordered n-tuples of distinct elements of $[n]$. That is, if $\mathbf{a} \in S_n$ then $\mathbf{a} = (a_1, \ldots, a_n)$ where $\{a_k\}_{k \in [n]} = [n]$. Given $\mathbf{a} \in S_n$ and $i \in [n]$, we write $\text{pos}(\mathbf{a}, i)$ for the *position* of i in $\mathbf{a}$, i.e. $\text{pos}(\mathbf{a}, i) = k$ if $a_k = i$. Given $1 \leq i < j \leq n$, the pair $\{i, j\}$ is said to be an *inversion* in $\mathbf{a}$ if $\text{pos}(\mathbf{a}, i) > \text{pos}(\mathbf{a}, j)$. We will write $\text{inv}(\mathbf{a})$ for the set of all inversions in $\mathbf{a}$. A pair $\{i, j\} \in \text{inv}(\mathbf{a})$ is *adjacent* in $\mathbf{a}$ if $\text{pos}(\mathbf{a}, i) = \text{pos}(\mathbf{a}, j) + 1$.

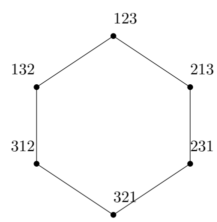
Definition. *Given a family of permutations $\mathcal{A} \subset S_n$, we say that:*

(i) $\mathcal{A}$ is a strong up-set *if given $\mathbf{a} \in \mathcal{A}$, any permutation obtained from $\mathbf{a}$ by swapping the elements in a pair $\{i, j\} \in \text{inv}(\mathbf{a})$ is also in $\mathcal{A}$.*

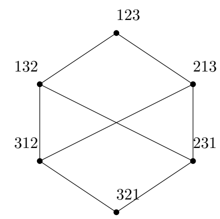
(ii) $\mathcal{A}$ is a weak up-set *if given $\mathbf{a} \in \mathcal{A}$, any permutation obtained from $\mathbf{a}$ by swapping the elements in an adjacent pair $\{i, j\} \in \text{inv}(\mathbf{a})$ is also in $\mathcal{A}$.*

We remark that both strong and weak up-sets have natural interpretations in the context of posets (see Chapter 2 of [4]). Given $\mathbf{a}, \mathbf{b} \in S_n$ write $\mathbf{a} \leq_s \mathbf{b}$ if $\mathbf{b}$ can be reached from $\mathbf{a}$ by repeatedly swapping inversions. We write $\mathbf{a} \leq_w \mathbf{b}$ if $\mathbf{b}$ can be reached from $\mathbf{a}$ by repeatedly swapping adjacent inversions. These relations give well-studied partial orders, known as the strong Bruhat order and the weak Bruhat order respectively. A (strong or weak) up-set is then simply a family which is closed upwards in the corresponding order[1].

[1]Note that this is in agreement with the usual notion of an up-set in $\mathcal{P}(X)$, starting from the poset $(\mathcal{P}(X), \subseteq)$.



The weak order for $n = 3$



The strong order for $n = 3$

It is clear that every strong up-set is also a weak up-set, but the opposite relation is not true. For $i, j \in [n]$ let $\mathcal{U}_{ij} = \{\mathbf{a} \in S_n : i \text{ occurs before } j \text{ in } \mathbf{a}\}$. If $i < j$, then $\mathcal{U}_{ij}$ is a weak up-set but not a strong up-set (except $\mathcal{U}_{1n}$). For example, when $n = 3$, the family $\mathcal{U}_{12} = \{123, 132, 312\}$ is a weak up-set but not a strong up-set (because 213 differs from 312 by swapping 2 and 3 into increasing order but $213 \notin \mathcal{U}_{12}$).

Our first result is that strong up-sets are positively correlated in the sense of Harris–Kleitman. That is, if $\mathcal{A}, \mathcal{B} \subset S_n$ are strong up-sets then

$$\frac{|\mathcal{A} \cap \mathcal{B}|}{n!} \geq \frac{|\mathcal{A}|}{n!} \times \frac{|\mathcal{B}|}{n!}.$$

As we will also consider non-uniform measures, we phrase this in a more probabilistic way. We will say that a probability measure μ on S_n is *positively associated* (for the strong order) if $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$ for all strong up-sets $\mathcal{A}$ and $\mathcal{B}$.

Theorem 1. *The uniform measure $\mu(\mathcal{A}) = \frac{|\mathcal{A}|}{n!}$ on S_n is positively associated (for the strong order).*

For weak up-sets the situation is more complicated. We saw that both $\mathcal{U}_{12}$ and $\mathcal{U}_{23}$ are weak up-sets and each has size $n!/2$. But $\mathcal{U}_{12} \cap \mathcal{U}_{23}$ is the family of permutations in which $1, 2, 3$ occur in ascending order. So, for $n \geq 3$ we have

$$\frac{|\mathcal{U}_{12} \cap \mathcal{U}_{23}|}{n!} = \frac{1}{6} < \frac{1}{4} = \frac{|\mathcal{U}_{12}|}{n!} \times \frac{|\mathcal{U}_{23}|}{n!}.$$

A plausible guess might be that every two up-sets $\mathcal{A}$ and $\mathcal{B}$ with size $n!/2$ achieve at least this level of correlation. Surprisingly, this turns out to be far from the truth; such an $\mathcal{A}$ and $\mathcal{B}$ can be almost disjoint.

Theorem 2. *Let $0 < \alpha, \beta < 1$ be fixed. Then there are weak up-sets $\mathcal{A}, \mathcal{B} \subset S_n$ with $|\mathcal{A}| = \lfloor \alpha n! \rfloor$, $|\mathcal{B}| = \lfloor \beta n! \rfloor$ and $|\mathcal{A} \cap \mathcal{B}| = \max(|\mathcal{A}| + |\mathcal{B}| - |S_n|, 0) + o(n!)$.*

The correlation given in Theorem 2 is (essentially) minimal, since *any* two families $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy $|\mathcal{A} \cap \mathcal{B}| \geq \max\big(|\mathcal{A}| + |\mathcal{B}| - |S_n|, 0\big)$.

Theorem 2 shows in quite a strong sense that the uniform measure on S_n is not positively associated under the weak order. Our next result will prove positive association for a wider collection of measures under the strong order, giving a different generalisation of Theorem 1.

Before describing these measures, we first give an alternative representation of elements of S_n (essentially the Lehmer encoding of permutations – see Chapter 11.4 of [10]). Given $\mathbf{a} \in S_n$ we can associate a vector $\mathbf{f}(\mathbf{a}) = (f_1, \ldots, f_n) \in G_n := [1] \times [2] \times \cdots [n]$, with

$$f_j := |\{i \in [j] : \mathrm{pos}(\mathbf{a}, i) \leq \mathrm{pos}(\mathbf{a}, j)\}|.$$

In other words, f_j describes where element j appears in the n-tuple $\mathbf{a}$ in relation to the elements from $[j]$. This gives a bijection between S_n and G_n, and our positively associated measures on S_n are built from this connection.

Definition. *Let $X_1, \ldots, X_n$ be independent random variables, where each X_k takes values in $[k]$. The* independently generated measure μ *defined by $\{X_k\}_{k \in [n]}$ is the following probability measure on S_n: given $\mathbf{a} \in S_n$ we have*

$$\mu(\mathbf{a}) := \prod_{k \in [n]} \mathbb{P}\big(X_k = \mathbf{f}(\mathbf{a})_k\big).$$

We simply say that μ is independently generated if this holds for some such collection of $\{X_k\}_{k \in [n]}$.

Our second positive result applies to independently generated measures.

Theorem 3. *Every independently generated probability measure on S_n is positively associated.*

We note that the uniform measure on S_n is independently generated, taking X_k to simply be uniform on $[k]$. Thus Theorem 3 implies Theorem 1.

We note that one special case of an independently generated measure is the Mallows measure [11]. Recalling the definition of $\mathrm{inv}(\mathbf{a})$ above, the Mallows measure with parameter $0 < q \leq 1$ is defined by setting

$$\mu(\mathbf{a}) \propto q^{|\,\mathrm{inv}(\mathbf{a})|}.$$

That is, $\mu(\mathbf{a}) = \left(\sum_{\mathbf{a}\in S_n} q^{|\,\mathrm{inv}(\mathbf{a})|}\right)^{-1} \cdot q^{|\,\mathrm{inv}(\mathbf{a})|}$.

Our results in fact go beyond independently generated measures, and it turns out that here a key idea is a notion of up-set that sits 'between' the weak and strong up-sets above. This notion, which we call 'grid up-sets' (defined in Section 2), provides an environment that is suitable for FKG-like inequalities. This approach will allow us to strengthen Theorem 3 to apply to measures satisfying more general conditions.

Before closing the introduction, we note that while we have stated our results for up-sets, it is easy to obtain equivalent down-set versions of Theorems 1–3 (for example, see Chapter 19 of [5]). These follow by noting that a set A in a partial order $(P, <)$ is an up-set if and only if $A^c = P \setminus A$ is a down-set. Indeed, if μ is a probability measure on P and $\mu(A \cap B) \geq \mu(A) \cdot \mu(B)$ then we obtain the complementary inequality

$$\begin{aligned}\mu(A^c \cap B^c) = 1 - \mu(A) - \mu(B) + \mu(A \cap B) &\geq 1 - \mu(A) - \mu(B) + \mu(A) \cdot \mu(B)\\ &= \mu(A^c) \cdot \mu(B^c).\end{aligned}$$

The plan of the paper is as follows. In Section 2 we prove our positive association results. Here we give a self-contained proof of Theorem 3. We also introduce grid up-sets and use them to extend Theorem 3. In Section 3 we prove Theorem 2, constructing weak up-sets with bad correlation properties. Section 4 gives some applications of our main results, to families of permutations defined with bounded 'displacements', sequential domination properties, as well as to left-compressed set systems. Finally, in Section 5, we raise some questions and directions for further work.

2 Correlation for strong up-sets

In this section we will prove Theorem 3. As noted in the introduction, the uniform case is an immediate corollary (Theorem 1). The proof will use induction on n. To relate a family of permutations of $[n]$ with a family of permutations of some smaller ground set, we 'slice' according to position of element n. Given a family $\mathcal{A} \subset S_n$ and $k \in [n]$ let $\mathcal{A}_k \subset S_{n-1}$ denote those permutations obtained by deleting the appearance of element 'n' from $\mathbf{a} \in \mathcal{A}$ with $\mathrm{pos}(\mathbf{a}, n) = k$. That is:

$$\mathcal{A}_k := \big\{(a_1, a_2, \ldots, a_{n-1}) \in S_{n-1} : (a_1, \ldots, a_{k-1}, n, a_k, \ldots, a_{n-1}) \in \mathcal{A}\big\}.$$

In the next simple lemma we collect two properties of the slice operation which will be useful later.

Lemma 4. *If $\mathcal{A} \subset S_n$ is a strong up-set and the slices $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \subset S_{n-1}$ are defined as above then:*

(i) *$\mathcal{A}_k$ is a strong up-set for all $k \in [n]$, and*

(ii) $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \cdots \subset \mathcal{A}_n$.

Proof. Part (i) is immediate. To see (ii), note that if $\mathbf{a} \in \mathcal{A}_k$ then we have $(a_1, \ldots, a_{k-1}, n, a_k, \ldots, a_{n-1}) \in \mathcal{A}$. Now, as $\mathcal{A}$ is a strong up-set and $n > a_k$, the pair $\{a_k, n\} \in \mathrm{inv}(\mathbf{a})$ and we find $(a_1, \ldots, a_k, n, a_{k+1}, \ldots, a_{n-1}) \in \mathcal{A}$, giving $\mathbf{a} \in \mathcal{A}_{k+1}$. $\square$

We will also need the following simple and standard arithmetic inequality, which will be used to relate the conditional probabilities of the slices in S_{n-1} to probabilities in S_n. We provide a proof for completeness.

Lemma 5. *Let $u_1, \ldots, u_n, v_1, \ldots, v_n, t_1, \ldots, t_n \in [0, \infty)$ with $u_1 \leq \ldots \leq u_n$, $v_1 \leq \ldots \leq v_n$ and $\sum_{k=1}^n t_k \leq 1$. Then*

$$\sum_{k=1}^n t_k u_k v_k \geq \left(\sum_{k=1}^n t_k u_k\right)\left(\sum_{k=1}^n t_k v_k\right).$$

Proof. For convenience set $u_0 = v_0 = 0$. Then, for all $k \in [n]$ set $x_k = u_k - u_{k-1}$ and $y_k = v_k - v_{k-1}$. Note that the conditions on u_k and v_k give $x_k, y_k \geq 0$. Now,

$$\sum_{k=1}^n t_k u_k v_k = \sum_{k=1}^n t_k(x_1 + \cdots + x_k)(y_1 + \cdots + y_k) = \sum_{i,j} r_{i,j} x_i y_j,$$

where

$$r_{i,j} = \begin{cases} t_i + \cdots + t_n & \text{if } i \geq j, \\ t_j + \cdots + t_n & \text{if } i \leq j. \end{cases}$$

Similarly,

$$\left(\sum_{k=1}^n t_k u_k\right)\left(\sum_{\ell=1}^n t_\ell v_\ell\right) = \left(\sum_{k=1}^n t_k \big(\sum_{i=1}^k x_i\big)\right)\left(\sum_{\ell=1}^n t_\ell \big(\sum_{j=1}^\ell y_j\big)\right) = \sum_{i,j} s_{i,j} x_i y_j,$$

where $s_{i,j} = (t_i + \cdots + t_n)(t_j + \cdots + t_n)$. As $t_k \geq 0$ for all $k \in [n]$ and $\sum_{k=1}^n t_k \leq 1$, we see that $r_{i,j} \geq s_{i,j}$ for all i, j and the result follows. $\square$

Proof of Theorem 3. We wish to show that if μ is an independently generated probability measure on S_n and $\mathcal{A}, \mathcal{B} \subset S_n$ are strong up-sets in S_n then $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$. We will prove this by induction on n. The statement is trivial for $n = 1$. Assuming that the statement holds for $n - 1$, we will prove it for n.

To begin, note that as μ is independently generated on S_n, it is defined by independent random variables $\{X_i\}_{i \in [n]}$. Take ν to denote the independently generated measure on S_{n-1} defined by the independent random variables $\{X_i\}_{i \in [n-1]}$. By definition of μ, if $\mathbf{a} = (a_1, \ldots, a_n) \in S_n$ with $a_k = n$ then setting $\mathbf{a}_{[n-1]} := (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \in S_n$ we have

$$\mu(\mathbf{a}) = \nu(\mathbf{a}_{[n-1]}) \cdot \mathbb{P}(X_n = k).$$

It follows that given any family $\mathcal{F} \subset S_n$ we have $\mu(\mathcal{F} | X_n = k) = \nu(\mathcal{F}_k)$. Note that the measure ν does not depend on k, which is important below.

With this in hand, suppose that $\mathcal{A}, \mathcal{B} \subset S_n$ are strong up-sets. Then

$$\mu\big(\mathcal{A} \cap \mathcal{B}\big) = \sum_{k \in [n]} \mathbb{P}(X_n = k) \mu\big(\mathcal{A} \cap \mathcal{B} | X_n = k\big) = \sum_{k \in [n]} \mathbb{P}(X_n = k) \nu\big((\mathcal{A} \cap \mathcal{B})_k\big),$$

where the second equality follows by the previous paragraph. Clearly we have $(\mathcal{A} \cap \mathcal{B})_k = \mathcal{A}_k \cap \mathcal{B}_k$. Moreover, as both $\mathcal{A}_k$ and $\mathcal{B}_k$ are strong up-sets by Lemma 4 (i) and ν is independently generated, by induction we have $\nu(\mathcal{A}_k \cap \mathcal{B}_k) \geq \nu(\mathcal{A}_k) \cdot \nu(\mathcal{B}_k)$. Applying this above gives

$$\mu(\mathcal{A} \cap \mathcal{B}) \geq \sum_{k \in [n]} \mathbb{P}(X_n = k) \cdot \nu(\mathcal{A}_k) \cdot \nu(\mathcal{B}_k) = \sum_{k \in [n]} t_k u_k v_k,$$

where $t_k = \mathbb{P}(X_n = k)$, $u_k = \nu(\mathcal{A}_k)$ and $v_k = \nu(\mathcal{B}_k)$. Note now from Lemma 4 (ii) that we have $u_1 \leq \ldots \leq u_n$, $v_1 \leq \ldots \leq v_n$ and $\sum_{k \in [n]} t_k = 1$. Thus the hypothesis of Lemma 5 applies, and this lemma gives

$$\begin{aligned}
\mu(\mathcal{A} \cap \mathcal{B}) \geq \sum_{k \in [n]} t_k u_k v_k &\geq \Big(\sum_{k \in [n]} t_k u_k\Big)\Big(\sum_{k \in [n]} t_k v_k\Big) \\
&= \Big(\sum_{k \in [n]} \mathbb{P}(X_n = k) \nu(\mathcal{A}_k)\Big)\Big(\sum_{k \in [n]} \mathbb{P}(X_n = k) \nu(\mathcal{B}_k)\Big) \\
&= \mu(\mathcal{A}) \cdot \mu(\mathcal{B}).
\end{aligned}$$

This completes the proof of the theorem. □

In contrast to this self-contained proof, our second proof will use the machinery of the FKG inequality in the following form.

Theorem 6 (FKG inequality [6])**.** *Let L be a finite distributive lattice and let μ be a probability measure on L satisfying*

$$\mu(x \wedge y) \cdot \mu(x \vee y) \geq \mu(x) \cdot \mu(y).$$

for all $x, y \in L$. Then any up-sets $\mathcal{A}, \mathcal{B} \subset L$ satisfy $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$.

To make use of Theorem 6 recall that the permutations S_n are in one-to-one correspondence with elements of the grid $G_n := [1] \times [2] \times \cdots \times [n]$, where $\mathbf{a} \in S_n$ is indentified with $\mathbf{f}(\mathbf{a}) \in G_n$. Using this correspondence we will transfer the 'grid' partial order $\leq_g$ on G_n to S_n, where $\mathbf{f} \leq_g \mathbf{g}$ for $\mathbf{f}, \mathbf{g} \in G_n$ if $\mathbf{f}_i \leq \mathbf{g}_i$ for all $i \in [n]$.

Definition. *The* grid order $\leq_g$ *on S_n is given by defining $\mathbf{a} \leq_g \mathbf{b}$ if $\mathbf{f}(\mathbf{a}) \leq_g \mathbf{f}(\mathbf{b})$ when viewed as elements of G_n. A family $\mathcal{A} \subset S_n$ is a* grid up-set *if whenever $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in S_n$ with $\mathbf{a} \leq_g \mathbf{b}$ then $\mathbf{b} \in \mathcal{A}$.*

We now use the FKG inequality to G_n to give a second proof of Theorem 3. In fact this approach strengthens the result in two ways: it applies to grid up-sets rather than just strong up-sets, and it applies to measures satisfying a more general FKG-type condition.

Let $\mathbf{a}, \mathbf{b} \in S_n$. As G_n is a distributive lattice we can define $\mathbf{a} \vee \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ in S_n in natural way: let $\mathbf{a} \vee \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$ be the unique elements of S_n with:

$$\mathbf{f}(\mathbf{a} \vee \mathbf{b})_k = \max\left\{\mathbf{f}(\mathbf{a})_k, \mathbf{f}(\mathbf{b})_k\right\}; \qquad \mathbf{f}(\mathbf{a} \wedge \mathbf{b})_k = \min\left\{\mathbf{f}(\mathbf{a})_k, \mathbf{f}(\mathbf{b})_k\right\}.$$

Theorem 7. *Suppose that μ is a probability measure on S_n with*

$$\mu(\mathbf{a} \vee \mathbf{b}) \cdot \mu(\mathbf{a} \wedge \mathbf{b}) \geq \mu(\mathbf{a}) \cdot \mu(\mathbf{b}) \tag{1}$$

for all $\mathbf{a}, \mathbf{b} \in S_n$. Then any grid up-sets $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$.

Proof of Theorem 7. Transfer μ from S_n to G_n, by setting $\mu(\mathbf{f}(\mathbf{a})) = \mu(\mathbf{a})$ for all $\mathbf{a} \in S_n$. As $\mathbf{f} : S_n \to G_n$ is a bijection this defines μ on G_n. By choice of the operations $\vee$ and $\wedge$ on S_n above, (1) implies that

$$\mu(\mathbf{f} \vee \mathbf{g}) \cdot \mu(\mathbf{f} \wedge \mathbf{g}) \geq \mu(\mathbf{f}) \cdot \mu(\mathbf{g})$$

for all $\mathbf{f}, \mathbf{g} \in G_n$. The result now follows by applying Theorem 6 to G_n. $\square$

To complete our second proof of Theorem 3, by Theorem 7, it is enough to show that (i) every strong up-set is a grid up-set and (ii) that (1) holds for independently generated measures. This is content of the next two lemmas.

Lemma 8. *If* $\mathbf{a}, \mathbf{b} \in S_n$ *with* $\mathbf{a} \leq_g \mathbf{b}$ *then* $\mathbf{a} \leq_s \mathbf{b}$. *Consequently, every strong up-set in* S_n *is also a grid up-set.*

Proof. Suppose that $\mathbf{a}, \mathbf{b} \in S_n$ where $(\mathbf{a}, \mathbf{b})$ is a covering relation in the grid order. That is, there is $i \in [n]$ with $\mathbf{f}(\mathbf{b})_i = \mathbf{f}(\mathbf{a})_i + 1$ and $\mathbf{f}(\mathbf{a})_j = \mathbf{f}(\mathbf{b})_j$ for all $j \neq i$. It suffices to show that $\mathbf{a} \leq_s \mathbf{b}$ by transitivity, since every relation in $\leq_g$ can be expressed as a sequence of covering relations.

Let $\mathrm{pos}(\mathbf{a}, i) = k$ and take $\ell > k$ minimal so that $\mathrm{pos}(\mathbf{a}, j) = \ell$ for some $j < i$; such a choice of ℓ must exist since $(\mathbf{a}, \mathbf{b})$ is a covering relation with $\mathbf{f}(\mathbf{b})_i = \mathbf{f}(\mathbf{a})_i + 1$. It is clear that $\mathbf{a}$ and $\mathbf{b}$ differ only in position k and ℓ, where $a_k = b_\ell = i$ and $a_\ell = b_k = j$. Thus $\{i, j\} \in \mathrm{inv}(\mathbf{a})$ and swapping these entries we obtain $\mathbf{b}$, i.e. $\mathbf{a} \leq_s \mathbf{b}$.

Lastly, if $\mathcal{A}$ is a strong up-set with $\mathbf{a} \in \mathcal{A}$ and $\mathbf{a} \leq_g \mathbf{b}$ then $\mathbf{a} \leq_s \mathbf{b}$, and so $\mathbf{b} \in \mathcal{A}$. Thus $\mathcal{A}$ is a grid up-set, as required. □

Lemma 9. *Inequality* (1) *holds for every independently generated probability measure* μ *on* S_n.

Proof. Suppose that μ is an independently generated probability measure on S_n, defined by the independent random variables $\{X_k\}_{k \in [n]}$. Then for every $\mathbf{a} \in S_n$ we have

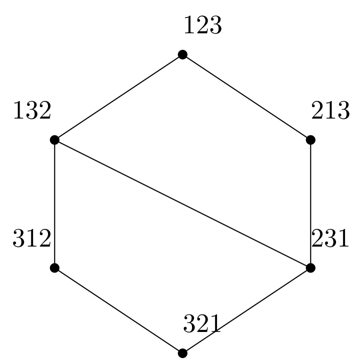
$$\mu(\mathbf{a}) = \prod_{k \in [n]} \mathbb{P}\big(X_k = \mathbf{f}(\mathbf{a})_k\big).$$

Then given $\mathbf{a}, \mathbf{b} \in S_n$ and $\mathbf{a} \vee \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ as above, we have

$$\begin{aligned}\mu(\mathbf{a} \vee \mathbf{b}) \cdot \mu(\mathbf{a} \vee \mathbf{b}) &= \prod_{k \in [n]} \Big[\mathbb{P}\big(X_k = \max(\mathbf{f}(\mathbf{a})_k, \mathbf{f}(\mathbf{b})_k)\big) \\ &\qquad\qquad \times \mathbb{P}\big(X_k = \min(\mathbf{f}(\mathbf{a})_k, \mathbf{f}(\mathbf{b})_k)\big)\Big] \\ &= \prod_{k \in [n]} \Big(\mathbb{P}\big(X_k = \mathbf{f}(\mathbf{a})_k\big) \cdot \mathbb{P}\big(X_k = \mathbf{f}(\mathbf{b})_k\big)\Big) = \mu(\mathbf{a}) \cdot \mu(\mathbf{b}).\end{aligned}$$

Thus (1) holds with equality for all $\mathbf{a}, \mathbf{b} \in S_n$, as required. □

Above we defined the grid order on S_n in such a way that it was isomorphic to the usual product ordering on G_n. Analysing the proof of Lemma 8 more carefully gives an alternative description of the grid order on S_n in terms of certain switches. Given $1 \leq i < j \leq n$, recall that $\{i, j\}$ is an inversion in $\mathbf{a}$ if $\mathrm{pos}(\mathbf{a}, j) = k < \ell = \mathrm{pos}(\mathbf{a}, i)$. We will say that $\{i, j\}$ is a *dominated inversion* in $\mathbf{a}$ if additionally $a_m \geq i, j$ for all $m \in [k, \ell]$. Then $\mathbf{a} \leq_g \mathbf{b}$ if $\mathbf{b}$ can be reached from $\mathbf{a}$ by a sequence of operations, each consisting of swapping the elements from a dominated inversion.



The grid order for $n = 3$

3 No correlation for weak up-sets

In this section we construct weak up-sets which are very far from being positively correlated. We will need the following simple concentration result.

Lemma 10. *Let* $0 < \gamma, \delta, \varepsilon < 1$. *Let* $U, V \subset [n]$ *with* $|U| = \gamma n$ *and* $|V| = \delta n$. *Select* $\mathbf{a} \in S_n$ *uniformly at random and consider the random variable* $N(\mathbf{a}) := |\{i \in U : a_i \in V\}|$. *Then* $\mathbb{P}\big(N > (\gamma + \varepsilon)|V|\big) \to 0$ *as* $n \to \infty$.

Proof. For each $i \in [n]$ let $1_i : S_n \to \{0, 1\}$ denote the Bernoulli random variable with $1_i(\mathbf{a}) = 1$ iff $a_i \in V$. Then $\mathbb{E}[1_i] = |V|/n$ for all $i \in [n]$. Noting that $N = \sum_{i \in U} 1_i$, linearity of expectation gives $\mathbb{E}[N] = \gamma|V|$.

To calculate the variance of N, note that $\mathbb{E}[1_i \cdot 1_j] \leq |V|^2/n^2$ for $i \neq j$. Since $N = \sum_{i \in U} 1_i$, this gives

$$\mathbb{E}[N^2] = \sum_{i \in U} \mathbb{E}[1_i^2] + \sum_{i \neq j \in U} \mathbb{E}[1_i 1_j] \leq \gamma|V| + \gamma^2|V|^2,$$

and so $\mathbb{V}\text{ar}(N) = \mathbb{E}[N^2] - \big(\mathbb{E}[N]\big)^2 \leq \gamma|V|$. Chebyshev's inequality then gives $\mathbb{P}(N > (\gamma+\varepsilon)|V|) \leq \mathbb{P}(|N - \mathbb{E}[N]| \geq \varepsilon|V|) \leq \gamma/\big(\varepsilon^2|V|\big) \to 0$ as $n \to \infty$. □

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. Given $0 < \alpha, \beta, \varepsilon < 1$, we require to find weak up-sets $\mathcal{A}, \mathcal{B} \subset S_n$ for large n, which satisfy $|\mathcal{A}| \geq \alpha n!$, $|\mathcal{B}| \geq \beta n!$ and $|\mathcal{A} \cap \mathcal{B}| \leq (\max(\alpha+\beta-1, 0) + 5\varepsilon)n!$. Indeed, by deleting minimal elements from such $\mathcal{A}$ and $\mathcal{B}$ we obtain weak up-sets of size $\lfloor \alpha n! \rfloor$ and $\lfloor \beta n! \rfloor$ as in the theorem.

To begin, set $m = \lceil (\frac{\alpha}{\alpha+\beta})n \rceil$ so that $\frac{m}{n-m} = \frac{\alpha}{\beta} + o(1)$. Consider the function $g : S_n \to [m]$ where $g(\mathbf{a})$ equals the number of elements from $[m]$ which do not appear after element m in $\mathbf{a}$. That is,

$$g(\mathbf{a}) := \big|\big\{ i \in [m] : \text{pos}(\mathbf{a}, i) \leq \text{pos}(\mathbf{a}, m) \big\}\big|.$$

Noting that g is non-decreasing under switching inversions, we see that $\mathcal{A} := \{\mathbf{a} \in S_n : g(\mathbf{a}) \geq (1-\alpha)m\}$ is a weak up-set in S_n. Also noting that the families $L_i = \{\mathbf{a} \in S_n : g(\mathbf{a}) = i\}$ for $i \in [m]$ partition S_n into equal-sized sets, we obtain $|\mathcal{A}| = \sum_{i \in [(1-\alpha)m, m]} |L_i| \geq \alpha n!$.

Our second family $\mathcal{B}$ is defined similarly. Let $h : S_n \to [n-m+1]$, where $h(\mathbf{a})$ equals the number of elements from $[m, n] := \{m, m+1, \ldots, n\}$ which do not appear before element m in $\mathbf{a}$. That is,

$$h(\mathbf{a}) := \big|\big\{ i \in [m, n] : \text{pos}(\mathbf{a}, i) \geq \text{pos}(\mathbf{a}, m) \big\}\big|.$$

Reasoning as above, we find $\mathcal{B} := \{\mathbf{a} \in S_n : h(\mathbf{a}) \geq (1-\beta)(n-m+1)\}$ is a weak up-set and $|\mathcal{B}| \geq \beta n!$.

Having defined both families, it only remains to upper bound $|\mathcal{A} \cap \mathcal{B}|$. Here it is helpful to consider two further families.

- For $\mathbf{a} \in S_n$ let $N_1(\mathbf{a}) := \big|\big\{ k \in U_1 : a_k \in V_1 \big\}\big|$, where $U_1 = [(1-\alpha-\varepsilon)n]$ and $V_1 = [m]$. Then $\mathcal{E}_1 := \big\{ \mathbf{a} \in S_n : N_1(\mathbf{a}) \geq (1-\alpha)|V_1| \big\}$.
- For $\mathbf{a} \in S_n$ let $N_2(\mathbf{a}) := \big|\big\{ k \in U_2 : a_k \in V_2 \big\}\big|$, where $U_2 = [(\beta+\varepsilon)n, n]$ and $V_2 = [m, n]$. Then $\mathcal{E}_2 := \big\{ \mathbf{a} \in S_n : N_2(\mathbf{a}) \geq (1-\beta)|V_2| \big\}$.

The functions N_1 and N_2 are defined as in Lemma 10, and so we have $|\mathcal{E}_1|, |\mathcal{E}_2| \leq \varepsilon n!$, provided $n \geq n_0(\alpha, \beta, \varepsilon)$.

We claim that every $\mathbf{a} \in \mathcal{C} := (\mathcal{A} \cap \mathcal{B}) \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ satisfies

$$\text{pos}(\mathbf{a}, m) \in I := \big[(1-\alpha-\varepsilon)n, (\beta+\varepsilon)n\big]. \tag{2}$$

Note that this will complete the proof of the theorem, since it gives

$$|\mathcal{A}\cap\mathcal{B}| \le |\mathcal{C}|+|\mathcal{E}_1|+|\mathcal{E}_2| \le \Big(\frac{|I|+1}{n}\Big)n!+2\varepsilon n! \le \big(\max(\alpha+\beta-1,0)+5\varepsilon\big)n!.$$

To prove the claim, take $\mathbf{a}\in\mathcal{A}\cap\mathcal{B}$. Note that if $\mathrm{pos}(\mathbf{a},m) < (1-\alpha-\varepsilon)n$ then $\mathbf{a}\in\mathcal{E}_1$ since

$$\begin{aligned} N_1(\mathbf{a}) = |\{k\in U_1 : a_k\in V_1\}| &= |\{i\in[m] : \mathrm{pos}(\mathbf{a},i) \le (1-\alpha-\varepsilon)n\}| \\ &\ge g(\mathbf{a}) \ge (1-\alpha)m = (1-\alpha)|V_1|, \end{aligned}$$

The first equality is by definition of N_1, the second equality holds by double counting, the first inequality follows from by definition of g and the fact that $\mathrm{pos}(\mathbf{a},m) \le (1-\alpha-\epsilon)n$, and the final inequality holds as $\mathbf{a}\in\mathcal{A}$.

Similarly, if $\mathrm{pos}(\mathbf{a},m) > (\beta+\varepsilon)n$ then $\mathbf{a}\in\mathcal{E}_2$, since

$$\begin{aligned} N_2(\mathbf{a}) = |\{k\in U_2 : a_k\in V_2\}| &= |\{i\in[m,n] : \mathrm{pos}(\mathbf{a},i) \ge (\beta+\varepsilon)n\}| \\ &\ge h(\mathbf{a}) \ge (1-\beta)(n-m+1) = (1-\beta)|V_2|. \end{aligned}$$

Again, the first two equalities hold by definition of N_2 and by double counting respectively. The first inequality follows from the definition of h and the fact that $\mathrm{pos}(\mathbf{a},m) > (\beta+\varepsilon)n$, and the final inequality holds as $\mathbf{a}\in\mathcal{B}$.

We have shown that if $\mathbf{a}\in(\mathcal{A}\cap\mathcal{B})\setminus(\mathcal{E}_1\cup\mathcal{E}_2)=\mathcal{C}$ then $\mathbf{a}$ satisfies (2) which, as described above, completes the proof. □

4 Examples and an application

Several natural families of permutations enjoy the property of being strong up-sets. In the first subsection we present a number of examples of these. Together these provide a wide variety of families for which positive correlation results can be deduced from Theorem 1 and Theorem 3. For instance, we shall see that for a random permutation $\mathbf{a}\in S_n$ (chosen uniformly or following an independently generated measure), the event that no element is displaced by more than a fixed distance d by $\mathbf{a}$ and the event that $\mathbf{a}$ contains at most k inversions are positively correlated. Likewise, each of these events is positively correlated with the event that at least u elements from $\{1,\ldots,v\}$ occur among the first w positions in $\mathbf{a}$.

In the second subsection we will give an application of Theorem 1 to the correlation of left-compressed set families.

4.1 Examples of strong up-sets

Layers

For each $k \in \left[0, \binom{n}{2}\right]$ let $\mathcal{L}_k := \{\mathbf{a} \in S_n : |\operatorname{inv}(\mathbf{a})| = k\}$. Then it is easily seen that the family $\mathcal{L}_{\geq k} := \cup_{i \geq k} \mathcal{L}_i$ is a strong up-set. In words, this is the set of all permutations which can be written as a product of at most $\binom{n}{2} - k$ adjacent transpositions.

Band-like permutations

Our next example is based on considering how much each element is moved by a permutation. Given a permutation $\mathbf{a} \in S_n$ and an element $i \in [n]$, the *displacement* of i in $\mathbf{a}$ is given by $\operatorname{disp}(\mathbf{a}, i) := |i - \operatorname{pos}(\mathbf{a}, i)|$. We will say $\mathbf{a}$ is a *t-band permutation* if $\operatorname{disp}(\mathbf{a}, i) \leq t$ for all $1 \leq i \leq n$.

Lemma 11. *The t-band permutations in S_n form a strong up-set.*

Proof. Suppose that $\mathbf{a} \in S_n$ is a t-band permutation and that $\{i, j\} \in \operatorname{inv}(\mathbf{a})$. Let $\mathbf{b}$ be the permutation obtained from $\mathbf{a}$ by swapping i and j. It is clear that $\operatorname{disp}(\mathbf{a}, k) = \operatorname{disp}(\mathbf{b}, k)$ for all $k \notin \{i, j\}$. A simple case check also gives

(a) $\operatorname{disp}(\mathbf{b}, i) + \operatorname{disp}(\mathbf{b}, j) \leq \operatorname{disp}(\mathbf{a}, i) + \operatorname{disp}(\mathbf{a}, j)$, and

(b) $|\operatorname{disp}(\mathbf{b}, i) - \operatorname{disp}(\mathbf{b}, j)| \leq |\operatorname{disp}(\mathbf{a}, i) - \operatorname{disp}(\mathbf{a}, j)|$.

As $\mathbf{a}$ is a t-band permutation we have $\operatorname{disp}(\mathbf{a}, i), \operatorname{disp}(\mathbf{a}, j) \leq t$ and so it follows that $\operatorname{disp}(\mathbf{b}, i), \operatorname{disp}(\mathbf{b}, j) \leq t$, i.e. $\mathbf{b}$ is also a t-band permutation. $\square$

In fact this argument shows rather more. Given $\mathbf{a} \in S_n$, the *displacement list* $\mathbf{d}(\mathbf{a})$ is the vector given by:

$$\mathbf{d}(\mathbf{a}) := \big(\operatorname{disp}(\mathbf{a}, 1), \ldots, \operatorname{disp}(\mathbf{a}, n)\big).$$

Now, given a set of vectors $\mathcal{D} \subset \{0, 1, \ldots, n-1\}^n$, we can form the family of permutations $\mathcal{A}(\mathcal{D}) := \{\mathbf{a} \in S_n : \mathbf{d}(\mathbf{a}) \in \mathcal{D}\} \subset S_n$. That is, those permutations in S_n whose displacement lists lie in $\mathcal{D}$.

Definition. *A set of permutations $\mathcal{A}$ is said to be* band-like *if $\mathcal{A} = \mathcal{A}(\mathcal{D})$ for some set $\mathcal{D} \subset \{0, 1, \ldots, n-1\}^n$ which is closed under:*

- *reordering the entries,*

- *decreasing any entry,*
- *replacing two entries of an element of* $\mathcal{D}$ *with new entries so that neither the sum or difference of these entries increases.*

The argument of Lemma 11 shows that:

Lemma 12. *Any band-like set of permutations in* S_n *is a strong up-set.*

In addition to t-band permutations, examples of band-like sets include $\{\mathbf{a} \in S_n : \sum_{i=1}^{n} \operatorname{disp}(\mathbf{a}, i) \le t\}$ and $\{\mathbf{a} \in S_n : \sum_{i=1}^{n} \operatorname{disp}(\mathbf{a}, i)^2 \le t\}$.

Sequentially dominating permutations

Our final example arises from assigning weight and thresholds as follows. Given a sequence of real weights $\mathbf{w} = (w_1, \dots, w_n)$ with $w_1 \ge w_2 \ge \cdots \ge w_n$ and thresholds $\mathbf{t} = (t_1, \dots, t_n)$ we consider the family

$$\mathcal{D}(\mathbf{w}, \mathbf{t}) := \Big\{ \mathbf{a} = (a_1, \dots, a_n) \in S_n : \sum_{i=1}^{m} w_{a_i} \ge t_m \text{ for all } m \Big\}.$$

Since the weights are decreasing, such families are closed under swapping inversions and so form strong up-sets.

Some common families arise in this way, including the families of permutations which satisfy 'at least a elements from $\{1, \dots, b\}$ occur among the first c positions'. Indeed, such families can be written as $\mathcal{D}(\mathbf{w}, \mathbf{t})$, where

$$\mathbf{w} = (\underbrace{1, \dots, 1}_{b}, \underbrace{0, \dots, 0}_{n-b}),$$

with $t_i = a$ if $i = c$, and $t_i = 0$ otherwise.

Many specific examples follow from these general families. For instance,

Corollary 13. *Let* $\mathbf{a}$ *be a random permutation chosen under an independently generated probability measure on* S_n*. Then, for any* $k, l, m, u, v, w \in \mathbb{N}$*, any two of the following events are positively correlated:*

- *There are at most* k *inversions in* $\mathbf{a}$*,*
- *No element is displaced by more than* l *by* $\mathbf{a}$*,*

- *The sum over all elements of the displacements in* $\mathbf{a}$ *is at most* m*,*
- *The first* w *positions of* $\mathbf{a}$ *contain at least* u *of the elements* $\{1,\ldots,v\}$.

□

Amusingly, the families of permutations constructed in the proof of Theorem 2 (our non-correlation result for weak up-sets) can be described using weights in a superficially similar way to a sequentially dominated family. Given a non-increasing sequence of weights and any thresholds, we may define the set of all permutations satisfying that the sum of all entries up to and including element m is at least t_m for all m. More precisely,

$$\mathcal{D}'(\mathbf{w},\mathbf{t}) := \Big\{\mathbf{a}=(a_1,\ldots,a_n)\in S_n : \sum_{i=1}^{m} w_{a_i} \geq t_{a_m} \text{ for all } m\Big\}.$$

In general this is not an up-set in the strong or weak sense. However, if we take weights $u_1 = u_2 = \ldots u_k = 1$, $u_{k+1} = \cdots = u_n = 0$ with threshold $s_k = k/2$ and weights $v_1 = v_2 = \ldots v_k = 0$, $v_{k+1} = \cdots = v_n = -1$ with threshold $t_k = -k/2$ then the two families $\mathcal{D}'(\mathbf{u},\mathbf{s})$ and $\mathcal{D}'(\mathbf{v},\mathbf{t})$ are precisely those constructed in the proof of Theorem 2.

4.2 Maximal chains and left-compressed up-sets

A family of sets $\mathcal{A}\subset\mathcal{P}(X)$ is *left-compressed* if for any $1\leq i<j\leq n$, whenever $A\in\mathcal{A}$ with $i\notin A, j\in A$ we also have $(A\setminus\{j\})\cup\{i\}\in\mathcal{A}$. See [5] for background and a number of useful applications of compressions. It is not hard to show that if $\mathcal{A}$ and $\mathcal{B}$ are left-compressed r-uniform families (that is, each consists of r-element subsets of $[n]$) then they are positively correlated in the sense that

$$\frac{|\mathcal{A}\cap\mathcal{B}|}{\binom{n}{r}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}} \times \frac{|\mathcal{B}|}{\binom{n}{r}}.$$

However, in general left-compressed families may not be positively correlated; indeed, they may simply be disjoint if the families have different sizes. Below we use Theorem 1 to give a natural measure of the similarity of non-uniform families from which positive correlation for left-compressed families follows.

A maximal chain in $\mathcal{P}(X)$ is a nested sequence of sets $C_0\subset C_1\subset\cdots\subset C_n$ with $C_i\subset X$ and $|C_i|=i$. A permutation $\mathbf{a}$ of X can be thought of as a maximal chain in $\mathcal{P}(X)$ by identifying $\mathbf{a}$ with the family of sets forming initial segments from $\mathbf{a}$; that is setting $C_i := \{a_1, \ldots, a_i\}$ for all $i \in [0, n]$.

If $\mathcal{A}$ is a family of sets, we write $c(\mathcal{A})$ for the number of maximal chains which contain an element of $\mathcal{A}$. Note that if $\mathcal{A}$ is r-uniform then the probability that a uniformly random maximal chain meets $\mathcal{A}$ is proportional to $|\mathcal{A}|$ and so in this case $c(\mathcal{A})/n! = |\mathcal{A}|/\binom{n}{r}$. If $\mathcal{A}$ and $\mathcal{B}$ are families of sets then we write $c(\mathcal{A}, \mathcal{B})$ for the number of maximal chains that meet both $\mathcal{A}$ and $\mathcal{B}$. We will use $c(\mathcal{A})$ as our measure of the size of $\mathcal{A}$ and $c(\mathcal{A}, \mathcal{B})$ as our measure of the intersection (or similarity) of $\mathcal{A}$ and $\mathcal{B}$. With this notion, the following Theorem can be interpreted as saying that left-compressed families are positively correlated.

Theorem 14. *If $\mathcal{A}$ and $\mathcal{B}$ are left-compressed families from $\mathcal{P}(X)$ then*

$$\frac{c(\mathcal{A}, \mathcal{B})}{n!} \geq \frac{c(\mathcal{A})}{n!} \times \frac{c(\mathcal{B})}{n!}.$$

Proof. Let $C(\mathcal{A})$ denote the set of all permutations of X which correspond to chains meeting $\mathcal{A}$ and $C(\mathcal{B})$ be the set of all permutations of X which correspond to chains meeting $\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are left-compressed $C(\mathcal{A})$ and $C(\mathcal{B})$ are strong up-sets in S_n. Applying Theorem 1 gives the result. □

We remark that while the functional $c(\mathcal{A})/n!$ is thought of as a measure $\mathcal{A}$, it is not a probability measure on $\mathcal{P}(X)$ since additivity fails (e.g. consider the partition $\mathcal{P}(X) = \cup_i \binom{X}{i}$).

A number of further variations on this result are possible (e.g. if $\mathcal{A}$ is left-compressed and $\mathcal{B}$ is right-compressed then $\mathcal{A}$ and $\mathcal{B}$ are negatively correlated). For example, given a family $\mathcal{F} \subset \mathcal{P}(X)$ and a maximal chain $\mathcal{C}$, let $N_{\mathcal{F}}(\mathcal{C}) := |\mathcal{C} \cap \mathcal{F}|$. Identifying permutations $\mathbf{a} \in S_n$ with maximal chains as above, we obtain the following.

Theorem 15. *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ be left-compressed families and suppose that $\mathcal{C}$ is a maximal chain from $\mathcal{P}(X)$ chosen uniformly at random. Then for any k, l we have*

$$\mathbb{P}\big(N_{\mathcal{A}}(\mathcal{C}) \geq k, N_{\mathcal{B}}(\mathcal{C}) \geq l\big) \geq \mathbb{P}\big(N_{\mathcal{A}}(\mathcal{C}) \geq k\big) \cdot \mathbb{P}\big(N_{\mathcal{B}}(\mathcal{C}) \geq l\big).$$

5 Open questions

One general question is to determine which other measures on S_n satisfy positive association. A particularly appealing class of measures to consider here are those given by a 1-dimensional spatial model. Spatial models of this kind are much studied in statistical physics. See [3, 2] for examples of such results.

Let $x(1), x(2), \ldots, x(n) \in \mathbb{R}$ with $x(1) \leq x(2) \leq \cdots \leq x(n)$. We will regard these as n particles placed in increasing order on the real line. A permutation $\mathbf{a} = (a_1, \ldots, a_n) \in S_n$ gives rise to a permutation of these particles. In this model, any point $x(i)$ is displaced by $|x(i) - x(\text{pos}(\mathbf{a}, i))|$. The total displacement is $\sum_i |x(i) - x(\text{pos}(\mathbf{a}, i))|$. We define the associated measure on S_n by

$$\mu(\mathbf{a}) \propto q^{\sum_i |x(i) - x(\text{pos}(\mathbf{a},i))|}.$$

More generally, given a non-decreasing function $V : \mathbb{R}^+ \to \mathbb{R}^+$, define

$$\mu(\mathbf{a}) \propto q^{\sum_i V(x(i) - x(\text{pos}(\mathbf{a},i)))}.$$

These definitions are special cases of the well-studied Boltzmann measures in which points are picked in $\mathbb{R}^d$ and more general functions in the exponent of q are allowed.

We suspect that all measures defined in this way have positive association. However we do not have a proof of this, even in special cases. The following three cases all seem interesting.

Question 1 (Equally spaced points)**.** *Is the measure μ defined by*

$$\mu(\mathbf{a}) \propto q^{\sum_i |i - \text{pos}(\mathbf{a},i)|}$$

positively associated?

This corresponds to taking $x(i) = i$ and $V(u) = |u|$.

Question 2 (Middle gap)**.** *Let $m(\mathbf{a}) = |\{k : 1 \leq k \leq n/2, n/2 < a_k \leq n\}|$ be the number of elements which are moved 'across the middle gap' by $\mathbf{a}$. Is the measure μ defined by*

$$\mu(\mathbf{a}) \propto q^{m(\mathbf{a})}$$

positively associated?

This corresponds to taking $x(i) = 0$ for $i \in [\frac{n}{2}]$ and $x(i) = 1$ if $i \in [\frac{n}{2}+1, n]$ and $V(u) = 1$ if $u < 0$ and $V(u) = 0$ otherwise.

Question 3 (Fixed points)**.** *Let* $f(\mathbf{a}) = |\{k : a_k = k\}|$ *be the number of fixed points of* $\mathbf{a}$*. Is the measure* μ *defined by*

$$\mu(\mathbf{a}) \propto q^{n-f(\mathbf{a})}$$

positively associated?

This corresponds to taking any distinct $\{x(i)\}_{i\in[n]}$ and setting $V(0) = 0$ and $V(u) = 1$ otherwise.

Lastly, the correlation behaviour seen in the strong and weak orders are extreme, with the first displaying Harris–Kleitman type correlation (Theorem 1) and the second displaying worst possible correlation (Theorem 2). It seems interesting to understand how correlation behaviour emerges between these extremes.

Definition. *Given* $t \in [n]$*, a family of permutations* $\mathcal{A} \subset S_n$ *is a* t-up-set *if given* $\mathbf{a} \in \mathcal{A}$*, any permutation obtained from* $\mathbf{a}$ *by swapping the elements in a pair* $\{i, j\} \in \mathrm{inv}(\mathbf{a})$ *with* $|\,\mathrm{pos}(\mathbf{a}, i) - \mathrm{pos}(\mathbf{a}, j)| \leq t$ *is also in* $\mathcal{A}$*.*

Note that if $t = 1$ then a t-up-set is simply a weak up-set. On the other hand, for $t = n$ then a t-up-set is a strong up-set. Thus we can think of t-up-sets as interpolating between the weak and strong notions as we increase $t \in [n]$. It seems natural to investigate the correlation behaviour of t-up-sets.

Question 4 (Correlation for t-up-sets)**.** *Given* $\alpha > 0$*, does there exist* $\beta > 0$ *such that the following holds: given* $n \in \mathbb{N}$ *and* $t = \lceil \alpha n \rceil$*, any two* t*-up-sets* $\mathcal{A}, \mathcal{B} \subset S_n$ *with* $|\mathcal{A}|, |\mathcal{B}| \geq \alpha n!$ *satisfy* $|\mathcal{A} \cap \mathcal{B}| \geq \beta n!$*.*

Acknowledgement. We are grateful to Ander Holroyd for bringing the Mallows measures to our attention. We would also like to thank the referees for their careful reading and helpful comments.

References

[1] R. Ahlswede and D.E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, *Probab. Theory Relat. Fields*, **43**(3) (1978), 183–185.

[2] V. Betz and D. Ueltschi, Spatial random permutations and infinite cycles, *Comm. Math. Phys.* **285** (2009), no. 2, 469–501.

[3] M. Biskup and T. Richthammer, Gibbs measures on permutations over one-dimensional discrete point sets, *Ann. Appl. Probab.* **25**, Number 2 (2015), 898–929.

[4] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, **231**, Springer, New York, 2005.

[5] B. Bollobás, *Combinatorics*, Cambridge Univ. Press, London/New York, 1986.

[6] C.M. Fortuin, P.W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets, *Comm. Math. Phys.* **22**(2) (1971), 89–103.

[7] T.E. Harris, A lower bound for the critical probability in a certain percolation, *Math. Proc. Cam. Phil. Soc.*, **56** (1960), 13–20.

[8] R. Holley, Remarks on the FKG inequalities, *Commun. Math. Phys.*, **36**(3), (1974), 227–231.

[9] D.J. Kleitman, Families of non-disjoint subsets, *J. Combin. Theory* **1** (1966), 153–155.

[10] M. Lothaire, *Algebraic combinatorics on words*, **90**, Cambridge University Press, 2002.

[11] C.L. Mallows, Non-null ranking models. I, *Biometrika*, **44** (1957), 114–130.