Convergence rates for variational regularization of statistical inverse problems

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Contents

Introduction

The field of inverse problems, although a rather young field of applied mathematics, has proven to be of great importance in a wide range of applications both in science and industry: Medical, astronomical and microscopic imaging, acoustics, geophysics, machine learning, finance, and many others. We will give an introduction to inverse problems in the first chapter. For now we just state that in all inverse problems it is usually the goal to approximate some quantity of interest (for example parts of the interior of the human body) by given observed data (for example X-ray measurements from different angles). To this end one applies so called regularization methods, that can give useful approximations even if the data is perturbed by noise.

This thesis is mainly concerned with the question how fast these approximations tend to the true (or exact) solution of an inverse problem if the noise tends to zero. In other words we want to determine the rate of convergence of the approximate to the exact solution. We will see in the following that such convergence rates can only hold under certain smoothness conditions on the true solution, also called source conditions, as generally convergence can be arbitrarily slow.

Convergence rates are both of theoretical and practical interest because of various questions: How do different regularization methods compare, when the noise is small? What are the limitations of a certain regularization method concerning convergence? Is it worth to invest effort and money into reducing the noise level, or is the payoff too small? What properties of the object of interest favor fast convergence?

Linear inverse problems on Hilbert spaces are very well understood. One can formulate source conditions that are both necessary and sufficient for certain convergence rates [51] and can show that rates are order optimal under such conditions [25]. For nonlinear problems or Banach space settings there are however still many open questions. The reason for this is that on general Banach spaces and for nonlinear operators one can not apply spectral theory, which is crucial for most of the results on Hilbert spaces. Instead of considering regularization operators and analyze them by spectral theory we will consider variational regularization. In variational regularization our approximation to the true solution is given by the solution to a minimization problem and we have to analyze our methods by variational methods like convex analysis. A large class of such variational regularization methods can be condensed in generalized Tikhonov regularization, a generalization of the famous and widely applied quadratic Tikhonov regularization.

An important step in the evolution of convergence rate theory for more general regularization methods has been the introduction of variational source conditions (VSCs) [37]. These generalize source conditions on Hilbert space, but can also be formulated for variational regularization on Banach spaces with general (nonlinear) forward operator. Under such a VSC one can prove convergence rates, however for many interesting problems it still remained an open question, whether and how such a VSC can actually be verified.

It was shown in [41] that at least in the Hilbert space the VSCs are necessary and sufficient for convergence rates and if one considers certain prominent examples of forward operators on L^2 spaces. In these settings the true solution satisfies a VSC if and only if it is smooth, with smoothness measured in Besov spaces.

Another problem connected to the VSCs is that they are limited in the sense that they can only yield a limited range of convergence rates, so that from a certain smoothness index on one does not profit from higher smoothness of the true solution as one would expect. This problem was attacked in [34] where a second order variational source condition (VSC^2) was introduced that yields faster convergence rates up to the best possible rates (saturation) for Tikhonov regularization. But again the VSC^2 is limited as the first order VSC and there are regularization methods that should exhibit faster convergence rates than what can be shown under the VSC^2. One example for such a method is Bregman iterated Tikhonov regularization (or Bregman iteration), which has attracted a lot of attention recently because of good numerical results, starting from [53]. This iteration can be seen as a generalization of iterated Tikhonov regularization on Hilbert spaces, for which it is known that it can converge faster that non iterated Tikhonov regularization. However in the Banach space setting this had not been shown.

A major new contribution of this work is that we introduce variational source conditions of arbitrary order (VSC^n) on Hilbert space that allow to prove higher order convergence rates, as well as a third order VSC in the more general Banach space setting. Under this VSC^3 we show new convergence rates for Bregman iteration that improve on the rates possible for Tikhonov regularization.

A special focus of this work lies on the case where the noise is not in the same (Hilbert) space as the data. This occurs for example in the distinguished cases, where the noise is given by a Gaussian white noise or a Poisson process. As these two noise models are extremely important in an abundance of applications there is of course already a large amount of literature on both of them. Convergence rates under Gaussian white noise are well understood again for linear problems on Hilbert spaces by spectral methods [6]. In this work we will reprove these results by variational methods allowing for generalizations to variational regularization. In a few aspects we can even improve the classical Hilbert space results as we can prove a deviation inequality on the regularization error instead of just convergence rates in expectation and we also show a saturation result for Gaussian white noise.

In case of Poisson data convergence rates have been shown under a VSC in [75]. However due to the limitations of the VSC again there is the questions whether faster convergence rates can be shown for Poisson data in the case of high smoothness of the true solution. Thus another large part of this thesis is devoted to overcoming the limitations of the VSC under statistical models. The idea is of course to use the VSC^2 but actually this condition is closely related to the data model, which complicates under the statistical models. Therefore we again introduce new source conditions that generalize the VSC^2 for the statistical models and lead to higher order convergence rates up to the saturation of Tikhonov regularization for Gaussian white noise.

To prove that all these abstract source conditions are useful we verify them for variants of generalized Tikhonov regularization under smoothness assumptions on the true solution and smoothing assumptions on the forward operator.

In addition to these major contributions this work also contains a new proof of upper and lower bounds for Bregman divergences. Bregman divergences are a crucial tool in our convergence analysis and also the main ingredient of the Bregman iteration. Further we show a result on boundedness of the regularized solution, which is important in the statistical setting and allows us to neglect the assumption $(H7)$ in [43, Assumption 4.1] that was necessary in the existing proofs of convergence rates under Poisson data.

This thesis is structured as follows.

- In Chapter 1 we will provide some background on inverse problems and regularization theory. Further we will motivate Tikhonov regularization and our noise model.
- Chapter 2 summarizes basic tools from convex analysis and contains our new proof for upper and lower bounds of the Bregman divergence.
- Chapter 3 gives an overview of variational regularization in the form of generalized Tikhonov regularization.
- Chapter 4 is the core of this work. We first introduce a generalized notion of the noise level. Then we prove a result on boundedness of the regularized solution, given stochastic noise. In Section 4.3 we recall how the VSC^1 can be applied in the deterministic setting and then show how it yields error estimates in the statistical setting. In Section 4.4 we introduce the VSC^2 and new second order source conditions that generalize the VSC^2 for stochastic data. Finally in Section 4.5 we introduce VSC^n for arbitrary $n \in \mathbb{N}$ in Hilbert space and further show how the VSC^3 can give error estimates for Bregman iterated Tikhonov regularization.
- Last but not least we verify the abstract source conditions in Chapter 5 under smoothness assumption for the true solution. The first section gives general strategies for verification. After that we consider in each section a more specialized regularization class, including maximum entropy regularization, Besov space regularization and Hilbert space regularization.

Chapter One

Inverse problems and regularization theory

"Could he write something original? He was not sure what idea he wished to express but the thought that a poetic moment had touched him took life within him like an infant hope. He stepped onward bravely."

Dubliners/A little cloud, J. Joyce

1.1 Inverse problems

As stated in the introduction we want to take a closer look on inverse problems. In general one could say that given observations of some scientific experiment the inverse problem consists of determining the causes responsible for these observations. The forward problem then instead is, given a certain cause, to find its corresponding effects. So in many physical experiments the inverse problem is to relate observations made in the present to a certain state of the physical system in the past, which usually turns out to be the more difficult problem than predicting future outcomes from the knowledge of all relevant parameters. However this definition is still quite vague and of course there are experiments like for example numerical integration/differentiation, where it is not obvious which problem one should call the inverse problem.

To model these kind of problems mathematically we define a forward operator $F\colon \mathcal{X} \to \mathcal{Y}$, which is a map from a certain set of causes $\mathcal{X}$ to the set of effects $\mathcal{Y}$. Given the data $g \in \mathcal{Y}$ the inverse problems then consists of finding a solution $f \in \mathcal{X}$ to the equation

$$F(f) = g. \tag{1.1}$$

One might try to compute the solution f to (1.1) by just applying the inverse operator F^{-1} (if it is defined), which gives another reason to call the problem an inverse problem. However in practice this can cause several problems, due to the forward operator being *ill-posed*. The definition of well- and ill-posedness goes back to Hadamard [36] and will allow us to give a more concrete definition of inverse problems.

Definition 1.1.1. *We say that a problem is well-posed if it fulfills the following three properties:*

(a) There exists a solution.
(b) The solution is unique.
(c) The solution depends continuously on the data (stability).

Instead a problem is called ill-posed if it is not well posed. That is if at least one of the three above conditions is violated.

Given these definitions we can now refine the scope of the field of inverse problems to finding approximate solutions to ill-posed problems of the form (1.1). We will especially focus on problems where the third condition of well-posedness is violated and one has to apply some form of regularization to restore the stability of the problem.

We want to give a definition of ill-posedness related to the operator equation (1.1). Clearly item (a) of Definition 1.1.1 is violated if F is not surjective, and item (b) is F is not injective. If F is bijective then item (c) is violated if F^{-1} is not continuous. However, for linear forward operators F on Banach spaces the bounded inverse theorem states that if F is bijective then F^{-1} is continuous, so item (c) is only violated if at least one of the other conditions is also violated. Until the 90s regularization theory focused mostly on linear forward operators and there exists a vast amount of experiments that can be modeled by a linear forward operator on Banach spaces. Thus we would like to define ill-posedness given by a violation of item (c) independent of the existence of F^{-1}. To this end we introduce the generalized inverse.

Definition 1.1.2 (generalized inverse)**.** *Let* $F\colon \mathcal{X} \to \mathcal{Y}$ *a mapping* $F^\dagger\colon \operatorname{dom}(F^\dagger) \to \mathcal{X}$ *with* $\operatorname{dom}(F^\dagger) \subset \operatorname{ran}(F)$ *is called* generalized inverse *of* F *if*

$$F \circ F^\dagger \circ F = F$$
$$F^\dagger \circ F \circ F^\dagger = F^\dagger.$$

If F is injective, then it is clear that a generalized inverse $F^\dagger$ with $\operatorname{dom}(F^\dagger) = \operatorname{ran}(F)$ exists and is given by the usual notion of inverse. For linear operators on Hilbert spaces there always exists a generalized inverse, given by the Moore-Penrose inverse, which is a linear operator with $\operatorname{dom}(F^\dagger) = \operatorname{ran}(F) + \operatorname{ran}(F)^\perp$. In this thesis we will consider inverse problems modeled by a forward operator $F\colon \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ that are ill-posed in the following sense.

Definition 1.1.3 (ill-posed inverse problem)**.** *We call the problem of solving* $F(f) = g$ *an ill-posed inverse problem if there does not exist a continuous generalized inverse of* F*.*

Note that to capture the ill-posedness of an inverse problem one has to consider infinite dimensional spaces $\mathcal{X}, \mathcal{Y}$. For a linear forward the reason for this is clear. A generalized inverse exists in the form of the Moore-Penrose inverse $F^\dagger$ and is linear. If either $\mathcal{X}$ or $\mathcal{Y}$ are finite dimensional then $\operatorname{ran} F^\dagger$ is finite dimensional and thus $F^\dagger$ is bounded, which is equivalent to being continuous. For a non-linear forward operator this is not obvious. However, usually one assumes for a non-linear operator that it is continuously differentiable in order to apply linearization. If one considers the linearized forward operator then again the previous argument applies. However, in general ill-posedness of a non linear forward operator is not necessarily related to ill-posedness of its linearizations as discussed in [26].

1.2 Regularization theory

In this section we will recall some basic results from regularization theory and to this end we of course firstly have to define a regularization method.

Definition 1.2.1 (Regularization method)**.** *Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces and the mapping $F\colon \operatorname{dom} F \to \mathcal{Y}$ be the forward operator. A family of continuous mappings $R_\alpha\colon \mathcal{Y} \to \mathcal{X}$ together with a parameter choice rule $\overline{\alpha}\colon (0,\infty) \times \mathcal{Y} \to (0,\infty)$ is called a (deterministic) regularization method if for all $g^\dagger \in \operatorname{ran} F$ there exists $f^\dagger \in \mathcal{X}$ with $F(f^\dagger) = g^\dagger$ such that*

$$\lim_{\delta \searrow 0} \sup\{\|R_{\overline{\alpha}(\delta, g^{\mathrm{obs}})}(g^{\mathrm{obs}}) - f^\dagger\| : g^{\mathrm{obs}} \in \mathcal{Y}, \|g^{\mathrm{obs}} - g^\dagger\| \leq \delta\} = 0.$$

Such a regularization method solves all three difficulties of an ill-posed problem as we demand $R_\alpha\colon \mathcal{Y} \to \mathcal{X}$ to be continuous. Further if the noise level δ is sufficiently small, then $R_{\overline{\alpha}(\delta, g^{\mathrm{obs}})}(g^{\mathrm{obs}})$ will be very close to the true solution $f^\dagger$ independent of how the noise g^{obs} exactly looks like. The main goal of this thesis is to derive convergence rates for certain regularization methods. That is we want even more than what is given by the above definition. We intend to show that for certain $g^\dagger \in \operatorname{ran}(F)$ there exists $f^\dagger \in \mathcal{X}$ with $g^\dagger = F(f^\dagger)$ such that the worst case error

$$\mathcal{E}(g^\dagger, f^\dagger, \delta) := \sup\{\|R_{\overline{\alpha}(\delta, g^{\mathrm{obs}})}(g^{\mathrm{obs}}) - f^\dagger\| : g^{\mathrm{obs}} \in \mathcal{Y}, \|g^{\mathrm{obs}} - g^\dagger\| \leq \delta\}$$

goes to zero as $\delta \to 0$ at a certain rate $\phi(\delta)$. However, for an ill-posed inverse problem given as in Definition 1.1.3 such a rate cannot hold uniformly for all $g^\dagger \in \operatorname{ran}(F)$ as the following theorem shows.

Theorem 1.2.2. *Assume for all $g^\dagger \in \operatorname{ran}(F)$ that there exists $f^\dagger(g^\dagger) \in \mathcal{X}$ with $F(f^\dagger) = g^\dagger$ and a regularization method $R_{\overline{\alpha}}$ such that $\mathcal{E}(g^\dagger, f^\dagger, \delta) \leq \phi(\delta)$, with $\lim_{\delta \searrow 0} \phi(\delta) = 0$. Define $F^\dagger\colon \operatorname{ran}(F) \to \mathcal{X}, g^\dagger \mapsto f^\dagger(g^\dagger)$. Then $F^\dagger$ is a generalized inverse of F which is continuous with respect to the norm topologies of $\mathcal{Y}$ and $\mathcal{X}$.*

Proof. It is clear from Definition 1.1.2 that $F^\dagger$ is a generalized inverse. Let $\varepsilon > 0$. As $\lim_{\delta \searrow 0} \phi(\delta) = 0$ we can find $\delta > 0$ such that $2\phi(\delta) \leq \varepsilon$. Now let $g_1, g_2 \in \operatorname{ran}(F)$ with $\|g_1 - g_2\|_{\mathcal{Y}} \leq \delta$, then we have

$$\|F^\dagger(g_1) - F^\dagger(g_2)\|_{\mathcal{X}} \leq \|F^\dagger(g_1) - R_{\overline{\alpha}(\delta, g_1)}(g_1)\|_{\mathcal{X}} + \|R_{\overline{\alpha}(\delta, g_1)}(g_1) - F^\dagger(g_2)\|_{\mathcal{X}} \leq 2\phi(\delta) \leq \varepsilon,$$

which proves that $F^\dagger$ is uniformly continuous. □

This theorem shows that for a general combination of $f^\dagger$ and $g^\dagger$ the convergence of the worst case error can be arbitrarily slow. Thus convergence rates can only be achieved on certain subsets of $\operatorname{ran}(F)$ respectively on subsets of $\mathcal{X}$ given by a-priori assumptions on $f^\dagger$ and $g^\dagger$. Define for a set $\mathcal{K} \in \mathcal{X}$ and a regularization method R the worst case error on $\mathcal{K}$ by

$$\mathcal{E}_R(\delta, F, \mathcal{K}) := \sup\{\|R(g^{\mathrm{obs}}) - f^\dagger\|_{\mathcal{X}} : f^\dagger \in \mathcal{K}, g^{\mathrm{obs}} \in \mathcal{Y}, \|g^{\mathrm{obs}} - F(f^\dagger)\|_{\mathcal{Y}} \leq \delta\}.$$

We will show that for all sets $\mathcal{K}$ and all reconstruction methods R this has a lower bound given by the modulus of continuity

$$\omega(\delta, F, \mathcal{K}) := \sup\{\|f_1 - f_2\|_{\mathcal{X}} : f_1, f_2 \in \mathcal{K}, \|F(f_1) - F(f_2)\|_{\mathcal{Y}} \leq \delta\}.$$

Theorem 1.2.3. *We have for all maps $R\colon \mathcal{Y} \to \mathcal{X}$ that*

$$\mathcal{E}_R(\delta, F, \mathcal{K}) \geq \frac{1}{2}\omega(2\delta, F, \mathcal{K}).$$

Proof. Let $f_1, f_2 \in \mathcal{K}$ such that $\|F(f_1) - F(f_2)\| \leq 2\delta$, then we have for all $g \in \mathcal{Y}$ that

$$\|f_1 - f_2\| \leq \|R(g) - f_1\| + \|R(g) - f_2\|.$$

Choosing $g = \frac{1}{2}(F(f_1) + F(f_2))$ we see that $\|F(f_i) - g\| \leq \delta$ for $i = 1, 2$ and thus

$$\|f_1 - f_2\| \leq 2\mathcal{E}_R(\delta, F, \mathcal{K}).$$

Taking the supremum over such f_1, f_2 gives the claim. □

Remark 1.2.4. *The last two theorems are generalizations of the classical results for linear forward operators [25, Propositions 3.10 and 3.11].*

For linear forward operators $F = T$ the modulus of continuity is also often defined by

$$\omega_{\text{lin}}(\delta, T, \mathcal{K}) := \sup\{\|f\|_{\mathcal{X}} : f \in \mathcal{K}, \|Tf\|_{\mathcal{Y}} \leq \delta\} \tag{1.2}$$

and we have for $\mathcal{K} = -\mathcal{K}$ that

$$\omega(\delta, Y, \mathcal{K}) = \omega_{\text{lin}}(\delta, T, 2\mathcal{K}).$$

Definition 1.2.5. *We say that a regularization method R has convergence rates of optimal order on $\mathcal{K}$ if there exists $\phi\colon [0, \infty) \to [0, \infty)$ and $C > 0$ such that*

$$\mathcal{E}_R(\delta, F, \mathcal{K}) \leq C\phi(\delta)$$

and there exists no other mapping $\tilde{R}\colon \mathcal{Y} \to \mathcal{X}$ such that

$$\mathcal{E}_{\tilde{R}}(\delta, F, \mathcal{K}) = o(\phi(\delta)), \qquad i.e. \qquad \lim_{\delta\to 0} \frac{\mathcal{E}_{\tilde{R}}(\delta, F, \mathcal{K})}{\phi(\delta)} = 0.$$

For linear operators T on Hilbert spaces $\mathcal{X}$,$\mathcal{Y}$ sufficient conditions for convergence rates are usually given by the so called Hölder source conditions, that is for some $\nu, \varrho > 0$ we consider

$$f^\dagger \in \mathcal{X}_{\nu,\varrho} = \{(T^*T)^\nu \omega : \omega \in \mathcal{X}, \|\omega\|_{\mathcal{X}} \leq \varrho\}, \tag{1.3}$$

where $(T^*T)^\nu$ is defined via the functional calculus (see [25, Sec. 2.3]). These conditions can be interpreted as smoothness conditions as usually T and T^* should be smoothing operators, so if $f^\dagger$ lies in the image of $(T^*T)^\nu$ it should be smooth and the degree of smoothness is increasing with ν. One can show [25, Proposition 3.15] for ill-posed T that there exist arbitrarily small values of δ such that

$$\omega(\delta, T, \mathcal{X}_{\nu,\varrho/2}) = \omega_{\text{lin}}(\delta, T, \mathcal{X}_{\nu,\varrho}) = \delta^{\frac{2\nu}{2\nu+1}}\varrho^{\frac{1}{2\nu+1}}.$$

Thus a regularization method is of optimal order on $\mathcal{X}_{\nu,\varrho}$ if

$$\mathcal{E}_R(\delta, \mathcal{X}_{\nu,\varrho}) \leq C\delta^{\frac{2\nu}{2\nu+1}}. \tag{1.4}$$

For each $\nu > 0$ there exist spectral regularization methods that are of optimal order on $\mathcal{X}_{\nu,\varrho}$ (see e.g. [25, Theorem 6.5]). Quadratic Tikhonov regularization, which will be introduced in the next section, has order optimal convergence rates on $\mathcal{X}_{\nu,\varrho}$ for $0 < \nu \leq 1$ [25, Sec. 5.1].

1.3 Tikhonov regularization[1]

We give a short probabilistic motivation for using generalized Tikhonov regularization. From here until the end of this section we will consider the finite-dimensional setting $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^m$. We start from equation (1.1), where $f \in \mathbb{R}^n$, $g^{\text{obs}} \in \mathbb{R}^m$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ is some function. Given the data g^{obs} we want an estimate for a possible solution f of (1.1), but recall that we cannot just apply the inverse operator F^{-1} as discussed in Section 1.1. Instead we might try to find an estimate $\hat{f}$ of f by maximizing the likelihood function $\mathcal{L}(f) = \mathbb{P}(g^{\text{obs}}|f)$, i.e. the probability that for a certain preimage f the data g^{obs} will occur. If we assume that our data is normally distributed with variance σ, then we can rearrange the problem by using the monotonicity of the logarithm, as well as the fact that neither additive nor multiplicative constants change the extremal point:

$$
\begin{aligned}
\hat{f}_{\text{ML}} &\in \arg\max_{f\in\mathbb{R}^n} \mathbb{P}(g^{\text{obs}}|f) \\
&= \arg\max_{f\in\mathbb{R}^n} \log\Big(\mathbb{P}(g^{\text{obs}}|f)\Big) \\
&= \arg\min_{f\in\mathbb{R}^n} -\log\Big(\mathbb{P}(g^{\text{obs}}|f)\Big) \\
&= \arg\min_{f\in\mathbb{R}^n} -\log\left(\frac{1}{\sqrt{2\pi}\sigma}\prod_{i=1}^{m}\exp\left(\frac{-(g_i^{\text{obs}} - F(f)_i)^2}{\sigma^2}\right)\right) \\
&= \arg\min_{f\in\mathbb{R}^n} \sum_{i=1}^{m}(g_i^{\text{obs}} - F(f)_i)^2 = \arg\min_{f\in\mathbb{R}^n} \|g^{\text{obs}} - F(f)\|_2^2. \qquad (1.5)
\end{aligned}
$$

So the maximum likelihood approach yields the well known least squares method. However even if this estimator $\hat{f}_{\text{ML}}$ is unique, this approach does not yield a regularization effect. In fact it is more reasonable to maximize $\mathbb{P}(f|g^{\text{obs}})$ instead of $\mathbb{P}(g^{\text{obs}}|f)$ as our goal should be to find the solution f which is most likely to have caused the observation g^{obs}, instead of just finding any f, which causes the observation g^{obs} with maximal probability. This seems like a subtle difference, but in practice it will be really important. Note that by maximizing $\mathbb{P}(f|g^{\text{obs}})$ we consider f as a random variable, which is the Bayesian perspective and in contrast to that of a Frequentist. By Bayes' theorem we have

$$
\mathbb{P}(f|g^{\text{obs}}) = \frac{\mathbb{P}(g^{\text{obs}}|f)\,\mathbb{P}(f)}{\mathbb{P}(g^{\text{obs}})} \Leftrightarrow \text{posterior} = \frac{\text{likelihood}\cdot\text{prior}}{\text{evidence}}
$$

and estimating f by maximizing the posterior $\mathbb{P}(f|g^{\text{obs}})$ is called maximum a posteriori probability (MAP) estimate. To actually find this maximum it will be important to understand the prior $\mathbb{P}(f)$. If we assume that for our solution f the entries f_j are normally

[1]The Bayesian motivation for Tikhonov regularization given in this section is not an original idea of the author. It can be found already in [44] and is also outlined for the infinite dimensional setting in [65]. Further it has literal overlap with Chapter 5 in [23] to which the author contributed.

distributed with mean $(f_0)_j \in \mathbb{R}$ and variance τ, then we find

$$\begin{aligned}
\hat{f}_{\text{MAP}} &\in \operatorname*{arg\,max}_{f\in\mathbb{R}^n} \mathbb{P}(f|g^{\text{obs}}) \\
&= \operatorname*{arg\,min}_{f\in\mathbb{R}^n} \Big[-\log\Big(\mathbb{P}(g^{\text{obs}}|f)\Big) - \log(\mathbb{P}(f))\Big] \\
&= \operatorname*{arg\,min}_{f\in\mathbb{R}^n} \frac{1}{\sigma}\left[\sum_{i=1}^{m}(g_i^{\text{obs}} - F(f)_i)^2 + \frac{\sigma}{\tau}\sum_{j=1}^{m}(f_j - (f_0)_j)^2\right] \\
&= \operatorname*{arg\,min}_{f\in\mathbb{R}^n} \Big[\tfrac{1}{2}\|g^{\text{obs}} - F(f)\|_2^2 + \tfrac{\alpha}{2}\|f - f_0\|_2^2\Big] =: \operatorname*{arg\,min}_{f\in\mathbb{R}^n} J_\alpha(f),
\end{aligned}$$

where $\alpha = \frac{\sigma}{\tau}$. The functional $J_\alpha(f)$ is actually nothing but the standard (quadratic) Tikhonov functional, introduced in 1963 [67], and therefore MAP and Tikhonov regularization coincide in this setting. Of course ending up with the functional $J_\alpha(f)$ was due to assuming normal distributions for both the likelihood and the prior, which might not always be appropriate.

Let us assume that the data g^{obs} are Poisson distributed rather than normally distributed. Remember that a random variable X with values in $\mathbb{N}_0$ is called Poisson distributed with rate parameter $\lambda \geq 0$ if

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!}e^{-\lambda}.$$

This already implies $\mathbb{E}[X] = \text{Var}[X] = \lambda$. So let for $j = 1, \ldots, n$ the data entries $g_j^{\text{obs}} =: k_j \in \mathbb{N}_0$ be Poisson distributed with rate parameter $\mathbb{E}[g_j^{\text{obs}}] =: g_j^\dagger$. Then the negative log-likelihood is given by

$$\begin{aligned}
-\log\Big(\mathbb{P}(g^{\text{obs}}|f)\Big) &= -\log\prod_{j=1}^{m}\frac{F(f)_j^{k_j}}{k_j!}e^{-F(f)_j} \\
&= \sum_{j=1}^{m} -g_j^{\text{obs}}\log(F(f)_j + \log(g_j^{\text{obs}}!) - F(f)_j.
\end{aligned}$$

If we replace the additive constant $\sum_{j=1}^m \log(g_j^{\text{obs}}!)$ by $\sum_{j=1}^m g_j^{\text{obs}}\log(g_j^{\text{obs}}) - g_j^{\text{obs}}$ (which is even a reasonable approximation by Stirling's formula, although we could also have chosen any other constant) then this equals the Kullback-Leibler divergence of g^{obs} and $F(f)$, given by

$$\text{KL}(g^{\text{obs}}, F(f)) := \sum_{i=1}^{m} g_i^{\text{obs}}\log\left(\frac{g_i^{\text{obs}}}{F(f)_i}\right) - g_i^{\text{obs}} + F(f)_i.$$

As any additive constant independent of f does not change the MAP estimator, this justifies considering generalized Tikhonov regularization in the form of

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f\in\mathbb{R}^n} \mathcal{S}(F(f)) + \alpha\mathcal{R}(f), \tag{1.6}$$

where the data fidelity term can be chosen as $\mathcal{S} = \text{KL}(g^{\text{obs}}, \cdot)$ for Poisson distributed data or as $\mathcal{S} = \frac{1}{2}\|g^{\text{obs}} - \cdot\|_2^2$ for normally distributed data. The penalty functional $\mathcal{R}$ depends on the prior respectively the a-priori information that we have on our solution.

1.4 Statistical noise models

Apart from the standard deterministic noise model, where $g^{\text{obs}} \in \mathcal{Y}$ and $\|g^\dagger - g^{\text{obs}}\|_{\mathcal{Y}} \leq \delta$ we will consider the following statistical noise model. Let the forward operator act between real Banach spaces $\mathcal{X}$ and $\mathcal{Y} \subset L^2(\mathbb{M})$, where $\mathbb{M} \subset \mathbb{R}^d$ is either a bounded Lipschitz domain or the d-dimensional torus $\mathbb{T}^d$. Notice that in contrast to the last section we now opt for continuous models, because these are the right ones to capture the ill-posedness of an inverse problem as stated in the introduction. Our goal is to estimate the true solution $f^\dagger \in \mathcal{X}$ given some observed data G^{obs}. We assume that the data is of the form

$$G^{\text{obs}} = g^\dagger + \varepsilon Z, \tag{1.7}$$

where $g^\dagger = F(f^\dagger)$, Z is some random variable that models the noise and $\varepsilon > 0$ corresponds to the noise level.

From now on we will make use of the concept of Besov spaces, which are a generalization of other function spaces such as Sobolev spaces. Some basic facts that the reader needs to know about them are found in the appendix. As mentioned in the introduction it was shown in [40] that for certain inverse problems convergence rates are equivalent to smoothness of $f^\dagger$ measured in Besov spaces. Another reason why they are a crucial tool in our analysis is that they give a sharp way to estimate the regularity of stochastic processes as for example Gaussian white noise, which we then can transfer to sharp error estimates.

For $\mathbb{M} = \mathbb{T}^d$ let $\mathcal{D}(\mathbb{M})$ denote the space of infinitely differentiable functions on $\mathbb{T}^d$, whereas for a bounded Lipschitz domain $\mathbb{M} \subset \mathbb{R}^d$ let $\mathcal{D}(\mathbb{M})$ denote the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support in $\mathbb{M}$. In both cases let $\mathcal{D}'(\mathbb{M})$ denote the corresponding dual space, called space of distributions on $\mathbb{M}$. We consider the error model (1.7) with a random variable $Z\colon \Omega \to \mathcal{D}'(\mathbb{M})$ from some probability space $(\Omega, \Sigma, \mathbb{P})$ to the space of distributions on $\mathbb{M}$. Additionally we have the following assumption on the Besov regularity of the noise.

Assumption 1.4.1. *Assume that for some $\gamma \geq 0$ and $p' \in [1, \infty]$ we have $Z \in B^{-\gamma}_{p',\infty}(\mathbb{M})$ almost surely and that there exist constants $C_Z, M_Z, \tau > 0$, such that*

$$\forall t > 0\colon \quad \mathbb{P}\left(\|Z\|_{B^{-\gamma}_{p',\infty}} > M_Z + t\right) \leq \exp(-C_Z t^\tau).$$

Now we introduce the two most important cases for Z.

1.4.1 Gaussian white noise

We work with the following definition from [72]. For a more general introduction to Gaussian processes we refer to [33].

Definition 1.4.2. *A random variable $W\colon \Omega \to \mathcal{D}'(\mathbb{M})$ is called Gaussian white noise if*

(a) for each $g \in \mathcal{D}(\mathbb{M})$, the random variable $\omega \mapsto \langle W(\omega), g\rangle$ is a centered Gaussian random variable,

(b) for all $g_1, g_2 \in \mathcal{D}(\mathbb{M})$ we have

$$\mathbb{E}(\langle W(\omega), g_1\rangle\langle W(\omega), g_2\rangle) = \langle g_1, g_2\rangle_{L^2(\mathbb{M})}.$$

Additionally we introduce the notion of Hilbert space processes, that has been used in [6] to study statistical inverse problems on Hilbert spaces.

Definition 1.4.3. *Let $\mathcal{Y}$ be a Hilbert space and $(\Omega, \Sigma, \mathbb{P})$ a probability space. A Hilbert process on $\mathcal{Y}$ is a bounded linear mapping*

$$\mathcal{W} \colon \mathcal{Y} \to L^2(\Omega).$$

$\mathbb{E}[\mathcal{W}] \in \mathcal{Y}$ is called expectation of $\mathcal{W}$ if

$$\langle \mathbb{E}[\mathcal{W}], g \rangle = \mathbb{E}[\mathcal{W}(g)], \quad \forall g \in \mathcal{Y},$$

and $\mathrm{Cov}[\mathcal{W}] \colon \mathcal{Y} \to \mathcal{Y}$ is called covariance operator of $\mathcal{W}$ if

$$\langle \mathrm{Cov}[\mathcal{W}] g_1, g_2 \rangle = \mathrm{Cov}[\mathcal{W}(g_1), \mathcal{W}(g_2)], \quad \forall g_1, g_2 \in \mathcal{Y}.$$

If $\mathrm{Cov}[\mathcal{W}] = \mathrm{id}$, where $\mathrm{id}(g) = g$ for all $g \in \mathcal{Y}$, then we call $\mathcal{W}$ a white noise process and if $\mathcal{W}(g)$ is a Gaussian random variable for all $g \in \mathcal{Y}$, then we call $\mathcal{W}$ a Gaussian process.

For Gaussian white noise both definitions coincide in the following way.

Proposition 1.4.4. *$W \colon \Omega \to \mathcal{D}'(\mathbb{M})$ being a Gaussian white noise is equivalent to the map $\mathcal{W} \colon L^2(\mathbb{M}) \to L^2(\Omega), g \mapsto \langle W, g \rangle$ being a Gaussian Hilbert space process with $\mathbb{E}(\mathcal{W}) = 0$ and $\mathrm{Cov}(\mathcal{W}) = \mathrm{id}$.*

Proof. Given Gaussian white noise W one can show that $\mathcal{W}$ is a Gaussian Hilbert process by using that $\mathcal{D}(\mathbb{M})$ is dense in $L^2(\mathbb{M})$ as in [72] (note that for $\mathbb{M}$, a bounded Lipschitz domain, density of $\mathcal{D}(\mathbb{M})$ in $L^2(\mathbb{M})$ is given by [71, Corollary 3.32]). The other implication follows from the definitions. □

Theorem 1.4.5. *For $Z = W$ Gaussian white noise Assumption 1.4.1 holds true for $\mathbb{M} = \mathbb{T}^d$, the d-dimensional torus, and all $p' \in [1, \infty)$ with $\gamma = d/2$, $\tau = 2$. If $\gamma < d/2$ then $Z \notin B^{-\gamma}_{p,q}(\mathbb{M})$ almost surely for all $1 \leq p, q \leq \infty$. Further if $p' = \infty$ then Assumption 1.4.1 instead holds true with $\gamma = d/2 + \varepsilon$ for all $\varepsilon > 0$.*

Proof. The first statements are given in [72, Theorem 3.4 and Corollary 3.7]. The last statement follows from Theorem A.2.5 as it gives the continuous embedding $B^{-d/2}_{d/\varepsilon,\infty} \subset B^{-d/2-\varepsilon}_{\infty,\infty}$. Thus

$$\|Z\|_{B^{-d/2-\varepsilon}_{\infty,\infty}} \leq C \|Z\|_{B^{-d/2}_{d/\varepsilon,\infty}}$$

and for the latter term we have the deviation inequality as $d/\varepsilon < \infty$. □

1.4.2 Poisson point process

We only give some basic properties and refer to [43, 48] for a more general overview.

Definition 1.4.6. *Let $g^\dagger \in L^1(\mathbb{M})$ with $g^\dagger \geq 0$. A point process $G = \sum_{i=1}^N \delta_{x_i}$ is called a Poisson point process or Poisson process with intensity $g^\dagger$ if*

(a) *For each choice of disjoint, measurable sets $A_1, \dots, A_n \subset \mathbb{M}$ the random variables $G(A_i) := \#\{x_i \in A_i : i = 1, \dots, N\}$ are stochastically independent,* *(b)* $\mathbb{E}[G(A)] = \int_A g^\dagger \mathrm{d}x$ *for each measurable set* $A \subset \mathbb{M}$.

By [47, Thm. 1.11.8] the number of event counts in each measurable subset of $\mathbb{M}$ is a Poisson distributed random variable.

Proposition 1.4.7. *Let G be a Poisson process with intensity $g^\dagger \in L^1(\mathbb{M})$. Then for each measurable $A \subset \mathbb{M}$ the random variable $G(A) = \#\{x_i \in A : i = 1, \dots, N\}$ is Poisson distributed with rate parameter $\lambda = \int_A g^\dagger$.*

For Poisson data we are interested in convergence rates as the observation time t or equivalently the number of counts N goes to infinity. Therefore we consider a spatio-temporal Poisson process $\tilde{G}$ on $\mathbb{M} \times [0, \infty)$ instead of just $\mathbb{M}$ and consider for each $t > 0$ a rescaled Poisson process as in [43, Definition 2.5].

Definition 1.4.8. *Let $\tilde{G} = \sum_j \delta_{x_j,t_j}$ be a (spatio-temporal) Poisson process on $\mathbb{M} \times [0, \infty)$ with temporally constant intensity $\tilde{g}^\dagger(x,t) = g^\dagger(x)$. Then a temporally normalized Poisson process $(G_t)_{t \geq 0}$ with intensity $g^\dagger$ is defined by*

$$G_t = \frac{1}{t} \int_0^t \tilde{G}(\cdot, \tau) \mathrm{d}\tau = \frac{1}{t} \sum_{t_j \leq t} \delta_{x_j,t_j}.$$

Theorem 1.4.9. *For $Z = \sqrt{t}(G_t - g^\dagger)$ Assumption 1.4.1 with $\tau = 1$, $p = 2$ holds true for all $\gamma > d/2$.*

Proof. Note that $\|Z\|_{B^{-\gamma}_{2,\infty}} \leq \|Z\|_{B^{-\gamma}_{2,2}} = \sup_{\|g\|_{H^\gamma} \leq 1} \langle Z, g \rangle$ (see Section A.2 in the appendix) and for this supremum the deviation inequality holds by [75, Theorem 2.1]. □

One would expect that for $Z = \sqrt{t}(G_t - g^\dagger)$ Assumption 1.4.1 also holds true with $\gamma = d/2$. However to our best knowledge this is only a conjecture up to now. If γ is chosen as $d/2 + \varepsilon$ with $\varepsilon > 0$ sufficiently small the analysis of this work can be done exactly in the same way as for Gaussian white noise and will result in slightly weaker convergence rates.

1.4.3 Discretization

Although a continuous model is the correct way to theoretically treat the ill-posedness of an inverse problem, in practice there has to be a discretization at some point. Even if one could come up with a continuous measurement device, any computer can only deal with a finite amount of data. Therefore we will show how the continuous noise models from above can be related back to the finite dimensional models that we shortly introduced in Section 1.3. Assume we have a partition $\mathbb{M} = \bigcup_{j=1}^m \mathbb{M}_j$ of our measurement domain, with disjoint subsets $\mathbb{M}_j \subset \mathbb{M}$, $|\mathbb{M}_j| > 0$. Under the Gaussian white noise model we then get the discrete data

$$g_j^{\mathrm{obs}} = \langle g^\dagger + \varepsilon W, \mathbb{1}_{M_j} \rangle, \quad \mathbb{1}_{M_j}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{M}_j \\ 0, & \text{else.} \end{cases}$$

Let $g_j^\dagger = \int_{M_j} g^\dagger$. Then g_j^{obs} is normally distributed with $\mathbb{E}[g_j^{\mathrm{obs}}] = g_j^\dagger$ and $\mathrm{Var}[g_j^{\mathrm{obs}}] = \varepsilon |\mathbb{M}_j|$ by Definition 1.4.2. Thus in oder to derive the negative log-likelihood as in (1.5) the bins $\mathbb{M}_j$ should all have the same size, resulting in a uniform variance for all g_j^{obs}.

Under the Poisson model we get the discrete data

$$g_j^{\mathrm{obs}} = \langle G_t, \mathbb{1}_{M_j} \rangle = \frac{1}{t} \#\{(x_i, t_i) : x_i \in M_j, t_i \leq t\}$$

and by Proposition 1.4.7 we have that $t g_j^{\mathrm{obs}}$ is Poisson distributed with rate parameter $t g_j^{\dagger}$. Therefore both continuous models give back the discrete models from Section 1.3. As any discretization usually causes a certain discretization error, convergence to a true underlying continuous function $f^{\dagger}$ can only by achieved by letting J go to infinity. This yields another argument as to why we rather consider the continuous model. Each estimate on the reconstruction error that one can obtain in the finite-dimensional setting will include a constant that depends on the dimension J whilst an error estimate for the continuous model also holds for all discretizations with a constant independent of J. In principle the discretization error can be controlled as outlined e.g. in [43, Section 2.5], however a convergence rate derived in the continuous setting more clearly illuminates the actual dependence on the noise as it only depends on the noise level and not on the dimension.

Notational conventions

We discuss a few conventions that will be assumed throughout this work. First of all C will always denote a generic positive constant that is allowed to change from one occurence to the next in the sense that for example $C = 5C$ is true. Some of the resulting constants depend on so many other constants that we will not always track all dependencies, but we will rather highlight special and possibly problematic dependencies individually and will leave it to the reader to put them together if she or he is interested. To point out that C depends on certain mathematical quantities $x_1, \dots x_n$ we will sometimes write $C_{x_1,\dots,x_n}$. We of course guarantee that no constants depend on the noise level or the regularization parameter and we are careful about dependencies on the true solution and related functions. These may still occur and in fact it is common in regularization theory that the constant of a convergence rate depends on $\varrho := \|f^\dagger\|_{\tilde{\mathcal{X}}}$, where $\tilde{\mathcal{X}} \subset \mathcal{X}$ is some space corresponding to a certain smoothness (compare (1.3)). We ensure that all constants depending on $f^\dagger$ can be bounded by some positive power of ϱ.

Regarding function spaces like $L^p(\Omega) \subset D'(\Omega)$ that are contained in the space of distributions we will often write $L^p(\Omega) = L^p$, when the underlying measure space Ω is clear. For a real number $1 \leq p \leq \infty$ the notation p' is always defined by the real number $1 \leq p' \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The notation $\langle f^*, f\rangle$ denotes the dual pairing $f^*(f)$, where $f^* \in \mathcal{X}^*$, $f^*\colon \mathcal{X} \to \mathbb{R}$, $f \in \mathcal{X}$ for some Banach space $\mathcal{X}$. For functions $f_1, f_2 \in D'(\Omega)$ the dual pairing denotes the standard L^2 dual pairing

$$\langle f_1, f_2\rangle = \int_\Omega f_1(x) f_2(x) \mathrm{d}x.$$

We will sometimes make use of the Landau notation $f(x) = \mathcal{O}\left(\phi(x)\right), x \to a$ with the meaning that $f(x) \leq C\phi(x)$ if $x \in \mathbb{R}$ is sufficiently close to $a \in \mathbb{R}$. The notation $f(x) \sim \phi(x)$ implies that there exist constants $c_1, c_2 > 0$ such that $c_1\phi(x) \leq f(x) \leq c_2\phi(x)$.

We try to use most symbols consistently with the same meaning. A list of common symbols can be found in Appendix A.3.

Chapter Two

Convex Analysis and the Bregman divergence

"To live, to err, to fall, to triumph, to recreate life out of life. A wild angel appeared to him, the angel of mortal youth and beauty, an envoy from the fair courts of life, to throw open before him in an instant of ecstasy the gates of all the ways of error and glory. On and on and on and on!"

A potrait of the artist as a young man, J. Joyce

In order to treat Tikhonov regularization of the form (1.6) in a very general way, we will make use of convexity, which induces several favorable properties. To this end we introduce some basic notions of the field of convex analysis. In Chapter 4 we will see that it is generally possible to obtain estimates on the reconstruction error, but with the error measured with respect to the so called Bregman divergence, rather than in the norm of $\mathcal{X}$. Thus another large part of this chapter is dedicated to the question when and how the Bregman divergence can be compared to the distance in the norm. In this chapter $\mathcal{X}$ will always be a real Banach space, $\mathcal{X}^*$ denotes its dual space and $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ some function.

2.1 Basics[1]

We allow that functions take the value ∞ as this in particular allows to rewrite constrained minimization problems of the form

$$\min_{x\in\mathcal{B}} \mathcal{F}(x),$$

with $\mathcal{B} \subset \mathcal{X}$ as unconstrained minimization problems

$$\min_{x\in X} \widetilde{\mathcal{F}}(x), \quad \widetilde{\mathcal{F}}(x) = \begin{cases} \mathcal{F}(x), & \text{if } x \in \mathcal{B} \\ \infty, & \text{else.} \end{cases}$$

Then of course we need some rules how to calculate in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.

[1] The proofs for all results in this section are standard and can be found e.g. in [24]. We give some of the proofs for the sake of self-containedness and to emphasize the simplicity of the variational approach.

Definition 2.1.1 (infinity). *We have*

$$
\begin{aligned}
&-\infty < \lambda < \infty, \quad \forall \lambda \in \mathbb{R} \\
&\lambda(\pm\infty) = \pm\infty, \quad \forall \lambda > 0 \\
&\lambda \pm \infty = \pm\infty + \lambda = \pm\infty, \quad \forall \lambda \in \mathbb{R} \\
&\pm\infty + (\pm)\infty = \pm\infty \\
&\infty + (-\infty) = (-\infty) + \infty = \infty.
\end{aligned}
$$

Definition 2.1.2 (convex set). *$A \subset \mathcal{X}$ is called convex if we have for all $x, y \in A$ that*

$$\{\lambda x + (1-\lambda)y : \lambda \in [0,1]\} \subset A.$$

In words: all line segments connecting two points in A are contained in A.

Definition 2.1.3 (epigraph). *The epigraph of a function $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$ is given by the set*

$$\operatorname{epi}(\mathcal{F}) = \{(x, \lambda) \in \mathcal{X} \times \mathbb{R} : \mathcal{F}(x) \leq \lambda\}.$$

Definition and Theorem 2.1.4 (convex function). *A functional $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$ is called convex if the following equivalent conditions hold true.*

(a) $\operatorname{epi}(\mathcal{F})$ *is convex.*

(b) For all $x, y \in \mathcal{X}$ and $\lambda \in [0,1]$ we have

$$\mathcal{F}(\lambda x + (1-\lambda)y) \leq \lambda \mathcal{F}(x) + (1-\lambda)\mathcal{F}(y). \tag{2.1}$$

(c) For all $x, y \in \mathcal{X}$ and $\lambda \in [1,\infty)$ we have

$$\mathcal{F}(\lambda x + (1-\lambda)y) \geq \lambda \mathcal{F}(x) + (1-\lambda)\mathcal{F}(y). \tag{2.2}$$

$\mathcal{F}$ is called strictly convex if (2.1) holds true with "$<$" for $x \neq y$, $\lambda \in (0,1)$. $\mathcal{F}$ is called (strictly) concave if $-\mathcal{F}$ is (strictly) convex.

Proof. $(a) \Leftrightarrow (b)$: $\operatorname{epi}(\mathcal{F})$ is convex if and only if for all $\lambda \in [0,1]$ and $(x,\alpha), (y,\beta) \in \operatorname{epi}(\mathcal{F})$ we have

$$\lambda(x,\alpha) + (1-\lambda)(y,\beta) \in \operatorname{epi}(\mathcal{F}) \qquad \Leftrightarrow \qquad \mathcal{F}(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\beta. \tag{2.3}$$

Now if $\mathcal{F}$ is convex, then the latter condition holds true by

$$\mathcal{F}(\lambda x + (1-\lambda)y) \leq \lambda \mathcal{F}(x) + (1-\lambda)\mathcal{F}(y) \leq \lambda\alpha + (1-\lambda)\beta.$$

Conversely if $\operatorname{epi}(\mathcal{F})$ is convex then (2.3) implies (2.1) for all $\lambda \in [0,1]$, $x, y \in \mathcal{X}$ where $\mathcal{F}(x), \mathcal{F}(y)$ are finite, by simply putting $\alpha = \mathcal{F}(x)$, $\beta = \mathcal{F}(y)$. If either $\mathcal{F}(x) = \infty$ or $\mathcal{F}(y) = \infty$ then (2.1) holds by Definition 2.1.1. Finally if $\mathcal{F}(x), \mathcal{F}(y) < \infty$ and either $\mathcal{F}(x) = -\infty$ or $\mathcal{F}(y) = -\infty$ then one can let α or β go to zero to also show $\mathcal{F}(\lambda x + (1-\lambda)y) = -\infty$.

$(b) \Leftrightarrow (c)$: (2.2) for all $x, y \in \mathcal{X}$ and $\lambda \in [1,\infty)$ can be equivalently rephrased as

$$\frac{1}{\lambda}\mathcal{F}(\lambda x + (1-\lambda)y) + \Big(1 - \frac{1}{\lambda}\Big)\mathcal{F}(y) \geq \mathcal{F}(x) = \mathcal{F}\Big(\frac{1}{\lambda}(\lambda x + (1-\lambda)y) + \Big(1-\frac{1}{\lambda}\Big)y\Big)$$

which is equivalent to (b) by putting $\lambda = \frac{1}{\lambda} \in (0,1]$ and $x = \lambda x + (1-\lambda)y$ in (2.1) as (2.1) holds trivially for $\lambda = 0$. ☐

Lemma 2.1.5. *Let $\mathcal{F}, \mathcal{G}\colon \to \overline{\mathbb{R}}$ convex, $\mathcal{H}\colon \to \overline{\mathbb{R}}$ strictly convex.*

(a) For $\alpha \geq 0$ the function $\alpha\mathcal{F}$ is convex, $\alpha\mathcal{H}$, strictly convex.
(b) The function $\mathcal{F} + \mathcal{G}$ is convex, $\mathcal{F} + \mathcal{H}$ strictly convex.
(c) For $T\mathcal{Y} \to \mathcal{X}$ linear the function $F \circ T\colon \mathcal{Y} \to \overline{\mathbb{R}}$ is convex.
(d) For $x \in \mathcal{X}$ the function $F(\cdot + x)$ is convex.

Proof. Exercise. □

Example 2.1.6. *Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is convex on an interval $(a, b) \subset \mathbb{R}$ if it is differentiable with increasing derivative or twice differentiable with non-negative second derivative. Thus we have*

(a) $f(x) = e^x$ is convex on $\mathbb{R}$.
(b) $f(x) = -\log(x)$ is convex on $(0, \infty)$ thus $f(x) = log(x)$ is concave.
(c) Let $\mathcal{X}$ be a normed space then $\mathcal{F}(x) = \|x\|_{\mathcal{X}}$ is convex.
(d) For $p > 1$ the function $f(x) = \frac{1}{p}|x|^p$ is convex since it can be written as $f = f_1 + f_2$, where $f_1 = \begin{cases} (-x)^p/p, & x \leq 0 \\ 0, & x > 0 \end{cases}$, $f_2 = \begin{cases} 0, & x \leq 0 \\ (x)^p/p, & x > 0 \end{cases}$ and both functions are differentiable with increasing derivative.
(e) Let $\mathcal{X}$ be a normed space and $p > 1$, then $\mathcal{F} : \mathcal{X} \to \mathbb{R}, x \mapsto \frac{1}{p}\|x\|^p$ is convex as $f(x) = \frac{1}{p}|x|^p$ is convex and increasing on $[0, \infty)$ and thus

$$\mathcal{F}(\lambda x + (1-\lambda)y) \leq f(\lambda\|x\| + (1-\lambda)\|y\|) \leq \lambda\mathcal{F}(x) + (1-\lambda)\mathcal{F}(y).$$

Definition and Theorem 2.1.7 (lower semi-continuous)**.** *A functional $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$ is called lower semi-continuous if the following equivalent conditions hold true.*

(a) For all $x \in \mathcal{X}$ we have $\liminf_{y\to x} \mathcal{F}(y) \geq \mathcal{F}(x)$.
(b) $\operatorname{epi}(\mathcal{F})$ is closed.

Proof. *(a)* ⇒ *(b)*: Let (x_n, λ_n) be a sequence in $\operatorname{epi}(\mathcal{F})$ converging to $(x, \lambda) \in \mathcal{X} \times \mathbb{R}$. Then $\mathcal{F}(x_n) \leq \lambda_n$ for all $n \in \mathbb{N}$ and $x_n \to x$ in $\mathcal{X}$, $\lambda_n \to \lambda$ in $\mathbb{R}$, thus

$$0 \geq \liminf_{n\in\mathbb{N}}(\mathcal{F}(x_n) - \lambda_n) = \liminf_{n\in\mathbb{N}} \mathcal{F}(x_n) - \lambda \geq \mathcal{F}(x) - \lambda.$$

Therefore $(x, \lambda) \in \operatorname{epi}(\mathcal{F})$.

(b) ⇒ *(a)*: *(a)* holds trivially if $\mathcal{F}(x) = -\infty$. Thus we can assume on the contrary that $\mathcal{F}(x) > \lambda$ for some $\lambda \in \mathbb{R}$. This means that $(x, \lambda) \notin \operatorname{epi}(\mathcal{F})$ and as $\operatorname{epi}(\mathcal{F})$ is closed there exists a neighborhood U_x of x with $U_x \subset \mathcal{X} \setminus \operatorname{epi}(\mathcal{F})$. This implies $\liminf_{y\to x} \mathcal{F}(y) > \lambda$ and this holds for all $\lambda < \mathcal{F}(x)$. □

Example 2.1.8. *Every continuous functional is lower semi-continuous. In Section 3.2.5 we will introduce the cross entropy functional which is lower semi-continuous, but nowhere continuous.*

Definition 2.1.9 (subgradient, subdifferential)**.** *$x^* \in \mathcal{X}^*$ is called a* subgradient *of a convex function $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$ at $x \in \mathcal{X}$ if $\mathcal{F}(x)$ is finite and*

$$\mathcal{F}(y) \geq \mathcal{F}(x) + \langle x^*, y - x\rangle\,, \tag{2.4}$$

for all $y \in \mathcal{X}$. The set of all subgradients of $\mathcal{F}$ at x is called the subdifferential *of $\mathcal{F}$ at x and denoted by $\partial\mathcal{F}(x)$.*

Example 2.1.10. *The absolute value function* $f:\mathbb{R}\to[0,\infty), x\mapsto|x|$ *has subdifferential*

$$\partial f(x)=\begin{cases}\{\operatorname{sgn}(x)\}, & x\neq 0\\ [-1,1], & x=0.\end{cases}$$

Theorem 2.1.11. *Let* $\mathcal{F}\colon\mathcal{X}\to\overline{\mathbb{R}}$ *be convex and Fréchet differentiable in* $x\in\mathcal{X}$*. Then* $\partial\mathcal{F}(x)=\{\mathcal{F}'[x]\}$*.*

Proof. From Definition A.1.2 we can see that

$$\mathcal{F}'[x]h=\lim_{t\searrow 0}\frac{F(x+th)-\mathcal{F}(x)}{t}.$$

As $\mathcal{F}$ is convex we have for $\lambda\in(0,1]$ that

$$\mathcal{F}(x+th)=\mathcal{F}\Big((1-\lambda)x+\lambda\Big(x+\frac{t}{\lambda}h\Big)\Big)\leq(1-\lambda)\mathcal{F}(x)+\lambda\mathcal{F}\Big(x+\frac{t}{\lambda}h\Big)$$

or equivalently

$$\frac{\mathcal{F}(x+th)-\mathcal{F}(x)}{t}\leq\frac{\mathcal{F}(x+(t/\lambda)h)-\mathcal{F}(x)}{t/\lambda}.$$

So the differential quotient is increasing in t and we have for all $h\in\mathcal{X}$ that

$$\langle\mathcal{F}'[x],h\rangle\leq\mathcal{F}(x+h)-\mathcal{F}(x),$$

which implies that $\mathcal{F}'[x]\in\partial\mathcal{F}(x)$. Alternatively if $x^*\in\partial\mathcal{F}(x)$ we have

$$\mathcal{F}(x+\lambda h)-\mathcal{F}(x)\geq\langle x^*,\lambda h\rangle$$

dividing by λ and taking the limit we thus get

$$\langle\mathcal{F}'[x]-x^*,h\rangle\geq 0.$$

This can only hold for all $h\in\mathcal{X}$ if $x^*=\mathcal{F}'[x]$. □

Proposition 2.1.12. *Let* $\mathcal{F}:\mathcal{X}\to\overline{\mathbb{R}}$ *be convex then we have*

$$\mathcal{F}(x)=\min_{y\in\mathcal{X}}\mathcal{F}(y)\qquad\textit{if and only if}\qquad 0\in\partial\mathcal{F}(x).$$

Every local minimum is also a global minimum and if $\mathcal{F}$ *is strictly convex, then there exists at most one solution to the minimization problem.*

Proof. The first claim follows from Definition 2.1.9. Let x be a local minimizer of $\mathcal{F}(y)$ and assume that there exists $\tilde{x}$ with $\mathcal{F}(\tilde{x})<\mathcal{F}(x)$. Then by convexity of $\mathcal{F}$ we have for all $\lambda\in(0,1]$ that

$$\mathcal{F}(\lambda x+(1-\lambda)\tilde{x})\leq\lambda\mathcal{F}(x)+(1-\lambda)\mathcal{F}(\tilde{x})<\mathcal{F}(x),$$

but this is a contradiction to $\mathcal{F}(x)$ being a local minimum. Therefore we must have $\mathcal{F}(x)\leq\mathcal{F}(y)$ for all $y\in\mathcal{X}$. If $\mathcal{F}$ is strictly convex, assume that there exist two minimizers $x_1\neq x_2$. Then we have for $\lambda\in(0,1)$ that

$$\mathcal{F}(\lambda x_1+(1-\lambda)x_2)<\lambda\mathcal{F}(x_1)+(1-\lambda)\mathcal{F}(x_2)=\mathcal{F}(x_1),$$

which is a contradiction. □

Definition 2.1.13 ((Fenchel/convex) conjugate)**.** *Let* $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$*. The* convex conjugate $\mathcal{F}^* : \mathcal{X}^* \to \overline{\mathbb{R}}$ *of* $\mathcal{F}$ *is defined by*

$$\mathcal{F}^*(x^*) = \sup_{x\in\mathcal{X}} \left[\langle x^*, x\rangle - \mathcal{F}(x)\right].$$

From these two definitions one can directly conclude the following *generalized Young (in)equality.*

Lemma 2.1.14 (generalized Young inequality)**.** *Let* $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$*. For all* $x \in \mathcal{X}, x^* \in \mathcal{X}^*$ *we have*

$$\langle x^*, x\rangle \leq \mathcal{F}(x) + \mathcal{F}^*(x^*). \tag{2.5}$$

Equality holds true if and only if $x^* \in \partial\mathcal{F}(x)$*.*

Example 2.1.15. *For* $1 < p, p' < \infty$ *such that* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $f\colon \mathbb{R}to[0,\infty), f(x) = \frac{1}{p}|x|^p$ *we have*

$$f^*(x^*) = \sup_{x\in\mathbb{R}}\left[x^*x - \frac{1}{p}|x|^p\right] = \sup_{\operatorname{sgn}(x^*)x\geq 0}\left[x^*x - \frac{1}{p}(\operatorname{sgn}(x^*)x)^p\right] = \frac{1}{p'}x^{p'},$$

as the unique maximum is assumed at $x = \operatorname{sgn}(x^*)|x^*|^{\frac{1}{p-1}}$ *and* $p' = \frac{p}{p-1}$*. This yields the classical Young inequality*

$$|x^*x| \leq \frac{1}{p}|x|^p + \frac{1}{p'}x^{p'}.$$

Let $F\colon \mathcal{X} \to [0,\infty), \mathcal{F}(x) = \frac{1}{p}\|x\|^p_{\mathcal{X}}$*, then we have*

$$\mathcal{F}^*(x^*) = \sup_{t\geq 0}\sup_{\|x\|_{\mathcal{X}}=t}\left[\langle x^*, x\rangle - \frac{1}{p}|t|^p\right] = \sup_{t\geq 0}\left[\|x^*\|_{\mathcal{X}^*}t - \frac{1}{p}|t|^p\right] = \frac{1}{p'}\|x^*\|^{p'}_{\mathcal{X}^*}. \tag{2.6}$$

As a special case of the Young inequality we also get the following lemma.

Lemma 2.1.16 (Peter-Paul-inequality)**.** *Let* $s, t \in \mathbb{R}$ *and* $1 < p, p' < \infty$ *such that* $\frac{1}{p} + \frac{1}{p'} = 1$ *then for all* $c > 0$ *we have*

$$|st| \leq c|s|^p + (pc)^{\frac{-p'}{p}}\frac{|t|^{p'}}{p'}. \tag{2.7}$$

Proof. Let $f(s) = \frac{1}{p}|s|^p$, then one can compute that $f^*(t) = \frac{1}{p'}|t|^{p'}$ so we can apply (2.5) to find

$$|st| = |s|(pc)^{\frac{1}{p}}(pc)^{\frac{-1}{p}}|t| \leq f\left((pc)^{\frac{1}{p}}|s|\right) + f^*\left((pc)^{\frac{-1}{p}}|t|\right) = c|s|^p + (pc)^{\frac{-p'}{p}}\frac{|t|^{p'}}{p'}.$$

□

If we have two sequences s_n, t_n with $\lim_{n\to\infty} s_n = 0$ then the above inequality allows us to treat the product s_nt_n in a way such that if we allow a large contribution from t_n we then can get an arbitrarily small contribution from s_n in exchange (rob Peter to pay Paul). This principle will be used quite frequently in this thesis.

Lemma 2.1.17. *Let $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$. Then $\mathcal{F}^*$ is a convex and lower semi-continuous function.*

Proof. For all $x \in X$ define the affine linear functional $f_x : \mathcal{X}^* \to \overline{\mathbb{R}}, x^* \mapsto \langle x^*, x\rangle - \mathcal{F}(x)$. Then

$$\begin{aligned}\operatorname{epi}(\mathcal{F}^*) &= \{(x^*, \lambda) \in \mathcal{X}^* \times \mathbb{R} : \mathcal{F}^*(x^*) \leq \lambda\} \\ &= \{(x^*, \lambda) \in \mathcal{X}^* \times \mathbb{R} : \langle x^*, x\rangle - \mathcal{F}(x) \leq \lambda, \forall x \in \mathcal{X}\} = \bigcap_{x\in\mathcal{X}} \operatorname{epi}(f_x).\end{aligned}$$

As all f_x are convex and lower semi-continuous, the epigraph of $\mathcal{F}^*$ is the intersection of convex and closed sets, which is convex and closed. □

Theorem 2.1.18. *Let $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$. Then*

$$\mathcal{F} \geq \mathcal{F}^{**} := (\mathcal{F}^*)^*, \tag{2.8}$$

where equality holds if and only if $\mathcal{F}$ is convex and lower semi-continuous.

Proof. The inequality (2.8) follows from the definition and Young's inequality (2.5) as

$$\mathcal{F}^{**}(x) := \sup_{x^*\in\mathcal{X}^*} [\langle x^*, x\rangle - \mathcal{F}^*(x^*)] \leq \sup_{x^*\in\mathcal{X}^*} \mathcal{F}(x).$$

Further if we have $F = \mathcal{F}^{**}$, then F is convex and lower semi-continuous by Lemma 2.1.17. For the other direction we refer to [24, Proposition 4.1]. □

Corollary 2.1.19. *Let $\mathcal{F} : \mathcal{X} \to \overline{\mathbb{R}}$ be convex and lower semi-continuous. Then $x^* \in \partial\mathcal{F}(x)$ if and only if $x \in \partial\mathcal{F}^*(x^*)$.*

Proof. By Lemma 2.1.14 and Theorem 2.1.18 we have

$$x^* \in \partial\mathcal{F}(x) \Leftrightarrow \mathcal{F}(x) + \mathcal{F}^*(x^*) = \langle x^*, x\rangle \Leftrightarrow \mathcal{F}^{**}(x) + \mathcal{F}^*(x^*) = \langle x^*, x\rangle \Leftrightarrow x \in \partial\mathcal{F}^*(x^*).$$

□

We will be especially interested in functionals $\mathcal{F}(x) = \frac{1}{p}\|x\|^p$ for some $p \geq 1$ and need to understand their subdifferentials. For some $p \geq 1$ the set-valued mapping $J_p : \mathcal{X} \to 2^{X^*}$ given by

$$J_p(x) = \left\{x^* \in \mathcal{X}^* : \langle x^*, x\rangle = \|x^*\|\|x\|, \|x^*\| = \|x\|^{p-1}\right\} \tag{2.9}$$

is called the *duality mapping* with respect to p of $\mathcal{X}$. The sets $J_p(x)$ are always non-empty. A mapping $j_p : \mathcal{X} \to \mathcal{X}^*$ is called *selection* of J_p if $j_p(x) \in J_p(x)$ for all $x \in \mathcal{X}$. If $\mathcal{F}(x) = \frac{1}{p}\|x\|^p$, then we have [18, Chap.1, Theorem 4.4]

$$\partial\mathcal{F}(x) = J_p(x).$$

2.2 Upper and lower bounds for the Bregman divergence[2]

In recent times the Bregman divergence (or Bregman distance) $\Delta_{\mathcal{F}}^{x^*}(y,x)$, introduced by Bregman in [9], has been used as a generalized distance measure in various branches of applied mathematics, for example optimization, inverse problems, statistics and computational mathematics, especially machine learning. To get an overview over the Bregman divergence and its possible applications in optimization and inverse problems we refer to [10, 15, 56]. In particular the Bregman divergence has been used for various algorithms in numerical analysis and also for convergence analysis of numerical methods and algorithms. In particular in regularization theory on Banach spaces the Bregman divergence has been used to measure the reconstruction error and likewise we will use it in this way.

Definition 2.2.1 (Bregman divergence)**.** *$\mathcal{F}: \mathcal{X} \to \overline{\mathbb{R}}$. For $x^* \in \partial\mathcal{F}(x)$, $x, y \in \mathcal{X}$ the* Bregman divergence *$\Delta_{\mathcal{F}}^{x^*}(y,x)$ is given by*

$$\Delta_{\mathcal{F}}^{x^*}(y,x) = \mathcal{F}(y) - \mathcal{F}(x) - \langle x^*, y - x\rangle.$$

We will sometimes just write $\Delta_{\mathcal{F}}(y,x)$ for the Bregman divergence if the subgradient is clear. If we also have $y^* \in \partial\mathcal{F}(y)$ then we can define the symmetric Bregman divergence

$$\Delta_{\mathcal{F}}^{sym}(x,y) = \Delta_{\mathcal{F}}^{x^*}(y,x) + \Delta_{\mathcal{F}}^{y^*}(x,y) = \langle y^* - x^*, y - x\rangle.$$

Lemma 2.2.2. *The Bregman divergence has the following properties.*

(a) $\Delta_{\mathcal{F}}^{x^*}(y,x) \geq 0$ for all $x, y \in \mathcal{X}$.
(b) $\Delta_{\mathcal{F}}^{x^*}(y,x)$ is convex in y.
(c) Let $x^* \in \partial\mathcal{F}(x)$, $y^* \in \partial\mathcal{F}(y)$, then

$$\Delta_{\mathcal{F}}^{x^*}(y,x) = \Delta_{\mathcal{F}^*}^{y}(x^*, y^*). \tag{2.10}$$

Proof. The first two properties follow from the definition of the subdifferential respectively Lemma 2.1.5. The third property follows from Young's equality from Lemma 2.1.14 as

$$\begin{aligned}\Delta_{\mathcal{F}}^{x^*}(y,x) &= \mathcal{F}(y) - \mathcal{F}(x) + \langle x^*, x\rangle - \langle x^*, y\rangle = \mathcal{F}^*(x^*) + \mathcal{F}(y) - \langle x^*, y\rangle \\ &= \mathcal{F}^*(x^*) - \mathcal{F}^*(y^*) - \langle x^* - y^*, y\rangle = \Delta_{\mathcal{F}^*}^{y}(x^*, y^*).\end{aligned}$$

□

Remark 2.2.3. *Although the Bregman divergence is non-negative, it is generally not symmetric and not positive definite. If $\mathcal{X}$ is a Hilbert space and $\mathcal{F} = \|\cdot\|_{\mathcal{X}}^2$ then $\{2x\} = \partial\mathcal{F}(x)$ for all $x \in \mathcal{X}$ and thus*

$$\Delta_{\mathcal{F}}^{x}(y,x) = \|y\|_{\mathcal{X}}^2 - \|x\|_{\mathcal{X}}^2 - \langle 2x, y - x\rangle = \|y - x\|_{\mathcal{X}}^2.$$

So in this case the square root of the Bregman gives a metric. To give an example, where the Bregman divergence is not very informative as a distance measure let $\mathcal{X} = \mathbb{R}$, $\mathcal{F}(x) = |x|$, then $\partial\mathcal{F}(x) = \{\operatorname{sgn}(x)\}$ for $x \neq 0$, so that

$$\Delta_{\mathcal{F}}^{\operatorname{sgn}(x)}(y,x) = |y| - |x| - \operatorname{sgn}(x)(y - x) = \begin{cases} 0, & \text{if } \operatorname{sgn}(y) = \operatorname{sgn}(x) \\ 2|y|, & \text{else.}\end{cases}$$

[2]Most of this section is taken literally from the article [63] of the author.

Especially when doing convergence analysis it is often crucial to have lower and upper bounds on the Bregman divergence in terms of norms. In [77] the authors prove upper and lower bounds for expressions

$$\|x+y\|^p - \|x\|^p - p\,\langle j_p(x), y\rangle =: \Delta_{\mathcal{F}}^{j_p(x)}(x+y, x), \tag{2.11}$$

where $j_p : \mathcal{X} \to \mathcal{X}^*$ is a selection of the duality mapping, under certain assumptions on the Banach space $\mathcal{X}$. As it turns out that (2.11) is the Bregman divergence corresponding to the functional $\mathcal{F} = \|\cdot\|^p$ these results have been used since then in many publications working with the Bregman divergence. However from the proofs of [77] it seems difficult to transfer the results to other functions $\mathcal{F}$. Thus we develop in this work a simple framework to find such bounds and in fact can apply it to give a short new proof of the results from [77] for $\mathcal{F}(x) = \|x\|^p, p > 1$.

Our approach is as follows: Proving upper bounds is rather simple if one sufficiently understands the smoothness of $\mathcal{F}$ as the Bregman divergence is basically a linearization error and linearization errors are related to differentiability by definition. In particular we will show that one can obtain upper bounds for the Bregman divergence corresponding to $\mathcal{F} = \phi(\|\cdot\|)$ if $\phi : \mathbb{R} \to \mathbb{R}$ is convex and sufficiently smooth.

Regarding lower bounds we will make use of $\mathcal{F}^*$, the convex conjugate of $\mathcal{F}$. Actually it can be shown that lower bounds for $\Delta_{\mathcal{F}}^{x^*}(y, x)$ correspond to upper bounds for $\Delta_{\mathcal{F}^*}^{x}(y^*, x^*)$. Note that this idea is not at all new. Already in [79, 5] this kind of connection between $\mathcal{F}$ and $\mathcal{F}^*$ was discussed in depth. So again one can just make use of the smoothness of $\mathcal{F}^*$ to conclude lower bounds for $\Delta_{\mathcal{F}}^{x^*}(y, x)$. One might argue that convex conjugates can be rather complicated functions and expecting differentiability is too optimistic. This is true to some extent, but actually reasonable lower bounds on $\Delta_{\mathcal{F}}^{x^*}(y, x)$ already imply differentiability of $\mathcal{F}^*$ at x^* (see [79, Theorem 2.1]). So if one has any hope on finding lower bounds then one might as well work with the convex conjugate.

If $\mathcal{F}$ is related to the norm $\|\cdot\|_{\mathcal{X}}$ it is necessary to understand smoothness and convexity of the space $\mathcal{X}$ and we therefore introduce the following definitions (see e.g. [49]):

Definition 2.2.4. *Let* $\dim \mathcal{X} \geq 2$. *Define* $S_{\mathcal{X}} := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} = 1\}$. *The modulus of convexity* $\delta_{\mathcal{X}} \colon [0, 2] \to [0, 1]$ *of the space* $\mathcal{X}$ *is defined by*

$$\delta_{\mathcal{X}}(\varepsilon) := \inf\{1 - \|y + \tilde{y}\|/2 : y, \tilde{y} \in S_{\mathcal{X}}, \|y - \tilde{y}\| = \varepsilon\}.$$

The modulus of smoothness $\rho_{\mathcal{X}} \colon [0, \infty) \to [0, \infty)$ *of* $\mathcal{X}$ *is defined by*

$$\rho_{\mathcal{X}}(\tau) := \sup\{(\|x + \tau y\| + \|x - \tau y\|)/2 - 1 : x, y \in S_{\mathcal{X}}\}.$$

The space $\mathcal{X}$ *is called uniformly convex if* $\delta_{\mathcal{X}}(\varepsilon) > 0$ *for every* $\varepsilon > 0$. *It is called uniformly smooth if* $\lim_{\tau\to 0} \rho_{\mathcal{X}}(\tau)/\tau = 0$. *The space* $\mathcal{X}$ *is called r-convex (or convex of power type r) if there exists a constant* $K > 0$ *such that* $\delta_{\mathcal{X}}(\varepsilon) \geq K\varepsilon^r$ *for all* $\varepsilon \in [0, 2]$. *Similarly, it is called s-smooth (or smooth of power type s) if* $\rho_{\mathcal{X}}(\tau) \leq K\tau^s$ *for all* $\tau > 0$.

The main result of this section is the following theorem

Theorem 2.2.5. *Let* $\mathcal{X}$ *be a Banach space and* $\mathcal{F}(x) = \frac{1}{p}\|x\|^p$ *for* $p > 1$ *then there exists constants* $C_1, C_2 > 0$ *such that for all* $x, y \in \mathcal{X}$ *we have*

$$\Delta_{\mathcal{F}}^{j_p(x)}(y, x) \leq C_1 \max\{\|x\|, \|y\|\}^p \rho_{\mathcal{X}}\left(\frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}}\right) \tag{2.12}$$

and

$$\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq C_2 \max\{\|x\|,\|y\|\}^p \delta_{\mathcal{X}}\left(\frac{\|x-y\|}{3\max\{\|x\|,\|y\|\}}\right). \tag{2.13}$$

If the space $\mathcal{X}$ is s-smooth, then there exists $C>0$ and for all $\overline{\tau}>0$ also $C_{\overline{\tau}}>0$ such that

$$\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \leq \begin{cases} C\|x-y\|^s, & p=s \\ C_{\overline{\tau}}\|x\|^{p-s}\|x-y\|^s, \; \textit{for } \frac{\|x-y\|}{\|x\|} \leq \overline{\tau}, & p \neq s. \end{cases} \tag{2.14}$$

If the space $\mathcal{X}$ is r-convex, then there exists $C_{p,\mathcal{Y}}>0$ and for all $\overline{\tau}>0$ also $\tilde{C}_{\overline{\tau}}>0$ such that

$$\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq \begin{cases} C_{p,\mathcal{Y}}\|x-y\|^r, & p=r \\ \tilde{C}_{\overline{\tau}}\|x\|^{p-r}\|x-y\|^r, \; \textit{for } \frac{\|x-y\|}{\|x\|} \leq \overline{\tau}, & p \neq r. \end{cases} \tag{2.15}$$

The moduli of smoothness and convexity have a well-developed theory, which is known in the literature for a long time and we will not discuss all their properties. However for our proofs we will need the following properties.

Lemma 2.2.6. *(a) We have for $\tau_1 \leq \tau_2$ that $\rho_{\mathcal{X}}(\tau_1)/\tau_1 \leq \rho_{\mathcal{X}}(\tau_2)/\tau_2$.*

(b) We have for all $\overline{\tau}>0$ that there exists a constant $C_{\overline{\tau}}$ such that for all Banach spaces $\mathcal{X}$ we have

$$\rho_{\mathcal{X}}(\tau) \geq (1+\tau^2)^{\frac{1}{2}} - 1 \geq C_{\overline{\tau}}\tau^2, \qquad \tau \leq \overline{\tau}.$$

(c) If $\delta_{\mathcal{X}}$ is extended by ∞ on $\mathbb{R}\setminus[0,2]$ then $(2\delta_{\mathcal{X}})^* = 2\rho_{\mathcal{X}^*}$.

(d) The space $\mathcal{X}$ is r-convex if and only if its dual $\mathcal{X}^*$ is r'-smooth.

(e) There exists a convex function f such that $\delta_{\mathcal{X}}(\tau/2) \leq f(\tau) \leq \delta_{\mathcal{X}}(\tau)$. In particular we have $\delta_{\mathcal{X}}^{**}(\tau) \geq \delta_{\mathcal{X}}(\tau/2)$.

Proof. All statements follow from [49, Ch. 1.e]. The function f in the last statement can be chosen as the Orlicz function $f(\tau) := (\rho_{\mathcal{X}^*})(\tau/2)$ by [49, Lemmata 1.e.6, 1.e.7]. □

For our purposes it will be more natural to introduce new definitions of the moduli of smoothness and convexity related to functionals instead of spaces.

Definition 2.2.7. *Let $\mathcal{F}\colon \mathcal{X} \to \overline{\mathbb{R}}$ be some arbitrary function, $x \in \mathcal{X}$, $\mathcal{F}(x) < \infty$ and $\xi \in \mathcal{X}^*$. Define the* linearization error functional *$\Delta_{\mathcal{F}}^{\xi}(y,x)$ by*

$$\Delta_{\mathcal{F}}^{\xi}(y,x) = \mathcal{F}(y) - \mathcal{F}(x) - \langle \xi, y-x \rangle .$$

The modulus of smoothness $\rho_{\mathcal{F},x}^{\xi}\colon [0,\infty) \to [0,\infty]$ of $\mathcal{F}$ in x with respect to ξ is defined by

$$\rho_{\mathcal{F},x}^{\xi}(\tau) := \sup_{y \in S_{\mathcal{X}}} |\mathcal{F}(x+\tau y) - \mathcal{F}(x) - \langle \xi, \tau y \rangle| = \sup_{\|x-y\|=\tau} \left|\Delta_{\mathcal{F}}^{\xi}(y,x)\right|.$$

The modulus of convexity $\delta_{\mathcal{F},x}^{\xi}\colon [0,\infty) \to [0,\infty]$ of $\mathcal{F}$ in x with respect to ξ is defined by

$$\delta_{\mathcal{F},x}^{\xi}(\tau) := \inf_{\|x-y\|=\tau} \left|\Delta_{\mathcal{F}}^{\xi}(y,x)\right|.$$

$\mathcal{F}$ is called r-convex (or convex of power type r) in x (w.r.t. ξ) if there exists $K,\overline{\tau}>0$ such that $\delta_{\mathcal{F},x}^{\xi}(\tau) \geq K\tau^r$ for all $0<\tau\leq\overline{\tau}$. Similarly, it is called s-smooth (or smooth of power type s) in x (w.r.t. ξ) if $\rho_{\mathcal{F},x}^{\xi}(\tau) \leq K\tau^s$ for all $0<\tau\leq\overline{\tau}$.

The quantities $\rho^{\xi}_{\mathcal{F},x}$, $\delta^{\xi}_{\mathcal{F},x}$ give us a reformulation of our basic problem: We want to find upper bounds for $\rho^{\xi}_{\mathcal{F},x}(\tau)$ and lower bounds for $\delta^{\xi}_{\mathcal{F},x}(\tau)$. Before we show some properties of these functions we should state some simple facts for their interpretation.

Remark 2.2.8. *We will mostly consider convex functions $\mathcal{F}$ with $\xi \in \partial\mathcal{F}(x)$ so that the linearization error functional is a Bregman divergence and one can neglect the absolute value.*

$\mathcal{F}$ is Fréchet-differentiable in x if and only if there exists $\xi \in \mathcal{X}^*$, such that $\rho^{\xi}_{\mathcal{F},x}(\tau)/\tau \to 0$ as $\tau \to 0$ (see Definition A.1.2). $\mathcal{F}$ being s-smooth in x, with $s \in (1, 2]$ then can be seen as a stronger form of differentiability, comparable to fractional derivatives, however $\mathcal{F}$ being 2-smooth is not equivalent to twice differentiability but rather to the notion of strong smoothness.

$\delta^{j(x)}_{\mathcal{F},x}(\tau) > 0$ for all x, τ implies strict convexity. As before r-convexity is an even stronger notion of convexity and 2-convexity is connected to strong convexity. In [13] the modulus of local (or total) convexity of $\mathcal{F}$, *$\nu_{\mathcal{F}}(x, \tau)$, was introduced and is basically given by $\delta^{\xi}_{\mathcal{F},x}(\tau)$ just that $\langle \xi, y - x\rangle$ is replaced by the right hand side derivative of $\mathcal{F}$ at x in direction $y - x$. If $\mathcal{F}$ is convex and Gâteaux-differentiable then $\nu_{\mathcal{F}}(x, \tau)$ coincides with $\delta^{\xi}_{\mathcal{F},x}(\tau)$, where $\xi = \mathcal{F}'(x)$. The modulus of total convexity has been studied in several papers, see e.g. [14]. There exist further definitions of moduli of convexity and smoothness related to functions (e.g. [56, 7]), but giving a complete overview over all such definitions goes beyond the scope of this work.*

It turns out that for functionals $\mathcal{F}$ that originate from the norm of $\mathcal{X}$ the moduli of the space and of the functions are closely related.

Proposition 2.2.9. *Let $\mathcal{F} = \|\cdot\|_{\mathcal{X}}$ and for all $x \in \mathcal{X}$ let $\xi_x \in \partial\mathcal{F}(x)$ be arbitrary. We have*

$$\rho \leq \sup_{x \in S_{\mathcal{X}}} \rho^{\xi_x}_{\mathcal{F},x} \leq 2\rho. \tag{2.16}$$

Proof. If we replace y by $-y$ in the definition of $\rho^{\xi_x}_{\mathcal{F},x}$ we see

$$\begin{aligned} 2\rho(\tau) &= \sup\left\{\mathcal{F}(x + \tau y) + \mathcal{F}(x - \tau y) - 2\mathcal{F}(x) + \langle \xi_x, \tau y - \tau y\rangle : x, y \in S_{\mathcal{X}}\right\} \\ &\leq 2 \sup_{x \in S_{\mathcal{X}}} \rho^{\xi_x}_{\mathcal{F},x}(\tau) \end{aligned}$$

and for all $x, y \in S_{\mathcal{X}}$ we have by the definition of the subdifferential and as $\mathcal{F}(x) = \|x\|_{\mathcal{X}} = 1$ that

$$\mathcal{F}(x + \tau y) - \mathcal{F}(x) - \langle \xi_x, \tau y\rangle \leq \mathcal{F}(x + \tau y) + \mathcal{F}(x - \tau y) - 2 \leq 2\rho(\tau).$$

□

So this already gives us an upper bound for $\rho^{\xi}_{\|\cdot\|_{\mathcal{X}},x}(\tau)$ if $x \in S_{\mathcal{X}}, \xi \in \partial\mathcal{F}(x)$. For generalizing this to all $x \in \mathcal{X}$ we use the following.

Proposition 2.2.10. *If the functional $\mathcal{F}$ is positively q-homogeneous then we have for all $x \in \mathcal{X} \setminus \{0\}$, $\xi \in \mathcal{X}^*$ that*

$$\|x\|^q \delta^{\xi/\|x\|^{q-1}}_{\mathcal{F},x/\|x\|}\left(\frac{\|x - y\|}{\|x\|}\right) \leq \left|\Delta^{\xi}_{\mathcal{F}}(y, x)\right| \leq \|x\|^q \rho^{\xi/\|x\|^{q-1}}_{\mathcal{F},x/\|x\|}\left(\frac{\|x - y\|}{\|x\|}\right)$$

and for all $\lambda > 0$ we have $\lambda^{p-1}\xi \in \partial\mathcal{F}(\lambda x)$ if and only if $\xi \in \partial\mathcal{F}(x)$.

Proof. If $\mathcal{F}$ is positively q-homogeneous we have

$$\begin{aligned} \left|\Delta_{\mathcal{F}}^{\xi}(y,x)\right| &= \|x\|^q \mathcal{F}\left(\frac{y}{\|x\|}\right) - \|x\|^q \mathcal{F}\left(\frac{x}{\|x\|}\right) - \|x\|^q \left\langle \frac{\xi}{\|x\|^{q-1}}, \frac{y}{\|x\|} - \frac{x}{\|x\|} \right\rangle \\ &= \|x\|^q \left| \Delta_{\mathcal{F}}^{\xi/\|x\|^{q-1}} \left(\frac{y}{\|x\|}, \frac{x}{\|x\|}\right)\right|, \end{aligned} \tag{2.17}$$

so that the first claim follows from Definition 2.2.7 . The second claim follows from multiplying (2.4) either by λ^q or λ^{-q}. □

For convex functions $\mathcal{F}$ one can show that both moduli are nondecreasing.

Proposition 2.2.11. *Let $\mathcal{F}$ be convex, $x \in \mathcal{X}$ and $\xi \in \partial\mathcal{F}(x)$. Then for $\lambda \geq 1$ one has*

$$\rho_{\mathcal{F},x}^{\xi}(\lambda\tau) \geq \lambda \rho_{\mathcal{F},x}^{\xi}(\tau), \qquad \delta_{\mathcal{F},x}^{\xi}(\lambda\tau) \geq \lambda \delta_{\mathcal{F},x}^{\xi}(\tau).$$

In particular $\delta_{\mathcal{F},x}^{\xi}, \rho_{\mathcal{F},x}^{\xi}$ are nondecreasing.

Proof. The idea is the same as in [13, sect. 2.4]. Let $\lambda \geq 1$. For all $y \in \mathcal{X}, \|y-x\| = \tau$ one can define $y_\lambda = \lambda y + (1-\lambda)x$, so $\|y_\lambda - x\| = \lambda\tau$. As $\lambda \geq 1$ we get by the convexity (see (2.2)) of $\mathcal{F}$ that

$$\frac{1}{\lambda}\Delta_{\mathcal{F}}^{\xi}(y_\lambda, x) = \frac{1}{\lambda}\Big(\mathcal{F}(\lambda y + (1-\lambda)x) - \mathcal{F}(x)\Big) - \langle \xi, y - x\rangle \geq \Delta_{\mathcal{F}}^{\xi}(y,x).$$

So for all $y \in \mathcal{X}, \|y-x\| = \tau$ we find

$$\Delta_{\mathcal{F}}^{\xi}(y,x) \leq \frac{1}{\lambda}\rho_{\mathcal{F},x}^{\xi}(\lambda\tau),$$

which gives the first inequality. Similarly for all $y \in \mathcal{X}, \|y-x\| = \lambda\tau$ one can define $\tilde{y}_\lambda = \frac{1}{\lambda}y + (1-\frac{1}{\lambda})x$, then $\|\tilde{y}_\lambda - x\| = \tau$ and again convexity of $\mathcal{F}$ can be used to show $\Delta_{\mathcal{F}}^{\xi}(y,x) \geq \lambda\Delta_{\mathcal{F}}^{\xi}(\tilde{y}_\lambda, x)$, which yields the other inequality. □

Lemma 2.2.12. *For $p > 1$ let $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{p}x^p$. Then $\rho_{f,1}^{1}$ is nondecreasing.*

Proof. We have

$$\rho_{f,1}^{1}(\tau) = \sup_{y\in\{-1,1\}} \frac{1}{p}\left|(1+y\tau)^p - 1 - py\tau\right| =: \max_{y\in\{-1,1\}} |h_y(\tau)|.$$

Now $|h_1| = h_1$ is nondecreasing for $\tau \geq 0$ as $h_1(0) = 0$ and $h_1^{'}(\tau) = (1+\tau)^{p-1} - 1 \geq 0$. h_{-1} is positive and nondecreasing for $0 \leq \tau \leq 2$ as $h_{-1}(0) = 0$ and $h_{-1}^{'}(\tau) = (1-\tau)^{p-1} + 1 \geq 0$. For $\tau = 2$ we already have $h_1(\tau) = 3^p - 2p - 1 \geq 2p - 2 = h_{-1}(\tau)$ for all $p > 1$. For $\tau \geq 2$ we have that $|h_{-1}|$ is decreasing until it gets 0 and from there on we have $h_1 \geq |h_{-1}| = -h_{-1}$ as $(\tau+1)^p \geq (\tau-1)^p + 2$ for all $\tau \geq 2$. Thus $\rho_{f,1}^{1}(\tau) = h_1(\tau)$ for $\tau \geq 2$ which is increasing. □

We also have a chain rule.

Proposition 2.2.13. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathcal{X}, \xi \in \mathcal{X}^*, t \in \mathbb{R}$ be such that $\rho_{f,\mathcal{F}(x)}^{t}$ is nondecreasing. Then for all $\tau \geq 0$ we have*

$$\rho_{f\circ\mathcal{F},x}^{t\xi}(\tau) \leq |t|\,\rho_{\mathcal{F},x}^{\xi}(\tau) + \rho_{f,\mathcal{F}(x)}^{t}\left(\|\xi\|\tau + \rho_{\mathcal{F},x}^{\xi}(\tau)\right).$$

Proof. Let $s = \mathcal{F}(x)$ and define functions R, r by

$$\begin{aligned} \mathcal{F}(x+y) - \mathcal{F}(x) &= \langle \xi, y\rangle + R(y) && \forall y \in \mathcal{X} \\ f(s+h) - f(s) &= th + r(h) && \forall h \in \mathbb{R}. \end{aligned}$$

Then we have for $\tau > 0$ and $y \in S_{\mathcal{X}}$ that

$$\begin{aligned} f\circ\mathcal{F}(x+\tau y) - f\circ\mathcal{F}(x) &= t\left(\langle \xi, \tau y\rangle + R(y)\right) + r\left(\langle \xi, \tau y\rangle + R(\tau y)\right) \\ &= \langle t\xi, \tau y\rangle + tR(\tau y) + r\left(\langle \xi, \tau y\rangle + R(\tau y)\right). \end{aligned}$$

Now the claim follows from $R(\tau y) \le \rho^{\xi}_{\mathcal{F},x}(\tau)$ and $r(h) \le \rho^{t}_{f,\mathcal{F}(x)}(|h|)$ together with the assumption that $\rho^{t}_{f,\mathcal{F}(x)}$ is a nondecreasing function. □

Propositions 2.2.9, 2.2.11 and 2.2.13 are already sufficient to find upper bounds on $\rho^{\xi}_{\mathcal{F},x}$ for $\mathcal{F} = f\left(\|x\|_{\mathcal{X}}\right)$ if f is convex and if we sufficiently understand the smoothness of both f and the space $\mathcal{X}$. Regarding lower bounds the following proposition will be our key instrument.

Proposition 2.2.14. *Let $\mathcal{F}$ be convex and x be such that there exists $\xi \in \partial\mathcal{F}(x)$. We have*

$$\left(\delta^{\xi}_{\mathcal{F},x}\right)^{*} = \rho^{x}_{\mathcal{F}^*,\xi}. \tag{2.18}$$

Further we have that $\mathcal{F}$ is p-convex in x w.r.t. ξ if and only if $\mathcal{F}^*$ is p'-smooth in ξ w.r.t. x.

Proof. We have

$$\begin{aligned} \rho^{x}_{\mathcal{F}^*,\xi}(\tau) &= \sup_{y^*\in S_{\mathcal{X}^*}} \left[\mathcal{F}^*(\xi+\tau y^*) - \mathcal{F}^*(\xi) - \langle \tau y^*, x\rangle\right] \\ &= \sup_{y^*\in S_{\mathcal{X}^*}} \sup_{y\in\mathcal{X}} \left[\langle \xi+\tau y^*, y\rangle - \mathcal{F}(y) - \mathcal{F}^*(\xi) - \langle \tau y^*, x\rangle\right] \\ &= \sup_{y\in\mathcal{X}} \left[\langle \xi, y\rangle - \mathcal{F}(y) - \mathcal{F}^*(\xi) + \tau\|y-x\|\right]. \end{aligned}$$

By Youngs equality (2.5) we then have

$$\begin{aligned} \rho^{x}_{\mathcal{F}^*,\xi}(\tau) &= \sup_{y\in\mathcal{X}} \left[\mathcal{F}(x) - \mathcal{F}(y) + \langle \xi, y-x\rangle + \tau\|y-x\|\right] \\ &= \sup_{\varepsilon\in\mathbb{R}^+_0} \sup_{y\in\mathcal{X},\|y-x\|=\varepsilon} \left[\varepsilon\tau - \Delta^{\xi}_{\mathcal{F}}(y,x)\right] = \left(\delta^{\xi}_{\mathcal{F},x}\right)^{*}(\tau). \end{aligned}$$

The second statement follows from (2.18), which gives that

$$\rho^{x}_{\mathcal{F}^*,\xi} = \left(\delta^{\xi}_{\mathcal{F},x}\right)^{*}, \qquad \delta^{\xi}_{\mathcal{F},x} \ge \left(\delta^{\xi}_{\mathcal{F},x}\right)^{**} = \left(\rho^{x}_{\mathcal{F}^*,\xi}\right)^{*}$$

and the fact that by Proposition 2.2.11 we have for $\tau > \overline{\tau}$ that $\rho^{\xi}_{\mathcal{F},x}(\tau) \ge \tau\rho^{\xi}_{\mathcal{F},x}(\overline{\tau})/\overline{\tau}$, $\delta^{\xi}_{\mathcal{F},x}(\tau) \ge \tau\delta^{\xi}_{\mathcal{F},x}(\overline{\tau})/\overline{\tau}$, so that in particular

$$\begin{aligned} \left(\delta^{\xi}_{\mathcal{F},x}\right)^{*}(\tau^*) &= \sup_{0\le\tau\le\overline{\tau}} \left[\tau^*\tau - \delta^{\xi}_{\mathcal{F},x}(\tau)\right], \text{ for } \tau^* \le \delta^{\xi}_{\mathcal{F},x}(\overline{\tau})/\overline{\tau} \\ \left(\rho^{\xi}_{\mathcal{F},x}\right)^{*}(\tau^*) &= \sup_{0\le\tau\le\overline{\tau}} \left[\tau^*\tau - \rho^{\xi}_{\mathcal{F},x}(\tau)\right], \text{ for } \tau^* \le \rho^{\xi}_{\mathcal{F},x}(\overline{\tau})/\overline{\tau}. \end{aligned}$$

Thus one can just put in the corresponding lower or upper bound and calculate the maximum, which completes the proof. □

From now on we will consider $\mathcal{F} = \frac{1}{p}\|\cdot\|^p$ for some $p > 1$ and use the framework outlined above to proof Theorem 2.2.5. Note that in light of Proposition 2.2.10 it is sufficient to understand $\delta_{\mathcal{F},x}^{j_p(x)}$ and $\rho_{\mathcal{F},x}^{j_p(x)}$ for $x \in S_{\mathcal{X}}$.

Theorem 2.2.15. *For some fixed $p > 1$ let $\mathcal{F} = \frac{1}{p}\|\cdot\|^p$.*

(a) For all $\overline{\tau} > 0$ exists a constant $C_{\overline{\tau},p} > 0$, such that for $x \in S_{\mathcal{X}}$ and $\tau \leq \overline{\tau}$ we have

$$\rho_{\mathcal{F},x}^{j_p(x)}(\tau) \leq C_{\overline{\tau},p}\rho_{\mathcal{X}}(\tau).$$

(b) If we have for $\overline{\tau} > 0, \tau \leq \overline{\tau}$ and all $x \in S_{\mathcal{X}}$ that

$$\rho_{\mathcal{F},x}^{j_p(x)}(\tau) \leq \phi(\tau),$$

then

$$\rho_{\mathcal{X}}(\tau) \leq p^{1/p-1}\phi(\tau) + C_{\overline{\tau}}\tau^2,$$

for $\tau \leq \overline{\tau}$. In particular if $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ fulfills $\lim_{\tau\to 0}\phi(\tau)/\tau = 0$, then $\mathcal{X}$ is uniformly smooth.

(c) Let $\frac{1}{p} + \frac{1}{p'} = 1$. For all $x \in S_{\mathcal{X}}, \overline{\tau} > 0$ we have

$$\delta_{\mathcal{F},x}^{j_p(x)}(\tau) \geq C_{\overline{\tau},p'}\delta_{\mathcal{X}}(\tau/C_{\overline{\tau},p'}), \qquad \tau \leq \frac{p'-1}{2}\overline{\tau}$$

where $C_{\overline{\tau},p'}$ is the constant from (a).

(d) If there exists $\overline{\tau} > 0$ such that we have for all $x \in S_{\mathcal{X}}$ and $\tau \leq \overline{\tau}$ that

$$\delta_{\mathcal{F},x}^{j_p(x)}(\tau) \geq \phi(\tau),$$

where $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and $\phi(\tau) > 0$ for $\tau > 0$, then $\mathcal{X}$ is uniformly convex.

Proof. *Claim (a)*: Note that $\mathcal{F} = f \circ \|\cdot\|$, with $f(t) = \frac{1}{p}t^p$ and $\rho_{f,\mathcal{F}(x)}^1$ nondecreasing by Lemma 2.2.12, so Proposition 2.2.13 gives

$$\rho_{\mathcal{F},x}^{j_p(x)}(\tau) \leq \rho_{\|\cdot\|,x}^{j_p(x)}(\tau) + \rho_{f,\mathcal{F}(x)}^1\left(\tau + \rho_{\|\cdot\|,x}^{j_p(x)}(\tau)\right).$$

We have by Taylor's theorem

$$\rho_{f,1}^1(\tau) = \sup_{\sigma\in\{-1,+1\}} \frac{p-1}{2}\tau^2 + r(\sigma\tau)\tau^2 \leq C\tau^2, \text{ for } \tau \leq 3\overline{\tau},$$

where the inequality holds for a constant depending on p and $\overline{\tau}$ as $\rho_{f,1}^1$ is always finite and so is the remainder r. We have $j_p(x) \in \partial\|\cdot\|(x)$ for $x \in S_{\mathcal{X}}$, so by Proposition 2.2.9 we have $\rho_{\|\cdot\|,x}^{j_p(x)}(\tau) \leq 2\rho_{\mathcal{X}}(\tau)$ and one can easily see that $\rho_{\mathcal{X}}(\tau) \leq \tau$. So we have

$$\rho_{\mathcal{F},x}^{j_p(x)}(\tau) \leq 2\rho_{\mathcal{X}}(\tau) + 9C\tau^2 \leq (2 + 9C/C_{\overline{\tau}})\rho_{\mathcal{X}}(\tau), \qquad \tau \leq \overline{\tau}$$

where the second inequality follows from item (b) of Lemma 2.2.6.

Claim (b): Note that $\|\cdot\| = f^{-1} \circ \mathcal{F}$ and $f^{-1}(t) = (pt)^{\frac{1}{p}}$ is concave, thus $-f^{-1}$ is convex and it is differentiable, so $-1 \in \partial\left(-f^{-1}\right)\left(\frac{1}{p}\right)$ and by Proposition 2.2.11 $\rho^{1}_{f^{-1},1/p} = \rho^{-1}_{-f^{-1},1/p}$ is nondecreasing. Then Proposition 2.2.13 gives for all $x \in S_{\mathcal{X}}$ that

$$\rho^{j_p(x)}_{\|\cdot\|,x}(\tau) \leq \rho^{j_p(x)}_{\mathcal{F},x}(\tau) + \rho^{1}_{f^{-1},1/p}\left(\tau + \rho^{j_p(x)}_{\mathcal{F},x}(\tau)\right) \leq \phi(\tau) + C_{\overline{\tau}}\tau^2,$$

where the second inequality follows by Taylors theorem as above and the fact that by Claim (a) we always have $\rho^{j_p(x)}_{\mathcal{F},x}(\tau) \leq C\tau$ for some $C > 0$. Thus Proposition 2.2.9 yields the claimed assertion.

Claim (c): First of all note that $\mathcal{F}^*(t) = \frac{1}{p'}t^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$. We have

$$\delta^{j_p(x)}_{\mathcal{F},x}(\tau) \geq \left(\delta^{j_p(x)}_{\mathcal{F},x}\right)^{**}(\tau) = \left(\rho^{x}_{\mathcal{F}^*,j_p(x)}\right)^{*}(\tau) = \sup_{r\geq 0}\left[\tau r - \rho^{x}_{\mathcal{F}^*,j_p(x)}(r)\right].$$

By Claim (a) we have for all $x \in S_{\mathcal{X}}$ that $\rho^{x}_{\mathcal{F}^*,j_p(x)}(r) \leq C_{\overline{\tau},p'}\rho_{\mathcal{X}^*}(r)$ for all $0 < r < \overline{\tau}$. We are only interested in the case $\tau \to 0$ so let $\tau \leq C_{\overline{\tau},p'}\rho_{\mathcal{X}^*}(\overline{\tau})/\overline{\tau}$, where $\rho_{\mathcal{X}^*}(\overline{\tau})/\overline{\tau} > 0$ by Lemma 2.2.6, (b). Then by Lemma 2.2.6, (a). we have $\tau r \leq C_{\overline{\tau},p'}\rho_{\mathcal{X}^*}(r)$ for $r \geq \overline{\tau}$ and thus find

$$\sup_{0\leq r}\left[\tau r - \rho^{x}_{\mathcal{F}^*,j_p(x)}(r)\right] \geq \sup_{0\leq r\leq\overline{\tau}}\left[\tau r - C_{\overline{\tau},p'}\rho_{\mathcal{X}^*}(r)\right] = \left(C\rho_{\mathcal{X}^*}\right)^*(\tau).$$

So we have by Lemma 2.2.6, (c) and (e) that

$$\delta^{j_p(x)}_{\mathcal{F},x}(\tau) \geq \left(C_{\overline{\tau},p'}\rho_{\mathcal{X}}\right)^*(\tau) = \frac{C_{\overline{\tau},p'}}{2}\left(2\delta_{\mathcal{X}}\right)^{**}\left(\frac{2\tau}{C_{\overline{\tau},p'}}\right) \geq C_{\overline{\tau},p'}\left(\delta_{\mathcal{X}}\right)\left(\frac{\tau}{C_{\overline{\tau},p'}}\right).$$

Finally note that $C_{\overline{\tau},p'} \geq \frac{p'-1}{2}C_{\overline{\tau}}^{-1}$, with $C_{\overline{\tau}}$ from item (b) of Lemma 2.2.6 and thus $C_{\overline{\tau},p'}\rho_{\mathcal{X}^*}(\overline{\tau})/\overline{\tau} \geq \frac{p'-1}{2}\overline{\tau}$.

Claim (d): By assumption we have $\delta^{j_p(x)}_{\mathcal{F},x}(\tau) \geq \phi(\tau)$ for $\tau \leq \overline{\tau}$ and by Proposition 2.2.11 we have for $\tau > \overline{\tau}$ that $\delta^{j_p(x)}_{\mathcal{F},x}(\tau) \geq \tau\delta^{j_p(x)}_{\mathcal{F},x}(\overline{\tau})/\overline{\tau}$ and thus $\delta^{j_p(x)}_{\mathcal{F},x}(\tau) \geq \tilde{\phi}(\tau)$ with

$$\tilde{\phi}(\tau) := \begin{cases} \phi(\tau), & \tau \leq \overline{\tau}, \\ \tau\phi(\overline{\tau})/\overline{\tau}, & \tau > \overline{\tau}. \end{cases}$$

So by Proposition 2.2.14 we have for all $x^* \in S_{\mathcal{X}^*}$ that

$$\rho^{j_p^*(x^*)}_{\mathcal{F}^*,x^*}(\tau) = \left(\delta^{x^*}_{\mathcal{F},j_p^*(x^*)}\right)^*(\tau) \leq \tilde{\phi}^*(\tau).$$

Now just observe that for $\tau < \phi(\overline{\tau})/\overline{\tau}$ we have

$$\frac{\tilde{\phi}^*(\tau)}{\tau} = \sup_{0\leq t}\left[t - \frac{\tilde{\phi}(t)}{\tau}\right] = \sup_{0\leq t\leq\overline{\tau}}\left[t - \frac{\phi(t)}{\tau}\right] \to 0,\ \tau \to 0,$$

as ϕ is nondecreasing. So by part (b) of the theorem we get that $\mathcal{X}^*$ is uniformly smooth from which it follows that $\mathcal{X}$ is uniformly convex [49, Prop. 1.e.2]. □

Remark 2.2.16. *One can see from the above proof that in the asymptotic case $\overline{\tau} \to 0$ one can choose the constant $C_{\overline{\tau},p}$ such that*

$$C_{\overline{\tau},p} \to \begin{cases} 2, & \mathcal{X} \text{ is not 2-smooth} \\ 1+p, & \mathcal{X} \text{ is 2-smooth.} \end{cases}$$

These constants are not sharp for every space $\mathcal{X}$, but at least in the asymptotic case the constants are much simpler than the ones given in [77]. For best known constants with respect to L^p spaces we refer to [76] and [78].

The above theorem combined with Proposition 2.2.10 gives us upper and lower bounds on the Bregman divergence for $\|x-y\| \leq \overline{\tau}\|x\|$. However as for large $\|x-y\|$ the Bregman divergence will be dominated by the term $\frac{1}{p}\|y\|^p$ it is not difficult to also find bounds that hold for all $x, y \in X$. Further one can additionally conclude bounds for the symmetric Bregman divergence,

$$\Delta_{\mathcal{F}}^{\mathrm{sym}}(x,y) := \Delta_{\mathcal{F}}^{j_p(x)}(y,x) + \Delta_{\mathcal{F}}^{j_p(y)}(x,y) = \langle j_p(x) - j_p(y), x-y \rangle \,,$$

from our theorem. These two claims are shown in the following two propositions.

Proposition 2.2.17. *For some fixed $p > 1$ let $\mathcal{F} = \frac{1}{p}\|\cdot\|^p$ and let $\phi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing. Let $V = \mathcal{X} \setminus \{0\} \times \mathcal{X}$ and define the statements:*

$$\exists C, c > 0 \forall (x,y) \in V, \|x-y\| \leq c\|x\| : \Delta_{\mathcal{F}}^{j_p(x)}(y,x) \leq C\|x\|^p \phi\left(\tfrac{\|x-y\|}{\|x\|}\right) \tag{a}$$

$$\exists C > 0 \forall (x,y) \in V : \Delta_{\mathcal{F}}^{\mathrm{sym}}(x,y) \leq C \max\{\|x\|, \|y\|\}^p \phi\left(\tfrac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}\right) \tag{b}$$

$$\exists C > 0 \forall (x,y) \in V : \Delta_{\mathcal{F}}^{j_p(x)}(y,x) \leq C \max\{\|x\|, \|y\|\}^p \phi\left(\tfrac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}\right) \tag{c}$$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Obviously one also has (c) $\Rightarrow$ {(a) with ϕ replaced by $\phi(2\cdot)$}.

Proof. We only show that (a) implies (b) as (b) $\Rightarrow$ (c) follows trivially. Without loss of generality let $c \leq 1$. First of all assume $\frac{\|x-y\|}{\|x\|} > c$. Then by

$$\frac{\|x-y\|}{\|x\|}\frac{\|x\|}{\|y\|} = \frac{\|x-y\|}{\|y\|} \geq \frac{\|y\|-\|x\|}{\|y\|} \geq 1 - \frac{\|x\|}{\|y\|}$$

one can see that regardless of whether we have $\|x\|/\|y\| > 1/2$ or $\|x\|/\|y\| \leq 1/2$ one always has $\frac{2\|x-y\|}{\|y\|} > c$. So by

$$\begin{aligned}\Delta_{\mathcal{F}}^{\mathrm{sym}}(x,y) = \langle j_p(x) - j_p(y), x-y \rangle &\leq \|x\|^p + \|y\|^p + \|x\|^{p-1}\|y\| + \|y\|^{p-1}\|x\| \\ &\leq 4\max\{\|x\|,\|y\|\}^p\end{aligned}$$

we find that

$$\Delta_{\mathcal{F}}^{\mathrm{sym}}(x,y) \leq \frac{4}{\phi\,(c)} \max\{\|x\|,\|y\|\}^p \phi\left(\frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}\right).$$

Now consider the case $\|x-y\|/\|x\| \leq c \leq 1$. In this range we can conclude (b) from (a) as $\|y\| \leq 2\|x\|$, so that

$$\phi\left(\frac{\|x-y\|}{\|x\|}\right) \leq \phi\left(\frac{2\|x-y\|}{\|y\|}\right).$$

□

Proposition 2.2.18. *For some fixed $p > 1$ let $\mathcal{F} = \frac{1}{p}\|\cdot\|^p$, let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing and $\phi(\tau) > 0$ for $\tau > 0$. Let $V = \mathcal{X} \setminus \{0\} \times \mathcal{X}$ and define the statements:*

$$\exists C, c > 0 \forall (x,y) \in V, \|x-y\| \leq c\|x\| : \Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq C\|x\|^p \phi\left(\tfrac{\|x-y\|}{\|x\|}\right) \tag{d}$$

$$\exists C > 0 \forall (x,y) \in V : \Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq C \max\{\|x\|,\|y\|\}^p \phi\left(\tfrac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right) \tag{e}$$

$$\exists C > 0 \forall (x,y) \in V : \Delta_{\mathcal{F}}^{\mathrm{sym}}(x,y) \geq C \max\{\|x\|,\|y\|\}^p \phi\left(\tfrac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right) \tag{f}$$

Then (d) $\Rightarrow$ (e) $\Rightarrow$ (f).

Proof. The proof is similar to the previous proof and we only show (d) $\Rightarrow$ (e) as the left implication follows trivially. Again w.l.o.g. $c \leq 1$. We now look at three different cases. We start with the case $\|x-y\|/\|x\| \to \infty$, say $\|x-y\|/\|x\| \geq N$, for some $N > 3$. Therefore we have $(N-1)\|x\| \leq \|y\|$ so that in particular $\max\{\|x\|,\|y\|\} = \|y\|$ and

$$\frac{\|x-y\|}{\|y\|} > \frac{1}{2}.$$

For N sufficiently large we find that

$$\begin{aligned}\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq C_{p,N}\|y\|^p \geq \frac{C_{p,N}}{\phi(1/2)}\|y\|^p\phi\left(\frac{\|x-y\|}{\|y\|}\right)\\ = C\max\{\|x\|,\|y\|\}^p\phi\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right).\end{aligned}$$

Now consider $c \leq \|x-y\|/\|x\| \leq N$. So $\|y\| \leq (N+1)\|x\|$ and thus

$$\frac{\|x-y\|}{\|x\|} \leq N+2, \qquad \phi\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right) \leq \phi(N+2).$$

By (d) we have for $\|x-y\| = c\|x\|$ that

$$\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq \rho_{\mathcal{F},x}^{j_p(x)}(c\|x\|) \geq C\|x\|^p\phi(c),$$

so that by Lemma 2.2.12 we have for $c \leq \|x-y\|/\|x\| \leq N$

$$\begin{aligned}\Delta_{\mathcal{F}}^{j_p(x)}(y,x) &\geq \rho_{\mathcal{F},x}^{j_p(x)}(\|x-y\|) \geq \rho_{\mathcal{F},x}^{j_p(x)}(c\|x\|) \geq C\|x\|^p\phi(c)\\ &= C\|x\|^p\phi(c)\frac{\phi(N+2)}{\phi(N+2)} \geq C\|x\|^p\frac{\phi(c)}{\phi(N+2)}\phi\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right)\\ &\geq C(N+1)^{-p}\|y\|^p\frac{\phi(c)}{\phi(N+2)}\phi\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right).\end{aligned}$$

Last and least we have for $\|x-y\|/\|x\| \leq c \leq 1$ that $\|y\| \leq 2\|x\|$ and thus by (d) that

$$\Delta_{\mathcal{F}}^{j_p(x)}(y,x) \geq C\|x\|^p\phi\left(\frac{\|x-y\|}{\|x\|}\right) \geq C2^{-p}\|y\|^p\phi\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right).$$

□

Proof of Theorem 2.2.5. By Proposition 2.2.10 we have

$$\|x\|^p \delta_{\mathcal{F},x/\|x\|}^{j_p(x)/\|x\|^{p-1}} \left(\frac{\|x-y\|}{\|x\|} \right) \leq \Delta_{\mathcal{F}}^{j_p(x)}(y,x) \leq \|x\|^p \rho_{\mathcal{F},x/\|x\|}^{j_p(x)/\|x\|^{p-1}} \left(\frac{\|x-y\|}{\|x\|} \right)$$

Thus item (a) of Theorem 2.2.15 and the s-smoothness show the bound (2.14) for $x \in \mathcal{X}$ and y such that $\|x-y\| \leq \overline{\tau}\|x\|$. Similarly item (c) of Theorem 2.2.15 and the r-convexity show (2.15) for $x \in \mathcal{X}$ and y such that $\|x-y\| \leq \frac{p'-1}{2}\overline{\tau}\|x\|$. As this holds true for all $\overline{\tau} > 0$ one can just replace $\tilde{\overline{\tau}} = \frac{2\overline{\tau}}{p'-1}$. Apply Proposition 2.2.17 and Proposition 2.2.18 to conclude from this the uniform bounds for all $x, y \in \mathcal{X}$. □

2.3 Kullback-Leibler divergence

In this section we will formally introduce the Kullback-Leibler divergence, which is actually also a Bregman divergence as we will see in Proposition 3.2.1. We have already motivated the Kullback-Leibler divergence in 1.3 as the negative log-likelihood of Poisson data up to a constant. Another reason for our interest in this special distance measure will be that it can also be applied as a penalty term in maximum entropy regularization, see Section 3.2.5.

Definition 2.3.1 (Kullback-Leibler divergence)**.** *We first define the function* $\mathrm{kl}\colon \mathbb{R}^2 \to \overline{\mathbb{R}}$ *by*

$$\mathrm{kl}(s,t) = \begin{cases} s\log\left(\frac{s}{t}\right) - s + t, & \text{if } s,t > 0 \\ t, & \text{if } s = 0, t \geq 0 \\ \infty, & \text{else.} \end{cases}$$

Then we can define on $\mathbb{M} \subset \mathbb{R}^d$ *the Kullback-Leibler divergence* $\mathrm{KL}\colon L^1(\mathbb{M}) \times L^1(\mathbb{M}) \to \overline{\mathbb{R}}$ *by*

$$\mathrm{KL}(g_1, g_2) = \int_{\mathbb{M}} \mathrm{kl}(g_1(x), g_2(x))\mathrm{d}x.$$

Lemma 2.3.2. *(a)* $\mathrm{KL}(g_1,g_2) \geq 0$ *and for* $g_1, g_2 \geq 0$ *we have* $\mathrm{KL}(g_1,g_2) = 0$ *if and only if* $g_1 = g_2$*.*

(b)

$$\|g_1 - g_2\|_{L^2}^2 \leq \left(\frac{2}{3}\|g_1\|_{L^\infty} + \frac{4}{3}\|g_2\|_{L^\infty} \right) \mathrm{KL}(g_1,g_2). \tag{2.19}$$

(c)

$$\|g_1 - g_2\|_{L^1}^2 \leq \left(\frac{2}{3}\|g_1\|_{L^1} + \frac{4}{3}\|g_2\|_{L^1} \right) \mathrm{KL}(g_1,g_2). \tag{2.20}$$

(d) For all $g_0 \in L^1, g_0 \geq 0$ *and all* $R > 0$ *we have for all* $g \in L^1, g \geq 0$ *with* $\|g\|_{L^1} \geq 2e^{2R+1}\|g_0\|$ *that*

$$\mathrm{KL}(g, g_0) \geq R\|g\|_{L^1}.$$

Proof. (a) It suffices to show the claim pointwise for $\mathrm{kl}(s,t)$. If $s=0$ the claim is clear. For $s>0$ we have $\mathrm{kl}(s,t) = s\,\mathrm{kl}(1,t/s)$. By the inequality $\log(t) \leq t-1$ for $t>0$ we find $\mathrm{kl}(1,t) = t-1-\log(t) \geq 0$, with equality only if $t=1$.

(b) The function

$$\phi(t) = \left(\frac{2t}{3} + \frac{4}{3}\right)(t\log(t) - t + 1) - (t-1)^2$$

is strictly convex on $(0,\infty)$ with $\min_{t\in(0,\infty)} \phi(t) = 0$ attained at $t=1$. Thus by $\phi(s/t) \geq 0$ we have

$$(s-t)^2 \leq \left(\frac{2}{3}s + \frac{4}{3}t\right)\mathrm{kl}(s,t). \tag{2.21}$$

(Note that this also holds true for $s=0$ as $1 \leq 4/3$). Integrating (2.21) gives the claim.

(c) We can take the square root in (2.21) and obtain by Cauchy-Schwarz

$$\begin{aligned}\left(\int_{\mathbb{M}} |g_1 - g_2|\mathrm{d}x\right)^2 &\leq \left(\int_{\mathbb{M}} \left(\frac{2}{3}g_1 + \frac{4}{3}g_2\right)^{\frac{1}{2}} \mathrm{kl}(g_1,g_2)^{\frac{1}{2}}\mathrm{d}x\right)^2 \\ &\leq \int_{\mathbb{M}} \left(\frac{2}{3}g_1 + \frac{4}{3}g_2\right)\mathrm{d}x \int_{\mathbb{M}} \mathrm{kl}(g_1,g_2)\mathrm{d}x.\end{aligned}$$

(d) Let $\|g\|_{L^1} \geq 2e^{2R+1}\|g_0\|$. Define $B := \{x \in \mathbb{M} : g(x) \geq e^{2R+1}g_0(x)\}$. Assume $\int_B g < \frac{1}{2}\|g\|_{L^1}$. This would imply

$$\|g\|_{L^1} = \int_B g + \int_{\mathbb{M}\setminus B} g < \frac{1}{2}\|g\|_{L^1} + e^{2R+1}\int_{\mathbb{M}\setminus B} g_0 < 2e^{2R+1}\|g_0\|,$$

which is a contradiction. Thus we have $\int_B g \geq \frac{1}{2}\|g\|_{L^1}$. This yields

$$\mathrm{KL}(g,g_0) \geq \int_B g\log\left(\frac{g}{g_0}\right) - g + g_0 \geq \int_B g\log(e^{2R+1}) - g \geq 2R\int_B g \geq R\|g\|_{L^1}.$$

□

Chapter Three

Generalized Tikhonov regularization

Let the forward operator $F\colon \mathcal{X} \to \mathcal{Y}$ be continuous between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. As motivated in Section 1.3 we will consider generalized Tikhonov regularization of the form

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f\in\mathcal{X}} \mathcal{T}(f) := \operatorname*{arg\,min}_{f\in\mathcal{X}} [\mathcal{S}(F(f)) + \alpha\mathcal{R}(f)], \tag{3.1}$$

where $\mathcal{S}\colon \mathcal{Y} \to \overline{\mathbb{R}}$, $\mathcal{R}\colon \mathcal{X} \to \overline{\mathbb{R}}$ are convex and lower semi-continuous. From now on we will have to distinguish between linear and nonlinear forward operators. To this end we will write the forward operator as $T\colon \mathcal{X} \to \mathcal{Y}$ whenever we assume it to be linear. If the forward operator is linear then the Tikhonov functional $\mathcal{T}$ is convex by Lemma 2.1.5. $\mathcal{T}$ is strictly convex if $\mathcal{R}$ is strictly convex. Convexity gives all the nice properties of Section 2.1, in particular Proposition 2.1.12. Further convexity gives us the strong tool of duality, which we will introduce in the following section.

3.1 Duality[1]

In this section we will introduce the so called dual problem of the minimization problem (3.1) for linear forward operators. For convenience we consider the equivalent problem

$$\inf_{f\in\mathcal{X}} \left[\frac{1}{\alpha}\mathcal{S}(Tf) + \mathcal{R}(f)\right] = \beta. \tag{3.2}$$

Duality is a well-known concept from optimization and the main reason why we consider it here is that it will give a useful characterization for solutions of (3.1), the so called Karush–Kuhn–Tucker conditions. First we have to introduce the perturbed problems

$$\inf_{f\in\mathcal{X}} \mathcal{P}(f,g) := \inf_{f\in\mathcal{X}} \left[\frac{1}{\alpha}\mathcal{S}(Tf-g) + \mathcal{R}(f)\right], \quad g \in \mathcal{Y},$$

so that minimizing $\mathcal{P}(f,0)$ gives back our original problem.

Definition 3.1.1 (primal and dual problem)**.** *We call (3.2) the primal problem and define the corresponding dual problem by*

$$\sup_{\xi\in\mathcal{Y}^*} [-\mathcal{P}^*(0,\xi)] = \gamma. \tag{3.3}$$

[1]Duality theory for optimization is well known and can be found e.g. in [24]. Still we give some proofs to illustrate the idea of duality as it will be an important concept later on.

We can compute the convex conjugate $\mathcal{P}^*$ with respect to $\mathcal{S}$ and $\mathcal{R}$ to get a more concrete formulation of the dual problem.

$$\begin{aligned}
\mathcal{P}^*(\phi,\xi) :&= \sup_{f\in\mathcal{X}, g\in\mathcal{Y}} \Big[\langle\phi, f\rangle + \langle\xi, g\rangle - \frac{1}{\alpha}\mathcal{S}(Tf-g) - \mathcal{R}(f)\Big] \\
&= \sup_{f\in\mathcal{X}} \sup_{\tilde{g}\in\mathcal{Y}} \Big[\langle\phi, f\rangle + \langle\xi, Tf-\tilde{g}\rangle - \frac{1}{\alpha}\mathcal{S}(\tilde{g}) - \mathcal{R}(f)\Big] \\
&= \sup_{f\in\mathcal{X}} [\langle\phi, f\rangle + \langle T^*\xi, f\rangle - \alpha\mathcal{R}(f)] + \frac{1}{\alpha}\sup_{\tilde{g}\in\mathcal{Y}}[\langle -\alpha\xi, \tilde{g}\rangle - \mathcal{S}(\tilde{g})] \\
&= \sup_{f\in\mathcal{X}} [\langle T^*\xi + \phi, f\rangle - \mathcal{R}(f)] + \frac{1}{\alpha}(\mathcal{S})^*(-\alpha\xi) \\
&= \frac{1}{\alpha}(\mathcal{S})^*(-\alpha\xi) + \mathcal{R}^*(T^*\xi + \phi).
\end{aligned}$$

Thus the dual problem is given by

$$\widehat{\xi}_\alpha \in \arg\max_{\xi\in\mathcal{Y}^*}\Big[-\frac{1}{\alpha}(\mathcal{S})^*(-\alpha\xi) - \mathcal{R}^*(T^*\xi)\Big], \ \sup_{\xi\in\mathcal{Y}^*}\Big[-\frac{1}{\alpha}(\mathcal{S})^*(-\alpha\xi) - \mathcal{R}^*(T^*\xi)\Big] = \gamma. \tag{3.4}$$

Equivalently we can write the dual problem as $\inf_{\xi\in\mathcal{Y}^*}\mathcal{P}^*(0,\xi)$ and then consider the bidual problem, that is the dual of the dual problem. According to Definition 3.1.1 this is given by

$$\sup_{f\in\mathcal{X}}[-\mathcal{P}^{**}(f,0)] = -\tilde{\beta}$$

and as $\mathcal{P}$ is convex and lower semi-continuous we have $\mathcal{P} = \mathcal{P}^{**}$ so that the bidual problem is equivalent to the primal problem with $\beta = \tilde{\beta}$. We have the following relations between primal and dual problem.

Theorem 3.1.2. *(a) Weak duality: $\beta \geq \gamma$.*

(b) We say that strong duality holds if there exists solutions $\widehat{f}_\alpha$ to the primal and $\widehat{\xi}_\alpha$ to the dual problem, as well as $\beta = \gamma$. Strong duality is equivalent to the (Karush-Kuhn-Tucker) extremal relations

$$T^*\widehat{\xi}_\alpha \in \partial\mathcal{R}(\widehat{f}_\alpha) \tag{3.5}$$

$$-\alpha\widehat{\xi}_\alpha \in \partial\mathcal{S}(T\widehat{f}_\alpha). \tag{3.6}$$

Proof. (a) By Young's inequality we have

$$\mathcal{P}(f,0) + \mathcal{P}^*(0,\xi) \geq \langle 0, f\rangle + \langle\xi, 0\rangle = 0.$$

Therefore $\beta - \gamma = \inf_{f\in\mathcal{X}}\mathcal{P}(f,0) - \sup_{\xi\in\mathcal{Y}^*} -\mathcal{P}^*(0,\xi) \geq 0$.

(b) Assume that strong duality holds with solutions $\widehat{f}_\alpha$ and $\widehat{\xi}_\alpha$. Then as $\beta = \gamma$ we have

$$\mathcal{P}(\widehat{f}_\alpha, 0) + \mathcal{P}^*(0, \widehat{\xi}_\alpha) = 0. \tag{3.7}$$

This is equivalent to

$$\Big[\mathcal{R}(\widehat{f}_\alpha) + \mathcal{R}^*\Big(T^*\widehat{\xi}_\alpha\Big) + \langle T^*\widehat{\xi}_\alpha, \widehat{f}_\alpha\rangle\Big] = -\frac{1}{\alpha}[\mathcal{S}(T\widehat{f}_\alpha) + (\mathcal{S})^*(-\alpha\widehat{\xi}_\alpha) + \langle -\alpha\widehat{\xi}_\alpha, T\widehat{f}_\alpha\rangle].$$

As both expressions in brackets are non-negative by Young's inequality they have to be zero. Then the necessary condition for equality in Lemma 2.1.14 gives (3.5) and (3.6).
On the contrary assume that the extremal relations (3.5) and (3.6) hold true. We have just seen that this is equivalent to (3.7). The necessary condition for equality in Lemma 2.1.14 gives $(0, \widehat{\xi}_\alpha)^T \in \partial \mathcal{P}(\widehat{f}_\alpha, 0)$ as well as $(\widehat{f}_\alpha, 0) \in \partial \mathcal{P}^*(0, \widehat{\xi}_\alpha)$ by Corollary 2.1.19. By the definition of the subdifferential this is equivalent to

$$\forall (f,g) \in \mathcal{X} \times \mathcal{Y} : \mathcal{P}(f,g) \geq \left\langle \begin{pmatrix} 0 \\ \widehat{\xi}_\alpha \end{pmatrix}, (f - \widehat{f}_\alpha, g) \right\rangle + \mathcal{P}(\widehat{f}_\alpha, 0)$$
$$\forall \begin{pmatrix} \phi \\ \xi \end{pmatrix} \in \mathcal{X}^* \times \mathcal{Y}^* : \mathcal{P}^*(\phi, \xi) \geq \left\langle \begin{pmatrix} \phi \\ \xi - \widehat{\xi}_\alpha \end{pmatrix}, (\widehat{f}_\alpha, 0) \right\rangle + \mathcal{P}^*(0, \widehat{\xi}_\alpha).$$

Setting $g = 0$, $\phi = 0$ we see that $\widehat{f}_\alpha$ is a solution to the primal and $\widehat{\xi}_\alpha$ to the dual problem. Further by (3.7) we have that $\beta = \gamma$.

□

If we want to make use of the extremal relations, we have to understand for which kind of problems strong duality holds. To this end we have the following theorem [24, Ch. III, Proposition 4.1].

Theorem 3.1.3. *Let there exist $f_0 \in \mathcal{X}$ such that the map $g \mapsto \mathcal{P}(f_0, g)$ is finite and continuous at $g = 0$. Then there exists a solution to the dual problem (3.3) and we have $\beta = \gamma$.*

Corollary 3.1.4. *(a) Let there exist $f_0 \in \mathcal{X}$ such that $\mathcal{R}(f_0) < \infty$, $\mathcal{S}(Tf_0) < \infty$ and $\mathcal{S}$ continuous at Tf_0. Then there exists a solution $\widehat{\xi}_\alpha$ to the dual problem and we have $\beta = \gamma$.*

(b) Let there exist $\xi_0 \in \mathcal{Y}^*$ such that $(\mathcal{S})^*(-\alpha\xi_0) < \infty$, $\mathcal{R}^*(T^*\xi_0) < \infty$ and $\mathcal{R}^*$ continuous at $T^*\xi_0$. Then there exists a solution $\widehat{f}_\alpha$ to the primal problem and we have $\beta = \gamma$.

Proof. (a) Immediate consequence of Theorem 3.1.3.

(b) Follows from the dual version of Theorem 3.1.3: If there exist $\xi_0 \in \mathcal{Y}^*$ such that $\phi \mapsto \mathcal{P}^*(\phi, \xi_0)$ is finite and continuous at $\phi = 0$, then there exists a solution to the bidual problem, which actually is the primal problem based on our assumptions, and we have $\beta = \gamma$.

□

Remark 3.1.5. *If both assumptions of Corollary 3.1.4 hold true, then we have strong duality. Actually, if we assume that the minimizer $\widehat{f}_\alpha$ exists, it is already sufficient for strong duality that (a) holds true. If we do not have existence of $\widehat{f}_\alpha$ then we do not have a regularization method, so existence is well understood for many interesting Tikhonov functionals.*

3.2 Examples

In this section we want to give an overview of some variants of generalized Tikhonov regularization, which will be most important in this thesis. To this end we separately introduce several data fidelity and penalty functionals, that can be in principle combined freely. We want to point out that we try to work as general as possible to ensure that our convergence analysis works for as many variants of generalized Tikhonov regularization as possible. In particular in the next chapter the results will hold for general penalty functionals or at least under general assumptions on the penalty, however we will have to distinguish the data fidelities at some points. On the contrary in Chapter 5 we will have to look closer at the different penalty functionals.

In this work we consider basically three different data fidelity functionals and two types of penalty functionals. The three data fidelities are first a simple deterministic data fidelity functional given by a norm power and second a least square data fidelity for random noise, which is of similar nature as the norm power data fidelity and is most interesting for Gaussian white noise. Last but not least we consider a Kullback-Leibler type data fidelity, which is tailored to the case of Poisson data. The first very general penalty term is again given by a norm power $\mathcal{R}\colon \mathcal{X} \to \mathbb{R}, \mathcal{R}(f) = \frac{1}{t}\|f\|_{\mathcal{X}}^t$ for some Banach space $\mathcal{X}$ and $t > 1$ whilst the second is the cross entropy $\mathcal{R}\colon L^1(\mathbb{M}) \to \overline{\mathbb{R}}, \mathcal{R}(f) = \mathrm{KL}(f, f_0)$, where $f_0 \in L^1(\mathbb{M})$ is an a-priori guess.

3.2.1 Deterministic data fidelity

If the data are given by $g^{\mathrm{obs}} \in \mathcal{Y}$, then we can define for $p \in (1, \infty)$ the data fidelity

$$\mathcal{S}^p_{g^{\mathrm{obs}}}(g) = \mathcal{S}_p(g - g^{\mathrm{obs}}) := \frac{1}{p}\|g - g^{\mathrm{obs}}\|^p_{\mathcal{Y}}.$$

The choice $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$ can be motivated for example as in Section 1.3, where it turned out that this is the “correct” choice under the prior knowledge that the data noise is normally distributed. More generally and simply put the choice $\mathcal{S} = \mathcal{S}^p_{g^{\mathrm{obs}}}$ will lead to an approximate solution $\widehat{f}_\alpha$ such that $F(\widehat{f}_\alpha)$ is close to the data in the norm of $\mathcal{Y}$. The most common and well studied example is $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$, where $\mathcal{Y}$ is a Hilbert space, corresponding to classical quadratic Tikhonov regularization.

As already shown in the last chapter $\mathcal{S}^p_{g^{\mathrm{obs}}}$ is convex (and it is obviously continuous). The convex conjugate is given by

$$(\mathcal{S}^p_{g^{\mathrm{obs}}})^*(\xi) = \sup_{g\in\mathcal{Y}}\Big[\langle \xi, g\rangle - \mathcal{S}_p(g - g^{\mathrm{obs}})\Big] = \sup_{g\in\mathcal{Y}}\Big[\langle \xi, g + g^{\mathrm{obs}}\rangle - \mathcal{S}_p(g)\Big] = \frac{1}{p'}\|\xi\|^{p'}_{\mathcal{Y}^*} + \langle \xi, g^{\mathrm{obs}}\rangle.$$

The subdifferential of $\mathcal{S}_p$ is given by the duality mapping (see Section 2.1)

$$J_p(g) = \left\{\xi \in \mathcal{Y}^* : \langle \xi, g\rangle = \|\xi\|\|g\|, \|\xi\| = \|g\|^{p-1}\right\}$$

and thus we have $\partial\mathcal{S}^p_{g^{\mathrm{obs}}}(g) = J_p(g - g^{\mathrm{obs}})$.

3.2.2 Least squares data fidelity for random noise

As motivated in Section 1.3 we would like to use a least squares approach for Gaussian white noise and thus our estimator will be given by

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f\in\mathcal{X}}\left[\frac{1}{2}\|F(f)\|^2_{L^2} - \left\langle G^{\mathrm{obs}}, F(f)\right\rangle + \alpha\mathcal{R}(f)\right]. \tag{3.8}$$

Notice that if we had $G^{\text{obs}} \in L^2(\mathbb{M})$ the above minimization would be equivalent to minimizing $\frac{1}{2}\|Tf - G^{\text{obs}}\|_{L^2}^2 + \alpha\mathcal{R}(f)$, however for Gaussian white noise we have $G^{\text{obs}} \notin L^2(\mathbb{M})$ with probability 1 by Theorem 1.4.5. Given the additive noise setting (1.7) we denote our data fidelity by

$$\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}(g) = \frac{1}{2\alpha}\|g\|_{L^2}^2 - \left\langle g^\dagger + \varepsilon Z, g\right\rangle.$$

For this subsection we suppose that Assumption 1.4.1 holds true for some $p' > 1$ and consider $\mathcal{Y} = B^{\gamma}_{p,1}(\mathbb{M})$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If we have that $\mathbb{P}(Z \in B^{-\gamma+\epsilon}_{p',\infty}(\mathbb{M})) = 0$ for all $\epsilon > 0$, which is true for Gaussian white noise and $\gamma = d/2$ by Theorem 1.4.5, then this choice of $\mathcal{Y}$ is necessary for the minimization functional to be well-defined (one could of course also set $\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}(g) = \infty$ if $g \notin \mathcal{Y} = B^{\gamma}_{p,1}(\mathbb{M})$ but this will complicate the duality). In the remaining two chapters we will still put $\mathcal{Y} = L^2$ for Gaussian white noise as it is not necessary for our analysis that $G^{\text{obs}} \in \mathcal{Y}$ and $F(f) \in B^{\gamma}_{p,1}(\mathbb{M})$ can still be assumed as a property of the forward operator. Note that the dual space to Y is $Y^* = B^{-\gamma}_{p',\infty}(\mathbb{M})$ by Theorem A.2.6. Assume now that the forward operator $F = T$ is linear. By Corollary 3.1.4 there exists a unique solution $\widehat{\xi}_\alpha$ of the dual problem as $\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}$ is continuous everywhere. The conjugate of the data fidelity is given by

$$\left(\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}\right)^*(\xi) = \sup_{h \in B^{\gamma}_{p,1}(\mathbb{M})}\left[\langle \xi, h\rangle - \frac{1}{2}\|h\|_{L^2}^2 + \langle g^\dagger + \varepsilon Z, h\rangle\right]. \tag{3.9}$$

Thus we have for all $g \in L^2$

$$\left(\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}\right)^*(g - \varepsilon Z) = \frac{1}{2}\|g + g^\dagger\|_{L^2}.$$

In particular $\left(\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}\right)^*(-\alpha\xi_0) < \infty$ for $\xi_0 = -\frac{1}{\alpha}(g - \varepsilon Z), g \in L^2$ so that we have strong duality if $\mathcal{R}^*$ is continuous at $T^*\xi_0$ for some g by Theorem 3.1.3. Under strong duality one has the extremal relations

$$T^*\widehat{\xi}_\alpha \in \partial\mathcal{R}(\widehat{f}_\alpha), \tag{3.10}$$
$$-\alpha\widehat{\xi}_\alpha \in \partial\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}(T\widehat{f}_\alpha). \tag{3.11}$$

As $\mathcal{S}$ is differentiable in $\mathcal{Y} = B^{\gamma}_{p,1}(\mathbb{M})$ we have that

$$\partial\mathcal{S}^{\text{LS}}_{G^{\text{obs}}}(T\widehat{f}_\alpha) = \left\{T\widehat{f}_\alpha - g^\dagger - \varepsilon Z\right\} \tag{3.12}$$

and for our error analysis we will decompose

$$\widehat{\xi}_\alpha = \frac{g^\dagger - T\widehat{f}_\alpha}{\alpha} + \frac{\varepsilon}{\alpha}Z =: \widehat{\xi}_{\alpha,L^2} + \widehat{\xi}_{\alpha,Z}. \tag{3.13}$$

3.2.3 Kullback-Leibler type data fidelity for random noise

If the data are given by a Poisson process we will choose, following [43], $\mathcal{S}(g)$ equal to

$$\mathcal{S}^{\text{KL}}_{G_t,\sigma}(g) := \begin{cases} \int_{\mathbb{M}} g\,\mathrm{d}x - \int_{\mathbb{M}} \log(g+\sigma)\,(\mathrm{d}G_t + \sigma\mathrm{d}x)\,, & \text{if } g > -\sigma \text{ a.e.,} \\ \infty, & \text{else,} \end{cases} \tag{3.14}$$

for some $\sigma \geq 0$, which is for $\sigma = 0$ the negative log-likelihood for Poisson data up to a constant (see [43, Section 2.2]). Notice that, as in the previous subsection, if we had $G_t \in L^1(\mathbb{M})$, then the data fidelity would equal the Kullback-Leibler divergence

$$\mathrm{KL}(G_t + \sigma, g + \sigma) = \int_{\mathbb{M}} g - G_t + (G_t + \sigma) \log\left(\frac{G_t + \sigma}{g + \sigma}\right)$$

up to a constant, so that we will call it Kullback-Leibler type data fidelity. In terms of our general additive noise model (1.7) $\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ is given by

$$\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(g) := \|g\|_{L^1} - \langle g^\dagger + \varepsilon Z + \sigma, \log(g + \sigma)\rangle,$$

which equals (3.14) by setting $\varepsilon = t^{-\frac{1}{2}}$ and $Z = t^{\frac{1}{2}}(G_t - g^\dagger)$. Concerning the penalty functional we can demand that $\mathcal{R} = \mathcal{R} + \chi_{\mathcal{B}}$, with some suitable subset $\mathcal{B} \subset \mathcal{X}$, so that our variational minimization problem is of the form

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f \in \mathcal{B}} \left[\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(F(f)) + \alpha \mathcal{R}(f)\right].$$

As $F(f^\dagger)$ is the density of the Poisson process $\mathcal{F}(\widehat{f}_\alpha)$ should be a density as well, thus we want to choose $\mathcal{B}$ such that

$$F(f) \geq 0 \qquad \forall f \in \mathcal{B}. \tag{3.15}$$

Then the offset parameter guarantees that we stay away from the singularity of the logarithm. Further reasons to introduce an offset parameter have been discusses in [43]. In Chapter 4 we will work with the relaxed condition $F(f) \geq -\sigma/2$ for all $f \in \mathcal{B}$ instead of (3.15). The reason for this is that we can then find a set $\mathcal{B}$, where $f^\dagger$ is in the interior (which will be important in Chapter 5) even if $f^\dagger$ takes or approaches the value 0 at some point. We again consider $\mathcal{Y} = B^\gamma_{p,1}(\mathbb{M})$ and $F = T$ linear for the rest of this subsection. The conjugate of $\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ is given by

$$\left(\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}\right)^*(\xi) = \sup_{\substack{g \in B^\gamma_{p,1}(\mathbb{M}) \\ g > -\sigma}} [\langle \xi, g\rangle - \|g\|_{L^1} + \langle G_t + \sigma, \log(g + \sigma)\rangle].$$

In particular we have

$$\left(\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}\right)^*((1 - \sigma) - G_t) = \sup_{\substack{g \in B^\gamma_{p,1}(\mathbb{M}) \\ g > -\sigma}} [\langle G_t + \sigma, \log(g + \sigma) - g\rangle] < \infty,$$

as $\log(x + \sigma) - x \leq \sigma - 1$ for all $x \in (-\sigma, \infty)$. Thus $\left(\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}\right)^*(-\alpha\xi_0) < \infty$ for $\xi_0 = -\frac{1}{\alpha}((1 - \sigma) - G_t)$ so that we have strong duality if $\mathcal{R}^*$ is continuous at $T^*\xi_0$ as for the least squares data fidelity. We have the same extremal relations as in the previous subsection and

$$\partial \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(T\widehat{f}_\alpha) = \left\{\mathbb{1} - \frac{\sigma}{T\widehat{f}_\alpha + \sigma} - \frac{1}{T\widehat{f}_\alpha + \sigma} G_t\right\}. \tag{3.16}$$

And similar as in the last subsection we decompose

$$\widehat{\xi}_\alpha = \frac{1}{\alpha}\left(\frac{g^\dagger - T\widehat{f}_\alpha}{T\widehat{f}_\alpha + \sigma}\right) + \frac{1}{\alpha(T\widehat{f}_\alpha + \sigma)}\left(G_t - g^\dagger\right) =: \widehat{\xi}_{\alpha,L^2} + \widehat{\xi}_{\alpha,Z}. \tag{3.17}$$

3.2.4 Norm power penalty

The penalty $\mathcal{R}(f) = \frac{1}{t}\|f\|_{\mathcal{X}}^t$ has similar motivation and properties as the data fidelity $\mathcal{S}^p_{g^{\text{obs}}}$, so there is some overlap with the above section on the deterministic data fidelity. The penalty $\mathcal{R} = \frac{1}{2}\|\cdot\|_{\mathcal{X}}^2$ can again be motivated as in Section 1.3, where it turned out that this is the "correct" choice under the prior knowledge that $f^\dagger$ is normally distributed. The idea behind using a norm power can also be that our main a-priori knowledge on the true solution is that $f^\dagger$ belongs to the space $\mathcal{X}$. So this penalty is in principle rather general, but one can make it stronger by choosing the smallest space that $f^\dagger$ belongs to. The exact value for the power t can again be motivated as in Section 1.3 or it can be simply chosen in a way such that $\mathcal{R}$ has favorable properties.

The most common and well studied example is $\mathcal{R}(f) = \frac{1}{2}\|f\|_{\mathcal{X}}^2$, where $\mathcal{X}$ is a Hilbert space, corresponding to classical quadratic Tikhonov regularization. We will also consider $\mathcal{R}(f) = \frac{1}{t}\|f\|_{B^0_{p,q}}^t$, where t is usually chosen either as $t = p = q$ or as $t = r$, with r being the convexity of the space $B^0_{p,q}$ (see for example [19, 50, 55, 74]).

As already shown in the last chapter $\mathcal{R}$ is convex (and it is obviously continuous) and the subdifferential of $\mathcal{R}$ is given by the duality mapping $J_t(x)$. The convex conjugate is given by $\mathcal{R}^*(f^*) = \frac{1}{t'}\|f^*\|_{\mathcal{X}^*}^{t'}$. As this is continuous everywhere in $\mathcal{X}^*$ we get the existence of $\widehat{f}_\alpha$ for linear forward operators by Corollary 3.1.4 if $\mathcal{S}^*$ is finite at some point.

3.2.5 Cross entropy penalty

We now consider $\mathcal{X} = L^1(\mathbb{M})$, where $\mathbb{M}$ is a bounded measurement manifold and assume that we are given an initial guess $f_0 \geq 0$ for the true solution $f^\dagger \geq 0$. Given our observed data G^{obs} we want to improve upon this initial guess. Considering probability distributions f, f_0 (or more general positive functions in $L^1(\mathbb{M})$) there is probabilistic motivation (see [62] or [25, Ch. 3.5]) that the correct way to do this is to find

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f \in L^1} \left[\mathcal{S}_{G^{\text{obs}}}(Tf) + \mathcal{R}_{f_0}(f)\right], \tag{3.18}$$

where $\mathcal{R}_{f_0}(f)$ is given by the cross entropy (or relative entropy or Kullback-Leibler) functional $\mathcal{R}_{f_0}(f) = \mathrm{KL}(f, f_0)$. Note that this is non-negativity enforcing as $\mathcal{R}_{f_0}(f) = \infty$ if we have $f < 0$ on a set of positive measure, so one could equivalently take the arg min over all non-negative functions.

Properties and the connection to Bregman divergences of the cross entropy functional are well known and can be found e.g. from the perspective of regularization in [57, 58]. For the sake of self-containedness we show the crucial properties that will be of interest for us.

Proposition 3.2.1. *(a) $\mathcal{R}_{f_0}$ is convex and lower semi-continuous, however nowhere continuous.*

(b) For $f^* \in L^\infty(\mathbb{M})$ we have $\mathcal{R}^*_{f_0}(f^*) = \int_{\mathbb{M}} f_0(x)(e^{f^*(x)} - 1)\mathrm{d}x$.

(c) Let $f_0 > 0$. The subdifferential $\partial\mathcal{R}_{f_0}(f)$ is non-empty if and only if $\log\left(\frac{f}{f_0}\right) \in L^\infty(\mathbb{M})$ and in this case we have

$$\partial\mathcal{R}_{f_0}(f) = \left\{\log\left(\frac{f}{f_0}\right)\right\},$$

as well as

$$\Delta_{\mathcal{R}_{f_0}}^{\log\frac{f}{f_0}}(h, f) = \mathrm{KL}(h, f) = \mathcal{R}_f(h),$$

for all $h \in L^1(\mathbb{M})$.

Proof. We can see that $\mathcal{R}_{f_0}$ is convex by looking pointwise at $\mathrm{kl}_{t_0}(t) = t\log\left(\frac{t}{t_0} - t + t_0\right)$, which has a non-negative second derivative. We calculate $\mathcal{R}^*_{f_0}$ also pointwise and find

$$\mathrm{kl}^*_{t_0}(t^*) = \sup t \geq 0[t^*t - \mathrm{kl}_{t_0}(t)] = t_0(e^{t^*} - 1),$$

as the unique maximum is attained at $t = t_0 e^{t^*}$. As $\mathbb{M}$ is bounded and $f^* \in L^\infty(\mathbb{M})$ we have that $\mathcal{R}_{f_0}$ is well defined at $f = f_0 e^{f^*}$ and it maximizes $f^*f - \mathrm{KL}(f, f_0)$ pointwise and thus also

$$\mathcal{R}^*_{f_0}(f^*) = \sup_{f\in L^1}[f^*f - \mathrm{KL}(f, f_0)] = \int_{\mathbb{M}} f_0(x)(e^{f^*(x)} - 1)\mathrm{d}x.$$

Similarly we can compute for $f \in L^1(\mathbb{M})$ that

$$\mathcal{R}^{**}_{f_0}(f) = \mathrm{KL}(f, f_0) = \mathcal{R}_{f_0}(f)$$

and therefore $\mathcal{R}_{f_0}$ is lower semi-continuous by Theorem 2.1.18.

By definition we have $f^* \in \partial\mathcal{R}_{f_0}(f)$ if the inequality

$$\int_{\mathbb{M}} \mathrm{kl}_{f_0(x)}(f(x) + h(x)) - \mathrm{kl}_{f_0(x)}(f(x)) - f^*(x)h(x)\mathrm{d}x \geq 0$$

holds for all $h \in \mathcal{X} = L^1$. This is equivalent to

$$\mathrm{kl}_{f_0(x)}(f(x) + h(x)) - \mathrm{kl}_{f_0(x)}(f(x)) - f^*(x)h(x) \geq 0$$

almost everywhere (otherwise just choose h as the characteristic function on the subset, where the above expression is negative). Now let $t_0 > 0$, then $\mathrm{kl}_{t_0}(t)$ is differentiable in $t > 0$ and thus has subdifferential $\partial\,\mathrm{kl}_{t_0}(t) = \left\{\log\frac{t}{t_0}\right\}$ by Theorem 2.1.11, whereas for $t \leq 0$ the subdifferential is empty. Thus we see that $f^* \in \partial\mathcal{R}_{f_0}(f) \subset L^\infty$ has to be of the form $f^* = \log\frac{f}{f_0}$ and exists only if this expression is in L^∞. If $f^* \in L^\infty$, then

$$\begin{aligned}\Delta^{f^*}_{\mathcal{R}_{f_0}}(h, f) &= \mathrm{KL}(h, f_0) - \mathrm{KL}(f, f_0) - \left\langle \log\frac{f}{f_0}, h - f\right\rangle \\ &= \int_{\mathbb{M}} h\log\frac{h}{f} - h + f\mathrm{d}x = \mathrm{KL}(h, f),\end{aligned}$$

if $h > 0$ almost everywhere, else both sides equal infinity.

Regarding continuity of $\mathcal{R}_{f_0}$ at some arbitrary $f \in L^1(\mathbb{M}), f \geq 0$ just note that f has to be bounded by say $B > 0$ on a subset $\mathbb{M}_B \subset \mathbb{M}$ and thus for every $\varepsilon > 0$ one can add a characteristic function $h = -2B\chi_{\mathbb{M}_\varepsilon}$ to f, where $\mathbb{M}_\varepsilon \subset \mathbb{M}_B, |\mathbb{M}_\varepsilon| \leq \frac{\varepsilon}{2B}$. Then $\mathcal{R}_{f_0}(f + h) = \infty$, $\|h\|_{L^1} \leq \varepsilon$ and thus $\mathcal{R}_{f_0}$ cannot be continuous at f. □

3.3 Bregman iteration[2]

Additionally to the plain generalized Tikhonov regularization (3.1) we consider Bregman iterated Tikhonov regularization, which is based on the simple idea that starting out with the first approximation $\widehat{f}_\alpha$ from (3.1) we could try to improve the approximation by using $\widehat{f}_\alpha$ as an initial guess.

Assumption 3.3.1. *Let the forward operator $T : \mathcal{X} \to \mathcal{Y}$ be linear and bounded, $\mathcal{S}(g) = \mathcal{S}^p_{g^{\text{obs}}}(g) =: \mathcal{S}_p(g - g^{\text{obs}})$, and $\mathcal{R}$ proper, convex and lower semi-continuous, with $\mathcal{R}^*$ continuous.*

Proposition 3.3.2. *Suppose Assumption 3.3.1 holds true. Let $\hat{f}^{(1)}_\alpha := \hat{f}_\alpha$ be the solution to $(P_1) :=$ (3.1) and set $\mathcal{R}_1 := \mathcal{R}$. Then for $n = 1, 2, \ldots$ we can define $\mathcal{R}_{n+1}(f) := \Delta^{f^*_n}_{\mathcal{R}}(f, \hat{f}^{(n)}_\alpha)$ as well as*

$$\hat{f}^{(n+1)}_\alpha \in \operatorname*{arg\,min}_{f \in \mathcal{X}} \left[\frac{1}{\alpha}\mathcal{S}_p(Tf - g^{\text{obs}}) + \mathcal{R}_{n+1}(f)\right], \qquad (P_{n+1})$$

where

$$f^*_n := \sum_{k=1}^{n} T^* \hat{\xi}^{(k)}_\alpha \in \partial\mathcal{R}\left(\hat{f}^{(n)}_\alpha\right),$$

with $\hat{\xi}^{(n)}_\alpha$ given by the unique solution

$$\hat{\xi}^{(n)}_\alpha \in \operatorname*{arg\,min}_{\xi \in \mathcal{Y}^*} \left[\frac{1}{\alpha}\mathcal{S}^*_p(-\alpha\xi) - \langle \xi, g^{\text{obs}}\rangle + \mathcal{R}^*_n(T^*p)\right] \qquad (P^*_n)$$

to the dual problem (P^*_n) of (P_n) for which strong duality holds.

Proof. The existence of $\hat{\xi}^{(1)}_\alpha$ as well as strong duality for $(P_1), (P^*_1)$ follows from Theorem 3.1.3 as both $\mathcal{S}^p_{g^{\text{obs}}}, (\mathcal{S}^p_{g^{\text{obs}}})^*$ are finite continuous, $\mathcal{R}$ is proper and $\mathcal{R}^*$ is continuous. By the extremal relation (3.5) we have

$$T^*\hat{\xi}^{(1)}_\alpha \in \partial\mathcal{R}(\hat{f}^{(1)}_\alpha),$$

so we can define $\hat{f}^{(2)}_\alpha$ as claimed. Now assume that the claim holds for $n = 1, 2, \ldots, m \in \mathbb{N}$. Then again the existence of $\hat{\xi}^{(m)}_\alpha$ as well as strong duality for $(P_m), (P^*_m)$ follows from Theorem 3.1.3 as $\mathcal{R}_m$ is still proper and $\mathcal{R}^*_m$ is just $\mathcal{R}^*$ with shifted argument and an additive constant, so still continuous. By the extremal relation (3.5) we have

$$T^*\hat{\xi}^{(m)}_\alpha \in \partial\mathcal{R}_m(\hat{f}^{(m)}_\alpha) = \partial\mathcal{R}(\hat{f}^{(m)}_\alpha) - \sum_{k=1}^{m-1} T^*\hat{\xi}^{(k)}_\alpha.$$

Therefore, we have $\sum_{k=1}^{m} T^*\hat{\xi}^{(k)}_\alpha \in \partial\mathcal{R}(\hat{f}^{(m)}_\alpha)$ and can define $\mathcal{R}_{m+1}$ in the way we claimed. □

[2] Most of this section is taken literally from the article [64] to which the author contributed.

We call this method Bregman iterated Tikhonov regularization or short Bregman iteration. It reduces to iterated Tikhonov regularization if $\mathcal{S}(g) = \|g\|_{\mathcal{Y}}^2$, and $\mathcal{R}(f) = \|f\|_{\mathcal{X}}^2$ by (2.2.3). There is a considerable literature on this type of iteration from which we can only give a few references here. Note that for $\mathcal{R}(f) = \|f\|_{\mathcal{X}}^2$ the iteration (P_{n+1}) can be interpreted as the proximal point method for minimizing $\mathcal{T}(f) := \mathcal{S}_p(Tf - g^{\text{obs}})$. In [16, 17, 21] generalizations of the proximal point method for general functions $\mathcal{T}$ on $\mathbb{R}^d$ were studied, in which the quadratic term is replaced by some Bregman divergence (also called D-function). For $\mathcal{T}(f) = \mathcal{S}_p(Tf - g^{\text{obs}})$ this leads to (P_{n+1}) and the references above discuss in particular the case of cross entropy functions $\mathcal{R}$ considered above. The maximum entropy case shows how canonical Bregman iteration actually is as we then have

$$\hat{f}_\alpha^{(n+1)} \in \operatorname*{arg\,min}_{f\in\mathcal{X}} \left[\frac{1}{\alpha}\mathcal{S}_p(Tf - g^{\text{obs}}) + \mathrm{KL}(f, \hat{f}_\alpha^{(n)})\right]$$

by Proposition 3.2.1, which is the only plausible way to iterate maximum entropy. In the context of total variation regularization of inverse problems, the iteration (P_{n+1}) was suggested in [53]. We emphasize that in contrast to all the references above, we consider only small fixed number of iterations as already $\hat{f}_\alpha^{(2)}$ should showcase improved convergence rates upon $\hat{f}_\alpha^{(1)}$. In particular we study convergence in the limit of $\alpha \to 0$ instead of $n \to \infty$. Low order convergence rates of this iterative method for quadratic data fidelity terms $\mathcal{S}$ and general penalty terms $\mathcal{R}$ were obtained in [12, 30, 31, 32].

A useful fact about the penalty functionals $\mathcal{R}_n$ is that their corresponding Bregman distances coincide for all $n \in \mathbb{N}$ as they only differ by an affine linear functional:

Lemma 3.3.3. *Let $f_0 \in \mathcal{X}$, $f_0^* \in \partial\mathcal{R}(f_0)$ and $\tilde{\xi} := \sum_{k=1}^{n-1} \hat{\xi}_\alpha^{(k)}$. Then we have for all $f \in \mathcal{X}$ that*

$$\Delta_{\mathcal{R}_n}^{f_0^* - T^*\tilde{\xi}}(f, f_0) = \Delta_{\mathcal{R}}^{f_0^*}(f, f_0).$$

Proof. By Proposition 3.3.2 we have $T^*\tilde{\xi} \in \partial\mathcal{R}(\hat{f}_\alpha^{(n-1)})$ so $f_0^* - T^*\tilde{\xi} \in \partial\mathcal{R}_n(f_0)$ and

$$\begin{aligned}\Delta_{\mathcal{R}_n}^{f_0^* - T^*\tilde{\xi}}(f, f_0) &= \Delta_{\mathcal{R}}^{T^*\tilde{\xi}}(f, \hat{f}_\alpha^{(n-1)}) - \Delta_{\mathcal{R}}^{T^*\tilde{\xi}}(f_0, \hat{f}_\alpha^{(n-1)}) - \left\langle f_0^* - T^*\tilde{\xi}, f - f_0\right\rangle \\ &= \mathcal{R}(f) - \mathcal{R}(f_0) - \langle f_0^*, f - f_0\rangle = \Delta_{\mathcal{R}}^{f_0^*}(f, f_0).\end{aligned}$$

□

The first step towards bounds on the error in the Bregman distance is given by the next lemma.

Lemma 3.3.4. *Suppose there exists $\xi^\dagger \in \mathcal{Y}^*$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$. With the notation of Proposition 3.3.2 define $s_\alpha^{(n)} := \xi^\dagger - \sum_{k=1}^{n-1} \hat{\xi}_\alpha^{(k)}$. Then*

$$\begin{aligned}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}\left(\hat{f}_\alpha^{(n)}, f^\dagger\right) \le \inf_{f\in\mathcal{X}} \Bigg[&\frac{1}{\alpha}\mathcal{S}_p\left(Tf - g^{\text{obs}}\right) + \left\langle s_\alpha^{(n)}, Tf - g^{\text{obs}}\right\rangle \\ &+ \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha s_\alpha^{(n)}\right) + \Delta_{\mathcal{R}}^{T^*\xi^\dagger}\left(f, f^\dagger\right)\Bigg].\end{aligned}$$

Proof. Due to the minimizing property of $\hat{f}_\alpha^{(n)}$ we have

$$\frac{1}{\alpha}\mathcal{S}_p\left(T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right) + \mathcal{R}_n\left(\hat{f}_\alpha^{(n)}\right) \leq \frac{1}{\alpha}\mathcal{S}_p\left(Tf - g^{\text{obs}}\right) + \mathcal{R}_n(f),$$

for all $f \in \mathcal{X}$, which is equivalent to

$$\mathcal{R}_n\left(\hat{f}_\alpha^{(n)}\right) - \mathcal{R}_n(f) \leq \frac{1}{\alpha}\mathcal{S}_p\left(Tf - g^{\text{obs}}\right) - \frac{1}{\alpha}\mathcal{S}_p\left(T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right). \tag{3.19}$$

As $T^* s_\alpha^{(n)} = T^*\xi^\dagger - f_{n-1}^* \in \partial\mathcal{R}(f^\dagger) - f_{n-1}^* = \partial\mathcal{R}_n(f^\dagger)$ by Proposition 3.3.2, it follows that

$$\begin{aligned}
\Delta_{\mathcal{R}_n}^{T^* s_\alpha^{(n)}}\left(\hat{f}_\alpha^{(n)}, f^\dagger\right) &= \mathcal{R}_n\left(\hat{f}_\alpha^{(n)}\right) - \mathcal{R}_n\left(f^\dagger\right) - \left\langle T^* s_\alpha^{(n)}, \hat{f}_\alpha^{(n)} - f^\dagger\right\rangle \\
&\leq \frac{1}{\alpha}\mathcal{S}_p(Tf - g^{\text{obs}}) - \frac{1}{\alpha}\mathcal{S}_p\left(T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right) \\
&\quad - \left\langle T^* s_\alpha^{(n)}, \hat{f}_\alpha^{(n)} - f^\dagger\right\rangle + \mathcal{R}_n(f) - \mathcal{R}_n\left(f^\dagger\right).
\end{aligned}$$

Due to the strong duality (Propostion 3.3.2) the extremal relation $-\alpha\hat{\xi}_\alpha^{(n)} \in \partial\mathcal{S}_p(T\hat{f}_\alpha^{(n)} - g^{\text{obs}})$ holds true, and thus the generalized Young equality yields

$$\begin{aligned}
-\frac{1}{\alpha}\mathcal{S}_p\left(T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right) &= \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha\hat{\xi}_\alpha^{(n)}\right) + \left\langle \hat{\xi}_\alpha^{(n)}, T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right\rangle \\
&= \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha s_\alpha^{(n)}\right) + \left\langle s_\alpha^{(n)}, T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right\rangle \\
&\quad - \frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}\left(-\alpha s_\alpha^{(n)}, -\alpha\hat{\xi}_\alpha^{(n)}\right) \\
&\leq \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha s_\alpha^{(n)}\right) + \left\langle s_\alpha^{(n)}, T\hat{f}_\alpha^{(n)} - g^{\text{obs}}\right\rangle
\end{aligned}$$

where we have used that the Bregman distance is non-negative. Combining this gives

$$\begin{aligned}
\Delta_{\mathcal{R}_n}^{T^* s_\alpha^{(n)}}\left(\hat{f}_\alpha^{(n)}, f^\dagger\right) &\leq \frac{1}{\alpha}\mathcal{S}_p(Tf - g^{\text{obs}}) + \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha s_\alpha^{(n)}\right) + \left\langle s_\alpha^{(n)}, Tf^\dagger - g^{\text{obs}}\right\rangle \\
&\quad + \mathcal{R}_n(f) - \mathcal{R}_n\left(f^\dagger\right) \\
&= \frac{1}{\alpha}\mathcal{S}_p(Tf - g^{\text{obs}}) + \left\langle s_\alpha^{(n)}, Tf - g^{\text{obs}}\right\rangle \\
&\quad + \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha s_\alpha^{(n)}\right) + \Delta_{\mathcal{R}_n}^{T^* s_\alpha^{(n)}}\left(f, f^\dagger\right).
\end{aligned}$$

Now the identity $\Delta_{\mathcal{R}_n}^{T^* s_\alpha^{(n)}}(f, f^\dagger) = \Delta_{\mathcal{R}}^{T^*\xi^\dagger}(f, f^\dagger)$ shown in Lemma 3.3.3 completes the proof. □

Given the above lemma all that is then left for proving an useful upper bound is to construct appropriate vectors f which approximately minimize the functional on the right hand side.

Chapter Four

Error estimates

> "What comforted his misapprehension?
> That as a competent keyless citizen he had proceeded energetically from the unknown to the known through the incertitude of the void."
>
> *Ulysses*, J. Joyce

In this chapter we will prove estimates on the reconstruction error for generalized Tikhonov regularization as introduced in the last chapter. As we have seen in Section 1.2 we need some kind of source conditions on the possible solutions to achieve convergence rates. We state these conditions in an abstract way to be as general as possible and will then focus in the next chapter on the verification of source conditions in more concrete settings. The first section of this chapter introduces a generalized notion of noise level. In the second section we will consider the boundedness of regularized solutions under statistical noise models. The last three sections introduce different types of source conditions that correspond to different ranges of convergence rates.

4.1 Effective noise level

As we will not always assume the simple deterministic model $\|g^{\mathrm{obs}} - g^\dagger\|_{\mathcal{Y}} \leq \delta$ with noise level $\delta > 0$, we need a more general notion of noise level. Recall that given measurements G^{obs} the data fidelity $\mathcal{S}_{G^{\mathrm{obs}}}$ should measure in some way how far the data are away from the argument. Thus one could consider $\mathcal{S}_{G^{\mathrm{obs}}}(g^\dagger)$ as noise level. However, we can change the data fidelity by an additive constant without changing the regularized solution, so the absolute value of $\mathcal{S}_{G^{\mathrm{obs}}}(g^\dagger)$ is not necessarily informative. Instead we will introduce the effective noise level, based on [75, 42], which is actually not a plain number, but a functional.

Definition 4.1.1 (Effective noise level). *Let $\mathcal{S}^\dagger \colon \mathcal{Y} \to [0, \infty)$ be some ("exact data fidelity") functional and let $C_{\mathrm{err}} \geq 0$ some constant. Then we define the effective noise level $\mathbf{err} \colon \mathcal{Y} \to \overline{\mathbb{R}}$ by*

$$\mathbf{err}(g) = \mathcal{S}_{G^{\mathrm{obs}}}(g^\dagger) - \mathcal{S}_{G^{\mathrm{obs}}}(g) + \frac{1}{C_{\mathrm{err}}}\mathcal{S}^\dagger(g).$$

Further for a subset $M \subset \mathcal{Y}$ we define $\mathbf{err}_M = \sup_{g \in M} \mathbf{err}(g)$.

The following lemma indicates the use of the effective noise level for convergence analysis.

Lemma 4.1.2. *Let $\widehat{f}_\alpha$ minimize (3.1), then we have*

$$\mathcal{R}(\widehat{f}_\alpha) - \mathcal{R}(f^\dagger) \leq \frac{\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big)}{\alpha} - \frac{1}{C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big)$$

Proof. As $\widehat{f}_\alpha$ minimizes the Tikhonov functional we have

$$\frac{1}{\alpha}\mathcal{S}_{G^{\mathrm{obs}}}\Big(F(\widehat{f}_\alpha)\Big) + \mathcal{R}(\widehat{f}_\alpha) \leq \frac{1}{\alpha}\mathcal{S}_{G^{\mathrm{obs}}}(g^\dagger) + \mathcal{R}(f^\dagger)$$

If we artificially add and subtract the exact data fidelity functional $\frac{1}{C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger$ we obtain the effective noise level

$$\mathcal{R}(\widehat{f}_\alpha) \leq \frac{1}{\alpha}\Big[\mathcal{S}_{G^{\mathrm{obs}}}(g^\dagger) - \mathcal{S}_{G^{\mathrm{obs}}}\Big(F(\widehat{f}_\alpha)\Big) + \frac{1}{C_{\mathrm{err}}}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big)\Big] - \frac{1}{C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big) + \mathcal{R}(f^\dagger).$$

□

For the simple deterministic model the effective noise level is nicely bounded by the usual noise level. However, in general finding upper bounds on **err** can be difficult.

Lemma 4.1.3. *(a) Let for $1 < p < \infty$ the data fidelity be given by $\mathcal{S}^p_{g^{\mathrm{obs}}}$ and choose $\mathcal{S}^\dagger(g) = \frac{1}{p}\|g - g^\dagger\|^p_{\mathcal{Y}}$, $C_{\mathrm{err}} = 2^{p-1}$. Then we have*

$$\mathbf{err}_{\mathcal{Y}} \leq \frac{2}{p}\delta^p.$$

(b) Let the data fidelity be given by $\mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ and choose $\mathcal{S}^\dagger(g) = \frac{1}{2}\|g - g^\dagger\|^2_{L^2}$, $C_{\mathrm{err}} = 1$. Then we have

$$\mathbf{err}(g) = \langle \varepsilon Z, g - g^\dagger \rangle \leq \varepsilon \|Z\|_{B^{-\gamma}_{p',\infty}} \|g - g^\dagger\|_{B^{\gamma}_{p,1}}.$$

(c) Let the data fidelity be given by $\mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ and choose $\mathcal{S}^\dagger(g) = \mathrm{KL}(g^\dagger + \sigma, g + \sigma)$, $C_{\mathrm{err}} = 1$. Then we have

$$\mathbf{err}(g) = \left\langle \varepsilon Z, \log\left(\frac{g+\sigma}{g^\dagger + \sigma}\right)\right\rangle \leq \varepsilon\|Z\|_{B^{-\gamma}_{p',\infty}} \left\|\log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\|_{B^{\gamma}_{p,1}}.$$

(d) For $a > \gamma$, $p \in [1,2]$ and all $c_1, c_2 > 0$, $r \geq 1$ we have

$$\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}} \|g\|_{B^{\gamma}_{p,1}} \leq C c_2^{-\frac{1-\gamma/a}{1+\gamma/a}} \left(c_1^{-\frac{\gamma}{ra}} \|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}\right)^{\frac{2}{1+\frac{\gamma}{ra}(2-r)}} + c_1\|g\|^r_{B^a_{p,2}} + c_2\|g\|^2_{L^2}.$$

Proof. (a) Note that by convexity of $|\cdot|^p$ we have for all $s, t \in \mathbb{R}$ that

$$|s-t|^p = |2(s/2) + (1-2)t|^p \geq 2|s/2|^p + (1-2)|t|^p = 2^{1-p}|s|^p - |t|^p.$$

This allows us to estimate the effective noise level by

$$\begin{aligned} p\,\mathbf{err}(g) &:= \|g^\dagger - g^{\mathrm{obs}}\|^p_{\mathcal{Y}} - \|g - g^{\mathrm{obs}}\|^p_{\mathcal{Y}} + 2^{1-p}\|g - g^\dagger\|^p_{\mathcal{Y}} \\ &\leq \delta^p - |\|g - g^\dagger\|_{\mathcal{Y}} - \|g^\dagger - g^{\mathrm{obs}}\|_{\mathcal{Y}}|^p + 2^{1-p}\|g - g^\dagger\|^p_{\mathcal{Y}} \leq 2\delta^p. \end{aligned}$$

As this holds for all $g \in \mathcal{Y}$ we have shown the claim.

(b) By definition we have

$$\begin{aligned}\mathbf{err}(g) &= \frac{1}{2}\|g^\dagger\|_{L^2}^2 - \langle g^\dagger + \varepsilon Z, g^\dagger\rangle - \frac{1}{2}\|g\|_{L^2}^2 + \langle g^\dagger + \varepsilon Z, g\rangle + \frac{1}{2}\|g - g^\dagger\|_{L^2}^2 \\ &= \langle \varepsilon Z, g - g^\dagger\rangle.\end{aligned}$$

The norm bound then follows from the duality given by Theorem A.2.6.

(c) By definition we have

$$\begin{aligned}\mathbf{err}(g) &= \|g^\dagger\|_{L^1} - \|g\|_{L^1} + \left\langle g^\dagger + \varepsilon Z + \sigma, \log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\rangle + \mathrm{KL}(g^\dagger + \sigma, g + \sigma) \\ &= \left\langle g^\dagger + \varepsilon Z + \sigma, \log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\rangle + \left\langle g^\dagger, \log\left(\frac{g^\dagger+\sigma}{g+\sigma}\right)\right\rangle \\ &= \left\langle \varepsilon Z, \log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\rangle\end{aligned}$$

and the norm bound follows as above.

(d) We have by Theorem A.2.7 that

$$\|g\|_{B^\gamma_{p,1}} \le C\|g\|_{B^0_{p,2}}^{1-\gamma/a}\|g\|_{B^a_{p,2}}^{\gamma/a} \le C\|g\|_{L^2}^{1-\gamma/a}\|g\|_{B^a_{p,2}}^{\gamma/a}$$

where we used the fact that $B^0_{p,2} \subset L^2$ continuously. Now we just have to apply the Peter-Paul inequality (2.7) two times, firstly with $q = \frac{2}{1-\gamma/a}$, $q' = \frac{2}{1+\gamma/a}$ such that

$$\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}\|g\|_{B^\gamma_{p,1}} \le Cc_2^{-\frac{1-\gamma/a}{1+\gamma/a}}\left(\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}\|g\|_{B^a_{p,2}}^{\gamma/a}\right)^{\frac{2}{1+\gamma/a}} + c_2\|g\|_{L^2}^2$$

and secondly with $q = \frac{r(\gamma+a)}{2\gamma}$, $q' = \frac{1+\gamma/a}{1+\frac{\gamma}{ra}(2-r)}$ to find

$$\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}\|g\|_{B^\gamma_{p,1}} \le Cc_2^{-\frac{1-\gamma/a}{1+\gamma/a}}\left(c_1^{-\frac{\gamma}{ra}}\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}\right)^{\frac{2}{1+\frac{\gamma}{ra}(2-r)}} + c_1\|g\|_{B^a_{p,2}}^r + c_2\|g\|_{L^2}^2. \qquad \square$$

4.2 Bounds on regularized solutions

Boundedness of $\widehat{f}_\alpha$ for deterministic data is not really an interesting question. If $g^{\mathrm{obs}} \mapsto \widehat{f}_\alpha$ is a regularization method, then $\widehat{f}_\alpha$ will be close to the solution for sufficiently small noise level δ and in particular bounded. For stochastic data it can however happen that $\widehat{f}_\alpha$ is not almost surely in any bounded set even though the distribution of $\widehat{f}_\alpha$ may be well concentrated around $f^\dagger$. As we will later need to bound the norm $\|\widehat{f}_\alpha\|$ at some points, we want to understand its dependence on the random noise. In the white noise setting this is straightforward.

Assumption 4.2.1. *Let $\gamma > 0$ and $p \in [1, 2]$ be as in Assumption 1.4.1.*

(A1) There exists $\widehat{f}_\alpha \in D(F)$ minimizing (3.1).

(A2) For some $a > \gamma$ we have $\|F(f_1) - F(f_2)\|_{B^a_{p,2}} \le L\|f_1 - f_2\|_X$ for all $f_1, f_2 \in D(F)$.

(A3) There exists $C_\mathcal{R}, \theta > 0$ such that $\|f\|_\mathcal{X} \le C_\mathcal{R}\mathcal{R}(f)$ for all $f \in \mathcal{X}$ with $\|f\|_\mathcal{X} \ge \theta$.

Remark 4.2.2. *For a linear forward operator $F = T$ (A1) follows for all penalty and data fidelity functionals that we discussed in the last chapter from Corollary 3.1.4. Regarding the existence of $\widehat{f}_\alpha$ for nonlinear forward operators we refer to [54, Theorem 1.6] in the general case, to [43, Proposition 4.2] for the case of Poisson data with $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ and to [74, Proposition 4.10] for the case of Gaussian white noise with $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$.*

Note that by (A.1) from the appendix (A2) is only slightly stronger than the assumption that $\mathcal{Y} = B^{\gamma}_{p,1}(\mathbb{M})$ (as in Sections 3.2.2 and 3.2.3) with a Lischpitz continuous forward operator.

(A3) obviously holds for norm powers $\mathcal{R} = \frac{1}{t}\|\cdot\|^t$ if if $t \geq 1$ and it holds for $\mathcal{R}(f) = \mathrm{KL}(f, f_0)$ by Lemma 2.3.2.

Proposition 4.2.3. *Let $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$. Let Assumption 4.2.1 hold true. Then we have*

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq \max(\theta, C_{\mathcal{R}}\mathcal{R}(\widehat{f}_\alpha)) \qquad \text{and} \qquad \mathcal{R}(\widehat{f}_\alpha) \leq 3\mathcal{R}(f^\dagger) + C\left(\frac{\|\varepsilon Z\|^2_{B^{-\gamma/2}_{p',\infty}}}{\alpha^{1+\gamma/a}}\right)^{\frac{1}{1-\gamma/a}}.$$

In particular for a parameter choice fulfilling $\alpha \geq c\varepsilon^{\frac{2}{1+\gamma/a}}$ we have

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq \max\left(\theta, 3C_{\mathcal{R}}\mathcal{R}(f^\dagger) + C_{\mathcal{R}}C\|Z\|^{\frac{2}{1-\gamma/a}}_{B^{-\gamma/2}_{p',\infty}}\right).$$

Proof. By Lemma 4.1.2 and Lemma 4.1.3 with $r = 1$, $c_1 = C_{\mathcal{R}}\alpha/(2L)$ and $c_2 = 1/2$ we have

$$\begin{aligned}
\alpha\mathcal{R}(\widehat{f}_\alpha) &\leq \mathbf{err}\Big(F(\widehat{f}_\alpha)\Big) - \frac{1}{2}\|F(\widehat{f}_\alpha) - g^\dagger\|^2_{L^2} + \alpha\mathcal{R}(f^\dagger) \\
&\leq C\Big(\alpha^{\frac{-\gamma}{a}}\|\varepsilon Z\|_{B^{-\gamma/2}_{p',\infty}}\Big)^{\frac{2}{1-\gamma/a}} + \frac{C_{\mathcal{R}}\alpha}{2L}\|F(\widehat{f}_\alpha) - g^\dagger\|_{B^a_{p,2}} + \alpha\mathcal{R}(f^\dagger) \\
&\leq C\Big(\alpha^{\frac{-\gamma}{a}}\|\varepsilon Z\|_{B^{-\gamma/2}_{p',\infty}}\Big)^{\frac{2}{1-\gamma/a}} + \frac{C_{\mathcal{R}}\alpha}{2}\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}} + \alpha\mathcal{R}(f^\dagger).
\end{aligned}$$

In absence of any source conditions we can only bound $\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}}$ by the triangle inequality, which leads to the claim by (A3) after subtracting $(\alpha/2)\mathcal{R}(\widehat{f}_\alpha)$ on both sides and dividing by α. □

The approach for the case of Poisson data is the same, but the logarithm leads to technical difficulties.

Proposition 4.2.4. *Let $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$. Let $g^\dagger \geq 0, F(\widehat{f}_\alpha) \geq -\sigma/2$, and let Assumption 4.2.1 hold true for $p = 2$ and $a \in \{1, 2\}$, $a > d/2$. Then we have*

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq \max(\theta, C_{\mathcal{R}}\mathcal{R}(\widehat{f}_\alpha)) \qquad \text{and} \qquad \mathcal{R}(\widehat{f}_\alpha) \leq 3\mathcal{R}(f^\dagger) + C_{g^\dagger,\sigma}\left(\frac{\|\varepsilon Z\|^2_{B^{-\gamma/2}_{2,\infty}}}{\alpha^{1+\gamma/a}}\right)^{\frac{1}{1-\gamma/a}}.$$

In particular for a parameter choice fulfilling $\alpha \geq c\varepsilon^{\frac{2}{1+\gamma/a}}$ we have

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq \max\left(\theta, 3C_{\mathcal{R}}\mathcal{R}(f^\dagger) + C_{\mathcal{R}}C_{g^\dagger,\sigma}\|Z\|^{\frac{2}{1-\gamma/a}}_{B^{-\gamma/2}_{2,\infty}}\right).$$

Proof. We want to apply Items (c) and (d) of Lemma 4.1.3, so we have to understand the Sobolev norm of the logarithm. By (A2) and Corollary A.2.12 we have for $a \in \{1,2\}$ that

$$\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|_{H^a} \leq C_{g^\dagger,\sigma,a}\left\|F(\widehat{f}_\alpha)-g^\dagger\right\|_{H^a} \leq LC_{g^\dagger,\sigma,a}\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}}.$$

Let $C_{g^\dagger,\sigma} = 2\frac{\|g^\dagger\|_{L^\infty}+\sigma}{\sigma}\|\frac{1}{g^\dagger+\sigma}\|_{L^\infty}$ be the constant from Corollary A.2.12. By Lemma 4.1.2 and Lemma 4.1.3 with the choices $c_2 = 3\left(10\|g^\dagger\|_{L^\infty}+6\sigma+4\log(\sigma)^2+2e^3\right)^{-1}C_{g^\dagger,\sigma}^{-1}$, $c_1 = C_{\mathcal{R}}\alpha/(2LC_{g^\dagger,\sigma,a})$ and $r=1$ we have

$$\begin{aligned}\alpha\mathcal{R}(\widehat{f}_\alpha) \leq& \,\mathbf{err}\left(F(\widehat{f}_\alpha)\right) - \mathrm{KL}\left(g^\dagger+\sigma, F(\widehat{f}_\alpha)+\sigma\right) + \alpha\mathcal{R}(f^\dagger)\\ \leq& \,C\left(\frac{\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}}{\alpha^{\gamma/a}}\right)^{\frac{2}{1-\gamma/a}} + c_2\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^2_{L^2} + \frac{C_{\mathcal{R}}\alpha}{2}\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}}\\ &- \mathrm{KL}\left(g^\dagger+\sigma, F(\widehat{f}_\alpha)+\sigma\right) + \alpha\mathcal{R}(f^\dagger).\end{aligned}$$

To estimate the integral

$$c_2\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^2_{L^2} - \mathrm{KL}\left(g^\dagger+\sigma, F(\widehat{f}_\alpha)+\sigma\right) \tag{4.1}$$

we cut the domain of integration in the two parts

$$\begin{aligned}A :&= \{x \in \mathbb{M} : F(\widehat{f}_\alpha)(x) \leq \max(2g^\dagger(x), \sigma, \log(\sigma)^2, e^3/2)\}\\ B :&= \{x \in \mathbb{M} : F(\widehat{f}_\alpha)(x) > \max\{2g^\dagger(x), \sigma, \log(\sigma)^2, e^3/2\}\}.\end{aligned}$$

In the first part we have the uniform bound $F(\widehat{f}_\alpha)(x) \leq 2\|g^\dagger\|_{L^\infty} + \sigma + \log(\sigma)^2 + \frac{1}{2}e^3$ so by (2.19) we have

$$\|g^\dagger - F(\widehat{f}_\alpha)\|^2_{L^2(A)} \leq \frac{1}{3}\left(10\|g^\dagger\|_{L^\infty}+6\sigma+4\log(\sigma)^2+2e^3\right)\mathrm{KL}_A\left(g^\dagger+\sigma, F(\widehat{f}_\alpha)+\sigma\right)$$

and by Corollary A.2.12 we have

$$\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^2_{L^2(A)} \leq C_{g^\dagger,\sigma}\|g^\dagger - F(\widehat{f}_\alpha)\|_{L^2(A)},$$

so on A the integral (4.1) is negative. On B we have $2g^\dagger(x), \sigma, \log(\sigma)^2, e^3/2 \leq F(\widehat{f}_\alpha)(x)$ so we can bound it by

$$\begin{aligned}&\int_B c_2\log\left(\frac{F(\widehat{f}_\alpha)(x)+\sigma}{g^\dagger(x)+\sigma}\right)^2 + g^\dagger(x) + (g^\dagger(x)+\sigma)\log\left(\frac{g^\dagger(x)+\sigma}{F(\widehat{f}_\alpha)(x)+\sigma}\right) - F(\widehat{f}_\alpha)(x)\mathrm{d}x\\ &\leq \int_B c_2\log\left(\frac{F(\widehat{f}_\alpha)(x)+\sigma}{g^\dagger(x)+\sigma}\right)^2 - \frac{1}{2}F(\widehat{f}_\alpha)(x)\mathrm{d}x\\ &= \int_B c_2\left(\log(F(\widehat{f}_\alpha)(x)+\sigma) - \log(g^\dagger(x)+\sigma)\right)^2 - \frac{1}{2}F(\widehat{f}_\alpha)(x)\mathrm{d}x\\ &\leq \int_B 2c_2\left(\log\left(2F(\widehat{f}_\alpha)(x)\right)^2 + \max\left(\log\left(2F(\widehat{f}_\alpha)(x)\right)^2, \log(\sigma)^2\right)\right) - \frac{1}{2}F(\widehat{f}_\alpha)(x)\mathrm{d}x,\end{aligned}$$

which is also negative as $c_2 \leq \frac{1}{10}$ and $F(\widehat{f}_\alpha)(x) \geq \frac{1}{2}e^3$. Thus we have

$$\alpha\mathcal{R}(\widehat{f}_\alpha) \leq C\Big(\alpha^{-\gamma/a}\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}\Big)^{\frac{2}{1-\gamma/a}} + \frac{C_{\mathcal{R}}\alpha}{2}\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}} + \alpha\mathcal{R}(f^\dagger).$$

and the claim follows as in the proof of Proposition 4.2.3. □

Remark 4.2.5. *The fact that the above result is only stated for $a = 1, 2$ comes from the fact that Theorem A.2.10 is only proven for $s = 1, 2$. However, we suppose that the above Proposition should hold for all $a > d/2$ and possibly also for all $p \in [1, 2]$ (compare Remark A.2.11).*

4.3 (First order) Variational source conditions

In this section we will show how to obtain estimates on the reconstruction error based on variational source conditions (VSCs). As we have seen in the introduction source conditions are necessary to obtain convergence rates and VSCs have become increasingly popular recently, because they are quite flexible in the sense that they yield convergence rates for general regularization methods on Banach spaces and non-linear forward operators.

Definition 4.3.1 (Index function)**.** *We call a function $\Phi\colon [0,\infty) \to [0,\infty)$ an index function if it is increasing and continuous with $\Phi(0) = 0$.*

Definition 4.3.2 (Variational source condition $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$)**.** *We say $f^\dagger$ satisfies a (first order) variational source condition $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$ if there exist $f^* \in \partial\mathcal{R}(f^\dagger)$ and a concave index function Φ such that for all $f \in \mathcal{X}$ we have*

$$\langle f^*, f^\dagger - f\rangle \leq \frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) + \Phi\Big(\mathcal{S}^\dagger(F(f))\Big). \tag{4.2}$$

Remark 4.3.3. *By definition of the Bregman divergence the inequality (4.2) is equivalent to*

$$\frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) \leq \mathcal{R}(f) - \mathcal{R}(f^\dagger) + \Phi\Big(\mathcal{S}^\dagger(F(f))\Big). \tag{4.3}$$

4.3.1 Regularization error bounds

Assuming $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$ one can get immediately error bounds in the Bregman divergence by the following Theorem (which is based on [42, Theorem 2.3]).

Theorem 4.3.4. *Let $f^\dagger$ fulfill a VSC with index function Φ, then the estimates*

$$\frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq \frac{\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big)}{\alpha} + (-\Phi)^*\Big(-\frac{1}{C_{\mathrm{err}}\alpha}\Big) \leq \frac{\mathbf{err}_{\operatorname{ran} F}}{\alpha} + (-\Phi)^*\Big(-\frac{1}{C_{\mathrm{err}}\alpha}\Big) \tag{4.4}$$

and

$$\frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq \frac{\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big)}{\alpha} - \frac{1}{2C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big) + (-\Phi)^*\Big(-\frac{1}{2C_{\mathrm{err}}\alpha}\Big) \tag{4.5}$$

hold true. For $\mathbf{err}_{\operatorname{ran} F} < \infty$ *the infimum of the right hand side of* (4.4) *is attained at* $\alpha = \bar{\alpha}$ *if*

$$-\frac{1}{C_{\mathrm{err}}\bar{\alpha}} \in \partial(-\Phi)^*(C_{\mathrm{err}}\,\mathbf{err}_{\operatorname{ran} F})$$

and in this case we have

$$\frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_{\bar{\alpha}}, f^\dagger) \leq \Phi(C_{\mathrm{err}}\,\mathbf{err}_{\operatorname{ran} F}).$$

Proof. By (4.3) we have

$$\frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \leq \mathcal{R}(\widehat{f}_\alpha) - \mathcal{R}(f^\dagger) + \Phi\Big(\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big)\Big)$$

and then Lemma 4.1.2 yields

$$\frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \leq \frac{\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big)}{\alpha} - \frac{1}{C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big) + \Phi\Big(\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big)\Big)$$

Introducing $\lambda \in [0,1]$ and writing $\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big) = \tau$ we find

$$\frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \leq \frac{\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big)}{\alpha} - \frac{\lambda}{C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\Big(F(\widehat{f}_\alpha)\Big) + \sup_{\tau\geq 0}\left[\frac{-(1-\lambda)\tau}{C_{\mathrm{err}}\alpha} - (-\Phi)(\tau)\right].$$

The supremum is by definition the convex conjugate, thus by choosing $\lambda = 0$ we get (4.4) and by $\lambda = 1/2$ we get (4.5). By Young's inequality we have

$$\Phi(C_{\mathrm{err}}\,\mathbf{err}_{\operatorname{ran} F}) \leq \frac{\mathbf{err}_{\operatorname{ran} F}}{\alpha} + (-\Phi)^*\Big(-\frac{1}{C_{\mathrm{err}}\alpha}\Big)$$

and equality holds if and only if $-(C_{\mathrm{err}}\bar{\alpha})^{-1} \in \partial(-\Phi)(C_{\mathrm{err}}\,\mathbf{err}_{\operatorname{ran} F})$. □

Thus we see that in the deterministic setting with data fidelity $\mathcal{S}^p_{g^{\mathrm{obs}}}$ a VSC immediately implies the convergence rate

$$\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_{\bar{\alpha}}, f^\dagger) \leq 2\Phi(C\delta^p),$$

by Theorem 4.3.4 and Lemma 4.1.3, where the speed of convergence is determined by the index function Φ. Before we come to the stochastic setting we want to give a short outlook on how to verify a VSC for a given solution $f^\dagger$.

4.3.2 Outlook: Verification of variational source conditions

As the VSC is a rather abstract condition we would like to understand, what it takes for a function to fulfill a VSC. We will actually verify the VSCs in complete detail for several settings in Chapter 5, but we want to give a short outlook on how to verify a VSC. For a linear forward operator T a VSC can be verified easily under the following condition (compare [37, Remark 4.3]).

Lemma 4.3.5. *Assume that* $\mathcal{S}^\dagger(g) = \frac{1}{p}\|g - g^\dagger\|^p$ *and that there exists* $\xi^\dagger$ *such that* $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$. *Then* $f^\dagger$ *fulfills a VSC with* $\Phi(t) = \|\xi^\dagger\|_{\mathcal{Y}^*}(pt)^{\frac{1}{p}}$.

Proof.

$$\langle f^*, f^\dagger - f\rangle = \langle \xi^\dagger, Tf^\dagger - Tf\rangle \le \|\xi^\dagger\|_{\mathcal{Y}^*}\|Tf^\dagger - Tf\|_{\mathcal{Y}} = \|\xi^\dagger\|_{\mathcal{Y}^*}\Big(p\mathcal{S}^\dagger(Tf)\Big)^{\frac{1}{p}}. \qquad \square$$

The preceding lemma uses a rather strong assumption ($T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$), which leads to a particular index function. More generally one can use the following strategy, which has been proposed in [40, 74, 73].

Proposition 4.3.6. *Assume that for some* $r \ge 2$ *and for all* $f_1, f_2 \in \mathcal{X}$, $f_2^* \in \partial\mathcal{R}(f_2)$ *we have*

$$\Delta_{\mathcal{R}}^{f_2^*}(f_1, f_2) \ge C_{\mathcal{R}}\|f_1 - f_2\|_{\mathcal{X}}^r. \tag{4.6}$$

Let $f^\dagger \in \mathcal{X}$ *and* $f^* \in \partial\mathcal{R}(f^\dagger)$. *Suppose that there exists a family of operators* $P_k\colon \mathcal{X}^* \to \mathcal{X}^*$ *for* $k \in \mathbb{N}_0$ *and constants* κ_k, ν_k *such that the following holds true for all* $k \in \mathbb{N}_0$:

$$\|(I - P_k)f^*\|_{\mathcal{X}^*} \le \kappa_k \qquad \textit{and} \qquad \inf_{k\in\mathbb{N}_0} \kappa_k = 0, \tag{4.7a}$$

$$\Big\langle P_k f^*, f^\dagger - f\Big\rangle \le \nu_k\Big\|F(f^\dagger) - F(f)\Big\|_{\mathcal{Y}}, \quad \textit{for all } f \in \mathcal{X} \textit{ with } \Big\|f^\dagger - f\Big\|_{\mathcal{X}} \le \Big(\tfrac{2}{C_{\mathcal{R}}}\|f^*\|_{\mathcal{X}^*}\Big)^{\frac{r'}{r}}. \tag{4.7b}$$

Then $f^\dagger$ *fulfills* $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$ *with the concave index function*

$$\Phi(\tau) = \inf_{k\in\mathbb{N}_0}\Big[\nu_k\tau^{1/2} + \frac{1}{r'}\Big(\tfrac{2}{C_{\mathcal{R}}}\Big)^{r'/r}\kappa_k^{r'}\Big]. \tag{4.8}$$

If we have instead of (4.6) *that* $\mathcal{R} = \frac{1}{t}\|\cdot\|_{\mathcal{X}}^t$, *for* $t > 1$ *and* $\mathcal{X}$ r*-convex, then there exists* $C_{\mathcal{R}} > 0$ *such that under* (4.7) $f^\dagger$ *fulfills a variational source condition* (4.2) *with the concave index function*

$$\Phi(\tau) = \inf_{k\in\mathbb{N}_0}\Big[\nu_k\tau^{1/2} + C_{\mathcal{X},t}\|f^\dagger\|_{\mathcal{X}}^{r'(r-t)/r}\kappa_k^{r'}\Big]. \tag{4.9}$$

Proof. We will not prove the case for general $\mathcal{R}$ fulfilling (4.6) and refer to [74, Theorem 3.3] for the original proof. Instead we will do the very similar proof for $\mathcal{R} = \frac{1}{t}\|\cdot\|_{\mathcal{X}}^t$, which is just slightly more complicated, because we have to work with Theorem 2.2.5 instead of (4.6). We distinguish two cases. First of all we assume that for some $B \ge 3$ to specify later we have $\|f - f^\dagger\| \ge B\|f^\dagger\|$. This in particular implies $\|f\| \ge (B-1)\|f^\dagger\| \ge 2\|f^\dagger\|$ and $\|f - f^\dagger\| \ge \frac{1}{2}\|f\|$ thus Theorem 2.2.5 gives that $\Delta_{\mathcal{R}}^{f^*}(f, f^\dagger) \ge C_{\mathcal{X},t}\|f\|^{t-r}\|f - f^\dagger\|^r \ge C_{\mathcal{X},t}2^{-r}\|f\|^t$, where $r \ge 2$. Therefore

$$\begin{aligned}\langle f^*, f^\dagger - f\rangle &\le 2\|f^*\|_{\mathcal{X}^*}\|f\|_{\mathcal{X}} = 2\|f^\dagger\|_{\mathcal{X}}^{t-1}\|f\|_{\mathcal{X}} \le 2\left(\frac{\|f^\dagger\|}{\|f\|}\right)^{t-1}\|f\|^2 \\ &\le 2(B-1)^{1-t}\|f\|^t \le \frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(f, f^\dagger),\end{aligned}$$

if B is chosen large enough. Consequently the variational inequality (4.2) holds for all f such that $\|f - f^\dagger\| \geq B\|f^\dagger\|$. Conversely assume now that $\|f - f^\dagger\| < B\|f^\dagger\|$. For $C_{\mathcal{R}} > 1$ sufficiently small this implies that (4.7b) holds true and we can estimate

$$\langle f^*, f^\dagger - f\rangle = \langle P_k f^*, f^\dagger - f\rangle + \langle (I - P_k) f^*, f^\dagger - f\rangle \leq \nu_k \left\| F(f^\dagger) - F(f)\right\|_{\mathcal{Y}} + \kappa_k \|f^\dagger - f\|.$$

By (2.15) from Theorem 2.2.5 we have $\Delta_{\mathcal{R}}^{f^*}(f, f^\dagger) \geq C_{\mathcal{X},t,B}\|f^\dagger\|^{t-r}\|f - f^\dagger\|^r$ and thus by the Peter-Paul inequality (2.7) that

$$\kappa_k\|f^\dagger - f\| \leq \Delta_{\mathcal{R}}^{f^*}(f, f^\dagger) + \left(2rC_{\mathcal{X},t,B}\|f^\dagger\|^{t-r}\right)^{\frac{-r'}{r}} \frac{\kappa_k^{r'}}{r'}.$$

Thus we see that the variational inequality holds true with Φ as in (4.9). To see that Φ is a concave index function note that it is given by an infimum over concave and increasing functions and hence is also increasing and concave, which implies in particular continuity. By (4.7a) we have $\Phi(0) = 0$, thus it is an index function. □

Remark 4.3.7. *For many choices of P_k, such as projections onto spaces of trigonometric polynomials or finite elements, Condition (4.7a) describes the smoothness of f^*, which is often closely related to the smoothness of $f^\dagger$. On the other hand (4.7b) describes the local ill-posedness of the problem (for more details we refer to [73]). In [29] it was shown that if $f^\dagger$ is the unique solution to $F(f^\dagger) = g^\dagger$ then $f^\dagger$ fulfills a variational source condition of the form (4.2) (possibly with a different constant) under mild assumptions on the forward operator and the penalty. Of course the index function Φ will strongly depend on the smoothness of $f^\dagger$ so that one still can end up with arbitrarily slow convergence.*

If one intends to use Proposition 4.3.6 with $\mathcal{R} = \frac{1}{t}\|\cdot\|^t$ to show convergence rates on a set of possible solutions one has to be careful if $t > r$ as the constant in the index function will explode if $\|f^\dagger\| \to 0$. So in this case a lower bound on $\|f^\dagger\|$ would be required to get a uniform constant.

At the end of this subsection we want to state a negative result on the range of index functions for which a VSC can be verified (compare [28, Prop. 12.10]).

Proposition 4.3.8. *Let the forward operator F be continuous. Assume that $\mathcal{S}^\dagger(g) = \frac{1}{p}\|g - g^\dagger\|^p$ and that $\mathcal{R}$ is Fréchet-differentiable. If a VSC with $\Phi(t) = o\left(t^{\frac{1}{p}}\right)$ holds true for $f^\dagger$ then $f^\dagger \in \arg\min \mathcal{R}$.*

Proof. The VSC yields

$$\langle f^*, f^\dagger - f\rangle \leq \frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(f, f^\dagger) + \Phi\left(\mathcal{S}^\dagger(F(f))\right) \leq \frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(f, f^\dagger) + \Phi\left(C\|f - f^\dagger\|^p\right), \quad (4.10)$$

because of the continuity of F. Now for some arbitrary $f_0 \in \mathcal{X}$, $t > 0$ set $f = f^\dagger + tf_0$ in (4.10) and divide by t to find

$$\langle f^*, f_0\rangle \leq \frac{1}{2t}\Delta_{\mathcal{R}}^{f^*}(f^\dagger + tf_0, f^\dagger) + \frac{1}{t}\Phi(Ct^p),$$

for all $t > 0$. As $\mathcal{R}$ is differentiable we have $\lim_{t\to 0} t^{-1}\Delta_{\mathcal{R}}^{f^*}(f^\dagger + tf_0, f^\dagger) = 0$ and by $\Phi(t) = o\left(t^{\frac{1}{p}}\right)$ the right hand side goes to 0 as $t \to 0$. As f_0 was arbitrary we have $f^* = 0$. By definition of the subdifferential this is equivalent to $f^\dagger \in \arg\min \mathcal{R}$. □

As $f^\dagger \in \arg\min \mathcal{R}$ is not an interesting situation, the above proposition limits the range of convergence rates that can be shown under VSC^1. In the deterministic setting VSC^1 thus yields convergence rates only up to $\Delta_{\mathcal{R}}^{f^*}(f_{\bar{\alpha}}, f^\dagger) = \mathcal{O}(\delta)$.

4.3.3 The stochastic setting

We have seen that under a certain parameter choice a VSC immediately yields convergence rates $\Phi(C_{\mathrm{err}}\,\mathbf{err}_{\mathrm{ran}\,F})$. However, this is only useful if $\mathbf{err}_{\mathrm{ran}\,F}$ can be bounded. For the data fidelity $\mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}$ we have seen that the effective noise level is of the form $\mathbf{err}(g) = \langle \varepsilon Z, g\rangle - \langle \varepsilon Z, g^\dagger\rangle$. Clearly in this case there is little chance for a uniform bound. If for example the forward operator is linear then $\mathrm{ran}\,F$ is unbounded and so is $\mathbf{err}(F(f))$. Even worse, if the noise is not in L^2 (compare 1.4.5) then $\mathbf{err}(g)$ might not be well defined for certain $g \in L^2$ so these should not lie in the image of F. Still we will show in this section that one can show order optimal convergence rates under the VSC^1 if additionally Assumption 1.4.1 holds true. Further we need the following assumption on F and $\mathcal{R}$.

Assumption 4.3.9. *Let $\gamma > 0$ and $p \in (1, 2]$ be as in Assumption 1.4.1.*

(B1) There exists $\widehat{f}_\alpha \in D(F)$ minimizing (3.1).

(B2) For some $a > \gamma$ we have $\|F(f_1) - F(f_2)\|_{B^a_{p,2}} \le L\|f_1 - f_2\|_X$ for all $f_1, f_2 \in D(F)$.

(B3) There exists $C_{\mathcal{R}}, \rho \ge 0$, $r \ge 2$ such that for all $B \ge 1$, $\|f_1\|_{\mathcal{X}}, \|f_2\|_{\mathcal{X}} \le B$, $f^* \in \partial\mathcal{R}(f_2)$ we have $\Delta_{\mathcal{R}}^{f^*}(f_1, f_2) \ge C_{\mathcal{R}} B^{-\rho}\|f_1 - f_2\|^r_{\mathcal{X}}$.

Remark 4.3.10. *(B1) and (B2) are as in Assumption 4.2.1. (B3) holds for norm powers $\mathcal{R} = \frac{1}{t}\|\cdot\|^t$ if $\mathcal{X}$ is r-convex with $\rho = \max(t - r, 0)$ by Theorem 2.2.5 and it holds for $\mathcal{R}(f) = \mathrm{KL}(f, f_0)$ by Proposition 3.2.1 and Lemma 2.3.2 with $r = 2$ and $\rho = 1$.*

Theorem 4.3.11. *Let Assumption 4.3.9 hold true. Let $B \ge 1$ such that $\|\widehat{f}_\alpha\|, \|f^\dagger\| \le B$. If $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ and $f^\dagger$ fulfills $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$ we have*

$$\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \le C \left(\frac{B^{\frac{2\gamma}{ra}\rho}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha^{1+\frac{\gamma}{a}}} \right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + 4(-\Phi)^*\left(-\frac{1}{2\alpha}\right),$$

where C is independent of any quantities on the right side and $f^\dagger$. If $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$, let Assumption 4.3.9 hold true with $p = 2$, $a > d/2$ and further assume $g^\dagger \ge 0, F(\widehat{f}_\alpha) \ge -\sigma/2$. If $f^\dagger$ fulfills $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathrm{KL}^\sigma_{g^\dagger})$ we have

$$\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \le C_{g^\dagger,\sigma} B^{\frac{a-\gamma}{a+\gamma}} \left(B^{\frac{2\gamma}{a}\left(\frac{\rho}{r} + \max(a-1,0)\right)} \alpha^{-1-\frac{\gamma}{a}} \|\varepsilon Z\|^2_{B^{-\gamma}_{2,\infty}} \right)^{\frac{1}{1+\frac{\gamma}{ra}(2-r)}} + (-\Phi)^*\left(-\frac{1}{2\alpha}\right),$$

where $C_{g^\dagger,\sigma}$ depends on $g^\dagger$ and σ but not on any other quantities of the right side.

Proof. By (4.5) from Theorem 4.3.4 we have

$$\frac{1}{2}\Delta_{\mathcal{R}}^{f^*}(\widehat{f}_\alpha, f^\dagger) \le \frac{\mathbf{err}\left(F(\widehat{f}_\alpha)\right)}{\alpha} - \frac{1}{2C_{\mathrm{err}}\alpha}\mathcal{S}^\dagger\left(F(\widehat{f}_\alpha)\right) + (-\Phi)^*\left(-\frac{1}{2C_{\mathrm{err}}\alpha}\right)$$

Let $\mathcal{S} = \mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}$, then by Lemma 4.1.2 we have for all $c_1, c_2 > 0$ that

$$\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big) \leq C_{c_2}\Big(c_1^{-\frac{2\gamma}{ra}}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}\Big)^{\frac{1}{1+\frac{\gamma}{ra}(2-r)}} + c_1\|F(\widehat{f}_\alpha) - g^\dagger\|^r_{B^a_{p,2}} + c_2\|F(\widehat{f}_\alpha) - g^\dagger\|^2_{L^2}.$$

Now use (B2) and choose $c_2 = \frac{1}{4}$ and $c_1 = \frac{\alpha}{4}C_{\mathcal{R}}B^{-\rho}L^{-r}$, then we have by (B3) that

$$\frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq C\Big(B^{\frac{2\gamma}{ra}\rho}\alpha^{-1-\frac{\gamma}{a}}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}\Big)^{\frac{1}{1+\frac{\gamma}{ra}(2-r)}} + \frac{1}{4}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) + (-\Phi)^*\Big(-\frac{1}{2\alpha}\Big).$$

Bringing $\frac{1}{4}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)$ to the other side gives the claim.

If $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ then by Lemma 4.1.2 we have c_1, c_2 that

$$\begin{aligned}\mathbf{err}\Big(F(\widehat{f}_\alpha)\Big) \leq & C_{c_2}\Big(c_1^{-\frac{\gamma}{ra}}\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}\Big)^{\frac{2}{1+\frac{\gamma}{ra}(2-r)}} \\ & + c_1\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^r_{H^a} + c_2\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^2_{L^2}.\end{aligned}$$

By Corollary A.2.12 and (B2) we have

$$\begin{aligned}\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|_{H^a} &\leq C_{g^\dagger,\sigma,a}B^{\max(a-1,0)}\left\|F(\widehat{f}_\alpha) - g^\dagger\right\|_{H^a} \\ &\leq LC_{g^\dagger,\sigma,a}B^{\max(a-1,0)}\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}}.\end{aligned}$$

Further by Corollary A.2.12 and Lemma 2.3.2 we have

$$\left\|\log\left(\frac{F(\widehat{f}_\alpha)+\sigma}{g^\dagger+\sigma}\right)\right\|^2_{L^2} \leq C^2_{g^\dagger,\sigma}\left\|F(\widehat{f}_\alpha) - g^\dagger\right\|^2_{L^2} \leq 2BC^2_{g^\dagger,\sigma}\,\mathrm{KL}(g^\dagger+\sigma, F(\widehat{f}_\alpha)+\sigma).$$

Thus by choosing $c_1 = \frac{\alpha}{4}C_{\mathcal{R}}B^{-\rho}(LC_{g^\dagger,\sigma,a}B^{\max(a-1,0)})^{-r}$ and $c_2 = (4BC^2_{g^\dagger,\sigma})^{-1}$ we find

$$\frac{1}{4}\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq C_{g^\dagger,\sigma}B^{\frac{a-\gamma}{a+\gamma}}\Big(B^{\frac{2\gamma}{a}\left(\frac{\rho}{r}+\max(a-1,0)\right)}\alpha^{-1-\frac{\gamma}{a}}\|\varepsilon Z\|^2_{B^{-\gamma}_{2,\infty}}\Big)^{\frac{1}{1+\frac{\gamma}{ra}(2-r)}} + (-\Phi)^*\Big(-\frac{1}{2\alpha}\Big).$$

□

Corollary 4.3.12. *Let both Assumption 4.2.1 and 4.3.9 hold true. Define* $Q_{\varepsilon,\alpha} := \left(\frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}}$.

(a) If $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ *and* $f^\dagger$ *fulfills* $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$ *we have under the parameter choice* $\alpha \geq C\varepsilon^{\frac{2}{1+\gamma/a}}$ *that*

$$\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq C\Big(\mathcal{R}(f^\dagger)^\rho + \|Z\|^{\frac{2a\rho}{a-\gamma}}_{B^{-\gamma}_{p',\infty}}\Big)(1 + \|Z\|_{B^{-\gamma}_{p',\infty}})^2 Q_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big). \tag{4.11}$$

Let additionally Assumption 1.4.1 hold true and let $c = 2 + \frac{2a\rho}{a-\gamma}$, *then we have under the parameter choice* $\alpha \geq C\varepsilon^{\frac{2}{1+\gamma/a}}$ *that*

$$\begin{aligned}&\mathbb{P}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) > C(1+x)Q_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-(2\alpha)^{-1}\Big)\Big) \leq \exp\Big(-C_Z x^{\frac{\tau}{c}}\Big),\\ &\mathbb{E}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)\Big) \leq CQ_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big),\end{aligned}$$

where the constant $C > 0$ *depends on* $\mathcal{R}(f^\dagger)$ *if* $\rho > 0$.

(b) *Let* $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$, $g^\dagger \geq 0, F(\widehat{f}_\alpha) \geq -\sigma/2$, $f^\dagger$ *fulfill* $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathrm{KL}^\sigma_{g^\dagger})$ *and* $a \geq a_0 > \max(d/2, \gamma)$, *where* $a_0 \in \{1, 2\}$. *Then there exists* $c > 0$ *such that under the parameter choice* $\alpha \geq C\varepsilon^{\frac{2}{1+\gamma/a_0}}$ *we have*

$$\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq C_{f^\dagger,g^\dagger,\sigma}\Big(1 + \|Z\|^c_{B^{-\gamma}_{2,\infty}}\Big)Q_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big).$$

Let additionally Assumption 1.4.1 hold true, then we have under $\alpha \geq C\varepsilon^{\frac{2}{1+\gamma/a_0}}$ *that*

$$\mathbb{P}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) > C(1+x)Q_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big)\Big) \leq \exp\Big(-C_Z x^{\frac{\tau}{c}}\Big)$$
$$\mathbb{E}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)\Big) \leq CQ_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big),$$

where the constant depends on $\mathcal{R}(f^\dagger)$, $g^\dagger$ *and* σ.

Proof. For $\mathcal{S} = \mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}$ we have by Proposition 4.2.3 and the choice $\alpha \geq c\varepsilon^{\frac{2}{1+\gamma/a}}$ that

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq C_{\mathcal{R},\theta}\Big(\mathcal{R}(f^\dagger) + \|Z\|^{\frac{2}{1-\gamma/a}}_{B^{-d/2}_{p',\infty}}\Big).$$

Thus we can choose $B = C_{\mathcal{R},\theta}(\mathcal{R}(f^\dagger) + \|Z\|^{\frac{2}{1-\gamma/a}}_{B^{-d/2}_{p',\infty}})$ in Theorem 4.3.11, so that (4.11) follows by

$$\begin{aligned}\Big(B^{\frac{2\gamma}{ra}\rho}\|Z\|^2_{B^{-\gamma}_{p',\infty}}\Big)^{\frac{1}{1+\frac{\gamma}{ra}(2-r)}} &\leq B^{\frac{2\gamma}{ra}\rho}\max(\|Z\|_{B^{-\gamma}_{p',\infty}}, 1)^2 \leq B^\rho \max(\|Z\|_{B^{-\gamma}_{p',\infty}}, 1)^2 \\ &\leq C\Big(\mathcal{R}(f^\dagger)^\rho + \|Z\|^{\frac{2a\rho}{a-\gamma}}_{B^{-\gamma}_{p',\infty}}\Big)(1 + \|Z\|_{B^{-\gamma}_{p',\infty}})^2\end{aligned}$$

Under Assumption 1.4.1 we have for all $x > 0$ and $c = 2 + \frac{2a\rho}{a-\gamma}$ that

$$\mathbb{P}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) > C(1+x^c)Q_{\varepsilon,\alpha} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big)\Big) \leq \exp(-C_Z x^\tau)$$

so replacing $x = \tilde{x}^{\frac{1}{c}}$ gives the claim. Let p_k denote

$$p_k = \mathbb{P}\Big(C(k+1)^{2+\rho}Q_{\varepsilon,\alpha} < \Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) - 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big) \leq C(k+2)^{2+\rho}Q_{\varepsilon,\alpha}\Big)$$

then

$$\begin{aligned}\mathbb{E}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)\Big) \leq& 4(-\Phi)^*\Big(-(2\alpha)^{-1}\Big) + \sum_{k=0}^\infty p_k C(k+2)^{2+\rho}Q_{\varepsilon,\alpha} \\ &+ \mathbb{P}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) - +4(-\Phi)^*\Big(-(2\alpha)^{-1}\Big) \leq CQ_{\varepsilon,\alpha}\Big)CQ_{\varepsilon,\alpha} \\ &\leq 4(-\Phi)^*\Big(-(2\alpha)^{-1}\Big) + CQ_{\varepsilon,\alpha}\Big(1 + \sum_{k=0}^\infty (k+2)^{2+\rho}\exp\Big(-C_Z k^{\frac{\tau}{\rho+2}}\Big)\Big),\end{aligned}$$

so that the claim follows as the sum is convergent.

For $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ the proof is basically the same. Just note that Proposition 4.2.4 dictates the parameter choice $\alpha \geq c\varepsilon^{\frac{2}{1+\gamma/a_0}}$ and then we have

$$\|\widehat{f}_\alpha\|_{\mathcal{X}} \leq 3\mathcal{R}(f^\dagger) + C_{g^\dagger,\sigma}\|Z\|^{\frac{2}{1-\gamma/a_0}}_{B^{-d/2}_{2,\infty}}.$$

Thus we can choose $B = C(\mathcal{R}(f^\dagger) + \|Z\|^{\frac{2}{1-\gamma/a_0}}_{B^{-d/2}_{p',\infty}})$ in Theorem 4.3.11 but the constant will already depend on $g^\dagger$ and σ. □

To see how the term $4(-\Phi)^*\left(-\frac{1}{2\alpha}\right)$ might behave we have the following lemma. Note that by Proposition 4.3.8 the strongest index function such that $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$ can hold true fulfills $\Phi(\tau) \leq C\tau^{\frac{1}{2}}$.

Lemma 4.3.13. *Let $\Phi(\tau) = \lambda\tau^\nu$ for $\nu \in (0,1)$, $\lambda > 0$ then we have*

$$(-\Phi)^*\left(-\frac{1}{2C_{\mathrm{err}}\alpha}\right) = \nu^{\frac{\nu}{1-\nu}}(1-\nu)\lambda^{\frac{1}{1-\nu}}(2C_{\mathrm{err}}\alpha)^{\frac{\nu}{1-\nu}} = C\alpha^{\frac{\nu}{1-\nu}}$$

Proof. We have for $x \geq 0$ that

$$(-\Phi)^*\left(-\frac{1}{x}\right) = \sup_{t\geq 0}\left[\Phi(t) - \frac{t}{x}\right]$$

as Φ is strictly concave the unique maximum is attained at $t = (\nu\lambda x)^{\frac{1}{1-\nu}}$ which gives

$$(-\Phi)^*\left(-\frac{1}{x}\right) = \lambda^{\frac{1}{1-\nu}}\left(\nu^{\frac{\nu}{1-\nu}}x^{\frac{\nu}{1-\nu}} - \nu^{\frac{1}{1-\nu}}x^{\frac{\nu}{1-\nu}}\right) = \nu^{\frac{\nu}{1-\nu}}(1-\nu)\lambda^{\frac{1}{1-\nu}}x^{\frac{\nu}{1-\nu}}.$$ □

4.4 Second order source conditions

We have seen in Lemma 4.3.5 and Proposition 4.3.8 that the strongest $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^p_{g^\dagger})$, which can be proven for non-trivial $f^\dagger$ and Frechet differentiable $\mathcal{R}$, fulfills $\Phi(\tau) \sim \tau^{\frac{1}{p}}$ and thus corresponds to deterministic convergence rates $\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) = \mathcal{O}(\delta)$. This implies that for quadratic Tikhonov regularization on Hilbert spaces the VSC^1 only covers convergence rates (1.4) with indices $\nu \in (0, 1/2]$, by $\Delta^{f^*}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) = \frac{1}{2}\|\widehat{f}_\alpha - f^\dagger\|^2$. We will call such rates first order convergence rates. Several alternatives to the formulation (4.2) of the source condition suffer from the same limitation: multiplicative variational source conditions [4, 45], approximate source conditions [28], and approximate variational source conditions [28]. Symmetrized version of multiplicative variational source conditions (see [4, eq. (6)] and [2, Ch. 4]) cover a larger range of ν, but have no obvious generalization to Banach space settings or non-quadratic $\mathcal{S}$ or $\mathcal{R}$. The limiting case $\Phi(\tau) = c\tau^{\frac{1}{p}}$ of $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^p_{g^\dagger})$ implies by [61, Prop. 3.38] the source condition

$$\exists \xi^\dagger \in \mathcal{Y}^*: \qquad T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger) \tag{4.12}$$

studied earlier in [11, 22] (and is thus equivalent to this condition by Lemma 4.3.5). To generalize also the Hölder rates (1.4) with $\nu > 1$ to the setting (3.1), one can impose a variational source condition on $\xi^\dagger$ [34], which turns out to be the solution of a Fenchel dual problem. Again the limiting case of this dual source condition, which we tag *second order source condition*, is equivalent to a simpler condition, $T\omega^\dagger \in \partial\mathcal{S}_p^*(\xi^\dagger)$, which was studied earlier in [52, 57, 59]. Hence, the second order source condition corresponds to the indices $\nu \in (1,2]$ in (1.4).

As this new condition strongly depends on the choice of the data fidelity term we have to consider the deterministic and stochastic cases separately.

4.4.1 The deterministic case

In this subsection we only consider the data fidelity term $\mathcal{S}^p_{g^{\mathrm{obs}}}(g) = \mathcal{S}_p(g - g^{\mathrm{obs}}) := \frac{1}{p}\|g - g^{\mathrm{obs}}\|_{\mathcal{Y}}^p$ for $p \in (1,\infty)$. First we give a definition of the second order source condition in Banach spaces based on [34, (4.2)].

Definition 4.4.1 (Variational source condition $\mathrm{VSC}^2(\Phi,\mathcal{R},\mathcal{S}_p)$)**.** *Let Φ be an index function and $\mathcal{R}$ a proper, convex, lower-semicontinuous functional on $\mathcal{X}$. We say that $f^\dagger \in \mathcal{X}$ satisfies the second order variational source condition* $\mathrm{VSC}^2(\Phi,\mathcal{R},\mathcal{S}_p)$ *if there exist $\xi^\dagger \in \mathcal{Y}^*$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$ and $\xi^* \in \partial\mathcal{S}_p^*(\xi^\dagger)$ such that*

$$\forall \xi \in \mathcal{Y}^*: \qquad \left\langle \xi^\dagger - \xi, \xi^* \right\rangle \le \frac{1}{2}\Delta^{\xi^*}_{\mathcal{S}_{p^*}}\left(\xi,\xi^\dagger\right) + \Phi\left(\Delta^{f^\dagger}_{\mathcal{R}^*}\left(T^*\xi, T^*\xi^\dagger\right)\right). \tag{4.13}$$

We have the following slight variant of [34, Theorem 4.4].

Theorem 4.4.2. *Let $\mathcal{Y}$ be p-smooth. Let there exist a minimizer $\widehat{f}_\alpha$ to (3.1). Let $f^\dagger$ fulfill the second order variational source condition with index function Φ. Then there exists $C > 0$ only depending on p and $\mathcal{Y}$ such that*

$$\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{4}\Delta^{\mathrm{sym}}_{\mathcal{S}_p^*}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger) \le C\frac{\delta^p}{\alpha} + \alpha^{p'-1}(-\Phi)^*\left(\frac{-1}{\alpha^{p'-1}}\right).$$

Remark 4.4.3. *The differences to [34, Theorem 4.4] are firstly that we allow the additional term $\frac{1}{2}\Delta^{\xi^*}_{\mathcal{S}_{p^*}}\left(\xi,\xi^\dagger\right)$ in the* VSC^2*, which makes the assumption formally weaker and will be important for the verification later on. Secondly we additionally bound the dual error $\frac{1}{4}\Delta^{\mathrm{sym}}_{\mathcal{S}_p^*}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger)$, which can be of use for example in Lemma 4.5.6 below and lastly our final bound is expressed differently.*

Proof. Strong duality holds by Corollary 3.1.4 as $\widehat{f}_\alpha$ exists and $\mathcal{S}_p$ is continuous everywhere. Thus we have the extremal relations $T^*\widehat{\xi}_\alpha \in \partial\mathcal{R}(\widehat{f}_\alpha)$ and $-\alpha\widehat{\xi}_\alpha \in \partial\mathcal{S}_p(T\widehat{f}_\alpha - g^{\mathrm{obs}})$ or equivalently $T\widehat{f}_\alpha - g^{\mathrm{obs}} \in \partial\mathcal{S}_p^*(-\alpha\widehat{\xi}_\alpha)$ by Corollary 2.1.19. Therefore we have

$$\begin{aligned}
\Delta^{\mathrm{sym}}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) &= \langle T^*\widehat{\xi}_\alpha - T^*\xi^\dagger, \widehat{f}_\alpha - f^\dagger\rangle = \langle \widehat{\xi}_\alpha - \xi^\dagger, T\widehat{f}_\alpha - g^\dagger\rangle \\
&= \langle \widehat{\xi}_\alpha - \xi^\dagger, T\widehat{f}_\alpha - g^{\mathrm{obs}}\rangle + \langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger\rangle \\
&= \langle \widehat{\xi}_\alpha - \xi^\dagger, -\alpha^{p'-1}\xi^*\rangle + \alpha^{p'-1}\langle \widehat{\xi}_\alpha - \xi^\dagger, \frac{T\widehat{f}_\alpha - g^{\mathrm{obs}}}{\alpha^{p'-1}} + \xi^*\rangle + \langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger\rangle \\
&= \alpha^{p'-1}\langle \xi^\dagger - \widehat{\xi}_\alpha, \xi^*\rangle - \alpha^{p'-1}\Delta^{\mathrm{sym}}_{\mathcal{S}_p^*}(\widehat{\xi}_\alpha, \xi^*) + \langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger\rangle,
\end{aligned}$$

where the last equality follows from the fact that $-\frac{T\widehat{f}_\alpha - g^{\mathrm{obs}}}{\alpha^{p'-1}} \in \partial \mathcal{S}_p^*(\widehat{\xi}_\alpha)$ by Proposition 2.2.10. Now we can apply VSC^2 on the first term to find

$$\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) \leq \alpha^{p'-1}\Phi\left(\Delta_{\mathcal{R}^*}^{f^\dagger}\left(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger\right)\right) - \frac{\alpha^{p'-1}}{2}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(\widehat{\xi}_\alpha, \xi^*) + \|\widehat{\xi}_\alpha - \xi^\dagger\|\|g^{\mathrm{obs}} - g^\dagger\|. \tag{4.14}$$

As $\mathcal{Y}$ is p-smooth we have that $\mathcal{Y}^*$ is p'-convex and thus we have by Theorem 2.2.5 that

$$\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(\widehat{\xi}_\alpha, \xi^*) \geq C\|\widehat{\xi}_\alpha - \xi^*\|^{p'}.$$

Consequently by the Peter-Paul inequality (2.7)

$$\|\widehat{\xi}_\alpha - \xi^\dagger\|\|g^{\mathrm{obs}} - g^\dagger\| - \frac{\alpha^{p'-1}}{4}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(\widehat{\xi}_\alpha, \xi^*) \leq C\alpha^{(p'-1)\frac{-p}{p'}}\delta^p = C\frac{\delta^p}{\alpha}.$$

Finally we have $\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) = \Delta_{\mathcal{R}}^{T^*\widehat{\xi}_\alpha}(f^\dagger, \widehat{f}_\alpha)$ by (2.10) and $\alpha^{p'-1}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(\widehat{\xi}_\alpha, \xi^*) = \Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger)$ by (2.17). Thus we can on both sides of (4.14) subtract $\Delta_{\mathcal{R}}^{T^*\widehat{\xi}_\alpha}(f^\dagger, \widehat{f}_\alpha)$ and add $\frac{1}{4}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger)$ to find

$$\begin{aligned}
&\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{4}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger) \\
&\leq C\frac{\delta^p}{\alpha} + \alpha^{p'-1}\Phi\left(\Delta_{\mathcal{R}^*}^{f^\dagger}\left(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger\right)\right) - \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) \\
&\leq C\frac{\delta^p}{\alpha} + \alpha^{p'-1}\sup_{\tau\geq 0}\left[-\frac{\tau}{\alpha^{p'-1}} - (-\Phi)(\tau)\right].
\end{aligned}$$

By the definition of the convex conjugate this gives the claim. □

4.4.2 The stochastic case

In this section we introduce a second order source condition that combines and generalizes the ideas of the second order VSC and the verification approach from Proposition 4.3.6. This condition can be used to show higher order convergence rates for statistical inverse problems.

Assumption 4.4.4 (Assumption on $f^\dagger, T$ and $\mathcal{R}$)**.** *Let $\gamma > 0$ and $p \in [1, 2]$ be as in Assumption 1.4.1.*

(D1) There exists $\widehat{f}_\alpha$ minimizing (3.1).

(D2) Assume that for some $a > \gamma$ we have $\|Tf\|_{B^a_{p,2}} \leq L\|f\|_X$.

(D3) There exists $C_{\mathcal{R}}, \rho > 0$, $r \geq 2$ such that for all $B \geq 1$, $\|f_1\|_{\mathcal{X}}, \|f_2\|_{\mathcal{X}} \leq B$, $f^* \in \partial\mathcal{R}(f_2)$ we have $\Delta_{\mathcal{R}}^{f^*}(f_1, f_2) \geq C_{\mathcal{R}}B^{-\rho}\|f_1 - f_2\|_{\mathcal{X}}^r$.

(D4) Let $\xi^\dagger \in L^2(\mathbb{M})$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$.

Suppose there exist for all $k \in \mathbb{N}_0$ maps $P_k \colon L^2 \to B^\gamma_{p,1}$, as well as constants $\nu_k \geq 1$, $\kappa_k \geq 0$, $\mu \geq 2$,$\beta \in \{0, 1\}$ such that we have

(D5) $\forall \xi \in B^{-\gamma}_{p',\infty} : \langle \xi^\dagger - \xi, P_k\xi^\dagger\rangle \leq \frac{1}{4}\|\xi^\dagger - \xi\|^2_{B^{-\gamma}_{p',\infty}} + \nu_k\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\beta\nu_k^2}{2}\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger).$

(D6) $\|(I-P_k)\xi^\dagger\|_{L^2} = \kappa_k$, *with* $\inf_{k\in\mathbb{N}_0}\kappa_k = 0$.
(D7) $\|P_k\xi^\dagger\|_{H_0^a} \le \nu_k$.

For $\mathcal{S}(g) = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(g)$ *let* $p=2$. *Instead of (D5) we need the following two conditions:*

(D8) *For* $B\ge 1, \sigma>0$ *and* $\mathcal{G}^a_{LB} = \{g\in H^a(\mathbb{M}) : \|g\|_{H^a}\le LB\}$ *assume for all* $\xi\in B^{-\gamma}_{p',\infty}$ *and* $g\in\mathcal{G}^a_{LB}$ *that for* $\nu_k = \nu_k(B,\sigma,|\mathbb{M}|)$ *we have*

$$\langle \xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger\rangle \le \frac{1}{4}\|(g+\sigma)(\xi^\dagger-\xi)\|^2_{B^{-\gamma}_{p',\infty}} + \nu_k\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\beta\nu_k^2}{2}\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger).$$

(D9) *Assume that* $a>d/2$ *and that* $\mathcal{R} = \mathcal{R}+\chi_{\mathcal{B}}$, *with* $Tf\ge -\sigma/2$ *for all* $f\in\mathcal{B}$.

Lemma 4.4.5. *Let* $p\in[1,2]$ *and let conditions (D2), (D3) and (D7) of Assumption 4.4.4 hold true. For* $B\ge 1$ *let* $\|f\|_{\mathcal{X}}, \|f^\dagger\|_{\mathcal{X}}\le B$. *Then we have the following interpolation inequality*

$$\|Tf - g^\dagger + \lambda P_k\xi^\dagger\|_{B^\gamma_{p,1}} \le C\|Tf-g^\dagger+\lambda P_k\xi^\dagger\|^{1-\frac{\gamma}{a}}_{L^2}\left(B^\rho\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(f,f^\dagger) + (\nu_k\lambda)^r\right)^{\frac{\gamma}{ar}} \tag{4.15}$$

for all $f\in\mathcal{X}$ *and* $\lambda\ge 0$. *If* $p=2$ *and (D9) holds true with* $f\in\mathcal{B}$ *then we also have*

$$\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|_{B^\gamma_{2,1}} \le CB^{\gamma(1+\frac{\rho}{ar})}\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|^{1-\frac{\gamma}{a}}_{L^2}\left(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(f,f^\dagger)+(\nu_k\lambda)^r\right)^{\frac{\gamma}{ar}} \tag{4.16}$$

Proof. Same reasoning as in [74, Lemma 4.7]. We only prove (4.16). By (D2) we have $\|Tf+\sigma\|_{H^a}\le BL+\sigma|\mathbb{M}|$, so by Theorem A.2.9 $\|(Tf+\sigma)^{-1}\|_{H^a}\le \sigma^{-\lceil a\rceil}(BL+\sigma|\mathbb{M}|)^a$. Further as $a>d/2$ we have by Theorem A.2.9 and Theorem A.2.8 that

$$\begin{aligned}\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|_{B^\gamma_{2,1}} &\le C\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|^{1-\frac{\gamma}{a}}_{L^2}\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|^{\frac{\gamma}{a}}_{H^a_0}\\ &\le C\left\|\frac{Tf-g^\dagger}{Tf+\sigma}+\lambda P_k\xi^\dagger\right\|^{1-\frac{\gamma}{a}}_{L^2}B^\gamma\left(\|Tf-g^\dagger\|_{H^a_0}+\lambda\|P_k\xi^\dagger\|_{H^a_0}\right)^{\frac{\gamma}{a}}.\end{aligned}$$

The claim follows by (D2), (D3) and (D7). □

Remark 4.4.6. *Let us discuss this abstract condition and relate it to other conditions. Conditions (D2),(D3) as well as the choice* $p\in[1,2]$ *are as in Assumption 4.3.9 and are necessary for the interpolation inequality (4.15). The condition* $T^*\xi^\dagger\in\partial\mathcal{R}(f^\dagger)$ *from (D4) was also necessary for the* VSC^2 *and is equivalent to the first order VSC with best possible index function (see (4.12)). As we intend to improve on the rates shown under this first order VSC it is natural to take this condition into our assumption. To see that* $\xi^\dagger\in L^2(\mathbb{M})$ *is not necessarily an additional constraint, we refer to [64, Corollary 5.3] where it is shown that in a specific setting the strongest first order VSC is equivalent to* $f^\dagger\in H^a_0(\mathbb{M})$ *and thus equivalent to the existence of* $\xi^\dagger\in L^2(\mathbb{M})$.

The inequality (D5) is a generalization of the second order VSC from Definition 4.4.1. As the noise is not in L^2 *almost surely we have to formulate the inequality with respect* *to the space $B^{-\gamma}_{p',\infty}$. However for the dual solution we only guarantee $\xi^\dagger \in L^2$, so we need to introduce the mappings P_k to make the dual products on the left hand side of (D5) well-defined. This approach is very similar to that of the verification strategy for VSCs in Proposition 4.3.6 so one can hope that it will still yield order optimal convergence rates, which in the end will turn out to be true. The inequalities (D6) and (D7) are similar to the assumptions of Proposition 4.3.6 and describe the smoothness of $\xi^\dagger$.*

Inequality (D8) is a variant of (D5) tailored to the case of $\mathcal{S}(g) = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(g)$. This inequality should not be stronger or weaker than (D5) as $g \in \mathcal{G}^a_{LB}$ is smoother than $P_k\xi^\dagger$. (D9) is only necessary for the case of $\mathcal{S}(g) = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(g)$ and similar assumptions were also necessary in [75] to show convergence rates for Poisson data. Different from [75] we assume only $Tf \geq -\sigma/2$ instead of $Tf \geq 0$, which would be the more natural assumption as Tf should be a density and thus positive. This has technical reasons as we will later require some sort of differentiability of $\mathcal{R}$ at $f^\dagger$, which is only possible if $f^\dagger$ lies in the interior of $\mathcal{B}$ and $f^\dagger$ might be equal or close to zero in some parts of $\mathbb{M}$. Later in Remark 5.1.7 we discuss how (D9) can be fulfilled.

With the notation of Assumption 4.4.4 we define the function $\Psi\colon [0,\infty) \to [0,\infty]$ by

$$\Psi(\alpha) = \inf_{\substack{k\in\mathbb{N}_0\\ \alpha\nu_k^2\leq 1}} \left[\alpha^{\mu'-1}\nu_k^{\mu'} + \kappa_k^2\right]. \tag{4.17}$$

The infimum of the empty set is $+\infty$. Note that $\Psi|_{[0,\max_k \nu_k^{-2}]} : [0, \max_k \nu_k^{-2}] \to [0,\infty)$ is a concave index function, i.e. increasing and continuous with $\Psi(0) = 0$. This follows from the fact that it is an infimum over concave, increasing functions, thus concave and increasing, so in particular continuous and by (D6) we have $\Psi(0) = 0$. The function Ψ will describe the speed of convergence in the following theorems.

First we consider $\mathcal{S}(g) = \frac{1}{2\alpha}\|g\|^2_{L^2} - \left\langle g^\dagger + \varepsilon Z, g\right\rangle$. We will show an upper bound on the error in the Bregman divergence in terms of ε, α and $\|Z\|_{B^{-\gamma}_{p',\infty}}$. Assumption 1.4.1 then immediately yields convergence rates in expectation and a deviation inequality on the reconstruction error.

Theorem 4.4.7. *Suppose (D1)-(D7) in Assumption 4.4.4 hold true. Let $B \geq 1$ such that $\|\widehat{f}_\alpha\|, \|f^\dagger\| \leq B$. Then we have for Ψ as in (4.17) and some constant $C > 0$ independent of Z, B, ε and α that*

$$C\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) \leq \left(\frac{B^{\frac{2\gamma\rho}{ar}}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + \frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha} + \alpha\Psi(\alpha). \tag{4.18}$$

Remark 4.4.8. *Note that for a fixed realization of the random variable $\|Z\|_{B^{-\gamma}_{p',\infty}}$ and $\varepsilon, \alpha \to 0$ we have*

$$\frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha} = \mathcal{O}\left(\left(\frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}}\right),$$

as $\gamma > 0$ and $r \geq 2$. Hence only the first term on the right side of (4.18) is important for the rate of convergence. But to formulate the error bound uniformly with a constant C independent of Z, in the case $r > 2$, we have to include also the other term. By definition of Ψ the above result only gives a useful upper bound if α is sufficiently small. However, Theorem 4.4.7 is most interesting in the asymptotic case, where $\varepsilon, \alpha \to 0$. Even if one is interested in a bound that holds for all α one could use the fact that by (D4) the first order VSC holds true, so Theorem 4.3.4 gives a bound which is even superior to (4.18) for large α.

Proof. As in the proof of Theorem 4.4.2 we begin by estimating the symmetric Bregman divergence. The extremal relations (3.5), (3.6) yield by (3.12) and by the decomposition (3.13) that

$$\begin{aligned}
\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) &= \langle T^*\widehat{\xi}_\alpha - T^*\xi^\dagger, \widehat{f}_\alpha - g^\dagger\rangle \\
&= \langle \widehat{\xi}_\alpha - \xi^\dagger, T\widehat{f}_\alpha - g^\dagger\rangle \\
&= \langle \widehat{\xi}_{\alpha,L^2} - \xi^\dagger, -\alpha\widehat{\xi}_{\alpha,L^2}\rangle + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle \\
&= \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \xi^\dagger\rangle - \alpha\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|_{L^2}^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle \\
&= \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, P_k\xi^\dagger\rangle + \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, (I - P_k)\xi^\dagger\rangle \\
&\quad - \alpha\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|_{L^2}^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle.
\end{aligned}$$

By

$$\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, (I - P_k)\xi^\dagger\rangle \leq \frac{1}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|_{L^2}^2 + \|(I - P_k)\xi^\dagger\|_{L^2}^2$$

we find

$$\begin{aligned}
\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) &\leq \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, P_k\xi^\dagger\rangle - \frac{3\alpha}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|_{L^2}^2 + \alpha\kappa_k^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle \\
&= \alpha\langle \xi^\dagger - \widehat{\xi}_\alpha, P_k\xi^\dagger\rangle - \frac{3\alpha}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|_{L^2}^2 + \alpha\kappa_k^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle.
\end{aligned}$$

The first term can be bounded by (D5), so that

$$\begin{aligned}
\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) &\leq \alpha\nu_k\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\alpha\beta\nu_k^2}{2}\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) \\
&+ \frac{\alpha}{4}\|\xi^\dagger - \widehat{\xi}_\alpha\|_{B^{-\gamma}_{p',\infty}}^2 - \frac{3\alpha}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|^2 + \alpha\kappa_k^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle.
\end{aligned} \tag{4.19}$$

Now note that

$$\begin{aligned}
\frac{\alpha}{4}\|\xi^\dagger - \widehat{\xi}_\alpha\|_{B^{-\gamma}_{p',\infty}}^2 &\leq \frac{\alpha}{2}\|\widehat{\xi}_{\alpha,Z}\|_{B^{-\gamma}_{p',\infty}}^2 + \frac{\alpha}{2}\|\xi^\dagger - \widehat{\xi}_{\alpha,L^2}\|_{B^{-\gamma}_{p',\infty}}^2 \\
&\leq \frac{1}{2\alpha}\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}}^2 + \frac{\alpha}{2}\|\xi^\dagger - \widehat{\xi}_{\alpha,L^2}\|_{L^2}^2
\end{aligned}$$

as well as $\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) = \Delta_{\mathcal{R}}^{T^*\widehat{\xi}_\alpha}(f^\dagger, \widehat{f}_\alpha)$ and by Young's inequality

$$\alpha\nu_k\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger)^{\frac{1}{\mu}} \leq \frac{1}{2}\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) + C(\alpha\nu_k)^{\mu'}. \tag{4.20}$$

From now on we assume that k is chosen such that $\alpha\beta\nu_k^2 \leq 1$ (this assumption is included in the function Ψ) so we can subtract $\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger)$ on both sides of (4.19) and together with (4.20) we find

$$\begin{aligned}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) \leq & C(\alpha\nu_k)^{\mu'} + \frac{1}{2\alpha}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}} + \alpha\kappa_k^2 \\ & + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle - \frac{\alpha}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|^2_{L^2}.\end{aligned} \tag{4.21}$$

The only remaining task of the proof is to bound $\langle \varepsilon Z, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle$ which can be seen as a version of the effective noise level $\mathbf{err}(T\widehat{f}_\alpha)$, which was already treated in Lemma 4.1.3 in a similar way as below. By (4.15) we have

$$\begin{aligned}&\frac{1}{\alpha}\langle \varepsilon Z, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle \\ \leq & \frac{1}{\alpha}\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}} \|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|_{B^{\gamma}_{p,1}} \\ \leq & \frac{C}{\alpha}\|\varepsilon Z\|_{B^{-\gamma}_{p',\infty}} \|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|_{L^2}^{1-\frac{\gamma}{a}} (B^\rho \Delta_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) + (\nu_k\alpha)^r)^{\frac{\gamma}{ra}} \\ \leq & C\Big(B^{\frac{\gamma\rho}{ar}}\varepsilon\alpha^{-1}\|Z\|_{B^{-\gamma}_{p',\infty}} \|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|_{L^2}^{1-\frac{\gamma}{a}}\Big)^{\frac{1}{1-\frac{\gamma}{ra}}} + \frac{1}{2}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{2}(\nu_k\alpha)^r.\end{aligned}$$

Notice that from (4.21) we still can take advantage of the term

$$\begin{aligned}-\frac{\alpha}{4}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|^2_{L^2} &\leq -\frac{\alpha}{8}\|\widehat{\xi}_{\alpha,L^2} - P_k\xi^\dagger\|^2_{L^2} + \frac{\alpha}{4}\|(I-P_k)\xi^\dagger\|^2_{L^2} \\ &= -\frac{1}{8\alpha}\|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|^2_{L^2} + \frac{\alpha}{4}\kappa_k^2,\end{aligned} \tag{4.22}$$

so that finally by Young's inequality with $q = \frac{2(1-\frac{\gamma}{ra})}{1-\frac{\gamma}{a}}$ and $q' = \frac{2(1-\frac{\gamma}{ra})}{1+\frac{\gamma}{a}-\frac{2\gamma}{ra}}$ we have

$$\begin{aligned}&\frac{1}{\alpha}\langle \varepsilon Z, T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\rangle - \frac{1}{8\alpha}\|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|^2_{L^2} \\ &\leq C\left(\frac{\|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|_{L^2}^{1-\frac{\gamma}{a}}}{\alpha^{\frac{1}{2}-\frac{d}{4a}}}\right)^{\frac{1}{1-\frac{\gamma}{ra}}}\left(\frac{B^{\frac{\gamma\rho}{ar}}\varepsilon\|Z\|_{B^{-\gamma}_{p',\infty}}}{\alpha^{\frac{1}{2}+\frac{d}{4a}}}\right)^{\frac{1}{1-\frac{\gamma}{ra}}} \\ &\quad - \frac{1}{8\alpha}\|T\widehat{f}_\alpha - g^\dagger + \alpha P_k\xi^\dagger\|^2_{L^2} + \frac{1}{2}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{2}(\nu_k\alpha)^r \\ &\leq C\left(B^{\frac{2\gamma\rho}{ar}}\frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + \frac{1}{2}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{2}(\nu_k\alpha)^r.\end{aligned}$$

Combining this with (4.21) and (4.22) we find

$$C\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) \leq \frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha} + \left(\frac{B^{\frac{2\gamma\rho}{ar}}\|\varepsilon Z\|^2_{B^{-\gamma}_{p',\infty}}}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + \alpha\kappa_k^2 + \max\Big((\alpha\nu_k)^{\mu'}, (\alpha\nu_k)^r\Big).$$

Taking the infimum over $k \in \mathbb{N}_0$ such that $\alpha\nu_k^2 \leq 1$ gives the claim as $r \geq 2$ and $\mu' \leq 2$, thus $(\alpha\nu_k)^{\mu'} \geq (\alpha\nu_k)^r$.

□

Now we consider $\mathcal{S}(g) = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}(g)$ and prove an analogous upper bound to Theorem 4.4.7.

Theorem 4.4.9. *Let Assumption 4.4.4 hold true with $p = 2$. Let $B \geq 1$ such that $\|\widehat{f}_\alpha\|, \|f^\dagger\| \leq B$. Then we have for Ψ as in (4.17) and some constant $C > 0$ independent of Z, B, ε and α that*

$$C\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) \leq \left(\frac{B^{2\gamma(1+\frac{\rho}{ar})}\|\varepsilon Z\|^2_{B^{-\gamma}_{2,\infty}}}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + \frac{\|\varepsilon Z\|^2_{B^{-\gamma}_{2,\infty}}}{\alpha} + \alpha\Psi(\alpha).$$

Proof. The proof is actually very similar to the previous proof. Basically the only difference is the additional factor $\frac{1}{T\widehat{f}_\alpha+\sigma}$ in $\widehat{\xi}_\alpha$. Still this causes some difficulties so for the sake of completeness we give the proof. We introduce the short notation $\hat{g}_\sigma = T\widehat{f}_\alpha + \sigma$. As before we begin by estimating the symmetric Bregman divergence. The extremal relations (3.5), (3.6) yield by (3.16) and the decomposition (3.17) that

$$\begin{aligned}
\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) &= \langle \widehat{\xi}_\alpha - \xi^\dagger, T\widehat{f}_\alpha - g^\dagger\rangle\\
&= \langle \widehat{\xi}_{\alpha,L^2} - \xi^\dagger, -\alpha\hat{g}_\sigma\widehat{\xi}_{\alpha,L^2}\rangle + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle\\
&= \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \hat{g}_\sigma\xi^\dagger\rangle - \alpha\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2} + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle\\
&= \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \hat{g}_\sigma P_k\xi^\dagger\rangle - \alpha\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2} + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle\\
&\quad + \alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \hat{g}_\sigma(I - P_k)\xi^\dagger\rangle
\end{aligned}$$

The Peter-Paul inequality gives

$$\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \hat{g}_\sigma(I - P_k)\xi^\dagger\rangle \leq \frac{1}{4}\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2} + \|(I - P_k)\xi^\dagger\|^2_{L^2}$$

and we have

$$\begin{aligned}
\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) \leq &\alpha\langle \xi^\dagger - \widehat{\xi}_{\alpha,L^2}, \hat{g}_\sigma P_k\xi^\dagger\rangle - \frac{3\alpha}{4}\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2} + \alpha\kappa_k^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger\rangle\\
=&\alpha\langle \xi^\dagger - \widehat{\xi}_\alpha, (T\widehat{f}_\alpha + \sigma)P_k\xi^\dagger\rangle - \frac{3\alpha}{4}\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2}\\
&+ \alpha\kappa_k^2 + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha\hat{g}_\sigma P_k\xi^\dagger\rangle.
\end{aligned}$$

The first term can be bounded by (D8), so that

$$\begin{aligned}
&\Delta_{\mathcal{R}}^{\mathrm{sym}}(\widehat{f}_\alpha, f^\dagger) \leq \alpha\nu_k\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\alpha\beta\nu_k^2}{2}\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\widehat{\xi}_\alpha, T^*\xi^\dagger) + \alpha\kappa_k^2\\
&+ \frac{\alpha}{4}\|\hat{g}_\sigma(\xi^\dagger - \widehat{\xi}_\alpha)\|^2_{B^{-\gamma}_{2,\infty}} - \frac{3\alpha}{4}\|\hat{g}_\sigma(\widehat{\xi}_{\alpha,L^2} - \xi^\dagger)\|^2_{L^2} + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha\hat{g}_\sigma P_k\xi^\dagger\rangle.
\end{aligned} \tag{4.23}$$

By the same reasoning as in the proof of the last theorem we have

$$\frac{\alpha}{4}\|\hat{g}_\sigma(\xi^\dagger - \widehat{\xi}_\alpha)\|^2_{B^{-\gamma}_{p',\infty}} \leq \frac{1}{4\alpha}\|t^{-\frac{1}{2}}Z\|^2_{B^{-\gamma}_{p',\infty}} + \frac{\alpha}{2}\|\hat{g}_\sigma(\xi^\dagger - \widehat{\xi}_{\alpha,L^2})\|^2_{L^2}$$

and as from (4.19) to (4.21) we find for k with $\alpha\beta\nu_k^2 \le 1$ that

$$\begin{aligned}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) \le & C(\alpha\nu_k)^{\mu'} + \frac{1}{4}\|t^{-\frac{1}{2}}Z\|^2_{B^{-\gamma}_{p',\infty}} + \alpha\kappa_k^2 \\ & + \langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha\hat{g}_\sigma P_k\xi^\dagger\rangle - \frac{\sigma\alpha}{8}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|^2_{L^2}.\end{aligned} \tag{4.24}$$

Bounding

$$\langle \widehat{\xi}_{\alpha,Z}, T\widehat{f}_\alpha - g^\dagger + \alpha\hat{g}_\sigma P_k\xi^\dagger\rangle = \frac{1}{\alpha}\langle \varepsilon Z, (T\widehat{f}_\alpha - g^\dagger)/\hat{g}_\sigma + \alpha P_k\xi^\dagger\rangle$$

is analogous to the last proof. By the interpolation inequality (4.16) we have

$$\begin{aligned}&\frac{1}{\alpha}\langle \varepsilon Z, (T\widehat{f}_\alpha - g^\dagger)/\hat{g}_\sigma + \alpha P_k\xi^\dagger\rangle \\ \le & \frac{1}{\alpha}\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}\left\|\frac{T\widehat{f}_\alpha - g^\dagger}{Tf+\sigma} + \alpha P_k\xi^\dagger\right\|_{B^{\gamma}_{2,1}} \\ \le & \frac{C}{\alpha}B^{\gamma(1+\frac{\rho}{ar})}\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}\left\|\frac{T\widehat{f}_\alpha - g^\dagger}{Tf+\sigma} + \alpha P_k\xi^\dagger\right\|_{L^2}^{1-\frac{\gamma}{a}}(\Delta_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) + (\nu_k\alpha)^r)^{\frac{\gamma}{ra}} \\ \le & C\left(\alpha^{-\frac{1}{2}-\frac{d}{4a}}B^{\gamma(1+\frac{\rho}{ar})}\|\varepsilon Z\|_{B^{-\gamma}_{2,\infty}}\|\sqrt{\alpha}(P_k\xi^\dagger - \widehat{\xi}_{\alpha,L^2})\|_{L^2}^{1-\frac{\gamma}{a}}\right)^{\frac{1}{1-\frac{\gamma}{ra}}} \\ & + \frac{1}{2}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{2}(\nu_k\alpha)^r.\end{aligned}$$

Notice that from (4.24) we still have

$$-\frac{\sigma\alpha}{8}\|\widehat{\xi}_{\alpha,L^2} - \xi^\dagger\|^2_{L^2} \le -\frac{\sigma\alpha}{16}\|\widehat{\xi}_{\alpha,L^2} - P_k\xi^\dagger\|^2_{L^2} + \frac{\sigma\alpha}{8}\|(I-P_k)\xi^\dagger\|^2_{L^2} \tag{4.25}$$

left to work with. By Young's inequality with $q = \frac{2(1-\frac{\gamma}{ra})}{1-\frac{\gamma}{a}}$ and $q' = \frac{2(1-\frac{\gamma}{ra})}{1+\frac{\gamma}{a}-\frac{2\gamma}{ra}}$ we obtain

$$\begin{aligned}&\frac{1}{\alpha}\langle \varepsilon Z, T\widehat{f}_\alpha - g^\dagger + \alpha P_k g_\sigma^\dagger\xi^\dagger\rangle - \frac{\sigma\alpha}{16}\|\widehat{\xi}_{\alpha,L^2} - P_k\xi^\dagger\|^2_{L^2} \\ \le\ & C\left(\frac{B^{2\gamma(1+\frac{\rho}{ar})}\|Z\|^2_{B^{-\gamma}_{2,\infty}}}{\alpha^{1+\frac{\gamma}{a}}t}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}} + \frac{1}{2}\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) + \frac{1}{2}(\nu_k\alpha)^r,\end{aligned}$$

which together with (4.24) and (4.25) gives the claim as in the last proof. □

Corollary 4.4.10. *Let Assumptions 1.4.1, 4.2.1 and 4.4.4 hold true and let* Ψ *be the function from* (4.17). *Define* $Q_{\varepsilon,\alpha} := \left(\frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}}\right)^{\frac{1}{1+\frac{\gamma}{ra}(r-2)}}$.

(a) If $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ *and* $c = 2 + \frac{2a\rho}{a-\gamma}$, *then we have under a parameter choice* $\alpha \ge C\varepsilon^{\frac{2}{1+\gamma/a}}$ *that*

$$\begin{aligned}&\mathbb{P}\Big(\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) > C(1+x)Q_{\varepsilon,\alpha} + \alpha\Psi(\alpha)\Big) \le \exp\Big(-C_Z x^{\frac{\tau}{c}}\Big), \\ &\mathbb{E}\Big(\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger)\Big) \le CQ_{\varepsilon,\alpha} + \alpha\Psi(\alpha),\end{aligned}$$

where the constant $C > 0$ *depends on* $\mathcal{R}(f^\dagger)$ *if* $\rho > 0$.

(b) Let $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ *and* $a \geq a_0 \geq \max(d/2, \gamma)$*, where* $a_0 \in \{1, 2\}$*. Under the parameter choice* $\alpha \geq C\varepsilon^{\frac{2}{1+\gamma/a_0}}$ *we have*

$$\mathbb{P}\Big(\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger) > C(1+x)Q_{\varepsilon,\alpha} + \alpha\Psi(\alpha)\Big) \leq \exp\Big(-C_Z x^{\frac{\tau}{c}}\Big)$$
$$\mathbb{E}\Big(\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(\widehat{f}_\alpha, f^\dagger)\Big) \leq CQ_{\varepsilon,\alpha} + \alpha\Psi(\alpha),$$

where the constant depends on $\mathcal{R}(f^\dagger)$*,* $g^\dagger$ *and* σ*.*

Proof. Just as the proof of Corollary 4.3.12. □

4.5 Higher order variational source conditions[1]

The aim of this section is to derive rates of convergence corresponding to indices $\nu > 2$ in (1.4), i.e. faster than $\|\hat{f}_\alpha - f^\dagger\| = \mathcal{O}(\delta^{2/3})$ in a Banach space setting. By the well-known saturation effect for Tikhonov regularization [35] such rates can occur in quadratic Tikhonov regularization only for $f^\dagger = 0$. Therefore, we consider Bregman iterated Tikhonov regularization as in Section 3.3. In Hilbert spaces we will obtain a full generalization of Hölder source conditions by introducing variational source conditions VSC^n for $n \in \mathbb{N}$. In Banach spaces things are more complicated but we still show that Bregman iteration can yield better convergence rates than non-iterated generalized Tikhonov regularization by introducing a $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$.

4.5.1 Hilbert spaces

First we consider the Hilbert space setting, where $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces and $\mathcal{R}(f) := \frac{1}{2}\|f\|^2_{\mathcal{X}}$, $\mathcal{S}_p(g) = \frac{1}{2}\|g\|^2_{\mathcal{Y}}$, to prove rates (1.4) for all $\nu > 0$ using variational source conditions, which are defined as follows:

Definition 4.5.1 (Variational source condition $\mathrm{VSC}^l(f^\dagger, \Phi)$)**.** *Let* $f^\dagger \in \mathcal{X}$*, let* Φ *be a concave index function, and let* $n \in \mathbb{N}$*. Then the statement*

$$\begin{aligned} &\exists \omega^\dagger_{n-1} \in \mathcal{X} \,:\, f^\dagger = (T^*T)^{n-1}\omega^\dagger_{n-1} \\ \wedge \quad &\forall f \in \mathcal{X} \,:\, \left\langle \omega^\dagger_{n-1}, f \right\rangle \leq \frac{1}{2}\|f\|^2 + \Phi\left(\|Tf\|^2\right) \end{aligned} \tag{4.26}$$

will be abbreviated by $\mathrm{VSC}^{2n-1}(f^\dagger, \Phi)$*, and the statement*

$$\begin{aligned} &\exists \xi^\dagger_n \in \mathcal{Y} \,:\, f^\dagger = (T^*T)^{n-1}T^*\xi^\dagger_n \\ \wedge \quad &\forall \xi \in \mathcal{Y} \,:\, \left\langle \xi, \xi^\dagger_n \right\rangle \leq \frac{1}{2}\|\xi\|^2 + \Phi\left(\|T^*\xi\|^2\right) \end{aligned} \tag{4.27}$$

will be abbreviated by $\mathrm{VSC}^{2n}(f^\dagger, \Phi)$*.* $\mathrm{VSC}^l(f^\dagger, \Phi)$ *for* $l \in \mathbb{N}$ *will be referred to as* variational source condition of order l with index function Φ *for (the true solution)* $f^\dagger$*.*

[1] This section is mostly taken literally from [64] to which the author contributed.

Note that that $\mathrm{VSC}^1(f^\dagger,\Phi)$ is the classical variational source condition, and $\mathrm{VSC}^2(f^\dagger,\Phi)$ coincides with the source condition from [34] up to the term $\frac{1}{2}\|\xi\|^2$, which implies that $\mathrm{VSC}^2(f^\dagger,\Phi)$ is formally weaker than the condition in [34]. It is well known that the spectral Hölder source conditions (1.3) with $\nu\in(0,1]$ imply $\mathrm{VSC}^1(f^\dagger, A\,\mathrm{id}^{\nu/(\nu+1)})$ for some $A>0$ (see [38]). Therefore, it is easy to see that for any $l\in\mathbb{N}$ and $\nu\in[0,1]$ the implication

$$f^\dagger\in\operatorname{ran}\left((T^*T)^{\frac{l-1+\nu}{2}}\right) \quad\Rightarrow\quad \exists A>0\,:\,\mathrm{VSC}^l\left(f^\dagger, A\,\mathrm{id}^{\frac{\nu}{\nu+1}}\right) \tag{4.28}$$

holds true. The converse implication is false for $\nu\in(0,1)$ as discussed in Section 5.4. For $\nu=1$ we have by [61, Prop. 3.35] that

$$f^\dagger\in\operatorname{ran}\left((T^*T)^{\frac{l}{2}}\right) \quad\Leftrightarrow\quad \exists A>0\,:\,\mathrm{VSC}^l\left(f^\dagger, A\sqrt{\cdot}\right). \tag{4.29}$$

The aim of this section is to prove error bounds for iterated Tikhonov regularization based on these source conditions:

Theorem 4.5.2. *Let $l,m\in\mathbb{N}$ with $m\geq l/2$, let Φ be an index function, and let $\psi(s):=\sup_{t\geq0}[st+\Phi(s)]$ denote the Fenchel conjugate of $-\Phi$. If $\mathrm{VSC}^l(f^\dagger,\Phi)$ holds true, there exists a constant $C>0$ depending only on l and m such that*

$$\|\hat f_\alpha^{(m)}-f^\dagger\|^2\leq C\left(\frac{\delta^2}{\alpha}+\alpha^{l-1}\psi\left(-\frac{1}{\alpha}\right)\right) \qquad \text{for all } \alpha,\delta>0. \tag{4.30}$$

Proof. We choose $n\in\mathbb{N}$ such that $l=2n$ or $l=2n-1$. Then $m\geq n$. In the following C will denote a generic constant depending only on m and l. The proof proceeds in four steps.
Step 1: Reduction to the case $m=n$. By Proposition 3.3.2 and the definition of the Bregman distance we have for all $k\geq2$ and $f\in\mathcal{X}$ that

$$\Delta_{\mathcal{R}_k}(\hat f_\alpha^{(k)},f)=\mathcal{R}_k(\hat f_\alpha^{(k)})-\mathcal{R}_k(f)-\left\langle f-\sum_{j=1}^{k-1}T^*\hat\xi_\alpha^{(j)},\hat f_\alpha^{(k)}-f\right\rangle.$$

By the optimality condition $\sum_{j=1}^{k-1}T^*\hat\xi_\alpha^{(j)}=\hat f_\alpha^{(k-1)}$ and the minimizing property of $\hat f_\alpha^{(k)}$ (3.19) we have

$$\begin{aligned}\frac{1}{2}\|\hat f_\alpha^{(k)}-f\|^2=\Delta_{\mathcal{R}_k}(\hat f_\alpha^{(k)},f)\leq&\frac{1}{2\alpha}\left(\|Tf-g^{\mathrm{obs}}\|^2-\|T\hat f_\alpha^{(k)}-g^{\mathrm{obs}}\|^2\right)\\&-\left\langle f-\hat f_\alpha^{(k-1)},\hat f_\alpha^{(k)}-f\right\rangle.\end{aligned} \tag{4.31}$$

Choosing $f=f^\dagger$ gives

$$\frac{1}{2}\|\hat f_\alpha^{(k)}-f^\dagger\|^2\leq\frac{1}{2\alpha}\|g^\dagger-g^{\mathrm{obs}}\|^2+\|\hat f_\alpha^{(k-1)}-f^\dagger\|\|\hat f_\alpha^{(k)}-f^\dagger\|.$$

Multiplying by four, subtracting $\|\hat f_\alpha^{(k)}-f^\dagger\|^2$ on both sides and completing the square we get

$$\|\hat f_\alpha^{(k)}-f^\dagger\|^2\leq\frac{2\delta^2}{\alpha}+4\|\hat f_\alpha^{(k-1)}-f^\dagger\|^2.$$

So it is enough to prove (4.30) for $m = n$ as this will then also imply the claimed error bound for all $m \geq n$ by the above inequality.

Step 2: Error decomposition based on Lemma 3.3.4. Both Assumptions (4.26) and (4.27) imply that there exist

$$\xi_1^\dagger, \ldots, \xi_{n-1}^\dagger \in \mathcal{Y}, \omega_1^\dagger, \ldots, \omega_{n-1}^\dagger \in \mathcal{X}$$

such that $f^\dagger = (T^*T)^{j-1}T^*\xi_j^\dagger, f^\dagger = (T^*T)^j\omega_j^\dagger$ for $j = 1, \ldots, n-1$. In the following we will write $\xi_1^\dagger = \xi^\dagger$ and $\omega_1^\dagger = \omega^\dagger$. We have $\partial\mathcal{R}(f^\dagger) = \{f^\dagger\} = \{T^*\xi^\dagger\}$, so Lemma 3.3.4 yields

$$\begin{aligned}\|\hat{f}_\alpha^{(n)} - f^\dagger\|^2 &\leq \frac{1}{\alpha}\left(\|Tf - g^{\text{obs}}\|^2 + 2\alpha\left\langle s_\alpha^{(n)}, Tf - g^{\text{obs}}\right\rangle + \|-\alpha s_\alpha^{(n)}\|^2\right) \\ &\quad + \|f - f^\dagger\|^2 \\ &= \frac{1}{\alpha}\|Tf - g^{\text{obs}} + \alpha\left(\xi^\dagger - \sum_{k=1}^{n-1}\hat{\xi}_\alpha^{(k)}\right)\|^2 + \|f - f^\dagger\|^2\end{aligned} \tag{4.32}$$

for $s_\alpha^{(n)} = \xi^\dagger - \sum_{k=1}^{n-1}\hat{\xi}_\alpha^{(k)}$ and all $f \in \mathcal{X}$.

We will choose $f = nf^\dagger - \alpha\omega^\dagger - \sum_{k=1}^{n-1}\hat{f}_\alpha^{(k)}$. As $T\omega^\dagger = \xi^\dagger$ and $T\hat{f}_\alpha^{(k)} - g^{\text{obs}} = -\alpha\hat{\xi}_\alpha^{(k)}$ by strong duality, we have

$$\|\hat{f}_\alpha^{(n)} - f^\dagger\|^2 \leq \frac{1}{\alpha}\|n\left(g^\dagger - g^{\text{obs}}\right)\|^2 + \|(n-1)f^\dagger - \alpha\omega^\dagger - \sum_{k=1}^{n-1}\hat{f}_\alpha^{(k)}\|^2. \tag{4.33}$$

It remains to bound the second term, which does not look favourable at first sight as we know that $\|\hat{f}_\alpha^{(k)} - f^\dagger\|$ should converge to zero slower than $\|\hat{f}_\alpha^{(n)} - f^\dagger\|$ for $k < n$. But it turns out that we have cancellation between the different $\hat{f}_\alpha^{(k)}$. Therefore, we will now introduce vectors $\sigma_k \in \mathcal{X}$ such that

$$\|(n-1)f^\dagger - \alpha\omega^\dagger - \sum_{k=1}^{n-1}\hat{f}_\alpha^{(k)}\| \leq \sum_{k=1}^{n-1}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\| \tag{4.34}$$

and then prove that all terms on the right hand side are of optimal order.

Let $(b_{k,j}) \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ denote the matrix given by Pascal's triangle,

$$(b_{k,j}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ 1 & 5 & 15 & 35 & 70 & 126 & \cdots \\ 1 & 6 & 21 & 56 & 126 & 252 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or equivalently $b_{k,j} = \binom{k+j-2}{j-1}$ for all $k, j \in \mathbb{N}$. We will need the identities

$$\sum_{k+j=n}(-1)^j b_{k,j} = -\delta_{n-2,0} \qquad \text{for all } n \geq 2, \tag{4.35}$$

which are equivalent to $(1-1)^{n-2} = \delta_{n-2,0}$ by the binomial theorem

$$(a+b)^n = \sum_{k+j=n+2} b_{k,j} a^{k-1} b^{j-1}.$$

Moreover, we need the defining property of the triangle,

$$b_{k,j} + b_{k-1,j+1} = b_{k,j+1}. \tag{4.36}$$

Using (4.35) we can add zero in the form

$$0 = \alpha\omega^\dagger + \sum_{l=1}^{n-1} \alpha^l \omega_l^\dagger \sum_{k+j=l+1} (-1)^j b_{k,j} = \alpha\omega^\dagger + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} (-1)^j b_{k,j} \alpha^{k+j-1} \omega_{k+j-1}^\dagger$$

to find that

$$(n-1)f^\dagger - \alpha\omega^\dagger - \sum_{k=1}^{n-1} \hat{f}_\alpha^{(k)} = \sum_{k=1}^{n-1} \left(f^\dagger - \hat{f}_\alpha^{(k)} + \sum_{j=1}^{n-k} (-1)^j b_{k,j} \alpha^{k+j-1} \omega_{k+j-1}^\dagger \right)$$

and by the triangle inequality this yields (4.34) with

$$\sigma_k := \sum_{j=1}^{n-k} (-1)^j b_{k,j} \alpha^{k+j-1} \omega_{k+j-1}^\dagger, \qquad k \in \mathbb{N}.$$

It will be convenient to set $\sigma_0 := -f^\dagger$ and $\hat{f}_\alpha^{(0)} := 0$.

Step 3: proof of (4.30) *for the case* $l = 2n-1$. In view of (4.33) and (4.34) it suffices to prove by induction that given $\mathrm{VSC}^{2n-1}(f^\dagger, \Phi)$ (4.26) we have

$$\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \le C \left(\frac{\delta^2}{\alpha} + \alpha^{2n-2} \psi\left(\frac{-1}{\alpha} \right) \right), \quad k = 0, 1, \ldots, n-1. \tag{4.37}$$

For $k = 0$ this is trivial. Assume now that (4.37) holds true for $k-1$ with $k \in \{1, \ldots, n-1\}$. Insert $f = f^\dagger + \sigma_k$ in (4.31) to get

$$\begin{aligned} \|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \le& \frac{1}{\alpha} \left(\|g^\dagger + T\sigma_k - g^{\mathrm{obs}}\|^2 - \|T\hat{f}_\alpha^{(k)} - g^{\mathrm{obs}}\|^2 \right) \\ & - 2 \left\langle f^\dagger + \sigma_k - \hat{f}_\alpha^{(k-1)}, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle. \end{aligned}$$

Now we add and subtract $\hat{f}_\alpha^{(k-1)} - f^\dagger - \sigma_{k-1}$ to the first term of the inner product to find

$$\begin{aligned} \|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \le& \frac{1}{\alpha} \left(\|g^\dagger + T\sigma_k - g^{\mathrm{obs}}\|^2 - \|T\hat{f}_\alpha^{(k)} - g^{\mathrm{obs}}\|^2 \right) \\ & - 2 \left\langle \sigma_k - \sigma_{k-1}, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle \\ & + 2\|\hat{f}_\alpha^{(k-1)} - f^\dagger - \sigma_{k-1}\| \|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\| \end{aligned}$$

The last term, denoted by

$$E := 2\|\hat{f}_\alpha^{(k-1)} - f^\dagger - \sigma_{k-1}\| \|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|,$$

will be dealt with at the end of this step. Because of the identity $T^*T\omega^{(l)} = \omega^{(l-1)}$ and (4.36) we have

$$\begin{aligned}\sigma_k - \sigma_{k-1} &= \alpha^{k-1}\omega_{k-1}^\dagger + \sum_{j=1}^{n-k}(-1)^j(b_{k,j} + b_{k-1,j+1})\alpha^{k+j-1}\omega_{k+j-1}^\dagger \\ &= -\frac{1}{\alpha}T^*T\sigma_k + (-1)^{n-k}b_{k,n-k+1}\alpha^{n-1}\omega_{n-1}^\dagger\end{aligned}$$

for $k > 1$, and it is easy to see that this also holds true for $k = 1$. Therefore,

$$\begin{aligned}\left\langle \sigma_k - \sigma_{k-1}, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle = \frac{1}{\alpha}\left\langle T\sigma_k, g^\dagger + T\sigma_k - g^{\mathrm{obs}} + g^{\mathrm{obs}} - T\hat{f}_\alpha^{(k)} \right\rangle \\ + \left\langle (-1)^{n-k}b_{k,n-k+1}\alpha^{n-1}\omega_{n-1}^\dagger, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle,\end{aligned}$$

which yields

$$\begin{aligned}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \leq \frac{1}{\alpha}\left(\|g^\dagger - g^{\mathrm{obs}}\|^2 - \|T\hat{f}_\alpha^{(k)} - T\sigma_k - g^{\mathrm{obs}}\|^2\right) + E \\ - (-1)^{n-k}2b_{k,n-k+1}\alpha^{n-1}\left\langle \omega_{n-1}^\dagger, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle\end{aligned} \tag{4.38}$$

For shortage of notation denote $b = 2b_{k,n-k+1}$. Apply $\mathrm{VSC}^{2n-1}(f^\dagger, \Phi)$ (4.26) with $f = (-1)^{n-k}(\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k)/(b\alpha^{n-1})$ and multiply by $(b\alpha^{n-1})^2$ to obtain

$$\begin{aligned}&- (-1)^{n-k}2b_{k,n-k+1}\alpha^{n-1}\left\langle \omega_{n-1}^\dagger, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle \\ &\qquad \leq \frac{1}{2}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 + (b\alpha^{n-1})^2\Phi\left((b\alpha^{n-1})^{-2}\|T\hat{f}_\alpha^{(k)} - g^\dagger - T\sigma_k\|^2\right).\end{aligned}$$

Combining this bound with (4.38) yields

$$\begin{aligned}\frac{1}{2}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \leq &\frac{1}{\alpha}\left(\|g^\dagger - g^{\mathrm{obs}}\|^2 - \|T\hat{f}_\alpha^{(k)} - T\sigma_k - g^{\mathrm{obs}}\|^2\right) + E \\ &+ (b\alpha^{n-1})^2\Phi\left((b\alpha^{n-1})^{-2}\|T\hat{f}_\alpha^{(k)} - g^\dagger - T\sigma_k\|^2\right).\end{aligned}$$

Then we have

$$\begin{aligned}\frac{1}{2}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \leq &\frac{1}{\alpha}\left(\|g^\dagger - g^{\mathrm{obs}}\|^2 - \|T\hat{f}_\alpha^{(k)} - T\sigma_k - g^{\mathrm{obs}}\|^2\right) + E \\ &+ (b\alpha^{n-1})^2\Phi\left((b\alpha^{n-1})^{-2}\|T\hat{f}_\alpha^{(k)} - g^\dagger - T\sigma_k\|^2\right) \\ \leq &\frac{\delta^2}{\alpha} - \frac{1}{\alpha}\|T\hat{f}_\alpha^{(k)} - T\sigma_k - g^\dagger\|^2 + E \\ &+ (b\alpha^{n-1})^2\Phi\left((b\alpha^{n-1})^{-2}\|T\hat{f}_\alpha^{(k)} - g^\dagger - T\sigma_k\|^2\right) \\ \leq &\ \frac{\delta^2}{\alpha} + b^2\alpha^{2n-2}\sup_{\tau\geq 0}\left[\frac{-\tau}{\alpha} - (-\Phi(\tau))\right] + E \\ = &\ \frac{\delta^2}{\alpha} + 4b_{k,n-k+1}^2\alpha^{2n-2}\psi\left(\frac{-1}{\alpha}\right) + E.\end{aligned}$$

To get rid of $E = 2\|\hat{f}_\alpha^{(k-1)} - f^\dagger - \sigma_{k-1}\|\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|$ subtract the term $\frac{1}{4}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2$ on both sides and use Young's inequality as well as the induction hypothesis (4.37).

Step 4: Proof of (4.30) *for the case* $l = 2n$. In view of (4.33) and (4.34) it suffices to prove by induction that given $\mathrm{VSC}^{2n}(f^\dagger, \Phi)$ (4.27) we have

$$\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \leq C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right), \qquad k = 0, \dots, n-1. \tag{4.39}$$

Again, the case $k = 0$ is trival. Assume that (4.39) holds true for all $j = 1, \dots, k-1$. Note that

$$\begin{aligned}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 &= \left\langle \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k, \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k \right\rangle \\ &\leq \left\langle \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k, \sum_{j=1}^{k}\left(\hat{f}_\alpha^{(j)} - f^\dagger - \sigma_j\right)\right\rangle \\ &\quad + \sum_{j=1}^{k-1}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|\|\hat{f}_\alpha^{(j)} - f^\dagger - \sigma_j\|.\end{aligned}$$

Then Young's inequality together with the induction hypothesis (4.39) gives

$$\begin{aligned}\frac{1}{2}\|\hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k\|^2 \leq &\left\langle \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k, \sum_{j=1}^{k}\left(\hat{f}_\alpha^{(j)} - f^\dagger - \sigma_j\right)\right\rangle \\ &+ C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right).\end{aligned} \tag{4.40}$$

A simple computation (for example another induction) shows that

$$-\alpha\omega^\dagger - \sum_{j=1}^{k}\sigma_j = \sum_{j=1}^{n-k-1}(-1)^j b_{k,j}\alpha^{k+j}\omega_{k+j}^\dagger =: \hat{\sigma}_k.$$

By $\mathrm{VSC}^{2n}(f^\dagger, \Phi)$ (4.27) we have $\sigma_k \in \operatorname{ran} T^*$ and by Proposition 3.3.2 we have $T^*\hat{\xi}_\alpha^{(k)} \in \partial\mathcal{R}_k(\hat{f}_\alpha^{(k)}) = \{\hat{f}_\alpha^{(k)} - \sum_{j=1}^{k-1} T^*\hat{\xi}_\alpha^{(j)}\}$ as well as $-\alpha\hat{\xi}_\alpha^{(j)} \in \partial\mathcal{S}_p(T\hat{f}_\alpha^{(j)} - g^{\mathrm{obs}}) = \{T\hat{f}_\alpha^{(j)} - g^{\mathrm{obs}}\}$ such that

$$\begin{aligned}&\left\langle \hat{f}_\alpha^{(k)} - f^\dagger - \sigma_k, \sum_{j=1}^{k}\left(\hat{f}_\alpha^{(j)} - f^\dagger - \sigma_j\right)\right\rangle \\ =&\left\langle \sum_{j=1}^{k} T^*\hat{\xi}_\alpha^{(j)} - T^*\xi^\dagger - T^*(T^{*-1}\sigma_k), \sum_{j=1}^{k}\hat{f}_\alpha^{(j)} - kf^\dagger + \alpha\omega^\dagger + \hat{\sigma}_k\right\rangle \\ =&\left\langle \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)} - \xi^\dagger - (T^{*-1}\sigma_k), \sum_{j=1}^{k}(-\alpha\hat{\xi}_\alpha^{(j)}) + \alpha T\omega^\dagger + T\hat{\sigma}_k + k(g^{\mathrm{obs}} - g^\dagger)\right\rangle \\ =&\alpha\left\langle \xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)}, \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)} - \xi^\dagger - \frac{T\hat{\sigma}_k}{\alpha}\right\rangle + kE,\end{aligned}$$

where $E := \left\langle \xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)}, g^\dagger - g^{\mathrm{obs}}\right\rangle$. On the right hand side of the scalar product we now exchange $\sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)} - \xi^\dagger$ by $(T^{*-1}\sigma_k)$ to find

$$\begin{aligned}&\left\langle \xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)}, \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)} - \xi^\dagger - \frac{T\hat{\sigma}_k}{\alpha}\right\rangle \\ =&\left\langle \xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)}, (T^{*-1}\sigma_k) - \frac{T\hat{\sigma}_k}{\alpha}\right\rangle - \|\xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}_\alpha^{(j)}\|^2\end{aligned}$$

and together with the identity

$$\begin{aligned}(T^{*-1}\sigma_k) - \frac{T\hat{\sigma}_k}{\alpha} &= \sum_{j=1}^{n-k}(-1)^j b_{k,j}\alpha^{k+j-1}\xi^\dagger_{k+j} + \sum_{j=1}^{n-k-1}(-1)^j b_{k,j}\alpha^{k+j-1}\xi^\dagger_{k+j}\\ &= (-1)^{n-k}b_{k,n-k}\alpha^{n-1}\xi^\dagger_n\end{aligned}$$

it follows that

$$\begin{aligned}\left\langle \hat{f}^{(k)}_\alpha - f^\dagger - \sigma_k, \sum_{j=1}^{k}\left(\hat{f}^{(j)}_\alpha - f^\dagger - \sigma_j\right)\right\rangle &= -\alpha\|\xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}^{(j)}_\alpha\|^2\\ &+ b_{k,n-k}\alpha^n\left\langle(-1)^{n-k}\left(\xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}^{(j)}_\alpha\right), \xi^\dagger_n\right\rangle + kE, \quad (4.41)\end{aligned}$$

so we are finally in a position to apply $\mathrm{VSC}^{2n-1}(f^\dagger, \Phi)$ (4.27). For shortage of notation denote $\tilde{b} = 4b_{k,n-k}$ and $\tilde{\xi} = \xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}^{(j)}_\alpha$. Choose $\xi = (-1)^{n-k}\tilde{\xi}/(\tilde{b}\alpha^{n-1})$, and multiply the inequality by $\alpha(\tilde{b}\alpha^{n-1})^2$ to obtain

$$\begin{aligned}&4\tilde{b}_{k,n-k}\alpha^n\left\langle(-1)^{n-k}\left(\xi^\dagger + (T^{*-1}\sigma_k) - \sum_{j=1}^{k}\hat{\xi}^{(j)}_\alpha\right), \xi^\dagger_n\right\rangle\\ &\leq \frac{\alpha}{2}\|\tilde{\xi}\|^2 + \tilde{b}^2\alpha^{2n-1}\Phi\left((\tilde{b}\alpha^{n-1})^{-2}\|T^*\tilde{\xi}\|^2\right).\end{aligned} \quad (4.42)$$

Now combine (4.40), (4.41) and (4.42) to find

$$\begin{aligned}2\|\hat{f}^{(k)}_\alpha - f^\dagger - \sigma_k\|^2 &\leq \tilde{b}^2\alpha^{2n-1}\Phi\left((\tilde{b}\alpha^{n-1})^{-2}\|f^\dagger + \sigma_k - \hat{f}^{(k)}_\alpha\|^2\right)\\ &+ \frac{\alpha}{2}\|\tilde{\xi}\|^2 - 4\alpha\|\tilde{\xi}\|^2 + 4kE + C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right). \quad (4.43)\end{aligned}$$

Completing the square, we get

$$\frac{\alpha}{2}\|\tilde{\xi}\|^2 - 4\alpha\|\tilde{\xi}\|^2 + 4kE = -\frac{7}{2}\alpha\|\tilde{\xi}\|^2 + 4\left\langle\tilde{\xi}, g^\dagger - g^{\mathrm{obs}}\right\rangle \leq \frac{8k^2\delta^2}{7\alpha}.$$

Now we subtract $\left\|\hat{f}^{(k)}_\alpha - f^\dagger - \sigma_k\right\|^2$ in (4.43) from both sides to find

$$\begin{aligned}\|\hat{f}^{(k)}_\alpha - f^\dagger - \sigma_k\|^2 &\leq \tilde{b}^2\alpha^{2n-1}\Phi\left((\tilde{b}\alpha^{n-1})^{-2}\|f^\dagger + \sigma_k - \hat{f}^{(k)}_\alpha\|^2\right)\\ &\quad - \|\hat{f}^{(k)}_\alpha - f^\dagger - \sigma_k\|^2 + C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right)\\ &\leq \tilde{b}^2\alpha^{2n-1}\sup_{\tau\geq 0}\left[\frac{-\tau}{\alpha} - (-\Phi(\tau))\right] + C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right)\\ &= 16b^2_{k,n-k}\alpha^{2n-1}\psi\left(\frac{-1}{\alpha}\right) + C\left(\frac{\delta^2}{\alpha} + \alpha^{2n-1}\psi\left(-\frac{1}{\alpha}\right)\right).\end{aligned}$$

□

Note that under a spectral source condition as on the left hand side of the implication (4.28), the VSC of the right hand side of (4.28) and Theorem 4.5.2 yield the error bound $C(\delta/\alpha^2 + \alpha^{l-1+\nu})$. For the choice $\alpha \sim \delta^{2/(l+\nu)}$ this leads to the optimal convergence rate $\left\|\hat{f}^{(m)}_\alpha - f^\dagger\right\| = \mathcal{O}\left(\delta^{(l-1+\nu)/(l+\nu)}\right)$. However, we have derived this rate under the weaker assumption $\mathrm{VSC}^l(f^\dagger, A\,\mathrm{id}^{\nu/(\nu+1)})$ using only variational, but no spectral arguments.

4.5.2 Banach spaces

In this section we will introduce a third order variational source condition and apply it to prove higher order convergence rates for two times Bregman iterated Tikhonov regularization (see Section 3.3).

Definition 4.5.3 (Variational source condition $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$)**.** *Let Φ be an index function. We say that $f^\dagger \in \mathcal{X}$ satisfies the third order variational source condition $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$ if there exist $\xi^\dagger \in \mathcal{Y}^*$ and $\omega^\dagger \in \mathcal{X}$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$ and $T\omega^\dagger \in \partial\mathcal{S}_p^*(\xi^\dagger)$ and if there exist constants $\beta \geq 0$, $\mu > 1$ and $\bar{t} > 0$ as well as $f_t^* \in \partial\mathcal{R}(f^\dagger - t\omega^\dagger)$ for all $0 < t \leq \bar{t}$ such that*

$$\begin{aligned}&\forall f \in \mathcal{X}\ \forall t \in (0, \bar{t}]:\\ &\left\langle f_t^* - T^*\xi^\dagger, f^\dagger - t\omega^\dagger - f\right\rangle \leq \Delta_{\mathcal{R}}^{f_t^*}\left(f, f^\dagger - t\omega^\dagger\right) + t^2\Phi\left(t^{-p}\|Tf - g^\dagger + tT\omega^\dagger\|^p\right) + \beta t^{2\mu}.\end{aligned}$$

Remark 4.5.4. *To see how VSC^2 and VSC^3 relate to other source conditions, recall from Section 4.4 that the strongest VSC^1 is equivalent to the existence of $\xi^\dagger \in \mathcal{Y}^*$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$. Similarly one can show that the strongest VSC^2 is equivalent to the existence of $T\omega^\dagger \in \partial\mathcal{S}_p^*(\xi^\dagger)$ (see [34, Lemma 5.1, 5.3]). So by assuming the existence of such $\xi^\dagger \in \mathcal{Y}^*$, $\omega^\dagger \in \mathcal{X}$ the VSC^3 is stronger than VSC^1 and VSC^2. Similarly, as discussed in the introduction of Section 4.4 the VSC^2 and VSC^3 are also stronger than the multiplicative variational source conditions in [4, 45] and approximate (variational) source conditions ([28]).*

Now let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $\mathcal{R}_2(f) := \frac{1}{2}\|f\|_{\mathcal{X}}^2$, $\mathcal{S}_2(g) = \frac{1}{2}\|g\|_{\mathcal{Y}}^2$. Then clearly the $\mathrm{VSC}^2(\Phi, \mathcal{R}_2, \mathcal{S}_2)$ is equivalent to $\mathrm{VSC}^2(f^\dagger, \Phi)$. We also have that the $\mathrm{VSC}^3(\Phi, \mathcal{R}_2, \mathcal{S}_2)$ is equivalent to $\mathrm{VSC}^3(f^\dagger, \Phi)$: In fact, for arbitrary $\beta \geq 0$ and $\mu > 1$ the condition $\mathrm{VSC}^3(f^\dagger, \Phi)$ is equivalent to

$$\forall f \in \mathcal{X}\ \forall t > 0: \qquad \left\langle \omega^\dagger, f\right\rangle \leq \frac{1}{2}\|f\|^2 + \Phi\left(\|Tf\|^2\right) + \beta t^{2\mu-2},$$

as the limit $t \to 0$ gives back the original inequality. Now we replace f by $\frac{f - f^\dagger + t\omega^\dagger}{t}$ and multiply by t^2 to see that this is equivalent to

$$\left\langle -t\omega^\dagger, f^\dagger - t\omega^\dagger - f\right\rangle \leq \frac{1}{2}\|f - f^\dagger + t\omega^\dagger\|^2 + t^2\Phi\left(\frac{\|Tf - g^\dagger + tT\omega^\dagger\|^2}{t^2}\right) + \beta t^{2\mu},$$

which is equivalent to $\mathrm{VSC}^3(\Phi, \mathcal{R}_2, \mathcal{S}_2)$.

We can now state the main result of this section:

Theorem 4.5.5. *Suppose Assumption 3.3.1 and that $\mathcal{Y}$ is p-smooth and r-convex. Further assume that $f^\dagger$ fulfills the $\mathrm{VSC}^3(f^\dagger, \Phi, \mathcal{R}, \mathcal{S}_p)$ with constants β, μ, and $\bar{t}$ and that $c^{-1}\delta \leq \alpha^{p'-1} \leq \bar{t}$ for some $c > 0$. Define $\widetilde{\Phi}(s) = \Phi(s^{p/r})$. Then the error is bounded by*

$$\Delta_{\mathcal{R}}\left(\hat{f}_\alpha^{(2)}, f^\dagger\right) \leq C\left(\frac{\delta^p}{\alpha} + \alpha^{2(p'-1)}\left(-\widetilde{\Phi}\right)^*\left(\frac{\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}}{-\alpha^{p'-1}}\right) + \beta\alpha^{2\mu(p'-1)}\right)$$

with constants $C, \tilde{C} > 0$ depending at most on p, r, c, and $\mathcal{Y}$.

The proof consists of the following three lemmata, in all of which we will tacitly assume that Assumption 3.3.1 holds true and $\mathcal{Y}$ is p-smooth and r-convex. By choosing $f = f^\dagger$ in Lemma 3.3.4 we show that $\Delta_{\mathcal{R}}(\hat{f}_\alpha^{(2)}, f^\dagger)$ is related to the Bregman distance $\Delta_{\mathcal{S}_p^*}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger)$, which is helpful as we will later actually use $\mathrm{VSC}^3(f^\dagger, \Phi, \mathcal{R}, \mathcal{S}_p)$ to prove convergence rates for $\widehat{\xi}_\alpha$. In fact the following lemma is already sufficient to show second order convergence rates for $\hat{f}_\alpha^{(2)}$ as Theorem 4.4.2 gives an upper bound on $\Delta_{\mathcal{S}_p^*}(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger)$.

Lemma 4.5.6. *If $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$, then there exists $C_{p',\mathcal{Y}^*} > 0$ such that*

$$\Delta_{\mathcal{R}}\left(\hat{f}_\alpha^{(2)}, f^\dagger\right) \leq \frac{2}{\alpha}\left(\mathcal{S}_p\left(g^\dagger - g^{\mathrm{obs}}\right) + C_{p',\mathcal{Y}^*}\Delta_{\mathcal{S}_p^*}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right)\right).$$

Proof. We apply Lemma 3.3.4 with $f = f^\dagger$ to find

$$\Delta_{\mathcal{R}}\left(\hat{f}_\alpha^{(2)}, f^\dagger\right) \leq \frac{1}{\alpha}\mathcal{S}_p\left(g^\dagger - g^{\mathrm{obs}}\right) + \left\langle \xi^\dagger - \widehat{\xi}_\alpha, g^\dagger - g^{\mathrm{obs}}\right\rangle + \frac{1}{\alpha}\mathcal{S}_p^*\left(-\alpha(\xi^\dagger - \widehat{\xi}_\alpha)\right).$$

The generalized Young inequality applied to the middle term yields

$$\Delta_{\mathcal{R}}\left(\hat{f}_\alpha^{(2)}, f^\dagger\right) \leq \frac{2}{\alpha}\left(\mathcal{S}_p\left(g^\dagger - g^{\mathrm{obs}}\right) + \mathcal{S}_p^*\left(-\alpha(\xi^\dagger - \widehat{\xi}_\alpha)\right)\right).$$

As $\mathcal{Y}$ is p-smooth, $\mathcal{Y}^*$ is p'-convex by Lemma 2.2.6, so we can apply Theorem 2.2.5 to obtain

$$\Delta_{\mathcal{R}}\left(\hat{f}_\alpha^{(2)}, f^\dagger\right) \leq \frac{2}{\alpha}\left(\mathcal{S}_p\left(g^\dagger - g^{\mathrm{obs}}\right) + C_{p',\mathcal{Y}^*}\Delta_{\mathcal{S}_p^*}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right)\right). \qquad \square$$

The next lemma shows convergence rates in the image space.

Lemma 4.5.7. *Suppose there exist $\xi^\dagger \in \mathcal{Y}^*$ and $\omega^\dagger \in \mathcal{X}$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$ and $T\omega^\dagger \in \partial\mathcal{S}_p^*(\xi^\dagger) = J_{\mathcal{Y}^*,p'}(\xi^\dagger)$. Then there exists a constant $C_p > 0$ depending only on p such that*

$$\|T\hat{f}_\alpha - g^\dagger\|_{\mathcal{Y}} \leq C_p\left(\delta + \alpha^{p'-1}\|T\omega^\dagger\|_{\mathcal{Y}}\right).$$

Proof. By Lemma 4.3.5 we get that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$ implies $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}_{g^\dagger}^p)$, with $\Phi(\tau) = \|\xi^\dagger\|_{\mathcal{Y}^*}(p\tau)^{\frac{1}{p}}$, which then implies by (4.5) in Theorem 4.3.4 that

$$\frac{1}{p}\|T\widehat{f}_\alpha - g^\dagger\|^p \leq 2C_{\mathrm{err}}\left(\mathbf{err}_{\mathcal{Y}} + \alpha(-\Phi)^*\left(\frac{-1}{2C_{\mathrm{err}}\alpha}\right)\right).$$

By Lemma 4.1.3 and Lemma 4.3.13 we thus get

$$\|T\widehat{f}_\alpha - g^\dagger\|^p \leq C_p\left(\delta^p + \|\xi^\dagger\|_{\mathcal{Y}^*}^{\frac{p}{p-1}}\alpha^{\frac{p}{p-1}}\right),$$

so that the claim follows from $\|T\omega^\dagger\|_{\mathcal{Y}} = \|\xi^\dagger\|_{\mathcal{Y}^*}^{p'-1}$ and $p/(p-1) = p'$, $p'/(p'-1) = p$. $\square$

The main part of the proof of Theorem 4.5.5 consists in the derivation of convergence rates for the dual problem:

Lemma 4.5.8. *Define* $\alpha_p := \alpha^{p'-1}$, $\widetilde{\Phi}(s) = \Phi(s^{p/r})$. *Let* $\mathrm{VSC}^3(f^\dagger, \Phi, \mathcal{R}, \mathcal{S}_p)$ *hold true with constants* β, μ, *and* $\bar{t}$. *If* α *is chosen such that* $c^{-1}\delta \leq \alpha_p \leq \bar{t}$, *for some* $c > 0$, *then*

$$\frac{1}{2\alpha}\Delta_{\mathcal{S}_p^*}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right) \leq C\frac{\delta^p}{\alpha} + \alpha_p^2\left(-\widetilde{\Phi}\right)^*\left(\frac{-\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}}{\alpha_p}\right) + \beta\alpha_p^{2\mu},$$

where $C, \tilde{C} > 0$ *depend at most on* p, r, c, $\mathcal{Y}$.

Proof. It follows from Proposition 2.2.10 and $T\omega^\dagger \in \partial\mathcal{S}_p^*(\xi^\dagger)$ that $-\alpha_p T\omega^\dagger \in \partial\mathcal{S}_p^*(-\alpha\xi^\dagger)$. By (3.6) together with Corollary 2.1.19 we can consider

$$\begin{aligned}\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right) &= \frac{1}{\alpha}\left\langle -\alpha\xi^\dagger - (-\alpha\widehat{\xi}_\alpha), -\alpha_p T\omega^\dagger - (T\hat{f}_\alpha - g^{\mathrm{obs}})\right\rangle \\ &= \left\langle T^*\widehat{\xi}_\alpha - T^*\xi^\dagger, f^\dagger - \alpha_p\omega^\dagger - \hat{f}_\alpha\right\rangle + \left\langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger\right\rangle.\end{aligned}$$

The second term $E := \left\langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger\right\rangle$ will be estimated later. Artificially adding zero in the form $f_{\alpha_p}^* - f_{\alpha_p}^*$ with $f_{\alpha_p}^* \in \partial\mathcal{R}(f^\dagger - \alpha_p\omega^\dagger)$, we find

$$\begin{aligned}\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right) = &\left\langle f_{\alpha_p}^* - T^*\xi^\dagger, f^\dagger - \alpha_p\omega^\dagger - \hat{f}_\alpha\right\rangle + E \\ &+ \left\langle T^*\widehat{\xi}_\alpha - f_{\alpha_p}^*, f^\dagger - \alpha_p\omega^\dagger - \hat{f}_\alpha\right\rangle.\end{aligned}$$

In view of (3.5) the last term is the negative symmetric Bregman distance $-\Delta_{\mathcal{R}}^{\mathrm{sym}}(\hat{f}_\alpha, f^\dagger - \alpha_p\omega^\dagger)$. The first term can be bounded using $\mathrm{VSC}^3(f^\dagger, \Phi, \mathcal{R}, \mathcal{S}_p)$ by choosing $f = \hat{f}_\alpha$ and $t = \alpha_p$:

$$\begin{aligned}\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}^{\mathrm{sym}}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right) &\leq \alpha_p^2\widetilde{\Phi}\left(\alpha_p^{-r}\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r\right) + \beta\alpha_p^{2\mu} \\ &\quad + \Delta_{\mathcal{R}}\left(\hat{f}_\alpha, f^\dagger - \alpha_p\omega^\dagger\right) - \Delta_{\mathcal{R}}^{\mathrm{sym}}(\hat{f}_\alpha, f^\dagger - \alpha_p\omega^\dagger) + E \\ &\leq \alpha_p^2\widetilde{\Phi}\left(\alpha_p^{-r}\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r\right) + \beta\alpha_p^{2\mu} + E.\end{aligned}$$

Now we use our joker. We subtract

$$\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}(-\alpha\xi^\dagger, -\alpha\widehat{\xi}_\alpha) = \frac{1}{\alpha}\Delta_{\mathcal{S}_p}(T\hat{f}_\alpha - g^{\mathrm{obs}}, -\alpha_p T\omega^\dagger)$$

(see (2.10)) from both sides leading to

$$\begin{aligned}\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}\left(-\alpha\widehat{\xi}_\alpha, -\alpha\xi^\dagger\right) \leq\ &\alpha_p^2\widetilde{\Phi}\left(\alpha_p^{-r}\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r\right) \\ &- \Delta_{\mathcal{S}_p}(T\hat{f}_\alpha - g^{\mathrm{obs}}, -\alpha_p T\omega^\dagger) + \beta\alpha_p^{2\mu} + E.\end{aligned} \tag{4.44}$$

So we need to bound $\Delta := \Delta_{\mathcal{S}_p}(T\hat{f}_\alpha - g^{\mathrm{obs}}, -\alpha_p T\omega^\dagger)$ from below. By Theorem 2.2.5 we have (as $p \leq r$)

$$\Delta \geq C_{p,\mathcal{Y}}\max\left(\|\alpha_p T\omega^\dagger\|, \|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\|\right)^{p-r}\|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\|^r.$$

Moreover, it follows from Lemma 4.5.7 and the choice $\delta \le c\alpha_p$ that

$$\begin{aligned}\|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\| &\le \|T\hat{f}_\alpha - g^\dagger\| + \|g^\dagger - g^{\mathrm{obs}}\| + \|\alpha_p T\omega^\dagger\| \\ &\le C_p\left(\delta + \alpha_p\|T\omega^\dagger\|\right) \le C_p\alpha_p\left(c + \|T\omega^\dagger\|\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\max\left(\|\alpha_p T\omega^\dagger\|, \|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\|\right) &\le \alpha_p \max\left(\|T\omega^\dagger\|, C_p\left(c + \|T\omega^\dagger\|\right)\right) \\ &\le \alpha_p \max\left(1, C_p\right)\left(c + \|T\omega^\dagger\|\right).\end{aligned}$$

Hence, there exists a constant $\tilde{C} > 0$ depending on p, $C_{p,\mathcal{Y}}$, and r such that

$$\frac{1}{\alpha}\Delta \ge 2^{r-1}\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}\alpha^{(p'-1)(p-r)-1}\|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\|^r.$$

Note that $(p'-1)(p-r)-1 = -(r-1)(p'-1)$. In order to replace g^{obs} by $g^\dagger$ on the right hand side we use the inequality

$$2^{1-r}\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r - \|T\hat{f}_\alpha - g^{\mathrm{obs}} + \alpha_p T\omega^\dagger\|^r \le \|g^\dagger - g^{\mathrm{obs}}\|^r$$

(see [61, Lemma 3.20]) leading to

$$-\frac{1}{\alpha}\Delta \le \frac{\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}}{\alpha_p^{r-1}}\left(-\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r + 2^{r-1}\delta^r\right).$$

Inserting this into (4.44) yields

$$\begin{aligned}\frac{1}{\alpha}\Delta_{\mathcal{S}_p^*}\left(-\alpha\hat{\xi}_\alpha, -\alpha\xi^\dagger\right) &\le \alpha_p^2\widetilde{\Phi}\left(\alpha_p^{-r}\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r\right) + \beta\alpha_p^{2\mu} + E \\ &\quad - \frac{\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}}{\alpha_p^{r-1}}\left(\|T\hat{f}_\alpha - g^\dagger + \alpha_p T\omega^\dagger\|^r - 2^{r-1}\delta^r\right) \\ &\le \alpha_p^2 \sup_{\tau\ge 0}\left[-\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}\alpha_p^{-1}\tau - \left(-\widetilde{\Phi}(\tau)\right)\right] \\ &\quad + \tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}2^{r-1}\frac{\delta^r}{\alpha_p^{r-1}} + E + \beta\alpha_p^{2\mu}.\end{aligned}$$

The supremum equals $\left(-\widetilde{\Phi}\right)^*\left(-\tilde{C}\left(c + \|T\omega^\dagger\|\right)^{p-r}\alpha_p^{-1}\right)$, by the definition of the convex conjugate . To deal with E let $C_{p',\mathcal{Y}^*}$ be the constant in (2.15) of Theorem 2.2.5. We use the generalized Young inequality

$$\begin{aligned}&\frac{1}{\alpha}\left\langle\left(\frac{C_{p',\mathcal{Y}^*}p'}{2}\right)^{\frac{1}{p'}}(\alpha\hat{\xi}_\alpha - \alpha\xi^\dagger), \left(\frac{C_{p',\mathcal{Y}^*}p'}{2}\right)^{\frac{-1}{p'}}(g^{\mathrm{obs}} - g^\dagger)\right\rangle \\ &\qquad\le \frac{C_{p',\mathcal{Y}^*}}{2\alpha}\|\alpha\hat{\xi}_\alpha - \alpha\xi^\dagger\|^{p'} + \frac{1}{p}\left(\frac{C_{p',\mathcal{Y}^*}p'}{2}\right)^{-\frac{p}{p'}}\frac{\delta^p}{\alpha}\end{aligned}$$

and apply Theorem 2.2.5, using that $\mathcal{Y}^*$ is p' convex, to find

$$E = \left\langle \widehat{\xi}_\alpha - \xi^\dagger, g^{\mathrm{obs}} - g^\dagger \right\rangle \le \frac{1}{2\alpha} \Delta_{\mathcal{S}_p^*} \left(-\alpha \widehat{\xi}_\alpha, -\alpha \xi^\dagger \right) + \frac{1}{p} \left(\frac{C_{p', \mathcal{Y}^*} p'}{2} \right)^{-\frac{p}{p'}} \frac{\delta^p}{\alpha}.$$

The assumption $\delta \le \alpha_p$, or equivalently $\delta^{r-p} \le \alpha_p^{r-p}$, implies $\frac{\delta^r}{\alpha_p^{r-1}} \le \frac{\delta^p}{\alpha_p^{p-1}} = \frac{\delta^p}{\alpha}$. Further $\left(c + \|T\omega^\dagger\| \right)^{p-r} \le c^{p-r}$, hence there exists a constant $C > 0$ depending on p, r, c, $C_{p,\mathcal{Y}}$, and $C_{p',\mathcal{Y}^*}$ such that

$$\frac{1}{2\alpha} \Delta_{\mathcal{S}_p^*} \left(-\alpha \widehat{\xi}_\alpha, -\alpha \xi^\dagger \right) \le C \frac{\delta^p}{\alpha} + \alpha_p^2 \left(-\widetilde{\Phi} \right)^* \left(-\tilde{C} \left(c + \|T\omega^\dagger\| \right)^{p-r} \alpha_p^{-1} \right) + \beta \alpha_p^{2\mu},$$

which completes the proof. □

Now Theorem 4.5.5 is an immediate consequence of Lemma 4.5.6 and Lemma 4.5.8.

Chapter Five

Verification of source conditions

> "For that (the rapt one warns) is what papyr is meed of, made of, hides and hints and misses in prints. Till ye finally (though not yet endlike) meet with the acquaintance of Mister Typus, Mistress Tope and all the little typtopies."
>
> *Finnegans Wake*, J. Joyce

In this chapter we will show how the different source conditions that we have introduced can be verified for many important inverse problems given smoothness of the exact solution $f^\dagger$. To this end we will derive abstract strategies that work in a general setting. These strategies are similar for all three orders of source conditions that we discuss. Firstly we always have a result similar to Proposition 4.3.6 which establishes the source condition for some function $h^\dagger$ (e.g. $f^\dagger$ or $\xi^\dagger$) under certain assumptions on the penalty function. However, the index function of the source condition will not be given in an explicit form yet, but rather as an infimum over terms related to projections of $h^\dagger$. These projections will be chosen for the rest of this thesis as the wavelet approximations from A.2.13. So secondly there will be a result that bounds these wavelet approximations and their tails in certain norms.

After the first section on abstract strategies and assumptions we will consider specific penalty functionals in each section. Note that against common praxis we do not start with the most understood example of Hilbert space regularization, but rather with maximum entropy regularization and Besov norm regularization. The reason for this order is that these two examples show best why we need certain assumptions in the general strategies and how we can profit from the variational approach.

The reader might consider to firstly read either Section 5.2 or Section 5.3 and then come back to the abstract Section 5.1 to have a better understanding of the necessity of the results.

5.1 General strategies and assumptions[1]

5.1.1 Verification of first order VSCs

In Section 4.3.2 we have already seen a general strategy to verify the $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$ for $\mathcal{S}^\dagger = \mathcal{S}^p_{g^\dagger}$. We now show how the assumptions (4.7a), (4.7b) of Proposition 4.3.6 can be fulfilled with the wavelet approximation projections from Proposition A.2.13.

Lemma 5.1.1. *For $p \in [1, 2], q \in [1, \infty]$ let $B^0_{p',q}(\mathbb{M}_0) \subset \mathcal{X}^*$, $\mathcal{Y} \subset L^2(\mathbb{M})$ with continuous embedding. Let P_k be given as in Proposition A.2.13. If for $s > 0$ we have $h^* \in B^s_{p',\infty}$ then we have*

$$\|(I - P_k)h^*\|_{\mathcal{X}^*} \leq C2^{-ks}\|h^*\|_{B^s_{p',\infty}}$$

Let $a > s$ and assume that

$$\|f_1 - f_2\|_{B^{-a}_{2,2}(\mathbb{M}_0)} \leq \|F(f_1) - F(f_2)\|_{L^2(\mathbb{M})}, \tag{5.1}$$

for all $f_1, f_2 \in \mathcal{X}$. Then we have

$$\langle P_k h^*, f_1 - f_2\rangle \leq C2^{(a-s)k}\|h^*\|_{B^s_{p',\infty}}\|F(f_1) - F(f_2)\|_{\mathcal{Y}}.$$

If, for a linear forward operator T, we assume instead of (5.1) that T^* viewed as a mapping $T^* \colon L^2(\mathbb{M}) \to B^a_{2,2}(\mathbb{M}_0)$ has a bounded right inverse $(T^*)^\dagger$ (i.e $T^* \circ (T^*)^\dagger = \mathrm{id}$) then we have

$$\|(T^*)^\dagger P_k h^*\|_{\mathcal{Y}^*} \leq C2^{(a-s)k}\|h^*\|_{B^s_{p',\infty}}.$$

Proof. The first inequality follows directly from Proposition A.2.13 as we have

$$\|(I - P_k)h^*\|_{\mathcal{X}^*} \leq C\|(I - P_k)h^*\|_{B^0_{p',q}(\mathbb{M}_0)} \leq C2^{-ks}\|h^*\|_{B^s_{p',\infty}(\mathbb{M}_0)}.$$

The second inequality follows again from Proposition A.2.13 by

$$\langle P_k h^*, f_1 - f_2\rangle \leq \|P_k h^*\|_{B^a_{p',2}}\|f_1 - f_2\|_{B^{-a}_{p,2}} \leq C2^{(a-s)k}\|h^*\|_{B^s_{p',\infty}}\|f_1 - f_2\|_{B^{-a}_{2,2}},$$

so that (5.1) gives the claim. Concerning the statement for linear forward operators notice that $L^2 \subset \mathcal{Y}^*$ with continuous embedding and $(T^*)^\dagger \colon B^a_{2,2} \to L^2$ is bounded so that we have

$$\|(T^*)^\dagger P_k h^*\|_{\mathcal{Y}^*} \leq C\|(T^*)^\dagger P_k h^*\|_{L^2(\mathbb{M})} \leq C\|P_k h^*\|_{B^a_{2,2}(\mathbb{M}_0)} \leq C\|P_k h^*\|_{B^a_{p',2}(\mathbb{M}_0)}.$$

Thus the claim follows from Proposition A.2.13. □

By the following lemma, we can conclude from the $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$ with $\mathcal{S}^\dagger = \mathcal{S}^p_{g^\dagger}$ the corresponding VSC with $\mathcal{S}^\dagger(g) = \mathrm{KL}^\sigma_{g^\dagger}(g) := \mathrm{KL}(g^\dagger + \sigma, g + \sigma)$.

[1]This section has some literal overlap with the article [64] to which the author contributed.

Lemma 5.1.2. *Let $f^\dagger$ fulfill* $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$. *If Assumption 4.3.9 holds true with $p = 2$, $a > d/2$ and $r > \rho + 1$, then there exists a constant $C_{f^\dagger,\mathcal{R},F,\sigma} > 0$ such that $f^\dagger$ fulfills* $\mathrm{VSC}^1(\Phi(C_{f^\dagger,\mathcal{R},F,\sigma}\cdot), \mathcal{R}, \mathrm{KL}^\sigma_{\mathrm{g}^\dagger})$. *If instead of (B3) of Assumption 4.3.9 we have $\mathcal{R}(f) = \mathcal{R}_{f_0}(f) = \mathrm{KL}(f, f_0)$ as in Section 3.2.5 with $f_0 \in L^1, f_0 \geq 0$ and $f^\dagger/f_0 \in L^\infty$ then there still exists $C_{f^\dagger,\mathcal{R},F,\sigma} > 0$ such that $f^\dagger$ fulfills* $\mathrm{VSC}^1(\Phi(C_{f^\dagger,\mathcal{R},F,\sigma}\cdot), \mathcal{R}_{f_0}, \mathrm{KL}^\sigma_{\mathrm{g}^\dagger})$.

Proof. As $f^\dagger$ fulfill $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^2_{g^\dagger})$ we have for all $f \in \mathcal{X}$ that

$$\langle f^*, f^\dagger - f\rangle \leq \frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) + \Phi\Big(\|F(f) - g^\dagger\|^2\Big)$$

and we want to verify $\mathrm{VSC}^1(\Phi(C\cdot), \mathcal{R}, \mathrm{KL}^\sigma_{\mathrm{g}^\dagger})$, i.e. that for all $f \in \mathcal{X}$ we have

$$\langle f^*, f^\dagger - f\rangle \leq \frac{1}{2}\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) + \Phi\Big(C\,\mathrm{KL}(g^\dagger + \sigma, F(f) + \sigma)\Big). \tag{5.2}$$

If (B3) holds true we have for all f with $\|f\|_{\mathcal{X}} > \max\Big(2\|f^\dagger\|_{\mathcal{X}}, \big(2^r C_{\mathcal{R}}^{-1}\|f^*\|_{\mathcal{X}^*}\big)^{\frac{1}{r-\rho-1}}\Big)$ that

$$\begin{aligned}\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) \geq C_{\mathcal{R}}\|f\|_{\mathcal{X}}^{-\rho}\|f - f^\dagger\|^r_{\mathcal{X}} &\geq 2^{-r+1}C_{\mathcal{R}}\|f\|_{\mathcal{X}}^{r-\rho-1}\|f - f^\dagger\|_{\mathcal{X}} \\ &\geq 2\|f^*\|_{\mathcal{X}^*}\|f - f^\dagger\|_{\mathcal{X}} \geq 2\langle f^*, f^\dagger - f\rangle\end{aligned}$$

so (5.2) holds true. Instead if $\mathcal{R}(f) = \mathcal{R}_{f_0}(f)$ then for f with $\|f\| > 2e^{8\|f^*\|_{L^\infty}+1}\|f^\dagger\|$ we have by Lemma 2.3.2 and Proposition 3.2.1 that

$$\Delta^{f^*}_{\mathcal{R}}(f, f^\dagger) = \mathrm{KL}(f, f^\dagger) \geq 4\|f^*\|_{L^\infty}\|f\|_{L^1} \geq 2\|f^*\|_{L^\infty}\|f - f^\dagger\|_{L^1} \geq 2\langle f^*, f^\dagger - f\rangle. \tag{5.3}$$

So in these cases the variational inequality holds trivially with $\Phi = 0$. Now assume on the contrary that $\|f\| \leq C$ and then we have by (B2) that

$$\|F(f)\|_{L^\infty} \leq C\|F(f)\|_{B^a_{2,2}} \leq CL\|f\|_{\mathcal{X}} + C\|F(0)\|_{B^a_{2,2}} \leq C.$$

Therefore we have by (2.19) that $\|F(f) + \sigma - g^\dagger - \sigma\|^2_{L^2} \leq C\,\mathrm{KL}(g^\dagger + \sigma, F(f) + \sigma)$, which gives the claim. □

5.1.2 Verification of second order source conditions

Proposition 5.1.3 (verification of $\mathrm{VSC}^2(\Phi, \mathcal{R}, \mathcal{S}_p)$). *Let $\mathcal{Y}$ be p-smooth. Assume that there exists $\xi^\dagger \in \mathcal{Y}^*$ such that $T^*\xi^\dagger = f^* \in \partial\mathcal{R}(f^\dagger)$ and let $\xi^* \in J_{p',\mathcal{Y}^*}(\xi^\dagger) \in \mathcal{Y}$. We distinguish two cases:*

(a) Assume that for some $\mu \geq 2$ and all $B \geq 1$ there exist constants $C_{\mu,B} > 0$ such that for all $x^* \in \mathcal{X}^*$ with $\|x^* - T^*\xi^\dagger\|_{\mathcal{X}^*} \leq B$ we have

$$\|x^* - T^*\xi^\dagger\|_{\mathcal{X}^*} \leq C_{\mu,B}\Delta^{f^\dagger}_{\mathcal{R}^*}(x^*, T^*\xi^\dagger)^{\frac{1}{\mu}}. \tag{5.4}$$

Set $V = \mathcal{X}$.

(b) Let $\widetilde{\mathcal{X}}$ be a Banach space continuously embedded in $\mathcal{X}$ and assume that $\mathcal{R}$ is continuously Fréchet-differentiable in a neighborhood of $f^\dagger \in \widetilde{\mathcal{X}}$ with respect to $\|\cdot\|_{\widetilde{\mathcal{X}}}$ and $\mathcal{R}' : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}^*$ is uniformly Lipschitz continuous with respect to $\|\cdot\|_{\widetilde{\mathcal{X}}}$ in this neighborhood. Further assume that $\mathcal{R}'[f^\dagger] = (T^*\xi^\dagger)|_{\widetilde{\mathcal{X}}}$. Set $V = \widetilde{\mathcal{X}}$ and $\mu = 2$.

Suppose that there exists a family of operators $P_k \in L(\mathcal{Y})$ indexed by $k \in \mathbb{N}$ such that $P_k \xi^* \in T(V)$ for all $k \in \mathbb{N}$ and let

$$\kappa_k := \|(I - P_k)\xi^*\|_{\mathcal{Y}}, \qquad \nu_k := \max\left\{\|T^{-1}P_k\xi^*\|_V, 1\right\}. \tag{5.5}$$

If $\lim_{k\to\infty} \kappa_k = 0$, then there exists $C > 0$ such that $\mathrm{VSC}^2(\Phi, \mathcal{R}, \mathcal{S}_p)$ *holds true with the index function*

$$\Phi(\tau) := C \inf_{k\in\mathbb{N}} \left[\nu_k \tau^{1/\mu} + \kappa_k^p\right]. \tag{5.6}$$

Proof. We show the $\mathrm{VSC}^2(\Phi, \mathcal{R}, \mathcal{S}_p)$, i.e.

$$\left\langle \xi^\dagger - \xi, \xi^* \right\rangle \le \frac{1}{2}\Delta^{\xi^*}_{\mathcal{S}_{p^*}}\left(\xi, \xi^\dagger\right) + \Phi\left(\Delta^{f^\dagger}_{\mathcal{R}^*}\left(T^*\xi, T^*\xi^\dagger\right)\right). \tag{5.7}$$

for all $\xi \in \mathcal{Y}^*$ by distinguishing three cases:

Case 1: $\xi \in \mathcal{A} := \{\xi \in \mathcal{Y}^* : \left\langle \xi^\dagger - \xi, \xi^* \right\rangle \le \frac{1}{A}\Delta_{\mathcal{R}^*}\left(T^*\xi, T^*\xi^\dagger\right)^{\frac{1}{2}}\}$, with a constant $A > 0$ whose exact value will be chosen later. For these ξ the inequality thus holds with $\Phi(\tau) = \frac{1}{A}\sqrt{\tau}$ which is smaller than (5.6) for $C \ge 1/A$.

Case 2: $\xi \in \mathcal{B} := \{\xi \in \mathcal{Y}^* : \|\xi^\dagger - \xi\|^{p'-1}_{\mathcal{Y}^*} \ge 2C^{-1}_{p',Y^*}\|\xi^*\|_{\mathcal{Y}}\}$, with constant C_{p',Y^*} from Theorem 2.2.5. $\mathcal{Y}^*$ is p'-convex as $\mathcal{Y}$ is p-smooth, so by Theorem 2.2.5 we have for all $\xi \in \mathcal{B}$ that

$$\left\langle \xi^\dagger - \xi, \xi^* \right\rangle \le \|\xi^\dagger - \xi\|\|\xi^*\| \le \frac{1}{2}\Delta_{\mathcal{S}^*_p}\left(\xi, \xi^\dagger\right),$$

so (5.7) also holds for $\xi \in \mathcal{B}$.

Case 3: $\xi \in \mathcal{Y}^* \setminus (\mathcal{A} \cup \mathcal{B})$. Assume that Item (a) holds true. As $\xi \notin \mathcal{B}$ we know that $\|\xi^\dagger - \xi\|$ is bounded, say $\|\xi^\dagger - \xi\| \le B$ for some $B \ge 1$ and thus $\|T^*\xi^\dagger - T^*\xi\| \le \|T^*\|B$. So we can apply (5.4) which yields

$$\begin{aligned}
\langle \xi^\dagger - \xi, \xi^* \rangle &= \langle \xi^\dagger - \xi, P_k\xi^* \rangle + \langle \xi^\dagger - \xi, (I - P_k)\xi^* \rangle \\
&\le \|T^*\xi^\dagger - T^*\xi\|\|T^{-1}P_k\xi^\dagger\| + \|\xi^\dagger - \xi\|\|(I - P_k)\xi^*\| \\
&\le C_{\mu,\|T^*\|B}\nu_k\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{1}{2}\Delta_{\mathcal{S}^*_p}\left(\xi, \xi^\dagger\right) + C_p\kappa_k^p,
\end{aligned}$$

where the last inequality follows by (5.4) and Young's inequality with $C_p = \frac{1}{p}\left(\frac{2}{p' c_{p',\mathcal{Y}^*}}\right)^{p/p'}$. Thus under Item (a) the claim holds true.

Now assume Item (b). By our regularity assumptions on $\mathcal{R}$, there exist constants $C_{f^\dagger}, c > 0$ such that for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \le C_{f^\dagger}$ we have the first order Taylor approximation

$$\mathcal{R}(f) \le \mathcal{R}(f^\dagger) + \left\langle T^*\xi^\dagger, f - f^\dagger \right\rangle + \frac{c}{2}\|f - f^\dagger\|^2_{\widetilde{\mathcal{X}}},$$

where c is the Lipschitz constant of $\mathcal{R}'$. Applying Young's inequalities $\mathcal{R}(f) + \mathcal{R}^*(T^*\xi) \ge \langle T^*\xi, f \rangle$ and $\mathcal{R}(f^\dagger) + \mathcal{R}^*(T^*\xi^\dagger) = \left\langle T^*\xi^\dagger, f^\dagger \right\rangle$, we find

$$\mathcal{R}^*(T^*\xi) \ge \mathcal{R}^*(T^*\xi^\dagger) + \left\langle T^*(\xi - \xi^\dagger), f^\dagger \right\rangle + \left\langle T^*(\xi - \xi^\dagger), f - f^\dagger \right\rangle - \frac{c}{2}\|f - f^\dagger\|^2_{\widetilde{\mathcal{X}}}$$

for all $\xi \in \mathcal{Y}^*$ and for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \le C_{f^\dagger}$, which is equivalent to

$$\left\langle T^*(\xi - \xi^\dagger), f - f^\dagger \right\rangle \le \Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger) + \frac{c}{2}\|f - f^\dagger\|_{\widetilde{\mathcal{X}}}^2 \tag{5.8}$$

for all $\xi \in \mathcal{Y}^*$ and for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \le C_{f^\dagger}$. We decompose the left hand side of (5.7) as follows:

$$\left\langle \xi^\dagger - \xi, \xi^* \right\rangle = \left\langle \xi^\dagger - \xi, P_k\xi^* \right\rangle + \left\langle \xi^\dagger - \xi, (I - P_k)\xi^* \right\rangle.$$

Now for some small $\varepsilon > 0$ choose f in (5.8) as $f = f^\dagger + \varepsilon T^{-1}P_k\xi^*$. Then we can conclude that

$$\left\langle \xi^\dagger - \xi, P_k\xi^* \right\rangle = \left\langle T^*(\xi^\dagger - \xi), T^{-1}P_k\xi^* \right\rangle \le \frac{1}{\varepsilon}\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger) + \frac{c\varepsilon}{2}\|T^{-1}P_k\xi^*\|_{\widetilde{\mathcal{X}}}^2.$$

Again as $\xi \notin \mathcal{B}$ we know that $\|\xi^\dagger - \xi\| \le B$. Now choose A from above as $A = \frac{C_{f^\dagger}}{B\|\xi^*\|}$. Then from $\xi \notin \mathcal{A}$ we know, that

$$\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}} \le A\|\xi^\dagger - \xi\|\|\xi^*\| \le AB\|\xi^*\| \le C_{f^\dagger}$$

so we can choose $\varepsilon = \Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}}/\nu_k$, which ensures $\|f - f^\dagger\| \le C_{f^\dagger}$. Therefore we have

$$\left\langle \xi^\dagger - \xi, P_k\xi^* \right\rangle \le \left(1 + \frac{c}{2}\right)\nu_k\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}}.$$

Combining everything and using Theorem 2.2.5 with $\mathcal{Y}^*$ being p'-convex we find

$$\begin{aligned}
\left\langle \xi^\dagger - \xi, \xi^* \right\rangle &\le \left(1 + \frac{c}{2}\right)\nu_k\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}} + \kappa_k\|\xi^\dagger - \xi\| \\
&\le \left(1 + \frac{c}{2}\right)\nu_k\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}} + C_p\kappa_k^p + \frac{1}{2}\Delta_{\mathcal{S}_p^*}\left(\xi, \xi^\dagger\right), \\
&\le \frac{1}{2}\Delta_{\mathcal{S}_p^*}\left(\xi, \xi^\dagger\right) + C\left(\nu_k\Delta_{\mathcal{R}^*}^{f^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}} + \kappa_k^p\right),
\end{aligned}$$

with $C_p = \frac{1}{p}\left(\frac{2}{p'c_{p',\mathcal{Y}^*}}\right)^{p/p'}$, $C = \max\left\{\frac{2+c}{2}, C_p, \frac{1}{A}\right\}$, which completes the proof. □

In view of (5.5) we need a variant of Lemma 5.1.1 in order to verify VSC^2 from smoothness assumptions on the true solution.

Lemma 5.1.4. *Let $V \subset \mathcal{X}$ be a normed space. For $p \in [1, 2], q \in [1, \infty]$ let $B_{p',q}^0(\mathbb{M}_0) \subset V$, $L^2(\mathbb{M}) \subset \mathcal{Y}$ with continuous embedding. Let P_k be given as in Proposition A.2.13. If for $s > 0$ we have $\xi^* \in B_{p',\infty}^s$ then we have*

$$\|(I - P_k)\xi^*\|_{\mathcal{Y}} \le C2^{-ks}$$

Let $a > s$ and assume a linear forward operator T which, viewed as $T\colon B_{p',q}^0(\mathbb{M}_0) \to B_{p',q}^a(\mathbb{M})$, has a bounded right inverse $T^\dagger$. Then we have

$$\|T^\dagger P_k\xi^*\|_V \le C2^{(a-s)k}.$$

If $g \in B_{p',q}^a$ and $s > d/2$ then also have

$$\|T^\dagger(g \cdot P_k\xi^*)\|_V \le C\|g\|_{B_{p',q}^a}2^{(a-s)k}.$$

Proof. By Proposition A.2.13 we have

$$\|(I-P_k)\xi^*\|_{\mathcal{Y}} \le C\|(I-P_k)\xi^*\|_{L^2} \le C\|(I-P_k)\xi^*\|_{B^0_{p',2}} \le C2^{-ks}\|\xi^*\|_{B^s_{p',\infty}},$$

as well as

$$\|T^\dagger P_k\xi^*\|_V \le C\|T^\dagger P_k\xi^*\|_{B^0_{p',q}} \le C\|P_k\xi^*\|_{B^a_{p',q}} \le C2^{(a-s)k}\|\xi^*\|_{B^s_{p',\infty}}.$$

If $g \in B^a_{p',q}$ and $s > d/2 \ge d/p'$ then similarly by Theorem A.2.8

$$\|T^\dagger(g\cdot P_k\xi^*)\|_V \le C\|gP_k\xi^*\|_{B^a_{p',q}} \le C\|g\|_{B^a_{p',q}}2^{(a-s)k}\|\xi^*\|_{B^s_{p',\infty}}. \qquad \square$$

The main task for verifying Assumption 4.4.4 lies in verifying (D5) respectively verifying (D8). We will actually verify both inequalities simultaneously as usually one of them should not be more difficult to prove than the other. However if one is only interested in showing (D5), then one can set $\mathcal{G}^a_R = \emptyset$ as the empty set to get a simplification.

Proposition 5.1.5 (verification of Assumption 4.4.4)**.** *Assume that there exists $\xi^\dagger \in L^2$ such that $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$. Let $\sigma > 0$ and for $R > 0$ let $\mathcal{G}^a_R = \{g \in H^a(\mathbb{M}) : \|g\|_{H^a} \le R\}$. We distinguish two cases:*

(a) Assume that for some $\mu \ge 2$ there exists a constant $C_\mu > 0$ such that

$$\forall x^* \in \mathcal{X}^* : \|x^* - T^*\xi^\dagger\|_{\mathcal{X}^*} \le C_\mu \Delta^{f^\dagger}_{\mathcal{R}^*}(x^*, T^*\xi^\dagger)^{\frac{1}{\mu}} \tag{5.9}$$

and that $P_k\xi^\dagger \in T(\mathcal{X}), (\mathcal{G}^a_R + \sigma)P_k\xi^\dagger \subset T(\mathcal{X})$. Set $\|\cdot\|_V := \|\cdot\|_{\mathcal{X}}$.

(b) Let $\widetilde{\mathcal{X}}$ be a Banach space continuously embedded in $\mathcal{X}$ and assume that $\mathcal{R}$ is continuously Fréchet-differentiable in a neighborhood of $f^\dagger \in \widetilde{\mathcal{X}}$ with respect to $\|\cdot\|_{\widetilde{\mathcal{X}}}$ and $\mathcal{R}' : \mathcal{X} \to \mathcal{X}'$ is uniformly Lipschitz continuous with respect to $\|\cdot\|_{\widetilde{\mathcal{X}}}$ in this neighborhood. Further assume that $\mathcal{R}'[f^\dagger] = (T^*\xi^\dagger)|_{\widetilde{\mathcal{X}}}$ and that $P_k\xi^\dagger \in T(\mathcal{X}), (\mathcal{G}^a_R + \sigma)P_k\xi^\dagger \subset T(\widetilde{\mathcal{X}})$. Set $\mu := 2$ and $\|\cdot\|_V := \|\cdot\|_{\widetilde{\mathcal{X}}}$.

Then there exists $C > 0$ such that (D5) and (D8) hold true for $\beta = 0$ (under (a)) or $\beta = 1$ (under (b)) and for any ν_k satisfying

$$\nu_k \ge C\max\left\{\sup_{g\in\{1-\sigma\}\cup\mathcal{G}^a_R}\|T^{-1}(g+\sigma)P_k\xi^\dagger\|_V, \|P_k\xi^\dagger\|_{B^\gamma_{p,1}}\right\} \tag{5.10}$$

In particular if $\xi^\dagger \in T(\mathcal{X}), (\mathcal{G}^a_R + \sigma)\xi^\dagger \subset T(\mathcal{X})$ one can choose $P_k = I$ for all $k \in \mathbb{N}_0$ and thus (D5) and (D8) hold true with $\nu_k = C$ for some $C > 0$ and we have $\kappa_k = 0$ such that Ψ as in (4.17) fulfills $\Psi(\alpha) \le C\alpha^{\mu'-1}$ for α sufficiently small.

Proof. Recall that we want to show the inequalities (D5), i.e.

$$\forall\xi \in B^{-\gamma}_{p',\infty} : \langle\xi^\dagger - \xi, P_k\xi^\dagger\rangle \le \frac{1}{4}\|\xi^\dagger - \xi\|^2_{B^{-\gamma}_{p',\infty}} + \nu_k\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\beta\nu_k^2}{2}\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger)$$

and (D8), i.e. for all $\xi \in B^{-\gamma}_{p',\infty}$, $g \in \mathcal{G}^a_{LB}$

$$\langle\xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger\rangle \le \frac{1}{4}\|(g+\sigma)(\xi^\dagger - \xi)\|^2_{B^{-\gamma}_{p',\infty}} + \nu_k\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}} + \frac{\beta\nu_k^2}{2}\Delta^{f^\dagger}_{\mathcal{R}^*}(T^*\xi, T^*\xi^\dagger).$$

of Assumption 4.4.4.

(a) By (5.9) we find for $g \in \{1-\sigma\} \cup \mathcal{G}_R^a$ that

$$\langle \xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger \rangle = \langle T^*\xi^\dagger - T^*\xi, T^{-1}(g+\sigma)P_k\xi^\dagger \rangle \leq \nu_k \|T^*\xi^\dagger - T^*\xi\|_{\mathcal{X}^*} \\ \leq C_\mu \nu_k \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{\mu}}.$$

Therefore (D5) and (D8) holds true for all $\xi \in B^{-\gamma}_{p',\infty}$ with $\beta = 0$. [2]

(b) The proof of this case is similar to the one of Proposition 5.1.3. For all $k \in \mathbb{N}_0$, $\xi \in B^{-\gamma}_{p',\infty}$ we distinguish two cases:

Case 1: We assume $\|P_k\xi^\dagger\|_{B^\gamma_{p,1}} \leq \nu_k \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}}$. Then we have by Young's inequality

$$\langle \xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger \rangle = \langle (g+\sigma)(\xi^\dagger - \xi), P_k\xi^\dagger \rangle \leq \|(g+\sigma)(\xi^\dagger - \xi)\|_{B^{-\gamma}_{p',\infty}} \|P_k\xi^\dagger\|_{B^\gamma_{p,1}} \\ \leq \frac{1}{4}\|(g+\sigma)(\xi^\dagger - \xi)\|^2_{B^{-\gamma}_{p',\infty}} + \nu_k^2 \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger)$$

Case 2: By our regularity assumptions on $\mathcal{R}$, there exist constants $C_{f^\dagger}, c > 0$ such that for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \leq C_{f^\dagger}$ we have the first order Taylor approximation

$$\mathcal{R}(f) \leq \mathcal{R}(f^\dagger) + \left\langle T^*\xi^\dagger, f - f^\dagger \right\rangle + \frac{c}{2}\|f - f^\dagger\|^2_{\widetilde{\mathcal{X}}},$$

where c is the Lipschitz constant of $\mathcal{R}'$. Applying Young's inequalities $\mathcal{R}(f) + \mathcal{R}^*(T^*\xi) \geq \langle T^*\xi, f \rangle$ and $\mathcal{R}(f^\dagger) + \mathcal{R}^*(T^*\xi^\dagger) = \left\langle T^*\xi^\dagger, f^\dagger \right\rangle$, we find

$$\mathcal{R}^*(T^*\xi) \geq \mathcal{R}^*(T^*\xi^\dagger) + \left\langle T^*(\xi - \xi^\dagger), f^\dagger \right\rangle + \left\langle T^*(\xi - \xi^\dagger), f - f^\dagger \right\rangle - \frac{c}{2}\|f - f^\dagger\|^2_{\widetilde{\mathcal{X}}}$$

for all $\xi \in \mathcal{Y}^*$ and for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \leq C_{f^\dagger}$, which is equivalent to

$$\langle T^*(\xi - \xi^\dagger), f - f^\dagger \rangle \leq \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger) + \frac{c}{2}\|f - f^\dagger\|^2_{\widetilde{\mathcal{X}}} \tag{5.11}$$

for all $\xi \in \mathcal{Y}^*$ and for all $f \in \widetilde{\mathcal{X}}$ with $\|f - f^\dagger\|_{\widetilde{\mathcal{X}}} \leq C_{f^\dagger}$. Now for some small $\varepsilon > 0$ choose f in (5.11) as $f = f^\dagger - \varepsilon T^{-1}(g+\sigma)P_k\xi^\dagger$. Then we can conclude that

$$\left\langle \xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger \right\rangle = \left\langle T^*(\xi^\dagger - \xi), T^{-1}(g+\sigma)P_k\xi^\dagger \right\rangle \\ \leq \frac{1}{\varepsilon}\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger) + \frac{c\varepsilon}{2}\|T^{-1}(g+\sigma)P_k\xi^\dagger\|^2.$$

Choose ν_k such that $\nu_k \geq C_{f^\dagger}^{-1}\|P_k\xi^\dagger\|_{B^\gamma_{p,1}}$. As we are not in Case 1 we have

$$\Delta_{\mathcal{R}^*}^{f^\dagger}(T^*\xi, T^*\xi^\dagger)^{\frac{1}{2}} \leq \frac{1}{\nu_k}\|P_k\xi^\dagger\|_{B^\gamma_{p,1}} \leq C_{f^\dagger}$$

so we can choose $\varepsilon = \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*p, T^*\xi^\dagger)^{\frac{1}{2}}/\nu_k$, which ensures $\|f - f^\dagger\| \leq C_{f^\dagger}$. Therefore we have

$$\left\langle \xi^\dagger - \xi, (g+\sigma)P_k\xi^\dagger \right\rangle \leq \left(1 + \frac{c}{2}\right)\nu_k \Delta_{\mathcal{R}^*}^{f^\dagger}(T^*p, T^*\xi^\dagger)^{\frac{1}{2}}. \qquad \square$$

[2] Under the slightly weaker assumption (5.4) instead of (5.9) we can still verify (D5) as for $4\|P_k\xi^\dagger\|_{B^\gamma_{p,1}} \leq \|\xi^\dagger - \xi\|_{B^\gamma_{p',\infty}}$ we have $\langle \xi^\dagger - \xi, P_k\xi^\dagger \rangle \leq \frac{1}{4}\|\xi^\dagger - \xi\|^2_{B^\gamma_{p',\infty}}$, thus we can assume $\|T^*\xi^\dagger - T^*\xi\|_{\mathcal{X}^*}$ to be bounded.

5.1.3 Verification of third order VSCs

Proposition 5.1.6 (verification of $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$)**.** *Let $\omega^\dagger \in \mathcal{X}$ as in Definition 4.5.3 exist. Let $0 < \bar{t} \leq 1$ and assume that for all $f^* \in \partial\mathcal{R}(f^\dagger)$ there exists some $\omega^* \in \mathcal{X}^*$ such that*

$$\|f^* - f_t^* - t\omega^*\|_{\mathcal{X}^*} \leq C_{\omega^\dagger} t^2, \tag{5.12}$$

whenever $0 < t \leq \bar{t}$ and $f_t^* \in \partial\mathcal{R}(f^\dagger - t\omega^\dagger)$. This last assumption follows for example from $\mathcal{R}$ being two times Fréchet-differentiable in $\mathcal{X}$ in a neighborhood of $f^\dagger$ with $\mathcal{R}'' \colon \mathcal{X} \to L(\mathcal{X}, \mathcal{X}^*)$ uniformly Lipschitz continuous in this neighborhood. Further assume

$$\Delta_{\mathcal{R}}(f_1, f_2) \geq C_\mu \|f_1 - f_2\|^\mu \tag{5.13}$$

for some $\mu > 1$, $C_\mu > 0$ and all $f_1, f_2 \in \mathrm{dom}(\mathcal{R})$. We have:

(a) If $\omega^* = T^*\xi_2^\dagger$ for some $\xi_2^\dagger \in \mathcal{Y}^*$, then $f^\dagger$ fulfills the $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$ with $\Phi(\tau) := \|\xi_2^\dagger\|\tau^{1/p}$.

(b) Suppose that $\mu \leq 2$, $\frac{1}{\mu} + \frac{1}{\mu'} = 1$, and that there exists a family of operators $P_k \in L(\mathcal{X}^*)$ indexed by $k \in \mathbb{N}$ such that $P_k\omega^* \in T^*\mathcal{Y}^*$ for all $k \in \mathbb{N}$, and let

$$\kappa_k := \|(I - P_k)\omega^*\|_{\mathcal{X}^*}, \qquad \nu_k := \|(T^*)^{-1}P_k\omega^*\|_{\mathcal{Y}^*}. \tag{5.14}$$

If $\lim_{j\to\infty} \kappa_k = 0$, then $f^\dagger$ fulfills the $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$ with the index function

$$\Phi(\tau) := \inf_{k\in\mathbb{N}} \left[\nu_k \tau^{1/q} + \frac{\kappa_k^{\mu'}}{\mu'(C_\mu \mu)^{\mu'/\mu}} \right]. \tag{5.15}$$

Proof. Recall that the $\mathrm{VSC}^3(\Phi, \mathcal{R}, \mathcal{S}_p)$ is of the form

$$\forall f \in \mathcal{X}\ \forall t \in (0, \bar{t}]:$$
$$\left\langle f_t^* - T^*\xi^\dagger, f^\dagger - t\omega^\dagger - f \right\rangle \leq \Delta_{\mathcal{R}}^{f_t^*}\left(f, f^\dagger - t\omega^\dagger\right) + t^2\Phi\left(t^{-p}\|Tf - g^\dagger + tT\omega^\dagger\|^p\right) + \beta t^{2\mu}.$$

Firstly, to prove that (5.12) is implied by two times differentiability with $\mathcal{R}''$ Lipschitz continuous, recall that $\partial\mathcal{R}(f) = \{\mathcal{R}'[f]\}$ if $\mathcal{R}$ is Fréchet-differentiable in $\mathcal{X}$. Then by the first order Taylor approximation of $t \mapsto \mathcal{R}'[f^\dagger - t\omega^\dagger]$ at $t = 0$ we have

$$\|\mathcal{R}'[f^\dagger - t\omega^\dagger] - \mathcal{R}'[f^\dagger] + t\mathcal{R}''[f^\dagger](\omega^\dagger, \cdot)\|_{\mathcal{X}^*} \leq Ct^2\|\omega^\dagger\|^2$$

for some $C > 0$. Thus (5.12) holds with $\omega^* = \mathcal{R}''[f^\dagger](\omega^\dagger, \cdot)$ and $C_{\omega^\dagger} = C\|\omega^\dagger\|^2$. Now let (5.12) hold true, then we have for all $f_t^* \in \partial\mathcal{R}(f^\dagger - t\omega^\dagger)$ that

$$\left\langle f_t^* - T^*\xi^\dagger, f^\dagger - t\omega^\dagger - f \right\rangle \leq -t\left\langle \omega^*, f^\dagger - t\omega^\dagger - f \right\rangle + C_{\omega^\dagger} t^2\|f^\dagger - t\omega^\dagger - f\|. \tag{5.16}$$

Then using (5.13) and Young's inequality, we find that

$$C_{\omega^\dagger} t^2\|f^\dagger - t\omega^\dagger - f\| \leq \gamma t^2 \Delta_{\mathcal{R}}\left(f, f^\dagger - t\omega^\dagger\right)^{\frac{1}{\mu}} \leq \Delta_{\mathcal{R}}\left(f, f^\dagger - t\omega^\dagger\right) + \beta t^{2\mu'}$$

with $\gamma := C_{\omega^\dagger} C_\mu^{-\frac{1}{\mu}}$ and $\beta := \frac{1}{\mu'}\mu^{-\frac{\mu'}{\mu}}\gamma^{\mu'}$.

So we only need to bound the first term on the right hand side of (5.16) and this is done in two ways based on the two different assumptions:

(a) If $\omega^* = T^*\xi_2^\dagger$, then

$$-t\left\langle \omega^*, f^\dagger - t\omega^\dagger - f\right\rangle = -t\left\langle \xi_2^\dagger, g^\dagger - tT\omega^\dagger - Tf\right\rangle$$
$$\leq t^2\|\xi_2^\dagger\|\left(t^{-p}\|Tf - g^\dagger + tT\omega^\dagger\|^p\right)^{1/p}.$$

Hence Assumption 4.5.3 holds true $\Phi(\tau) := \|\xi_2^\dagger\|\tau^{1/p}$.

(b) In the second case we have for all $k \in \mathbb{N}$ with $c_\mu := \frac{1}{\mu'(C_\mu\mu)^{\mu'/\mu}}$ that

$$\begin{aligned}
&- t\left\langle \omega^*, f^\dagger - t\omega^\dagger - f\right\rangle \\
&= -t\left\langle P_k\omega^*, f^\dagger - t\omega^\dagger - f\right\rangle - t\left\langle (I - P_k)\omega^*, f^\dagger - t\omega^\dagger - f\right\rangle \\
&\leq t\nu_k\|Tf - g^\dagger + tT\omega^\dagger\| + t\kappa_k\|f^\dagger - t\omega^\dagger - f\| \\
&\leq t^2\left(\nu_k t^{-1}\|Tf - g^\dagger + tT\omega^\dagger\| + c_\mu t^{\mu'-2}\kappa_k^{\mu'}\right) + C_\mu\|f^\dagger - t\omega^\dagger - f\|^\mu \\
&\leq t^2\left(\nu_k t^{-1}\|Tf - g^\dagger + tT\omega^\dagger\| + c_\mu\kappa_k^{\mu'}\right) + \Delta_{\mathcal{R}}\left(f, f^\dagger - t\omega^\dagger\right)
\end{aligned}$$

for $t \leq \bar{t} \leq 1$ as $\mu' \geq 2$. Substituting $\tau = t^{-p}\|Tf - g^\dagger + tT\omega^\dagger\|^p$ and taking the infimum over k shows Assumption 4.5.3 with Φ given by (5.15). It follows as in [41, Thm. 2.1] that Φ is an index function.

□

5.1.4 Required properties of penalty functionals

We give a short summary of the properties that we have to require from the penalty functional in order to verify the source conditions and perform the convergence analysis as in the last chapter.

(a) One has to understand the subdifferential $\partial\mathcal{R}(f^\dagger)$ in order to transfer smoothness properties of $f^\dagger$ to the elements in the subdifferential.

(b) One has to be able to show that the assumptions of either Proposition 4.3.6, Proposition 5.1.3, Proposition 5.1.5 or Proposition 5.1.6 on $\mathcal{R}$ hold true.

(c) For the statistical setting we need that (A3) of Assumption 4.2.1 and (B3) of Assumption 4.3.9 hold true.

(d) For higher order rates we need strong duality for which it is sufficient that $\mathcal{R}^*$ is continuous, given the data fidelities of Section 3.2.

We give a short exemplary overview how these properties can be shown for the penalties that we are especially interested in.

Squared norm on Hilbert spaces The first case where all of these restrictions are met in the most simple and beautiful way is of course the standard case of $\mathcal{X}$ being a Hilbert space and $\mathcal{R} = \frac{1}{2}\|\cdot\|_{\mathcal{X}}^2$. Then the differential at $f^\dagger$ coincides with $f^\dagger$ and $\mathcal{R}^* = \mathcal{R}$ such that the assumption of Proposition 4.3.6, case (a) of Proposition 5.1.5 and (B3) hold true by

$$\Delta_{\mathcal{R}}^{f_2}(f_1, f_2) = \frac{1}{2}\|f_1 - f_2\|^2.$$

In the case of the Kullback-Leibler type data fidelity term we would like to incorporate the constraint $f \in \mathcal{B}$ for a convex set $\mathcal{B}$ into the penalty function. So let

$$\widetilde{\mathcal{R}}(f) := \frac{1}{2}\|f\|_{\mathcal{X}}^2 + \chi_{\mathcal{B}}(f) := \begin{cases} \frac{1}{2}\|f\|_{\mathcal{X}}^2 & \text{if } f \in \mathcal{B} \\ \infty & \text{else.} \end{cases}$$

If we have $f^\dagger$ in the interior of $\mathcal{B}$ then we still have that $\widetilde{\mathcal{R}}$ is differentiable at $f^\dagger$ with derivative $f^\dagger$ and

$$\widetilde{\mathcal{R}}^*(f) = \frac{1}{2}\|f\|_{\mathcal{X}}^2 - \inf_{\tilde{f}\in\mathcal{B}}\left[\frac{1}{2}\|f - \tilde{f}\|_{\mathcal{X}}^2\right]$$

is continuous everywhere. Further $\widetilde{\mathcal{R}}$ satisfies the condition (b) of Proposition 5.1.5 and we still have

$$\Delta_{\widetilde{\mathcal{R}}}^{f^\dagger}(f, f^\dagger) = \frac{1}{2}\|f - f^\dagger\|^2 + \chi_{\mathcal{B}}(f) \geq \frac{1}{2}\|f - f^\dagger\|^2.$$

In particular $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathcal{S}^\dagger)$ trivially implies $\mathrm{VSC}^1(\Phi, \widetilde{\mathcal{R}}, \mathcal{S}^\dagger)$.

Remark 5.1.7. *For many operators it is possible to find a convex and closed subset $\mathcal{B} \in \mathcal{X}$ such that (D9) holds true with $f^\dagger$ in the interior of $\mathcal{B}$ if $f^\dagger \geq 0$. In particular this is the case if T is positivity preserving, i.e. $Tf \geq 0$ for all $f \geq 0$ and T fulfills (D2) with $a \geq d/p$. Then let $f =: f_+ + f_-$, with $f_+ = \max(f, 0), f_- = \min(f, 0)$ and for some sufficiently small $C_\sigma > 0$ choose $\mathcal{B} = \{f \in \mathcal{X} : \|f_-\| \leq C_\sigma\}$. Then $f^\dagger$ is in the interior of $\mathcal{B}$ and we have for all $f \in \mathcal{B}$ that $\|Tf - Tf_+\|_{L^\infty} \leq C\|Tf_-\|_{B^a_{p,2}} \leq CLC_\sigma \leq \sigma/2$ for C_σ chosen appropriately, by (D2) and Theorem A.2.5. Thus we also have $Tf \geq -\sigma/2$.*

Note that in many applications with Poisson data the true solution $f^\dagger$ will be some density and thus positive. $Tf \geq 0$ for all $f \geq 0$ holds true for example for a convolution operator with positive kernel.

Cross-entropy penalty Higher order convergence rates for the choice $\mathcal{X} = L^1(\mathbb{M})$, $\mathcal{R}(f) = \mathrm{KL}(f, f_0)$ for some initial guess $f_0 \in \mathcal{X}$ have already been discussed in [64] and unsurprisingly this choice of $\mathcal{R}$ also works in the statistical setting of this paper. We have $\partial\mathcal{R}(f^\dagger) = \log\frac{f^\dagger}{f_0}$ so that under suitable assumptions on the initial guess the smoothness of $f^\dagger$ transfers to the subdifferential. Case (b) of Proposition 5.1.5 is actually tailored to the cross entropy functional. (A3) of Assumption 4.2.1 and (B3) of Assumption 4.3.9 are fulfilled by Lemma 2.3.2. Finally $\mathcal{R}^*$ is continuous in $L^\infty(\mathbb{M})$ by Proposition 3.2.1.

Note that if $f_0 \geq 0$ then $\mathcal{R} = \mathcal{R} + \chi_{\mathcal{B}}$, where $\mathcal{B} = \{f \in L^1(\mathbb{M}) : f \geq 0\}$ so (D9) holds for a positivity preserving forward operator and there is no necessity for additional constraints.

Wavelet Besov norm penalty In [74] deterministic and statistical convergence rates under a VSC were shown for (3.8) with $\mathcal{X} = B^0_{p,q}$ a Besov space and $\mathcal{R} = \frac{1}{t}\|\cdot\|^t_{B^0_{p,q}}$. Smoothness of the subgradient was a key result of [74] and is given by Theorem 5.3.1. By [46] the space $B^0_{p,q}$ is $r = \max(2, p, q)$-convex, so if one chooses $t = r$, then by Theorem 2.2.5

$$\|f_1 - f_2\|^r \leq C_r\Delta_{\mathcal{R}}^{f_2^*}(f_1, f_2). \tag{5.17}$$

Further $\mathcal{R}^* = \frac{1}{r'}\|\cdot\|^{r'}_{B^0_{p',q'}}$ is continuous and as $B^0_{p',q'}$ is $\max(2,p',q') =: \mu$ convex by Lemma 2.2.6 we have

$$\|x^* - T^*\xi^\dagger\|^\mu_{\mathcal{X}^*} \le C_\mu \Delta^{f^\dagger}_{\mathcal{R}^*}(x^*, T^*\xi^\dagger), \tag{5.18}$$

for x^* sufficiently close to $T^*\xi^\dagger$ by Theorem 2.2.5 so case (a) of Proposition 5.1.3 holds true. If $t = \mu$, then also case (a) of Proposition 5.1.5 holds true.

5.1.5 a-smoothing forward operator

Concerning the forward operator its smoothing property is most important for the verification of source conditions. We will give a short summary of the properties that we require for the forward operator. Notice that we will only consider linear forward operators as we are mostly interested in higher order convergence rates and for these we need duality and thus linearity of the forward operator. However, verification is similarly possible given a non linear forward operator, see [74, Section 4] and [39].

Note that in the following we will only consider finitely smoothing forward operators. This excludes severely ill-posed inverse problems [25, Section 2.2]. The reason for this neglect of infinitely smoothing forward operators is that a first order variational source condition is typically sufficient to yield the complete range of convergence rates for these operators (with the exception of extremely smooth true solutions). For an example of convergence rates for a severely ill-posed inverse problems under variational source conditions we again refer to [74].

Assumption 5.1.8. *Let $T\colon \mathcal{X} \to \mathcal{Y} \subset L^2(\mathbb{M})$ be linear and bounded, let T^* denote the L^2-adjoint of T, let $p \in [1,2]$ as in Assumption 1.4.1, $\mathbb{M}, \mathbb{M}_0 \subset \mathbb{R}^d$ bounded Lipschitz domains or d-dimensional tori and $a > 0$. Assume that T is a-smoothing in the sense that it has some of the following properties.*

(T1) T viewed as a mapping $T\colon \mathcal{X} \to B^a_{p,2}(\mathbb{M})$ is bounded.
(T2) $T^*\colon L^2(\mathbb{M}) \to B^a_{2,2}(\mathbb{M}_0)$ has a bounded right inverse $(T^*)^\dagger$.
(T3) $T\colon B^0_{p',q}(\mathbb{M}_0) \to B^a_{p',q}(\mathbb{M})$ has a bounded right inverse $T^\dagger$ for all $q \in [1,\infty]$.
(T4) $T^*\colon B^s_{p',\infty}(\mathbb{M}) \to B^{s+a}_{p',\infty}(\mathbb{M}_0)$ has a bounded right inverse $(T^*)^\dagger$ for all $s \in (0,a)$.
(T5) $T\colon B^s_{p',\infty}(\mathbb{M}_0) \to B^{s+a}_{p',\infty}(\mathbb{M})$ and $T^*\colon B^{s+a}_{p',\infty}(\mathbb{M}) \to B^{s+2a}_{p',\infty}(\mathbb{M}_0)$ have bounded right inverses for all $s \in (0,a)$.
(T6) $T\colon B^{-a}_{2,2}(\mathbb{M}_0) \to L^2(\mathbb{M})$ is bounded.
(T7) $T\colon B^0_{p,q}(\mathbb{M}_0) \to B^a_{p,q}(\mathbb{M})$ has a bounded inverse T^{-1} and $T\colon B^s_{p,\infty}(\mathbb{M}_0) \to B^{s+a}_{p,\infty}(\mathbb{M})$ is bounded and surjective for all $s \in (0,a)$.

The first property is necessary for the stochastic error analysis in Chapter 4 (see e.g. (D2) of Assumption 4.4.4). The second respectively third property is required so that we can apply Lemma 5.1.1 respectively 5.1.4. Properties (T4) and (T5) are used in order to get smoothness of preimages of $f^* \in B^{s+a}_{p',\infty}(\mathbb{M}_0)$ respectively of $f^* \in B^{s+2a}_{p',\infty}(\mathbb{M}_0)$. The last two properties will allow us to show optimality of convergence rates in Section 5.3.

Example 5.1.9. *Note that for $s \in \mathbb{R}$, $p,q \in [1,\infty]$ the operator $(\rho\,\mathrm{id} - \Delta)^{-k}\colon B^s_{p,q}(\mathbb{T}^d) \to B^{s+2k}_{p,q}(\mathbb{T}^d)$, where Δ is the Laplace-Beltrami operator, is an isomorphism if $\rho > 0$ is sufficiently large by [70, Theorem 7.4.3]. Therefore Assumption 5.1.8 holds true for all*

$p \in [1,2]$, $a = 2k$ and $\mathbb{M} = \mathbb{M}_0 = \mathbb{T}^d$ if $T = (\rho\,\mathrm{id} - \Delta)^{-k}$. More generally this assumption can be shown for convolution operators, for which the convolution kernel has a certain type of singularity at 0, boundary integral operators, injective elliptic pseudodifferential operators, and compositions of such operators.

Remark 5.1.10. *As laid out in [39, Example 2.7] one can also consider the Radon transform on a bounded domain $\mathbb{M}_0 \subset \mathbb{R}^d$ as an a-smoothing Forward operator with $a = \frac{d-1}{2}$. The Radon transform appears as forward operator in computed tomography (CT) and positron emission tomography (PET), among others. However, for the Radon transform the measurement manifold $\mathbb{M}$ has to be chosen as $S^{d-1} \times \mathbb{R}$, where $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere, which does not quite fit into the setting of Assumption 5.1.8. Considering more general manifolds in our framework is only a technical problem: One would have to ensure that the results of the appendix remain true and verify the deviation inequality of Assumption 1.4.1 in order to treat statistical noise.*

In Section 5.4 we will work with a different definition of a-smoothing, which is simpler and slightly more general, but of the same nature as the above assumption.

5.2 Maximum entropy regularization[3]

In this section we will verify our source conditions in the case where the penalty term is chosen as a cross-entropy term given by the Kullback-Leibler divergence

$$\mathcal{R}(f) := \mathrm{KL}(f, f_0) := \int_{\mathbb{M}_0} \left[f \ln \frac{f}{f_0} - f + f_0 \right] dx \tag{5.19}$$

for $\mathbb{M}_0 \subset \mathbb{R}^d$ a bounded Lipschitz domain or $\mathbb{T}^d$. Here f_0 is some a-priori guess of f, possibly constant. Under the source condition (4.12) convergence rates of order $\|\hat{f}_\alpha - f^\dagger\|_{L^1} = \mathcal{O}(\sqrt{\delta})$ were shown in [22] by variational methods and in [27] by a reformulation as Tikhonov regularization with quadratic penalty term for a nonlinear forward operator. In [57] the faster rate $\|\hat{f}_\alpha - f^\dagger\|_{L^1} = \mathcal{O}(\delta^{2/3})$ was obtained under the source condition $T^*T\omega^\dagger \in \partial\mathcal{R}(f^\dagger)$.

5.2.1 Convergence rates under Hölder-Zygmund smoothness assumptions

Let $\mathcal{Y} = L^2(\mathbb{M})$ and $T\colon L^1(\mathbb{M}_0) \to L^2(\mathbb{M})$ linear and bounded. We apply generalized Tikhonov regularization in the form of maximum entropy regularization

$$\hat{f}_\alpha \in \operatorname*{arg\,min}_{f \in L^1(\mathbb{M}_0)} \left[\mathcal{S}(Tf) + \alpha\,\mathrm{KL}(f, f_0) \right], \tag{5.20}$$

respectively Bregman iteration given by

$$\hat{f}_\alpha^{(n)} \in \operatorname*{arg\,min}_{f \in L^1(\mathbb{M}_0)} \left[\mathcal{S}(Tf) + \alpha\,\mathrm{KL}\left(f, \hat{f}_\alpha^{(n-1)}\right) \right]. \tag{5.21}$$

[3]This section has some overlap with the article [64] to which the author contributed.

As we consider $\mathcal{X} = L^1(\mathbb{M}_0)$ we have $\mathcal{X}^* = L^\infty(\mathbb{M}_0)$ so that in view of Lemma 5.1.1 it will be natural to measure the smoothness of the true solution in the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{M}_0) := B^s_{\infty,\infty}(\mathbb{M}_0)$. Note that the definition of these spaces (see e.g. [69]) is much more straightforward and easier to relate to differentiability than the definitions of Besov spaces in the appendix.

Theorem 5.2.1. *Let $\mathbb{M}, \mathbb{M}_0 \subset \mathbb{R}^d$ be bounded Lipschitz domains or $\mathbb{T}^d$,$\mathcal{X} = L^1(\mathbb{M}_0)$, $\mathcal{Y} = L^2(\mathbb{M})$. Let $f_0 \in L^1(\mathbb{M}_0), f_0 \geq 0$. Suppose that T is a-smoothing in the sense that Assumption 5.1.8 (T1)-(T5) hold true for $p = 1$. Moreover, suppose there exists $c > 0$ such that*

$$c \leq \frac{f^\dagger}{f_0} \leq c^{-1} \qquad \text{a.e. in } \mathbb{M}. \tag{5.22}$$

We distinguish the different orders of convergence rates:

(a) Assume that

$$\frac{f^\dagger}{f_0} \in B^s_{\infty,\infty}(\mathbb{M}_0) \qquad \text{for some } s \in (0, a)\,.$$

Then there exists $C > 0$ such $f^\dagger$ fulfills $\mathrm{VSC}^1(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}^2_{g^\dagger})$ with

$$\Phi(\tau) = C\tau^{\frac{s}{s+a}}.$$

(b) Assume that

$$\frac{f^\dagger}{f_0} \in B^s_{\infty,\infty}(\mathbb{M}_0) \qquad \text{for some } s \in (a, 2a)\,.$$

Then there exists $C > 0$ such that $\mathrm{VSC}^2(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}_2)$ holds true with

$$\Phi(\tau) = C\tau^{\frac{s-a}{s}}.$$

Further (D5)-(D8) in Assumption 4.4.4 hold true and for Ψ as in (4.17) we have

$$\Psi(\tau) \leq C\tau^{\frac{s-a}{s}}$$

(c) Assume additionally to (5.22) that $f^\dagger \geq \rho$ and

$$f^\dagger, f_0 \in B^s_{\infty,\infty}(\mathbb{M}_0) \qquad \text{for some } s \in (2a, 3a)\,. \tag{5.23}$$

Let $B \geq 2\|f^\dagger\|_{L_1}$ and χ_B be the characteristic function of $\|f\|_{L^1} \leq B$. Then there exists $C > 0$ such that $f^\dagger$ fulfills $\mathrm{VSC}^3(\Phi, \mathrm{KL}(\cdot, f_0) + \chi_B, \mathcal{S}_2)$ with $\mu = 2$ and

$$\Phi(\tau) = C\tau^{\frac{s-2a}{s-a}}.$$

Let $\hat{f}^{(2)}_\alpha$ be given by (5.21). For $s \in (a, 2a) \cup (2a, 3a)$ we obtain under the deterministic model with $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$ for the parameter choice $\alpha \sim \delta^{\frac{2a}{s+a}}$ the convergence rates

$$\mathrm{KL}\left(\hat{f}^{(2)}_\alpha, f^\dagger\right) = \mathcal{O}\left(\delta^{\frac{2s}{s+a}}\right), \qquad \delta \to 0. \tag{5.24}$$

Let $\widehat{f}_\alpha$ be given by (5.20). Under the first and second order smoothness assumptions ($s \in (0,a) \cup (a, 2a)$) we get

$$\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\delta^{\frac{2s}{s+a}}\right), \qquad \delta \to 0. \tag{5.25}$$

under the deterministic noise model with $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$. Under the Gaussian white noise model with $\mathbb{M} = \mathbb{T}^d$ and $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ we get for $a > \gamma > d/2$ and a parameter choice $\alpha \sim \varepsilon^{\frac{2a}{s+a+\gamma}}$ that

$$\mathbb{E}\Big(\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) = \mathcal{O}\left(\varepsilon^{\frac{2s}{s+a+\gamma}}\right), \qquad \varepsilon \to 0. \tag{5.26}$$

Proof. (a) We want to apply Proposition 4.3.6 to verify $\mathrm{VSC}^1(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}^2_{g^\dagger})$, however there is the slight complication that (4.6) does not hold uniformly for all $f \in L^1$. Still, the variational inequality (4.2) holds obviously for non positive f (as $\mathcal{R}$ is just infinity then) and as in (5.3) the variational inequality holds trivially if $\|f\|_{L^1} > C_{f^\dagger}$. For $\|f\|_{L^1} \leq C_{f^\dagger}$ we have (4.6) with $r = 2$ by Lemma 2.3.2, thus we can then apply Proposition 4.3.6.

Let $f^* = \log\Big(\frac{f^\dagger}{f_0}\Big)$. By Theorem A.2.9 we have $f^* \in B^s_{\infty,\infty}(\mathbb{M})$. We apply Proposition 4.3.6 with P_k as in Proposition A.2.13, $\nu_k = \|(T^*)^\dagger P_k f^*\|_{\mathcal{Y}^*}$ and $\kappa_k = \|(I - P_k) f^*\|_{\mathcal{X}^*}$ to find that $f^\dagger$ fulfills $\mathrm{VSC}^1(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}^2_{g^\dagger})$ with

$$\Phi(\tau) = C \inf_{k \in \mathbb{N}_0} \left[\nu_k \tau^{1/2} + \kappa_k^2\right] \leq C \inf_{k \in \mathbb{N}_0} \left[2^{(a-s)k} \tau^{1/2} + 2^{-2ks}\right],$$

where the inequality follows from Lemma 5.1.1 as $B^0_{\infty,2}(\mathbb{M}_0) \subset L^\infty(\mathbb{M}_0) = \mathcal{X}^*$. Choosing $2^{-k} \sim \tau^{\frac{1}{2(s+a)}}$ gives $\Phi(\tau) \leq C\tau^{\frac{s}{s+a}}$. The deterministic rate (5.25) then follows from Theorem 4.3.4 and the stochastic rate (5.26) follows from Corollary 4.3.12 and Lemma 4.3.13, which give

$$\mathbb{E}\Big(\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) \leq C \frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big) \leq C\left(\frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}} + \alpha^{\frac{s}{a}}\right),$$

where γ has to be chosen such that $\gamma > d/2$ by Theorem 1.4.5 as $\mathcal{X} = L^1$ and we thus only have $T : \mathcal{X} \to B^a_{p,2}$ bounded only for $p = 1$ by Theorem A.2.5. The parameter choice $\alpha \sim \varepsilon^{\frac{2a}{s+a+\gamma}}$ then yields (5.26).

(b) Note that due to (5.22) the functional $\mathcal{R}$ is Fréchet differentiable at $f^\dagger$ in $L^\infty(\mathbb{M}_0)$ and

$$\mathcal{R}'[f^\dagger](g) = \int_{\mathbb{M}_0} \ln\left(\frac{f^\dagger}{f_0}\right) g\,dx, \quad \mathcal{R}''[f^\dagger](g,h) = \int_{\mathbb{M}_0} \frac{1}{f^\dagger} hg\,dx \tag{5.27}$$

Local Lipschitz continuity of $\mathcal{R}'$ w.r.t. $\widetilde{\mathcal{X}} := L^\infty(\mathbb{M}_0)$ follows from local boundedness of $\mathcal{R}''$ w.r.t. $\widetilde{\mathcal{X}}$. As above we have $f^* \in B^s_{\infty,\infty}(\mathbb{M}_0)$ and by (T4) $(T^*)^\dagger f^* = \xi^\dagger \in B^{s-a}_{\infty,\infty}(\mathbb{M})$ exists. Thus we can now apply both Proposition 5.1.3 and Proposition 5.1.5 with P_k as in Proposition A.2.13 in order to verify $\mathrm{VSC}^2(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}_2)$ respectively (D5)-(D8) of Assumption 4.4.4. By Lemma 5.1.4 we have $\kappa_k \leq C2^{-k(s-a)}$

and $\nu_k \leq C2^{k(2a-s)}$ (note that regarding (D8) the constant for ν_k depends on $\mathcal{G}_R^a$ and σ). Thus $\mathrm{VSC}^2(\Phi, \mathrm{KL}(\cdot, f_0), \mathcal{S}_2)$ holds with

$$\Phi(\tau) \leq C \inf_{k\in\mathbb{N}_0} \left[2^{(2a-s)k}\tau^{1/2} + 2^{-2k(s-a)}\right] \leq C\tau^{\frac{s-a}{s}},$$

where the second inequality follows from $2^{-k} \sim \tau^{\frac{1}{2s}}$. By Theorem 4.4.2 and Lemma 4.5.6 we have the deterministic estimates

$$\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(f, f^\dagger) = \mathcal{O}\left(\frac{\delta^2}{\alpha} + \alpha(-\Phi)^*\left(\frac{-1}{\alpha}\right)\right) = \mathcal{O}\left(\frac{\delta^2}{\alpha} + \alpha^{\frac{s}{a}}\right),$$

for both $f = \widehat{f}_\alpha$ and $f = \hat{f}_\alpha^{(2)}$. The parameter choice $\alpha \sim \delta^{\frac{2a}{s+a}}$ gives the claim. Under the white noise model we have for Ψ as in (4.17) that

$$\Psi(\alpha) = \inf_{\substack{k\in\mathbb{N}_0 \\ \alpha\nu_k^2\leq 1}} \left[\alpha\nu_k^2 + \kappa_k^2\right] \leq \begin{cases} C\alpha^{\frac{s-a}{a}} \text{ if } C\alpha^{\frac{s-a}{a}} \leq 1 \\ \infty \text{ else.} \end{cases} \tag{5.28}$$

Thus the white noise rate (5.26) follows from Corollary 4.4.10.

(c) Assumption (5.13) of Proposition 5.1.6 is satisfied for $\widetilde{\mathcal{R}} = \mathrm{KL}(\cdot, f_0) + \chi_B$ with $\mu = 2$ due to Lemma 2.3.2. By Theorem A.2.8 and Theorem A.2.9 we have $f^* \in B^s_{\infty,\infty}(\mathbb{M}_0)$. By (T5) there exists $\omega^\dagger = (T^*)^\dagger T^\dagger f^* \in B^{s-2a}_{2,\infty}(\mathbb{M}_0) \subset L^\infty(\mathbb{M}_0)$. In particular, $\mathcal{R}$ is Fréchet-differentiable at $f^\dagger - t\omega^\dagger > 0$ w.r.t. L^∞ for $t < \bar{t} := c/\|\omega^\dagger\|_{L^\infty}$ with $f_t^* := \mathcal{R}'[f^\dagger - t\omega^\dagger]$ given by $\langle f_t^*, h\rangle = \langle \ln(f^\dagger - t\omega^\dagger) - \ln f_0, h\rangle$ (see (5.27)). Therefore, assumption (5.12) of Proposition 5.1.6 is satisfied with $\omega^* = \frac{\omega^\dagger}{f^\dagger}$ as then

$$\|f^* - f_t^* - t\omega^*\|_{L^\infty} = \left\|\log\left(1 + \frac{t\omega^\dagger}{f^\dagger - t\omega^\dagger}\right) - t\frac{\omega^\dagger}{f^\dagger}\right\|_{L^\infty} \leq C_{\omega^\dagger} t^2.$$

Again by Theorem A.2.8 and Theorem A.2.9 we obtain $\omega^* \in B^{s-2a}_{p,\infty}(\mathbb{M})$. Thus Proposition 5.1.6 with P_k as in Proposition A.2.13 and Lemma 5.1.1 show that $f^\dagger$ satisfies $\mathrm{VSC}^3(\Phi, \widetilde{\mathcal{R}}, \mathcal{S}_p)$ with $\mu = 2$ and

$$\Phi(\tau) \leq C \inf_{k\in\mathbb{N}_0} \left[2^{k(3a-s)}\sqrt{\tau} + 2^{-2k(s-2a)}\right] \leq C\tau^{\frac{s-2a}{s-a}},$$

which follows from $2^{-k} \sim \tau^{\frac{1}{2(s-a)}}$.

Now instead of $\widetilde{\mathcal{R}}$ we consider iterated Tikhonov with $\mathcal{R} = \mathrm{KL}(\cdot, f_0)$ however for $\delta \leq 1$ both $\|\hat{f}_\alpha^{(1)}\|_{L^1}$ and $\|\hat{f}_\alpha^{(2)}\|_{L^1}$ are bounded (for example by Lemma 4.1.2 and Lemma 2.3.2) so that we can choose $B = \max\left(\|\hat{f}_\alpha^{(1)}\|_{L^1}, \|\hat{f}_\alpha^{(2)}\|_{L^1}\right)$ so that both choices for $\mathcal{R}$ give the same minimizers. Finally we apply Theorem 4.5.5 with $r = q = \mu = 2$ and note that $\widetilde{\Phi} = \Phi$, $(-\Phi)^*(x) = C(-x)^{(2a-s)/a}$ for $x < 0$ and $(-\Phi)^*(x) = \infty$ else. Hence, $\mathrm{KL}\left(f^\dagger, \hat{f}_\alpha^{(2)}\right) \leq C(\delta^2/\alpha + \alpha^{s/a} + \beta\alpha^4)$, and the choice $\alpha \sim \delta^{\frac{2a}{s+a}}$ leads to (5.24).

□

Remark 5.2.2. *Although the Hölder-Zygmund smoothness assumptions are simple and natural it is possible to show the same convergence rates by considering weaker smoothness* *assumptions with respect to the Nikolskii scale* $B^s_{2,\infty}(\mathbb{M}_0)$ *(note that* $B^s_{\infty,\infty}(\mathbb{M}_0) \subset B^s_{2,\infty}(\mathbb{M}_0)$ *by Theorem A.2.5) if one additionally assumes boundedness w.r.t* $L^\infty(\mathbb{M}_0)$ *of the minimizer* $\hat{f}_\alpha$*. This will be shown in the next subsection.*

We omitted $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$*, respectively the case of Poisson data, in the above theorem. In principle we can verify* $\mathrm{VSC}^1(\Phi, \mathcal{R}, \mathrm{KL}^\sigma_{g^\dagger})$ *by Lemma 5.1.2 and (D8) in Assumption 4.4.4 can be shown as in the above Theorem. However, we concentrated our error analysis for* $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ *in Chapter 4 on the case* $p = 2$ *in Assumption 1.4.1, whereas for* $\mathcal{R} = \mathrm{KL}(f, f_0)$ *we would need* $p = 1$*. The reasons for only considering* $p = 2$ *is that for Poisson data we only have the deviation inequality 1.4.9 for* $p = 2$ *and the boundedness result of Proposition 4.2.4 also works only for* $p = 2$ *because of Theorem A.2.10. Both of these problems are of technical nature and should be solved in future research. Alternatively the next section provides a workaround given boundedness of the minimizer.*

5.2.2 Convergence rates under boundedness assumptions

In this subsection we will assume that $f^\dagger \in L^\infty(\mathbb{M}_0)$. Therefore it makes sense to restrict regularization to bounded functions so we will consider $\mathcal{R} = \mathcal{R}_{f_0} + \chi_{\mathcal{B}}$, where $\mathcal{B} \subset L^1(\mathbb{M}_0)$ is closed and convex and for some constant $R > 0$ we have $\sup_{f\in\mathcal{B}} \|f\|_{L^\infty(\mathbb{M}_0)} \leq R$. With the true solution and the minimizers bounded we can consider $\mathcal{X} = L^2(\mathbb{M}_0)$ and as a forward operator the restriction of the forward operator from the last section to $\mathcal{X}$. Then generalized Tikhonov regularization is of the form

$$\hat{f}_\alpha \in \operatorname*{arg\,min}_{f\in\mathcal{B}} \left[\mathcal{S}(Tf) + \alpha\, \mathrm{KL}\,(f, f_0)\right]. \tag{5.29}$$

If all iterates are in the interior of $\mathcal{B}$, Bregman iteration is given by

$$\hat{f}^{(n)}_\alpha \in \operatorname*{arg\,min}_{f\in\mathcal{B}} \left[\mathcal{S}(Tf) + \alpha\, \mathrm{KL}\left(f, \hat{f}^{(n-1)}_\alpha\right)\right], \tag{5.30}$$

otherwise the iteration formula may involve an element of the normal cone of $\mathcal{B}$ at $\hat{f}^{(n-1)}_\alpha$.

Theorem 5.2.3. *Let* $\mathbb{M}, \mathbb{M}_0 \subset \mathbb{R}^d$ *be bounded Lipschitz domains or* $\mathbb{T}^d$*,* $\mathcal{X} = L^2(\mathbb{M}_0)$*,* $\mathcal{Y} = L^2(\mathbb{M})$*. Let* $f_0 \in L^1(\mathbb{M}_0), f_0 \geq 0$*. Suppose that* T *is a-smoothing in the sense that Assumption 5.1.8 (T1)-(T5) hold true for* $p = 2$*. Moreover, suppose there exists* $c > 0$ *such that*

$$c \leq \frac{f^\dagger}{f_0} \leq c^{-1} \qquad \textit{a.e. in } \mathbb{M}. \tag{5.31}$$

We distinguish the different orders of convergence rates:

(a) Assume that

$$\log\left(\frac{f^\dagger}{f_0}\right) \in B^s_{2,\infty}(\mathbb{M}_0) \qquad \textit{for some } s \in (0, a)\,.$$

Then there exists $C > 0$ *such* $f^\dagger$ *fulfills both* $\mathrm{VSC}^1(\Phi, \mathrm{KL}(\cdot, f_0) + \chi_{\mathcal{B}}, \mathcal{S}^2_{g^\dagger})$ *and* $\mathrm{VSC}^1(\Phi, \mathrm{KL}(\cdot, f_0) + \chi_{\mathcal{B}}, \mathrm{KL}^\sigma_{g^\dagger})$ *with*

$$\Phi(\tau) = \mathcal{O}\left(\tau^{\frac{s}{s+a}}\right).$$

(b) *Assume additionally to* (5.31) *that* $f^\dagger \geq \rho$ *and*

$$f^\dagger, f_0 \in B^s_{2,\infty}(\mathbb{M}_0) \qquad \text{for some } s \in (2a + d/2, 3a)\,.$$

Then there exists $C > 0$ *such that* $f^\dagger$ *fulfills* $\mathrm{VSC}^3(\Phi, \mathrm{KL}(\cdot, f_0) + \chi_{\mathcal{B}}, \mathcal{S}_2)$ *with* $\mu = 2$ *and*

$$\Phi(\tau) = C\tau^{\frac{s-2a}{s-a}}.$$

Let $\widehat{f}_\alpha$ *be given by* (5.29)*. Under the first order smoothness assumption* ($s \in (0, a)$) *we get*

$$\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\delta^{\frac{2s}{s+a}}\right), \qquad \delta \to 0. \tag{5.32}$$

under the deterministic noise model with $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$*. Under the Gaussian white noise model with* $\mathbb{M} = \mathbb{T}^d$, $a > d/2$ *and* $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$ *we get for a parameter choice* $\alpha \sim \varepsilon^{\frac{2a}{s+a+d/2}}$ *that*

$$\mathbb{E}\Big(\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) = \mathcal{O}\left(\varepsilon^{\frac{2s}{s+a+d/2}}\right), \qquad \varepsilon \to 0. \tag{5.33}$$

Under the Poisson data model with $\mathbb{M}$ *a bounded Lipschitz domain,*$a \geq a_0 > \gamma$ *(with* $a_0 \in \{1, 2\}$*) and* $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$ *we get that for all* $\gamma > d/2$ *and a parameter choice* $\alpha \sim t^{\frac{-a}{s+a+\gamma}}$ *that*

$$\mathbb{E}\Big(\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) = \mathcal{O}\left(t^{-\frac{s}{s+a+\gamma}}\right), \qquad t \to \infty. \tag{5.34}$$

For $\widehat{f}^{(2)}_\alpha$ *as in* (5.30) *and* $s \in (2a + d/2, 3a)$ *we obtain under the deterministic model with* $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$ *for the parameter choice* $\alpha \sim \delta^{\frac{2a}{s+a}}$ *the convergence rates*

$$\mathrm{KL}\left(\widehat{f}^{(2)}_\alpha, f^\dagger\right) = \mathcal{O}\left(\delta^{\frac{2s}{s+a}}\right), \qquad \delta \to 0. \tag{5.35}$$

Proof. The proof has obviously some similarities to the proof of the last subsection. But actually it simplifies at some points due to the boundedness of the minimizers.

(a) Let $f^* = \log\left(\frac{f^\dagger}{f_0}\right)$. By assumption we have $f^* \in B^s_{2,\infty}(\mathbb{M})$. Due to $\sup_{f\in\mathcal{B}}\|f\|_{L^\infty(\mathbb{M}_0)} \leq R$ we have that (4.6) holds true by (2.19). So we can apply Proposition 4.3.6 with P_k as in Proposition A.2.13, $\nu_k = \|(T^*)^\dagger P_k f^*\|_{\mathcal{Y}^*}$ and $\kappa_k = \|(I - P_k)f^*\|_{\mathcal{X}^*}$ to find

$$\Phi(\tau) = C \inf_{k\in\mathbb{N}_0}\left[\nu_k\tau^{1/2} + \kappa_k^2\right] \leq C \inf_{k\in\mathbb{N}_0}\left[2^{(a-s)k}\tau^{1/2} + 2^{-2ks}\right],$$

where the inequality follows from Lemma 5.1.1 as $B^0_{2,2}(\mathbb{M}_0) = L^2\infty(\mathbb{M}_0) = \mathcal{X}^*$. Choosing $2^{-k} \sim \tau^{\frac{1}{2(s+a)}}$ gives $\Phi(\tau) \leq C\tau^{\frac{s}{s+a}}$. The deterministic rate (5.32) then follows from Theorem 4.3.4 and the stochastic rates (5.33) and (5.34) follow from Corollary 4.3.12 and Lemma 4.3.13, which give

$$\mathbb{E}\Big(\mathrm{KL}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) \leq C\frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}} + 4(-\Phi)^*\Big(-\frac{1}{2\alpha}\Big) \leq C\Big(\frac{\varepsilon^2}{\alpha^{1+\frac{\gamma}{a}}} + \alpha^{\frac{s}{a}}\Big),$$

where γ for Gaussian white noise can be chosen as $\gamma = d/2$ by Theorem 1.4.5, whereas for Poisson noise we have to choose $\gamma > d/2$ by 1.4.9. The parameter choice $\alpha \sim \varepsilon^{\frac{2a}{s+a+\gamma}}$ then yields (5.26) and if we replace $\varepsilon = t^{-1/2}$ we get (5.34).

(b) Assumption (5.13) of Proposition 5.1.6 is satisfied for $\mathcal{R} = \mathrm{KL}(\cdot, f_0) + \chi_{\mathcal{B}}$ with $\mu = 2$ by (2.19) as $\sup_{f\in\mathcal{B}}\|f\|_{L^\infty(\mathbb{M}_0)} \leq R$. By Theorem A.2.8 and Theorem A.2.9 we have $f^* \in B^s_{2,\infty}(\mathbb{M}_0)$. By (T5) there exists $(T^*)^\dagger T^\dagger f^* = \omega^\dagger \in B^{s-2a}_{2,\infty}(\mathbb{M}_0) \subset L^\infty(\mathbb{M}_0)$ by Theorem A.2.5 as $s - 2a > d/2$. In particular, $\mathcal{R}$ is Fréchet-differentiable at $f^\dagger - t\omega^\dagger > 0$ w.r.t. L^∞ for $t < \bar{t} := c/\|\omega^\dagger\|_{L^\infty}$ with $f_t^* := \mathcal{R}'[f^\dagger - t\omega^\dagger]$ given by $\langle f_t^*, h\rangle = \langle \ln(f^\dagger - t\omega^\dagger) - \ln f_0, h\rangle$ (see (5.27)). Therefore, assumption (5.12) of Proposition 5.1.6 is satisfied with $\omega^* = \frac{\omega^\dagger}{f^\dagger}$ as then

$$\|f^* - f_t^* - t\omega^*\|_{L^2} = \left\|\log\left(1 + \frac{t\omega^\dagger}{f^\dagger - t\omega^\dagger}\right) - t\frac{\omega^\dagger}{f^\dagger}\right\|_{L^2} \leq C_{\omega^\dagger} t^2.$$

Again by Theorem A.2.8 and Theorem A.2.9 we obtain $\omega^* \in B^{s-2a}_{2,\infty}(\mathbb{M}_0)$. Thus Proposition 5.1.6 with P_k as in Proposition A.2.13 and Lemma 5.1.1 show that $f^\dagger$ satisfies $\mathrm{VSC}^3(\Phi, \widetilde{\mathcal{R}}, f_0), \mathcal{S}_p)$ with $\mu = 2$ and

$$\Phi(\tau) \leq C \inf_{k\in\mathbb{N}_0}\left[2^{k(3a-s)}\sqrt{\tau} + 2^{-2k(s-2a)}\right] \leq C\tau^{\frac{s-2a}{s-a}},$$

which follows from $2^{-k} \sim \tau^{\frac{1}{2(s-a)}}$. Finally we apply Theorem 4.5.5 with $r = q = \mu = 2$ and note that $\widetilde{\Phi} = \Phi$, $(-\Phi)^*(x) = C(-x)^{(2a-s)/a}$ for $x < 0$ and $(-\Phi)^*(x) = \infty$ else. Hence, $\mathrm{KL}\left(f^\dagger, \hat{f}_\alpha^{(2)}\right) \leq C(\delta^2/\alpha + \alpha^{s/a} + \beta\alpha^4)$, and the choice $\alpha \sim \delta^{\frac{2a}{s+a}}$ leads to (5.35).

□

Remark 5.2.4. *The rates (5.32), (5.33) and (5.35) are of optimal order, see [41]. We did not consider second order rates in the above Theorem as $\chi_{\mathcal{B}}$ with $\sup_{f\in\mathcal{B}}\|f\|_{L^\infty(\mathbb{M}_0)} \leq R$ cannot be differentiable w.r.t. $L^2(\mathbb{M}_0)$ so that we cannot verify an assumption like (b) of Proposition 5.1.3. We might have differentiability w.r.t. $L^\infty(\mathbb{M}_0)$, but then we would have to choose $V = \tilde{\mathcal{X}} = L^\infty(\mathbb{M}_0)$ in (5.5) of Proposition 5.1.3, which would result in no improvement compared to Theorem 5.2.1. In [64, Theorem 5.7] the authors claimed second order rates but oversaw this problem.*

5.2.3 Numerical results

In this section we give some numerical results for the iterated maximum entropy regularization.

Test problem: We choose $T\colon L^1(\mathbb{T}) \to L^2(\mathbb{T})$ to be the periodic convolution operator $(Tf)(x) := \int_0^1 k(x-y)f(y)\,dy$ with kernel

$$k(x) = \sum_{j=-\infty}^{\infty} \exp(|x-j|/2) = \left(\sinh\frac{1}{4}\right)^{-1}\cosh\frac{2x - 2\lfloor x\rfloor - 1}{4}, \qquad x \in \mathbb{R}$$

where $\lfloor x\rfloor := \max\{n \in \mathbb{Z} \colon n \leq x\}$. Then integration by parts shows that $T = (-\partial_x^2 + (1/4)I)^{-1}$, and hence T satisfies the assumptions of Theorem 5.2.1 with $a = 2$. We choose $f_0 = 1$ and the true solution $f^\dagger$ such that $f^\dagger - 1$ is the standard B-spline B_5 of order 5 with $\mathrm{supp}(B_5) = [0, 1]$ and equidistant knots. Then we have $f^\dagger \in B^{5.5}_{2,\infty}(\mathbb{T})$, i.e. $s = 5.5$ in Theorem 5.2.3. (To see this note that piecewise constant functions belong to $B^{0.5}_{2,\infty}(\mathbb{T})$ using

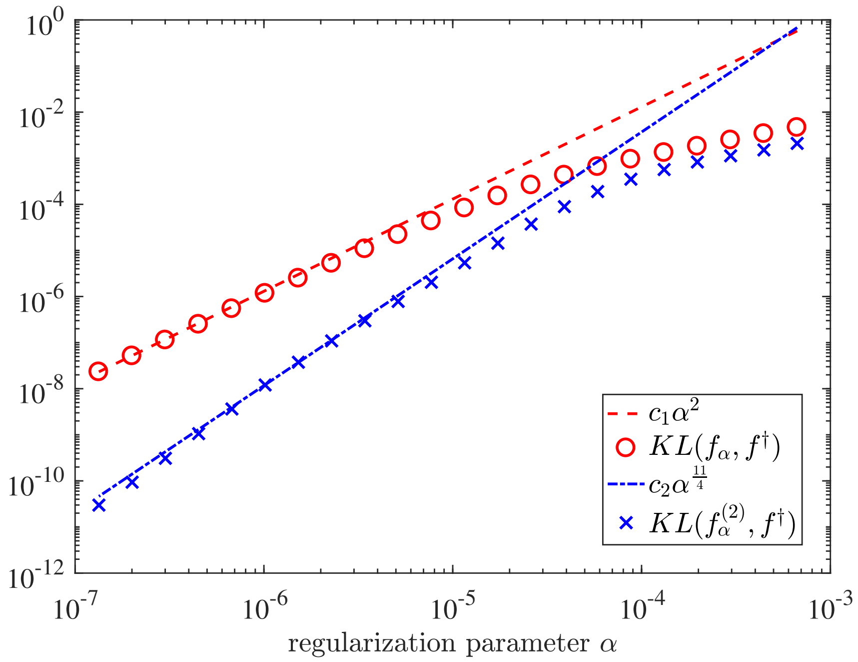

FIGURE 5.1: Predicted and computed approximation error for standard and iterated maximum entropy regularization.

the defintion of this space via the modulus of continuity.) Hence, according to Theorem 5.2.1 a third order variational source condition condition $\mathrm{VSC}^3(A\tau^{3/7}, \mathrm{KL}(\cdot, 1) + \chi_{\mathcal{B}}, \mathcal{S}_2)$ is satisfied for some $A > 0$. Note that by Theorem A.2.5 we also have $f^\dagger \in B^5_{\infty,\infty}(\mathbb{T})$, thus one could also apply Theorem 5.2.1. This results in a worse index function for the VSC^3, however we can see that the strongest VSC^2 should be fulfilled and thus expect the maximal rate for non-iterated maximum entropy regularization.

Implementation: The operator T is discretized by sampling k and f on an equidistant grid with 480 points. Then matrix-vector multiplications with $T = T^*$ can be implemented efficiently by FFT. The minimizers $\hat{f}_\alpha$ and $\hat{f}_\alpha^{(2)}$ are computed by the Douglas-Rachford algorithm. To be consistent with our theory, we consider the constraint set $\mathcal{B} := \{f \in L^1(\mathbb{T}) \colon 0 \leq f \leq 5 \text{ a.e.}\}$. We checked that for none of the unconstrained minimizers the bound constraints were active such that an explicit implementation of these constraints was not required for our test problem.

To check the predicted convergence rates with respect to the noise level δ the regularization parameter α was chosen by an a-priori rule of the form $\alpha = c\delta^\sigma$ with an optimal exponent $\sigma > 0$ and a constant c chosen to minimize the constants for the upper bound given in the figures. As we bound the worst case errors in our analysis we tried to approximate the worst case noise. Let $G_\delta := \{g^\dagger + \delta \sin(2\pi k\cdot) : k \in \mathbb{N}\}$. For each value of δ we found $g^{\mathrm{obs}} \in G_\delta$ such that the reconstruction error gets maximal. This in particular yielded larger propagated data errors than discrete white noise.

Discussion of the results: Figure 5.1 shows the approximation error as a function of α, i.e. $\mathrm{KL}(f_\alpha, f^\dagger)$ where f_α and $f_\alpha^{(2)}$, respectively, are the reconstructions for exact data $g^{\mathrm{obs}} = g^\dagger$. The two dashed lines indicate the corresponding asymptotic convergence rates predicted by our theory, which are in good agreement with the empirical results. Note that

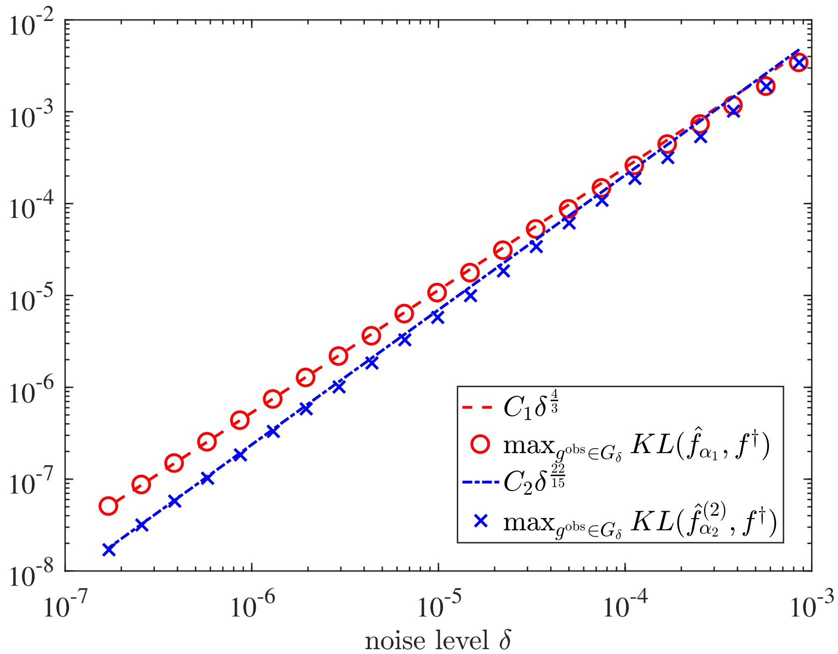

FIGURE 5.2: Predicted and computed convergence rates for standard and iterated maximum entropy regularization.

the saturation effect limits the convergence of the standard maximum entropy estimator $f_\alpha = f_\alpha^{(1)}$ to the maximal rate $\mathrm{KL}(f_\alpha, f^\dagger) = \mathcal{O}(\alpha^2)$. Iterating maximum entropy estimation yields a clear improvement to $\mathrm{KL}(f_\alpha^{(2)}, f^\dagger) = \mathcal{O}(\alpha^{s/a}) = \mathcal{O}(\alpha^{11/4})$.

Figure 5.2 displays the convergence rates with respect to the noise level δ for the a-priori choice rule of α described above. Of course, in practice one would rather use some a-posteriori stopping rule such as the Lepskii balancing principle, but this is not in the scope of this paper. Again, we observe very good agreement of the empirical rate $\mathrm{KL}(\hat{f}_\alpha, f^\dagger) = \mathcal{O}(\delta^{4/3})$ with the maximal rate for non-iterated maximum entropy regularization, as well as agreement of the rate $\mathrm{KL}(\hat{f}_\alpha^{(2)}, f^\dagger) = \mathcal{O}(\delta^{2s/(s+a)}) = \mathcal{O}(\delta^{22/15})$ of the Bregman iterated estimator $\hat{f}_\alpha^{(2)}$ with the rate predicted by Theorem 5.2.3.

5.3 Besov norm regularization[4]

Let $\mathcal{X} = B^0_{p,q}(\mathbb{M}_0)$ with $\mathbb{M}_0$ a bounded Lipschitz domain in $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$. We consider generalized Tikhonov regularization with penalty

$$\mathcal{R}(f) = \tfrac{1}{t}\|f\|^t_{B^0_{p,q}(\mathbb{M}_0)},$$

for $1 < p, q, t < \infty$ so that

$$\hat{f}_\alpha \in \operatorname*{arg\,min}_{f\in B^0_{p,q}(\mathbb{M}_0)} \left[\mathcal{S}(Tf) + \frac{\alpha}{t}\|f\|^t_{B^0_{p,q}(\mathbb{M}_0)}\right]. \tag{5.36}$$

[4]This section has some overlap with the article [74] to which the author contributed.

5.3.1 Upper bounds

To apply the verification strategies it will be crucial to understand the smoothness of the subgradients of our penalty at $f^\dagger$. To this end we cite the following result from [74].

Theorem 5.3.1. *Let $\mathcal{R}$ be given by (5.3), with $1 < p, q, t < \infty$. Let $f^\dagger \in B^s_{p,\infty}(\mathbb{M}_0)$ and $f^* \in \partial\mathcal{R}(f^\dagger)$. Then we have $f^* \in B^{s(q-1)}_{p',\infty}(\mathbb{M}_0)$ with*

$$\|f^*\|_{B^{s(q-1)}_{p',\infty}} = \|f^\dagger\|^{t-q}_{B^0_{p,q}} \|f^\dagger\|^{q-1}_{B^s_{p,\infty}}.$$

Theorem 5.3.2. *Let $\mathbb{M}_0, \mathbb{M} \subset \mathbb{R}^d$ be bounded Lipschitz domains or $\mathbb{T}^d$, $\mathcal{Y} = L^2(\mathbb{M})$. Let $p \in (1, 2]$, $q \in (1, \infty)$. Suppose that T is a > 0 times smoothing in the sense that Assumption 5.1.8 (T1)-(T5) hold true. We distinguish the different orders of convergence rates:*

(a) Let $s \in \left(0, \frac{a}{q-1}\right)$ and let $q \leq t \leq r = \max(p, q, 2)$. There exists $C > 0$ such that for all $f^\dagger \in B^s_{p,\infty}(\mathbb{M}_0)$ with $\|f^\dagger\|_{B^s_{p,\infty}} \leq \varrho$ the $\mathrm{VSC}^1(\Phi, \frac{1}{t}\|\cdot\|^t_{B^s_{p,q}}, \mathcal{S}^2_{g^\dagger})$ holds true with

$$\Phi(\tau) = C\tau^\nu \quad \textit{where} \quad \nu = \begin{cases} \frac{s(q-1)}{s(q-1)+a}, & q \leq 2, \\ \frac{qs}{2(s+a)}, & q \geq 2. \end{cases}$$

For $\widehat{f}_\alpha$ as in (5.36) this yields with $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$ the deterministic convergence rates

$$\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) = \begin{cases} \mathcal{O}\left(\delta^{\frac{2s(q-1)}{s(q-1)+a}}\right), & q \leq 2, \ \textit{if } \alpha \sim \delta^{\frac{2a}{s(q-1)+a}} \\ \mathcal{O}\left(\delta^{\frac{qs}{s+a}}\right), & q \geq 2, \ \textit{if } \alpha \sim \delta^{\frac{2a+(2-q)s}{s+a}} \end{cases} \tag{5.37}$$

as $\delta \searrow 0$ and under the Gaussian white noise model with $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$, $a > d/2$ that

$$\mathbb{E}\left(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)\right) = \begin{cases} \mathcal{O}\left(\varepsilon^{\frac{2s(q-1)}{s(q-1)+a+d/2}}\right), & q \leq 2, \ \textit{if } \alpha \sim \varepsilon^{\frac{2a}{s(q-1)+a+d/2}} \\ \mathcal{O}\left(\varepsilon^{\frac{qs}{s+a+d/2}}\right), & q \geq 2, \ \textit{if } \alpha \sim \varepsilon^{\frac{2a+(2-q)s}{s+a+d/2}}, \end{cases} \tag{5.38}$$

as $\varepsilon \searrow 0$.

(b) Let $s \in \left(\frac{a}{q-1}, \frac{2a}{q-1}\right)$ and $t \geq q$. Define $\mu = \max(p', q')$. There exists $C > 0$ such that for all $f^\dagger \in B^s_{p,\infty}(\mathbb{M}_0)$ with $\|f^\dagger\|_{B^s_{p,\infty}} \leq \varrho$ the $\mathrm{VSC}^2(\Phi, \frac{1}{t}\|\cdot\|^t_{B^s_{p,q}}, \mathcal{S}_2)$ holds true with

$$\Phi(\tau) = C\tau^{\frac{2(s(q-1)-a)}{\mu s(q-1)}}.$$

This yields for $\mathcal{S} = \mathcal{S}^2_{g^{\mathrm{obs}}}$ the deterministic rates

$$\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger) = \begin{cases} \mathcal{O}\left(\delta^{\frac{qs}{s+a}}\right), & q \leq p, \alpha \sim \delta^{\frac{2a+(2-q)s}{s+a}} \\ \mathcal{O}\left(\delta^{\frac{p's(q-1)}{(p'-1)(q-1)s+a}}\right), & q > p, \alpha \sim \delta^{\frac{2a+(p'-2)(q-1)s}{(p'-1)(q-1)s+a}}. \end{cases} \tag{5.39}$$

Let $t = q$ and $q \leq p$, then there exists $C > 0$ such that for all $f^\dagger \in B^s_{p,\infty}(\mathbb{M}_0)$ with $\|f^\dagger\|_{B^s_{p,\infty}} \leq \varrho$ (D5)-(D8) in Assumption 4.4.4 hold true and for Ψ as in (4.17) we have

$$\Psi(\tau) \leq C\tau^{\frac{2(s(q-1)-a)}{(\mu-2)s(q-1)+2a}},$$

for τ sufficiently small. This yields under the Gaussian white noise model with $\mathcal{S} = \mathcal{S}^{LS}_{G\text{obs}}$, $a > d/2$ and for a parameter choice $\alpha \sim \varepsilon^{\frac{2a+(2-q)s}{s+a+\frac{d}{2}\left(1+\frac{(2-q)s}{2a}\right)}}$ that

$$\mathbb{E}\Big(\Delta^{T^*\xi^\dagger}_{\mathcal{R}}(\widehat{f}_\alpha, f^\dagger)\Big) = \mathcal{O}\left(\varepsilon^{\frac{qs}{s+a+\frac{d}{2}\left(1+\frac{(2-q)s}{2a}\right)}}\right).$$

Proof. (a) By Theorem 5.3.1 we have $f^* \in B^{s(q-1)}_{p',\infty}(\mathbb{M}_0)$ with

$$\|f^*\|_{B^{s(q-1)}_{p',\infty}} = \|f^\dagger\|^{t-q}_{B^0_{p,q}} \|f^\dagger\|^{q-1}_{B^s_{p,\infty}} \le C\varrho^{t-1},$$

as $t - q \ge 0$. As $\mathcal{X}$ is r-convex we can apply Proposition 4.3.6 with P_k as in Proposition A.2.13, $\nu_k = \|(T^*)^\dagger P_k f^*\|_{\mathcal{Y}^*}$ and $\kappa_k = \|(I-P_k)f^*\|_{\mathcal{X}^*}$ to find that $f^\dagger$ fulfills $\mathrm{VSC}^1(\Phi, \frac{1}{t}\|\cdot\|^t_{B^s_{p,q}}, \mathcal{S}^2_{g^\dagger})$ with

$$\Phi(\tau) = \inf_{k\in\mathbb{N}_0}\left[\nu_k\tau^{1/2} + C_{\mathcal{X},t}\|f^\dagger\|^{r'(r-t)/r}_{\mathcal{X}}\kappa^{r'}_k\right] \le C_\varrho \inf_{k\in\mathbb{N}_0}\left[2^{(a-s(q-1))k}\tau^{1/2} + 2^{-kr's(q-1)}\right],$$

where the inequality follows from Lemma 5.1.1 and the constant depends only on ϱ and not in another way on $f^\dagger$ only because we have $q \le t \le r$. Choosing $2^{-k} \sim \tau^{\frac{1}{2(s(q-1)(r'-1)+a)}}$ gives

$$\Phi(\tau) \le C\tau^{\frac{s(q-1)r'}{2(s(q-1)(r'-1)+a)}}.$$

Now use that for $q \le 2$ we have $r = r' = 2$ and for $q \ge 2$ we have $r = q$ and $r' = q'$, as well as $q'(q-1) = q$ and $(q-1)(q'-1) = 1$. The deterministic rate (5.37) then follows from Theorem 4.3.4 and the stochastic rate (5.38) follows from Corollary 4.3.12 and Lemma 4.3.13, which give

$$\begin{aligned}\mathbb{E}\Big(\Delta^{f^*}_{\mathcal{R}}\left(\widehat{f}_\alpha, f^\dagger\right)\Big) &\le C\left(\frac{\varepsilon^2}{\alpha^{1+\frac{d}{2a}}}\right)^{\frac{1}{1+\frac{d(r-2)}{2ar}}} + 4(-\Phi)^*\left(-\frac{1}{2\alpha}\right)\\ &\le \begin{cases} C\frac{\varepsilon^2}{\alpha^{1+\frac{d}{2a}}} + C\alpha^{\frac{s(q-1)}{a}}, & q \le 2\\ C\left(\frac{\varepsilon^2}{\alpha^{1+\frac{d}{2a}}}\right)^{\frac{1}{1+\frac{d(q-2)}{2aq}}} + C\alpha^{\frac{qs}{2a+(2-q)s}}, & q \ge 2,\end{cases}\end{aligned}.$$

We obtain (5.38) by the parameter choices $\alpha \sim \varepsilon^{\frac{2a}{s(q-1)+a+d/2}}$ for $q \le 2$ and $\alpha \sim \varepsilon^{\frac{2a+(2-q)s}{s+a+d/2}}$ for $q \ge 2$.

(b) By Theorem 5.3.1 we have $f^* \in B^{s(q-1)}_{p',\infty}(\mathbb{M}_0)$ with $\|f^*\|_{B^{s(q-1)}_{p',\infty}} = \|f^\dagger\|^{t-q}_{B^0_{p,q}}\|f^\dagger\|^{q-1}_{B^s_{p,\infty}}$. Note that because of this the assumption $t \ge q$ is crucial to get a uniform constant for all $\|f^\dagger\|_{B^s_{p,\infty}} \le \varrho$. As $\mathcal{X}^* = B^0_{p',q'}(\mathbb{M}_0)$ is μ-convex with $\mu = \max(p', q', 2) = \max(p', q')$ we have by (2.15) from Theorem 2.2.5 that (5.4) holds true (note that $t' \le \mu$) and thus we can apply Proposition 5.1.3 with P_k as in Proposition A.2.13. Together with Lemma 5.1.4 and $\xi^* \in B^{s(q-1)-a}_{p',\infty}$ we see that $\mathrm{VSC}^2(\Phi, \frac{1}{t}\|\cdot\|^t_{B^s_{p,q}}, \mathcal{S}_2)$ holds true with

$$\begin{aligned}\Phi(\tau) = C\inf_{k\in\mathbb{N}_0}\left[\nu_k\tau^{1/\mu} + \kappa^q_k\right] &\le C\inf_{k\in\mathbb{N}_0}\left[2^{k(2a-s(q-1))}\tau^{1/\mu} + 2^{-2k(s(q-1)-a)}\right]\\ &\le C\tau^{\frac{2(s(q-1)-a)}{\mu s(q-1)}},\end{aligned}$$

where the last inequality follows from $2^{-k} \sim \tau^{\frac{1}{\mu s(q-1)}}$.
By Theorem 4.4.2 and Lemma 4.5.6 we have the deterministic estimates

$$\Delta_{\mathcal{R}}^{T^*\xi^\dagger}(f,f^\dagger) = \mathcal{O}\left(\frac{\delta^2}{\alpha} + \alpha(-\Phi)^*\left(\frac{-1}{\alpha}\right)\right) = \mathcal{O}\left(\frac{\delta^2}{\alpha} + \alpha^{\frac{\mu s(q-1)}{2a+(\mu-2)(q-1)s}}\right) = \mathcal{O}\left(\delta^{\frac{\mu s(q-1)}{a+(\mu-1)(q-1)s}}\right),$$

where the last bound follows from the parameter choice $\alpha \sim \delta^{\frac{2a+(\mu-2)(q-1)s}{a+(\mu-1)(q-1)s}}$. If $q \le p$, so $\mu = q'$ the exponent simplifies significantly to $\frac{qs}{a+s}$ by $q'(q-1) = q$ and $(q'-1)(q-1) = 1$.
Under a stochastic model with $t = q$ and $q \le p$ we have by Theorem 2.2.5 that (5.9) holds true as $t' = \mu$ and thus Proposition 5.1.5 gives that (D5)-(D8) in Assumption 4.4.4 hold true. For Ψ as in (4.17) and sufficiently small α we have that

$$\Psi(\alpha) = \inf_{\substack{k\in\mathbb{N}_0 \\ \alpha\nu_k^2 \le 1}} \left[\alpha^{\mu'-1}\nu_k^{\mu'} + \kappa_k^2\right] \le C\alpha^{\frac{2(s(q-1)-a)}{2a+(\mu-2)s(q-1)}} \tag{5.40}$$

where the inequality follows from $2^{-k} \sim \alpha^{\frac{1}{2a+(\mu-2)s(q-1)}}$. This leads to the stochastic error bound

$$\mathbb{E}\left(\Delta_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right)\right) \le C\frac{\varepsilon^2}{\alpha^{1+\frac{d}{2a}}} + \alpha\Psi(\alpha) \le C\frac{\varepsilon^2}{\alpha^{1+\frac{d}{2a}}} + C\alpha^{\frac{\mu s(q-1)}{2a+(\mu-2)(q-1)s}}.$$

With the parameter choice $\alpha \sim \varepsilon^{\frac{2a+(2-q)s}{s+a+\frac{d}{2}\left(1+\frac{(2-q)s}{2a}\right)}}$ we obtain the claimed rate as $\mu' = q'$, $q'(q-1) = q$ and $(q'-1)(q-1) = 1$.

□

5.3.2 Lower bounds

In the following we want to discuss the sharpness of the estimates obtained in the previous theorem. Recall that for a linear forward operator lower bounds for the reconstruction error measured in the norm can be formulated in terms of the modulus of continuity ω_{lin} introduced in (1.2). One can actually also show lower bounds in the Bregman divergence, see [73, Lemma 4.1].

Theorem 5.3.3. *Let $\mathcal{R} = \frac{1}{t}\|\cdot\|_{\mathcal{X}}^t$ for some $t > 1$ and $\mathcal{K} \subset \mathcal{X}$ such that $\mathcal{K} = -\mathcal{K}$ and let j_t be a selection of the duality mapping J_t, with $j_t(-f) = -j_t(f)$. Then we have for any reconstruction method $R\colon \mathcal{Y} \to \mathcal{X}$ the lower bound*

$$\inf_R \sup\left\{\Delta_{\mathcal{R}}^{j_t(f^\dagger)}(R(g^{\text{obs}}), f^\dagger) : f^\dagger \in \mathcal{K}, g^{\text{obs}} \in \mathcal{Y}, \|Tf^\dagger - g^{\text{obs}}\|_{\mathcal{Y}} \le \delta\right\} \ge \frac{1}{t'}\omega_{\text{lin}}(\delta, T, \mathcal{K})^t$$

Proof. Let $f \in \mathcal{K}$ such that $\|Tf\| \le \delta$. Setting $g = 0$ we see that $\|Tf - g\| \le \delta$ is fulfilled. By symmetry of $\mathcal{K}$ we also have $-f \in \mathcal{K}$ with $\|T(-f) - g\| \le \delta$. Hence,

$$\begin{aligned}
&2\sup\left\{\Delta_{\mathcal{R}}^{j_t(f^\dagger)}(R(g^{\text{obs}}), f^\dagger) : f^\dagger \in \mathcal{K}, g^{\text{obs}} \in \mathcal{Y}, \|Tf^\dagger - g^{\text{obs}}\|_{\mathcal{Y}} \le \delta\right\} \\
&\ge \sup_{f\in\mathcal{K}, \|Tf\|\le\delta} \left[\Delta_{\mathcal{R}}^{j_t(f)}(R(0), f) + \Delta_{\mathcal{R}}^{j_t(-f)}(R(0), -f)\right] \\
&= \sup_{f\in\mathcal{K}, \|Tf\|\le\delta} 2[\mathcal{R}(R(0)) - \mathcal{R}(f) + \langle j_t(f), f\rangle] = 2\mathcal{R}(R(0)) + 2\sup_{f\in\mathcal{K}, \|Tf\|\le\delta} \mathcal{R}^*(j_t(f)),
\end{aligned}$$

where the last equality is Young's equality. By definition of the duality mapping we have

$$\mathcal{R}^*(j_t(f)) = \frac{1}{t'}\|j_t(f)\|_{\mathcal{X}^*}^{t'} = \frac{1}{t'}\|f\|_{\mathcal{X}}^{t'(t-1)} = \frac{1}{t'}\|f\|^t.$$

As $\inf_R \mathcal{R}(R(0)) = 0$ the claim follows by (1.2). □

A lower bound for the modulus of continuity can be shown under Assumption 5.1.8, see [74, Theorem 4.12].

Theorem 5.3.4. *Let $\mathcal{K} = \{f \in B^s_{p,\infty} : \|f\|_{B^s_{p,\infty}} \leq \varrho\}$ and let T be a-smoothing in the sense that (T6) of Assumption 5.1.8 holds true. Then we have $\omega_{\mathrm{lin}}(\delta, T, \mathcal{K}) \geq C_\varrho \delta^{\frac{s}{s+a}}$.*

In the light of this result we see that the first order rates from Theorem 5.3.2 are order optimal in the Bregman divergence for $q \geq 2$ and the second order rates are order optimal in the Bregman divergence if $t = q$ and $q \leq p$.

Lower bounds for the statistical convergence rates can be concluded from results in [20]. Instead of the continuous Gaussian white noise model they consider an n-dimensional *normal means model*. However as their results in [20, Thms. 7 and 9] do not depend on the dimension n one can send n to infinity, so that the Le Cam distance of the two models goes to zero (compare [33, Ch. 1]) and thus conclude for general estimators $S = S(g^\dagger + \varepsilon W) \in B^{s^*}_{p,q}$:

Theorem 5.3.5. *We have*

$$\inf_S \sup_{\|g^\dagger\|_{B^{s^{**}}_{p,\infty}} \leq \varrho} \mathbb{E}\left(\left\|g^\dagger - S(g^\dagger + \varepsilon W)\right\|_{B^{s^*}_{p,q}}\right) \geq c\varrho^{\frac{s^*+d/2}{s^{**}+d/2}} \varepsilon^{\frac{s^{**}-s^*}{s+d/2}},$$

with c depending on s^*, s^{**}, p, q.

Combining all results we see that we get order optimal rates in the norm for $q \geq 2$, both for deterministic data and for the Gaussian white noise model.

Corollary 5.3.6. *Let the assumptions of Theorem 5.3.2 hold true and let T fulfill Assumption 5.1.8. Let $s \in \left(0, \frac{a}{q-1}\right)$. There exists $C > 0$ such that for all $f^\dagger \in B^s_{p,\infty}(\mathbb{M})$ with $\|f^\dagger\|_{B^s_{p,\infty}} \leq \varrho$ we have the deterministic convergence rates*

$$\|\widehat{f}_\alpha - f^\dagger\|_{B^0_{p,q}} \leq C\delta^{\frac{s}{s+a}}$$

and under the Gaussian white noise model for $\mathbb{M} = \mathbb{T}^d$

$$\mathbb{E}\left(\|\widehat{f}_\alpha - f^\dagger\|_{B^0_{p,q}}\right) \leq C\delta^{\frac{s}{s+a+d/2}}.$$

These rates are order optimal.

Proof. The rates in the norm follow immediately from Theorem 5.3.2 by Theorem 2.2.5 and Jensen's inequality. The optimality of the rates for deterministic data follows from Theorem 1.2.3 together with Theorem 5.3.4. To see that also the convergence rates under the white noise model are of optimal order we apply Theorem 5.3.5. By setting $s^* = a$ and $s^{**} = s + a$ in Corollary 5.3.5 we find that

$$\inf_S \sup_{\|Tf^\dagger\|_{B^{s+a}_{p,\infty}(\mathbb{M})} \le \varrho} \mathbb{E}\left(\left\|Tf^\dagger - S(Tf^\dagger + \varepsilon W)\right\|_{B^a_{p,q}(\mathbb{M})}\right) \ge C\varrho^{\frac{a+d/2}{s+a+d/2}}\varepsilon^{\frac{s}{s+a+d/2}}.$$

For all reconstruction methods R we have by (T7) that

$$\left\|f^\dagger - R(Tf^\dagger + \varepsilon W)\right\|_{B^0_{p,q}(\mathbb{M}_0)} \ge C\left\|Tf^\dagger - TR(Tf^\dagger + \varepsilon W)\right\|_{B^a_{p,q}(\mathbb{M})}.$$

Thus we get again by (T7) a lower bound

$$\begin{aligned}&\inf_R \sup_{\|f^\dagger\|_{B^s_{p,\infty}(\mathbb{M}_0)} \le C\varrho} \mathbb{E}\left(\left\|f^\dagger - R(Tf^\dagger + \varepsilon W)\right\|_{B^0_{p,q}(\mathbb{M}_0)}\right)\\ \ge &\inf_R \sup_{\|Tf^\dagger\|_{B^{s+a}_{p,\infty}(\mathbb{M})} \le \varrho} \mathbb{E}\left(\left\|f^\dagger - R(Tf^\dagger + \varepsilon W)\right\|_{B^0_{p,q}(\mathbb{M}_0)}\right) \ge C\varrho^{\frac{a+d/2}{s+a+d/2}}\varepsilon^{\frac{s}{s+a+d/2}}. \qquad \square\end{aligned}$$

Remark 5.3.7. *Although we did achieve order optimal second order convergence rates in the Bregman divergence for $t = q$, $q \le p$ this does not yield order optimal rates in the norm. Therefore it would be interesting to find an approach which avoids the Bregman divergence. For $q = 1$ such an approach was recently outlined in [39] and resulted in optimal convergence rates. Closing the gap between upper and lower bounds for first order rates with $q \in (1, 2)$ remains an open problem.*

5.4 Quadratic regularization on Hilbert spaces[5]

In this section we consider Tikhonov regularization (3.1) and iterated Tikhonov regularization (P_{n+1}) on Hilbert spaces with $\mathcal{R} = \frac{1}{2}\|\cdot\|^2$. We give some new results for statistical inverse problems, like higher order convergence rates given Poisson data, a saturation result for Gaussian white noise and the deviation inequality for the reconstruction error. But we also want to see how variational source conditions compare to other source conditions on Hilbert spaces. To this end we do not consider Hölder source conditions, but rather more general conditions due to Neubauer [51], that are necessary and sufficient for rates of convergence of spectral regularization methods. Let $E^{T^*T}_\lambda := 1_{[0,\lambda)}(T^*T)$, $\lambda \ge 0$, denote the spectral projections (see [25, Sec. 2.3]) for the operator T^*T with the characteristic function $1_{[0,\lambda)}$ of the interval $[0, \lambda)$. For an index function ψ we define

$$\mathcal{X}^T_\psi := \left\{f \in \mathcal{X} \colon \|f\|_{\mathcal{X}^T_\psi} < \infty\right\}, \qquad \|f\|_{\mathcal{X}^T_\psi} := \sup_{\lambda > 0} \frac{1}{\psi(\lambda)}\|E^{T^*T}_\lambda f\|_{\mathcal{X}}. \tag{5.41}$$

If we want to compare this to Hölder source conditions (1.3), then the function ψ corresponds to the function in $f^\dagger \in \operatorname{ran}(\psi(T^*T))$. The corresponding convergence rate function is

$$\Phi_\psi(t) := \psi\left(\Theta_\psi^{-1}(\sqrt{t})\right)^2, \qquad \Theta_\psi(\lambda) := \sqrt{\lambda}\psi(\lambda).$$

In particular $\Phi_{\mathrm{id}^\nu} = \mathrm{id}^{\frac{2\nu}{2\nu+1}}$. Note that $f^\dagger \in \operatorname{ran}\psi(T^*T)$ implies $f^\dagger \in \mathcal{X}^T_\psi$, but not vice versa and it was shown in [51] that $f^\dagger \in \mathcal{X}^T_\psi$, with $\psi(t) = t^{\frac{s}{2a}}$, $s \in (0, 2a]$ is equivalent to convergence rates $\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}} \le C\delta^{\frac{s}{s+a}}$ for deterministic quadratic Tikhonov regularization.

[5]This section has some overlap with the article [64] to which the author contributed.

5.4.1 Deterministic convergence rates

The following theorem is a generalization of [41, Thm. 3.1], where the first order VSC was equivalently characterized by the set $\mathcal{X}_\psi^T$, to VSCs of arbitrary order.

Theorem 5.4.1. *Let ψ be an index function such that $t \mapsto \psi(t)^2/t^{1-\mu}$ is decreasing for some $\mu \in (0,1)$, $\psi \cdot \psi$ is concave, and ψ is decaying sufficiently rapidly such that*

$$C_\psi := \sup_{0<\lambda\leq\|T^*T\|} \frac{\sum_{k=0}^\infty \psi(2^{-k}\lambda)^2}{\psi(\lambda)^2} < \infty. \tag{5.42}$$

Moreover, let $l \in \mathbb{N}_0$ and define $\psi_l(t) = \psi(t)t^{l/2}$. Then

$$f^\dagger \in \mathcal{X}_{\psi_l}^T \quad \Leftrightarrow \quad \exists A > 0 \,:\, \mathrm{VSC}^{l+1}(f^\dagger, A\Phi_\psi). \tag{5.43}$$

Note that condition (5.42) holds true for all power functions $\psi(t) = t^\nu$ with $\nu > 0$, but not for logarithmic functions $\psi(t) = (-\ln t)^{-p}$ with $p > 0$. The first two conditions on the other hand imply that ψ must not decay to 0 too rapidly. They are both satisfied for power functions $\psi(t) = t^{\nu/2}$ if and only if $\nu \in (0,1)$. We point out that for the case $l = 0$ the condition (5.42) is not required.

Proof. We first show for all $l \in \mathbb{N}$ that

$$f^\dagger \in \mathcal{X}_{\psi_l}^T \qquad \Leftrightarrow \qquad \exists\, \omega_{\frac{l}{2}}^\dagger \in \mathcal{X}_\psi^T \,:\, f^\dagger = (T^*T)^{\frac{l}{2}}\omega_{\frac{l}{2}}^\dagger \tag{5.44}$$

which together with the special case $l = 0$ from [41, Thm 3.1] already implies (5.43) for even l:

(i) Assume there exists $\omega_{\frac{l}{2}}^\dagger \in \mathcal{X}_\psi^T$ such that $f^\dagger = (T^*T)^{\frac{l}{2}}\omega_{\frac{l}{2}}^\dagger$. Define $\int_0^{\lambda+} := \lim_{\varepsilon\searrow 0}\int_0^{\lambda+\varepsilon}$. The following notation is based on [25, Sec. 2.3]. We have

$$\begin{aligned}
\|f^\dagger\|_{\mathcal{X}_{\psi_l}^T}^2 &= \sup_{\lambda>0} \frac{1}{\psi_l(\lambda)^2} \|E_\lambda^{T^*T}(T^*T)^{\frac{l}{2}}\omega_{\frac{l}{2}}^\dagger\|^2 \\
&= \sup_{\lambda>0} \frac{1}{\psi_l(\lambda)^2} \int_0^{\lambda+} \tilde\lambda^l \,\mathrm{d}\|E_{\tilde\lambda}\omega_{\frac{l}{2}}^\dagger\|^2 \\
&\leq \sup_{\lambda>0} \frac{1}{\psi_l(\lambda)^2} \int_0^{\lambda+} \lambda^l \,\mathrm{d}\|E_{\tilde\lambda}\omega_{\frac{l}{2}}^\dagger\|^2 \\
&= \sup_{\lambda>0} \frac{1}{\psi(\lambda)^2} \int_0^{\lambda+} \mathrm{d}\|E_{\tilde\lambda}\omega_{\frac{l}{2}}^\dagger\|^2 = \|\omega_{\frac{l}{2}}^\dagger\|_{\mathcal{X}_\psi^T}^2 < \infty.
\end{aligned}$$

(ii) Now assume that $f^\dagger \in \mathcal{X}_{\psi_l}^T$. It follows that

$$\begin{aligned}
\frac{1}{\psi(\lambda)^2} \int_0^{\lambda+} \tilde\lambda^{-l} \,\mathrm{d}\|E_{\tilde\lambda} f^\dagger\|^2 &= \frac{1}{\psi(\lambda)^2} \sum_{k=0}^\infty \int_{2^{-k-1}\lambda}^{2^{-k}\lambda+} \tilde\lambda^{-l} \,\mathrm{d}\|E_{\tilde\lambda} f^\dagger\|^2 \\
&\leq \frac{1}{\psi(\lambda)^2} \sum_{k=0}^\infty \int_{2^{-k-1}\lambda}^{2^{-k}\lambda+} (2^{-k-1}\lambda)^{-l} \,\mathrm{d}\|E_{\tilde\lambda} f^\dagger\|^2 \\
&\leq \frac{1}{\psi(\lambda)^2} \sum_{k=0}^\infty \frac{\psi(2^{-k}\lambda)^2}{\psi(2^{-k}\lambda)^2(2^{-k-1}\lambda)^l} \int_0^{2^{-k}\lambda+} \mathrm{d}\|E_{\tilde\lambda} f^\dagger\|^2
\end{aligned}$$

$$= \lim_{\varepsilon \searrow 0} \frac{2^l}{\psi(\lambda)^2} \sum_{k=0}^{\infty} \frac{\psi(2^{-k}\lambda)^2}{\psi_l(2^{-k}\lambda)^2} \|E_{2^{-k}\lambda+\varepsilon} f^\dagger\|^2$$
$$\leq \frac{2^l}{\psi(\lambda)^2} \sum_{k=0}^{\infty} \psi(2^{-k}\lambda)^2 \|f^\dagger\|^2_{\mathcal{X}^T_{\psi_l}} \leq 2^l C_\psi \|f^\dagger\|^2_{\mathcal{X}^T_{\psi_l}}.$$

This shows that $\omega^\dagger_{l/2} := \int_0^\infty \tilde{\lambda}^{-l/2}\, \mathrm{d}E_{\tilde{\lambda}} f^\dagger$ is well defined. Moreover, we have that $\omega^\dagger_{l/2} = (T^*T)^{-(l/2)} f^\dagger$ and $\|\omega^\dagger_{l/2}\|_{\mathcal{X}^T_\psi} < \infty$.

To prove the theorem in the case of odd l we use the polar decomposition $T = U(T^*T)^{1/2}$ with a partial isometry U satisfying $N(U) = N(T)$ and set $\xi^\dagger_{\frac{l+1}{2}} := U\omega^\dagger_{\frac{l}{2}}$. As $U : \mathcal{X}^T_\psi \to \mathcal{Y}^{T^*}_\psi$ is an isometry, (5.44) implies

$$f^\dagger \in \mathcal{X}^T_{\psi_l} \qquad \Leftrightarrow \qquad \exists\, \xi^\dagger_{\frac{l+1}{2}} \in \mathcal{Y}^{T^*}_\psi \;:\; f^\dagger = (T^*T)^{\frac{l-1}{2}} T^* \xi^\dagger_{\frac{l+1}{2}}. \tag{5.45}$$

Applying (5.43) for $l = 0$ from [41, Thm. 3.1] to $\mathcal{Y}$ and TT^* yields (5.43) for the case of odd l. □

The equivalence (5.43) together with the equivalence in [2, Prop. 4.1] also shows that in Hilbert spaces higher order variational source conditions are equivalent to certain symmetrized multiplicative variational source conditions.

We have already seen at the end of Section 4.5.1 that $\mathrm{VSC}^l(f^\dagger, A\,\mathrm{id}^{\nu/(\nu+1)})$ implies the order optimal convergence rate $\|\hat{f}^{(m)}_\alpha - f^\dagger\| = \mathcal{O}\left(\delta^{(l-1+\nu)/(l+\nu)}\right)$ for an optimal choice of α and $m \geq l/2$. It follows from [41] and Theorem 5.4.1 that $\mathrm{VSC}^l\left(f^\dagger, A\,\mathrm{id}^{\nu/(\nu+1)}\right)$, with $\nu \in (0,1)$ is not only a sufficient condition for this rate of convergence, but in contrast to spectral Hölder source conditions also a necessary condition:

Corollary 5.4.2. *Let $l \in \mathbb{N}$, $m \geq l/2$, and $\nu \in (0,1)$. Moreover, let $f^\dagger \neq 0$ and let $\hat{f}^{(m)}_\alpha = \hat{f}^{(m)}_\alpha(g^{\mathrm{obs}})$ denote the m-times iterated Tikhonov estimator. Then the following statements are equivalent:*

(a)
$$\exists A > 0 \;:\; \mathrm{VSC}^l\left(f^\dagger, A\,\mathrm{id}^{\nu/(\nu+1)}\right)$$

(b)
$$\exists C > 0\; \forall \delta > 0 \;:\; \sup_{\delta>0} \inf_{\alpha>0} \sup_{\|g^{\mathrm{obs}} - Tf^\dagger\| \leq \delta} \|\hat{f}^{(m)}_\alpha(g^{\mathrm{obs}}) - f^\dagger\| \leq C\delta^{\frac{l-1+\nu}{l+\nu}}$$

For operators which are a-times smoothing in the sense specified below, higher order variational source conditions can be characterized in terms of Besov spaces in analogy to first order variational source conditions (see [41]):

Corollary 5.4.3. *Assume that $\mathbb{M}$ is a bounded Lipschitz domain or a d-dimensional torus $\mathbb{T}^d$ and that $T : H^s(\mathbb{M}) \to H^{s+a}(\mathbb{M})$ is bounded and boundedly invertible for some $a > 0$ and all $s \in \mathbb{R}$. Then for all $f^\dagger \in L^2(\mathbb{M})$, all $l \in \mathbb{N}$ and all $\nu \in (0,1)$ we have*

$$\exists A > 0 \;:\; \mathrm{VSC}^l\left(f^\dagger, A\,\mathrm{id}^{\frac{\nu}{\nu+1}}\right) \qquad \Leftrightarrow \qquad f^\dagger \in B^{(l-1+\nu)a}_{2,\infty}(\mathbb{M}), \tag{5.46a}$$
$$\exists A > 0 \;:\; \mathrm{VSC}^l\left(f^\dagger, A\sqrt{\cdot}\right) \qquad \Leftrightarrow \qquad f^\dagger \in B^{la}_{2,2}(\mathbb{M}) = H^{la}(\mathbb{M}). \tag{5.46b}$$

5.4.2 Statistical convergence rates

In Section 5.1.5 we have already discussed properties that define an a-smoothing operator. From now until the end of this chapter we will assume the following slightly more general definition of a-smoothing.

Definition 5.4.4 (a-smoothing forward operator)**.** *We call a linear operator $T\colon \mathcal{X} \to L^2(\mathbb{M})$ an a-smoothing operator if*

(a) TT^* as a map $TT^*\colon L^2(\mathbb{M}) \to H_0^{2a}(\mathbb{M})$ is bounded with bounded inverse,
(b) T as a map $T : \mathcal{X} \to H_0^a(\mathbb{M})$ is bounded with bounded inverse.

Lemma 5.4.5. *Let $\mathbb{M} = \mathbb{T}^d$. If the linear operator T is injective and fulfills item (a) of Definition 5.4.4, then item (b) holds as well.*

Proof. Let Δ be the Laplace-Beltrami operator, with $(I-\Delta)^{-a}\colon L^2(\mathbb{T}^d) \to H^{2a}(\mathbb{T}^d)$ being an isometric isomorphism. Then item (a) of Definition 5.4.4 is equivalent to the existence of $C_1, C_2 > 0$ such that for all $g \in H^{2a}(\mathbb{T}^d)$ we have

$$C_1\|(I-\Delta)^a g\|_{L^2} = C_1\|g\|_{H^{2a}} \le \|(TT^*)^{-1}g\|_{L^2} \le C_2\|g\|_{H^{2a}} = C_2\|(I-\Delta)^a g\|_{L^2}.$$

Thus by the inequality of Heinz [25, Proposition 8.21] we conclude that for all $g \in H^a(\mathbb{T}^d)$ we have

$$C_1^{\frac{1}{2}}\|g\|_{H^a} = C_1^{\frac{1}{2}}\|(I-\Delta)^{\frac{a}{2}}g\|_{L^2} \le \|(TT^*)^{-\frac{1}{2}}g\|_{L^2} \le C_2^{\frac{1}{2}}\|(I-\Delta)^{\frac{a}{2}}g\|_{L^2} = C_2^{\frac{1}{2}}\|g\|_{H^a}$$

and thus $(TT^*)^{1/2}\colon L^2(\mathbb{T}^d) \to H_0^a(\mathbb{T}^d)$ is bounded with bounded inverse. By [66, Theorem 3.1.5] there exists a polar decomposition $T^* = U(TT^*)^{1/2}$ with a partial isometry $U\colon L^2(\mathbb{M}) \to \mathcal{X}$, $\ker(U) = \ker(T^*) = 0$. Therefore we have $T = (TT^*)^{1/2}U^*$ with an isometry U and the claim holds. □

Note that first order statistical convergence rates for an a-smoothing forward operator follow from $\mathrm{VSC}^1(f^\dagger, \Phi)$ by Corollary 4.3.12 (together with Lemma 5.1.2 for Poisson data). But for second order convergence rates we still have to show how to verify Assumption 4.4.4.

Theorem 5.4.6. *Let T be an a-smoothing forward operator in the sense of Definition 5.4.4 with $a > \gamma$ and let $f^\dagger \in \mathcal{X}_\psi^T$ with $\psi(t) = Ct^{\frac{s}{2a}}$, $s \in (a, 2a)$. Then (D1)-(D8) of Assumption 4.4.4 hold true with $p = 2, \mu = 2$, $r = 2$ and there exists $C_\Psi > 0$ such that the index function Ψ in (4.17) satisfies*

$$\Psi(\alpha) \le C_\Psi \alpha^{\frac{s-a}{a}}, \text{ for all } \alpha \text{ such that } C_\Psi \alpha^{\frac{s-a}{a}} \le 1. \tag{5.47}$$

Moreover, if $f^\dagger \in \operatorname{ran} T^*T$, then (D1)-(D8) hold with $\Psi(\alpha) \le C\alpha$ for α sufficiently small.

Proof. Note that (D1),(D2) and (D3) are fulfilled by our choice of $\mathcal{R}$ and T. By (5.45) we see that $f^\dagger \in \mathcal{X}_\psi^T$ is equivalent to the existence of $\xi^\dagger \in L^2(\mathbb{M})$ such that $\xi^\dagger \in L^2(\mathbb{M})_{\tilde{\psi}}^{T^*}$, where $\tilde{\psi}(t) = Ct^{\frac{s-a}{2a}}$. From [3, Proposition 2.2] we see that

$$L^2(\mathbb{M})_{\tilde{\psi}}^{T^*} = (L^2(\mathbb{M}), TT^*L^2(\mathbb{M}))_{\frac{s-a}{2a},\infty} = (L^2(\mathbb{M}), H_0^{2a}(\mathbb{M}))_{\frac{s-a}{2a},\infty} = B_{2,\infty}^{s-a}(\mathbb{M}),$$

where the second inequality follows from Definition 5.4.4 and the third by Theorem A.2.7. Thus we have found $\xi^\dagger$ such that (D4) holds true and we even have $\xi^\dagger \in B^{s-a}_{2,\infty}$. Now we want to apply Proposition 5.1.3. Let P_k be given by the projections defined in Proposition A.2.13. We have for $g \in \{1\} \cup \mathcal{G}^a_R$ that

$$\|T^{-1} g P_k \xi^\dagger\|_{\mathcal{X}} \leq C\|gP_k\xi^\dagger\|_{H^a_0} \leq C\|g\|_{H^a}\|P_k\xi^\dagger\|_{H^a_0} \leq CR\|P_k\xi^\dagger\|_{H^a_0}.$$

By (A.6) in the appendix we see that

$$\|P_k\xi^\dagger\|_{H^a_0} \leq C2^{k(2a-s)}\|\xi^\dagger\|_{B^{s-a}_{2,\infty}} \leq C2^{k(2a-s)},$$

thus we can choose $\nu_k := C2^{k(2a-s)}$. Finally by (A.7) we obtain

$$\kappa_k = \|(I-P_k)\xi^\dagger\|_{L^2} \leq C2^{-k(s-a)}\|\xi^\dagger\|_{B^{s-a}_{2,\infty}} \leq C2^{-k(s-a)}.$$

Let $\overline{\alpha} \geq 1$ then we can for all $\alpha \in (0, \overline{\alpha}]$ choose $k \in \mathbb{N}_0$ such that $2^k \sim \alpha^{\frac{-1}{2a}}$ with the implicit constants depending on $\overline{\alpha}$. With this choice of k we have

$$\alpha\nu_k^2 + \kappa_k^2 \leq C\alpha^{\frac{s-a}{a}} =: C_\Psi \alpha^{\frac{s-a}{a}}.$$

Now if $C_\Psi\alpha^{\frac{s-a}{a}} \leq 1$ then we also have $\alpha\nu_k^2 \leq 1$, which implies

$$\Psi(\alpha) = \inf_{\substack{l\in\mathbb{N}_0 \\ \alpha\nu_l^2 \leq 1}} \left[\alpha\nu_l^2 + \kappa_l^2\right] \leq \alpha\nu_k^2 + \kappa_k^2 \leq C_\Psi\alpha^{\frac{s-a}{a}}.$$

The last statement in the theorem follows directly from Proposition 5.1.3 as $g\xi^\dagger \in H^a_0 = T\mathcal{X}$ and thus one can choose $P_k = I$ for all $k \in \mathbb{N}_0$. □

Combining Theorems 5.4.1 and 5.4.6 together with the deviation inequality from Assumption 1.4.1 and Corollaries 4.3.12 and 4.4.10 yields optimal (or almost optimal) convergence rates in expectation.

Corollary 5.4.7. *Let T be an a-smoothing forward operator in the sense of Definition 5.4.4 with $a > \gamma$ and let $f^\dagger \in \mathcal{X}^T_\psi$ with $\psi(t) = Ct^{\frac{s}{2a}}$, $s \in (0,a) \cup (a, 2a)$. We have for $\mathcal{S} = \mathcal{S}^{LS}_{G^{\mathrm{obs}}}$, with $Z = W$ Gaussian white noise and a parameter choice $\alpha \sim \varepsilon^{\frac{2a}{s+a+d/2}}$ that*

$$\forall x > 0: \qquad \mathbb{P}\Big(\|\widehat{f}_\alpha - f^\dagger\|_{L^2} > (C+x)\varepsilon^{\frac{s}{s+a+d/2}}\Big) \leq \exp\Big(-C_W x^2\Big),$$

for some constants $C, C_W > 0$ which implies for all $q \geq 1$ that

$$\mathbb{E}\Big(\|\widehat{f}_\alpha - f^\dagger\|^q_{L^2}\Big)^{\frac{1}{q}} \leq C\varepsilon^{\frac{s}{s+a+d/2}}.$$

Assume additionally (D9), with $f^\dagger$ in the interior of $\mathcal{B}$, and that $a \geq a_0 > d/2$ for $a_0 \in \{1, 2\}$, $s \geq a\frac{d}{2a_0} - d/2$. Then we get for $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$, $Z = \sqrt{t}(G_t - g^\dagger)$, $\varepsilon = t^{-1/2}$, $\mathcal{R} = \frac{1}{2}\|\cdot\|_{L^2} + \chi_{\mathcal{B}}$ and a parameter choice $\alpha \sim t^{\frac{-a}{s+a+d/2}}$ that for all $0 < \epsilon$ we have

$$\forall x > 0: \qquad \mathbb{P}\Big(\|\widehat{f}_\alpha - f^\dagger\|_{L^2} > (C+x)t^{\frac{-s}{2s+2a+d+\epsilon}}\Big) \leq \exp(-C_{G_t}x).$$

This implies for all $q \geq 1$ that

$$\mathbb{E}\Big(\|\widehat{f}_\alpha - f^\dagger\|^q_{L^2}\Big)^{\frac{1}{q}} \leq Ct^{\frac{-s}{2s+2a+d+\epsilon}}.$$

Proof. For $s \in (0,a)$ we have by Theorem 5.4.1 that $\mathrm{VSC}^1(f^\dagger,\Phi)$ holds true with $\Phi(\tau) = C\tau^{\frac{s}{s+a}}$ and by Lemma 5.1.2 also $\mathrm{VSC}^1(\Phi(C\cdot),\frac{1}{2}\|\cdot\|^2,\mathrm{KL}^\sigma_{g^\dagger})$ holds true. For $s \in (a,2a)$ we have by Theorem 5.4.6 that (D1)-(D8) of Assumption 4.4.4 hold true with $p=2, \mu=2$, $r=2$ and index function as in (5.47). The bounds for the quadratic Tikhonov regularization with Gaussian white noise follow directly from Corollaries 4.3.12 and 4.4.10 together with Lemma 4.3.13 which give

$$\forall x > 0: \qquad \mathbb{P}\left(\|\widehat{f}_\alpha - f^\dagger\|_{L^2} > C\sqrt{1+x}\left(\frac{\varepsilon}{\alpha^{\frac{1}{2}+\frac{d}{4a}}} + \alpha^{\frac{s}{2a}}\right)\right) \le \exp(-C_W x),$$

so that the claim follows by the parameter choice $\alpha \sim \varepsilon^{\frac{2a}{s+a+d/2}}$ and replacing x by x^2. (Note that under (5.47) we would strictly speaking need α sufficiently small. However, if $s \in (a,2a)$ then in particular the first order VSC holds true with $\Phi = C\sqrt{\cdot}$ and for large α this actually gives a stronger bound than the above, compare Remark 4.4.8. In other words: large α respectively ε are not interesting.)

For the case of Kullback-Leibler regularization with Poisson data we have $\varepsilon := t^{-1/2}$ and convergence rates follow likewise from Corollaries 4.3.12 and 4.4.10 together with Lemma 4.3.13. The main differences to the white noise case are the different deviation inequality (Theorem 1.4.9), as well as the additional assumption that

$$\alpha \ge C t^{\frac{-1}{1+\gamma/a_0}}, \tag{5.48}$$

where $a \ge a_0 \ge \gamma$, $a_0 \in \{1,2\}$. Because of the deviation inequality we want to choose $\gamma = d/2+\varepsilon$ with $\varepsilon > 0$ sufficiently small. Under the parameter choice $\alpha \sim t^{\frac{-a}{s+a+d/2}}$ we have (5.48) if and only if $s \ge \frac{ad}{2a_0} - \frac{d}{2}$. □

5.4.3 Converse and saturation results for Gaussian white noise

In this subsection we will only consider an a-smoothing forward operator T with $a > d/2$ as in Definition 5.4.4 and quadratic Tikhonov regularization given by

$$\widehat{f}_\alpha \in \operatorname*{arg\,min}_{f\in\mathcal{X}} \left[\frac{1}{2\alpha}\|Tf\|^2_{\mathcal{X}} - \left\langle G^{\mathrm{obs}}, Tf\right\rangle + \frac{1}{2}\|f\|^2_{L^2}\right], \tag{5.49}$$

with a Hilbert space $\mathcal{X}$ and $G^{\mathrm{obs}} = g^\dagger + \varepsilon W$, where W is Gaussian white noise. The following theorem shows equivalence of the standard second order VSC and some conditions of Assumption 4.4.4, which in particular implies that converse results from the literature are applicable.

Theorem 5.4.8. *Let T be a-smoothing in the sense of Definition 5.4.4 with $a > d/2$ and $\mathbb{M} = \mathbb{T}^d$. For $s \in (a,2a)$ the following statements are equivalent:*

(a) (D4), (D5), (D6), (D7) hold true with $p=\nu=2$, $\beta=0$ and the index function $\Phi(t) := \inf_{k\in\mathbb{N}_0} \frac{1}{2}\left[\nu_k\sqrt{t}+\kappa_k^2\right] = \mathcal{O}\left(t^{\frac{s-a}{s}}\right)$, $t \searrow 0$.

(b) (D4), (D6), (D7), (D8) hold true with $p=\nu=2$, $\beta=0$ and the index function $\Phi(t) := \inf_{k\in\mathbb{N}_0} \frac{1}{2}C_{R,\sigma,\mathbb{M}}\left[\nu_k\sqrt{t}+\kappa_k^2\right] = \mathcal{O}\left(t^{\frac{s-a}{s}}\right)$, $t \searrow 0$, with a constant $C_{R,\sigma,\mathbb{M}} > 0$ depending on R,σ and $\mathrm{vol}(\mathbb{M})$.

(c) *There exists* $\xi^\dagger \in L^2(\mathbb{M})$ *such that* $f^\dagger = T^*\xi^\dagger$. $\xi^\dagger$ *fulfills the standard (Hilbert space) second order VSC*

$$\forall \xi \in L^2(\mathbb{M}) : \langle \xi, \xi^\dagger \rangle \leq \frac{1}{2}\|\xi\|_{L^2}^2 + \Phi\Big(\|T^*\xi\|_{L^2}^2\Big)$$

with $\Phi(t) = \mathcal{O}\left(t^{\frac{s-a}{s}}\right)$, $t \searrow 0$.

(d) $f^\dagger \in \mathcal{X}_\psi^T$ *with* $\psi(t) = \mathcal{O}\left(t^{\frac{s}{2a}}\right)$, $t \searrow 0$.

(e) $\left(\inf_{\alpha>0} \mathbb{E}\Big(\|\widehat{f}_\alpha - f^\dagger\|_{\mathcal{X}}^2\Big)\right)^{\frac{1}{2}} = \mathcal{O}\left(\varepsilon^{\frac{s}{s+a+d/2}}\right)$, $\varepsilon \searrow 0$.

Proof. (a) or (b) $\Rightarrow$ (c): We first assume (a) and thus have for all $\xi \in L^2(\mathbb{T}^d)$ and $k \in \mathbb{N}_0$ by applying (D5) for $\xi = \xi^\dagger - \xi$ that

$$\langle \xi, P_k\xi^\dagger \rangle \leq \frac{1}{4}\|\xi\|_{B^{-d/2}_{2,\infty}}^2 + \frac{\nu_k}{2}\|T^*\xi\|_{L^2}.$$

Thus together with (D6) we have

$$\begin{aligned}
\langle \xi, \xi^\dagger \rangle &= \langle \xi, P_k\xi^\dagger \rangle + \langle \xi, (I - P_k)\xi^\dagger \rangle \\
&\leq \frac{1}{4}\|\xi\|_{B^{-d/2}_{2,\infty}}^2 + \frac{\nu_k}{2}\|T^*\xi\|_{L^2} + \|\xi\|_{L^2}\|(I - P_k)\xi^\dagger\|_{L^2} \\
&\leq \frac{1}{4}\|\xi\|_{L^2}^2 + \frac{\nu_k}{2}\|T^*\xi\|_{L^2} + \kappa_k\|\xi\|_{L^2} \\
&\leq \frac{1}{2}\|\xi\|_{L^2}^2 + \frac{\nu_k}{2}\|T^*\xi\|_{L^2} + \frac{1}{2}\kappa_k^2.
\end{aligned}$$

Taking the infimum over k gives the claim. The other implication works analogously by choosing $g \in \mathcal{G}_R^a$ constantly so that this only gives an additional scalar factor depending on R and $\mathrm{vol}(\mathbb{M})$. The resulting different constant in front of $\frac{1}{2}\|\xi\|_{L^2}^2$ can be changed by rescaling ξ.

(c) $\Leftrightarrow$ (d): As in the last section we see that (d) is equivalent to the existence of $\xi^\dagger \in L^2(\mathbb{M})_{\tilde{\psi}}^{T^*}$, with $\tilde{\psi}(t) = \psi(t)t^{-\frac{1}{2}}$ by (5.45). $\xi^\dagger \in L^2(\mathbb{M})_{\tilde{\psi}}^{T^*}$ is equivalent to (c) by Theorem 5.4.1.

(d) $\Rightarrow$ (a) and (b): This follows from the proof of Theorem 5.4.6 as we then have $\nu_k \leq C2^{k(2a-s)}$ and $\kappa_k \leq C2^{-k(s-a)}$, so choosing $2^{-k} \sim t^{\frac{1}{2s}}$ gives the claim.

(d) $\Leftrightarrow$ (e): This follows from the general equivalence theorems 3.3 and 5.1 in [41] as in the proof of [41, Theorem 7.1]. □

Finally we will prove a saturation result in the setting of this subsection. As we assume $a > d/2$ we can compute the minimizer in (5.49) as usual by

$$\widehat{f}_\alpha = (T^*T + \alpha I)^{-1}T^*G^{\mathrm{obs}} \tag{5.50}$$

because the Tikhonov functional is differentiable, strictly convex and has the above solution as unique zero of the derivative. Further we can show that for a nontrivial exact solution $f^\dagger$ one can bound each parameter choice rule $\bar{\alpha}(\varepsilon, G^{\mathrm{obs}})$ in the following way.

Lemma 5.4.9. *Let* T *be* a*-smoothing, with* $a > d/2$, *let* $f^\dagger \neq 0$ *and* $\alpha_0 > 0$. *Then for every parameter choice* $\bar{\alpha}(\varepsilon, G^{\mathrm{obs}}) \leq \alpha_0$ *we have*

$$\mathbb{E}\Big(\bar{\alpha}(\varepsilon, G^{\mathrm{obs}})^2\Big) = \mathcal{O}\left(\varepsilon^2 + \mathbb{E}\Big(\|\widehat{f}_{\bar{\alpha}} - f^\dagger\|^2\Big)\right).$$

Proof. For easier notation we just write $\bar\alpha$ for $\bar\alpha(\varepsilon, G^{\mathrm{obs}})$. By (5.50) we find

$$(T^*T + \bar\alpha I)(f^\dagger - \widehat{f}_{\bar\alpha}) = \bar\alpha f^\dagger - \varepsilon T^*W.$$

In particular as $f^\dagger \neq 0$ we have

$$\bar\alpha \leq \frac{\|T^*W\|}{\|f^\dagger\|}\varepsilon + \frac{\|T^*T + \bar\alpha I\|}{\|f^\dagger\|}\|\widehat{f}_{\bar\alpha} - f^\dagger\|$$

and thus by Young's inequality

$$\bar\alpha^2 \leq 2\left(\frac{\|T^*W\|}{\|f^\dagger\|}\varepsilon\right)^2 + 2\left(\frac{\|T^*T + \bar\alpha I\|}{\|f^\dagger\|}\|\widehat{f}_{\bar\alpha} - f^\dagger\|\right)^2.$$

As T is a-smoothing we have $\mathbb{E}(\|T^*W\|_{L^2}^2) \leq \mathbb{E}\left(C\|W\|^2_{B^{-d/2}_{2,\infty}}\right) < \infty$, so the claim follows by monotonicity of the expected value. □

Now we can prove the saturation result, which shows that the rate we obtain for $f^\dagger \in \operatorname{ran} T^*T$ is actually the best rate, that one can achieve by quadratic Tikhonov regularization with nontrivial solution $f^\dagger$ and an a-priori parameter choice for α.

Theorem 5.4.10. *Let T be a-smoothing with $a > d/2$ and let $\alpha_0 > 0$. Assume that we have*

$$\inf_{0<\alpha\leq\alpha_0} \mathbb{E}\left(\|\widehat{f}_\alpha - f^\dagger\|^2\right) = o\left(\varepsilon^{\frac{4}{3+d/2a}}\right),$$

then $f^\dagger = 0$.

Proof. Let f_α be the solution to (5.49) for exact data, i.e. $\varepsilon = 0$. From (5.50) we can conclude that

$$\|\widehat{f}_\alpha - f^\dagger\|^2 = \|f_\alpha - f^\dagger\|^2 + 2\langle \widehat{f}_\alpha - f_\alpha, f_\alpha - f^\dagger\rangle + \|\widehat{f}_\alpha - f_\alpha\|^2.$$

Taking the expected value on both sides yields the bias-variance decomposition

$$\mathbb{E}\left(\|\widehat{f}_\alpha - f^\dagger\|^2\right) = \|f_\alpha - f^\dagger\|^2 + \mathbb{E}\left(\|\widehat{f}_\alpha - f_\alpha\|^2\right),$$

as $\mathbb{E}(W) = 0$ and $\widehat{f}_\alpha - f_\alpha$ depends linearly on W. By assumption there exists a parameter choice $\bar\alpha(\varepsilon)$, such that $\mathbb{E}\left(\|\widehat{f}_{\bar\alpha} - f^\dagger\|^2\right) = o\left(\varepsilon^{\frac{4}{3+d/2a}}\right)$ which leads to

$$\mathbb{E}\left(\|\widehat{f}_{\bar\alpha} - f_{\bar\alpha}\|^2\right) = o\left(\varepsilon^{\frac{4}{3+d/2a}}\right). \tag{5.51}$$

Now assume that $f^\dagger \neq 0$. By Lemma 5.4.9 we then have

$$\bar\alpha^2 = o\left(\varepsilon^{\frac{4}{3+d/2a}}\right). \tag{5.52}$$

We will show that such a parameter choice $\bar\alpha$ cannot exist. To this end we consider a different "true solution" $h^\dagger$ with $h^\dagger = T^*T\omega$, $\|\omega\|_{\mathcal{X}} \leq 1$ and the corresponding regularized solutions $\widehat{h}_{\bar\alpha} = (T^*T + \bar\alpha I)^{-1}T^*(Th^\dagger + \varepsilon W)$ for noisy data, $h_{\bar\alpha} = (T^*T + \bar\alpha I)^{-1}T^*Th^\dagger$ for exact data. Note that $\widehat{h}_{\bar{\alpha}} - h_{\bar{\alpha}} = \varepsilon(T^*T + \bar{\alpha}I)^{-1}T^*W = \widehat{f}_{\bar{\alpha}} - f_{\bar{\alpha}}$ does not depend on the true solution thus we have by the bias-variance decomposition and (5.51) that

$$\mathbb{E}\Big(\|\widehat{h}_{\bar{\alpha}} - h^\dagger\|^2\Big) = \|h_{\bar{\alpha}} - h^\dagger\|^2 + o\left(\varepsilon^{\frac{4}{3+d/2a}}\right).$$

Now by classical regularization theory (or by Theorem 5.4.6 and Theorem 4.4.7 with $W = 0$) we easily see that $\|h_{\bar{\alpha}} - h^\dagger\|^2 \leq C\bar{\alpha}^2$ so together with (5.52) we have

$$\mathbb{E}\Big(\|\widehat{h}_{\bar{\alpha}} - h^\dagger\|^2\Big) = o\left(\varepsilon^{\frac{4}{3+d/2a}}\right), \tag{5.53}$$

which is a contradiction to the optimal rates that one can show under the source condition $h^\dagger = T^*T\omega$. (See [6, Section 5.3] for $\mathcal{X} = L^2(\mathbb{M})$. If $\mathcal{X} \neq L^2(\mathbb{M})$ then $\mathcal{X}$ still has to be infinite dimensional as T is a-smoothing, so there exists an isometric isomorphism $i_{\mathcal{X}} : L^2(\mathbb{M}) \to \mathcal{X}$ and one can consider the inverse problem related to $\widetilde{T} = T \circ i_{\mathcal{X}}$. As $\widetilde{T}$ is still a-smoothing one can conclude the same rate (5.53) for $h^\dagger = \widetilde{T}^*\widetilde{T}\omega \not\equiv 0$, this also leads to a contradiction.) Thus (5.52) cannot hold and we must have $f^\dagger = 0$.

□

Discussion and outlook

Whilst we successfully have derived higher order convergence rates for statistical inverse problems and even introduced variational source conditions of third and higher order, there emerge several open problems from what we achieved.

The most nail-biting one is how variational source conditions of fourth and higher order for Banach spaces should look like. This problem has already been discussed in [64, Section 7]. Usually a mathematician would expect that if he understands the first two or three instances of an inductive problem then he will see some pattern and find a generalization. However, in this case the second order VSC is already more complicated and restrictive than the first order VSC and the third order VSC even more so. Thus it is possible that there is no VSC^4 which is as general as the VSC^3 and that the Hilbert space VSC^4 is actually almost the most general version that one can find. Nonetheless we would be glad to be proved wrong here.

Another big and non obvious problem is to show higher order rates for nonlinear forward operators F under higher order VSCs. Since VSCs have been originally introduced to relax the strong assumptions necessary on F it is a natural question whether higher order VSCs also work for nonlinear F. As pointed out in [34] this comes down to the problem that duality requires a linear forward operator, so that this concept has to be generalized.

It would be interesting to see whether the higher order VSCs could be verified under less restrictive assumptions on $\mathcal{R}$. At the moment we require some kind of differentiability of $\mathcal{R}$ (note that item (a) of Proposition 5.1.3 implies differentiability of $\mathcal{R}$ at $f^\dagger$ by Proposition 2.2.14). This is the reason why we did not show a VSC^2 under Nikolskii smoothness assumptions for maximum entropy regularization in Theorem 5.2.3 and also a VSC^3 for Besov norm regularization in Theorem 5.3.2 is not obvious from Proposition 5.1.6 as the norm is not twice differentiable for $p < 2$ or $q < 2$. Especially in the case of maximum entropy (Theorem 5.2.3) it would be remarkable if it would be impossible to show a VSC^2 under Nikolskii smoothness assumptions although we can show a VSC^3. Therefore we suspect that it should be possible to relax the assumptions required of $\mathcal{R}$.

There are some open questions that seem to be of more technical nature and will hopefully be resolved soon. This includes firstly an optimal deviation inequality for Poisson processes, i.e. Assumption 1.4.1 with $\gamma = d/2$ and ideally also for all $p \in [1, 2]$. Secondly it should be possible to show inequalities of the kind of Theorem A.2.10 for all smoothness indices s and also for Besov spaces (where again integrability indices $p \in [1, 2]$ would be most interesting in order to consider entropy and Besov penalty terms for inverse problems with Poisson data, compare Remark 5.2.2). Finally one can most likely consider more general measurement manifolds $\mathbb{M}$. To this end one would have to ensure that the necessary results in Section A.2 of the appendix still hold true and to deal with statistical data one would need to verify the deviation inequality on the manifold of interest.

Concerning Theorem 5.3.2 for Besov norm regularization the error bounds (5.37) for $q < 2$ can probably be improved. Similarly the second order rates (5.39) are not optimal in the Bregman divergence so one would expect that some improvement is possible.

By Definition A.2.1 of the Besov spaces for bounded Lipschitz domains $\mathbb{M}$ and the assumption $F(f) \in B^{a}_{p,2}(\mathbb{M})$ for $f \in \mathcal{X}$, $a > d/2$, that we consider for Gaussian white noise, we restrict us to exact data $g^\dagger$ that vanish on the boundary of $\mathbb{M}$. If one is interested in inverse problems with non trivial boundary values then one has to define $B^s_{p,q}(\mathbb{M}) = \{f \in \mathcal{D}'(\mathbb{M}) : f = g|_{\mathbb{M}}, g \in B^s_{p,q}(\mathbb{R}^d)\}$ for $s > 0$. The main difficulty then is to find a wavelet system such that Theorem A.2.4 holds true. This problem is extensively covered in [71, Ch. 5]. For the simple case of $\Omega = [0, 1]$ one can consider boundary-corrected wavelet bases as in [33, Sec. 4.3.5].

We did not consider the problem of showing statistical convergence rates under the third order VSC. The reason why our approach in the deterministic case cannot easily be generalized to solve also the statisical case is simply put that duality becomes much more complicated for statistical noise. More concretely we cannot conlcude an analogue of Lemma 4.5.6 for $\mathcal{S} = \mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}$ (or $\mathcal{S} = \mathcal{S}^{\mathrm{KL}}_{G^{\mathrm{obs}},\sigma}$) by Lemma 3.3.4 as the quantity $\frac{1}{\alpha}\left(\mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}\right)^*\left(-\alpha(\xi^\dagger - \widehat{\xi}_\alpha)\right)$ almost surely equals ∞ if $Z \notin L^2$. To see this note that by (3.9) and (3.13) we have

$$\left(\mathcal{S}^{\mathrm{LS}}_{G^{\mathrm{obs}}}\right)^*(-\alpha(\xi^\dagger - \widehat{\xi}_\alpha)) = \sup_{h \in B^{\gamma}_{p,1}(\mathbb{M})} \left[\langle -\alpha\xi^\dagger + 2g^\dagger - T\hat{f}_\alpha, h\rangle - \frac{1}{2}\|h\|^2_{L^2} + 2\langle \varepsilon Z, h\rangle\right].$$

Whilst the first two terms can be bounded by $\|h\|_{L^2}$ the last term can not. Thus there is a sequence $(h_n) \subset B^{\gamma}_{p,1}(\mathbb{M})$ with $\|h_n\|_{L^2} \le C$ and $\langle \varepsilon Z, h_n\rangle \to \infty$.

Last but not least there are of course a lot of regularization methods, penalty and data fidelity terms that we did not consider in this work. It would be interesting to investigate on which of those the methods that we outlined can be applied.

Appendix

A.1 Normed vector spaces

Theorem A.1.1 (Bounded inverse theorem)**.** *Let $\mathcal{X}, \mathcal{Y}$ Banach spaces and $T\colon \mathcal{X} \to \mathcal{Y}$ linear, bounded and bijective. Then the inverse operator T^{-1} is also linear and bounded.*

Proof. [60, Cor. 2.12]. □

Definition A.1.2. *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces. $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$ is called Fréchet differentiable in $x \in \mathcal{X}$ if there exists a bounded linear operator $\mathcal{F}'[x]\colon \mathcal{X} \to \mathcal{Y}$ such that*

$$\lim_{h\to 0} \frac{1}{\|h\|_{\mathcal{X}}} \|\mathcal{F}(x+h) - \mathcal{F}(x) - \mathcal{F}'[x]h\| = 0.$$

A.2 Function spaces

Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are function spaces that allow to measure smoothness s and integrability p of a function and generalize other common function spaces. To this end one has the additional fine tune parameter q. For example Sobolev spaces are given by $B^s_{2,2}(\mathbb{R}^d) = H^s_2(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ and in particular $B^0_{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

We do not intend to give a full overview over Besov spaces, but will rather state the theorems, that we need for our proof. In particular we will give no definition of $B^s_{p,q}(\mathbb{R}^d)$ as we actually are interested in $B^s_{p,q}(\mathbb{M})$ for a bounded Lipschitz domain $\mathbb{M}$ or $\mathbb{M} = \mathbb{T}^d$, the d-dimensional torus. However it is the standard approach to Besov spaces to start with $B^s_{p,q}(\mathbb{R}^d)$ and then go to other domains and we refer the reader who wants to have a thorough understanding of Besov spaces to [33, 69]. We also neglect the d-dimensional torus, however it can be defined by a wavelet basis as in Definition A.2.3 below (see [33] or [74]) and properties can be transfered from $B^s_{p,q}(\mathbb{R}^d)$ to $B^s_{p,q}(\mathbb{T}^d)$ as described in [69, Ch. 9].

Let from now on $\mathbb{M} \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We will define $B^s_{p,q}(\mathbb{M})$ in two ways: first via restrictions from $\mathbb{R}^d$ to $\mathbb{M}$ and secondly via wavelet coefficients.

Definition A.2.1. *Let $1 \le p, q \le \infty$, define*

$$B^s_{p,q}(\mathbb{M}) = \begin{cases} \{f \in \mathcal{D}'(\mathbb{M}) : f = g|_{\mathbb{M}}, g \in B^s_{p,q}(\mathbb{R}^d)\} & \text{if } s \le 0 \\ \{f \in \mathcal{D}'(\mathbb{M}) : f = g|_{\mathbb{M}}, g \in B^s_{p,q}(\mathbb{R}^d), \operatorname{supp} g \subset \overline{\mathbb{M}}\} & \text{if } s > 0. \end{cases}$$

and

$$\|f\|_{B^s_{p,q}(\mathbb{M})} = \inf \|g\|_{B^s_{p,q}(\mathbb{R}^d)},$$

where the infimum is taken over all extensions g as above. For $s > 0$ denote $H_0^s(\mathbb{M}) = B_{2,2}^s(\mathbb{M})$ and define the spaces

$$\tilde{B}_{p,q}^s(\mathbb{M}) = \{f \in \mathcal{D}'(\mathbb{M}) : f = g|_{\mathbb{M}}, g \in B_{p,q}^s(\mathbb{R}^d)\}$$

with the norm given again by the infimum over the norm of all such extensions. Denote $H^s(\mathbb{M}) = \tilde{B}_{2,2}^s(\mathbb{M})$.

We only introduce the Sobolev spaces $H_0^s(\mathbb{M})$, $H^s(\mathbb{M})$ as special cases of the defined Besov spaces, but one can also show that this definition coincides with more intrinsic definitions for example by distributional derivatives for $s \in \mathbb{N}$ or by moduli of smoothness (see [71, Remark 2.3]).

Theorem A.2.2. *Let $k \in \mathbb{N}_0$. An equivalent norm on $H^s(\mathbb{M})$ is given by*

$$\|f\| = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2}.$$

Definition A.2.3. *Let for $u \in \mathbb{N}$,*

$$\left\{\Phi_l^j : j \in \mathbb{N}_0, l = 1, \dots N_j\right\} \text{with } N_j \in \mathbb{N} \cup \{\infty\}$$

be an orthonormal wavelet basis of $L^2(\mathbb{M})$ (as in [71, Def. 2.4,2.31 and Thm. 2.33]). Then we can define

$$B_{p,q}^{s,W}(\mathbb{M}) = \{f \in \mathcal{D}'(\mathbb{M}) : f = \sum_{j=0}^{\infty} \sum_{l \in N_j} \lambda_l^j \Phi_l^j, \|f\|_{B_{p,q}^{s,W}(\mathbb{M})} < \infty\},$$

where

$$\|f\|_{B_{p,q}^{s,W}(\mathbb{M})} = \left(\sum_{j=0}^{\infty} 2^{jsq} 2^{jd(\frac{1}{2} - \frac{1}{p})q} \left(\sum_{l \in N_j} |\lambda_l^j|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},$$

where one has to replace the l_q (or l_p) norm by the l_∞ norm for $q = \infty$ (or $p = \infty$).

Actually both definitions give the same space for a suitable wavelet basis by [71, Thm. 3.23, Cor. 3.25].

Theorem A.2.4. *Let (Φ_k^j) be a u-wavelet basis as in [71, Def. 2.4,2.31 and Thm. 2.33]. Then we have for all $1 \leq p, q \leq \infty, |s| < u$ that $B_{p,q}^s(\mathbb{M}) = B_{p,q}^{s,W}(\mathbb{M})$ with equivalent norms.*

We have the following embedding results.

Theorem A.2.5. *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, then*

$$B_{p_1,q_1}^{s_1}(\mathbb{M}) \subset B_{p_2,q_2}^{s_2}(\mathbb{M}) \begin{cases} \text{if} & s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2} \text{ and } q_1 = q_2 \\ \text{or if} & s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}. \end{cases}$$

Let $1 \leq p_1 \leq p_2 \leq \infty$, $q \in [1, \infty]$ and $s \in \mathbb{R}$ then

$$B_{p_2,q}^s(\mathbb{M}) \subset B_{p_1,q}^s(\mathbb{M}).$$

Let $p \in [1, \infty]$, then

$$B_{p,\min(p,2)}^0 \subset L^p(\mathbb{M}) \subset B_{p,\max(p,2)}^0.$$

All above embeddings are continuous.

Proof. As in [69, Theorem 3.3.1] the first embedding follows directly from the definition and the corresponding embeddings for $B^s_{p,q}(\mathbb{R}^d)$ and the second basically by boundedness of $\mathbb{M}$, compare [69, Lemma 3.3.1]. The final embedding is given by [69, Proposition 3.2.4] together with the fact that $F^0_{p,2}(\mathbb{M}) = L^p(\mathbb{M})$, compare [69, 3.4.2/(1)]. □

From the above embeddings we can in particular conclude that the Besov spaces form scales with respect to the smoothness and fine tune parameter: For $s \in \mathbb{R}$, $p \in [1,\infty]$, $1 \le q_1 \le q_2 \le \infty$ and $\varepsilon > 0$ we have

$$B^{s+\varepsilon}_{p,\infty} \subset B^s_{p,1} \subset B^s_{p,q_1} \subset B^s_{p,q_2} \subset B^s_{p,\infty} \subset B^{s-\varepsilon}_{p,1}. \tag{A.1}$$

It is crucial for our work that we have the following duality result [71, Thm. 3.30].

Theorem A.2.6. *Let $1 \le p, q < \infty, s \in \mathbb{R}$, then*

$$\left(B^s_{p,q}(\mathbb{M})\right)^* = B^{-s}_{p',q'}(\mathbb{M}).$$

Furthermore we need the following interpolation inequalities.

Theorem A.2.7. *Let $1 < p < \infty$, $-\infty < s_1 < s < s_2 < \infty$, $q, q_1, q_2 \in [1,\infty]$ and let $\theta = \frac{s-s_1}{s_2-s1}$ then*

$$\left(B^{s_1}_{p,q_1}(\mathbb{M}), B^{s_2}_{p,q_2}(\mathbb{M})\right)_{\theta,q} = B^s_{p,q}(\mathbb{M})$$

and there exists $C_{q,\theta} > 0$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{M})} \le C_{q,\theta} \|f\|^{1-\theta}_{B^{s_1}_{p,q_1}(\mathbb{M})} \|f\|^{\theta}_{B^{s_2}_{p,q_2}(\mathbb{M})}.$$

Proof. By [71, Thm. 4.19] we get the interpolation statement, which gives the inequality by [68, Thm. 1.33]. □

For the case of the Kullback-Leibler fidelity term we also need the following two results.

Theorem A.2.8. *Let $f \in B^s_{p,q}(\mathbb{M})$, $g \in \tilde{B}^s_{p,q}(\mathbb{M})$ with $1 \le p, q \le \infty$ and $s > d/p$, then $fg \in B^s_{p,q}(\mathbb{M})$ and there exists $C > 0$ such that*

$$\|fg\|_{B^s_{p,q}(\mathbb{M})} \le C\|f\|_{B^s_{p,q}(\mathbb{M})} \|g\|_{\tilde{B}^s_{p,q}(\mathbb{M})}.$$

Proof. For $\mathbb{M} = \mathbb{R}^d$ the statement follows from [69, Thm. 2.83]. To transfer it to bounded Lipschitz domains we use Definition A.2.1. We have $s > d/p > 0$, so we know that there exist $\tilde{f}, \tilde{g} \in B^s_{p,q}(\mathbb{R}^d), \operatorname{supp}(\tilde{f}) \subset \overline{\mathbb{M}}$ such that $f = \tilde{f}|_{\mathbb{M}}$ and $g = \tilde{g}|_{\mathbb{M}}$. As we have $\tilde{f}\tilde{g} \in B^s_{p,q}(\mathbb{R}^d)$ with $\operatorname{supp}(\tilde{f}\tilde{g}) \subset \overline{\mathbb{M}}$ and $fg = \tilde{f}\tilde{g}|_{\mathbb{M}}$ we know that $fg \in B^s_{p,q}(\mathbb{M})$. The norm is given by

$$\begin{aligned}
\|fg\|_{B^s_{p,q}(\mathbb{M})} &= \inf\{\|h\|_{B^s_{p,q}(\mathbb{R}^d)} : h \in B^s_{p,q}(\mathbb{R}^d), fg = h|_{\mathbb{M}}, \operatorname{supp}(h) \subset \overline{\mathbb{M}}\} \\
&\le \inf\{\|\tilde{f}\tilde{g}\|_{B^s_{p,q}(\mathbb{R}^d)} : \tilde{f}, \tilde{g} \in B^s_{p,q}(\mathbb{R}^d), fg = \tilde{f}\tilde{g}|_{\mathbb{M}}, \operatorname{supp}(\tilde{f}\tilde{g}) \subset \overline{\mathbb{M}}\} \\
&\le \inf\{C\|\tilde{f}\|_{B^s_{p,q}(\mathbb{R}^d)} \|\tilde{g}\|_{B^s_{p,q}(\mathbb{R}^d)} : \tilde{f}, \tilde{g} \in B^s_{p,q}(\mathbb{R}^d), f = \tilde{f}|_{\mathbb{M}}, g = \tilde{g}|_{\mathbb{M}}, \operatorname{supp} \tilde{f} \subset \overline{\mathbb{M}}\} \\
&= C\|f\|_{B^s_{p,q}(\mathbb{M})} \|g\|_{\tilde{B}^s_{p,q}(\mathbb{M})}.
\end{aligned}$$

□

Theorem A.2.9. *(a) Let $f \in H^s(\mathbb{M})$ with $s \geq d/2$ and $f \geq c > 0$. Then $\log(f), 1/f \in H^s(\mathbb{M})$. Further let $m-1 < s \leq m$ with $m \in \mathbb{N}$, then*

$$\|\log(f)\|_{H^s(\mathbb{M})} \leq C \min(1,c)^{-m}\Big(\|f-1\|_{H^s} + \|f-1\|_{H^s}^{\max(1,s)}\Big)$$
$$\|1/f\|_{H^s(\mathbb{M})} \leq C \min(1,c)^{-m-1}(\|f\|_{H^s} + \|f\|_{H^s}^s).$$

(b) For $p \in \{2,\infty\}$ let $f \in \tilde{B}^s_{p,\infty}(\mathbb{M})$ for some $s > d/p$ with $f \geq c > 0$. Then $\log(f), 1/f \in \tilde{B}^s_{p,\infty}(\mathbb{M})$.

Proof. Let $\tilde{h}\colon [c-1,\infty) \to \mathbb{R}, x \mapsto \log(1+x)$, then $\tilde{h}$ can for all $n \in \mathbb{N}$ be continued to a function $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and there exists $C > 0$ such that $\|h'\|_{C^{n-1}} \leq C\min(1,c)^{-n}$ (with constant independent of $c > 0$). Similarly let $\tilde{g}\colon [c,\infty) \to \mathbb{R}, x \mapsto 1/x$, then $\tilde{g}$ can for all $n \in \mathbb{N}$ be continued to a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and there exists $C > 0$ such that $\|g'\|_{C^{n-1}} \leq C\min(1,c)^{-n-1}$.

(a) We first prove the statements for $f \in H^s(\mathbb{R}^d)$. By Theorem A and the following remarks in [1] we have

$$\|h \circ (f-1)\|_{H^s(\mathbb{R}^d)} \leq C\|h'\|_{C^{m-1}}\Big(\|f-1\|_{H^s(\mathbb{R}^d)} + \|f-1\|_{H^s(\mathbb{R}^d)}^{\max(1,s)}\Big).$$

Let now $\tilde{f} \in H^s(\mathbb{M})$, then for all $\varepsilon > 0$ there exists $f \in H^s(\mathbb{R}^d)$ with $\tilde{f} = f|_{\mathbb{M}}$ and $\|f\|_{H^s(\mathbb{R}^d)} \leq \|\tilde{f}\|_{H^s(\mathbb{M})} + \varepsilon$. As above we have $\log(f) \in H^s(\mathbb{R}^d)$ with

$$\|\log(f)\|_{H^s(\mathbb{R}^d)} \leq C\min(1,c)^{-m}\Big(\|\tilde{f}-1\|_{H^s(\mathbb{M})} + \varepsilon + (\|\tilde{f}-1\|_{H^s(\mathbb{M})} + \varepsilon)^{\max(1,s)}\Big).$$

Of course $\log(\tilde{f}) = \log(f)|_{\mathbb{M}}$, thus $\log(f) \in H^s(\mathbb{M})$ and we can let ε go to zero to show the claimed inequality. The same argument can be applied for the reciprocal.

(b) We have by [8, Thm. 6] (for $0 < s < 1$) and [8, Thm. 10] (for $s > \max(d/p, 1)$) that for $f \in B^s_{p,\infty}(\mathbb{R}^d)$ we have $h \circ f, g \circ f \in B^s_{p,\infty}(\mathbb{R}^d)$. The claim for $\mathbb{M}$ instead of $\mathbb{R}^d$ follows simply from Definition A.2.1.

□

Theorem A.2.10. *Let $s \in \{0,1,2\}$, $f \in H^s(\mathbb{M})$, $f \geq c > 0$ and $\mathbb{M}$ given by the d-dimensional torus or a bounded Lipschitz domain in $\mathbb{R}^d$. In the latter case assume that $f = h + c$, where $h \in H^s_0(\mathbb{M})$. Then $\log(f), 1/f \in H^s(\mathbb{M})$ and*

$$\|\log(f)\|_{H^s(\mathbb{M})} \leq C\min(1,c)^{-1}\|f-1\|_{H^s}$$
$$\|1/f\|_{H^s(\mathbb{M})} \leq C\min(1,c)^{-2}\|f\|_{H^s}.$$

Proof. We will only consider the more interesting case of small $c \leq 1$, as the other case should then be clear. We show the claim for the equivalent norm from Theorem A.2.2. We will use the short notation $\partial_i = \frac{\partial}{\partial x_i}$. For both $s = 0, 1$ the claims are quite clear. We have $|\log(x)| \leq \max\Big(x-1, \frac{1-x}{c}\Big) \leq \frac{1}{c}|x-1|$.

$$\|\log(f)\|_{L^2}^2 \leq \frac{1}{c^2}\int_{\mathbb{M}} |f-1|^2 = \frac{1}{c^2}\|f-1\|_{L^2}^2. \tag{A.2}$$

Further by $1/f = f/f^2$ we have $\|1/f\|_{L^2} \le 1/c^2 \|f\|_{L^2}$ and by

$$\int_{\mathbb{M}} \left(\frac{\partial_i f}{f^n}\right)^2 \mathrm{d}x \le \frac{1}{c^{2n}} \int_{\mathbb{M}} |\partial_i f|^2 \mathrm{d}x$$

we get both $\|\log(f)\|_{H^1(\mathbb{M})} \le Cc^{-1}\|f-1\|_{H^1}$ and $\|1/f\|_{H^1(\mathbb{M})} \le Cc^{-2}\|f\|_{H^1}$. The more interesting part are the estimates for $s=2$. Basically we have to understand $\|\partial_i\partial_j \log(f)\|_{L^2}$ (the estimates for the reciprocal $1/f$ then follow by pulling out $\|1/f\|_{L^\infty}$ of the norm). We have

$$\|\partial_i\partial_j \log(f)\|_{L^2} = \left\|\frac{(\partial_i\partial_j f)}{f} - \frac{(\partial_i f)(\partial_j f)}{f^2}\right\|_{L^2} \le \frac{1}{c}\|\partial_i\partial_j f\|_{L^2} + \left\|\frac{(\partial_i f)(\partial_j f)}{f^2}\right\|_{L^2}.$$

The first term is good, but we need to find a bound for the second. We have

$$\begin{aligned}
\int_{\mathbb{M}} \frac{(\partial_i f)^2(\partial_j f)^2}{f^4}\mathrm{d}x &= \int_{\mathbb{M}} (\partial_i f)\left(\left[\frac{(\partial_i f)(\partial_j f)^2}{f^4} - \frac{2(\partial_j f)(\partial_i\partial_j f)}{3f^3}\right] + \frac{2(\partial_j f)(\partial_i\partial_j f)}{3f^3}\right)\mathrm{d}x \\
&= \int_{\mathbb{M}} (\partial_i^2 f)\frac{(\partial_j f)^2}{3f^3}\mathrm{d}x + \frac{2}{3}\int_{\mathbb{M}} \frac{(\partial_i f)(\partial_j f)(\partial_i\partial_j f)}{f^3}\mathrm{d}x \\
&\le \frac{1}{3}\left(\frac{1}{2}\int_{\mathbb{M}} \frac{(\partial_i^2 f)^2}{f^2}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{M}} \frac{(\partial_j f)^4}{f^4}\mathrm{d}x\right) \\
&\quad + \frac{2}{3}\left(\frac{1}{2}\int_{\mathbb{M}} \frac{(\partial_i\partial_j f)^2}{f^2}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{M}} \frac{(\partial_i f)^2(\partial_j f)^2}{f^4}\mathrm{d}x\right).
\end{aligned}$$

The second equality follows from integration by parts, where the boundary terms vanish due to periodicity, and the last inequality by two times Young's inequality. For $i=j$ we can conclude

$$\int_{\mathbb{M}} \frac{(\partial_j f)^4}{f^4}\mathrm{d}x \le \int_{\mathbb{M}} \frac{(\partial_j^2 f)^2}{f^2}\mathrm{d}x$$

and with this also

$$\begin{aligned}
\left\|\frac{(\partial_i f)(\partial_j f)}{f^2}\right\|_{L^2}^2 = \int_{\mathbb{M}} \frac{(\partial_i f)^2(\partial_j f)^2}{f^4}\mathrm{d}x &\le \frac{1}{4}\int_{\mathbb{M}} \frac{(\partial_i^2 f)^2 + (\partial_j^2 f)^2 + 2(\partial_i\partial_j f)^2}{f^2} \\
&\le \frac{1}{4c^2}\left(\|\partial_i^2 f\|_{L^2}^2 + \|\partial_j^2 f\|_{L^2}^2 + 2\|\partial_i\partial_j f\|_{L^2}^2\right),
\end{aligned}$$

which concludes the proof. □

Remark A.2.11. *The above theorem improves the estimates from Theorem A.2.9 both in the constant $c>0$ and it allows to forget the additive norm power for $s=2$. This is crucial in Proposition 4.2.4 and also generally the constants obtained for Poisson data could improve a lot by estimates of this form . However, even though we suppose that these estimates should hold for all $s>0$, it is not known to us whether this is true. Also the extension to $\tilde{B}^s_{p,q}$ is an open problem.*

Corollary A.2.12. *Let $\sigma > 0, B \geq 1$ and $s \geq d/2$. Let $g, g^\dagger \in H^s(\mathbb{M})$, with $g^\dagger \geq 0, g \geq -\sigma/2$, $\|g\|_{H^s}, \|g^\dagger\|_{H^s} \leq B$. Then we have*

$$\left\|\log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\|_{L^2} \leq C_{g^\dagger,\sigma}\left\|g-g^\dagger\right\|_{L^2}, \text{ with } C_{g^\dagger,\sigma} = 2\frac{\|g^\dagger\|_{L^\infty}+\sigma}{\sigma}\left\|\frac{1}{g^\dagger+\sigma}\right\|_{L^\infty}, \tag{A.3}$$

$$\left\|\log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\|_{H^s} \leq \begin{cases} C_{g^\dagger,\sigma,s}\left\|g-g^\dagger\right\|_{H^s} & \text{if } s \in \{1,2\} \\ C_{g^\dagger,\sigma,s}B^{\max(1,s)-1}\left\|g-g^\dagger\right\|_{H^s} & \text{else,} \end{cases} \tag{A.4}$$

$$\left\|\frac{g^\dagger+\sigma}{g+\sigma}\right\|_{H^s} \leq C_{\sigma,s}B^{\max(1,s)-1}\left\|g-g^\dagger\right\|_{H^s}. \tag{A.5}$$

Proof. Again we only consider small $\sigma \leq 1$. In the other case the constants simplify. We have $\frac{g+\sigma}{g^\dagger+\sigma} \geq \frac{\sigma}{2(\|g^\dagger\|_{L^\infty}+\sigma)}$ and thus by (A.2) that A.2.10

$$\left\|\log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\|_{L^2} \leq \frac{2(\|g^\dagger\|_{L^\infty}+\sigma)}{\sigma}\left\|\frac{g+\sigma}{g^\dagger+\sigma}-1\right\|_{L^2} \leq 2\frac{\|g^\dagger\|_{L^\infty}+\sigma}{\sigma}\left\|\frac{1}{g^\dagger+\sigma}\right\|_{L^\infty}\left\|g-g^\dagger\right\|_{L^2}.$$

By Theorems A.2.9 and A.2.8 we have for $m-1 < s \leq m$ that

$$\begin{aligned}\left\|\log\left(\frac{g+\sigma}{g^\dagger+\sigma}\right)\right\|_{H^s} &\leq C\left(\frac{\sigma}{\|g^\dagger\|_\infty+\sigma}\right)^{-m}\left(\left\|\frac{g+\sigma}{g^\dagger+\sigma}-1\right\|_{H^s}+\left\|\frac{g+\sigma}{g^\dagger+\sigma}-1\right\|_{H^s}^s\right)\\ &\leq C\left(\frac{\sigma}{\|g^\dagger\|_\infty+\sigma}\right)^{-m}\left(\left\|\frac{1}{g^\dagger+\sigma}\right\|_{H^s}\left\|g-g^\dagger\right\|_{H^s}+\left\|\frac{1}{g^\dagger+\sigma}\right\|_{H^s}^s\left\|g-g^\dagger\right\|_{H^s}^s\right)\\ &\leq C\left(\frac{\sigma}{\|g^\dagger\|_\infty+\sigma}\right)^{-m}\left\|\frac{1}{g^\dagger+\sigma}\right\|_{H^s}\left(1+\left\|\frac{B}{g^\dagger+\sigma}\right\|_{H^s}^{s-1}\right)\left\|g-g^\dagger\right\|_{H^s}\end{aligned}$$

and Theorem A.2.10 gives the improved constants for $s = 1, 2$. Finally we have again by Theorems A.2.8 and A.2.9 that

$$\left\|\frac{g^\dagger+\sigma}{g+\sigma}\right\|_{H^s} \leq C\left\|\frac{1}{g+\sigma}\right\|_{H^s}\left\|g-g^\dagger\right\|_{H^s} \leq C\sigma^{-m-1}(\|g+\sigma\|_{H^s}+\|g+\sigma\|_{H^s}^s)\left\|g-g^\dagger\right\|_{H^s},$$

which concludes the proof. □

The final result gives the projections P_k that we need for the verification of our assumptions and crucial inequalities on them.

Proposition A.2.13. *Let for $s < t$ and $u > |s|, |t|$ an u-wavelet basis as in Theorem A.2.4 be given by (Φ_l^j). Define $P_k : B_{p,q}^s(\mathbb{M}) \to B_{p,q}^s(\mathbb{M})$ by*

$$P_k f = \sum_{j=0}^{k}\sum_{l \in N_j}\lambda_l^j\Phi_l^j, \text{ for } f = \sum_{j=0}^{\infty}\sum_{l \in N_j}\lambda_l^j\Phi_l^j.$$

Then we have for $s < t$ and $\tilde{q} \in [1, \infty]$ the Bernstein and Jackson inequalities

$$\|P_k f\|_{B_{p,q}^t(\mathbb{M})} \leq C2^{kq(t-s)}\|f\|_{B_{p,\tilde{q}}^s(\mathbb{M})} \tag{A.6}$$

$$\|(I-P_k)f\|_{B_{p,q}^s(\mathbb{M})} \leq C2^{kq(s-t)}\|f\|_{B_{p,\tilde{q}}^t(\mathbb{M})}. \tag{A.7}$$

Proof. As in [74, Example 3.9]. □

A.3 List of symbols

In the following we list some of the symbols that recur throughout this thesis.

Spaces and domains

$\mathcal{X}$	Banach space containing causes of an inverse problem
$\mathcal{X}^*$	dual space of $\mathcal{X}$
$\mathcal{Y}$	Banach space containing effects of an inverse problem
$\overline{\mathbb{R}}$	$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{T}^d$	d-dimensional Torus
$\mathbb{M}$	generic domain (usually either a bounded Lipschitz domain in $\mathbb{R}^d$ or $\mathbb{T}^d$)
$\mathbb{M}_0$	see Assumption 5.1.8
$B^s_{p,q}(\mathbb{M})$	Besov space, see Section A.2
$\tilde{B}^s_{p,q}(\mathbb{M})$	see Definition A.2.1
$H^s(\mathbb{M})$	Sobolev space, see Section A.2
$H^s_0(\mathbb{M})$	see Definition A.2.1

Functions

F	Forward operator of an inverse problem, $F\colon \mathcal{X} \to \mathcal{Y}$
T	linear forward operator, $T\colon \mathcal{X} \to \mathcal{Y}$
T^*	adjoint operator of T, $T^*\colon \mathcal{Y}^* \to \mathcal{X}^*$
f	generic function $f \in \mathcal{X}$
g	generic function $g \in \mathcal{Y}$
$f^\dagger$	true solution of an inverse problem
$\hat{f}_\alpha$	Tikhonov minimizer, see Chapter 3
$\widehat{\xi}_\alpha$	dual Tikhonov minimizer, see (3.4)
$\xi^\dagger$	source element $T^*\xi^\dagger \in \partial\mathcal{R}(f^\dagger)$
$\omega^\dagger$	source element $T\omega^\dagger \in \partial\mathcal{S}^*_p(\xi^\dagger)$
$g^\dagger$	exact data of an inverse Problem, $g^\dagger = F(f^\dagger)$
g^{obs}	noisy data of an inverse Problem (deterministic)
G^{obs}	random variable modelling the data, see Section 1.4
Z	random variable modelling the noise, $G^{\mathrm{obs}} = g^\dagger + Z$
W	Gaussian white noise, see Definition 1.4.2
G_t	temporally normalized Poisson process, see Definition 1.4.8

$\mathcal{F}$	generic Functional, $\mathcal{F}\colon \mathcal{X} \to \overline{\mathbb{R}}$
$\mathcal{F}^*$	convex conjugate of $\mathcal{F}$, see Definition 2.1.13
$\partial\mathcal{F}(x)$	subdifferential of $\mathcal{F}$ at x, see Definition 2.1.9
$\Delta_{\mathcal{F}}^{f^*}(\cdot,\cdot)$	Bregman divergence, see Definition 2.2.1
J_p	duality mapping, see (2.9)
$\mathcal{R}$	Tikhonov penalty functional, see Chapter 3
$\mathcal{S}$	Tikhonov data fidelity functional, see Chapter 3
$\mathcal{S}_p(g)$	$\mathcal{S}_p(g) = \frac{1}{p}\|g\|_{\mathcal{Y}}^p$, see Section 3.2.1
$\mathcal{S}_{\tilde{g}}^p(g)$	$\mathcal{S}_{\tilde{g}}^p(g) = \mathcal{S}_p(g-\tilde{g})$, see Section 3.2.1
$\mathcal{S}_{G^{\mathrm{obs}}}^{\mathrm{LS}}$	least squares data fidelity for statistical noise, see Section 3.2.2
KL	Kullback-Leibler divergence, see Section 2.3
$\mathrm{KL}_{\mathrm{g}^\dagger}^{\sigma}(g)$	$\mathrm{KL}_{\mathrm{g}^\dagger}^{\sigma}(g) := \mathrm{KL}(g^\dagger+\sigma, g+\sigma)$ for $\sigma \geq 0$
$\mathcal{S}_{G^{\mathrm{obs}},\sigma}^{\mathrm{KL}}$	Kullback-Leibler data fidelity for statistical noise, see Section 3.2.3
$\mathcal{S}^\dagger$	"exact data fidelity", see Definition 4.1.1
err	effective noise level, see Definition 4.1.1
Φ	generic index function, see Definition 4.3.1
Ψ	index function for statistical second order rates, see (4.17)
$\chi_{\mathcal{B}}$	characteristic function of $\mathcal{B} \subset \mathcal{X}$, $\chi_{\mathcal{B}}(f) = \begin{cases} 0 \text{ if } f \in \mathcal{B} \\ \infty \text{ else} \end{cases}$

Constants

α	regularization parameter, $\alpha > 0$
a	(Besov) regularity of the image of F or T, see e.g. Assumption 5.1.8
a_0	restriction of a to values $\{1,2\}$ emerging from Proposition 4.2.4
γ	(Besov) regularity of the noise Z, see Assumption 1.4.1
δ	deterministic noise level, $\delta > 0$
d	dimension of measurement domain $\mathbb{M}$, $d \in \mathbb{N}$
ε	statistical noise constant, see (1.7)
κ_k, ν_k	see either Proposition 4.3.6, Assumption 4.4.4, Prop. 5.1.3 or Prop. 5.1.6
σ	offset parameter in $\mathcal{S}_{G^{\mathrm{obs}},\sigma}^{\mathrm{KL}}$, $\sigma \geq 0$, see Section 3.2.3
s	(Besov) regularity of the true solution $f^\dagger$, $s > 0$

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