2020

Discrete Morse Theory by Vector Fields: A Survey and New Directions

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by

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An Alternate Plan Paper Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Science

In

Mathematics and Statistics Minnesota State University, Mankato

Mankato, Minnesota May 2020

Discrete Morse Theory by Vector Fields: A Survey and New Directions

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This Alternate Plan Paper has been examined and approved by the following members of the defense committee.

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Abstract

We synthesize some of the main tools in discrete Morse theory from various sources. We do this in regards to abstract simplicial complexes with an emphasis on vector fields and use this as a building block to achieve our main result which is to investigate the relationship between simplicial maps and homotopy. We use the discrete vector field as a catalyst to build a chain homotopy between chain maps induced by simplicial maps.

Contents

1 Introduction

We present a survey of the fundamental ideas of discrete Morse theory where we combine the notions of discrete Morse theory on abstract simplicial complexes which has not been discussed by many in detail. Forman [5] has provided the basics and foundation to discrete Morse theory. The only known comprehensive published treatments of discrete Morse theory on abstract simplicial complexes are from a new text by Scoville [16] and work with combinatorial algebraic topology by Kozlov [12] which were published in 2019 and 2008 respectively. We will discuss discrete Morse theory in a way which synthesizes the original approach of Forman together with the innovations of Kozlov and Scoville. This will be done with a focus on discrete vector fields and their associated maps. After the survey we construct a chain homotopy between chain maps induced by simplicial maps. The discrete vector field map will be used to construct this chain homotopy. Our major innovation is to introduce a restricted sense of homotopy induced by pivoting a vertex. We will prove that functions differing by a vertex pivot induce chain homotopic maps and should be thought of as natural analogues of a specific type of homotopic functions. However, before we dive into the survey, we first discuss some historical development of discrete Morse theory.

Morse Theory has been one of the most indispensable tools in mathematics. John Milnor's famous book Morse Theory [14], which is based off of [15], gives us the development of the subject as well as applications. What Morse Theory allows us to do is find the homology of a manifold by analyzing differentiable functions regarding that manifold. Specifically, we can find CW-complexes which relate to the critical points of the manifold which gives us information about the homology on that manifold. The tools of classic Morse theory have not only been useful in differential topology, but in other areas of mathematics and have been used in other fields other than mathematics.

Keeping this in mind, recently a discrete analogue of Morse Theory has been developed. Robin Forman laid the foundations of the subject with his development, which we now call Discrete Morse Theory, in his paper Morse Theory for Cell Complexes [5]. The goal of his paper was to develop such a theory in order to prove the discrete analogs of the main theorems of Morse Theory. Just as classical Morse theory is a tool to calculate homology on manifolds, discrete Morse theory has gained popularity to calculate homology of cell complexes or discrete spaces. For an example of the applications of this form of discrete Morse theory, one can look at chapter 5 of Scoville [16] which discusses the uses of discrete Morse theory in persistent homology. Extending on this, Forman also wrote a survey paper on the matter as well as extended the theory in general [6], [7], [8].

In the development of discrete Morse Theory, Forman's main object of use was CW-complexes with some mention of PL-manifolds and simplicial complexes (where the simplicial complexes are primarily considered to be subsets of a topological space). Though it is not how Forman developed discrete Morse theory, we will focus on a development based on abstract simplicial complexes. An abstract simplicial complex can be thought of as a set of points, lines, and triangles, but an abstract simplicial complex is defined purely in terms of finite sets. (see figure 1).

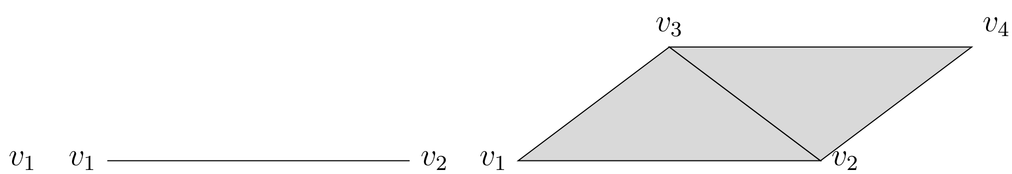


Figure 1: Examples of basic simplicial complexes

2 Abstract Simplicial Complexes

Abstract simplicial complexes will be the main objects which we work with and we will discuss the operations and properties we need. One can consult Kozlov [12] for a full treatment. Note that abstract simplicial complexes are not the same as what Hatcher [10] refers to as Δ-complexes. That is, we are not worried about the vectors formed from the vertices of the Δ-complex and whether the barycentric coordinates sum to 1. Abstract simplicial complexes are a completely combinatorial analog of this and as such there are no coordinates or vectors. As previously described, we can think of abstract simplicial complexes as set of vertices, lines, triangles, tetrahedrons, etc. Consider figure 2 for an example.

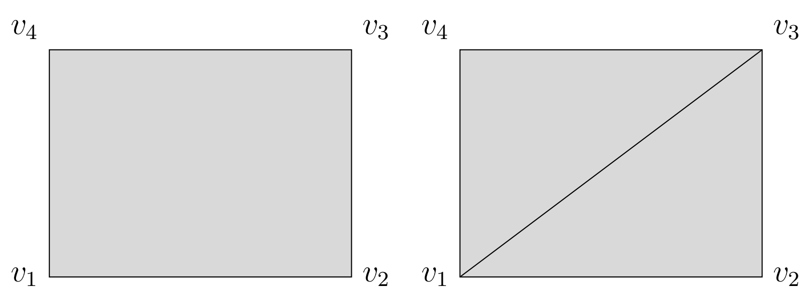


Figure 2: The diagram on the left is not a simplicial complex. The diagram on the right is a simplicial complex

In figure 2 we see that the left diagram is a solid square. If we consider the all possible sets which v_1, v_2, v_3 and v_4 make, we get

$$\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}.$$

Where the singleton sets are the vertices, the sets with two elements are the line segments, and the set with four elements represents the solid square. Why this fails to be a simplicial complex is because we are not accounting for the set $\{v_1, v_3\}$ which would divide that solid square into two solid triangles $\{v_1, v_3, v_4\}$ and $\{v_1, v_2, v_3\}$. If we do this division of the solid square we get the diagram to the right which is a simplicial complex. To make this more precise, consider the following definition.

Definition 2.1. *An **abstract simplicial complex** K on a finite set $V(K) := \{v_0, ..., v_n\}$ where $n \geq 0$ is an integer, is a collection of subsets of $V(K)$, not including the empty set, such that*

1. *If set $\sigma \in K$ and $\gamma \subseteq \sigma$, then $\gamma \in K$.*
2. *$\{v_i\} \in K$ for all $v_i \in V(K)$*

Definition 2.2. *Consider a cell σ of an abstract simplicial complex K. The **dimension** of σ is given by one less than its cardinality.*

$$\dim(\sigma) = |\sigma| - 1$$

If a cell has dimension p, we denote this as $\sigma^{(p)}$.

The elements of $V(K)$, the vertex set of K, are **vertices** of K. In the same vein as vertices, the sets σ in K are called **cells** of K. This allows us to talk about an abstract simplicial complex as a set of its vertices. Therefore, each cell can be talked about in terms of the vertices of which it is composed of. For simplicity we can regard the abstract simplicial complex as a set of its cells. We will talk about the cells in terms of their vertices if such an occasion does arise. That is, when a set of vertices are in our abstract simplicial complex, we can think of the cell which is defined by those vertices is also in the complex as well as all of the nonempty subsets defined by the set of vertices. For those cells not contained in any other cell in K we call **maximal**. A

subcomplex L of K, where $L \subseteq K$, is a subset of K such that L is also an abstract simplicial complex. An important type of simplex is the m-**simplex** Δ^m. One may think of an m-simplex as the convex hull of the m-dimensional region formed by m+1 vertices. That is, we defined the m-simplex to be the abstract simplicial complex with a vertex set of $m+1$ vertices and the cells consisting of all nonempty subsets.

Example 2.3. *Consider the collection of sets*

$$\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$$

This collection of sets is, indeed, an abstract simplicial complex. This can be viewed as a hollow triangle. We have the sets of vertices as well as the sets of line segments the vertices create but not the middle portion. That is, if we wanted to fill in the triangle we would also need the set $\{v_1, v_2, v_3\}$ *which would give us the 2-simplex* Δ^2.

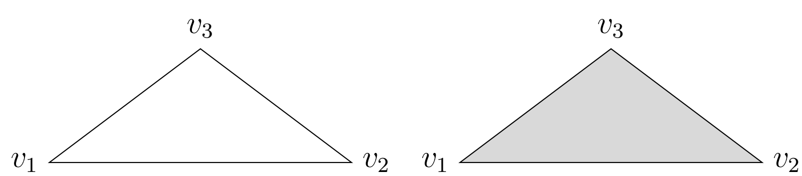


Figure 3: The diagram on the left is the simplicial complex given in the example and the diagram on the right is if we added the set $\{v_1, v_2, v_3\}$ to the set given in the example.

For two cells σ and τ in an abstract simplicial complex K with $\sigma \subseteq \tau$, we say that σ is a **face** of τ. A special case of this, which we will use often, is given in the following definition.

Definition 2.4. *Let* σ *and* τ *be cells where* $\tau = \sigma \cup \{v\}$. *Then we say that* σ *is a* ***co-dimension one*** *face of* τ. *We denote this by* $\sigma <_1 \tau$

Definition 2.5. *A* ***d-skeleton*** *of* K, *denoted by* $K^{(d)}$, *is the abstract simplicial complex formed by the collection* $\{\sigma \in K | \dim(\sigma) \leq d\}$

For example the 0-skeleton $K^{(0)}$ is the set of vertices of K. Note that the 0-skeleton is not the vertex set of K but the set of the vertices as singleton sets. The 1-skeleton $K^{(1)}$ of K is the set of vertices and edges, and the 2-skeleton $K^{(2)}$ is the set of vertices, edges, and triangles.

We will make use of the following discussion in later sections. We have discussed properties of abstract simplicial complexes so the next natural step would be to consider functions between abstract simplicial complexes.

Definition 2.6. *Let K and L be two abstract simplicial complexes. A* ***simplicial map*** *from K to L is a set map $f : V(K) \to V(L)$ on the vertex sets of K and L with the property that if σ is a cell of K, then $f(\sigma)$ is a cell of L. We will write $f : K \to L$ to refer to the simplicial map between K and L.*

Let $f : K \to L$ be a simplicial map. Consider a cell σ of an abstract simplicial complex K where $\sigma = v_0v_1 \cdots v_i$. A simplicial map has the property that $f(\sigma) = f(v_0)f(v_1)\cdots f(v_i)$ is a cell in L. That is, we essentially have that vertices get mapped to vertices. Therefore, simplicial maps can be determined by their "effects" on vertices. Consider the following example.

Example 2.7. *We illustrate a simplicial map by a simple example. Let K be the 1-simplex $\Delta^1 = \{\{v_0\}, \{v_1\}, \{v_0, v_1\}\}$ and $L = \{\{a\}, \{b\}\}$ be the abstract simplicial complex defined by two disconnected vertices. Consider the following two mappings:*

1. *$f(v_0) = a$ and $f(v_1) = a$*
2. *$f(v_0) = a$ and $f(v_1) = b$*

We can see that the first mapping is a constant map to the vertex a. Therefore, $f(v_0)f(v_1)$ is indeed a cell in L, it is just $\{a\}$ and hence is a simplicial map. The *second mapping on the other hand is not a simplicial map as $f(v_0)f(v_1)$ is not a cell in L because $\{a\}$ and $\{b\}$ are disconnected vertices.*

Now that we have a seen an example of a simplicial map, we wonder if compositions of simplicial maps are also simplicial maps. They are indeed, consider the following proposition.

Proposition 2.8. *The composition of two simplicial maps is simplicial map.*

Proof. Let K_1, K_2, K_3 be abstract simplicial complexes and $f : K_1 \to K_2$, $g : K_2 \to K_3$ be simplicial maps. We will prove that $(g \circ f) : K_1 \to K_3$ is also a simplicial map. Since f is a simplicial map, for any cell $\sigma \in K_1$, we have $f(\sigma) = \sigma_f$ as a cell in K_2. Since g is a simplicial map, for any cell $\sigma' \in K_2$, we have $g(\sigma')$ as a cell in K_3. Hence $(g \circ f)(\sigma) = g(f(\sigma)) = g(\sigma_f)$. Therefore, since $\sigma_f \in K_2$ we have $g(\sigma_f) \in K_3$. Hence, composition of simplicial maps is simplicial. $\square$

These basic properties of definitions of abstract simplicial complexes will serve as the foundation for our development of discrete Morse theory.

3 The Discrete Morse Function and Discrete Vector Fields

We now introduce the main definitions of discrete Morse theory. For a full treatment of discrete Morse theory on simplicial complexes see Scoville [16]. Though, we follow the development of Forman [5] we emphasize the development of the discrete vector fields which is stressed more by Scoville [16] and Kozlov [12]. As before, assume that K is an abstract simplicial complex.

Definition 3.1. *A* ***discrete Morse function*** f *on* K *is a function*

$$f : K \to \mathbb{R}$$

which satisfies, for all $\sigma^{(p)}$ *in* K

(i) $|\{\tau^{(p+1)} > \sigma^{(p)} | f(\tau) \leq f(\sigma)\}| \leq 1$

and

(ii) $|\{\gamma^{(p-1)} < \sigma^{(p)} | f(\gamma) \geq f(\sigma)\}| \leq 1$

Example 3.2. *We illustrate a simple example. Consider the hollow triangle and the following value assignments to each cell.*

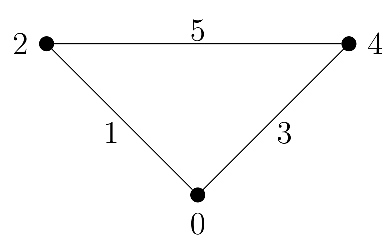


Figure 4: A discrete Morse function on the hollow triangle.

That is, a discrete Morse function locally assigns higher values to higher dimensional simplicies with at most one exception. By locally, we can think of codimension-one cells. So, for some p-dimensional cell σ in K we want to compare the values of σ to its codimension-one simplicies $\gamma^{(p-1)}$ and $\tau^{(p+1)}$. We check to see if the value assignment agrees with the above definition.

Our next main definition is about critical points of a discrete Morse function.

Definition 3.3. *A cell $\sigma^{(p)}$ of K is a **critical point** if both of the following hold.*

(i) $|\{\tau^{(p+1)} > \sigma^{(p)} | f(\tau) \leq f(\sigma)\}| = 0$

and

(ii) $|\{\gamma^{(p-1)} < \sigma^{(p)} | f(\gamma) \geq f(\sigma)\}| = 0$

*We then say that σ is a critical point of **index** p (its dimension). We also say that $f(\sigma)$ is the **critical value**.*

Again, we are looking at the cells of codimension-one of σ. Interestingly, by definition 3.1 and definition 3.3 the minimum value of a discrete Morse function on an abstract simplicial complex must occur on a 0-cell (vertex); however, this may not be true when given a classic Morse function the geometric realization of the abstract simplicial complex. Furthermore, definition 3.3 gives us the following lemma.

Lemma 3.4. *For a non-critical cell $\sigma^{(p)}$ only one of the following conditions can be true:*

(i) There exists $\tau^{(p+1)} > \sigma^{(p)}$ such that $f(\tau) \leq f(\sigma)$.

(ii) There exists $\gamma^{(p-1)} < \sigma^{(p)}$ such that $f(\gamma) \geq f(\sigma)$.

Proof. Assume $p > 1$ and, by way of contradiction, that conditions (i) and (ii) are both true. So we have $\sigma <_1 \tau$ and $\gamma <_1 \sigma$ such that $f(\tau) \leq f(\sigma) \leq f(\gamma)$. Furthermore, from definition 3.1 there is another p-cell $\bar{\sigma}^{(p)}$ such that $\gamma <_1 \bar{\sigma} <_1 \tau$ such that

$f(\gamma) < f(\bar{\sigma})$ since $\gamma <_1 \bar{\sigma}$ with $f(\gamma) \geq f(\sigma)$. Similarly, we have $f(\bar{\sigma}) < f(\tau)$. Therefore,

$$f(\tau) \leq f(\sigma) \leq f(\gamma) < f(\bar{\sigma}) < f(\tau)$$

which is a contradiction. Hence, only one of the conditions may hold. $\square$

Given an abstract simplicial complex, how would one assign values to each cell in the abstract simplicial complex? We could arbitrarily assign numbers to each cell and check to make sure the conditions are satisfied; however, this can turn into an extremely tedious task. To do so we specify the function in terms of its gradient. In the classic sense this not efficient; however, discrete vector fields are defined in a way so that they are determined by relatively few arrows.

Definition 3.5. *A **discrete vector field** V is a collection of ordered sets of cells called **arrows** $(\sigma^{(p)}, \tau^{(p+1)})$ such that*

(i) The arrow $(\sigma, \tau) \in V$ means $\sigma <_1 \tau$.

(ii) Each $\sigma \in K$ belongs to at most one arrow in V.

The cell σ is the tail of the arrow while τ is the head.

The notion of a discrete gradient vector field is what allows us to implicitly keep track of the heights of cells.

Definition 3.6. *A discrete Morse function f induces a **discrete gradient vector field** V_f which is defined by*

$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)}) | \sigma <_1 \tau,\ f(\sigma) \geq f(\tau)\}$$

We will refer to discrete gradient vector field as gradient vector field from now on.

Remark 3.7. *By lemma 3.4 and we see that each cell has an assignment by the gradient vector field. That is, each cell is the tail of an arrow, the head of an arrow, or is not in the gradient vector field. If a cell is not in the gradient vector field, then it is a critical cell. More precisely, let* σ *be a cell of* K *and* f *a discrete Morse function on* K*. Then only one of the following holds:*

(i) σ *is the tail of exactly one arrow.*

(ii) σ *is the head of exactly one arrow.*

(iii) σ *is neither the head nor the tail of an arrow. That is,* σ *is a critical cell.*

Naturally, we could ask when does the converse of the above statement grant us a gradient vector field? The partition of the cells of K *given in the three conditions leads to the definition of discrete vector field.*

Example 3.8. *We draw the corresponding gradient vector field of the discrete Morse function given in example 3.2.*

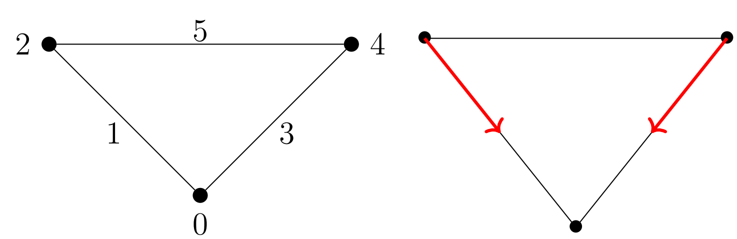


Figure 5: A discrete Morse function on the hollow triangle and its corresponding gradient vector field.

In the illustration above, the critical cells are $f^{-1}(0)$ *and* $f^{-1}(5)$*. As such, they are not on an arrow. The other cells, however; are on arrows because the vertex* $f^{-1}(2)$ *has a higher value than the edge* $f^{-1}(1)$ *so* $f^{-1}(2)$ *is the tail of an arrow and* $f^{-1}(1)$ *is the head of that arrow. Similarly with the case of cells* $f^{-1}(4)$ *and* $f^{-1}(3)$*. Note that* *we usually will not refer to cells as the inverse of their value assignment. In more complicated simplicial complexes, we could have many cells with the same value and as such it would be difficult to refer to cells in this manner.*

Note that in a discrete vector field, the arrows are "pointing" downward; however, we know that a gradient vector field in terms of a differential gradient will "point" up. It is common for one to keep track of the negative gradient so that we have a flow pushing downwards. In our development we do not have a differential, so we define the gradient to point down.

Remark 3.9. *We illustrate that not every discrete vector field comes from a gradient of a discrete Morse function. Consider the following figure.*

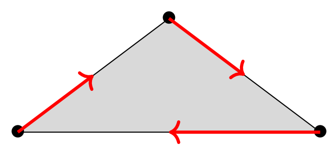


Figure 6: A discrete vector field on Δ^2

If the above did, in-fact, come from a discrete Morse function, then as we follow the arrows the heights of the cells would need to decrease. However, there is an immediate issue with this. We would have a contradiction of the cell having a smaller height than itself. So we see that this discrete vector field did not come from a discrete Morse function.

Gradient vector fields keep track of the pairs (arrows) of cells of higher dimension which would be assigned lower values than one of its faces of one dimension less. If a cell is not in a pair then it is a critical cell. Later, we wish to seamlessly change from discussions about discrete Morse functions to gradient vector fields and vice versa. We will prove that there is an equivalence between discrete Morse functions and gradient vector fields which will allow us to do so. We will; however, need some more tools to prove this equivalence.

We see that discrete vector fields differ from gradient vector fields. Discrete vector fields is a collection of arrows without the property of the discrete Morse function that the gradient vector fields follow. Given a collection of arrows, from a discrete vector field or gradient vector filed, we can follow these arrows as a path along K. This can be described as follows.

Definition 3.10. *Let V be a discrete vector field on K. A p+1 dimensional **V-path** is a sequence of cells*

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \tau_2^{(p+1)}, ..., \sigma_r^{(p)}, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)}$$

Where for each $i = 0, ..., r$ such that $(\sigma_i^{(p)}, \tau_i^{(p+1)}) \in V$ and $\tau_i > \sigma_{i+1}^{(p)} \neq \sigma_i^{(p)}$.
*We say the V-path is a **non-trivial closed path** if $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. Note that if the path is not closed, σ_{r+1} need not be in a pair. The **length** of the path is $r+1$, which refers to the number of arrows on the V-path.*

Suppose that we are given a discrete Morse function on an abstract simplicial complex K. We know how to obtain the gradient vector field V_f which the discrete Morse function induces. The V-paths of a gradient vector field will each give a sequence of cells in K where the values of the cells given by f, as we go along the path, will decrease. This discussion comes from the following result.

Lemma 3.11. *Let V_f be a gradient vector field for a discrete Morse function f on K. A sequence of cells*

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \tau_2^{(p+1)}, ..., \sigma_r^{(p)}, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)}$$

is a V-path if and only if

$$f(\sigma_0) \geq f(\tau_0) > f(\sigma_1) \geq f(\tau_1) > f(\sigma_2) \geq f(\tau_2) > ... > f(\sigma_r) \geq f(\tau_r) > f(\sigma_{r+1})$$

Proof. First, assume that the sequence of cells given is a V-path. Recall that the definition of a V-path says the following pairs $(\sigma_i^{(p)}, \tau_i^{(p+1)})$ are in the gradient vector field V_f as well as $\sigma_{i+1} <_1 \tau_i$. Therefore the pairs of cells follow $f(\sigma_i) \geq f(\tau_i)$. Furthermore, since cells σ_{i+1} and τ_i are not paired but $\sigma_{i+1} <_1 \tau_i$ we must have $f(\tau_i) > f(\sigma_{i+1})$. Hence, we obtain

$$f(\sigma_0) \geq f(\tau_0) > f(\sigma_1) \geq f(\tau_1) > f(\sigma_2) \geq f(\tau_2) > ... > f(\sigma_r) \geq f(\tau_r) > f(\sigma_{r+1})$$

Now assume that we have a sequence of cells such that

$$f(\sigma_0) \geq f(\tau_0) > f(\sigma_1) \geq f(\tau_1) > f(\sigma_2) \geq f(\tau_2) > ... > f(\sigma_r) \geq f(\tau_r) > f(\sigma_{r+1}).$$

Since f is a discrete Morse function, we have for each $f(\sigma_i) > f(\tau_i)$, in the above string of inequalities, $(\sigma_i, \tau_i) \in V_f$. Recall that if we are given a discrete Morse function, we pair cells in this manner to obtain the gradient vector field. Since f is a discrete Morse function, τ_i is the unique cell for which $\sigma_i <_1 \tau_i$ we have $f(\sigma_i) > f(\tau_i)$. Thus, each σ_i and τ_i will occur in only one pair. Hence, the cells being evaluated by f form a V-path. $\square$

We are in a good place to prove the equivalence of the gradient vector field and discrete Morse function. This equivalence comes from the gradient (V-path) not having any closed paths. Note that V-paths are the discrete analog to integral curves for discrete vector fields. Furthermore, lemma 3.11 is the analog of integral curves having no loops. The following theorem says that heights along V-paths of a discrete Morse function decrease. Note that the first part of the theorem follows directly from lemma 3.11. The second part of the theorem is a bit more involved. We create a discrete Morse function for the discrete vector field with no closed V-paths by defining heights of cells such that they decrease along V-paths. We do this by starting at lower dimensions and working our way up.

Theorem 3.12. *A discrete vector field is a gradient vector field of a discrete Morse function if and only if the discrete vector field contains no non-trivial closed V-paths.*

Proof. Let K be an abstract simplicial complex of dimension n. First we will prove the necessary condition. Assume a discrete vector field on K is a gradient vector field V_f of a discrete Morse function f on K. Then each cell σ of K is either a tail of an arrow, head of an arrow, or not on an arrow based on f. Let the following be an arbitrary V-path in K

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \tau_2^{(p+1)}, ..., \sigma_r^{(p)}, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)}$$

Then, for each pair $(\sigma_i^{(p)}, \tau_i^{(p+1)}) \in V_f$ where $\sigma <_1 \tau$, we have $f(\sigma_i) \geq f(\tau_i)$. By lemma 3.11 we obtain the following string of inequalities

$$f(\sigma_0) \geq f(\tau_0) > f(\sigma_1) \geq f(\tau_1) > \cdots > f(\sigma_r) \geq f(\tau_r) > f(\sigma_{r+1}).$$

Thus, the V-path is not closed. If it were closed we would have the following contradiction $f(\sigma_1) > f(\sigma_{r+1}) = f(\sigma_1)$. Hence, because the choice of V-path was arbitrary, there are no non-trivial closed V-paths.

Now we will prove the sufficient condition. We will prove this by building a discrete Morse function inductively on the d-skeletons of K. That is, we build functions

$f_d : K^{(d)} \to \mathbb{R}$ such that

- The function f_d is a discrete Morse function on $K^{(d)}$.
- If V^d is the collection of all arrows that only include cells on $K^{(d)}$, then V^d is a gradient vector field for f_d.
- The image of f_d is contained in the interval $(-1, d]$

We will prove that each function has these properties inductively on d. To do so, consider the following. Define f_0 to be the constant function which sends all cells to 0. Given a definition for f_{d-1}, define f_d as follows

- If σ is of dimension less than $d-1$, or if σ is of dimension $d-1$ and is either a critical cell or on a $(d-1)$ dimensional V-path, then we let

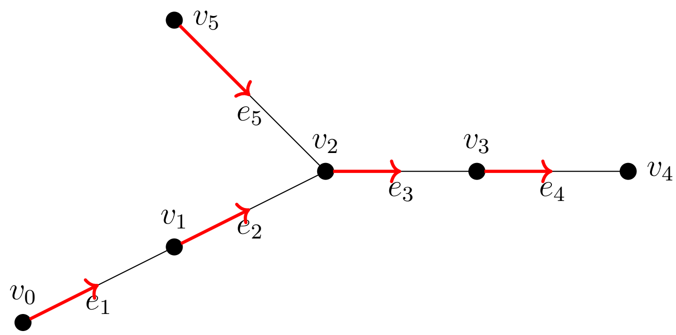
$$f_d(\sigma) = f_{d-1}(\sigma).$$

- If σ is of dimension d and is critical, we let $f_d(\sigma) = d$.
- If σ is on a V-path of dimension d, we let:

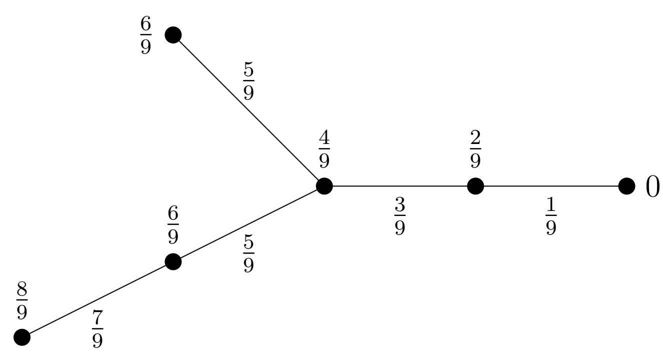
$$f_d(\sigma) = (d-1) + \frac{\delta_\sigma + 1}{\eta + 1}.$$

Where η is defined to be double the maximum length $d-$dimensional V-path and δ_σ is defined to be the number of cells which occur after σ on the maximal V-path which contains σ (not necessarily the maximum length V-path). We will demonstrate how to calculate δ_σ and η via a quick example.

Consider the following simplicial complex with a given vector field.



Note that we are working with 1-dimensional V-paths. First observe that the maximal length V-path is of length 8. Now we calculate δ_σ for v_0. Note that v_0 is only contained in one V-path, which has 8 cells that are on arrows. Since 7 cells appear on arrows after v_0 in the V-path, $\delta_{v_0} = 7$. Now, lets calculate $f_1(v_0)$. Since the highest dimensional cell is 1, $d = 1$. What we have is $f_1(v_0) = 0 + \dfrac{7+1}{8+1}$, so $f_1(v_0) = \dfrac{8}{9}$. In a similar manner, $f_1(e_1) = \dfrac{7}{9}$, $f_1(v_1) = \dfrac{6}{9}$, and $f_1(e_2) = \dfrac{5}{9}$. Now we want to evaluate v_2. However, v_2 is on multiple different V-paths. Recall that η takes values from the V-path with maximum length. So $f_1(v_2) = \dfrac{4}{9}$. Recall that since v_4 is not on an arrow it is critical, so it will take a value of 0. The following figure shows the finished calculations.



We can see that the algorithm leaves most of the heights from f_{d-1} alone; however, we need to define heights for all d-dimensional cells since they did not appear in the $(d-1)$-skeleton. Furthermore, the heights of the $(d-1)$-dimensional cells which now appear on d-dimensional paths need to be updated to obtain the desired discrete Morse function, otherwise they will have height $(d-1)$. However, if the $(d-1)$-dimensional cells now appear on a d-dimensional path, the cells earlier on the path need to have greater height than the cells which appear later via lemma 3.11. The constants η and δ_σ are chosen to do exactly that.

Before we prove that these functions have the desired properties, we need to show that the constant δ_σ is well defined. Since η depends only on the dimension and there is no cycles in our V-paths, it is well defined. On the other hand, since V-paths are of finite length and there are no non-trivial closed paths, δ_σ is always defined as well. However, it may be possible that σ is on multiple different V-paths all of the same length. We will show that δ_σ is still defined in this case. We must show that the number of cells appearing after σ in each V-path must always be the same.

Assume, by way of contradiction, that $\widetilde{\sigma}$ is on two d-dimensional V-paths of maximal length and appears at a different position along both paths. Recall, that since a cell can only be on one arrow that the V-paths may join into single V-path. Let the first V-path VP_1 be the following sequence of cells:

$$\sigma_0^{(d-1)}, \tau_0^{(d)}, \sigma_1^{(d-1)}, \tau_1^{(d)}, \ldots \widetilde{\sigma}_i^{(d-1)}, \widetilde{\tau}_i^{(d)}, \ldots \sigma_r^{(d-1)}, \tau_r^{(d)}, \sigma_{r+1}^{(d-1)}$$

and the second V-path VP_2 be the following sequence of cells

$$\hat{\sigma}_0^{(d-1)}, \hat{\tau}_0^{(d)}, \hat{\sigma}_1^{(d-1)}, \hat{\tau}_1^{(d)}, \ldots \widetilde{\sigma}_j^{(d-1)}, \widetilde{\tau}_j^{(d)}, \ldots, \hat{\sigma}_r^{(d-1)}, \hat{\tau}_r^{(d)}, \hat{\sigma}_{r+1}^{(d-1)}.$$

Not only is the cell $\widetilde{\sigma}$ in a different position on both V-paths, but so is the arrow it is on. That is, since $\widetilde{\sigma}$ is in a different position on both V-paths, so it the cell it is paired with which we denoted $\widetilde{\tau}$. So, without loss of generality assume that $i < j$. That is, $\widetilde{\sigma}$ appears in an earlier position in VP_1 than VP_2 since they are of the same length. If this is the case, then there are more cells after $\widetilde{\sigma}$ in VP_1 than before it and there are more cells before $\widetilde{\sigma}$ in VP_2 than there are after. With this we can piece together another V-path from the cells before $\widetilde{\sigma}$ in VP_2 and the cells after $\widetilde{\sigma}$ in VP_1. The new V-path is of the form

$$\hat{\sigma}_0^{(d-1)}, \hat{\tau}_0^{(d)}, \hat{\sigma}_1^{(d-1)}, \hat{\tau}_1^{(d)}, ...\widetilde{\sigma}_m^{(d-1)}, \widetilde{\tau}_m^{(d)}, ..., \sigma_r^{(d-1)}, \tau_r^{(d)}, \sigma_{r+1}^{(d-1)}$$

What we have is the first j arrows from VP_2 and added $r-i$ arrows from VP_1. So the new V-path has $j-i+r$ arrows. Since $i<j$ we have more arrows in the new V-path than we have in the previous two. Therefore, the length of the new V-path is larger than VP_1 and VP_2 which is a contradiction. Hence, the same number of cells must come after σ in both V-paths.

Now that we have proven η and δ_σ are well defined, we now prove that the quantity $\dfrac{\delta_\sigma+1}{\eta+1} \in (0,1)$. We wish to do this because when we calculate heights for cells on a d-dimensional V-path, we want their values to be strictly between $d-1$ and d. This will also give us the desired range of $(-1,d]$ for each f_d. By definition of δ_σ and η, $\delta_\sigma+1$ and $\eta+1$ are both positive. Furthermore, $\delta_\sigma<\eta$ Hence, $\dfrac{\delta_\sigma+1}{\eta+1} \in (0,1)$.

Now we use induction to prove that the functions f_d have the desired properties, i.e. they are discrete Morse functions on the d-skeletons, that they induce the arrows on cells of dimension d or less for their gradient vector field, and that their heights are contained in the interval $(-1,d]$.

When $d=0$ we assign all cells a value of 0. Note that any functions on a collection of disjoint 0-cells will automatically be a discrete Morse function with every cell being critical. Since there are no arrows only on dimension 0, the function has the correct gradient vector field. We also see that since all cells have a height of 0, this is in the range $(-1,0]$.

Now suppose that f_{d-1} has the desired properties and consider f_d. We will break this down into cases based on the properties of the cells. That is, we want to prove the following: the heights of cells fall in the range (-1, d], and heights decrease along $d-$ dimensional V-paths on the d-skeleton. First we show the heights fall in the interval $(-1, d]$.

- If σ is of dimension less than $d-1$, then $f_d(\sigma) = f_{d-1}(\sigma)$. So, $f_d(\sigma) \in (-1, d-1]$ and the desired property for heights would come from f_{d-1} already having these properties.

- If σ is a $d-1$ dimensional cell and not on a d-dimensional V-path. Then we wont change the height, that is $f_d(\sigma) = f_{d-1}(\sigma)$, so we know that $f_d \in (-1, d)$. Now we verify that $f_d(\sigma) < f_d(\tau)$ whenever $\sigma <_1 \tau$. By our previous results of δ_σ and η, we know that if τ is on a d-dimensional V-path, then its height is strictly between $d-1$ and d. If τ is critical, it attains a height of d. Since we know that $f(\sigma) \leq d-1$, the result follows.

- If σ is a critical d-cell, or a d-cell on an arrow with a (d+1)-cell on the $d+1$-skeleton (which is still critical on the d-skeleton), then its height is exactly d. Which is in the range $(-1, d]$. Since we have chosen heights for cells of lower dimension to be strictly less than d, we have the desired properties.

- We can consider both cells of dimension $d-1$ and d which are on a d-dimensional path. Since the quantity $\dfrac{\delta_\sigma + 1}{\eta + 1}$ is strictly between 0 and 1, the heights of these cells are strictly between $d-1$ and d. Thus the heights of these cells are in the range $(-1, d]$.

With the discussion above we can say that the height of each cell is contained in the interval $(-1, d]$. Therefore, the range of f_d is $(-1, d]$. What we have so far is that

f_d is almost a discrete Morse function. What we need to show now is that heights decrease along V-paths.

Claim: Cells of dimension d and $d-1$ which are on a d-dimensional V-path have their heights decrease as we go along the V-path. We will have to check the following 4 cases: multiple V-paths of different length join together, a single V-path splitting into multiple different V-paths of different length, multiple V-paths of different length joining and then splitting where the V-paths could be of different length, and a single V-path which does not split nor joins other V-paths.

- We start with the case of a single V-path which does not split nor join other V-paths. This one is the simplest case as this follows from the construction of f_d.

- Consider multiple V-paths of different length joining into one V-path. We proved earlier that the number of cells appearing after σ is the same in each V-path. That is, σ is the cell which comes right after the joining of these V-paths. We know that σ will attain its height from the V-path of largest length. What we need to show is that $f_d(\sigma)$ is less than the heights of all adjacent cells which come before σ on these V-paths. Note, what we need to focus on is $\dfrac{\delta_\sigma + 1}{\eta + 1}$ since the constant $d-1$ is being added to each. The quantity $\delta_\sigma + 1$ will be just one less than $\delta_\tau + 1$. With the above information, $\dfrac{\delta_\sigma + 1}{\eta + 1} < \dfrac{\delta_\tau + 1}{\eta + 1}$. Hence, the heights decrease along these V-paths.

- Suppose we have a single V-path splitting into multiple V-paths of different length. In a similar fashion of V-paths joining, one can prove that the number of cells before the V-path splits are the same for each V-path. However, we will continue with our proof. Let τ be the cell on the V-path before the split. What we need to show is that $f_d(\tau)$ is greater than the heights of the cells σ after the splitting of the V-path which are not on the longest path. Note that since the denominator $\eta+1$ is always the same and we know that $\delta_\tau > \delta_\sigma + 1$. The result follows: $f_d(\tau) > f_d(\sigma)$.

- We could have a case where there is a joining and splitting. One can prove that the cells between the join and splitting. Therefore, we can combine the two arguments above and obtain our result.

With critical $d-1$ dimensional cells taking a height of $d-1$, critical d-dimensional cells taking a height of d, and the heights of cells on V-paths, which include the possibility of the V-path just being a single arrow, are strictly between $d-1$ and d. Hence, we see that f_d is a discrete Morse function. We now need to check that f_d induces the arrows on cells of dimension d or less for their gradient vector field. So, we will show the following: when $\sigma <_1 \tau$ then $f_d(\sigma) > f_d(\tau)$ if and only if there is an arrow from σ to τ.

First, let σ be a $d-1$ dimensional cell and τ be a d-dimensional cell. Assume, by way of contradiction, that when $\sigma <_1 \tau$ we have $f_d(\sigma) > f_d(\tau)$ and that there is no arrow from σ to τ. We will proceed by cases.

- If σ and τ are not on any arrow, then by construction of f_d we have $d-1 > d$ which is a contradiction.

- If σ is a head of an arrow with a $d-2$ dimensional cell γ. By f_{d-1} already having the desired properties and that $f_d(\gamma) = f_{d-1}(\gamma)$, we have that $f_d(\gamma) > f_d(\sigma) > f_d(\tau)$. Whether τ be a critical d cell or on a d-dimensional V-path, since the value of $f_d(\tau) > d-1$ and $f_d(\gamma) = f_{d-1}(\gamma) < d-1$ we have a contradiction.

- If σ the tail on an arrow with another d-dimensional cell $\widetilde{\tau}$. We have that $f_d(\sigma) > f_(\widetilde{\tau})$ by our construction. Therefore we have $f_d(\sigma) > f_(\widetilde{\tau})$ and $f_d(\sigma) > f_(\tau)$.

Since we proved that f_d was a discrete Morse function, this is a contradiction.

- We have a similar argument for the case when τ is the head of an arrow with another $d-1$ dimensional cell $\widetilde{\sigma}$. That is, $f_d(\tau) < f_d(\widetilde{\sigma})$ and $f_d(\tau) < f_d(\sigma)$ which gives us a contradiction of f_d being a discrete Morse function.

Hence, there must be an arrow from σ to τ.

Conversely, when there is an arrow from σ to τ we have $\sigma <_1 \tau$ by definition of arrow. Furthermore, by construction of f_d we have $f_d(\sigma) > f_d(\tau)$ which comes from f_{d-1} already having that property along with the constants δ_σ and η. Therefore, f_d maintains the desired gradient vector field. Therefore, by induction each f_d satisfies the desired properties. In particular, if K is an n-dimensional abstract simplicial complex, then $f_n : K \to \mathbb{R}$ is a discrete Morse function which induces V as a gradient vector field. $\square$

We see that a discrete vector field and gradient vector fields are similar but the gradient gradient vector field has a differing property. The discrete vector field is just a collection of arrows on K, where as the gradient vector field is also a collection of arrows on K with the additional condition that the arrows follow the decreasing values of the discrete Morse function on K as we go up dimensions. That is, we are referring to the negative of the gradient vector field in the smooth sense. We investigate the differences of the discrete and gradient vector fields in the following section.

4 Discrete Flow

Now we start our development of relating the vector field to the notion of homotopy. To do this we will develop a discrete notion of gradient flow by using the vector field map as a chain homotopy. We follow Forman's [5] approach and develop the discrete analog of gradient flow. We than see how this discrete flow relates to homotopy more generally. For the following discussion, let f be a fixed discrete Morse function on an abstract simplicial complex K. The discrete gradient flow is defined in terms of chains groups, so we remind the reader of the notion of simplicial chain group with integer coefficients as well as the usual boundary operator.

Definition 4.1. *The* ***chain group*** $C_n(K)$ *consists of formal sums of oriented cells of* K *of dimension* n*. An ordered cell can be listed in the form* $[w_0, w_1, ..., w_n]$ *where* $w_0, w_1, ..., w_n$ *are the vertices of the cell. The cells are oriented so that* $[w_0, w_1, ..., w_n] = (-1)^k[w_{i_0}, w_{i_1}, ..., w_{i_n}]$ *where where* $i_0, i_1, \ldots, i_n$ *is a permutation of* $0, 1, \ldots, n$ *and* k *is* 0 *if this permutation is even, otherwise* k *is* 1*.*

Definition 4.2. *The boundary operator* $\partial : C_n(K) \to C_{n-1}(K)$ *is homomorphism defined by*

$$\partial([v_0, v_1, v_2, ..., v_n]) = \sum_{k=0}^{n} (-1)^k [v_0, v_1, ..., \hat{v_k}, ..., v_{n-1}, v_n]$$

Where the ˆ *over the vertex* v_k *means we remove that vertex from the sequence.*

We define an equivalent boundary map in terms of incidence numbers. The previous definition is beneficial for calculation where as the following will be more beneficial for our theory. Fix an orientation on K by fixing an ordering on the vertices of K and two orderings of the vertices on a cell will have the same orientation if and only if they differ by an even permutation. Let σ be a p-cell in K. Then we have

$$\partial\sigma = \sum_{\gamma^{(p-1)}} \varepsilon(\sigma,\gamma)\gamma$$

where $\varepsilon(\sigma,\gamma)$ is the incidence number of γ in the boundary of σ. That is, an **incidence number** $\varepsilon(\sigma,\gamma)$ is defined as follows;

1. $\varepsilon(\sigma,\gamma) = 0$ if γ is not in the boundary of σ.

2. $\varepsilon(\sigma,\gamma) = 1$ if γ has a positive orientation in the boundary of σ.

3. $\varepsilon(\sigma,\gamma) = -1$ if γ has a negative orientation in the boundary of σ.

Making this more convenient, define an inner product $\langle,\rangle$ on C_* by setting the positively oriented cells of K to be an orthonormal basis. Now we can write

$$\partial\sigma = \sum_{\gamma^{(p-1)}} \langle\partial\sigma,\gamma\rangle\gamma.$$

We can now define the map for the vector field V as follows.

Definition 4.3. *Let σ be a p-cell of K with a fixed orientation. If there is a (p+1)-cell τ such that $\sigma <_1 \tau$ where the arrow $(\sigma,\tau) \in V$, then set*

$$V(\sigma) = -\langle\partial\tau,\sigma\rangle\tau.$$

If there is no such τ then we set $V(\sigma) = 0$. Extending V linearly to a map for each p, we obtain the ***vector field map***

$$V : C_p(K) \to C_{p+1}(K)$$

With the extra condition that $f(\tau) \leq f(\sigma)$, where the arrow $(\sigma, \tau) \in V_f$ we have the ***gradient vector field map***

$$V_f : C_p(K) \to C_{p+1}(K)$$

We see that the vector field map is adding an additional vertex to the p-dimensional cells that are on arrows. We can write this out as follows. Let $\sigma = [v_0, ..., v_p]$ be on the tail of an arrow. We write

$$V(\sigma) = (-1)[w, v_0, ..., v_p]$$

for the cell which is the head of the arrow.

The difference between the maps V and V_f will be given by context and we will refer to the vector field and gradient vector field maps as V. Furthermore, we when we add a vertex to a cell via V, we multiply by negative one because this guarantees that when we take the boundary of $V(\sigma)$ we will receive σ with the opposite orientation which agrees with the $V(\sigma) = -\langle \partial\tau, \sigma \rangle \tau$ formula. We are now in a position to see how the gradient flow ϕ should be defined. Forman [5] gives us the following discussion for motivation of the defining the gradient flow. Consider the vertices of K. For the first case, if a vertex v is a critical vertex then it should remain fixed under the gradient flow. That is, for a critical vertex v we have $V(v) = 0$ and $\phi(v) = v$. If the vertex is not critical and $V(v) = \pm e$, for some edge e, then v should flow to the other vertex adjacent to e. That is $\phi(v) = v + \partial(V(v))$. Defining ϕ for any cell in K we obtain the following.

Definition 4.4. *For any oriented p-cell σ we define the **gradient flow** as follows*

$$\phi(\sigma) = \sigma + \partial V(\sigma) + V(\partial\sigma)$$

which extends linearly to the following map for each p

$$\phi : C_p(K) \to C_p(K)$$

Note that we can also write the gradient flow function as

$$\phi = 1 + \partial V + V\partial$$

Thus, we see that the gradient vector field V is a chain homotopy between the gradient flow ϕ and the identity map 1. We will prove main properties of V and ϕ. First we prove the following properties of V.

Proposition 4.5. *1. $V \circ V = 0$.*

2. If σ is and oriented p-cell of K, then

$$|\{\gamma^{(p-1)} | V(\gamma) = \pm\sigma\}| \leq 1.$$

3. If σ is a oriented p-cell of K, then

$$\sigma \textit{ is critical } \iff \sigma \notin im(V) \textit{ and } V(\sigma) = 0$$

Proof. 1. If $V(\gamma^{(p-1)}) = \pm\sigma^{(p)}$, then $\gamma <_1 \sigma$ and $f(\sigma) < f(\gamma)$. So by lemma 3.4 there cannot exist a (p+1)-cell τ such that $\sigma <_1 \tau$ where $f(\tau) \leq f(\sigma)$. Therefore,

$$\begin{aligned} V \circ V(\gamma) &= V(V(\gamma)) \\ &= V(\sigma) \\ &= 0 \end{aligned}$$

We can make a similar argument using our facts regarding arrows. Since a cell can not be on more than on arrow, we have $V \circ V = 0$.

2. Let σ be an oriented p-cell. In a similar style as the previous proof, if $V(\gamma^{(p-1)}) = \pm\sigma^{(p)}$ where $\gamma <_1 \sigma$ and $f(\sigma) \leq f(\gamma)$ then, by the first condition of definition 3.1, γ must be the only cell which satisfies said condition. Again, we could make a similar argument using what we know about arrows in the gradient vector field. Because each cell can only be on one arrow, the cells $\gamma^{(p-1)}$ and $\sigma^{(p)}$ can only be in one arrow $(\gamma, \sigma) \in V_f$.

3. Let σ be an oriented p-cell. Definition 3.3 states that σ is critical if and only if

(i) $|\{\tau^{(p+1)} > \sigma^{(p)} | f(\tau) \leq f(\sigma)\}| = 0$ and

(ii) $|\{\gamma^{(p-1)} < \sigma^{(p)} | f(\gamma) \geq f(\sigma)\}| = 0$

Condition (i) is equivalent to saying that there is no $\tau^{(p+1)}$ where $V(\sigma) = \pm\tau$. Therefore, $V(\sigma) = 0$. Condition (ii) is equivalent to saying that there is no $\gamma^{(p-1)}$ with $V(\gamma) = \pm\sigma$. Therefore, $\sigma \notin \text{im}(V)$. Equivalently, if σ is not on an arrow we consider two cases. There is no (p-1)-cell γ such that $\gamma <_1 \sigma$ where $(\gamma, \sigma) \in V_f$. That is, since σ is not a head of an arrow $\sigma \notin \text{im}(V)$. The other case us that there is no (p+1)-cell τ such that $\sigma <_1 \tau$ where $(\sigma, \tau) \in V_f$. That is, since σ is not a tail of an arrow $V(\sigma) = 0$. $\square$

Now we prove properties of the gradient flow map ϕ.

Proposition 4.6. *The gradient flow map commutes with the boundary operator, that is $\partial\phi = \phi\partial$. Furthermore, if $\sigma_1, ..., \sigma_r$ are oriented p-cells of K, write*

$$\phi(\sigma_i) = \sum_j a_{ij}\sigma_j.$$

1. *For every i, $a_{ii} = 0$ or 1, and $a_{ii} = 1$ if and only if σ_i is critical.*

2. *If $i \neq j$ and $a_{ij} \neq 0$, then $f(\sigma_j) < f(\sigma_i)$*

Proof. First we prove that the boundary operator commutes with the gradient flow map.

$$\phi\partial = (1 + \partial V + V\partial)\partial = \partial + \partial V\partial + V\partial^2 = \partial + \partial V\partial$$

$$\partial\phi = \partial(1 + \partial V + V\partial) = \partial + \partial^2 V + \partial V\partial = \partial + \partial V\partial.$$

We now prove *1* and *2*. Recall from proposition 4.5 that a p-cell σ satisfies exactly one of the following: (i) σ is critical, (ii) $\pm\sigma \in \text{im}(V)$, or (iii) $V(\sigma) \neq 0$. We will consider each of these cases.

(i) First, suppose that σ is critical. Since σ is critical we have that $V(\sigma) = 0$ and for any (p-1)-cell γ where $\gamma <_1 \sigma$ we have $f(\gamma) < f(\sigma)$. Note that for each of these (p-1)-cells we have either $V(\gamma) = 0$ or $V(\gamma) = \pm\widetilde{\sigma}$ where $f(\widetilde{\sigma}) \leq f(\gamma) < f(\sigma)$. With this discussion in mind, observe the following.

$$\begin{aligned}
\phi(\sigma) &= \sigma + 0 + V(\partial\sigma) \\
&= \sigma + \sum_{\gamma <_1 \sigma} \langle \partial\sigma, \gamma \rangle V(\gamma) \\
&= \sigma + \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}}\widetilde{\sigma}
\end{aligned}$$

Where all values $a_{\widetilde{\sigma}}\widetilde{\sigma}$ with $a_{\widetilde{\sigma}} \neq 0$ satisfy $f(\widetilde{\sigma}) < f(\sigma)$.

(ii) Next, suppose that $\sigma \in \mathrm{im}(V)$. Then there exists a (p-1)-cell κ such that $V(\kappa) = \sigma$. Recall that $V \circ V = 0$. Therefore,

$$\begin{aligned}\phi(\sigma) &= \sigma + \partial V(\sigma) + V(\partial\sigma) \\ &= \sigma + \partial V(V(\kappa)) + V(\partial\sigma) \\ &= \sigma + V(\partial\sigma) \\ &= \sigma + \sum_{\gamma <_1 \sigma} \langle \partial\sigma, \gamma \rangle V(\gamma)\end{aligned}$$

Where $\gamma <_1 \sigma$. Recall from proposition 4.5, that κ is a unique face of σ such that $V(\kappa) = \pm\sigma$ and $\langle \partial\sigma, \kappa \rangle V(\kappa) = -\sigma$. It follows that

$$\begin{aligned}\phi(\sigma) &= \sigma + \sum_{\gamma <_1 \sigma} \langle \partial\sigma, \gamma \rangle V(\gamma) \\ &= \sigma + \langle \partial\sigma, \kappa \rangle V(\kappa) + \sum_{\kappa \neq \gamma <_1 \sigma} \langle \partial\sigma, \gamma \rangle V(\gamma) \\ &= \sum_{\kappa \neq \gamma <_1 \sigma} \langle \partial\sigma, \gamma \rangle V(\gamma)\end{aligned}$$

Moreover, for any other face γ of σ, $V(\gamma) = 0$ or $V(\gamma) = \widetilde{\sigma}$ where $f(\widetilde{\sigma}) \leq f(\gamma) < f(\sigma)$. Therefore,

$$\phi(\sigma) = \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}}\widetilde{\sigma}.$$

(iii) Now, suppose that $V(\sigma) \neq 0$, that is $V(\sigma) = -\langle \partial\tau, \sigma \rangle \tau$ where $\sigma <_1 \tau$. Recall that $\sigma \notin \mathrm{im}(V)$. So, for any face γ of σ where $\gamma <_1 \sigma$ we have $V(\gamma) = 0$ or $V(\gamma) = \pm\widetilde{\sigma}$ where $f(\widetilde{\sigma}) \leq f(\gamma) \leq f(\sigma)$. We have $\phi(\sigma) = \sigma + \partial V(\sigma) + V(\partial\sigma)$. Note that,

$$V(\partial\sigma) = \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}}\widetilde{\sigma}$$

and

$$\begin{aligned}\partial V(\sigma) &= -\langle \partial\tau, \sigma\rangle \partial\tau \\ &= -\langle \partial\tau, \sigma\rangle^2 \sigma + \sum_{\widetilde{\sigma}^{(p)}} b_{\widetilde{\sigma}} \widetilde{\sigma} \\ &= -\sigma + \sum_{\widetilde{\sigma}^{(p)}} b_{\widetilde{\sigma}} \widetilde{\sigma}.\end{aligned}$$

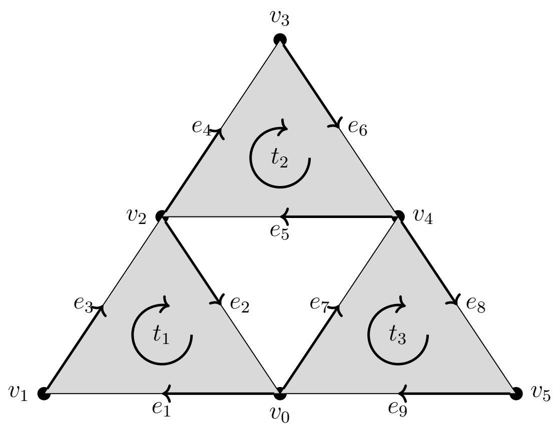
Therefore,

$$\begin{aligned}\phi(\sigma) &= \sigma + \partial V(\sigma) + V(\partial\sigma) \\ &= \sigma - \sigma + \sum_{\widetilde{\sigma}^{(p)}} b_{\widetilde{\sigma}} \widetilde{\sigma} + \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}} \widetilde{\sigma} \\ &= \sum_{\widetilde{\sigma}^{(p)}} d_{\widetilde{\sigma}} \widetilde{\sigma}\end{aligned}$$

where $d_{\widetilde{\sigma}} \neq 0$ implies $f(\widetilde{\sigma}) < f(\sigma)$. Note that the only time in the three cases when $a_{ii} = 1$ was when σ was critical. □

In the above proposition the gradient flow map is a chain map since it commutes with the boundary operator. Furthermore, *2.* from the above proposition says that the gradient flow traverses decreasing heights. Note that the gradient flow and V-paths are note the same. Recall that V-paths is a sequence of cells/arrows of decreasing height within two dimensions p and $p+1$ where as the gradient flow can take into account arrows in many different dimensions.

Example 4.7. *We will illustrate how the gradient flow works with an example. This will show us how the gradient flow works algebraically as well as geometrically. We will need a simplicial complex with some orientation and a vector field on it as well. Note that the vector field need not be a gradient vector field. Consider the following* *simplicial complex with the given orientation and then the following figure will be the vector field we place on it. Note that the orientation of the cells and the arrows on the vector field will determine the incidence number.*



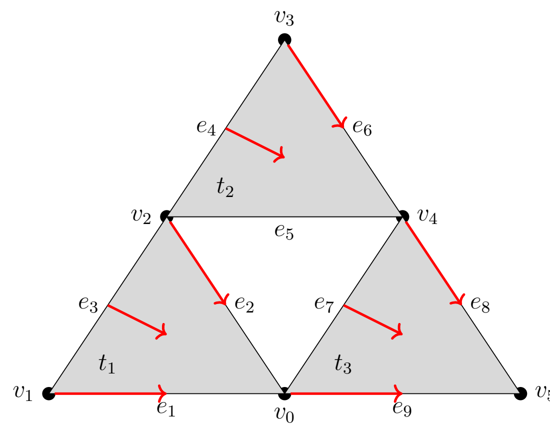
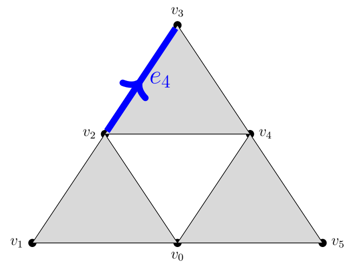


Figure 7: Left: Simplicial complex with a given orientation. Right: the same simplicial complex with a given vector field.

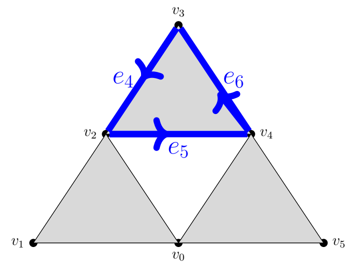
Consider the edge e_4*. We will find* $\phi(e_4)$*. Observe that*

$$\begin{aligned}
\phi(e_4) &= e_4 + \partial V(e_4) + V(\partial e_4) \\
&= e_4 + \partial(-\langle \partial t_2, e_4\rangle t_2) + V(v_3 - v_2) \\
&= e_4 - \partial t_2 + V(v_3) - V(v_2) \\
&= e_4 - e_6 + e_5 - e_4 + (-\langle \partial e_6, v_3\rangle e_6) - (-\langle \partial e_2, v_2\rangle e_2) \\
&= e_4 - e_6 + e_5 - e_4 + e_6 + e_2 \\
&= e_2 + e_5
\end{aligned}$$

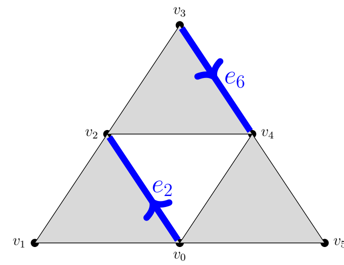
This is what is happening on the simplicial complex itself.



(a) Orientation of e_4.



(b) $\partial V(e_4)$



(c) $V(\partial e_4)$

What we have shown above is what is happening at each part of

$$\phi(e_4) = e_4 + \partial V(e_4) + V(\partial e_4)$$

on the simplicial complex. Combining these figures together, we see what the gradient flow of e_4 is.

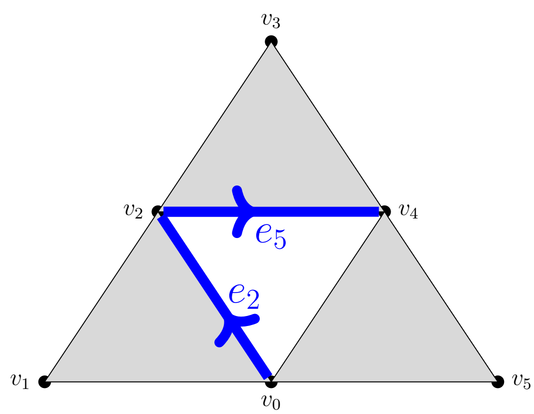
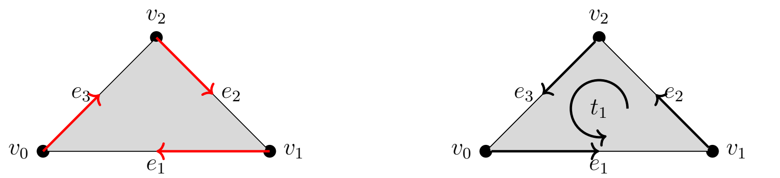


Figure 9: The gradient flow of e_4: $\phi(e_4)$

A quick observation based on the example above that is also true in general is that $V(\partial\sigma)$ is the portion of ϕ which is tangent to σ and $\partial V(\sigma)$ is the portion of ϕ which can be though of as transversal to σ.

Now that we have seen an example of the gradient flow of a gradient vector field, consider the following example of the gradient flow of a non-gradient discrete vector field.

Example 4.8. *We bring back the discrete vector field on Δ^2 from before.*



(a) A DVF on Δ^2 (b) The orientation on Δ^2.

Observe the following two calculations.

$$\begin{aligned}\phi(v_1) &= v_1 + \partial V(v_1) + V(\partial v_1)\\ &= v_0\end{aligned}$$

and

$$\begin{aligned}\phi(e_1) &= e_1 + \partial V(e_1) + V(\partial e_1)\\ &= e_3.\end{aligned}$$

We see from the above calculations in our example that if there is a cycle in our discrete vector field, the flow will simply take a cell to the next same dimensional cell in the cycle.

These are the main properties of the vector field and gradient flow maps. One can take these ideas further and talk about the Morse complex of an abstract simplicial complex as [5] and [16] do. Delving even deeper is Forman in [9].

We now change gears to discuss simple homotopy type; however, we will bring the discussion back around to discrete Morse theory. That is, we will show how simple homotopy type and discrete Morse theory behave together. Doing this will put us in a place where we can go into our final results. That is, we will show how simple homotopy works within Morse theory and for our finale we will show how to construct a homotopy using discrete Morse theory.

5 Simple Homotopy and Discrete Morse Theory

Here, we introduce one way homotopy and discrete Morse theory intermingle. By introducing simple homotopy type, we may discuss some familiar results from classical Morse theory in our discrete setting. We do this so we can discuss homotopy in terms of homotopies between functions as well as homotopy equivalence between abstract simplicial complexes. First we discuss the notion of simple homotopy and then we see an example of the relationship between simple homotopy and discrete Morse theory. A full treatment of simple homotopy is given by Cohen [3].

Definition 5.1. *Let K be an abstract simplicial complex. Let σ, $\tau \in K$ be cells such that*

1. $\sigma <_1 \tau$.

2. *τ is maximal and no other maximal cell contains σ.*

An ***elementary collapse*** *of K is the removal of the set σ, τ from K and denoted by $K \searrow K - \{\sigma, \tau\}$. Suppose $\{\sigma^{(p)}, \tau^{(p+1)}\}$ is a pair of cells not in K where $\sigma <_1 \tau$ and the rest of the faces of τ are in K, then $K \cup \{\sigma^{(p)}, \tau^{(p+1)}\}$ is called an* ***elementary expansion*** *of K, denoted $K \nearrow K \cup \{\sigma^{(p)}, \tau^{(p+1)}\}$.*

That is, first find a cell which is maximal in K, say σ_m (not necessarily the only one, this is just a choice). Find all cells contained in σ_m which are not contained in another cell. These faces are called **free faces** and we remove these cells contained in σ_m. On the other hand, if we have a pair of cells such that $\sigma <_1 \tau$ and σ is contained in no other face we call this a **free pair**. This is what we collapse for elementary collapses or expand to for elementary expansions.

Definition 5.2. *Let K_1 and K_2 be two abstract simplicial complexes. If we can perform a sequence of elementary collapses and expansions on K_1 which leads to K_2, we say that K_1 and K_2 have the same **simple homotopy type**. The sequence of elementary collapses and expansions from K_1 to K_2 is called a **formal deformation**. We say that if K has a formal deformation to a vertex v, then we say K is **collapsible**.*

$$K = K_0 \searrow K_1 \searrow \cdots \searrow K_{n-1} \searrow K_n = \{v\}$$

So how does simple homotopy type relate to discrete Morse theory? Recall from classical Morse theory that homotopy type is discussed in terms of critical points. Consider, for example, theorems 3.1 and 3.2 from Milnor's text [14]. Though, we will only focus on the discrete version of the former theorem mentioned. In classical Morse theory we talk about sublevel sets, where as in discrete Morse theory we consider the sublevel complex.

Definition 5.3. *Let K be an abstract simplicial complex and $f : K \to \mathbb{R}$ be a discrete Morse function. For any $c \in \mathbb{R}$, the **sublevel complex** $K(c)$ is the subcomplex of K consisting of all cells σ with $f(\sigma) \leq c$ as well as their faces. That is,*

$$K(c) = \bigcup_{f(\sigma)\leq c} \bigcup_{\gamma <_1 \sigma} \gamma$$

The following theorem is the discrete analog, in regards to simplicial complexes, of theorem 3.1 from Milnor [14]. The theorem will be stated without proof; however, one may find the proof in Scoville [16].

Theorem 5.4. *Let K be an abstract simplicial complex and $f : K \to \mathbb{R}$ a discrete Morse function. If $[a, b] \subseteq \mathbb{R}$ is an interval which contains no critical values, then $K(b)$ and $K(a)$ differ by a sequence of elementary collapses.*

In fact, the gradient vector field gives us an outline of such collapses. What this theorem tells us is that the two sublevel complexes $K(b)$ and $K(a)$ have the same simple homotopy type. Therefore, one may only want consider the sublevel complexes which are induced by the critical values.

As revealed at the beginning of this section, we wish to take a different look at this relationship between homotopy and discrete Morse theory. This will be our finale. We now construct the chain homotopy between chain maps which were induced by simplicial maps.

6 Pivot Induced Discrete Vector Fields

As mentioned previously, the survey work before this section was set up for our final results. Here we will build a chain homotopy between chain maps which are induced by simplicial maps together with arrows. We will prove that these arrows, which are generated by these simplicial maps, form a discrete vector field in the sense of Forman. That is, we are using our previous discussion of discrete Morse theory to build this chain homotopy. First we will briefly introduce the notion of simplicial homology as a reminder to the reader since this is where we base our discussion; consult Hatcher [10] for details.

Recall that a sequence of homomorphisms of chain groups is called a chain complex.

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Given the boundary operator from definition 4.2 we know that $\partial_n \partial_{n-1} = 0$ so we obtain $\mathrm{im}\partial_n \subset \ker \partial_{n-1}$. We define the n$^{\text{th}}$ **homology group** to be the quotient group $H_n = \ker \partial_n / \mathrm{im}\partial_{n+1}$. The results of chain homotopies are what we are after. Recall that a chain map is a map $f_\sharp : C_n(X) \to C_n(Y)$ which is induced from a map $f : X \to Y$ and satisfies $\partial f_\sharp = f_\sharp \partial$. That is, a chain map sends boundaries to boundaries and cycles to cycles. We are mainly interested in the following results of homology.

Theorem 6.1. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism on homology groups, that is, $f_* = g_* : H_n(X) \to H_n(Y)$.*

For a proof of the above theorem, one can consult Hatcher [10]. Recall that a chain homotopy between chain maps is defined to be a map, say, H which satisfies $\partial H + H\partial = g_\sharp - f_\sharp$. Recall from section 4 that Forman constructed a chain homotopy from the gradient flow map to the identity map using the vector field map as the chain homotopy, i.e. $\phi = 1 + \partial V + V\partial$. Forman was interested in the following result from algebraic topology as he used it in his construction of the Morse complex, for more information on his construction one can consult [5]. We are also interested in the following result for our construction as it will show we can induce the same homomorphism on the homology groups of abstract simplicial complexes using simplicial maps, which is the goal of this section.

Proposition 6.2. *Chain homotopic chain maps induce the same homomorphism on homology.*

Proof. Let H be a chain homotopy between $g_\sharp$ and $f_\sharp$. Let $\alpha \in H_n(C_*)$. Recall that elements of H_n are cosets of $\mathrm{im}\partial_{n+1}$ and $\partial\alpha = 0$. Therefore,

$$\begin{aligned} g_\sharp(\alpha) - f_\sharp &= \partial H(\alpha) + H\partial(\alpha) \\ &= \partial H(\alpha) + 0 \end{aligned}$$

Therefore, $g_\sharp(\alpha) - f_\sharp(\alpha) = \partial H(\alpha) \in \mathrm{im}\partial_{n+1}$. Hence, $g_\sharp = f_\sharp$. $\square$

We will follow this development of chain maps and chain homotopies in the following discussion. Recall that the objective is to construct a chain homotopy between chain maps which are induced by simplicial maps and not continuous maps as in the development found in Hatcher.

Let K and L be abstract simplicial complexes and $f : K \to L$ be a simplicial map. Let $f_\sharp : C_n(K) \to C_n(L)$ be a function defined by

$$f_\sharp([v_0, ..., v_n]) = \begin{cases} [f(v_0), , ..., f(v_n)] & \text{if } f(v_0), ..., f(v_n) \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$

We will prove that this function is a chain map induced by the simplicial map f.

Proposition 6.3. *$f_\sharp$ defined above is a chain map induced by f.*

Proof. What we must show is that $f_\sharp \circ \partial = \partial \circ f_\sharp$. We will need to consider some cases.

- First, consider when the image of all vertices under f are distinct. Observe that

$$\begin{aligned} f_\sharp \circ \partial([v_0, v_1, ..., v_n]) &= f_\sharp\left(\sum_k (-1)^k [v_0, v_1, ..., \hat{v_k}, ..., v_n]\right) \\ &= \sum_k (-1)^k f_\sharp([v_0, v_1, ..., \hat{v_k}, ..., v_n]) \\ &= \sum_k (-1)^k [f(v_0), f(v_1), ..., f(\hat{v_k}), ..., f(v_n)] \\ &= \partial \circ f_\sharp([v_0, v_1, ..., v_n]) \end{aligned}$$

- Now consider the case when two vertices have the same image. That is, $f(v_j) = f(v_i)$. We know that we will have $\partial \circ f_\sharp = 0$. Now we need to show that $f_\sharp \circ \partial([v_0, v_1, ..., v_n]) = 0$. Observe that

$$\begin{aligned} f_\sharp \circ \partial([v_0, v_1, ..., v_n]) &= \sum_k (-1)^k f_\sharp [v_0, ..., \hat{v_k}, ..., v_n] \\ &= (-1)^j [f(v_0), ..., f(v_{j-1}), f(v_{j+1}), ..., f(v_i), ..., f(v_n)] \\ &+ (-1)^i [f(v_0), ..., f(v_j), ..., f(v_{i-1}), f(v_{i+1}), ..., f(v_n)] \\ &= (-1)^j [f(v_0), ..., f(v_{j-1}), f(v_j), f(v_{j+1}), ..., f(v_n)] \\ &+ (-1)^i [f(v_0), ..., f(v_{j-1}), f(v_j), f(v_{j+1}), ..., f(v_n)] \end{aligned}$$

We will now show that the sum we left off with cancels out. We will proceed by cases. Without loss of generality, assume that $j < i$. We proceed by a parity argument in regards to i and j.

- Case 1: Both i and j are even. We then know that $(-1)^j = (-1)^i = 1$. Now we focus on the term which swapped $f(v_i)$ for $f(v_j)$. Note that $f(v_i)$ moved i-j-1 positions. Therefore, since both i and j are even, $i - j - 1$ will be odd. Since $f(v_i)$ shifted an odd amount of positions, the resulting cell will have a negative orientation. Therefore, we obtain $f_\sharp \circ \partial = 0$.
- Case 2: Both i and j are odd. Therefore, $(-1)^i = (-1)^j = -1$. Similarly, $i - j - 1$ will be odd once again. Because $f(v_i)$ shifted an odd amount of positions, the resulting cell will have the opposite orientation. Therefore, we obtain $f_\sharp \circ \partial = 0$.
- Case 3: Assume, without loss of generality, that i is odd and j is even. Then we know $(-1)^i = -1$ and $(-1)^j = 1$, so the cells have opposite orientations. The quantity $i - j - 1$ will be even, therefore the shifting of $f(v_i)$ will not change the orientation of the resulting cell. Therefore, we obtain $f_\sharp \circ \partial = 0$.

Hence, $\partial \circ f_\sharp = f_\sharp \circ \partial$.

- Now we consider the case where there are more than two vertices with the same image under $f_\sharp$. Note that when this occurs, the boundary wont be able to take out each of these overlapping vertices so there will be non-distinct vertices in the sum $\sum_k (-1)^k [f(v_0), ..., f(\hat{v_k}), ..., f(v_n)]$. Therefore, $f_\sharp$ will send this to 0. Furthermore, We know we will have $\partial \circ f_\sharp = 0$. Therefore, $f \circ \partial = \partial \circ f_\sharp$.

Hence, we have shown that $f_\sharp$ is a chain map $\square$

Now that we have our chain maps we can start constructing the chain homotopy. We wish to model a notion of closeness between two simplicial maps. This is where our arrows come into play. However, before we get into that consider the following definition.

Definition 6.4. *Let K and L both be abstract simplicial complexes and $f, g : K \to L$ be simplicial maps. We say that f and g differ by a **vertex pivot** if the following occurs. For a cell $\sigma \in K$*

- $f(\sigma) = g(\sigma)$.
- $f(\sigma) = (g(\sigma) - \{u\}) \cup \{v\}$ *Where* $\{u\} \in g(\sigma)$ *and* $\{v\} \in f(\sigma)$ *are vertices in* L *and* $f(\sigma) \cup \{v\}$ *is a cell in* L.

Furthermore, the second condition requires that $f(\sigma)$ and $g(\sigma)$ share cell. If two simplicial maps f and g differ by vertex pivot we will denote this by $f \sim_p g$

We illustrate the previous definition with an example.

Example 6.5. *In this example f and g are simplicial maps. We will map the 2-simplex to the tetrahedron. Consider the following figure.*

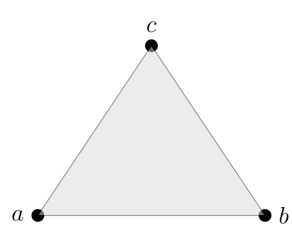


Figure 11: The 2-simplex with labeled vertices

We map the 2-simplex according to the mapping rule visualized by the following figure. In the figure, the shaded regions reflect that f and g mapped the 2-simplex to

2-dimensional cells on the tetrahedron. We see that the images of the vertices a and b under f and g are the same. Notice that the images of c under f and g differ.

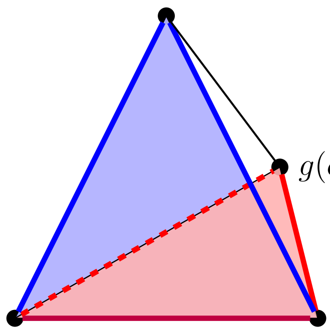


Figure 12: A solid tetrahedron with the images of f and g

In the figure of the tetrahedron, we see that $f(\Delta^2) = (g(\Delta^2) - \{g(\{c\})\}) \cup \{f(\{c\})\}$ *as well as* $g(\Delta^2) = (f(\Delta^2) - \{f(\{c\}\})) \cup \{g(\{c\})\}$.

In definition 6.4 and the above example we see that the images of f and g differ by vertex. The idea here is, we can slide a vertex in the image of f in one move to get to a vertex in the image of g. We elaborate more on what me mean by slide later. Definition 6.4 almost makes an equivalence relation, so we extend the definition as follows.

Definition 6.6. *We say that two simplicial maps f, g are **pivot homotopic** if there exists a sequences of maps*

$$f = f_1 \sim_p f_2 \sim_p \cdots \sim_p f_n = g$$

Now we are in a place to prove that pivot homotopic maps define an equivalence relation.

Proposition 6.7. *Pivot homotopic maps define an equivalence relation. We will say that if functions differ by a simplicial pivot, then they are in the same pivot class.*

Proof.

- First we show reflexivity. By definition we have $f(\sigma) = f(\sigma) \in L$.
- Now we show symmetry. Assume $f \sim_p g$. We will show $g \sim_p f$. Note that when $g(\sigma) = f(\sigma)$ this follows right away. Now we need to check the other condition. We have that $f(\sigma) = (g(\sigma) - \{u\}) \cup \{v\}$. Since $\{u\}$ is in the image of g but not in the image of f and $\{v\}$ is in the image of f but not g, we have $g(\sigma) = (f(\sigma) - \{v\}) \cup \{u\}$. Hence, $g \sim_p f$.
- Now we prove transitivity. Assume that $f \sim_p g$ and $g \sim_p h$. Let the vertices $\{v\}$, $\{u\}$, and $\{w\}$ be in the images of f, g, and h respectively. Since $f \sim_p g$, we have $f(\sigma) = (g(\sigma) - \{u\}) \cup \{v\}$. Furthermore, since $g \sim_p h$ we obtain

$$\begin{aligned} f(\sigma) &= (((h(\sigma) - \{w\}) - \{u\}) \cup \{u\}) \cup \{v\} \\ &= (h(\sigma) - \{w\}) \cup \{v\} \end{aligned}$$

  Therefore, $f \sim_p h$.

Hence, pivot homotopic maps form an equivalence relation. □

We now bring discrete Morse theory into the development. We define arrows based on simplicial pivots. We will then show that these set of arrows will define a vector field in the sense of Forman.

Definition 6.8. *For two simplicial maps where $f \sim_p g$ we construct* ***pivot arrows*** *in the following way.*

- *If $f(\sigma) = g(\sigma)$, then $f(\sigma)$ will not be on an arrow.*

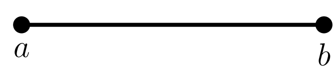
- *If $f(\sigma) = (g(\sigma) - \{u\}) \cup \{v\}$, then the pair $(f(\sigma), f(\sigma) \cup \{u\})$ is an arrow where $f(\sigma)$ is the tail and $f(\sigma) \cup \{u\}$ is the head.*

We will prove that the collection of arrows from the above definition forms a discrete vector field.

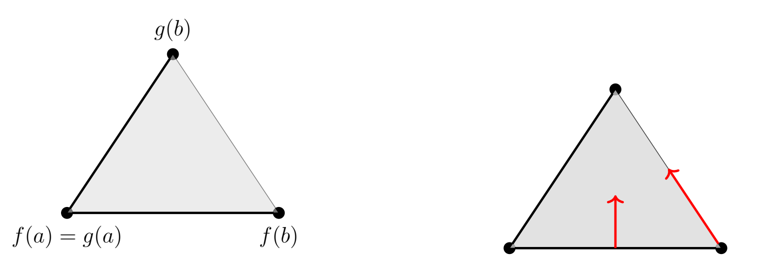
Proposition 6.9. *The set of pivot arrows form a discrete vector field.*

Proof. We need to show that each cell is on at most one arrow. Note that the only cells which are on arrows are cells which are contained in $f(\sigma)$ and $f(\sigma) \cup \{u\}$ where $f(\sigma)$ does not overlap with $g(\sigma)$ and $\{u\}$ is in the image of g and not f. We know each $f(\sigma)$ will be a tail of an arrow and $f(\sigma) \cup \{u\}$ a head. With these observations combined, we see that each $f(\sigma)$ will be on at most one tail of an arrow and never a head. Similarly, each $f(\sigma) \cup \{u\}$ will be on at most one head of an arrow and never a tail. Hence, the collection of pivot arrows on an abstract simplicial complex forms a discrete vector field. □

Example 6.10. *We will illustrate what this induced vector field may look like in a simple example. Consider the line segment.*



Consider the following 2-simplex with the image of f and g defined on it with the induced vector field beside it.



(a) Images of f and g on Δ^2

(b) Induced vector field.

The idea here is that we are sliding the image of f to the image of g where there is a common overlap in their images. Here the common overlap is a vertex. We can think of the vertex as the fixed position as we pivot the image of f to the image of g.

Recall that the goal is to show that if two simplicial maps f and g differ by a vertex pivot then there is a chain homotopy between $f_\sharp$ and $g_\sharp$. Our candidate for chain homotopy is $H = V \circ f_\sharp$ where V is the vector field map associated with the discrete vector field induced by vertex pivots.

Theorem 6.11. *$H = V \circ f_\sharp$ is a chain homotopy from $f_\sharp$ to $g_\sharp$. That is,*
$\partial H + H\partial = g_\sharp - f_\sharp$

Proof. We first prove the case when the image of f and g differ by a vertex. Suppose, without loss of generality, that the vertices have been ordered such that $f(v_i) = g(v_i)$ if $i \neq n$ and $f(v_n) \neq g(v_n)$. Let $f(v_n) = v$ and $g(v_n) = u$. Note that in the following calculations $[u, f(v_0)..., f(v_{n-1}), \hat{v}] = g_\sharp$ and
$[\hat{u}, f(v_0), .., f(v_{n-1}), v] = f_\sharp$. Observe the following:

$$\begin{aligned}
\partial H + H\partial &= \partial(V f_\sharp) + (V f_\sharp)\partial \\
&= \partial(V f_\sharp) + V(\partial f_\sharp) \\
&= -\langle \partial(f_\sharp \cup \{u\}), f_\sharp\rangle \, \partial(f_\sharp \cup \{u\}) - \langle \partial(\partial f_\sharp \cup \{u\}), f_\sharp \cup \{u\}\rangle \, \partial f_\sharp \cup \{u\} \\
&= (-1)^v g_\sharp - (-1)^u f_\sharp - \sum_{k \neq v,u} (-1)^k [u, f(v_0), ..., f(\hat{v_k}), ..., f(v_{n-1}), v] \\
&+ \sum_{k \neq u,v} (-1)^k [u, f(v_0), ..., f(\hat{v_k}), ..., f(v_{n-1}), v] \\
&= g_\sharp - f_\sharp
\end{aligned}$$

Some explanation may be in order for the last two equalities. So we know from section 4 and our definition of pivot arrows that g is not in the image of V so we are justified for $k \neq u, v$ in the last part of the sum. The last equality comes from how v and u are ordered in regards to $(-1)^v$ and $(-1)^u$.

Now we consider the case when the image of f is the same as the image of g. Note that $f_\sharp$ and $g_\sharp$ are not in the image of V. Observe the following:

$$\begin{aligned}
\partial H + H\partial &= \partial(V f_\sharp) + (V f_\sharp)\partial \\
&= \partial(V f_\sharp) + V(\partial f_\sharp) \\
&= 0 + 0 \\
&= 0 \\
&= g_\sharp - f_\sharp
\end{aligned}$$

Hence, H is a chain homotopy between $g_\sharp$ and $f_\sharp$. $\square$

7 Vertex Pivots and Contiguous Maps

In the previous section we developed a chain homotopy between chain maps which were induced by simplicial maps. Though this idea is not new, it has not been discussed in detail under the scope of discrete Morse theory. The reader may have noticed that simplicial maps which differ by a vertex pivot is a close notion of contiguous maps.

Definition 7.1. *Let K and L be abstract simplicial complexes and $f, g : K \to L$ simplicial maps. If for every cell $\sigma \in K$ we have that $f(\sigma) \cup g(\sigma)$ is a cell in L, then we say f and g are* ***contiguous****. Denote contiguous maps by $f \sim_c g$.*

We see that simplicial maps which differ by a vertex pivot is a stronger notion than contiguous maps. In fact, simplicial maps as described in definition 6.4 are contiguous maps; however, not all contiguous maps are of definition 6.4. Making use contiguous maps, Barmak and Minian [1] introduce the notion of strong homotopy.

Definition 7.2. *Two simplicial maps f, g are* ***strongly homotopic****, $f \sim g$, if there exists a sequence of contiguous maps joining f and g.*

$$f = f_0 \sim_c f_1 \sim_c f_2 \sim_c \cdots \sim_c f_{n-1} \sim_c f_n = g$$

Scoville [16] has a chapter dedicated to the combination of discrete Morse theory and strong homotopy theory. This chapter summarizes the collaborative work of himself and others in Fernández-Ternero et al. [4] in a more digestible way. The motivation here is that elementary collapses may not induce a simplicial map; however, a composition of elementary collapses might be. With this motivation, it is then showed exactly when a composition of elementary collapses induces a simplicial map.

As subtly mentioned in section 5, one of the main ideas of discrete Morse theory is, in our case, every abstract simplicial complex can be broken down or built up using elementary expansions or collapses. Strong homotopy theory would allow the notion of a strong elementary collapses to be added to the mix. However, we need one more definition before we give the definitions of strong elementary collapses and expansions.

Definition 7.3. *Let K be an abstract simplicial complex. A vertex v' is **dominated** by a vertex v if every maximal cell of v' also contains v.*

Example 7.4. *In this example, we see that each maximal cell which contains v' also contains v but not the other way around. Here we say that v dominates v' or v' is dominated by v.*

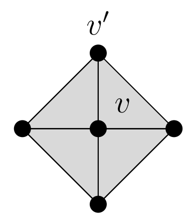


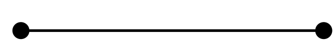
Figure 14: Example of dominating vertices.

Definition 7.5. *Let K be an abstract simplicial complex with v' and v as vertices. If v' is dominated by v, then the removal of v' is an **elementary strong collapse** which we denote by $K \searrow\searrow K - \{v'\}$. An **elementary strong expansion** is the addition of a dominated vertex which we denote by $K \nearrow\nearrow K \cup \{v'\}$. We also denote sequences of elementary strong collapses/expansions by $\searrow\searrow$ and $\nearrow\nearrow$ respectively. Let L be another abstract simplicial complex. If there is a sequence of strong collapses and expansions from K to L, then K and L have the same **strong homotopy type**. If there is a sequence of elementary strong collapses from K to a vertex, then K is **strongly collapsible**.*

We see that the above definition is extremely similar to that of definition 5.2. What Scoville [16] does is add the notion of elementary strong collapses and expansions as another tool into discrete Morse theory. So why mention all of this after the main result was proven? Well, to generalize the notion of vertex pivots we would like to use contiguous functions and instead of pivot homotopic maps we would like to use strongly homotopic maps. However, there are some issues if we try to extend our development to include general contiguous functions.

Consider the following as one example. We could have images of functions which differ by more than one vertex pivot. Some cases within this issue include the possibility of not being able to slide the image of f to the image of g and therefore we would not have induced vector fields. In the cases where we could slide the image of f to that of g we do not have a general way to choose which cells would be on arrows. Another example of what could go wrong is when the image of f is smaller than the image of g. Here we could have a couple different cases. We would slide the image of f to g; however, we again run into the issue of how to choose arrows and the other case would, again, be there is no possible way to slide f onto g. The main point here being that we have issues with not always being able to collapse f onto g. We illustrate some of these issues with one final example where the dimension of the image of f is smaller than that of the image of g.

Example 7.6. *Let K be the abstract simplicial complex $\{\{a\},\{b\}\}$ and L be the abstract simplicial complex $\{\{x\},\{y\},\{x,y\}\}$. Let $f,g : K \to L$ be simplicial maps such that $f(a) = f(b) = x = g(a)$ and $g(b) = y$. Visually we have*



$f(a) = f(b) = x = g(a)$ $\quad$ $g(b) = y$

We have two possible induced vector fields

1. One of the vector fields

2. The other vector field

Now we calculate H *for both cases. First we will consider the chain map* $g_\sharp$*. Observe the following two calculations which correspond to the vector field 2.*

$$H([a]) = V(g_\sharp([a])) = V([x]) = 0$$

and

$$H([b]) = V(g_\sharp([b]) = V([y]) = -[x,y] = [y,x]$$

So we obtain $\partial H([a]) + H\partial([a]) = 0$ *and* $\partial H([b]) + H\partial([b]) = [x] - [y] = f_\sharp([b]) - g_\sharp([b])$*. On the other hand, when we consider* $f_\sharp$*. Observe the following calculations with respect to vector field 1.*

$$H([a]) = V(f_\sharp([a])) = V([x]) = -[y,x] = [x,y]$$

and

$$H([b]) = V(f_\sharp([b])) = V([x]) = -[y,x] = [x,y]$$

Therefore, $\partial H([a]) + H\partial([a]) = [y] - [x] = g_\sharp([b]) - f_\sharp([a])$ *and* $\partial H([b]) + H\partial([b]) = [y] - [x] = f_\sharp([b]) - g_\sharp([b])$*.*

We see that H *is a chain homotopy from* $g_\sharp$ *to* $f_\sharp$ *but not vice versa. That is, we* *see that it is impossible to obtain a chain homotopy from* $f_\sharp$ *to* $g_\sharp$ *unless we take* $-H$ *but this would not correspond to an arrow as* $-V$ *does not define a vector field map. Another issue is that* H *cannot distinguish between* $[a]$ *and* $[b]$ *as we see with the calculations with respect to the first vector field. In other words, we do not get a chain homotopy from* K *to* L *for all elements in the domain. The issue here is that composing* V *with* $f_\sharp$ *can generate some homotopies but cannot generate all of them.*

We see that in example 7.6 where our development in section 6 needs some modification to work. Notice that in example 7.6 that f and g are both contiguous functions but composing the vector field map with one of the chain maps did not give us a chain homotopy in this case. It was noted in the example that it is impossible to obtain the chain homotopy by the way we defined H. However, not all hope is lost. We did find, in this specific case, that we were able to build a chain homotopy for one direction. There may be a modification which we could make in order to fix the issue we have. As a future avenue of research which extends from what we have, one could possibility define a vector field between two abstract simplicial complexes which would allow us to define vector field maps between abstract simplicial complexes as well. The hope here is that we can, in the future, extend this idea of simplicial maps which differ by a vertex pivot to that of contiguous maps.

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