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A Natural Rank Problem for Homogeneous Polynomials and Connections with the Theory of Functions of Several Complex Variables

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Abstract: A Natural Rank Problem for Homogeneous Polynomials and Connections with the Theory of Functions of Several Complex Variables

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We study a natural extremal problem about the vector space consisting of all homogeneous polynomials of degree d in $n+1$ variables with real coefficients, together with the zero polynomial. We define the *rank* of a polynomial to be the number of distinct monomials appearing in the polynomial with non-zero coefficient. We are particularly interested in those homogeneous polynomials whose quotient with the homogeneous polynomial $x_0 + x_1 + \cdots + x_n$ is a polynomial of degree $d-1$ with maximal rank. For each degree d, we seek the minimum rank for an element of this subfamily and we seek to describe those polynomials with minimum rank. We call such polynomials *sharp* polynomials.

These problems have a simple solution for polynomials in one and two variables. The three-variable case is interesting and non-trivial, but well-understood. This research question has its roots in the study of proper polynomial mappings between balls in complex Euclidean spaces of different dimensions and the degree estimates problem. D'Angelo, Kos and Riehl [13] and Lebl and Peters [24] used a graph-theoretic approach to solve this problem in the case of proper monomial mappings. Lebl and Peters give a minimum rank estimate that answers our question in the three variable case. A family of sharp polynomials was described by D'Angelo and has been extensively studied. Brooks and Grundmeier [3] provided a new proof of the minimum rank theorem in the three-variable case using a commutative algebra approach. They reformulate the problem as a question about homogeneous ideals and address it by studying the Hilbert function and the graded Betti number of certain ideals.

Using the same method as Brooks and Grundmeier, we give a sharp estimate for the minimum rank of homogeneous polynomials of our subfamily in four variables as well as a family of sharp polynomials. Moreover, we state a general result on the minimum rank for polynomials in $n+1$ variables. Although this estimate is sharp in the three- and four- variable cases, the estimate is not sharp when the number of variables is greater than four.

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Chapter 1

Introduction

1.1 A Natural Question about Homogeneous Polynomials

Many important and interesting problems in mathematics are extremal or optimization problems. Extremal problems have the following characterization: we give some restrictions on a collection of objects and ask how big or small the objects are under those conditions. Perhaps one of the most famous examples is Zorn's lemma, which states that a partially-ordered set containing upper bounds for every chain must contain at least one maximal element. Another example is the problem of finding the orthogonal projection of an element x of a Hilbert space $\mathcal{V}$ onto a closed subspace M. This problem is solved by finding the element in M that minimizes the distance between x and M. This element always exists and is unique.

We consider an extremal problem about the vector space consisting of all homogeneous polynomials of degree d in $n+1$ variables with real coefficients, together with the zero polynomial. We denote this space by $\mathcal{H}_{n+1,d}$. Given a polynomial $P \in \mathcal{H}_{n+1,d}$, the rank of P, denoted by $\rho(P)$, is the number of distinct monomials appearing in P with non-zero coefficient. If every possible monomial of degree d in $n+1$ variables appears in P with non-zero coefficient, we say P is *full*. For instance, in the space $\mathcal{H}_{3,2}$, the polynomial $P_1(x_0, x_1, x_2) = x_0^2 + x_0x_1 + 3x_1^2 + x_0x_2 + 2x_1x_2 + x_2^2$ is full, while the polynomial $P_2(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2$ is not full.

We now formulate our research questions. Let $S = x_0 + x_1 + x_2 + \cdots + x_n$. Let P be a homogeneous polynomial of degree d such that $P = SQ$ for some homogeneous polynomial Q. Question 1: If Q is a full polynomial of degree $d-1$, what is the minimum rank for P? Question 2: What are those polynomials with the minimum rank? These polynomials are called *sharp* polynomials. The answers to these questions are known for $n+1 = 1, 2$ and 3, and we will discuss them in the next two sections. For

$n+1 = 2,3,4$, we use x, y, z, w instead of x_0, x_1, x_2, x_3 to avoid subscripts.

1.2 The One- and Two-Variable Cases

First, note that Question 1 stated above for polynomials in one variable has a trivial answer. In fact, the non-zero homogeneous polynomials of degree d in one variable are of the form $P(x) = cx^d$ with $c \in \mathbb{R}$. Then $S = x$, $Q = cx^{d-1}$, and $\rho(P) = 1$.

For polynomials in two variables, this question is more interesting; however, it can be easily addressed.

Lemma 1.1. *Let $P_d(x,y) = x^d + (-1)^{d-1}\, y^d$, with $d \geqslant 1$. Then $P_d(x,y)$ is divisible by $S = x+y$. Moreover, the quotient is full.*

Proof. We claim that the quotient P_d/S is the polynomial

$$Q_d(x,y) = \sum_{j=0}^{d-1} (-1)^j x^{d-1-j} y^j \tag{1.1}$$

Indeed,

$$\begin{aligned}
S(x,y)Q_d(x,y) &= (x+y)\sum_{j=0}^{d-1}(-1)^j x^{d-1-j}y^j \\
&= \sum_{j=0}^{d-1}(-1)^j x^{d-j}y^j + \sum_{j=0}^{d-1}(-1)^j x^{d-1-j}y^{j+1} \\
&= \sum_{j=0}^{d-1}(-1)^j x^{d-j}y^j + \sum_{j'=1}^{d}(-1)^{j'-1} x^{d-j'}y^{j'}, \qquad j' := j+1 \\
&= x^d + \sum_{j=1}^{d-1}(-1)^j x^{d-j}y^j + (-1)^{d-1}y^d + \sum_{j'=1}^{d-1}(-1)^{j'-1}x^{d-j'}y^{j'} \\
&= x^d + (-1)^{d-1}y^d + \sum_{j=1}^{d-1}[(-1)^j + (-1)^{j-1}]x^{d-j}y^j \\
&= x^d + (-1)^{d-1}y^d.
\end{aligned}$$

Therefore,

$$Q_d(x,y) = \frac{x^d + (-1)^{d-1}y^d}{x+y} = \frac{P_d(x,y)}{S(x,y)}.$$

Observe that Q_d is a full homogeneous polynomial of degree $d-1$. Thus, the statement is proved.

$\square$

The above lemma shows that for any d, there is a homogeneous polynomial P of rank two with full quotient when divided by $S = x+y$. Clearly, a monomial of degree d in two variables is not divisible by $x+y$. Hence, for those polynomials P in the class $\mathcal{H}_{2,d}$ such that $P = SQ$, with $S = x+y$ and Q full, the minimum rank is two and the polynomials P_d defined above are sharp.

1.3 The Three-Variable Case

The three-variable case is interesting and non-trivial, and the minimum rank and sharp polynomials are well-understood. As we will see in Chapter 2, this problem has its roots in the study of functions of several complex variables, specifically in the study of proper polynomial mappings between balls. The proof has two different approaches - the graph-theoretic approach ([12],[13],[24]) and the algebraic approach [3]. We are interested in the algebraic approach.

The following theorem is due to Lebl and Peters [24], with an independent proof by Brooks and Grundmeier [3].

Theorem 1.1. *Suppose $R = \mathbb{R}[x,y,z]$ and Q is a full homogeneous polynomial of degree $d-1$ in R. If $S = x+y+z$,*

$$\rho(SQ) \geqslant \frac{d+5}{2},$$

and the inequality is sharp.

A family of sharp polynomials is given below.

For d odd:

$$
\begin{aligned}
F_1(x,y,z) &= x+y+z \\
F_3(x,y,z) &= x^3+y^3+z^3-3xyz \\
F_5(x,y,z) &= x^5+y^5+z^5-5x^3yz+5xy^2z^2 \\
F_7(x,y,z) &= x^7+y^7+z^7-7x^5yz+14x^3y^2z^2-7xy^3z^3 \\
F_9(x,y,z) &= x^9+y^9+z^9-9x^7yz+27x^5y^2z^2-30x^3y^3z^3+9xy^4z^4 \\
F_{11}(x,y,z) &= x^{11}+y^{11}+z^{11}-11x^9yz+44x^7y^2z^2-77x^5y^3z^3+55x^3y^4z^4-11xy^5z^5 \\
F_{13}(x,y,z) &= x^{13}+y^{13}+z^{13}-13x^{11}yz+65x^9y^2z^2-156x^7y^3z^3+182x^5y^4z^4 \\
&\quad -91x^3y^5z^5+13xy^6z^6 \\
&\ \ \vdots
\end{aligned}
$$

For d even:

$$
\begin{aligned}
F_2(x,y,z) &= x^2-y^2-z^2-2yz \\
F_4(x,y,z) &= x^4-y^4-z^4-4x^2yz+2y^2z^2 \\
F_6(x,y,z) &= x^6-y^6-z^6-6x^4yz+9x^2y^2z^2-2y^3z^3 \\
F_8(x,y,z) &= x^8-y^8-z^8-8x^6yz+20x^4y^2z^2-16x^2y^3z^3+2y^4z^4 \\
F_{10}(x,y,z) &= x^{10}-y^{10}-z^{10}-10x^8yz+35x^6y^2z^2-50x^4y^3z^3+25x^2y^4z^4-2y^5z^5 \\
F_{12}(x,y,z) &= x^{12}-y^{12}-z^{12}-12x^{10}yz+54x^8y^2z^2-112x^6y^3z^3+105x^4y^4z^4 \\
&\quad -36x^2y^5z^5+2y^6z^6 \\
&\ \ \vdots
\end{aligned}
$$

Note that when d is even, $\frac{d+5}{2}$ is not an integer. However, this estimate is essentially sharp in the sense that the minimum rank is the smallest natural number satisfying the inequality from Theorem 1.1. These polynomials are homogenized versions of the polynomials $f_d(x,y)-1$ discovered by D'Angelo and will be discussed in Chapter 2. The polynomials $f_d(x,y)$ have interesting properties. For example, they have integer coefficients which can be expressed as sums of binomial coefficients and $f_d(x,y)=1$ when $x+y=1$.

1.4 Main Results

Building on results already achieved in the three-variable case and generalizing the approach used in the proof of Theorem 1.1, we give a bound for the corresponding polynomials belonging to the class $\mathcal{H}_{4,d}$. The result is the following:

Theorem 1.2. *Let $R = \mathbb{R}[x, y, z, w]$. Let $S = x + y + z + w$, and let $Q \in R$ be a full homogeneous polynomial of degree $d - 1$. Then*

$$\rho(SQ) \geqslant 2d + 2.$$

Moreover, this inequality is sharp.

A family of sharp polynomials is given by

$$P_d(x, y, z, w) = (x + y)^d + (-1)^{d-1}(z + w)^d$$

Next, we state the related result for polynomials in $\mathcal{H}_{n+1,d}$.

Theorem 1.3. *Let $R = \mathbb{R}[x_0, x_1, ..., x_n]$. Let $S = x_0 + x_1 + \cdots + x_n$, and let Q be a full homogeneous polynomial of degree $d - 1$ in R. If $n + 1 \geqslant 4$, then*

$$\rho(SQ) \geqslant n(n + 1)\frac{d - 1}{6} + (n + 1).$$

This estimate is not sharp but is nonetheless nontrivial.

Chapter 2

Relationship with Several Complex Variables

This chapter introduces all the basic theory that motivates our research questions. We define proper mappings in general, and then we focus on proper polynomial mappings between balls in complex Euclidean spaces of different dimensions. We will see that there is a relationship between the domain dimension, the target dimension, and the degree of the proper mapping. This relationship has been studied extensively and there are many results concerning this problem. The development in this chapter about the theory of proper mappings closely follows [6] and [8].

2.1 Proper Mappings Between Topological Spaces

Definition 2.1. A continuous map $f : X \longrightarrow Y$ between topological spaces is **proper** if, for every compact subset K of Y, the subset $f^{-1}(K)$ is compact in X.

Example 2.1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the mapping defined by $f(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$. Note that this map is continuous. Now, the set $\{0\}$ is compact, but $f^{-1}(\{0\}) = \mathbb{R}^2$ is not bounded (and hence non-compact). Therefore, f is not a proper map.

The next example describes how easy it is for a continuous map to be proper if the domain is compact.

Example 2.2. Let $f : X \longrightarrow Y$ be a continuous map between topological spaces. If X is compact and Y is a Hausdorff space, then f is a proper map.

To prove this statement, take a compact set $K \subseteq Y$. Because Y is a Hausdorff space, K is a closed subset of Y. Now, because f is a continuous map, $f^{-1}(K)$ is a closed subset of X. We know X is compact and because closed subsets of compact sets are compact, we conclude that $f^{-1}(K)$ is compact. Therefore, f is a proper map.

The following theorem gives an alternative description of proper maps. We will use this theorem in the next section when we study proper polynomial mappings between balls in complex Euclidean spaces.

Theorem 2.1. *Let X be a bounded domain in $\mathbb{C}^n$ and let Y be a bounded domain in $\mathbb{C}^N$. Let $f : X \longrightarrow Y$ be continuous. Then f is proper if and only if the following condition holds: whenever $\{z_j\}$ tends to the boundary bX, $\{f(z_j)\}$ tends to the boundary bY.*

Proof. We will prove the contrapositive of each statement. Suppose that f is not a proper mapping. Then there is a compact set $K \subseteq Y$ such that $f^{-1}(K)$ is not compact in X. Therefore, we can find a sequence $\{z_j\}$ in $f^{-1}(K)$ such that $z_j \longrightarrow z$ with $z \in bX$. Now, because K is compact and $\{f(z_j)\}$ is a sequence in K, the continuity of f implies that $f(z_j) \longrightarrow f(z)$, where $f(z) \in K$. It follows that if $f : X \longrightarrow Y$ is not proper, we can find a sequence $\{z_j\}$ that tends to the boundary of X for which $\{f(z_j)\}$ does not tend to the boundary of Y.

On the other hand, suppose that there is a sequence $\{z_j\}$ tending to bX such that $\{f(z_j)\}$ does not tend to bY. Then there exists $K \subseteq Y$ compact and a subsequence $\{f(z_{j_k})\}$ in K such that $f(z_{j_k}) \longrightarrow w$ and $w \in K$. Note that $\{z_{j_k}\}$ is a subsequence of $\{z_j\}$ in $f^{-1}(K)$ such that $\{z_{j_k}\}$ tends to bX. Therefore, $f^{-1}(K)$ is not compact. This implies that f is not a proper mapping. $\square$

Corollary 2.1. *Let X be a bounded domain in $\mathbb{C}^n$ and let Y be a bounded domain in $\mathbb{C}^N$. Let $f : \overline{X} \longrightarrow \overline{Y}$ be a continuous map such that $f(X) \subseteq Y$. Then $f|_X : X \longrightarrow Y$ is proper if and only if $f(bX) \subseteq bY$.*

Proof. By Theorem 2.1, $f|_X$ is proper if and only if whenever $\{z_j\}$ tends to the boundary bX, $\{f(z_j)\}$ tends to the boundary bY. Suppose that $f|_X$ is proper. If $z \in bX$, there exists a sequence of points $\{z_j\}$ in X such that $z_j \longrightarrow z$. Because f is continuous, $f(z_j) \longrightarrow f(z)$. Since $\{f(z_j)\}$ tends to the boundary bY, $f(z) \in bY$. Therefore, if $f|_X$ is proper, then $f(bX) \subseteq bY$.

Conversely, suppose that $f(bX) \subseteq bY$. Let K be compact in Y. If $f^{-1}(K)$ is not compact in X, there exists a sequence $\{z_j\}$ in $f^{-1}(K)$ such that $z_j \longrightarrow z$ and $z \in bX$. Observe that $\{f(z_j)\}$ is a sequence of points in K such that $f(z_j) \longrightarrow f(z)$. By hypothesis $f(z) \in bY$, which contradicts the fact that K is compact in Y. Consequently, if $f(bX) \subseteq bY$, then $f|_X$ is proper. $\square$

2.2 Proper Polynomial Mappings between Balls

Let B_n denote the unit ball in $\mathbb{C}^n$ and let B_N denote the unit ball in $\mathbb{C}^N$. We may ask how we determine whether a holomorphic map $f : B_n \longrightarrow B_N$ is proper. Following the idea from the last section, we could say that f is proper if and only if whenever a sequence tends to the boundary bB_n, the image sequence tends to the boundary bB_N.

We are interested in proper (holomorphic) polynomial mappings between balls, so that each component f_i of f is a polynomial. Thus, these mappings are also defined on the boundary sphere. Corollary 2.1 implies that proper polynomial mappings send the boundary to the boundary.

Consequently, a polynomial mapping $f : B_n \longrightarrow B_N$ is proper if and only if it is non-constant and sends the sphere S^{2n-1} to S^{2N-1}, that is,

$$\|f(z)\| = 1 \quad \text{whenever} \quad \|z\| = 1 \quad \text{for all } z \in S^{2n-1},$$

where

$$\|f(z)\|^2 = \sum_{j=1}^{N} |f_j(z)|^2 \quad \text{and} \quad \|z\|^2 = \sum_{j=1}^{n} |z_j|^2.$$

Example 2.3. Consider the polynomial mapping $f : B_2 \longrightarrow B_3$ defined by $f(z_1, z_2) = (z_1^2, \sqrt{2}z_1z_2, z_2^2)$. Clearly f is nonconstant, and if $\|z\| = 1$ (that is, if $|z_1|^2 + |z_2|^2 = 1$), then

$$\begin{aligned}\|f(z_1, z_2)\|^2 &= \|(z_1^2, \sqrt{2}z_1z_2, z_2^2)\|^2 \\ &= |z_1|^4 + 2|z_1z_2|^2 + |z_2|^4 \\ &= (|z_1|^2 + |z_2|^2)^2 \\ &= 1.\end{aligned}$$

Therefore, f is a proper polynomial mapping between B_2 and B_3.

Example 2.4. The map $f : B_2 \longrightarrow B_4$ defined by $f(z_1, z_2) = (z_1^2, \sqrt{2}z_1^2z_2, \sqrt{2}z_1z_2^2, z_2^2)$

is a proper polynomial mapping. To see this, suppose that $|z_1|^2 + |z_2|^2 = 1$. Then

$$\begin{aligned}
\|f(z_1, z_2)\|^2 &= \|(z_1^2, \sqrt{2}z_1^2 z_2, \sqrt{2}z_1 z_2^2, z_2^2)\|^2 \\
&= |z_1|^4 + 2|z_1|^4|z_2|^2 + 2|z_1|^2|z_2|^4 + |z_2|^4 \\
&= |z_1|^4 + 2|z_1 z_2|^2(|z_1|^2 + |z_2|^2) + |z_2|^4 \\
&= |z_1|^4 + 2|z_1 z_2|^2 + |z_2|^4 \\
&= (|z_1|^2 + |z_2|^2)^2 \\
&= 1.
\end{aligned}$$

Suppose that $f : B_n \longrightarrow B_N$ is a proper holomorphic mapping. Let us classify these maps for $n = N$, $n > N$, and $n < N$. First, assume that $n = 1 = N$. We have the following classical result. For a proof, see, for instance [8].

Theorem 2.2. *Let $f : B_1 \longrightarrow B_1$ be a proper holomorphic mapping. Then f is a finite Blaschke product. That is, there are points $a_1, a_2, ..., a_d \in B_1$ with positive integer multiplicities m_j, and a point $e^{i\theta}$ such that*

$$f(z) = e^{i\theta} \prod_{j=1}^{d} \left(\frac{a_j - z}{1 - \bar{a}_j z} \right)^{m_j}.$$

Furthermore, if $f^{-1}(\{0\}) = \{0\}$ or f is a polynomial, then $f(z) = e^{i\theta} z^m$ for some positive integer m.

Therefore, in the one complex variable setting, proper rational mappings from the disk to the disk can be of arbitrarily large degree, but, as we are about to see, the situation is rather different in several complex variables.

Suppose next that $n = N$ and $n \geqslant 2$. In 1977, Alexander [1] proved that such proper holomorphic mappings are automorphisms. It turns out that these mappings are equivalent to the individual factors that appear in the Blaschke product above. Rudin [26] gave an explicit description of these automorphisms. Thus, given a proper holomorphic mapping $f : B_n \longrightarrow B_n$ $(n \geqslant 2)$ we have that

$$f(z) = U \frac{w - L_w z}{1 - \langle z, w \rangle},$$

where $w \in B_n$, U is a unitary matrix, and L_w is the linear map

$$L_w(z) = sz + \frac{\langle z, w \rangle}{s+1} w \qquad \text{with} \qquad s^2 = 1 - \|w\|^2.$$

Let us consider next the situation when $n \neq N$. The next proposition states that there are no non-constant proper holomorphic mappings from B_n to B_N when $n > N$. For the proof, see [6].

Proposition 2.1. *Let $f : B_n \longrightarrow B_N$ be a non-constant proper holomorphic mapping. Then necessarily $N \geqslant n$.*

In light of this result, we focus on $n < N$. In order to describe results in this case, we need the next definition.

Definition 2.2. Let $f : B_n \longrightarrow B_N$ and $g : B_n \longrightarrow B_N$ be proper polynomial mappings. f and g are said to be **spherically equivalent** if there are automorphisms φ and ψ of the domain and target balls such that $f\varphi = \psi g$.

Faran [16] proved that if $f : B_2 \longrightarrow B_3$ is a proper polynomial mapping, then f is spherically equivalent to one of the following four maps:

$$\begin{aligned}
(z, w) &\mapsto (z^3, w^3, \sqrt{3}zw) \\
(z, w) &\mapsto (z, zw, w^2) \\
(z, w) &\mapsto (z^2, \sqrt{2}zw, w^2) \\
(z, w) &\mapsto (z, w, 0)
\end{aligned}$$

Note that these proper polynomial mappings are monomial mappings of degree at most 3.

Later, D'Angelo [11] proved that if $f : B_2 \longrightarrow B_4$ is a proper monomial map, then f is spherically equivalent to one of the fifteen maps listed below:

$$
\begin{aligned}
&(z,w) \mapsto (z,w,0,0)\\
&(z,w) \mapsto (z^2, zw, w, 0)\\
&(z,w) \mapsto (z^2, \sqrt{2}zw, w^2, 0)\\
&(z,w) \mapsto (z^3, \sqrt{3}zw, w^3, 0)\\
&(z,w) \mapsto (z^3, \sqrt{3}z^2w, \sqrt{3}zw^2, w^3)\\
&(z,w) \mapsto (z^3, z^2w, zw, w)\\
&(z,w) \mapsto (z^3, z^2w, zw^2, w)\\
&(z,w) \mapsto (z^2, \sqrt{2}z^2w, \sqrt{2}zw^2, w^2)\\
&(z,w) \mapsto (z^3, \sqrt{3}z^2w, \sqrt{2}zw^2, w^2)\\
&(z,w) \mapsto (z, z^2w, \sqrt{2}zw^2, w^3)\\
&(z,w) \mapsto (z^4, z^3w, \sqrt{3}zw, w^3)\\
&(z,w) \mapsto (z^4, \sqrt{3}z^2w, zw^3, w)\\
&(z,w) \mapsto (z^5, \sqrt{5}z^3w, \sqrt{5}zw^2, w^5)\\
&(z,w) \mapsto (z, \cos(\theta)w, \sin(\theta)zw, \sin(\theta)w^2)\\
&(z,w) \mapsto (z^2, \sqrt{(1+\cos^2(\theta))}zw, \cos(\theta)w^2, \sin(\theta)w).
\end{aligned}
$$

Thus the maximum degree possible is five. The classification of proper holomorphic mappings has been the subject of research for a long time. This endeavor has connections with CR geometry, representation theory of unitary finite groups, graph theory, and of course commutative algebra, as we will see later.

2.3 The Degree Estimates Problem

As we have seen, there is a relationship between the domain and target dimensions of a proper holomorphic mapping between balls. Now, is there any connection between the degree of such a mapping and the dimension of the domain and the target? The answer is yes, and this is called a *degree estimates problem*.

Specifically, let $p : B_n \longrightarrow B_N$ be a proper rational map of degree d. A degree estimates problem refers to the problem of bounding the degree d of the proper rational map p in terms of some function of n and N. Thus, the aim is to find a function $u(n, N)$ such that $d \leqslant u(n, N)$. Recall that if $n > N$, proper holomorphic mappings from B_n to B_N are constants and so of degree zero. In 1982, Faran [16] proved that for $n \leqslant N \leqslant 2n-2$, every proper rational map from B_n to B_N is of degree one; moreover, if $N \leqslant 3$, every proper rational map from B_2 to B_N is of degree at most three. Seven years later, Forstneric [17] showed that a function, in terms of n and N, bounding the degree of the map p exists.

Concerning proper monomial mappings, D'Angelo [12] conjectured that the best possible bound for the degree of the map $p : B_n \longrightarrow B_N$ is $d \leqslant 2N-3$ if $n=2$ and $d \leqslant \frac{N-1}{n-1}$ if $n \geqslant 3$. D'Angelo, Kos and Riehl [12] proved that for $n=2$ this statement holds and is sharp.

In 2006, Meylan [25] proved a general statement about proper rational holomorphic mapppings. She showed that if $p : B_2 \longrightarrow B_N$ is a proper holomorphic rational map of degree d, then

$$d \leqslant \frac{n(n-1)}{2}.$$

The same year, Huang, Ji, and Xu [21] proved that if $4 \leqslant n \leqslant N \leqslant 3n-4$, then the degree of p is at most two.

One year later, D'Angelo, Lebl, and Peters [13] stated an estimate about proper monomial mappings. They proved that if $p : B_n \longrightarrow B_N$ is a proper monomial mapping and $n \geqslant 2$, then

$$d \leqslant \frac{2n(2N-3))}{3n^2-3n-2} \leqslant \frac{4(2N-3)}{3(2n-3)}.$$

Furthermore, if n is sufficiently large compared with d, $d \leqslant \frac{N-1}{n-1}$. The sharp degree estimate theorem for proper monomial mappings is known:

Theorem 2.3 (D'Angelo, Kos, and Riehl [12]; Lebl and Peters [24])**.** *Let $p : B_n \longrightarrow B_N$ be a proper monomial mapping of degree d. Then*

$$d \leqslant \begin{cases} 2N-3 & n=2 \\ \frac{N-1}{n-1} & n>2, \end{cases}$$

and the inequality is sharp.

For $n=2$, the inequality is sharp in the sense that the for a fixed d, the minimum target dimension N is the smallest natural number satisfying the inequality. For $n=3$, the inequality is sharp [24]. For $n \geqslant 4$, the inequality above is sharp [23] and sharp polynomials are called Whitney polynomials. Furthermore, it was shown that if $n \geqslant 4$, these polynomials are the only polynomials for which $d = \frac{N-1}{n-1}$.

2.4 Reduction to Homogeneous Polynomials

Let us restrict our attention to proper monomial mappings. In this section, we will explain carefully how we are led to the study of polynomials in n real variables with nonnegative coefficients taking the value of one when $x_1 + x_2 + \cdots + x_n = 1$. Then we will use a standard homogenization process to instead study a family of homogeneous polynomials in $n+1$ real variables that vanish when $x_0 + x_1 + \cdots + x_n = 0$. This procedure was already introduced in Chapter 1 but only for polynomials in three real variables.

Let $r : B_n \longrightarrow B_N$ be a proper polynomial mapping such that all components are monomials. Thus, each component of r has the form $a_\alpha z^\alpha$, where $z = (z_1, ..., z_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$, and each $\alpha_i \in \mathbb{N} \cup \{0\}$. By definition of proper monomial mapping, we must have that

$$\|r(z)\|^2 = \sum |a_\alpha|^2 \prod_{j=1}^{n} |z_j|^{2\alpha_j} = 1 \quad \text{whenever} \quad \|z\|^2 = \sum_{j=1}^{n} |z_j|^2 = 1.$$

Let us replace $(|z_1|^2, ..., |z_n|^2)$ with $(x_1, ..., x_n)$ and $|a_\alpha|^2$ with c_α, and consider the polynomial $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $p(x) = \sum c_\alpha x^\alpha$, where $x = (x_1, ..., x_n)$. Here we are using multi-index notation. Consequently,

$$p(x) = \sum c_\alpha x^\alpha = 1 \;\; \text{whenever} \;\; \sum_{j=1}^{n} x_j = 1. \tag{2.1}$$

Now, we will reformulate Theorem 2.3, so that instead of considering proper monomial mappings we will consider a family of real polynomials. Let $R = \mathbb{R}[x_1, ..., x_n]$. Let $\mathcal{P}(n)$ be the class of polynomials in R with non-negative coefficients taking the value 1 when $x_1 + x_2 + \cdots + x_n = 1$. The set $\mathcal{P}(n)$ is closed under multiplication and convex combinations. For $p \in \mathcal{P}(n)$, let N denote the number of distinct monomials in p. Then Theorem 2.3 can be reformulated as follows:

Theorem 2.4 (D'Angelo, Kos, and Riehl [12]; Lebl and Peters [24]). *Let $p \in \mathcal{P}(n)$ of degree d. Then*

$$d \leqslant \begin{cases} 2N-3 & n=2 \\ \frac{N-1}{n-1} & n>2, \end{cases}$$

and the inequality is sharp. For $n = 2$, equality only holds for odd degree d.

Consider the polynomial p as defined in (2.1). Let us homogenize the polynomial $p - 1$ with x_0. We obtain the polynomial

$$\tilde{p}(x) = (x_0)^d \; [p(\tfrac{x_1}{x_0}, ..., \tfrac{x_n}{x_0}) - 1].$$

Next, we replace x_0 with $-x_0$ to get the polynomial

$$P(x_0, x_1, ..., x_n) = (-x_0)^d \left[p(\tfrac{x_1}{-x_0}, ..., \tfrac{x_n}{-x_0}) - 1\right].$$

It follows that $P(x_0, x_1, ..., x_n) = 0$ provided that $\frac{x_1}{-x_0} + \cdots + \frac{x_n}{-x_0} = 1$, and this implies that $P(x_0, x_1, ..., x_n) = 0$ whenever $x_0 + x_1 + \cdots + x_n = 0$. Therefore, instead of studying polynomials of n real variables with nonnegative coefficients as above, we study homogeneous polynomials in $n+1$ real variables that vanish when $\sum_{j=0}^{n} x_j = 0$.

Let $\widehat{\mathcal{P}}(n)$ denote the set of polynomials $P \in \mathbb{R}[x_0, ..., x_n]$ such that $P(x_0, ..., x_n) = 0$ on $x_0 + x_1 + \cdots + x_n = 0$. We finish this section with two important definitions about the class of polynomials $\widehat{\mathcal{P}}(n)$.

Definition 2.3. Let $P \in \widehat{\mathcal{P}}(n)$. We say P has p**-degree** d if d is the smallest integer such that there exists a monomial x^α (where α is a multi-index and $x = (x_0, ..., x_n)$) and a polynomial $R(x)$ of degree d so that $P(x) = x^\alpha R(x)$.

Observe that if the monomials in P have no common divisor, then we take $\alpha = (0, 0, ..., 0)$ and the p-degree of P is equal to the degree of P.

Definition 2.4. The polynomial $P \in \widehat{\mathcal{P}}(n)$ is **indecomposable** if P cannot be written as the sum of two non-trivial polynomials P_1 and P_2 in $\widehat{\mathcal{P}}(n)$ with no monomials in common.

2.5 A Closer Look at the Case n=2

As we saw above, D'Angelo, Kos, and Riehl [12] proved that if $p \in \mathcal{P}(2)$, the number of distinct monomials in p, N, is related to the degree d of the mapping as follows:

$$d \leqslant 2N - 3,$$

and equality is only possible for odd degree d. Their proof is quite difficult and it uses a graph-theoretic approach. A family of sharp polynomials is given by the following recurrence formula.

Let $r_0(x, y) = x$, $r_1(x, y) = x^3 + 3xy$, and for $k \geqslant 0$, let

$$r_{k+2}(x,y) = (x^2+2y)r_{k+1}(x,y) - y^2 r_k(x,y).$$

Finally, set

$$f_{2k+1}(x,y) = r_k(x,y) + y^{2k+1}. \tag{2.2}$$

Therefore,

$$\begin{aligned}
f_1(x,y) &= x+y \\
f_3(x,y) &= x^3+3xy+y^3 \\
f_5(x,y) &= x^5+5x^3y+5xy^2+y^5 \\
f_7(x,y) &= x^7+7x^5y+14x^3y^2+7xy^3+y^7 \\
f_9(x,y) &= x^9+9x^7y+27x^5y^2+30x^3y^3+9xy^4+y^9 \\
&\vdots
\end{aligned}$$

Set $\mathcal{C} = \{f_d : d \text{ is a positive odd integer }\}$. This family of polynomials has interesting group-invariant, combinatorial, and number-theoretic properties (see [8],[9]). They appear for the first time in [10]. D'Angelo discovered that these polynomials are invariant under certain finite unitary groups. The polynomials f_d must have at least $\frac{d-1}{2}$ mixed terms and at least two pure terms. Furthermore, Lebl and Lichtblau [22] addressed the question of whether, for each d, f_d is the unique sharp polynomial. In general, there is not a unique sharp polynomial of each degree. The main properties of these polynomials are compiled in the following proposition. For a proof, see for example [12].

Proposition 2.2 (D'Angelo [12])**.** *Let f_d be defined as in (2.2). Then f_d is the unique polynomial satisfying:*

1. *$f_d(0,0) = 0$.*
2. *$f_d(x,y) = 1$ when $x+y=1$.*
3. *f_d has degree d.*
4. *Each non-zero coefficient of f_d is a positive integer.*
5. *$f_d(\omega x, \omega^2 y) = f_d(x,y)$ for ω a primitive d-th root of unity.*

Furthermore, f_d has the interesting property that $f_d(x,y) \cong x^d + y^d$ if and only if d is prime.

The real polynomials f_d correspond to a certain class of group invariant monomial mappings between balls $\phi_d : B_2 \longrightarrow B_N$, where N is the smallest possible target dimension. The squared norm of these maps is given by

$$\|\phi_d(z_1,z_2)\|^2 = (|z_1|^2)^d + (|z_2|^2)^d + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} K_{d,s}(|z_1|^2)^{d-2s}(|z_2|^2)^s, \tag{2.3}$$

where the coefficients $K_{d,s}$ are defined as follows.

$$K_{d,s} \;=\; \binom{d-s}{s} + \binom{d-s-1}{s-1} \quad \text{for} \quad 1 \leqslant s \leqslant \lfloor \tfrac{d}{2} \rfloor,$$

and $K_{d,0} = 0$. After replacing $(|z_1|, |z_2|)$ with (x, y), we obtain the family of real polynomials f_d. Even though the even degree polynomials do not correspond to proper maps between balls, we modify (2.3) to define a family of polynomials $\{f_d\}$ for both even and odd degree. We define

$$f_d(x, y) \;=\; x^d - (-y)^d + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} K_{d,s}\; x^{d-2s} y^s. \tag{2.4}$$

Therefore,

$$\begin{aligned}
f_1(x, y) &= x + y \\
f_2(x, y) &= x^2 - y^2 + 2y \\
f_3(x, y) &= x^3 + 3xy + y^3 \\
f_4(x, y) &= x^4 - y^4 + 4x^2y + 2y^2 \\
f_5(x, y) &= x^5 + 5x^3y + 5xy^2 + y^5 \\
f_6(x, y) &= x^6 - y^6 + 6x^4y + 9x^2y^2 + 2y^3 \\
&\vdots
\end{aligned}$$

These polynomials generate an interesting family of homogeneous polynomials in three variables whose quotient with $x + y + z$ is a homogeneous polynomial of degree $d - 1$. To obtain this new set of polynomials, Lebl and Peters [24] implemented the homogenization process explained in the previous section. For this purpose, we take a polynomial $f_d \in \mathcal{C}$ of degree d and homogenize $f_d - 1$ with z. Lastly, we replace z with $-z$. Thus, we obtain a homogeneous polynomial $F_d(x, y, z)$ of degree d such that $F_d(x, y, z) = (x + y + z)Q(x, y, z)$, where Q is a homogeneous polynomial of degree $d - 1$. Furthermore, $F_d(x, y, -1) + 1 = f_d(x, y)$.

Definition 2.5. Let $\mathcal{I} = \{F_d(x, y, z) : \; F_d(x, y, -1) = f_d(x, y) - 1 \text{ and } f_d(x, y) \in \mathcal{C}\}$.

Next, we give an explicit formula for the polynomials F_d.

$$\begin{aligned}
F_1(x, y, z) &= x + y + z \\
F_3(x, y, z) &= x^3 + y^3 + z^3 - 3xyz \\
F_5(x, y, z) &= x^5 + y^5 + z^5 - 5x^3yz + 5xy^2z^2 \\
F_7(x, y, z) &= x^7 + y^7 + z^7 - 7x^5yz + 14x^3y^2z^2 - 7xy^3z^3 \\
&\vdots
\end{aligned}$$

By (2.4), we have that

$$F_d(x,y,z) \;=\; x^d - (-y)^d - (-z)^d + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} (-1)^s K_{d,s}\; x^{d-2s} y^s z^s. \tag{2.5}$$

These polynomials are precisely the sharp polynomials defined in Theorem 1.1. Because these polynomials belong to $\mathcal{I}$, it is expected they have many interesting properties. For instance, $F_d(x,y,z) = x^d + y^d + z^d$ in $\mathbb{Z}_d[x,y,z]$ if and only if d is prime.

As Brooks [2] points out, the fact that F_d is divisible by $x+y+z$ and the quotient is a full polynomial seems to be a known result; however, there is not any proof of this in the literature. Brooks proved that the quotient is defined by

$$\begin{aligned} Q(x,y,z) = {} & \sum_{j=1}^{d-1} (-1)^j \sum_{s=0}^{\min\{d-1-j,j-1\}} \binom{d-1-j}{s} x^{d-1-s-j} [z^j y^s + y^j z^s] \\ & + \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^j \binom{d-1-j}{j} x^{d-1-2j} y^j z^j. \end{aligned}$$

This discussion shows how the original research question stated in Chapter 1 is connected with long-standing open questions in several complex variables. Lebl and Peters showed that the number of monomials in P, $\rho(P)$, satisfies $\rho(P) \geqslant \frac{d+5}{2}$. However, they imposed certain conditions on the polynomial P. This is because they were pursuing the degree estimates problem discussed in Section 2.3. The first condition is that the polynomial P must be indecomposable. The second hypothesis is that P must have p-degree d. Now, we will see that the hypotheses of indecompisibility and p-degree d are necessary.

The first example shows that the hypothesis of indecomposable is necessary.

Example 2.5. Let $P = x^d + x^{d-1}y + x^{d-1}z + xy^{d-1} + y^d + y^{d-1}z + xz^{d-1} + yz^{d-1} + z^d$, where d is an arbitrary positive integer. Hence, $P = SQ$, where $S = x+y+z$ and $Q = x^{d-1} + y^{d-1} + z^{d-1}$ and so $P \in \widehat{\mathcal{P}}(2)$. Moreover, $P = P_1 + P_2 + P_3$, where $P_1 = (x+y+z)x^{d-1}$, $P_2 = (x+y+z)y^{d-1}$, and $P_3 = (x+y+z)z^{d-1}$. This implies that $P_1, P_2, P_3 \in \widehat{\mathcal{P}}(2)$. Note that $\rho(P) = 9$, but the degree d can be arbitrarily large. Therefore, no degree estimate is possible.

Our next example shows why the concept of p-degree is important.

Example 2.6. Let d be an arbitrary positive integer. Let $P = x^{d+1} + x^d y + x^d z = x^d(x+y+z)$. Then P is in the class $\widehat{\mathcal{P}}(2)$. Furthermore, P has p-degree 1. However, $\rho(P) = 3$. Hence, no degree estimate is possible.

Next, we state Lebl and Peters' theorem.

Theorem 2.5 (Lebl and Peters [24]). *Let $P(x,y,z)$ be a homogeneous polynomial of degree d such that $P(x,y,z) = (x+y+z)Q(x,y,z)$, where Q is a homogeneous polynomial of degree $d-1$. Suppose that P is indecomposable and of p-degree d. Then*

$$\rho(P) \geqslant \frac{d+5}{2},$$

and the inequality is sharp.

Lebl and Peters proved this rank estimate by associating a graph with the quotient polynomial Q. This graph is called a Newton diagram. We discuss this in more detail in Section 4.2.

Brooks and Grundmeier ([3]), looking for a different and natural approach to prove the degree estimates problem, provided an elegant proof of Theorem 2.5 using a commutative algebra approach. It used numerical invariants called graded Betti numbers of some ideals associated with the polynomial Q to state a general result for the number of monomials $\rho(P)$. They mention two main advantages of this new approach. First, the Newton diagram of P is not easy to visualize when the number of variables is greater than three. Second, it is not possible to address the degree estimates problem for general polynomial mappings using the graph-theoretic approach; however graded Betti numbers are defined in general for any homogeneous ideal. Thus, this new proof seems to be very useful for the task of generalizing this result.

Chapter 3

Commutative Algebra Framework

In Chapter 1, we mentioned an algebraic approach led by Brooks and Grundmeier [3] to give an alternative proof of the degree estimates theorem for proper monomial mappings. This chapter introduces the concepts from commutative algebra that will be used in the remainder of this document. The development in this chapter closely follows [15], [5] and [14].

3.1 Graded Rings and Modules

Definition 3.1. A ring R is called **graded** if there exists a family of subgroups R_k of R such that $R = \oplus_{k\in\mathbb{Z}} R_k$ and $R_k \cdot R_m \subseteq R_{k+m}$. The elements in R_k are called homogeneous elements of degree k in the grading.

Consider the polynomial ring $R = \mathbb{R}[x_0, x_1, ..., x_n]$. We will use multi-index notation. Thus we write the monomial $x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as x^α, where $x = (x_0, x_1, ..., x_n)$, $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$, and each α_j is a non-negative integer. The degree of the monomial x^α is $\sum \alpha_j$ and is denoted by $|\alpha|$. For example, $x_0^2 x_1^2 x_2 x_4^5$ is a monomial of degree 10 in the variables x_0, x_1, x_2, x_3, x_4 associated with the multi-index $(2, 2, 1, 0, 5)$.

The polynomial ring R is a graded ring (graded by degree) because

$$R = \bigoplus\nolimits_{k\geqslant 0} R_k,$$

where

$$\begin{aligned} R_k &= \operatorname{span}\{x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ : \ \alpha_0 + \alpha_1 + \cdots + \alpha_n = k\} \\ &= \operatorname{span}\{x^\alpha : |\alpha| = k\}. \end{aligned}$$

Given a collection of polynomials $h_1, h_2, ..., h_J$ in R, we denote by $\langle h_1, ..., h_J \rangle$ the smallest ideal in R containing $h_1, ..., h_J$. That is,

$$\langle h_1, ..., h_J \rangle = \{c_1 h_1 + \cdots + c_J h_J \ : \ c_i \in R, \quad \text{for } 1 \leqslant i \leqslant J\}.$$

Recall that a polynomial h is said to be **homogeneous** if all its monomials with non-zero coefficient have the same degree.

Definition 3.2. The ideal $I = \langle h_1, ..., h_J \rangle \subseteq R$ is a **homogeneous ideal** if each h_j is a homogeneous polynomial.

Example 3.1. The ideal $I_1 = \langle x_0^4 + x_1^2 x_2^2,\ x_2^2 + x_3 x_4,\ x_0^3 + x_3 x_4^2 \rangle$ is a homogeneous ideal, whereas $I_2 = \langle x_0^2 + x_2, x_2^3 \rangle$ is not homogeneous.

In this research, we will work with monomial ideals, that is, ideals generated by monomials in the polynomial ring R. Hence, all the ideals we work with are homogeneous.

Definition 3.3. Let R be a ring and let M be an abelian group together with an operation of R on M called scalar multiplication. We say that M is an R**-module** (technically a left R-module) if, for all $a, b, c \in R$ and $f, g, h \in M$, the following properties hold:

(i) $a(g + h) = ag + bh$.

(ii) $(b + c)f = bf + cf$.

(iii) $(bc)f = a(bf)$.

(iv) If 1 is the multiplicative identity in R, $1f = f$.

The most natural examples of modules are the vector spaces. If V is a vector space over a field F, then V is an abelian group under addition of vectors. Furthermore, the multiplication of a vector in V by a scalar (scalar multiplication) is well-defined. Hence, V is an F-module.

Observe that an ideal I in the ring R is an R-module. In general, any subset of a module M that is an R-module under the operations induced from M is said to be a submodule of M. Therefore, a nonempty subset of M is a submodule if it is closed under addition and closed under multiplication by elements of R.

Next, we generalize the concept of group homomorphism to the case of R-modules.

Definition 3.4. Let M and N be two R-modules. The map $\varphi : M \longrightarrow N$ is an R**-homomorphism** if, for all $a \in R$ and $f, g \in M$,

$$\varphi(f + g) = \varphi(f) + \varphi(g) \quad \text{and} \quad \varphi(af) = a\varphi(f).$$

We define the kernel of φ, denoted by $\ker(\varphi)$, to be the set:

$$\ker(\varphi) = \{f \in M : \varphi(f) = 0\},$$

and the image of φ, denoted by $\text{im}(\varphi)$, to be the set

$$\text{im}(\varphi) = \{g \in N : \exists\ f \in M \text{ with } \varphi(f) = g\}.$$

Some of the important properties of an R-homomorphism $\varphi : M \longrightarrow N$ are summarized below:

(i) $\ker(\varphi)$ is a submodule of M.

(ii) $\text{im}(\varphi)$ is a submodule of N.

(iii) $\ker(\varphi) = \{0\}$ if and only if φ is injective.

(iv) $\varphi(0) = 0$.

Recall, φ is said to be an **isomorphism** if it is both injective and surjective. The two R-modules M and N are called isomorphic and we write $M \cong N$.

Given a vector space V, there exists a set of elements B such that every element in V can be written in a unique way as a finite linear combination of elements of B. The set B is called a *basis* for the vector space V. Although we can not assure the existence of a basis for a module over an arbitrary ring, the modules that do have a basis are very important in the theory of rings and modules.

Definition 3.5. Let M be an R-module. M is called a **free module** if there exists a subset $X \subseteq M$ such that each element $f \in M$ can be expressed uniquely as a finite sum $f = a_1x_1 + \cdots + a_kx_k$, where $a_1, ..., a_k \in R$ and $x_1, ..., x_k \in X$. In other words, M is a free module if M has a basis.

Example 3.2. Let R be a ring. The polynomial ring $R[x]$ is a free module with a possible basis $\{1, x, x^2, ...\}$.

Example 3.3. Let R be a ring. Then the matrix ring $M_{mn}(R)$ is a free R-module with basis $e_{i,j}$, $i = 1, ..., m$, $j = 1, ..., n$, where $e_{i,j}$ is the matrix which has 1 at the position (i, j) and zeros elsewhere.

Example 3.4. Given an R-module M and an element $f \in M$, the set $Rf = \{rf : r \in R\}$ is called the cyclic submodule generated by f. In general, let X be a subset of M. Define $\langle X \rangle$ to be the collection of all $f \in M$ such that $f = a_1x_1 + \cdots + a_kx_k$, where $a_j \in R$ and $x_j \in X$ for all j. Thus, $\langle X \rangle$ is the smallest submodule of M containing X, and we call it the submodule generated by X. If the set X is finite, M is said to be *finitely generated.*

Definition 3.6. Let M and N be two R-modules. The direct sum $M \oplus N$ is defined as follows:

$$M \oplus N = \{(f, g) : f \in M \text{ and } g \in N\}.$$

It is an immediate consequence that $M \oplus N$ is an R-module under the component-wise sum and scalar multiplication operations. More generally, we can consider the direct sum of finitely many R-modules $M_1, ..., M_k$, denoted $\oplus_{j=1}^{k} M_j$. If each M_j is isomorphic to M, we write $\oplus_{j=1}^{k} M_j = M^k$.

Observe that if the ring R is viewed as an R-module, then R^k, the direct sum $R^k = R \oplus \cdots \oplus R$ of R with itself k times, is a free module with basis $\varepsilon_1 = (1, 0, 0, ..., 0)$, $\varepsilon_2 = (0, 1, 0, ..., 0)$,..., $\varepsilon_k = (0, 0, 0, ..., 1)$.

The definition of graded module is exactly analogue to the definition of graded ring. The most basic examples of graded modules are the homogeneous ideals. In fact, $R = \oplus_k R_k$, and so if we let $I_k = I \cap R_k$, then $I = \oplus_k I_k$. From now on, R will denote a ring with multiplicative identity element.

3.2 Hilbert Functions

In this section, we will study Hilbert functions. These functions are used to measure the growth of the dimension of the homogeneous components of a graded ring or module.

Definition 3.7. Let $M = \oplus_{k \in \mathbb{Z}} M_k$ be a finitely-generated graded module over the polynomial ring $R = \mathbb{R}[x_0, x_1, ..., x_n]$. The **Hilbert function** of M is the map $H_M : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by:

$$H_M(k) = \dim_{\mathbb{R}}(M_k).$$

Example 3.5. Consider $R = \mathbb{R}[x_0, ..., x_n]$ to be a graded R-module over itself. Recall $R = \oplus_{m \in \mathbb{Z}} R_m$, where R_m is the vector space of all homogeneous polynomials of degree m in $n+1$ variables with coefficients in $\mathbb{R}$, together with the zero polynomial. The standard basis for this vector space consists of the set $B = \{x^\alpha \ : \ |\alpha| = m\}$. So the dimension of this space is equal to the number of monomials in B.

The problem of counting monomials can be reduced to the problem of counting balls in boxes. Hence, let each variable be represented by a box and let the degree of each variable be the number of balls in its respective box. For example, $x_0^2 x_1 x_2^3$ (6 balls into 3 boxes) is represented by

$$| \bullet\bullet | \bullet | \bullet\bullet\bullet |.$$

Thus, counting the number of monomials in $R = \mathbb{R}[x_0, ..., x_n]$ of degree m is equivalent to counting the number of ways to place m balls into $n+1$ boxes. Observe that there are in total $n+1+m+1$ symbols, but the first and the last symbol are fixed. Then out of the remaining $n+m$ symbols, we must choose m to be dots. Therefore the number of monomials in R of degree m is $\binom{n+m}{m} = \binom{m+n}{n}$. It follows that for $m \geqslant 0$,

$$H_R(m) = \dim_{\mathbb{R}}(R_m) = \binom{m+n}{n}.$$

Note also that if $m < 0$, $\binom{m+n}{n} = 0$. Thus, the above formula holds for all $m \in \mathbb{Z}$.

Example 3.6. Consider the homogeneous ideal $I = \langle x^3, x^2y, xy^2 \rangle$ of $R = \mathbb{R}[x, y, z]$. Let us find the first few values for the Hilbert function of I. Observe that $I = \bigoplus_{m \geqslant 0} I_m$, where

$$I_m = \operatorname{span}\{x^i y^j z^k : i + j + k = m \text{ and } x^i y^j z^k \in I\}.$$

Since no monomial of the ideal I has a total degree 0, 1, or 2, we must have $I_0 = I_1 = I_2 = \{0\}$. Furthermore $I_3 = \operatorname{span}\{x^3, x^2y, xy^2\}$ and hence

$$I_4 = \operatorname{span}\{x^4,\ x^3y,\ x^3z,\ x^2y^2,\ x^2yz,\ xy^3,\ xy^2z\}.$$

Therefore, $H_I(0) = H_I(1) = H_I(2) = 0$, $H_I(3) = 3$, $H_I(4) = 7$. Now,

$$I_5 = \operatorname{span}\{x^5,\ x^4y,\ x^4z,\ x^3y^2,\ x^3yz,\ x^3z^2,\ x^2y^3,\ x^2y^2z,\ x^2yz^2,\ xy^4,$$
$$xy^3z,\ xy^2z^2\}.$$

Then $H_I(5) = \dim_{\mathbb{R}}(I_5) = 12$.

In Example 3.6, all generators of I are of degree 3. In our next example, we consider an ideal in which not all generators are of the same degree.

Example 3.7. Let $R = \mathbb{R}[x, y, z]$. Consider the homogeneous ideal $I = \langle x^2, xy^2, xyz^2 \rangle$. Let us find the Hilbert function values $H_I(m)$ for $0 \leq m \leq 4$.

Since no monomial of the ideal I has total degree 0 or 1, we must have $I_0 = I_1 = \{0\}$. On the other hand,

$$I_2 = \operatorname{span}\{x^2\}, \quad I_3 = \operatorname{span}\{x^3,\ x^2y,\ x^2z,\ xy^2\},$$

and

$$I_4 = \text{span}\{x^4,\ x^3y,\ x^3z,\ x^2y^2,\ x^2yz,\ x^2z^2,\ xy^3,\ xy^2z,\ xyz^2\}.$$

Thus, $H_I(0) = H_I(1) = 0$, $H_I(2) = 1$, $H_I(3) = 4$, $H_I(4) = 9$.

3.3 Syzygies

It is not necessarily true that every R-module is free. To see this, let M be a finite abelian group of order n. Clearly M is a $\mathbb{Z}$-module, and if $x \in M$, then $nx = 0$. Therefore, $\{x\}$ is linearly dependent for every $x \in M$. It follows that M does not have nonempty linearly independent subsets, and hence M is not a free $\mathbb{Z}$-module. This consideration motivates the following definition. From now on, R will denote the polynomial ring $\mathbb{R}[x_0, x_1, ..., x_n]$.

Definition 3.8. Let $M = \langle f_1, ..., f_k \rangle$ be an R-module with generators $f_1, ...f_k$. The k-tuple of elements $(a_1, ..., a_k)$ of R is called a **syzygy** between the generators if $a_1f_1 + \cdots + a_kf_k = 0$.

We agree that a syzygy is a relation given between the elements of the generating set of a finitely-generated R-module. Note that if $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$ are two syzygies and $r \in R$, then:

$$(ra_1 + b_1)f_1 + (ra_2 + b_2)f_2 + \cdots + (ra_k + b_k)f_k = 0.$$

Thus, $(ra_1 + b_1, ..., ra_k + b_k)$ is also a syzygy between the generators. In consequence we have the following proposition.

Proposition 3.1. *Let $M = \langle f_1, ..., f_k \rangle$ be a finitely-generated R-module. The set of all syzygies is an R-submodule of R^k, called the first syzygy module of M.*

Our next aim is to find a set of generators for the first syzygy module of a finitely-generated R-module M.

A monomial m in R^k is an element of the form $x^a\varepsilon_i$, where ε_i is the i-th element of the standard basis of R^k and $a = (a_1, ..., a_k)$. Furthermore, each element in R^k can be written uniquely as a linear combination of monomials m_i. The best way to visualize this definition is through an example. Let $R = \mathbb{R}[x, y]$ and let $V = (3x^2 + 4y, 5 - y^3, 4xy - 7y^8 + 2) \in \mathbb{R}[x, y]^3$. Then,

$$\begin{aligned} V &= 3(x^2, 0, 0) + 4(y, 0, 0) + 5(0, 1, 0) - (0, y^3, 0) + 4(0, 0, xy) - 7(0, 0, y^8) + 2(0, 0, 1) \\ &= 3x^2\varepsilon_1 + 4y\varepsilon_1 + 5\varepsilon_2 - y^3\varepsilon_2 + 4xy\varepsilon_3 - 7y^8\varepsilon_3 + 2\varepsilon_3. \end{aligned}$$

Let $m_1 = x^a \varepsilon_i$ and $m_2 = x^b \varepsilon_j$ be two monomials. We say that the monomial m_1 involves the basis element ε_i and that m_2 involves the basis element ε_j. If $i = j$, the quotient m_1/m_2 is defined by $m_1/m_2 := x^a/x^b = x^{a-b}$. Furthermore, we define the least common multiple and greatest common divisor of m_1 and m_2, denoted $\mathrm{lcm}(m_1, m_2)$ and $\gcd(m_1, m_2)$, respectively, to be the least common multiple and greatest common divisor of x^a and x^b , times ε_i. These definitions are important to get a set of generators for the fisrs syzygy module of a finitely-generated R-module.

3.4 Free Resolutions

As we saw in the last section, not every R-module M is free. However, we use a tool called a free resolution to extract important information about the module M.

Definition 3.9. A sequence of R-modules and homomorphisms of the form

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots,$$

is called **exact** if $\mathrm{im}(\varphi_{i+1}) = \ker(\varphi_i)$ for all i.

Next, we have the following notations. A map $M \xrightarrow{\varphi} 0$ will represent the trivial homomorphism, that is, $\varphi(x) = 0$ for all $x \in M$. A map $0 \xrightarrow{\varphi} N$ will represent the trivial embedding. Let $\varphi : M \longrightarrow N$ be a homomorphism. Recall that $\mathrm{coker}(\varphi) = N/\mathrm{im}(\varphi)$, and so the map $N \xrightarrow{\psi} \mathrm{coker}(\varphi)$ given by $\psi(y) = y + \mathrm{im}(\varphi)$ for all $y \in N$ is the canonical homomorphism onto the quotient module $N/\mathrm{im}(\varphi)$.

Some properties of R-homomorphisms are fully described using exact sequences. For instance,

(a) An R-homomorphism $\varphi : M \longrightarrow N$ is surjective if and only if the sequence $M \xrightarrow{\varphi} N \xrightarrow{\varphi'} 0$ is exact, i.e., if and only if $\mathrm{im}(\varphi) = N = \ker(\varphi')$.

(b) An R-homomorphism $\varphi : M \longrightarrow N$ is injective if and only if the sequence $0 \xrightarrow{\varphi'} M \xrightarrow{\varphi} N$ is exact, i.e., if and only if $\mathrm{im}(\varphi') = \{0\} = \ker(\varphi)$.

(c) An R-homomorphism $\varphi : M \longrightarrow N$ is an isomorphism if and only if the sequence $0 \xrightarrow{\varphi_2} M \xrightarrow{\varphi} N \xrightarrow{\varphi_1} 0$ is exact. Again the proof of this fact is straightforward.

(d) Let $\varphi : M \longrightarrow N$ be an R-homomorphism. Then the sequence

$$0 \xrightarrow{\varphi_4} \ker(\varphi) \xrightarrow{\varphi_3} M \xrightarrow{\varphi} N \xrightarrow{\varphi_2} \operatorname{coker}(\varphi) \xrightarrow{\varphi_1} 0,$$

is exact. Here φ_3 represents the inclusion mapping.

Definition 3.10. Let M be an R-module. A **free resolution** of M is an exact sequence of the form

$$\cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

where each F_i is a free module. If there exists m such that $F_{m+1} = F_{m+2} = \cdots = 0$ and $F_m \neq 0$, the resolution is finite of length m. Thus, the properties of M can be studied by analyzing the structure of a free resolution.

It is an immediate consequence that in the finite free resolution

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

$\ker(\varphi_{m-1})$ is a free module. Indeed, by definition of exact sequence, φ_m is injective and so $\operatorname{im}(\varphi_m) \subseteq F_{m-1}$ is isomorphic to F_m, and hence a free module. Therefore, $\ker(\varphi_{m-1}) = \operatorname{im}(\varphi_m)$ is a free module.

A natural question arises when we are dealing with finite free resolutions: Does every R-module have a finite free resolution?. The answer is affirmative and is addressed in the theorem below. However, a finite free resolution is not necessarily unique.

Theorem 3.1 (Hilbert Syzygy Theorem)**.** *Every finitely-generated R-module has a finite free resolution of length at most n.*

The proof of this theorem uses some tools from algebraic geometry, and we will omit it. For the proof, see for example [5].

Definition 3.11. Let M and N be graded R-modules. A **graded homomorphism** of degree d is a homomorphism $\varphi : M \longrightarrow N$ such that $\varphi(M_k) \subseteq N_{k+d}$ for all $k \in \mathbb{Z}$.

Proposition 3.2. *Let M be a graded R-module, and let d be an integer. Let $M(d)$ denote the shift of M by d, so that $M(d)_k = M_{k+d}$. Then the direct sum*

$$M(d) := \bigoplus_{k \in \mathbb{Z}} M(d)_k$$

is also a graded R-module.

Definition 3.12. Let M be a graded R-module, and let d be an integer. The module $M(d)$, as defined above, is called the d-th *shift* or *twist* of M.

The graded R-module $M(d)$ is isomorphic to M as a module. However, we have changed the graded module M by shifting its grading d steps. Because we are interested in graded homomorphisms that take the grading of one to the grading of the other with a shift of degrees, we are now able to treat these homomorphisms as graded homomorphisms of degree zero.

If R is a graded module and d is any integer, the shifted graded free module $R(d)^n$ has as basis the standard basis ε_j defined above, however each vector ε_j is now considered to be a homogeneous element of degree $-d$.

We also define the Hilbert function of the shifted module $R(d) = \oplus_{m\in\mathbb{Z}} R(d)_m$ by

$$H_{R(d)}(m) = \dim_{\mathbb{R}}(R(d)_m) = \dim_{\mathbb{R}}(R_{m+d}) = \binom{m+d+n}{n}, \quad \text{for all } m \in \mathbb{Z}.$$

Theorem 3.2. [5]. *Let $M = \langle f_1, ..., f_k\rangle$ be a graded R-module. Suppose that the homogeneous element f_j has degree d_j. Then the graded homomorphism*

$$\varphi : R(-d_1) \oplus \cdots \oplus R(-d_k) \longrightarrow M,$$

defined by $\varphi(\varepsilon_j) = f_j$, where ε_j is the generator or standard basis element of degree d_j, has degree zero. Moreover, φ is a surjective map.

Proof. Let $S = R(-d_1) \oplus \cdots \oplus R(-d_k)$. Clearly

$$S_j = R(-d_1)_j \oplus \cdots \oplus R(-d_k)_j = R_{j-d_1} \oplus \cdots \oplus R_{j-d_k}$$

defines a graded module on S. We need to show that $\varphi(S_j) \subseteq M_j$. To see this, let $a = (a_1, ..., a_k) \in S_j$. Then $a = a_1\varepsilon_1 + \cdots + a_k\varepsilon_k$ and by definition of φ, $\varphi(a) = a_1f_1 + \cdots + a_kf_k \in M$. Note that $\deg(a_if_i) = \deg(a_i) + \deg(f_i) = j - d_i + d_i = j$ for all $i = 1, ..., k$. Therefore, $\varphi(a) \in M_j$ and the first assertion is proved.

Let us prove that φ is surjective. Take $g \in M$. Then $g = g_1f_1 + \cdots + g_kf_k$ for some $g_1, ..., g_k \in R$. Consider the element $g' = g_1\varepsilon_1 + \cdots + g_k\varepsilon_k$. Clearly $g' \in R(-d_1) \oplus \cdots \oplus R(-d_k)$ and $\varphi(g') = g$. Since g is arbitrary, it follows that φ is surjective.

□

Definition 3.13. Let M be a graded R-module. A resolution

$$\cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

where each F_m is a free shifted module and each homomorphism φ_m is a graded homomorphism of degree zero is called a **graded resolution** of M.

As expected, we have a version of the Hilbert Syzygy Theorem for finitely-generated graded R-modules:

Theorem 3.3 (Graded Hilbert Syzygy Theorem)**.** *Every finitely-generated graded R-module has a finite graded resolution of length at most n.*

Definition 3.14. Let M be a finitely-generated graded R-module. A graded free solution of M

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

is called **minimal** if and only if for each j, φ_j takes a basis of F_j to a minimal set of generators G for $\varphi_j(F_j)$. That is, no proper subset of G generates the module $\varphi_j(F_j)$.

Now we are able to give a set of generators for the first syzygy module of a finitely-generated R-module. To do this, we have the following proposition whose proof we omit.

Proposition 3.3. *Let $M = \langle m_1, ..., m_k \rangle$ be an R-module generated by the monomials $m_1, ..., m_k$. Thus, each m_i has the form $m_i = x^a$, where $x = (x_0, x_1, ..., x_n)$ and $a = (a_0, a_1, ..., a_n)$. Let $\{\varepsilon(a)\}$ be the set of generators for the free module F_0 in the minimal resolution of M, so $\varphi_0(\varepsilon(a)) = x^a = m_i$. Then the first syzygy module of M is generated by the syzygies*

$$\sigma(a, b) = \frac{x^b}{gcd(x^a, x^b)}\varepsilon(a) - \frac{x^a}{gcd(x^a, x^b)}\varepsilon(b).$$

The elements $\sigma(a, b)$ are called *divided Koszul relations*. Observe that these relations do not in general give a *minimal* set of generatos for the first syzygy module $\ker(\varphi_0)$. Now that we know how to obtain a set of generators for the first syzygy module of a finitely generated R-module whose generators are monomials, we define a second syzygy to be a syzygy between the generators of the fist syzygy module. The collection of all second syzygies is a submodule of R^k called the second syzygy module of M. We need a set of generators for the second syzygy module, then we get the third syzygies and so on.

3.5 Betti Numbers

Definition 3.15. Let

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a graded minimal free resolution of a finitely-generated graded module M. The number of generators of F_i in degree j is called a **graded Betti number** and it is denoted by $\beta_{i,j}$. These numbers are uniquely determined and form a set of invariants of M as a graded R-module. We arrange the Betti numbers in a compact way called a *Betti table*.

Definition 3.16. The graded Betti numbers are organized into a table, called the **Betti table**, where the entry in column i and row j is $\beta_{i,i+j}$:

$j \setminus i$	0	1	$\cdots$	n
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{n,n}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{n,n+1}$
$\vdots$			$\vdots$	

The next theorem relates the graded Betti numbers to the Hilbert function.

Theorem 3.4. *Let M be a graded R-module. Given a graded free resolution of M*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

we have

$$H_M(k) = \sum_{i=0}^{m} (-1)^i H_{F_i}(k).$$

Proof. This proof repeatedly uses two fundamental facts. First, it uses the rank-nullity theorem for vector spaces, which states that if V and W are two vector spaces, V is finite dimensional, and $T : V \longrightarrow W$ is a linear transformation, then $\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \dim(V)$. Second, it uses the assumption that the above sequence is exact, so that $\mathrm{im}(\varphi_{i+1}) = \ker(\varphi_i)$ for all i.

First, we will prove the following result: If $V_0, V_1, ..., V_m$ are finite-dimensional vector spaces over $\mathbb{C}$ and

$$0 \xrightarrow{\varphi_{m+1}} V_m \xrightarrow{\varphi_m} V_{m-1} \cdots \longrightarrow V_1 \xrightarrow{\varphi_1} V_0 \xrightarrow{\varphi_0} 0,$$

is an exact sequence of linear mappings, then

$$\sum_{i=0}^{m} (-1)^i \dim_{\mathbb{R}}(V_i) = 0.$$

We know that

$$\dim_{\mathbb{R}}(V_m) = \dim_{\mathbb{R}}(\mathrm{im}(\varphi_m)) + \dim_{\mathbb{R}}(\ker(\varphi_m)).$$

Now, $\ker(\varphi_m) = \{0\} = \mathrm{im}(\varphi_{m+1})$. Thus, $\dim_{\mathbb{R}}(V_m) = \dim_{\mathbb{R}}(\mathrm{im}(\varphi_m))$. Therefore,

$$\begin{aligned}\dim_{\mathbb{R}}(V_{m-1}) &= \dim_{\mathbb{R}}(\ker(\varphi_{m-1})) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-1}))\\ &= \dim_{\mathbb{R}}(\mathrm{im}(\varphi_m)) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-1}))\\ &= \dim_{\mathbb{R}}(V_m) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-1})).\end{aligned}$$

Consequently,

$$\dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-1})) = \dim_{\mathbb{R}}(V_{m-1}) - \dim_{\mathbb{R}}(V_m).$$

It follows that

$$\begin{aligned}\dim_{\mathbb{R}}(V_{m-2}) &= \dim_{\mathbb{R}}(\ker(\varphi_{m-2})) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-2}))\\ &= \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-1})) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-2}))\\ &= \dim_{\mathbb{R}}(V_{m-1}) - \dim_{\mathbb{R}}(V_m) + \dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-2})).\end{aligned}$$

Hence,

$$\dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-2})) = \dim_{\mathbb{R}}(V_{m-2}) - \dim_{\mathbb{R}}(V_{m-1}) + \dim_{\mathbb{R}}(V_m).$$

Iterating gives

$$\dim_{\mathbb{R}}(\mathrm{im}(\varphi_{m-j})) = \sum_{i=0}^{j}(-1)^i \dim_{\mathbb{R}}(V_{m-j+i}).$$

Thus,

$$0 = \dim_{\mathbb{R}}(\mathrm{im}(\varphi_0)) = \sum_{i=0}^{m}(-1)^i \dim_{\mathbb{R}}(V_i).$$

Now we proceed to prove the theorem. First, we restrict all R-homomorphisms $\varphi_{m+1}, ..., \varphi_0$ to the degree k homogeneous parts of the graded modules. Therefore, we get an exact sequence

$$0 \longrightarrow (F_m)_k \xrightarrow{\varphi_m} (F_{m-1})_k \cdots \longrightarrow (F_1)_k \xrightarrow{\varphi_1} (F_0)_k \xrightarrow{\varphi_0} M_k \longrightarrow 0$$

By the same argument used above, we obtain

$$\dim_{\mathbb{R}} M_k = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{R}}(F_i)_k.$$

Thus,

$$H_M(k) = \sum_{i=0}^{m} (-1)^i H_{F_i}(k).$$

$\square$

Corollary 3.1. *Let M be a graded R-module. If M has a graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

with $F_i = \bigoplus_j R(-d_{i,j})$, then

$$H_M(k) = \sum_{i=0}^{m} (-1)^i \sum_j \binom{k - d_{i,j} + n}{n} \tag{3.1}$$

Moreover, if $B_j = \sum_i (-1)^i \beta_{i,j}$,

$$H_M(k) = \sum_j B_j \binom{k - j + n}{n}$$

Proof. Recall

$$H_{R(-d)}(k) = \dim_{\mathbb{R}}(R(-d)_k) = \dim_{\mathbb{R}}(R_{k-d}) = \tbinom{k-d+n}{n}, \quad \text{for all } k \in \mathbb{Z}.$$

It follows that

$$H_{F_i}(k) = \sum_j H_{R(-d_{i,j})}(k) = \sum_j \binom{k - d_{i,j} + n}{n}.$$

As a result,

$$H_M(k) = \sum_{i=0}^{m} (-1)^i \sum_j \binom{k - d_{i,j} + n}{n}$$

Since the number of generators of F_i of degree j is by definition the graded Betti number $\beta_{i,j}$, we write equation (3.1) as

$$H_M(k) = \sum_j B_j \binom{k - j + n}{n},$$

where $B_j = \sum_{i=0}^{m} (-1)^i \beta_{i,j}$. This completes the proof of the corollary. $\square$

3.6 Examples

Let us consider two examples to illustrate how we can obtain a minimal free resolution and Betti table for a homogeneous ideal. In the first example we analyze the homogeneous ideal $I = \langle x^5, x^4y, x^4z, x^3y^2 \rangle$ in $R = \mathbb{R}[x, y, z]$, whereas the second example considers the quotient R/I, where $I = \langle x^3, x^2y, xyz \rangle$.

Example 1. Let $R = \mathbb{R}[x, y, z]$. We will obtain a minimal free resolution and Betti table of the homogeneous ideal $I = \langle x^5, x^4y, x^4z, x^3y^2 \rangle$.

The ideal I is a homogeneous ideal with four homogeneous generators $h_1 = x^5$, $h_2 = x^4y$, $h_3 = x^4z$, and $h_4 = x^3y^2$. First, we will find the generators of the first syzygy module of I. We have the following nontrivial relationships among the generators of I:

$$\begin{aligned}
&(1)\ \ -y(x^5) + x(x^4y) + 0(x^4z) + 0(x^3y^2) = 0\\
&(2)\ \ -z(x^5) + 0(x^4y) + x(x^4z) + 0(x^3y^2) = 0\\
&(3)\ \ -y^2(x^5) + 0(x^4y) + 0(x^4z) + x^2(x^3y^2) = 0\\
&(4)\ \ 0(x^5) - z(x^4y) + y(x^4z) + 0(x^3y^2) = 0\\
&(5)\ \ 0(x^5) - y(x^4y) + 0(x^4z) + x(x^3y^2) = 0\\
&(6)\ \ 0(x^5) + 0(x^4y) - y^2(x^4z) + xz(x^3y^2) = 0.
\end{aligned}$$

These are the $\binom{4}{2} = 6$ divided Koszul relations among the generators. Let us multiply equation (1) by y and equation (5) by x. Then if we combine the resulting equations, we get equation (3). Similarly if we multiply equation (4) by $-y$ and equation (5) by z an add the resulting equations, we obtain equation (6). Because the four relations $(-y, x, 0, 0), (-z, 0, x, 0), (0, -z, y, 0)$, and $(0, -y, 0, x)$ are independent, they define a minimal set of generators for the first syzygy module of I. To find the second syzygy module of I, note that the only nontrivial relationship among the generators of the first syzygy module is given by

$$z(-y, x, 0, 0) - y(-z, 0, x, 0) + x(0, -z, y, 0) + 0(0, -y, 0, x) = 0.$$

Thus, $(z, -y, x, 0)$ is the generator for the second syzygy module of I.

Let $F_0 = R^4(-5)$ and define $\varphi_0 : F_0 \longrightarrow I$ by $\varphi(\varepsilon_j^0) = h_j$, where ε_j^0 is the generator of the $j-$th summand of F_0. Hence, $\varphi_0(\varepsilon_1^0) = x^5$, $\varphi_0(\varepsilon_2^0) = x^4y$, $\varphi_0(\varepsilon_3^0) = x^4z$, and $\varphi_0(\varepsilon_4^0) = x^3y^2$. Furthermore,

$$\begin{aligned}\ker(\varphi_0) &= \{s \in R^4(-5) \ : \ \varphi_0(s) = 0\} \\ &= \langle -y\varepsilon_1^0 + x\varepsilon_2^0, \ -z\varepsilon_1^0 + x\varepsilon_3^0, \ -z\varepsilon_2^0 + y\varepsilon_3^0, \ -y\varepsilon_2^0 + x\varepsilon_4^0\rangle.\end{aligned}$$

In this way, $\ker(\varphi_0)$ is precisely the first syzygy module of I.

Let $F_1 = R^4(-6)$ with generators $\varepsilon_1^1, \varepsilon_2^1, \varepsilon_3^1, \varepsilon_4^1$. Define the map $\varphi_1 : F_1 \longrightarrow F_0$ by $\varphi_1(\varepsilon_1^1) = -y\varepsilon_1^0 + x\varepsilon_2^0$, $\varphi_1(\varepsilon_2^1) = -z\varepsilon_1^0 + x\varepsilon_3^0$, $\varphi_1(\varepsilon_3^1) = -z\varepsilon_2^0 + y\varepsilon_3^0$, and $\varphi_1(\varepsilon_4^1) = -y\varepsilon_2^0 + x\varepsilon_4^0$.

It follows that $\mathrm{im}(\varphi_1) = \ker(\varphi_0)$. Also,

$$\ker(\varphi_1) = \langle z\varepsilon_1^1 - y\varepsilon_2^1 + x\varepsilon_3^1\rangle.$$

So $\ker(\varphi_1)$ is the second syzygy module of I. Finally, let $F_2 = R(-7)$ with generator ε_1^2 and define the map $\varphi_2 : F_2 \longrightarrow F_1$ by $\varphi_2(\varepsilon_1^2) = z\varepsilon_1^1 - y\varepsilon_2^1 + x\varepsilon_3^1$. Then $\mathrm{im}(\varphi_2) = \ker(\varphi_1)$. In addition, $\ker(\varphi_2) = \langle 0\rangle$. Therefore, a minimal free resolution for I is given by

$$0 \longrightarrow R(-7) \xrightarrow{\varphi_2} R^4(-6) \xrightarrow{\varphi_1} R^4(-5) \xrightarrow{\varphi_0} I \longrightarrow 0.$$

Furthermore, we have the Betti numbers $\beta_{0,5} = 4$, $\beta_{1,6} = 4$ and $\beta_{2,7} = 1$ and so the Betti table associated with I is:

	0	1	2	3
0	–	–	–	–
1	–	–	–	–
2	–	–	–	–
3	–	–	–	–
4	–	–	–	–
5	4	4	1	–

.

Example 2. Let $R = \mathbb{R}[x, y, z]$. Consider the homogeneous ideal $I = \langle x^3, x^2y, xyz\rangle$. We will find generators for the first and second syzygy modules, we will obtain a minimal free resolution for R/I, and finally we will construct the Betti table.

Let $I = \langle x^3, x^2y, xyz\rangle$ and let $h_1 = x^3$, $h_2 = x^2y$, $h_3 = xyz$. To find generators of the first syzygy module of I, consider the following three relations:

$$\begin{gathered}-y(x^3) + x(x^2y) + 0(xyz) = 0 \\ 0(x^3) - z(x^2y) + x(xyz) = 0 \\ -yz(x^3) + 0(x^2y) + x^2(xyz) = 0.\end{gathered}$$

Observe that

$$-yz(x^3)+0(x^2y)+x^2(xyz) = z[-y(x^3)+x(x^2y)+0(xyz)]+x[0(x^3)-z(x^2y)+x(xyz)].$$

Hence, $\{(-y, x, 0), (0, -z, x)\}$ is a minimal set of generators for the first syzygy module of I. However, we do not have non-trivial relationships among these two sysygies. It follows that there is no second syzygy module for I.

Consider the quotient R/I. Set $F_0 = R$ and consider the canonical map $\varphi_0 : R \longrightarrow R/I$. Then $M_1 = \ker(\varphi_0) = I$. Let $F_1 = R^3(-3)$ with generators $\varepsilon_1^1, \varepsilon_2^1, \varepsilon_3^1$ and define $\varphi_1 : F_1 \longrightarrow F_0$ by $\varphi_1(\varepsilon_j^1) = h_j$. Now, $\mathrm{im}(\varphi_1) = M_1$ and $\ker(\varphi_1)$ is precisely the set of syzygies of I whose homogeneous generators are $-y\varepsilon_1^1 + x\varepsilon_2^1, -z\varepsilon_2^1 + x\varepsilon_3^1$.

Let $F_2 = R^2(-4)$ with generators $\varepsilon_1^2, \varepsilon_2^2$ and define $\varphi_2 : F_2 \longrightarrow F_1$ by $\varphi_2(\varepsilon_1^2) = -y\varepsilon_1^1 + x\varepsilon_2^1$ and $\varphi_2(\varepsilon_2^2) = -z\varepsilon_2^1 + x\varepsilon_3^1$. Then $M_2 = \mathrm{im}(\varphi_2) = \ker(\varphi_1)$, that is, $\varphi_2(F_2) = M_2$. Note that $\ker(\varphi_2) = \langle 0 \rangle$ because there is no second syzygy module for I. Thus, a minimal free resolution for R/I is given by:

$$0 \longrightarrow R^2(-4) \xrightarrow{\varphi_2} R^3(-3) \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/I \longrightarrow 0.$$

Furthermore, we have the Betti numbers $\beta_{0,0} = 1$, $\beta_{1,3} = 3, \beta_{2,4} = 2$, and hence the Betti table associated with R/I is:

	0	1	2
0	1	–	–
1	–	–	–
2	–	3	2

.

3.7 Macaulay2

We use the free computer algebra system Macaulay2 [18] to verify the results we obtained in the previous section.

The next sequence of commands will produce the results from Example 1. The i means input and the o means output. For example i10 is the tenth input and o10 is the tenth output. First, we define a polynomial ring R with coefficients in $\mathbb{Q}$ in the variables $x, y, , z$ by entering the following command:

```
i1 : R = QQ[x,y,z]
o1 = R
o1 : PolynomialRing
```

Next, we define the ideal I from Example 1 over the ring R:

```
i2 : I = ideal(x^5,x^4*y,x^4*z,x^3*y^2)

               5    4     4     3 2
o2 = ideal (x , x y, x z, x y )
o2 : Ideal of R
```

To obtain the free resolution of the module I, we use the command *resolution*. We call this resolution MR.

```
i3 : MR = resolution I

      1      4      4      1
o3 = R  <-- R  <-- R  <-- R  <-- 0

     0      1      2      3      4

o3 : ChainComplex
```

If we want to see the maps together with the syzygy modules, we use the command *.dd*. The generators of the first syzygy module are the columns of the matrix in the second map. The generators of the second syzygy module are the columns of the matrix in the third map, and so on. Since we call the above resolution MR, we enter the command MR.dd:

```
i4 : MR.dd

          1                             4
o4 = 0 : R  <-------------------------- R  : 1
               | x5 x4y x3y2 x4z |

          4                             4
     1 : R  <-------------------------- R  : 2
               {5} | -y 0  -z 0  |
               {5} | x  -y 0  -z |
               {5} | 0  x  0  0  |
               {5} | 0  0  x  y  |

          4                 1
     2 : R  <-------------- R  : 3
               {6} | z  |
```

```
                 {6} | 0  |
                 {6} | -y |
                 {6} | x  |

           1
3 : R  <-------- 0 : 4
              0

o4 : ChainComplexMap
```

Finally, to calculate the Betti table of the module I, we enter the command *betti res module*.

```
i7 : betti res module I

                0 1 2
o7 = total:     4 4 1
             5: 4 4 1

o7 : BettiTally
```

To verify the results from the second example, we use the same commands except for the Betti table where we use the command *betti res I* which yield the Betti table of the quotient R/I.

```
i1 : betti res I

                0 1 2
o1 = total:     1 3 2
             0: 1 . .
             1: . . .
             2: . 3 2

o1 : BettiTally
```

Chapter 4

Proof of the Three-Variable Case

In this chapter, we present the proof given by Brooks and Grundmeier [3] solving our research question for polynomials in three variables. Therefore, we give the minimum rank of the polynomials P in the class $\mathcal{H}_{3,d}$ such that $P = SQ$ for $S = x_0 + x_1 + x_2$ and Q a full homogeneous polynomial of degree $d-1$. We exhibit a family of sharp polynomials as well. As we mentioned in Chapter 2, Brooks and Grundmeier used a commutative algebra approach.

4.1 Algebraic Setting

Let $R = \mathbb{R}[x_0, x_1, ..., x_n]$. R is a graded ring, graded by degree. Let $Q(x) = \sum_a c_a x^a \in R$ be a full homogeneous polynomial of degree $d-1$ and let $S = x_0 + x_1 + \cdots + x_n$. Let $\mathcal{A}$ be the set of multi-indices a for which $c_a > 0$ and let $\mathcal{B}$ be the set of multi-indices for which $c_a < 0$. We generate the ideals $I^+ = \langle x^a \rangle_{a \in \mathcal{A}}$, $I^- = \langle x^a \rangle_{a \in \mathcal{B}}$, and $I = \langle x^a \rangle_{a \in \mathcal{A} \cup \mathcal{B}}$. Next, we introduce the numerical invariants associated with these ideals. Let $\{\alpha_{i,j}\}$, $\{\beta_{i,j}\}$, and $\{\gamma_{i,j}\}$ be the sets of graded Betti numbers associated with I^+, I^-, and I, respectively.

The aim is to obtain a lower bound for the number of distinct monomials appearing in SQ with non-zero coefficient. We denote this number by $\rho(SQ)$. The next proposition gives a lower bound for $\rho(SQ)$ in terms of the graded Betti numbers introduced above.

Proposition 4.1 (Brooks and Grundmeier [3]). *Let $Q(x) = \sum_a c_a x^a \in R$ be a full homogeneous polynomial of degree $d-1$ and let $S = x_0 + x_1 + \cdots + x_n$. Consider the ideals and the associated graded Betti numbers as described above. Then*

$$\rho(SQ) \geqslant (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha_{1,d} + \beta_{1,d}.$$

Proof. Each of the ideals I, I^+, and I^- can be seen as a graded R-module. For example

$$I = \bigoplus_{m\in\mathbb{Z}} I_m, \quad \text{where} \quad I_m = \text{span}\{x^a : |a| = m \text{ and } x^a \in I\}.$$

Observe that the generators of I, I^+, and I^- are monomials of degree $d-1$. Hence the monomials potentially appearing in SQ are those monomials in I_d. We are able to count the number of monomials in I_d through the Hilbert function. The total number of such monomials is $H_I(d)$. However, we cannot assume that all monomials in I_d appear in SQ with non-zero coefficient. We can get a lower bound on $\rho(SQ)$ by counting the monomials that we are sure appear in SQ. These are monomials in I_d generated by only elements of I^+ or by only elements of I^-. That is,

$$\rho(SQ) \geqslant H_I(d) - H_{I^+}(d) + H_I(d) - H_{I^-}(d) = 2H_I(d) - H_{I^+} - H_{I^-}(d).$$

By Corollary 3.1,

$$H_I(d) = \sum_j C_j \binom{d-j+n}{n}, \quad \text{where } C_j = \sum_i (-1)^i \gamma_{i,j}.$$

$$H_{I^+}(d) = \sum_j A_j \binom{d-j+n}{n}, \quad \text{where } A_j = \sum_i (-1)^i \alpha_{i,j}.$$

$$H_{I^-}(d) = \sum_j B_j \binom{d-j+n}{n}, \quad \text{where } B_j = \sum_i (-1)^i \beta_{i,j}.$$

Because all monomials in I are of degree at least $d-1$, we have that $C_j = 0$ for $j = 0, ..., d-2$. Also, note that if $j > d$, $\binom{d-j+n}{n} = 0$. Thus, $C_{d-1} = \gamma_{0,d-1}$ and $C_d = -\gamma_{1,d}$. Hence

$$\begin{aligned} H_I(d) &= \gamma_{0,d-1}\binom{d-(d-1)+n}{n} - \gamma_{1,d}\binom{d-d+n}{n} \\ &= (n+1)\gamma_{0,d-1} - \gamma_{1,d}. \end{aligned}$$

In a similar way, we get expressions for $H_{I^+}(d)$ and $H_{I^-}(d)$. Because $\gamma_{0,d-1} = \alpha_{0,d-1} + \beta_{0,d-1}$, it follows that

$$\begin{aligned} \rho(SQ) &\geqslant 2H_I(d) - H_{I^+} - H_{I^-}(d) \\ &= 2\left[\gamma_{0,d-1}(n+1) - \gamma_{1,d}\right] - \left[\alpha_{0,d-1}(n+1) - \alpha_{1,d}\right] - \left[\beta_{0,d-1}(n+1) - \beta_{1,d}\right] \\ &= 2(n+1)\gamma_{0,d-1} - 2\gamma_{1,d} - (n+1)\alpha_{0,d-1} + \alpha_{1,d} - (n+1)\beta_{0,d-1} + \beta_{1,d} \\ &= (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha_{1,d} + \beta_{1,d}. \end{aligned}$$

□

Let $\alpha_{1,d} := \alpha$ and $\beta_{1,d} := \beta$. Then

$$\rho(SQ) \geqslant (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta. \tag{4.1}$$

Thus, the goal is to obtain a sharp lower bound for $\alpha + \beta$.

Example 3.5 implies that $\gamma_{0,d-1} = \binom{d-1+n}{n}$ and that $H_I(d) = \binom{d+n}{n}$. By Corollary 3.1, $H_I(d) = (n+1)\gamma_{0,d-1} - \gamma_{1,d}$. Therefore,

$$\begin{aligned}
\gamma_{1,d} &= (n+1)\gamma_{0,d-1} - H_I(d) \\
&= (n+1)\binom{d+(n-1)}{n} - \binom{d+n}{n} \\
&= n\binom{d+(n-1)}{n} + \binom{d+(n-1)}{n} - \binom{d+n}{n} \\
&= d\binom{d+(n-1)}{n-1} - \binom{d+(n-1)}{n-1} \\
&= (d-1)\binom{d+n-1}{n-1}.
\end{aligned}$$

Consequently, there are $(d-1)\binom{d+n-1}{n-1}$ independent elements that generates the first syzygy module of I.

4.2 Newton Diagrams

Let us suppose that $n+1 = 3$. We have a visual tool that helps to represent homogeneous polynomials in 3 variables of any degree d. This tool is called the *Newton diagram*. It is also used to represent polynomials in 4 variables. However, for more than four variables, it is hard to visualize.

The Newton diagram of a full homogeneous polynomial $f(x_0, x_1, x_2)$ of degree $d-1$ is a graph consisting of one vertex for each monomial appearing in f. We label each vertex with the coefficient of the corresponding monomial. In this case, the diagram will resemble a big triangle pointing down whose leftmost, rightmost, and bottommost vertices will be x_0^{d-1}, x_1^{d-1}, and x_2^{d-1}, respectively. We join the two vertices associated with the monomials m_1 and m_2 if and only if $x_i m_1 = x_j m_2$ for $0 \leqslant i \neq j \leqslant 2$. We place the coefficients of the monomials into the graph so that higher rows correspond to coefficients of monomials with lower powers of x_2. We prefer to use x, y, z instead of x_0, x_1, x_2 to avoid subscripts. Therefore, the powers of z increase as we move down, the powers of x increase as we move left, and the powers of y increase as we move right.

Recall, we are interested in homogeneous polynomials $P \in \mathcal{H}_{n+1,d}$ such that $P = SQ$ and Q is a full homogeneous polynomial. Thus, we will usually represent the Newton diagram of the quotient Q and we will say that it is the Newton diagram corresponding to P.

Remark: Edges on the Newton diagram correspond to first syzygies of degree d, specifically to the divided Koszul relations.

Example 4.1. To illustrate the definition of a Newton diagram, we will construct the Newton diagram that corresponds to the polynomial $P(x, y, z) = x^3 - 2xy^2 - y^3 + 3x^2z - 2y^2z + xz^2 - 2yz^2 - z^3$. Hence, we must construct the Newton diagram of $Q(x, y, z) = x^2 - xy - y^2 + 2xz - yz - z^2$.

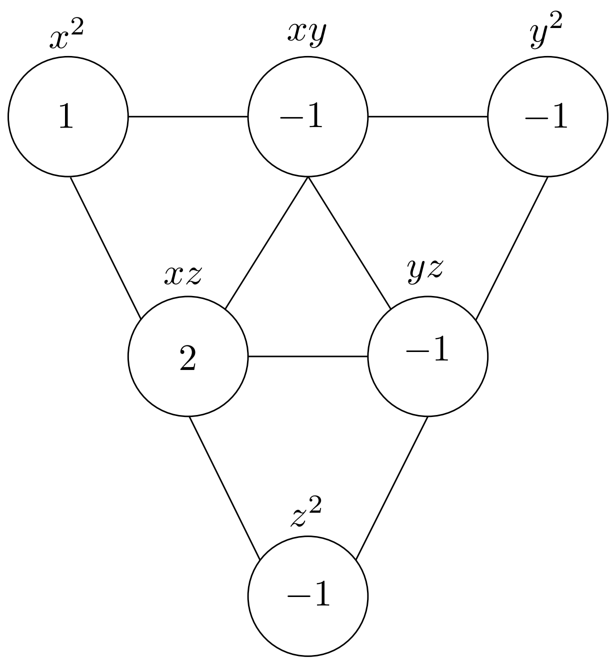


Figure 4.1: Newton Diagram of $Q(x, y, z)$.

Now that we have the Newton diagram of Q, it is possible to represent the polynomial $P = SQ$, where $S = x + y + z$. To see this, let m be a monomial in Q. The monomial xm is located to the left of the monomial m inside a triangle pointing up, so that the right vertex intersects the vertex corresponding to m (circle enclosing the monomial m). The figure below helps to illustrate this situation.

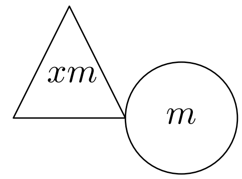


Figure 4.2: The monomial xm.

Similarly, the monomial ym will be placed in a triangle to the right of m and the monomial zm in a triangle below m. Therefore, we can represent the monomials xm, ym, zm with the figure 4.3 below.

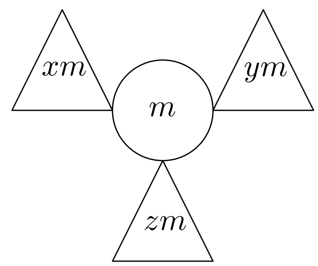


Figure 4.3: Monomials xm, ym, zm.

Doing the same with each monomial m in Q and labeling each triangle with the coefficient resulting from adding the coefficients of the monomials in Q that the triangle intersects, we end up with the figure 4.4 below.

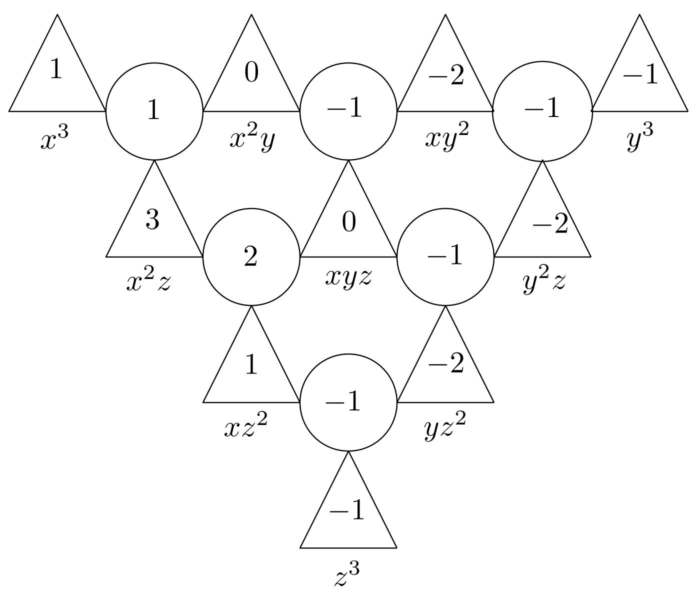


Figure 4.4: Diagram of $P = SQ$.

Remark: The vertices of a Newton diagram are label only with the coefficients of the corresponding monomials. However, we have labeled the monomials in the above graphs just to clarify the structure of a Newton diagram.

Next, we connect the ideas discussed in the previous section with Newton diagrams. Let $Q(x,y,z) = x^2 - xy - y^2 + 2xz - yz - z^2$. Then $I = \langle x^2, xy, xz, y^2, yz, z^2 \rangle$, $I^+ = \langle x^2, xz \rangle$, and $I^- = \langle xy, y^2, yz, z^2 \rangle$. Recall that to obtain a lower bound for SQ, we need to count the monomials generated by only elements of I^+ or by only elements of I^-. Looking at the last figure, we can see, for example, that the monomial x^2z has coefficient 3, which is coming from adding the positive coefficients of x^2 and xz. Likewise, the monomial xy^2 has coefficient -2, which comes from adding the negative coefficients of xy and y^2. Doing the same with all elements generating $I_3 = \langle x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3 \rangle$, we have that two monomials (x^3 and x^2z) are coming from only elements of I^+ and five other monomials ($xy^2, y^3, y^2z, yz^2,$ and z^3) are coming from only elements of I^-. Therefore, Proposition 4.1 gives the lower bound $\rho(SQ) \geqslant 7 - 2 + 7 - 5 = 7$. Of course in this example, $\rho(SQ) = 8$.

4.3 Counting Syzygies

When $n+1 = 3$, we know that the number of independent syzygies among generators of I is $\gamma_{1,d} = (d-1)\binom{d+1}{1} = d^2 - 1$. This number is certainly smaller than the number of edges (first syzygies or divided Koszul relations) we see on the Newton diagram. Thus, we want to understand how to obtain an independent set from the full set of first syzygies. It would be particularly nice if we could identify groups of edges on the Newton diagram that represent independent syzygies and groups that are dependent. It turns out it will be convenient to consider multi-indices associated with vertices on the *border* of the diagram and multi-indices associated with vertices in the *interior* of the diagram. We call these multi-indices border or interior multi-indices, respectively. Specifically, let $\mathbf{B} = (B_0, B_1, B_2)$ be a multi-index of length $d-1$. Then $\mathbf{B}$ is a border multi-index if $B_j = 0$ for at least one j and $\mathbf{B}$ is an interior multi-index if $B_j > 0$ for all j.

Let $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, and $e_2 = (0,0,1)$. Let $\mathbf{A} = (A_0, A_1, A_2)$ be a multi-index of length d. Suppose that $A_j = 0$ for exactly one j. A border syzygy is a syzygy between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$, for some $i \in \{0,1,2\}$. Equivalently, it is a syzygy between two monomials associated with border multi-indices. Suppose that $A_j > 0$ for all j. An interior syzygy is a syzygy between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$, for some $i \in \{0,1,2\}$. To get the number of independent interior syzygies, we will use the following lemma. Recall that the ideal I is generated by all monomials of degree $d-1$ in three variables.

Lemma 4.1. *Let $\boldsymbol{A} = (A_0, A_1, A_2)$ be a multi-index of length d such that $A_0, A_1, A_2 > 0$. Then $\sigma(\boldsymbol{A} - e_0, \boldsymbol{A} - e_1)$, $\sigma(\boldsymbol{A} - e_0, \boldsymbol{A} - e_2)$, and $\sigma(\boldsymbol{A} - e_1, \boldsymbol{A} - e_2)$ are dependent but any pair are independent.*

Proof. By definition,

$$\begin{aligned}\sigma(\mathbf{A} - e_0, \mathbf{A} - e_1) &= x_0\varepsilon(\mathbf{A} - e_0) - x_1\varepsilon(\mathbf{A} - e_1)\\ \sigma(\mathbf{A} - e_0, \mathbf{A} - e_2) &= x_0\varepsilon(\mathbf{A} - e_0) - x_2\varepsilon(\mathbf{A} - e_2)\\ \sigma(\mathbf{A} - e_1, \mathbf{A} - e_2) &= x_1\varepsilon(\mathbf{A} - e_1) - x_2\varepsilon(\mathbf{A} - e_2)\end{aligned}$$

Observe that

$$\sigma(\mathbf{A} - e_0, \mathbf{A} - e_2) - \sigma(\mathbf{A} - e_0, \mathbf{A} - e_1) = \sigma(\mathbf{A} - e_1, \mathbf{A} - e_2).$$

Therefore, the set of three divided Koszul relations is dependent. Let us prove that any pair is independent. First, we prove that $\sigma(\mathbf{A} - e_0, \mathbf{A} - e_1)$ and $\sigma(\mathbf{A} - e_0, \mathbf{A} - e_2)$ are independent. Let $c_1, c_2 \in \mathbb{R}[x, y, z]$. Suppose that

$$c_1\sigma(\mathbf{A} - e_0, \mathbf{A} - e_1) + c_2\sigma(\mathbf{A} - e_0, \mathbf{A} - e_2) = (0, 0, 0).$$

Then

$$c_1(x_0\varepsilon(\mathbf{A} - e_0) - x_1\varepsilon(\mathbf{A} - e_1)) + c_2(x_0\varepsilon(\mathbf{A} - e_0) - x_2\varepsilon(\mathbf{A} - e_2)) = (0, 0, 0).$$

Hence

$$(c_1 + c_2)x_0\varepsilon(\mathbf{A} - e_0) - c_1x_1\varepsilon(\mathbf{A} - e_1) - c_2x_2\varepsilon(\mathbf{A} - e_2) = (0, 0, 0).$$

Equivalently,

$$(-c_2x_2, -c_1x_1, (c_1 + c_2)x_0) = (0, 0, 0).$$

Therefore, $c_1 = 0 = c_2$. It follows that $\sigma(\mathbf{A} - e_0, \mathbf{A} - e_1)$ and $\sigma(\mathbf{A} - e_0, \mathbf{A} - e_2)$ are independent. The proof for the other two pairs is analogous.

□

Let $\mathbf{A} = (A_0, A_1, A_2)$ be a multi-index of length d with no zero components. Let T^+ be the number of triples $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2\}$ in which all three multi-indices are in $\mathcal{A}$, let T^- be the number of triples $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2\}$ in which all three multi-indices belong to $\mathcal{B}$, and let T^0 be the number of triples in which not all multi-indices are in the same set. Now, we consider multi-indices $\mathbf{A} = (A_0, A_1, A_2)$ with exactly one zero component, so there exist $i, j \in \{0, 1, 2\}$ distinct with $\mathbf{A} - e_i, \mathbf{A} - e_j$ both valid multi-indices. We define E^+ to be the number of pairs of the form $\{\mathbf{A} - e_i, \mathbf{A} - e_j\}$

for which the two multi-indices are in $\mathcal{A}$. As above, we define E^- and E^0.

Note that the number of interior syzygies is three times the number of multi-indices $\mathbf{A} = (A_0, A_1, A_2)$ of length d with no zero components. Because there are $T^+ + T^- + T^0$ such multi-indices, Lemma 4.1 implies that there are $2(T^+ + T^- + T^0)$ independent interior first syzygies. On the other hand, the number of edges on the border of the diagram corresponds to the number of multi-indices $\mathbf{A} = (A_0, A_1, A_2)$ of length d with precisely one zero component. Since there are $E^+ + E^- + E^0$ such multi-indices and the number of independent syzygies is $d^2 - 1$, it follow that

$$d^2 - 1 = E^+ + E^- + E^0 + 2(T^+ + T^- + T^0). \tag{4.2}$$

Lemma 4.2.

$$\alpha + \beta = E^+ + E^- + 2T^+ + 2T^- + T^0.$$

Proof. First, we analyze the terms with T's. Let $\mathbf{A} = (A_0, A_1, A_2)$ be a multi-index of length d with all non-zero components. Consider the triple $\{\mathbf{A} - e_0, \mathbf{A} - e_1, \mathbf{A} - e_2\}$. If all three multi-indices are in $\mathcal{A}$, Lemma 4.1 implies that there are 2 independent syzygies contributing to α. Since there are T^+ such triples, we get $2T^+$ independent syzygies that contributes to α. Likewise, we have $2T^-$ independent syzygies contributing to β. Now, suppose that not all elements of the triple $\{\mathbf{A} - e_0, \mathbf{A} - e_1, \mathbf{A} - e_2\}$ belong to the same set. Therefore, at least one multi-index belong to $\mathcal{A}$ and at least one multi-index belong to $\mathcal{B}$. Thus, we have the following six cases:

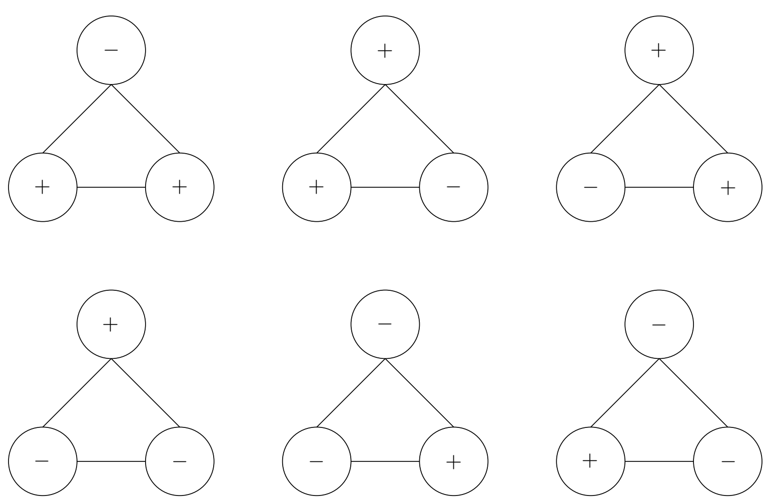


Figure 4.5: All six cases.

However, it is enough to consider only two cases, since the other four cases are the same up to a permutation of the variables. Therefore, we consider the two situations:

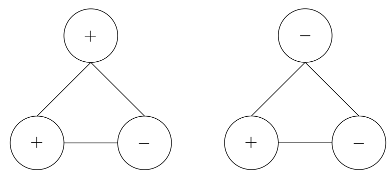


Figure 4.6: Case one and two.

Case 1: two positive and one negative coefficient. From the three divided Koszul relations associated to $\{\mathbf{A} - e_0, \mathbf{A} - e_1, \mathbf{A} - e_2\}$, only one is between two elements of $\mathcal{A}$ and none is between two elements of $\mathcal{B}$. Hence, we get one independent syzygy contributing to α.

Case 2: two negative and one positive coefficients. This case is analogous to case 1 and we omit the details. Here we end up with one independent syzygy contributing to β.

Since there are T^0 triples with this characteristic, we have T^0 independent syzygies contributing to $\alpha + \beta$. We have examined all terms with T's.

By definition of E^+, E^- and E^0, it is clear that there are $E^+ + E^-$ independent syzygies contributing to $\alpha + \beta$. This completes the proof of the lemma. $\square$

As a check of our work, we proceed to count the number of interior first syzygies. Then we count the number of independent border syzygies. For the first case, we must count multi-indices $\mathbf{A}$ with length d and no zero components. This is the same as counting multi-indices $\mathbf{A'} = (\mathbf{A}_0 - 1, \mathbf{A}_1 - 1, \mathbf{A}_2 - 1)$ of length $d - 3$. Note that there are $\binom{d-3+2}{2} = \binom{d-1}{2}$ multi-indices of length $d - 3$ with three components. It follows by Lemma 4.1 that there are

$$2\binom{d-1}{2} = d^2 - 3d + 2$$

independent interior syzygies. In addition, there are another $3(d-1)$ independent syzygies between pairs of monomials associated with border multi-indices. In conclusion, there are $d^2 - 3d + 2 + 3(d-1) = d^2 - 1$ independent first syzygies. This agrees with the result given before about $\gamma_{1,d}$.

4.4 The Minimum Rank Estimate

In order to obtain a sharp lower bound for $\rho(SQ)$, we need a sharp lower bound for $\alpha+\beta$. By equation 4.1 and Lemma 4.2,

$$\alpha+\beta=\gamma_{1,d}-(E^0+T^0)=\gamma_{1,d}-\frac{1}{2}E^0-\frac{1}{2}(E^0+2T^0). \tag{4.3}$$

Thus, we need to estimate E^0 and E^0+2T^0. Observe that the quantity E^0+2T^0 is the total number of independent syzygies between monomials for which the corresponding two multi-indices are such that one multi-index is in $\mathcal{A}$ and one multi-index is in $\mathcal{B}$. To get this estimate, we will count these syzygies in a different way.

Let $\mathbf{B}$ be a multi-index of length $d-2$. Consider the triple $\{\mathbf{B}+e_0, \mathbf{B}+e_1, \mathbf{B}+e_2\}$ and the resulting three divided Koszul relations:

$$\begin{aligned}
\sigma(\mathbf{B}+e_0,\mathbf{B}+e_1)&=x_1\varepsilon(\mathbf{B}+e_0)-x_0\varepsilon(\mathbf{B}+e_1)\\
\sigma(\mathbf{B}+e_0,\mathbf{B}+e_2)&=x_2\varepsilon(\mathbf{B}+e_0)-x_0\varepsilon(\mathbf{B}+e_2)\\
\sigma(\mathbf{B}+e_1,\mathbf{B}+e_2)&=x_2\varepsilon(\mathbf{B}+e_1)-x_1\varepsilon(\mathbf{B}+e_2).
\end{aligned}$$

Then the three divided Koszul relations are linearly independent. Furthermore, at most two of the three divided Koszul relations (syzygies) can be associated with one element in $\mathcal{A}$ and one element in $\mathcal{B}$. We know that the number of independent syzygies associated with multi-indices in which not all multi-indices are in the same set $\mathcal{A}$ or $\mathcal{B}$ is E^0+2T^0. Note that there are $\binom{d-2+2}{2}=\binom{d}{2}=\frac{1}{2}(d^2-d)$ multi-indices of length $d-2$ with three components. Therefore,

$$E^0+2T^0\leqslant 2(\tfrac{d^2-d}{2})=d^2-d.$$

As we mentioned above, there are $3(d-1)$ independent syzygies between pairs of monomials associated with border multi-indices. This implies that $E^0\leqslant 3(d-1)$. It follows from equation 4.1 that

$$\begin{aligned}
\rho(SQ)&\geqslant 3\gamma_{0,d-1}-2\gamma_{1,d}+\alpha+\beta\\
&\geqslant 3\gamma_{0,d-1}-2\gamma_{1,d}+\gamma_{1,d}-\frac{1}{2}E^0-\frac{1}{2}(E^0+2T^0)\\
&=H_I(d)-\frac{1}{2}E^0-\frac{1}{2}(E^0+2T^0)\\
&\geqslant\binom{d+2}{2}-\frac{3}{2}(d-1)-\frac{1}{2}(d^2-2)\\
&=\frac{d+5}{2}.
\end{aligned}$$

Therefore, for a fixed integer d, the minimum rank of a polynomial $P \in \mathcal{H}_{3,d}$ such that $P = SQ$, where Q is a full polynomial in $\mathcal{H}_{3,d-1}$ and $S = x + y + z$, is $\frac{d+5}{2}$. Moreover, this inequality is sharp when d is odd, as we shall see in the next section.

4.5 Family of Sharp Polynomials

As discussed in Chapter 2, a family of sharp polynomials is given by

$$F_d(x,y,z) \;=\; x^d - (-y)^d - (-z)^d + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} (-1)^s K_{ds}\; x^{d-2s} y^s z^s.$$

Hence,

$$\begin{aligned}
F_1(x,y,z) &= x + y + z \\
F_3(x,y,z) &= x^3 + y^3 + z^3 - 3xyz \\
F_5(x,y,z) &= x^5 + y^5 + z^5 - 5x^3yz + 5xy^2z^2 \\
F_7(x,y,z) &= x^7 + y^7 + z^7 - 7x^5yz + 14x^3y^2z^2 - 7xy^3z^3 \\
F_9(x,y,z) &= x^9 + y^9 + z^9 - 9x^7yz + 27x^5y^2z^2 - 30x^3y^3z^3 + 9xy^4z^4 \\
&\;\;\vdots
\end{aligned}$$

And the quotient is defined by

$$\begin{aligned}
Q_d(x,y,z) = &\sum_{j=1}^{d-1} (-1)^j \sum_{s=0}^{\min\{d-1-j,j-1\}} \binom{d-1-j}{s} x^{d-1-s-j} [z^j y^s + y^j z^s] \\
&+ \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^j \binom{d-1-j}{j} x^{d-1-2j} y^j z^j.
\end{aligned}$$

We exhibit the Newton diagrams for the quotients Q_3, Q_5 and Q_7 of F_3, F_5 and F_7, respectively. In this way we can see how the non-zero coefficients in F_3, F_5 and F_7 are obtained.

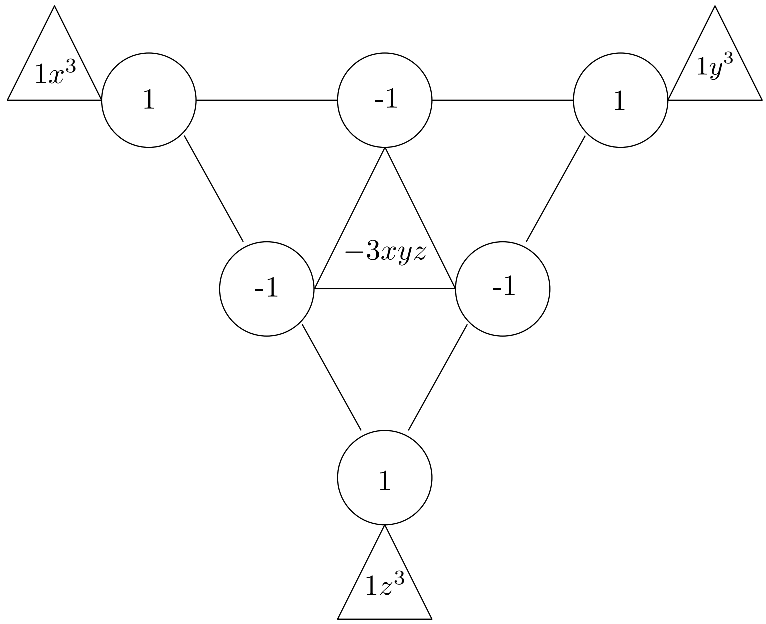


Figure 4.7: Newton diagram for Q_3 together with F_3.

Brooks [2] showed that when $d \equiv 1, 3 \bmod 6$ we can get new sharp polynomials. Furthermore, these polynomials are *symmetric*, which means that they are invariant under any permutation of variables. The polynomials are listed below,

$$\begin{aligned}
S_1(x,y,z) &= x + y + z \\
S_3(x,y,z) &= x^3 + y^3 + z^3 - 3xyz \\
S_7(x,y,z) &= x^7 + y^7 + z^7 - 7x^3y^3z - 7x^3yz^3 - 7xy^3z^3 \\
S_9(x,y,z) &= x^9 + y^9 + z^9 + 9x^4y^4z + 9x^4yz^4 + 9xy^4z^4 - 30x^3y^3z^3 \\
S_{13}(x,y,z) &= x^{13} + y^{13} + z^{13} + 13x^6y^6z + 13x^6yz^6 + 13xy^6z^6 \\
&\quad - 91x^5y^5z^3 - 91x^5y^3z^5 - 91x^3y^5z^5 \\
&\vdots
\end{aligned}$$

Thus, Theorem 1.1 has been proved.

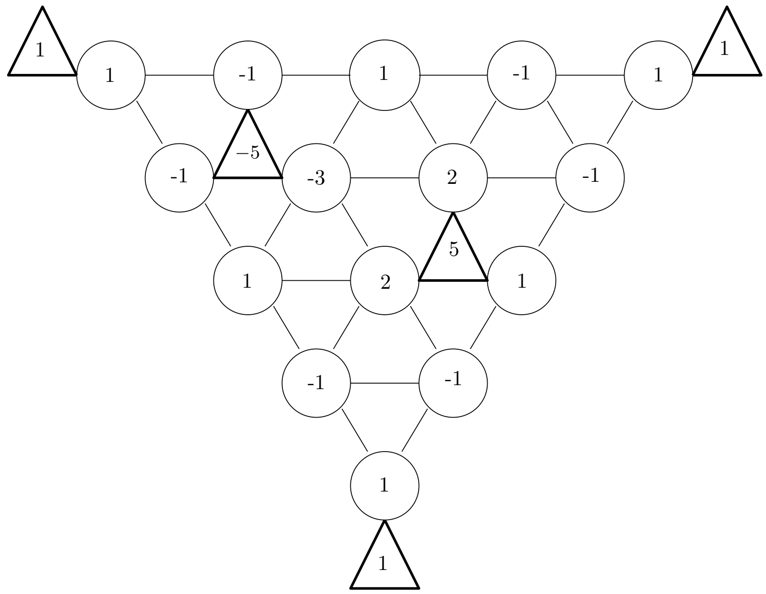


Figure 4.8: Newton diagram for Q_5 together with F_5.

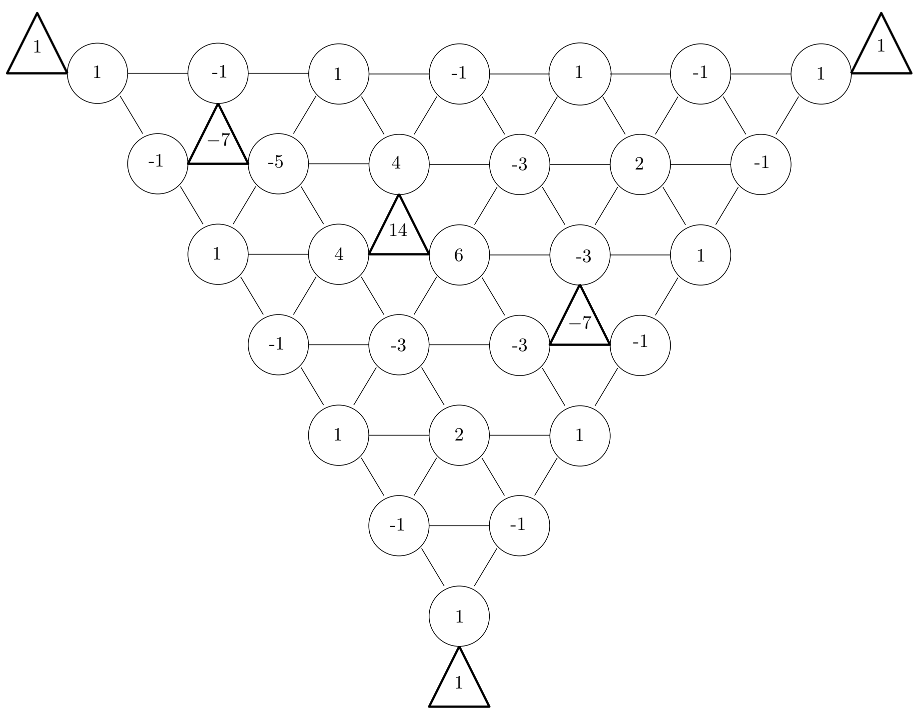


Figure 4.9: Newton diagram for Q_7 together with F_7.

Chapter 5

Proof of the Four-Variable Case

In this chapter, we apply the technique developed in the last chapter to polynomials P in $\mathcal{H}_{4,d}$ such that $P = SQ$, where $Q(x) = \sum_a c_a x^a \in R = \mathbb{R}[x_0, x_1, x_2, x_3]$ is a full homogeneous polynomial of degree $d-1$ and $S = x_0 + x_1 + x_2 + x_3$. As in Chapter 4, let $\mathcal{A}$ be the set of multi-indices a for which $c_a > 0$ and let $\mathcal{B}$ be the set of multi-indices for which $c_a < 0$. Furthermore, consider the ideals $I^+ = \langle x^a \rangle_{a \in \mathcal{A}}$, $I^- = \langle x^a \rangle_{a \in \mathcal{B}}$, and $I = \langle x^a \rangle_{a \in \mathcal{A} \cup \mathcal{B}}$ and the sets of graded Betti numbers $\{\alpha_{i,j}\}$, $\{\beta_{i,j}\}$, and $\{\gamma_{i,j}\}$ associated with I^+, I^-, and I, respectively.

There are two main reasons why we discuss the four-variable case separately from the general case. First, we obtain a family of sharp polynomials. Second, we are still able to visualize the Newton diagram of the quotient polynomial Q. Recall that in the three-variable case, the Newton diagram of a polynomial can be visualized on a triangular array. In the four-variable case, the Newton diagram is visualized on an array that is a tetrahedron. The faces of the tetrahedron represent terms in the polynomial that involve only three of the four variables.

We seek the minimum rank of the polynomial $P \in \mathcal{H}_{4,d}$ when $P = SQ$ for full quotient Q. By proposition 4.1,

$$\rho(SQ) \geqslant 4\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta.$$

We already obtained expressions for $\gamma_{0,d-1}$ and $\gamma_{1,d}$. As in Chapter 4, we must find a sharp lower bound for $\alpha + \beta$. In order to obtain a sharp lower bound on $\alpha + \beta$, we need to study carefully the first syzygy modules for the three ideals described above.

5.1 Counting Syzygies

Newton diagrams for polynomials in four variables are now tetrahedrons. A tetrahedron has 4 faces and 6 edges. Furthermore, if the degree of the polynomial is greater than or equal to five, a small tetrahedron is formed in the interior of the Newton diagram. We will see this in detail in Section 5.4, where we exhibit the Newton diagrams of sharp polynomials. As in the previous chapter, we need to identify groups of edges on the Newton diagram that represent independent syzygies and grousp that are dependent. To do this, we consider multi-indices associated to either vertices (monomials) on the *interior*, *face* or *border* of the diagram. We call these multi-indices: interior, face or border mult-indices, respectively. More precisely, let $\mathbf{B} = (B_0, B_1, B_2, B_3)$ be a multi-index of length $d-1$. $\mathbf{B}$ is an interior multi-index if $B_j > 0$ for all j. $\mathbf{B}$ is a face multi-index if $B_j = 0$ for at least one j. Finally, $\mathbf{B}$ is a border multi-index if $B_j = 0$ for at least two j's. Note that under these definitions, border multi-indices are also face multi-indices.

Let $e_0 = (1, 0, 0, 0)$, $e_1 = (0, 1, 0, 0)$, $e_2 = (0, 0, 1, 0)$, and $e_3 = (0, 0, 0, 1)$. Let $\mathbf{A} = (A_0, A_1, A_2, A_3)$ be a multi-index of length d. Suppose that $A_j = 0$ for exactly two j's. A border syzygy is a syzygy between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$ for some $i \in \{0, 1, 2, 3\}$. Suppose that $A_j = 0$ for exactly one j. A face syzygy is a syzygy between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$ for some $i \in \{0, 1, 2, 3\}$. Lastly, suppose that $A_j > 0$ for all j. An interior syzygy is a syzygy between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$ for some $i \in \{0, 1, 2, 3\}$. To count the number of interior first syzygies, we need the following lemma. Recall that the ideal I is generated by all monomials of degree $d-1$ in four variables.

Lemma 5.1. *Let $\boldsymbol{A} = (A_0, A_1, A_2, A_3)$ be a multi-index of length d such that $A_j > 0$ for every j. Let $e_0 = (1, 0, 0, 0)$, $e_1 = (0, 1, 0, 0)$, $e_2 = (0, 0, 1, 0)$, and $e_3 = (0, 0, 0, 1)$. Then the set*

$$D = \{\sigma(\boldsymbol{A} - e_l, \boldsymbol{A} - e_m) \ : \ 0 \leqslant l < m \leqslant 3\}$$

is dependent, and the largest independent subset has three elements.

Proof. The number of distinct elements in D is six. So there are six divided Koszul relations associated with the multi-index $\mathbf{A}$. These relations correspond to the six edges of the tetrahedron in the figure below. We need to show that only three of these six divided Koszul relations are independent.

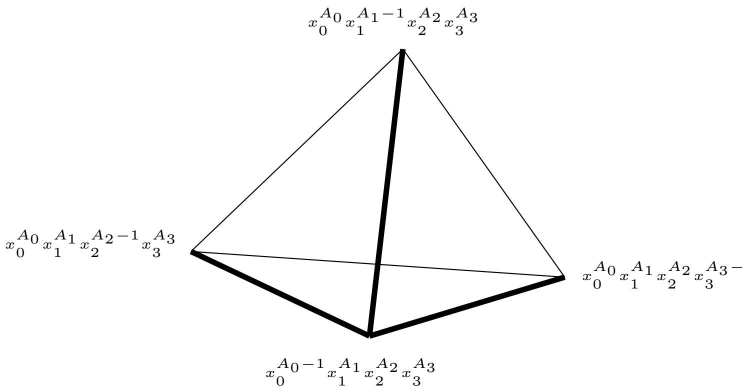


Figure 5.1: The six syzygies.

By definition we have that

$$\begin{aligned}
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_1) &= x_0\varepsilon(\mathbf{A}-e_0) - x_1\varepsilon(\mathbf{A}-e_1)\\
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_2) &= x_0\varepsilon(\mathbf{A}-e_0) - x_2\varepsilon(\mathbf{A}-e_2)\\
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_3) &= x_0\varepsilon(\mathbf{A}-e_0) - x_3\varepsilon(\mathbf{A}-e_3)\\
\sigma(\mathbf{A}-e_1,\ \mathbf{A}-e_2) &= x_1\varepsilon(\mathbf{A}-e_1) - x_2\varepsilon(\mathbf{A}-e_2)\\
\sigma(\mathbf{A}-e_1,\ \mathbf{A}-e_3) &= x_1\varepsilon(\mathbf{A}-e_1) - x_3\varepsilon(\mathbf{A}-e_3)\\
\sigma(\mathbf{A}-e_2,\ \mathbf{A}-e_3) &= x_2\varepsilon(\mathbf{A}-e_2) - x_3\varepsilon(\mathbf{A}-e_3).
\end{aligned}$$

We claim that the subset

$$D' = \{\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_1), \sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_2), \sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_3)\}$$

is a maximal independent subset of D. Because $\varepsilon(\mathbf{A}-e_j)$ and $\varepsilon(\mathbf{A}-e_k)$ are distinct generators of F_0 if $j \neq k$, the elements in D' are linearly independent. Observe that

$$\begin{aligned}
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_2) - \sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_1) &= \sigma(\mathbf{A}-e_1,\ \mathbf{A}-e_2)\\
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_3) - \sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_1) &= \sigma(\mathbf{A}-e_1,\ \mathbf{A}-e_3)\\
\sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_3) - \sigma(\mathbf{A}-e_0,\ \mathbf{A}-e_2) &= \sigma(\mathbf{A}-e_2,\ \mathbf{A}-e_3).
\end{aligned}$$

That is, the remaining three elements of D can be expressed as linear combinations of the elements of D'. The elements in D' correspond to the three selected edges in the tetrahedron above.

$\square$

Let $\mathbf{A} = (A_0, A_1, A_2, A_3)$ be a multi-index of length d with no zero component. Let H^+ be the number of quadruples $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2, \mathbf{A}-e_3\}$ in which all four multi-indices are in $\mathcal{A}$, let H^- be the number of quadruples $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2, \mathbf{A}-e_3\}$ such that all four multi-indices are in $\mathcal{B}$, and let H^0 be the number of quadruples for which not all four multi-indices $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2, \mathbf{A}-e_3\}$ belong to the same set. Now we will consider multi-indices $\mathbf{A}$ of length d with precisely one zero component, say A_l. We define T^+ to be the number of triples of the form $\{\mathbf{A}-e_i, \mathbf{A}-e_j, \mathbf{A}-e_k\}$ where $0 \leqslant i \neq j \neq k \neq l \leqslant 3$, in which all three multi-indices are in $\mathcal{A}$, with analogous definitions for T^- and T^0. Finally, we consider multi-indices $\mathbf{A}$ of length d with precisely two zero components, say A_k, A_l. Let E^+ be the number of pairs of multi-indices $\{\mathbf{A}-e_i, \mathbf{A}-e_j\}$, where $0 \leqslant i \neq j \neq k, l \leqslant 3$, such that both multi-indices belonging to $\mathcal{A}$. Likewise, we define E^- and E^0.

Observe that the number of interior syzygies is six times the number of multi-indices $\mathbf{A} = (A_0, A_1, A_2, A_3)$ of length d with no zero components. Since there are $H^+ + H^- + H^0$ such multi-indices, Lemma 5.1 implies that there are $3(H^+ + H^- + H^0)$ independent interior first syzygies. Now, the number of face syzygies is three times the number of multi-indices $\mathbf{A}$ of length d with exactly one zero component. Because there are $T^+ + T^- + T^0$ such multi-indices, Lemma 4.1 implies that there are $2(T^+ + T^- + T^0)$ independent face syzygies. Finally, the number of edges on the border of the diagram corresponds to the number of multi-indices $\mathbf{A}$ of length d with exactly two zero components. Because there are $E^+ + E^- + E^0$ such multi-indices, it follows that the number of independent border syzygies is $E^+ + E^- + E^0$. Therefore, the number of independent first syzygies $\gamma_{1,d}$ can be written as

$$\gamma_{1,d} = 3(H^+ + H^- + H^0) + 2(T^+ + T^- + T^0) + E^+ + E^- + E^0.$$

The next lemma says how many of these contribute to $\alpha + \beta$.

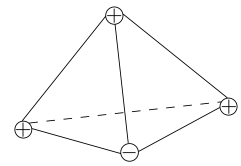
Lemma 5.2.

$$\alpha + \beta = 3H^+ + 3H^- + 2H^0 + 2T^+ + 2T^- + T^0 + E^+ + E^-.$$

Proof. Let $\mathbf{A} = (A_0, A_1, A_2, A_3)$ be a multi-index with no zero components and with length d. Let us start by analyzing the terms with the H's. If all four multi-indices $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2, \mathbf{A}-e_3\}$ belong to $\mathcal{A}$, we have 3 independent syzygies contributing to α. Because there are H^+ quadruples with this property, we have $3H^+$ independent syzygies that contributes to α. Likewise, we have $3H^-$ independent syzygies that contributes to β.

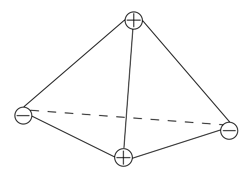
Suppose now that the quadruple $\{\mathbf{A}-e_0, \mathbf{A}-e_1, \mathbf{A}-e_2, \mathbf{A}-e_3\}$ is such that at least one of the four multi-indices belongs to $\mathcal{A}$ and at least one of the four multi-indices belongs to $\mathcal{B}$. Let us arrange this quadruple in such a way that the coefficient of the corresponding monomial is the vertex of a tetrahedron. Regarding the sign of each coefficient we have the following 3 cases:

Case 1: three positive and one negative coefficients.



Let us prove that it is not possible to have three independent syzygies associated with the 3 multi-indices belonging to $\mathcal{A}$. Without loss of generality, we assume that $\mathbf{A} - e_3 \in \mathcal{B}$. Consider the triple $\{\mathbf{A} - e_0, \mathbf{A} - e_1, \mathbf{A} - e_2\}$. Then all multi-indices $\mathbf{A} - e_j$, with $j = 0, 1, 2$, have equal A_3 component. Therefore, the multi-index $\mathbf{A}$ can be written as $\mathbf{A} = (A', A_3)$, where A' is a triple. By Lemma 4.1, the set of three divided Koszul relations associated with these multi-indices is dependent, but any set of two of them is independent. Hence, we have two independent syzygies contributing to α.

Case 2: two positive and two negative coefficients.



It is evident that we have one syzygy associated with the two multi-indices belonging to $\mathcal{A}$ and one syzygy associated with the two multi-indices belonging to $\mathcal{B}$. The other four divided Koszul relations involve an element of $\mathcal{A}$ and an element of $\mathcal{B}$ and do not contribute to the sum $\alpha + \beta$. Thus, we have two independent syzygies contributing to $\alpha + \beta$.

Case 3: one positive and three negative coefficients. This case is analogous to case 1 and we omit the details.

Since there are H^0 such quadruples, we have $2H^0$ independent syzygies contributing to $\alpha + \beta$. So we have examined all terms with H's.

Let us now analyze the terms with T's. Assume that $A_l = 0$ for some $0 \leqslant l \leqslant 3$. Consider the triple $\{A - e_i, A - e_j, A - e_k\}$, where $0 \leqslant i \neq j \neq k \neq l \leqslant 3$. If all three multi-indices belong to $\mathcal{A}$, Lemma 4.1 guarantees that there are two independent syzygies contributing to α. Because there are T^+ such triples, we have $2T^+$ independent syzygies contributing to α. Analogously, there are $2T^-$ independent syzygies contributing to β.

We now suppose that at least one of the three multi-indices belongs to $\mathcal{A}$ and at least one of the three multi-indices belongs to $\mathcal{B}$. As above, we arrange the multi-indices in a way that each multi-index corresponds to the vertex of a triangle. We have two possibilities:

Case 1: two positive and one negative coefficients. Note that from the three associated divided Koszul relations, only one involves two elements of $\mathcal{A}$, and none involve two elements of $\mathcal{B}$. Hence, we have one independent syzygy contributing to α.

Case 2: one positive and two negative coefficients. By a similar argument, in this case we have one syzygy contributing to β.

In either case, since the number of triples with this property is T^0, there are T^0 independent syzygies contributing to $\alpha + \beta$.

Finally, we consider the terms with E's. Recall that if $\mathbf{A}$ has exactly two zero components, E^+ is defined to be the number of pairs of border multi-indices both belonging to $\mathcal{A}$, with an analogous definitions for E^- and E^0. Then there are $E^+ + E^-$ independent syzygies contributing to $\alpha + \beta$.

$\square$

Next, we will verify that the number of independent syzygies agree with the above calculation for $\gamma_{1,d}$. To do this, we count the number of independent syzygies in different categories and verify that the total is indeed $\gamma_{1,d}$. This is not a necessary part of the argument but is nonetheless instructive.

Let us count the number of independent interior syzygies. We must count multi-indices $\mathbf{A} = (A_0, A_1, A_2, A_3)$ of length d such that $A_j > 0$ for every $j = 0, 1, 2, 3$. Observe that this is equivalent to counting multi-indices $\mathbf{A'} = (A_0 - 1, A_1 - 1, A_2 - 1, A_3 - 1)$ of length $d - 4$. Since there are $\binom{d-4+3}{3} = \binom{d-1}{3}$ multi-indices of length $d-4$ with four components, we get

$$3\binom{d-1}{3} = 3\frac{(d-1)(d-2)(d-3)}{3!} = \frac{d^3 - 6d^2 + 11d - 6}{2}$$

independent interior syzygies. Now, we count independent face syzygies. As we saw before, in each face (triangle) there are

$$2\binom{d-1}{2} = d^2 - 3d + 2$$

independent syzygies. Because a tetrahedron has 4 faces, we have $4(d^2 - 3d + 2)$ independent face syzygies. Finally, since there are six edges, we get $6(d-1)$ independent border syzygies. Therefore, there are in total

$$\frac{1}{2}d^3 - 3d^2 + \frac{11}{2}d - 3 + 4(d^2 - 3d + 2) + 6(d-1) = \frac{1}{2}d^3 + d^2 - \frac{1}{2}d - 1$$

independent first syzygies. This result agrees with the calculation of $\gamma_{1,d}$ given above.

5.2 The Minimum Rank Estimate

Lemma 5.2 implies that

$$\begin{aligned}\alpha+\beta &= 3H^+ + 3H^- + 2H^0 + 2T^+ + 2T^- + T^0 + E^+ + E^- \\ &= 3H^+ + 3H^- + 3H^0 + 2T^+ + 2T^- + 2T^0 + E^+ + E^- + E^0 - (E^0 + T^0 + H^0) \\ &= \gamma_{1,d} - (E^0 + T^0 + H^0).\end{aligned}$$

Now,

$$E^0 + T^0 + H^0 = \tfrac{1}{2}E^0 + \tfrac{1}{2}(E^0 + 2T^0) + H^0.$$

Thus,

$$\alpha + \beta = \gamma_{1,d} - \tfrac{1}{2}E^0 - \tfrac{1}{2}(E^0 + 2T^0) - H^0.$$

Let us focus on only one of the faces of the tetrahedron. In Section 4.4, we had a similar expression $E^0 + 2T^0$ to estimate. In that case, this quantity represented the number of independent syzygies involving an element of $\mathcal{A}$ and an element of $\mathcal{B}$, and we gave the estimate $E^0 + 2T^0 \leqslant d^2 - d$. Now this quantity represents the number of face or border syzygies between monomials associated with multi-indices that involve an element of $\mathcal{A}$ and an element of $\mathcal{B}$. In the Newton diagram, each edge representing a border syzygy is part of two faces. Thus the last expression counts each edge contributing to E^0 twice. In conclusion, we have that the number of syzygies obtained from the four faces that can be associated with one element of $\mathcal{A}$ and one element of $\mathcal{B}$ is at most $4(d^2 - d) - E^0$. That is,

$$E^0 + 2T^0 \leq 4(d^2 - d) - E^0.$$

Since there are $\frac{1}{2}d^3 - 3d^2 + \frac{11}{2}d - 3$ independent interior syzygies,

$$3H^0 \leq \tfrac{1}{2}d^3 - 3d^2 + \tfrac{11}{2}d - 3.$$

Hence,

$$H^0 \leq \tfrac{1}{6}d^3 - d^2 + \tfrac{11}{6}d - 1.$$

Furthermore, $E^0 \leq 6(d-1)$. It follows that

$$\begin{aligned}
\rho(SQ) &\geqslant 4\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta \\
&\geqslant 4\gamma_{0,d-1} - 2\gamma_{1,d} + \gamma_{1,d} - \frac{1}{2}E^0 - \frac{1}{2}(E^0 + 2T^0) - H^0 \\
&= H_I(d) - \frac{1}{2}E^0 - \frac{1}{2}(E^0 + 2T^0) - H^0 \\
&\geqslant \binom{d+3}{3} - \frac{1}{2}E^0 - \frac{1}{2}(4d^2 - 4d) + \frac{1}{2}E^0 - \frac{1}{6}d^3 + d^2 - \frac{11}{6}d + 1 \\
&= 2d + 2.
\end{aligned}$$

It follows that for fixed $d \geqslant 1$, the minimum rank for polynomials $P \in \mathcal{H}_{4,d}$ for which $P = SQ$, where Q is a full polynomial of degree $d-1$ and $S = x_0+x_1+x_2+x_3$, is $2d+2$.

5.3 Family of Sharp Polynomials

In this section, we prove that the above inequality $\rho(SQ) \geqslant 2d+2$ is sharp. We use x, y, z, w as the variables to avoid subscripts. We claim that for $d \geqslant 1$, the family of polynomials

$$P_d(x, y, z, w) = (x+y)^d + (-1)^{d-1}(z+w)^d$$

is sharp. We need to show that the rank of P_d is $2d+2$, that P_d is divisible by $S = x+y+z+w$, and that the quotient Q is full. Observe that the Binomial Theorem implies that

$$(x+y)^d = \sum_{k=0}^{d} \binom{d}{k} x^{d-k} y^k.$$

Therefore, the polynomial $(x+y)^d$ has $d+1$ non-zero terms. Similarly, $(z+w)^d$ has $d+1$ non-zero terms. Hence the rank of P_d is $2d+2$. For the remaining two properties of P_d we proceed as in Lemma 1.1.

Lemma 5.3. *The quotient*

$$\frac{P_d}{S} = \frac{(x+y)^d + (-1)^{d-1}(z+w)^d}{x+y+z+w}$$

is given by the polynomial

$$Q = \sum_{j=0}^{d-1} (-1)^j (x+y)^{d-1-j} (z+w)^j.$$

Moreover, Q is a full polynomial of degree $d-1$.

Proof. Let $a = x+y$ and $b = z+w$. Then $P_d = a^d + (-1)^{d-1}\ b^d$. Observe that

$$Q = \sum_{j=0}^{d-1} (-1)^j a^{d-1-j}\ b^j,$$

and so

$$\begin{aligned}
SQ &= (x+y+z+w) \sum_{j=0}^{d-1} (-1)^j\ (x+y)^{d-1-j}\ (z+w)^j \\
&= (x+y) \sum_{j=0}^{d-1} (-1)^j\ (x+y)^{d-1-j}\ (z+w)^j + (z+w) \sum_{j=0}^{d-1} (-1)^j\ (x+y)^{d-1-j}\ (z+w)^j \\
&= a \sum_{j=0}^{d-1} (-1)^j\ a^{d-1-j}\ b^j + b \sum_{j=0}^{d-1} (-1)^j\ a^{d-1-j}\ b^j \\
&= a^d + (-1)^{d-1}\ b^d \\
&= (x+y)^d + (-1)^{d-1}\ (z+w)^d.
\end{aligned}$$

Consequently $P_d = SQ$.

Let $x^k y^l z^m w^n$ be an arbitrary monomial with $k+l+m+n = d-1$. Note that

$$x^k y^l z^m w^n = x^{(k+l)-l} y^l\ z^{(m+n)-n} w^n.$$

Thus, $x^k y^l$ and $z^m w^n$ can be seen as terms in $(x+y)^{k+l}$ and $(z+w)^{m+n}$, respectively. As a result, the product $(x^k y^l) \cdot (z^m w^n)$ is a term of $(x+y)^{k+l}(z+w)^{m+n}$, which is a term in $\sum_j (-1)^j (x+y)^{d-1-j} (z+w)^j$. In particular, it is part of the term associated with $j = m+n$.

Consequently, each monomial in four variable of degree $d-1$ appears in Q. This implies that Q is a full polynomial of degree $d-1$. Therefore, the family of polynomials P_d is sharp.

$\square$

With the proof of Lemma 5.3 we complete the proof of Theorem 1.2.

5.4 Newton Diagram Examples

We exhibit a planar representation of the Newton diagrams of the quotients

$$Q_5 = \frac{P_5}{S} = \frac{(x+y)^5 + (z+w)^5}{x+y+z+w}$$

and

$$Q_6 = \frac{P_6}{S} = \frac{(x+y)^6 - (z+w)^6}{x+y+z+w}.$$

For the first polynomial, the interior of the Newton diagram only consists of the monomial $4xyzw$, whereas for the second polynomial, we get a small tetrahedron in the interior composed of those monomials involving all four variables of degree five. A tetrahedron appears in the interior of the Newton diagram of a polynomial P when the degree is greater than four. It happens because the vertices on the faces of the Newton diagram correspond to monomials composed of at most three variables. In both cases, we give a flattened-out view of the tetrahedron, and we fold along the bold lines to make the 3D figure. Furthermore, we show the two faces (triangles) that contributes to the non-zero coefficients of P_5 and P_6.

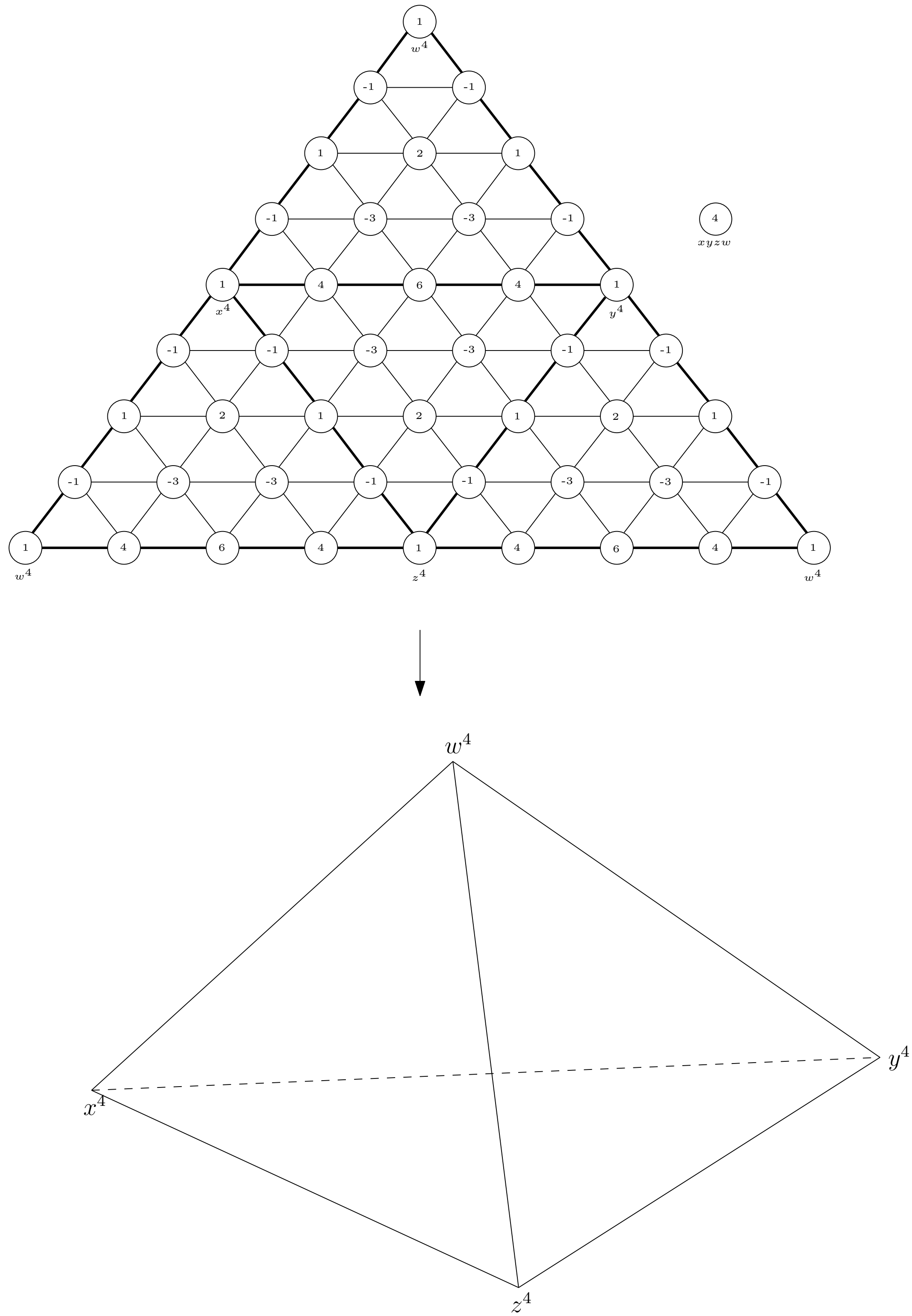


Figure 5.2: Newton diagram of Q_5.

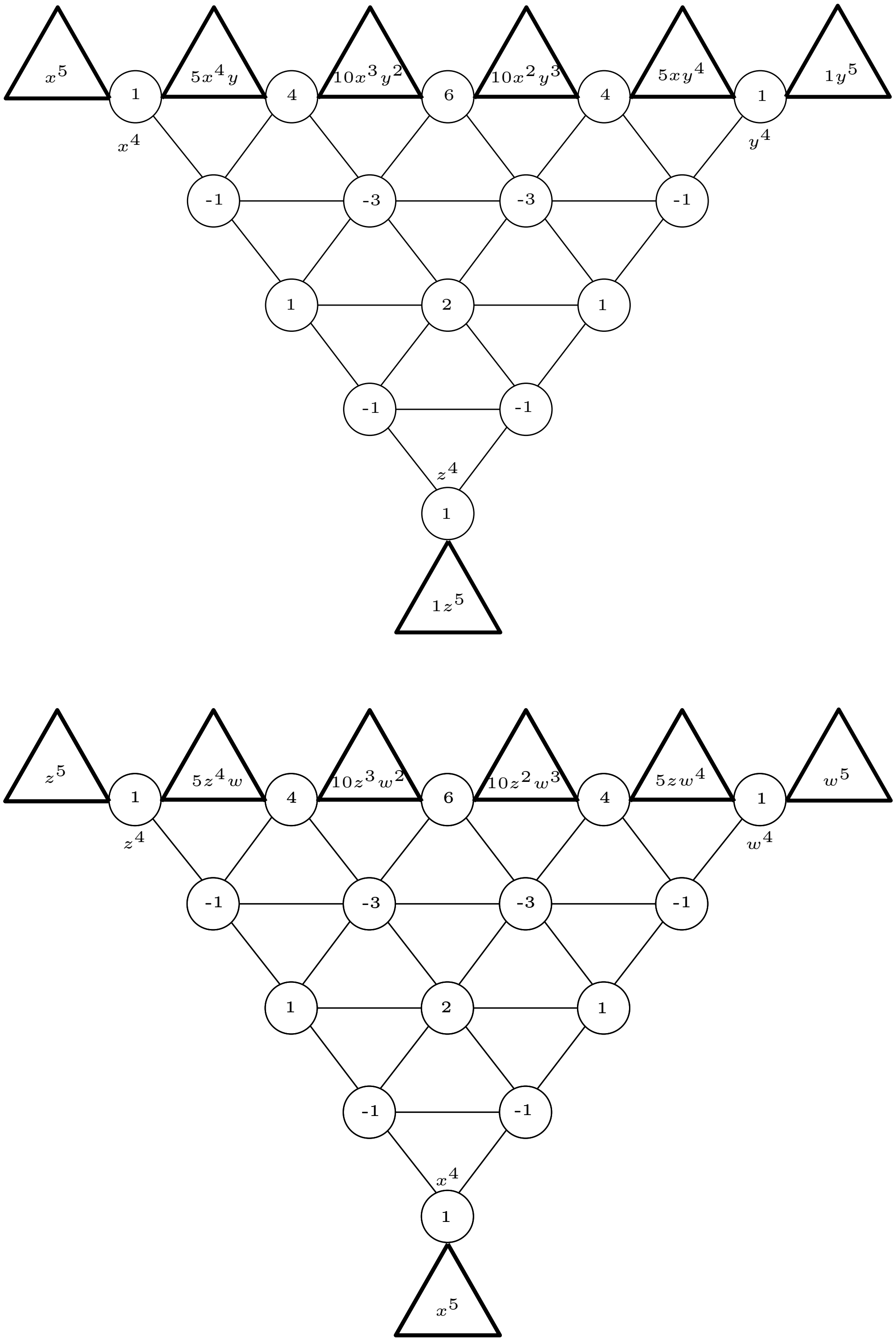
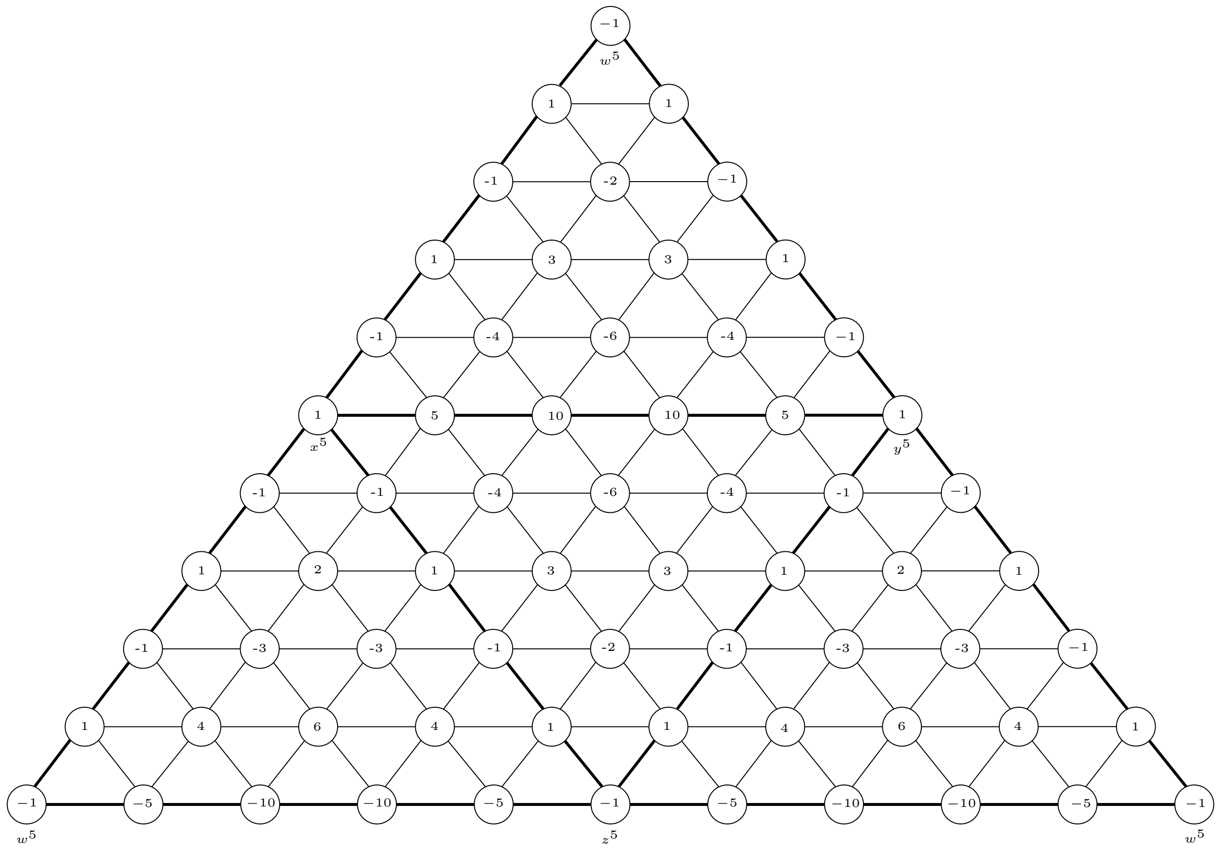


Figure 5.3: Faces xyz and xzw.



Interior

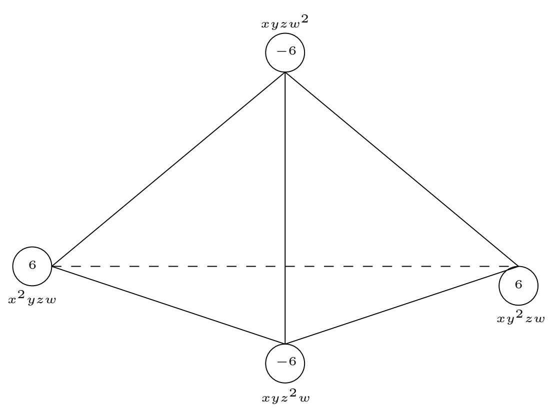


Figure 5.4: Newton diagram of Q_6.

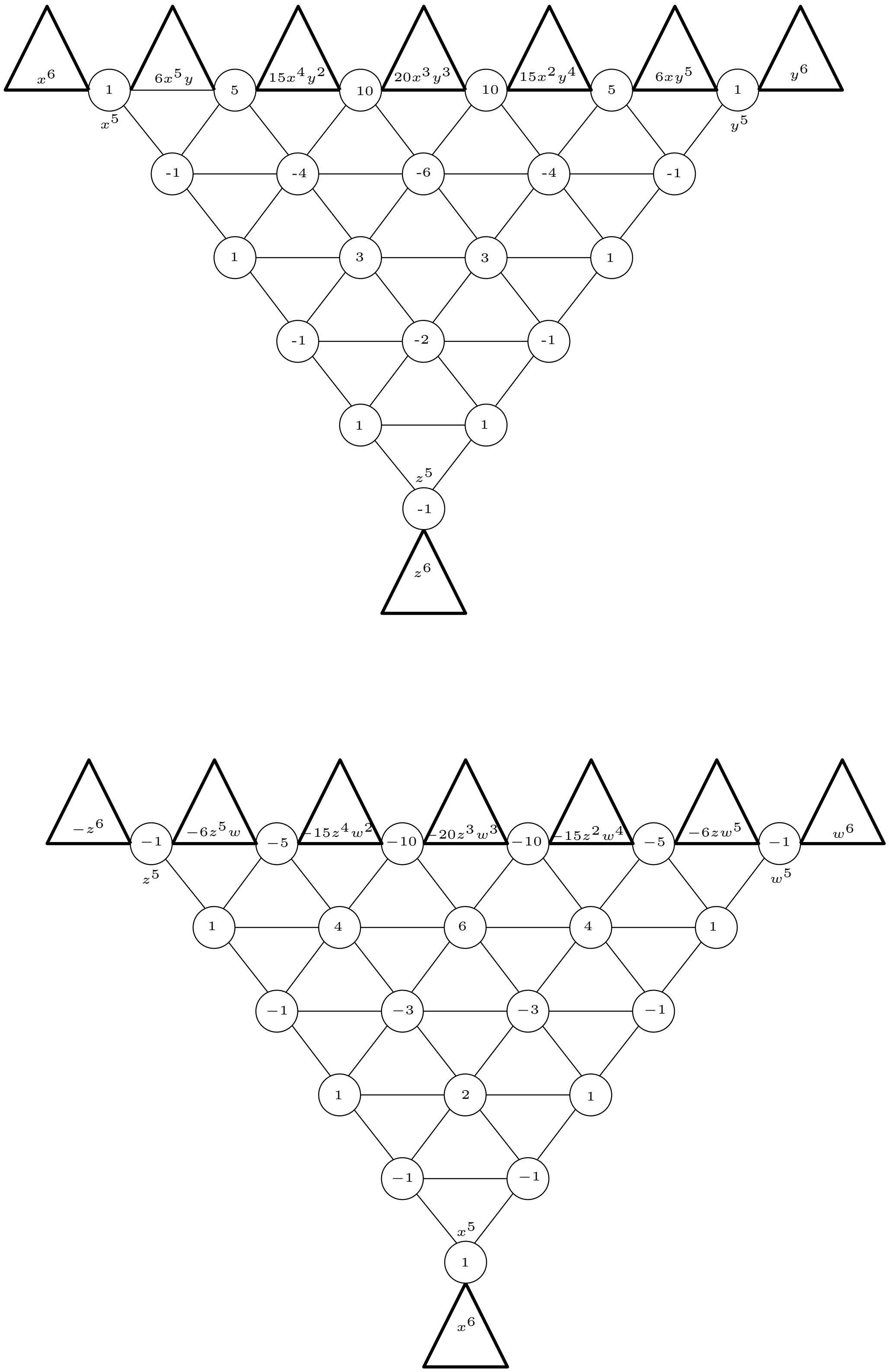


Figure 5.5: Faces xyz and xzw.

Chapter 6

Proof of the General Case

We use the technique provided by Brooks and Grundmeier [3] to prove Theorem 1.3. Although the estimate is sharp for $n+1=4$, we will see that the estimate is not sharp when $n+1 \geqslant 5$.

Suppose that $R = \mathbb{R}[x_0, x_1, ..., x_n]$ and that $S = x_0 + x_1 + \cdots + x_n$. Let Q be a full homogeneous polynomial of degree $d-1$ in R. We must show that

$$\rho(SQ) \geqslant n(n+1)\frac{d-1}{6} + (n+1).$$

Once again, we define $\mathcal{A}$ to be the set of multi-indices a for which $c_a > 0$ and $\mathcal{B}$ to be the set of multi-indices for which $c_a < 0$. As usual, we consider the ideals $I^+ = \langle x^a \rangle_{a \in \mathcal{A}}$, $I^- = \langle x^a \rangle_{a \in \mathcal{B}}$, $I = \langle x^a \rangle_{a \in \mathcal{A} \cup \mathcal{B}}$, and the associated sets of graded Betti numbers $\{\alpha_{i,j}\}$, $\{\beta_{i,j}\}$, and $\{\gamma_{i,j}\}$.

By Proposition 4.1, our starting point is

$$\rho(SQ) \geqslant (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta.$$

Thus, the aim is to find a sharp lower bound for $\alpha + \beta$.

6.1 Counting Syzygies

We will address the problem as in the three- and four-variable cases. Hence, we will partition the syzygies into categories. Although we are not able to visualize the polynomials through a Newton diagram, we are still using a general argument based on the number of non-zero components of a multi-index of length d. When we look at a multi-index $\mathbf{A}$ of length d, we are looking at a multi-index associated with a monomial in the product SQ. The multi-indices $\mathbf{A} - e_j$, where e_j is the $(n+1)$-tuple with all entries 0 except for the j-th entry which is 1 and $j \in \{0, 1, ..., n\}$, are associated with the monomials in Q that contribute to that monomial in SQ, and we are looking at the syzygies among those monomials. Thus, we start by partitioning the terms in the product SQ according to how many zeros appear in the corresponding multi-index.

We start by counting the number of independent syzygies associated with multi-indices $\mathbf{A} - e_j$, where $\mathbf{A}$ is a multi-index of length d with only non-zero components. To do this, we need the next two lemmas.

Lemma 6.1. *Let $\boldsymbol{A} = (A_0, A_1, ..., A_n)$ be a multi-index of length d, with $A_j > 0$ for all j. Then the set*

$$D = \{\sigma(\boldsymbol{A} - e_l, \boldsymbol{A} - e_m) \ : \ 0 \leqslant l < m \leqslant n\}$$

is dependent and the largest independent subset has n elements.

Proof. The number of distinct elements in D is $\binom{n+1}{2}$, so there are $\frac{n(n+1)}{2}$ divided Koszul relations. We want to prove that only n are independent. Set

$$\sigma_{l,m} := \sigma(\mathbf{A} - e_l, \mathbf{A} - e_m) = x_l \varepsilon(\mathbf{A} - e_l) - x_m \varepsilon(\mathbf{A} - e_m).$$

We claim that the subset

$$D' = \{\sigma_{0,k} \ : \ 1 < k \leqslant n\}$$

is a maximal independent subset of D. Let $c_1, c_2, ..., c_n \in \mathbb{R}[x_0, x_1, ..., x_n]$. Suppose that

$$c_1 \sigma_{0,1} + c_2 \sigma_{0,2} + \cdots + c_n \sigma_{0,n} = (0, 0, ..., 0).$$

Then

$$(c_1 + c_2 + \cdots + c_n) x_0 \varepsilon(\mathbf{A} - e_0) - c_1 x_1 \varepsilon(\mathbf{A} - e_1) - \cdots - c_n x_n \varepsilon(\mathbf{A} - e_n) = (0, 0, ..., 0).$$

Therefore, $-c_j x_j = 0$ for all $1 \leqslant j \leqslant n$. Thus, $c_1 = c_2 = \cdots = c_n = 0$. It follows that the elements in D' are linearly independent.

On the other hand, for all $0 < r < s \leqslant n$, we have that

$$\sigma_{0,s} - \sigma_{0,r} = x_r \varepsilon(\mathbf{A} - e_r) - x_s \varepsilon(\mathbf{A} - e_s) = \sigma_{r,s}.$$

and so every other element of D can be expressed as linear combinations of the elements of D'. $\square$

Let $\mathbf{A} = (A_0, A_1, ..., A_n)$ be a multi-index of length d. Suppose that $\mathbf{A}$ has exactly k zero components. Define E^+_{n+1-k} to be the number of such multi-indices $\mathbf{A}$ for which all the multi-indices $\mathbf{A} - e_j$ are in $\mathcal{A}$. Let E^-_{n+1-k} be the number of multi-indices $\mathbf{A}$ such that all multi-indices $\mathbf{A} - e_j$ are in $\mathcal{B}$, and let E^0_{n+1-k} be the number of multi-indices $\mathbf{A}$ for which not all multi-indices $\mathbf{A} - e_j$ belong to the same set.

Note that the number of multi-indices $\mathbf{A} = (A_0, A_1, ..., A_n)$ of length d, with exactly k zero components is $E^+_{n+1-k} + E^-_{n+1-k} + E^0_{n+1-k}$. Lemma 6.1 implies that there are $(n-k)(E^+_{n+1-k} + E^-_{n+1-k} + E^0_{n+1-k})$ independent syzygies between pairs of monomials associated with those multi-indices. It follows that

$$\gamma_{1,d} = \sum_{j=2}^{n+1} (j-1)(E^+_j + E^-_j + E^0_j).$$

As in the proof in the four-variable case, we now want to express $\alpha + \beta$ in terms of the numbers E^+_{n+1-k}, E^-_{n+1-k}, and E^0_{n+1-k}. The next lemma is the analogous of Lemma 5.1 from the last chapter.

Lemma 6.2.

$$\alpha + \beta = \sum_{j=2}^{n+1} \left[(j-1)(E^+_j + E^-_j) + (j-2)E^0_j \right].$$

Proof. Let $0 \leqslant k \leqslant n-1$ be arbitrary. Suppose that $\mathbf{A} = (A_0, A_1, ..., A_n)$ is a multi-index of length d with exactly k zero components. If all $(n+1-k)$ multi-indices $\mathbf{A} - e_j$, where j is such that $A_j \neq 0$, belong to $\mathcal{A}$, Lemma 6.1 guarantees that there are $n-k$ independent syzygies between pairs of monomials associated with these multi-indices. Because there are E^+_{n+1-k} multi-indices with this property, we have $(n-k)E^+_{n+1-k}$ independent syzygies contributing to α. A similar argument shows that there are $(n-k)E^-_{n+1-k}$ independent syzygies contributing to β.

On the other hand, suppose that at least one of the $(n+1-k)$ multi-indices $\mathbf{A} - e_j$ belongs to $\mathcal{A}$ and at least one multi-index belongs to $\mathcal{B}$. Let us assume that there are i elements in $\mathcal{A}$ and $(n+1-k)-i$ elements in $\mathcal{B}$. Observe that all i multi-indices $\mathbf{A} - e_j$ have equal A_l components, where l is such that $\mathbf{A} - e_l \in \mathcal{B}$. Lemma 6.1 implies that only $i-1$ of the $\binom{i}{2}$ divided Koszul relations are independent. Therefore, we have $i-1$ independent syzygies contributing to α. Analogously, there are $n-k-i$ independent syzygies contributing to β corresponding to multi-indices $\mathbf{A} - e_j$ that belong to $\mathcal{B}$. Because the number of multi-indices $\mathbf{A}$ of length d and exactly k zero components for which not all multi-indices $\mathbf{A} - e_j$ belong to the same set $\mathcal{A}$ or $\mathcal{B}$ is E^0_{n+1-k}, it follows that there are $(n-k-1)E^0_{n+1-k}$ independent syzygies between pairs of monomials associated with these multi-indices $\mathbf{A}$ contributing to $\alpha + \beta$. This completes the proof of the lemma.

□

As in the previous cases, we verify that the number of independent syzygies agree with the calculation of $\gamma_{1,d}$ given in Section 4.1. Although this is not a necessary part of the argument, it is nonetheless instructive.

First, suppose that $\mathbf{A} = (A_0, A_1, ..., A_n)$ is a multi-index of length d such that $A_j > 0$ for all j. Let us count the number of independent syzygies between pairs of monomials associated with multi-indices of the form $\mathbf{A} - e_i$. Thus, we need to count multi-indices $\mathbf{A}$ of length d with no zero components. This is the same as counting multi-indices $\mathbf{A'} = (A_0 - 1, A_1 - 1, ..., A_n - 1)$ of length $d - (n+1)$. We know that there are $\binom{d-(n+1)+n}{n} = \binom{d-1}{n}$ multi-indices of length $d - (n+1)$ in $n+1$ variables. Observe that there are $E^+_{n+1} + E^-_{n+1} + E^0_{n+1}$ multi-indices $\mathbf{A}$ as described above. Let $E_{n+1} = E^+_{n+1} + E^-_{n+1} + E^0_{n+1}$. Then

$$E_{n+1} = \tbinom{d-1}{n}.$$

This, together with Lemma 6.1 implies that there are

$$n\binom{d-1}{n} = n(E_{n+1})$$

independent syzygies between pairs of monomials associated with multi-indices $\mathbf{A} - e_i$, where $\mathbf{A}$ is a multi-index of length d with no zero components.

Next, suppose that $\mathbf{A} = (A_0, A_1, ..., A_n)$ is a multi-index of length d such that $A_j = 0$ for exactly one j. We proceed to count independent syzygies between pairs of monomials associated with multi-indices $\mathbf{A} - e_i$. Hence, we must count the number of multi-indices $\mathbf{A}$ with exactly one zero component. We can choose which component is zero in $\binom{n+1}{1} = n+1$ different ways. Without loss of generality, we suppose $A_n = 0$. Note that the number of multi-indices $\mathbf{A} = (A_0, A_1, ..., A_{n-1}, 0)$ of length d is equivalent to the number of multi-indices $\mathbf{A'} = (A_0 - 1, A_1 - 1, ..., A_{n-1} - 1, 0)$ of length $d - n$. Again, there are $E^+_n + E^-_n + E^0_n$ of such multi-indices $\mathbf{A}$. Let $E_n = E^+_n + E^-_n + E^0_n$. Then

$$E_n = \tbinom{d-n+n-1}{n-1} = \tbinom{d-1}{n-1}.$$

On the other hand, Lemma 6.1 assures that only $n-1$ of the $\binom{n}{2}$ divided Koszul relations are independent. Then there are

$$(n+1)(n-1)\binom{d-1}{n-1}$$

independent syzygies between pairs of monomials associated with multi-indices $\mathbf{A} - e_i$, where $\mathbf{A}$ is a multi-index of length d with exactly one zero component. Generalizing this argument, we have that for all $2 \leqslant k \leqslant n+1$, the number of multi-indices $\mathbf{A} = (A_0, A_1, ..., A_n)$ of length d with $n+1-k$ zero components, denoted E_k, is

$$\binom{n+1}{k}\binom{d-1}{k-1}.$$

Furthermore, $E_k = E_k^+ + E_k^- + E_k^0$.

Finally, we consider multi-indices $\mathbf{A} = (A_0, A_1, ..., A_n)$ of length d with n zero components. Thus, the monomials associated to these multi-indices consist of all pure terms of degree d in $n+1$ variables $x_0^d, x_1^d, ..., x_n^d$. Hence, $E_1 = n+1$. Clearly,

$$\sum_{k=1}^{n+1} E_k = \sum_{k=1}^{n+1} \binom{n+1}{k}\binom{d-1}{k-1} = \binom{d+n}{n}.$$

Furthermore, the number of independent syzygies between pairs of monomials associated with multi-indices $\mathbf{A} - e_i$, where $\mathbf{A}$ is a multi-index of length d with $n+1-k$ zero components is

$$(k-1)\binom{n+1}{k}\binom{d-1}{k-1} = (k-1)E_k.$$

Because $E_k = E_{n+1-k}^+ + E_{n+1-k}^- + E_{n+1-k}^0$,

$$(k-1)\binom{n+1}{k}\binom{d-1}{k-1} = (k-1)(E_{n+1-k}^+ + E_{n+1-k}^- + E_{n+1-k}^0).$$

Observe that

$$\begin{aligned}
\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}\binom{d-1}{k-1} &= \sum_{k=2}^{n+1} k\binom{n+1}{k}\binom{d-1}{k-1} - \sum_{k=2}^{n+1}\binom{n+1}{k}\binom{d-1}{k-1} \\
&= \sum_{k=2}^{n+1}(n+1)\binom{n}{k-1}\binom{d-1}{k-1} - \sum_{k=2}^{n+1}\binom{n+1}{k}\binom{d-1}{k-1} \\
&= \sum_{k=1}^{n}(n+1)\binom{n}{k}\binom{d-1}{k} - \sum_{k=1}^{n}\binom{n+1}{k+1}\binom{d-1}{k} \\
&= (n+1)\left[\binom{d+(n-1)}{n} - 1\right] - \left[\binom{d+n}{n} - (n+1)\right] \\
&= (n+1)\binom{d+(n-1)}{n} - \binom{d+n}{n}.
\end{aligned}$$

This shows that there are in total $(n+1)\binom{d+(n-1)}{n} - \binom{d+n}{n}$ independent first syzygies of degree d. This result agrees with the earlier calculation of $\gamma_{1,d}$ from Section 4.1.

Note that

$$\begin{aligned}\alpha+\beta &= \sum_{k=2}^{n+1}\left[(k-1)(E_k^+ + E_k^-) + (k-2)E_k^0\right]\\ &= \sum_{k=2}^{n+1}(k-1)(E_k^+ + E_k^- + E_k^0) - \sum_{k=2}^{n+1} E_k^0\\ &= \gamma_{1,d} - \sum_{k=2}^{n+1} E_k^0.\end{aligned}$$

To get a good lower bound on $\alpha+\beta$, we need a good upper bound on E_k^0. In our earlier proofs, we obtained this upper bound by rewriting $E_2^0 + E_3^0 = \frac{1}{2}E_2^0 + \frac{1}{2}(E_2^0 + 2E_3^0)$. We want to find the best upper bound for $E_2^0 + E_3^0$ in the general case.

Lemma 6.3. *Let $n+1 \geqslant 4$. Then*

$$E_2^0 + E_3^0 \leqslant \frac{n(n+1)}{2}(d-1)\left[\frac{2}{3} + \frac{(n-1)(d-2)}{6}\right].$$

Proof. First, note that

$$E_2^0 \leqslant \binom{n+1}{2}(d-1) \quad \text{and} \quad 2E_3^0 \leqslant 2\binom{n+1}{3}\binom{d-1}{2}.$$

It is hard to visualize Newton diagrams for polynomials in more than four variables. However, the Newton diagram of a polynomial in $n+1$ variables consists of $\binom{n+1}{3}$ "faces" and $\binom{n+1}{2}$ "borders". Vertices on the faces of the Newton diagram correspond to those monomials of degree $d-1$ with at most three non-zero components, or equivalently, to monomials involving at most three distinct variables. Likewise, vertices on the borders correspond to monomials involving at most two distinct variables. Now, each border is part of $n-1$ distinct faces. As we saw in Chapter 4, the number of independent syzygies on each face that comes from one element of $\mathcal{A}$ and one element of $\mathcal{B}$ is at most d^2-d. It follows that

$$E_2^0 + 2E_3^0 \leq \binom{n+1}{3}(d^2-d).$$

However, in the above expression each edge contributing to E_2^0 was counted $n-1$ times. Therefore, a better bound is possible, namely

$$E_2^0 + 2E_3^0 \leq \binom{n+1}{3}(d^2-d) - (n-2)E_2^0.$$

Generalizing the reasoning we used in the three- and four- variable cases, we write

$$E_2^0 + E_3^0 = (1-\lambda)\, E_2^0 + \lambda(E_2^0 + 2E_3^0) + (1-2\lambda)\, E_3^0, \quad 0 \leqslant \lambda \leqslant 1/2.$$

Hence,

$$\begin{aligned} E_2^0 + E_3^0 &\leqslant (1-\lambda)\, E_2^0 + \lambda \left[\binom{n+1}{3}(d^2-d) - (n-2)E_2^0 \right] + (1-2\lambda)\, E_3^0 \\ &= [1-\lambda(n-1)]E_2^0 + \lambda \binom{n+1}{3}(d^2-d) + (1-2\lambda)\, E_3^0. \end{aligned}$$

Observe that the first term in the last equality could be positive or negative. If it is positive ($0 \leqslant \lambda \leqslant 1/(n-1)$), the best estimate we can hope for this term is the one obtained by replacing E_2^0 with $\binom{n+1}{2}(d-1)$. If, on the other hand, the coefficient of this first term is negative ($1/(n-1) \leqslant \lambda \leqslant 1/2$), then the best upper bound we can give for this first term is 0.

For the first case, we have that for $0 \leqslant \lambda \leqslant \frac{1}{n-1}$,

$$\begin{aligned} E_2^0 + E_3^0 &\leqslant [1-\lambda(n-1)] \binom{n+1}{2}(d-1) + \lambda \binom{n+1}{3}(d^2-d) \\ &\quad + (1-2\lambda) \binom{n+1}{3}\binom{d-1}{2} \\ &= [1-\lambda(n-1)]\frac{n(n+1)}{2}(d-1) + \lambda\frac{n(n+1)(n-1)}{6}(d^2-d) \\ &\quad + (1-2\lambda)\, \frac{n(n+1)(n-1)(d-1)(d-2)}{12} \\ &= \frac{n(n+1)}{2}(d-1)\left[1-\lambda(n-1) + \frac{\lambda(n-1)d}{3} + \frac{(1-2\lambda)(n-1)(d-2)}{6}\right] \\ &= \frac{n(n+1)}{2}(d-1)\left[1-\lambda(n-1)\left(1-\frac{1}{3}d+\frac{1}{3}(d-2)\right) + \frac{(n-1)(d-2)}{6}\right] \\ &= \frac{n(n+1)}{2}(d-1)\left[1-\lambda(n-1)\left(\frac{1}{3}\right) + \frac{(n-1)(d-2)}{6}\right]. \end{aligned}$$

The last expression is smallest when $\lambda = \frac{1}{n-1}$. Therefore,

$$E_2^0 + E_3^0 \leqslant \frac{n(n+1)}{2}(d-1)\left[\frac{2}{3} + \frac{(n-1)(d-2)}{6}\right].$$

For the second case, we have that for $1/(n-1) \leqslant \lambda \leqslant 1/2$,

$$\begin{aligned}
E_2^0 + E_3^0 &\leqslant \lambda \binom{n+1}{3}(d^2-d) + (1-2\lambda)\binom{n+1}{3}\binom{d-1}{2} \\
&= \lambda \frac{n(n+1)(n-1)}{6}(d^2-d) + (1-2\lambda)\frac{n(n+1)(n-1)(d-1)(d-2)}{12} \\
&= \frac{n(n+1)}{6}(d-1)\left[\lambda(n-1)d + \frac{(1-2\lambda)(n-1)(d-2)}{2}\right] \\
&= \frac{n(n+1)}{6}(d-1)\left[2\lambda(n-1) + \frac{(n-1)(d-2)}{2}\right].
\end{aligned}$$

The last expression is smallest when $\lambda = \frac{1}{n-1}$. Therefore,

$$\begin{aligned}
E_2^0 + E_3^0 &\leqslant \frac{n(n+1)}{6}(d-1)\left[2 + \frac{(n-1)(d-2)}{2}\right] \\
&= \frac{n(n+1)}{2}(d-1)\left[\frac{2}{3} + \frac{(n-1)(d-2)}{6}\right].
\end{aligned}$$

This complete the proof of the lemma. $\square$

6.2 The Minimum Rank Estimates

By Proposition 4.1,

$$\rho(SQ) \geqslant (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta.$$

Furthermore,

$$\alpha + \beta = \gamma_{1,d} - \sum_{k=2}^{n+1} E_k^0 \quad \text{and} \quad H_I(d) = (n+1)\gamma_{0,d-1} - \gamma_{1,d}.$$

Hence,

$$\begin{aligned}
\rho(SQ) &\geqslant (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \alpha + \beta \\
&= (n+1)\gamma_{0,d-1} - 2\gamma_{1,d} + \gamma_{1,d} - \sum_{k=2}^{n+1} E_k^0 \\
&= (n+1)\gamma_{0,d-1} - \gamma_{1,d} - \sum_{k=2}^{n+1} E_k^0 \\
&= \binom{d+n}{n} - \sum_{k=2}^{n+1} E_k^0 \\
&= \binom{d+n}{n} - (E_2^0 + E_3^0) - \sum_{k=4}^{n+1} E_k^0 \\
&\geqslant \binom{d+n}{n} - (E_2^0 + E_3^0) - \sum_{k=4}^{n+1} E_k \\
&= \binom{d+n}{n} - (E_2^0 + E_3^0) - \left(\binom{d+n}{n} - (E_1 + E_2 + E_3)\right) \\
&= E_1 + E_2 + E_3 - (E_2^0 + E_3^0).
\end{aligned}$$

Lemma 6.3 implies that

$$\begin{aligned}
\rho(SQ) &\geqslant n+1 + \binom{n+1}{2}(d-1) + \binom{n+1}{3}\binom{d-1}{2} - \frac{2}{3}\left[\frac{n(n+1)}{2}(d-1)\right] \\
&\quad - \frac{n(n+1)(n-1)(d-1)(d-2)}{12} \\
&= n+1 + \frac{n(n+1)}{2}(d-1) + \frac{n(n+1)(n-1)(d-1)(d-2)}{12} \\
&\quad - \frac{2}{3}\left[\frac{n(n+1)}{2}(d-1)\right] - \frac{n(n+1)(n-1)(d-1)(d-2)}{12} \\
&= (n+1) + \frac{n(n+1)}{6}(d-1).
\end{aligned}$$

6.3 Sharpness in the General Case

In this section, we address the question of whether the minimum rank estimate from Theorem 1.3 is sharp. We showed that this estimate is sharp for $n+1=4$. We will demonstrate that the estimate is not sharp when $n+1 \geqslant 5$.

We use simple MATLAB code to find the minimum rank for a homogeneous polynomial $P \in \mathcal{H}_{5,d}$, for $d=2,3$ and such that $P=SQ$, where Q is full of degree $d-1$ and $S=x_0+\cdots+x_4$. Also, we give explicit formulas for these two sharp polynomials. Based on the formulas of the sharp polynomials of degree 2 and 3 in five variables, we give a family of polynomials P_d for d arbitrary, with the same properties as above which we conjecture are sharp. As we will see below, the code requires one to find the ranks of a very large number of large matrices, and so it only runs in a sensible amount of time for $d=2$ and 3. We will list the polynomials thus obtained for $d=2$ and d=3. Even though this is very limited data, it will be enough to see that the estimate we have proved is not sharp and it will be enough to make a conjecture about the sharp polynomials. I would like to thank Javier Perez and Brad Ochocki for helping me with the implementation of the MATLAB code.

Before showing the MATLAB code, we explain through an example the idea behind the code. Consider $n+1=4$ and use x,y,z,w instead of x_0,x_1,x_2,x_3 to avoid subscripts. Let $Q \in \mathcal{H}_{4,1}$ be arbitrary and let $S=x+y+z+w$. Then $Q=Ax+By+Cz+Dw$ for some $A,B,C,D \in \mathbb{R}$ and

$$\begin{aligned} SQ = & Ax^2+(A+B)xy+(A+C)xz+(A+D)xw+By^2+(B+C)yz+(B+D)yw \\ & +Cz^2+(C+D)zw+Dw^2. \end{aligned}$$

We write the vector containing coefficients of the polynomial SQ as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Recall that the aim is to find the minimum number of terms in SQ. That is, we are trying to make as many coefficients as possible equal to zero. Because Q is full, the polynomial SQ must have the pure terms Ax^2, By^2, Cz^2, and Dw^2. Thus, we consider the vector containing coefficients of SQ that do not correspond to pure terms

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}.$$

For the full matrix T, the system

$$T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution. So we are looking for submatrices T' of T obtained by deleting some rows for which the corresponding system

$$T' \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \mathbf{0}$$

has non-trivial solutions. Since T has rank four, we are interested in submatrices T' of rank three. As we will see below, submatrices with 5 rows do not work. However, there are submatrices T' with 4 rows such that the system

$$T' \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has non-trivial solutions. The MATLAB code is presented in detail in the Appendix A.

Let us check the code with the matrix T. The input and output are as follows:

```
>> T=[1 1 0 0; 1 0 1 0; 1 0 0 1; 0 1 1 0; 0 1 0 1; 0 0 1 1];
>> newcheck(T,5)
No combinations found!
Number of combinations tried:
     6
```

The code is looking for submatrices of T with 5 rows ($k = 5$) and rank 3. The code did not find any combination with this property. The number of combinations tried was 6. That makes sense because from 6 rows we choose 5 rows in $\binom{6}{5} = 6$ different ways. Next, we check for submatrices with 4 rows ($k = 4$) and rank 3.

```
>> newcheck(T,4)
Found a solution on iteration:
     5

     2     3     4     5

    -1
    -1
     1
     1
```

The code found a combination on iteration 5. The submatrix T' composed of rows 2,3,4 and 5 has rank 3. Hence,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = T' \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $A = -1$, $B = -1$, $C = 1$, and $D = 1$. The remaining two rows, 1 and 6, yield two non-zero terms in SQ. These two terms, together with the four pure terms give a total of 6 non-zero terms in SQ:

$$SQ = -x^2 - 2xy - y^2 + z^2 + 2zw + w^2 = -(x + y)^2 + (z + w)^2.$$

In Chapter 5, we proved that if $P \in \mathcal{H}_{4,d}$, $P = SQ$, Q is full of degree $d - 1$, and $S = x_0 + \cdots + x_3$, the minimum rank is given by the formula $\rho(P) = 2d + 2$. The above polynomial has degree $d = 2$, so $\rho(SQ) = 2(2) + 2 = 6$. The result agrees with the one obtained by the MATLAB code.

Now that we understand how the MATLAB code works, we present the sharp polynomials we obtained with the MATLAB code for polynomials $P_d \in \mathcal{H}_{5,d}$ for $d = 1, 2, 3$, where $P_d = SQ$, Q is full of degree $d - 1$, and $S = x_0 + \cdots + x_4$. We use variables x, y, z, w, t, instead of x_0, x_1, x_2, x_3, x_4 to avoid subscripts.

$$\begin{aligned} P_1(x, y, z, w, t) &= x + y + z + w + t \\ P_2(x, y, z, w, t) &= (x + y)^2 - (z + w)^2 - t^2 - 2t(z + w) \\ P_3(x, y, z, w, t) &= (x + y)^3 + (z + w)^3 + t^3 - 3t(x + y)(z + w) \end{aligned}$$

The matrix approach is not feasible for $d \geqslant 4$. For instance, if $d = 4$, the corresponding matrix T has size 65×35. If $k = 57$, there are $\binom{65}{57}$ different ways to choose submatrices T' with 57 rows. This is too expensive to calculate with the MATLAB Code. If it works, we would have polynomials with 13 non-zero terms, which is not possible because we showed above that if $d = 4$, $\rho(P_d) = 15$. Thus, we need a number $k < 57$, which makes the code even more expensive.

Following the pattern of the Newton diagrams of the above polynomials, we conjecture

$$
\begin{aligned}
P_4(x,y,z,w,t) &= (x+y)^4 - (z+w)^4 - t^4 - 4t(x+y)^2(z+w) + 2t^2(z+w)^2 \\
P_5(x,y,z,w,t) &= (x+y)^5 + (z+w)^5 + t^5 - 5t^3(x+y)(z+w) + 5t(x+y)^2(z+w)^2 \\
P_6(x,y,z,w,t) &= (x+y)^6 - (z+w)^6 - t^6 - 6t(x+y)^4(z+w) + 9t^2(x+y)^2(z+w)^2 \\
&\quad - 2t^3(z+w)^3 \\
P_7(x,y,z,w,t) &= (x+y)^7 + (z+w)^7 + t^7 - 7t^5(x+y)(z+w) + 14t^3(x+y)^2(z+w)^2 \\
&\quad - 7t(x+y)^3(z+w)^3 \\
&\vdots
\end{aligned}
$$

The minimum rank estimate obtained in the previous section assures that if $R = \mathbb{R}[x_0, x_1, x_2, x_3, x_4]$, $S = x_0 + \cdots + x_4$, and $Q \in R$ is full of degree $d-1$, then

$$\rho(SQ) \geqslant \frac{10d+5}{3}.$$

If this were sharp, our potential polynomials would satisfy $\rho(P_1) = 5$, $\rho(P_2) = \lfloor \frac{25}{3} \rfloor = 8$, $\rho(P_3) = \lfloor \frac{35}{3} \rfloor = 11$. However, the MATLAB code found a degree 3 polynomial P_3 with $\rho(P_3) = 13$ and that there are not polynomials P_3 3 with less than 13 terms. Therefore, the general estimate for polynomials in $n+1$ variables

$$\rho(SQ) \geqslant (n+1) + \frac{n(n+1)}{6}(d-1).$$

is not sharp.

Let us consider the family of sharp polynomials in three variables introduced in Section 2.5:

$$F_d(a,b,c) \;=\; a^d - (-b)^d - (-c)^d + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} (-1)^s K_{ds}\, a^s b^s c^{d-2s},$$

where the coefficients K_{ds} are defined as follows. Set $K_{d0} = 0$ and

$$K_{ds} \;=\; \binom{d-s}{s} + \binom{d-s-1}{s-1} \quad \text{for} \quad 1 \leqslant s \leqslant \lfloor \tfrac{d}{2} \rfloor.$$

For d odd, we recall the first few polynomials are

$$\begin{aligned}
F_1(a,b,c) &= a+b+c \\
F_3(a,b,c) &= a^3+b^3+c^3-3abc \\
F_5(a,b,c) &= a^5+b^5+c^5-5abc^3+5a^2b^2c \\
F_7(a,b,c) &= a^7+b^7+c^7-7abc^5+14a^2b^2c^3-7a^3b^3c \\
&\vdots
\end{aligned}$$

For d even, consider

$$F_d(a,b,c) \;=\; a^d-(-b)^d-(-c)^d+\sum_{s=1}^{\lfloor \frac{d}{2}\rfloor}(-1)^s K_{ds}\; a^{d-2s}b^sc^s,$$

Then

$$\begin{aligned}
F_2(a,b,c) &= a^2-b^2-c^2-2bc \\
F_4(a,b,c) &= a^4-b^4-c^4-4a^2bc+2b^2c^2 \\
F_6(a,b,c) &= a^6-b^6-c^6-6a^4bc+9a^2b^2c^2-2b^3c^3 \\
&\vdots
\end{aligned}$$

Observe that if we set $a=x+y$, $b=z+w$, and $c=t$, we get the above polynomials P_d. The previous approach and the results already obtained for polynomials in $\mathcal{H}_{n+1,d}$, with $n+1=1,2,3,4$ suggest the following conjectures.

Conjecture 6.1. *Let $P\in\mathcal{H}_{n+1,d}$. Let Q be a full polynomial of degree $d-1$ and let $S=x_0+x_1+\cdots+x_n$ such that $P=SQ$. If $n+1=2k$ for some positive integer k, a family of sharp polynomials is given by*

$$P=\left(\sum_{j=0}^{k-1}x_j\right)^d+(-1)^{d-1}\left(\sum_{j=k}^{2k-1}x_j\right)^d.$$

If there is a positive integer k such that $n+1=2k+1$, a family of sharp polynomials is given by

$$\begin{aligned}
P \;=\; &\left(\sum_{j=0}^{k-1}x_j\right)^d-\left(-\left(\sum_{j=k}^{2k-1}x_j\right)\right)^d-(-x_{2k})^d \\
&+\sum_{s=1}^{\lfloor \frac{d}{2}\rfloor}(-1)^sK_{ds}\;\left(\sum_{j=0}^{k-1}x_j\right)^s\;\left(\sum_{j=k}^{2k-1}x_j\right)^s\;x_{2k}^{d-2s},
\end{aligned}$$

where the coefficients K_{ds} are defined as follows. Set $K_{d0} = 0$ and

$$K_{ds} \;=\; \tbinom{d-s}{s} + \tbinom{d-s-1}{s-1} \quad \textit{for} \quad 1 \leqslant s \leqslant \lfloor \tfrac{d}{2} \rfloor.$$

Conjecture 6.2. *Let $R = \mathbb{R}[x_0, x_1, ..., x_n]$. Let $S = x_0 + \cdots + x_n$ and let Q be a full homogeneous polynomial of degree $d-1$. If $n+1 \geqslant 4$ and $n+1 = 2k$ for some positive integer k,*

$$\rho(SQ) \geqslant 2\binom{d+k-1}{k-1}.$$

If $n+1 = 2k+1$ for some positive integer k and d is odd,

$$\rho(SQ) \geqslant 2\binom{d+k-1}{k-1} + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} \binom{s+k-1}{k-1}^2 + 1.$$

If d is even,

$$\rho(SQ) \geqslant 2\binom{d+k-1}{k-1} + \sum_{s=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-2s+k-1}{k-1}\binom{s+k-1}{k-1} + 1.$$

The conjecture is for $n+1 = 5$ and 6 at least.

Appendix A

MATLAB Code

```
function newcheck(A, k)

[n, m] = size(A);

nextCombination = zeros(1, n);
nextCombination(1:k) = 1;

found_solution = 0;
progress_bar = waitbar(0, 'Please wait...');
total_combinations = nchoosek(n, k);


for i = 1:total_combinations

    if mod(i, 1000) == 0
        waitbar(1/total_combinations, progress_bar, i)
    end

    % perform calculation for current combination
    rows = find(nextCombination);

    B = A(rows, :);
    Z = null(B, 'r');

    if rank(B)==(m-1) && nnz(Z)==m
        found_solution = 1;
        break
    end
```

```
    % increment combination
    nextCombination = nextComb(nextCombination);
end

if found_solution
    disp('Found a solution on iteration:')
    disp(i)
    disp(rows)
    disp(Z)
else
    disp('No combinations found!')
    disp('Number of combinations tried:')
    disp(total_combinations)
end

close(progress_bar)
```

The function nextComb is defined by

```
function [nextc] = nextComb(oldc)
   o = find(oldc, 1);                    %// find the first one
   z = find(~oldc(o+1:end), 1) + o;    %// find the first zero
                                           *after* the first one

   nextc = oldc;
   nextc(1:z-1) = 0;
   nextc(z) = 1;                       %// make the first zero a one
   nextc(1:nnz(oldc(1:z-2))) = 1;    %// move previous ones to
                                           the beginning

end
```

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