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Caleb Huber

University of Montana, Missoula, fox14_gamer@hotmail.com

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Introductory Calculus: Through the Lenses of Covariation and Approximation

Caleb Huber

University of Montana

caleb1.huber@umontana.edu

Abstract

Over the course of a year, I investigated reformative approaches to the teaching of calculus. My research revealed the substantial findings of two educators, Michael Oehrtman and Pat Thompson, and inspired me to design a course based upon two key ideas, covariation and approximation metaphors.

Over a period of six weeks, I taught a course tailored around these ideas and documented student responses to both classroom activities and quizzes. Responses were organized into narratives, covariation, rates of change, limits, and delta notation. Covariation with respect to rates of change was found to be incredibly complex, and students would often see it as a series of steps rather than a simultaneous occurrence. With regards to rates of change, students went from seeing the average rate of change as some mean of variation to a change in y divided by the change in x within some acceptable error bound. Limits were a new concept to students, and they ended the course with an understanding of limits as finding an approximation for some value within an acceptable bound. Similar to limits, delta notation was also new to the students. Although it helped students better articulate their thoughts, the context in which students used it to describe change was oftentimes not mathematically rigorous.

Besides these four narratives, evidence was also shown that students may gain deeper insights from problems based outside of the traditional physics context, such as velocity. These findings resulted in a list of suggestions of how the course might be implemented in the future so as to better ensure that students have a deeper conceptual understanding of derivatives.

Introductory Calculus: Through the Lense of Covariation and Approximation

As mathematics education continues to advance instruction, care must be made to not focus all advancements on the “simpler” math subjects. This fact is only further emphasized by the tendency of calculus to function as a barrier to higher mathematics for students entering their first undergraduate years (Bressoud, 2015). An article by Larsen et. al. (2017) noted that although there has been research on the understanding of students’ alternate conceptions over the past several decades, very little research has been done on the practical application of this research in the classroom. Two exceptions are Pat Thompson and Michael Oehrtman. Not only have both designed and implanted their curriculums into the working classroom, but they have published substantial work on the research which eventually led to their respective curriculum’s core values.

When studying these two curriculums, I noticed that both touted the potential for students to understand the concepts of calculus, rather than simply mastering procedures. It was because of this that I became interested in seeing potential insights into how they impact a calculus classroom first hand. Specifically, I wanted to see how Oehrtman’s curriculum, called CLEAR Calculus (<http://clearcalculus.okstate.edu>), could be used to teach students formal calculus notation in a meaningful way and how Thompson’s work could be used to support these endeavors. Thus when I was given the opportunity to teach a calculus course of my own design for a period of six weeks, I built the course primarily around Oehrtman’s CLEAR calculus and set the formal definition of the derivative as the overall theme. Specifically, I hoped that students would develop a rich *concept image* of the formal definition of the derivative that matches the *concept definition*.

Davis and Vinner (1986) defined concept images as the knowledge representation structures or frames generated by students as they come across problems. Concept definitions are the formal definitions of mathematical concepts. Definitions lie in contrast to images in that every definition invokes certain images, but the converse is not necessarily true. Research found that many inconsistencies in student understanding can be traced back to a contradiction between their concept images and the concept definition (Vinner, 1983; Dreyfus and Vinner, 1982). In many ways Oehrtman and Thompson’s work can be seen as the study of how to bridge these inconsistencies.

My preference for Oehrtman’s work as the primary resource came from not only its design but the research it was built upon. Davis and Vinner (1986) had found that students do not learn limits from a blank state. Rather, they come with specific concept images in the form of metaphors, including conceptualizing limits as an approximation or boundary. Oehrtman (2009) built off this research and

found that the metaphors utilized by students to tackle new concepts had a variety of effectiveness. The most effective of these was found to be approximation. Thus he built his curriculum to continually utilize this metaphor. Specifically, students are continually asked the following questions: What are we approximating? What are the approximations? What is the error? What is the error bound? Can we generalize our approximation to be within any given error bound? (Oehrtman, 2008) I believed these questions to be essential to understanding the limit process in the definition of the derivative.

Note that the formal definition of the derivative does indeed contain a limit process, but it also contains a ratio representing covariation and rates of change. Thompson found that student's do not see derivatives as rates of change and covariation. Rather, they learn derivatives to be procedural in nature. Thus I needed Thompson's curriculum to address these concepts.

Research Questions:

1. How can students be guided to understand calculus in such a way that their concept image of the formal definition of the derivative indeed matches the concept definition.
2. What do students learn about rates of change and limits in the process?

Literature Review

Thompson: Variation and Covariation

Most students first learn rates of change in its simplest form as "rise over run", but such a simplistic understanding hides just how complex it can be to define either "rise" or "run", let alone the ratio between them. For example: To calculate the "run", one must first select two x values. These values, being in one dimension, rely on an understanding of how x-axis may vary. Simultaneously, these x values will have a covariational relationship with some y values, and the distance between these y values forms the "rise". Thus, varying these two choices of x necessarily varies the "rise over run" ratio and, therefore, implies another form of covariational relationship. Thus we see understanding rates of change to be incredibly complex, and derivatives, being themselves rates of change, suffer from the same complexity. Furthermore, such understandings have been argued to be "epistemologically necessary for students and teachers to develop useful and robust conceptions of functions" (Thompson and Carlson, 2017, p. 423).

Covariation as we know it today is relatively new to the mathematical landscape. As Kleiner (1989) noted, there are four eras of functional mathematical understanding. The first era concerned proportional reasoning. This era was defined by geometric relationships which, being oftentimes viewed as moment by moment instances, represented motion statically. The second era was that of the equation. It was "characterized by the use of equations to represent constrained variation in related

quantities' values" (Thompson and Carlson, 2017, p. 422). For example, an equation such as $3x+4y=1$ shows that y varies $-3/4$ for every increase of 1 for x . The third era was represented by continuous variation and the development of function notation such as $f(x)$. The fourth era was defined by "values of one variable being determined uniquely by values of another" (Thompson and Carlson, 2017, p. 422). This era is significant in that it is still the era we are in today. Since functions such as Dirichlet's function where $f(x)$ is 0 unless x is rational are most representative of this era, it could be argued that the modern function era is dominated by mathematical analysis.

Although covariational change presents itself as early as the second era (about 1000 A.D.), its use was always under some tacit understanding rather than through an explicit definition (Thompson and Carlson, 2017, p. 423). This has changed as researchers have sought to fundamentally define covariation as a theoretical construct. Covariation as a construct entails two essential attributes. The first is the idea of variation over individual quantities. The second is the idea of variation over two or more quantities simultaneously (Thompson and Carlson, 2017, p. 423). A simple example modeling this behavior is the idea of runner's location on a track. A common approach to this problem is to assign some variable, d , to the runner's distance along the track. When a student thinks of the d varying, the student may be only envisioning the runner moving away from the starting line (Thompson and Carlson, 2017, p. 424). True covariational understanding occurs when the student understands that any change in d guarantees a simultaneous change in some other quantity, such as time.

Thompson and Carlson (2017) note that research recognizes six levels of covariational understanding.

1. **Smooth Continuous** - Students see a change in one variable simultaneously changing another.
2. **Chunky Continuous** - Students see variation as an event happening simultaneously over several variables, but they only understand variation as isolated, discrete events.
3. **Coordination of values** - Students coordinate a set of singular variable (x) and then a set of another (y). Next, they join these two sets to create a discrete collection (x,y).
4. **Gross coordination of values** – Students see values as loosely changing together but lack any form of quantitative measurement. For example, a student with this understanding may use statements such as: "They both go up."
5. **Precoordination of values** - Students see variables taking turns in their variation. One varies, then the other, then the first, etc.

6. **No coordination** – *The student sees no coordination of variation. Either one variable varies or the other. (Thompson and Carlson, 2017, p. 435)*

Similarly, they note six levels of variational understanding.

1. **Smooth Continuous** – *Students see a variable as varying smoothly and continuously.*
2. **Chunky Continuous** – *Students see a variable's variation as isolated, discrete events.*
3. **Gross Variation** – *Students see a variable as increasing or decreasing, but they have no form of quantitative understanding.*
4. **Discrete Variation** – *Students see a variable as alternating over specific points but fail to recognize the existence of any points lying within them.*
5. **No Variation** – *Students see a variable as fixed. Any changes to the variable are, in fact, just alternate fixed cases.*
6. **Variable as Symbol** – *Students see variables as a symbol with no variational significance.*

Although this list seems rather daunting, Thompson and Carlson (2017) argue that covariational reasoning is not something that takes many years to learn (p. 445). Furthermore, such reasoning is essential “for students to learn advanced mathematical ideas” (Thompson and Carlson, 2017, p. 449).

Oehrtman: Pedagogy

One approach to calculus, unique to Micheal Oehrtman, is based upon spontaneous ideas of approximation. The curriculum is focused around the use of limits to rigorously and effectively approximating phenomena. Furthermore it, addresses how to design instruction on limit concepts, identifies crucial alternative conceptions, and utilizes frequent approximations to develop rigorous understanding.

Oehrtman (2009) notes that the study of teaching abstract mathematical ideas, such as limits, lends itself to the research of John Piaget (Piaget, 1970a, 1970b, 1975, 1980, 1985, 1997)., who saw teaching mathematics as “*actions or coordinations of actions on physical or mental objects*” (as cited by Oehrtman, p. 66). Not only must such actions recur repeatedly, but feedback must be given and incorporated at every iteration. Thus Oehrtman (2008) sees instructional activity to have three key responsibilities. First, any desired outcomes must be modeled by the activities for which students are expected to perform (Oehrtman, p. 66). Note that this does not mean only using formal definitions, nor does it mean using only informal language to describe behavior. Rather, “conceptual structures that already make sense to students” must be built, internalized, and then used to tackle the formally abstract (Oehrtman, 2008, p. 66). Second, instruction should coordinate student actions in such a way that the constraints of the system provide continual and repetitive feedback (Oehrtman, 2008, p. 66).

Finally, the limit concept should become a familiar tool, one which is revisited throughout the entirety of the course (Oehrtman, 2008, p. 66). This stands in stark contrast to more traditional curriculums which tend to teach calculus in chunks (limits, derivatives, integrals,...).

Oehrtman's (2008) curriculum relies on the five design principles listed below:

1. *Identify the mathematical structures that must be reflected in the instructional activities.*
2. *Identify a structurally equivalent conceptual system and language base that is accessible to students.*
3. *Develop, test, and refine instructional activities in which students apply the framework to particular applications.*
4. *Repeat Step 3 for a variety of applications of the concept.*
5. *Design tasks to foster formalization as an end result. This includes naming or symbolizing a structure that has already been abstracted and can lead to discussion and use of formal definitions and proofs.*

One immediate aspect of these principles is the exposure to the reasoning behind formal definitions and proofs. Many calculus textbooks include a sample of complex proofs. One of the most common examples is the epsilon-delta proof. However, Oehrtman (2008) notes that "since most introductory calculus courses are not intended to provide a rigorous treatment of analysis" and thus suffer from such instances being all too brief to have any meaningful impact (p. 67). Through Oehrtman's (2008) curriculum, students learn about limits through a natural progression of abstraction. This is in sharp contrast to the standard curriculum which always presents the formal definition first (Oehrtman, 2008, p. 67). Furthermore, setting formalization as the end goal allows for coherence between steps, as each step is clearly a natural progression from the preceding one. As Oehrtman (2008) states, "the treatment should be mutually reinforcing across the entire calculus curriculum" (p. 70).

Oehrtman's (2014) work can also be seen as an expansion of Swinyard and Larson's (2012) work on the progression of student understanding as a limiting process (as referenced in Oerthman, Swinyard, and Martin, p. 134). Such research has suggested that students learn limits through an iterative process. Students visualize a value, a_n , being approximately close to their desired outcome or limit, L . They then visualize a new graph where a_n is now closer to the desired outcome than it was before. Note that for each iteration the graph is static. Sfard (1992) noted that such an understanding may be useful in that it allows students to see the properties of what they are working with on a more manageable scale, but inversely, such an understanding may prevent students from understanding how limits can terminate to

a specific value (as cited by Oehrtman, Swinyard, and Martin, 2014, p. 135). Oehrtman (1992) makes the claim that, "students must develop a condensed image of all instantiations of intervals of possible variation in order to recognize the possibility of universal quantification on ϵ and to incorporate it into their understanding of formal limit definitions" (p. 135). Such a process can be found repeatedly in Oehrtman's curriculum. When a student has found a bound and an error, they are asked how they might generalize their solution to an arbitrary case. If what they have calculated is not accurate enough, the student must try yet again. Ever present is the goal for generalization. Oehrtman (2012) calls such a process guided reinvention and notes that "The students' cognitive progress in the guided reinvention was more true to the process of doing mathematics through constructing a meaningful and useful definition by resolving issues that were truly problematic to them" (p. 146.).

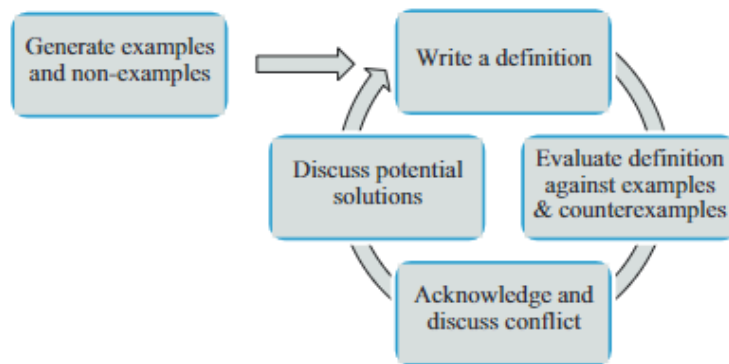


Fig. 2. Iterative refinement in the process of guided reinvention of a formal definition.

(Oertman, Swinyard, and Martin, 2014, p. 135)

Thus we see not only the psychology behind Oehrtman's curriculum but the reason why it places so much emphasis on student discovery as well.

Oehrtman: The Importance of Language

Of the five design principles presented as essential to Oehrtman's curriculum, principle two could be viewed as one of the most essential. Furthermore, just as covariation is a key theme to Thompson's research, accessible language can similarly be viewed as the bread and butter of Oehrtman's work. "Williams (1991) found students' exhibited strongly held sets of beliefs typically surrounding the contexts in which they were first exposed to limits and that their viewpoints were extremely resistant to change, even in response to explicit discussions about contradictory examples. (as cited by Oehrtman, 2008, p. 73)" Furthermore, Maxwell Black (1962a, 1977) noted that these student beliefs could be seen as metaphors defined by their emphasis, "commitment by the producer" and resonance, "support for high degrees of elaborative implication" (as cited by Oehrtman, 2008, p. 396).

Thus Oehrtman (2009) concludes that it is necessary to understand both the structure and function of students' metaphors (p. 399).

Oehrtman's work on limit metaphors yielded eight common clusters. Three of which were considered to be lacking in emphasis and resonance. The first of these metaphors was Motion Imagery and Interpretations of "Approaching". Students using these metaphors oftentimes included the words "approaching" or "tends to" in their descriptions yet tended to actually describe sequential instances (Oehrtman, 2009, p.405). The second weak metaphor was Zooming Imagery and Interpretations of Local Linearity. This metaphor can be used to describe zooming in infinitely close to a curve to determine behavior such as differentiation. Although research has been done suggesting such metaphors might be useful, Oehrtman (2009) found that most students did not bother using this metaphor even after seeing it in lecture (p. 406). The last and perhaps most interesting weak metaphor was Interpretations of Arbitrary and Sufficient. To students with a firm mathematical background, "arbitrary" can be seen as a universal quantifier such as "for any" and sufficient can be seen as the existential quantifier "there exists", yet students most often saw them as a progression of degree, such as "small" to "very very small" (Oehrtman, 2009, p. 408).

Of the eight metaphor clusters Oehrtman identified, five were considered to be "strong metaphors". The first of which was the metaphor of "collapsing dimensions". "These metaphors all involved an image of a multidimensional object varying in size along one of its dimensions. Corresponding to the independent variable in the limit going to zero, this dimension was ultimately imagined to vanish, resulting in a 'collapsed' object of reduced dimension." (Oehrtman, 2009, p. 410) The collapsing dimension metaphor became exceptionally powerful when describing paradoxes such as "Torricelli's trumpet" having finite volume. In such examples, students visualize a cross section approaching zero and essentially pinching off the three dimensional shape.

The next strong metaphor was the approximation metaphor. From a historical perspective, the approximation metaphor can be viewed as the backbone of historical calculus. The metaphor entails students recognizing complex calculus concepts in terms of error, bound, and approximation. Such understanding lends itself exceptionally well to series, sequences, and derivatives. For example, students learning Taylor series used approximation metaphors to justify that the maximum error for any finite expansion was merely the next term (Oehrtman, 2009, p. 415).

The third metaphor was the proximity metaphor. The proximity metaphor was unique in that it could be viewed as either incredibly strong or incredibly weak but such discernment is masked by students' lack of articulation when expressing such metaphors (Oehrtman, 2009, p. 416). For example,

students discussing a Taylor approximation to $\sin(x)$ used terminology such as “more and more loosely fitted around the curve” to describe adding terms approximated $\sin(x)$ (Oehrtman, 2009, p. 417). Another interesting example was the tendency of students to describe points becoming closer to something for both x and y . Such language can imply a degree of sophisticated covariational understanding, yet it can also lead to students making incorrect assumptions such as: “if two points x and y are close together, then the function values $f(x)$ and $f(y)$ will also be close” (Oehrtman, 2009, p. 415).

The fourth metaphor was that of infinity as a number. In this metaphor, students learned to treat infinity as a very large number which could then be used to solve problems algebraically. A unique aspect of such an approach is in how it lends itself to a student understanding of infinitesimals (Oehrtman, 2009, p.417). Rather than memorizing that the limit of $1/n$ as n approaches infinity is zero, students see the problem as a infinitesimal quantities which as infinity grows become essentially “nonexistent in size” (Oehrtman, 2009, p. 417). Students using this metaphor also gain a sense of the different sets of infinity. For example: When solving L’Hospital’s rule, students in Oehrtman’s (2009) study saw the indeterminate form $\frac{\infty}{\infty}$ as two functions growing at different rates (p. 418). Thus the idea of $\frac{\infty}{\infty} = 0$ seems reasonable if the denominator was growing “fast enough”. Such understanding could easily be extended to discuss the dimensionality of a set in later more advanced analytical courses.

The fifth and final metaphor was physical limitation metaphors. Such metaphors placed physical limitations on objects, proclaiming that “there is a scale beyond which nothing can be observed, be measured, or even exist” (Oehrtman, 2009, p. 418). For example, one student justified the finite volume of Torricelli’s trumpet by asserting that “at some point, a single molecule would plug up the container, allowing the rest to fill” (Oehrtman, 2009, p. 419).

Of all these metaphors, Oehrtman (2009) chose to focus his calculus curriculum primarily on the use of approximation metaphors. Not only did they have the most resonance as students struggled to understand complex ideas, but they tended to demonstrate significant emphasis in their equivalence to “epsilon-delta and epsilon-N arguments typically considered beyond the comprehension of students in introductory calculus” (Oehrtman, 2009, p. 421). In comparison to other strong metaphors, approximation metaphors tend to also not rely on visual metaphors such as “filling a region with paint” which tend to oversimplify the problem in question. Thus Oehrtman (2009) decided to base much of his curriculum around the following five questions: (a) What is being approximated? (b) What are the approximations? (c) What are the errors? (d) Given an approximation how can you find the bound on

the error? and (e) Given a desired bound on error, how can you generate an approximation with that level of accuracy? (p. 421)

Through Oehrtman’s significant work on metaphors, one sees the importance of proper language to student understanding. Such language should still be accessible to students, as described in step two, but should be comprised of strong metaphors which are most likely to resonate throughout the curriculum. Furthermore, initial activities should include substantial scaffolding. As students work through the curriculum, such scaffolding should slowly be replaced with an expectation that students “begin to remember or develop appropriate strategies to solve increasingly more sophisticated problems” (Oehrtman, 2008, p. 74).

Oehrtman: An Example

Consider the following example from Oehrtman’s curriculum:

Instructions: You will approximate the instantaneous rate of change for one of the situations below by answering each of the following questions algebraically, numerically, and by representing each answer in your diagram:

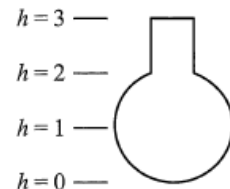
0. Draw a large picture of the physical situation for the value of the variable given.
1. Imagine how things are changing in this situation. List all of the quantities that you think are changing. Describe how they are changing.
2. On the same picture, draw several “snapshots” of the situation.
3. Label the changing and constant quantities in your drawing.
4. Describe in more detail what you have been asked to approximate.
5. What can you use for approximations?
6. What are the errors?
7. Find an approximation and a bound for the error. What is the resulting range of possible values for your instantaneous rate?
8. How can you find an approximation with error smaller than a predetermined bound?

Context 1: An object is falling according to the equation $h(t) = 100 - 16t^2$ feet (with t measured in seconds). Approximate the speed when $t = 2$ seconds.

Context 2: Approximate the instantaneous rate of change of the area of a circle with respect to its radius when the radius is 3 cm.

Context 3: The force of gravity between two objects is inversely proportional to the square of the distance separating them. Approximate the instantaneous rate of change of the gravitational force with respect to distance when two objects are 230 km apart. (Note that all of your answers will involve the constant of proportionality.)

Context 4: Approximate the rate of change of the height of water in this bottle with respect to the volume of water when the height is 1.5. (Note that your answers will involve the size of the spherical portion of the bottle.)



Context 5: The half-life of Iodine-123, used in some medical radiation treatments, is about 13.2 hours. Thus a sample that originally has $6.4 \mu\text{g}$ of Iodine-123 will decay so that the amount left after t hours will be roughly $I(t) = 6.4\left(\frac{1}{2}\right)^{t/13.2} \mu\text{g}$. Approximate the instantaneous rate at which the Iodine-123 is decaying after 5 hours.

(Oehrtman, 2008, p. 76)

Several important features immediately become apparent. There is a natural progression from an informal drawing to the formal analytical solution of finding an “approximation with error smaller than a predetermined bound”. In fact, Oehrtman (2008) notes that an emphasis is always placed on finding “the size of errors” and making them “smaller than any predetermined bound” (p. 78). Such efforts not only emphasizes coherence but resulted in some students making sense “out of the epsilon-delta definition in terms of their approximation language, at which point they began interchanging language and symbols related to approximation and the formal definition” (Oehrtman, 2008, p. 78). Also note that students are asked to invoke their mental approximation of the problem through a picture for step zero. Furthermore, they are expected to describe what is changing rather than immediately calculate the numerical rate. Finally, the problem is applied to a variety of contexts so as to invoke coherent instruction and continual, repetitive feedback. Thus we see how Oehrtman effectively utilizes rigorous questioning to tie pedagogical research into every problem presented.

Research Methods

Setting

This study took place in a 6 weeks summer Upward Bound course for high school students. Six students participated in the course. Classes ran for fifty minutes four to five days a week, depending on the occasional scheduled extracurricular field trip. All of the students had taken two years of high school algebra, but I soon learned that most students were not fluent with many Algebra II concepts and skills.

Design

The goal of the course was to help students understand the formal definition of the derivative. Although I had personal hopes to make it farther than the formal definition of the derivative, students’ lack of experience outside of a non-traditional classroom suggested that planning that far might be unrealistic. At the same time, my experience as a high school teacher had shown me that the proper intrinsic work ethic amongst students could result in us making it as far as simple optimization. It was with these observations in hand that I designed my study using an action research approach. I began with a rough plan, in the form of a conjectured learning trajectory (see Table 1). I then used daily plan-implement-reflect cycles. Immediately prior to every class I would plan activities and lecture material, which I would then implement in class. Immediately after every class, I reflected on what had worked, what hadn’t worked, and how these results might alter my plans for the next day. My plans were guided by my overall plan as well as my reflections. In this way, they were malleable to the last lessons

insights, and I regularly modified the homework tasks and quizzes to be implemented based entirely upon how the students had performed on the homework from the day before.

Rather than traditional lecture, the course was built on student’s active engagement in mathematical activity. Thus, on a given day, students spent the majority of the class time solving problems and working on labs. During this time, they were broken into either pairs or groups of three, and the majority of work was done on whiteboards. These activities were followed by summative mini lectures. In the mini lectures, I summarized the main take-aways of the activities, with a particular emphasis on connecting students’ work with formal language and symbols.

Data collection

Throughout this period, I collected data through a variety of primarily qualitative means. The first was through the collection of semi-daily homework assignments and weekly quizzes. The second was through the recording of student scratch work. The third was through a final summative test focused on breaking down the formal definition of the derivative into its various components and asking students to define the various rates of change discussed throughout the course. Daily journal entries were also recorded. Finally, data collected also included a quantitative pre and post test on pre calculus concepts, including rates of change and limits.

Data analysis

First, I compared and contrasted pre and post test results. These results identified differences in student knowledge both prior to and upon completion of the course. Next, I organized photos of student work by date so as to coincide with my daily journal entries. I made a new journal which traced the learning of the four identified students over the course of every daily entry. I took special care to document any changes in their concept images over any given day. Upon completion of this new journal, I organized the student’s concept images into a data display, in the form of a flowchart containing four specific threads. These threads were average rates of change, covariation, limits, and delta notation. Key landmarks on each thread were identified along with the means of support for each one. The final version of this flowchart can be found in Appendix A.

Table 1:

Conjectured Learning Trajectory (Excluding daily homework and weekly quizzes subject to change)

Phase	Student Learning:	Means of support:
0:	Instructional starting point: A pre test will be given to establish what level of mathematical background students are bringing to the class. The first class will also begin with an open ended racing problem to introduce students to the idea of problem solving.	

<p>1: Rates of change</p>	<ul style="list-style-type: none"> • Students will be able to Interpret constant, average, and instantaneous rates of change • Students will learn to use $\Delta y/\Delta x$ form <p>LANDMARK: Understand average rate of change as slope between any two points</p>	<p>CLEAR Calculus Lab 1: Discuss distance and velocity from a complex multivariable graph</p>
<p>2: Rates of change</p>	<ul style="list-style-type: none"> • Students will be able to expand on constant, average, and instantaneous rates of change through graphical representations • Students will be able to define inflection points • Students will gain covariational understanding of rates of change <p>LANDMARK: Be able to loosely approximate slope behavior using average rates of change</p>	<p>CLEAR Calculus Lab 2: Utilize a variety of 3 dimensional graphical objects filling with water to model rates of change</p>
<p>3: Limit of a function</p>	<ul style="list-style-type: none"> • Students will learn how to graph functions with geogebra • Students will learn how to define an algebraic "hole" • Students will be able to express the limit process with limit notation • Students will learn limits as a form of approximation <p>LANDMARK: Understand limits through the approximation of a graphical hole</p>	<p>CLEAR Calculus Lab 3: Utilize complex rational functions with algebraic holes to gain a sense of the process and purpose behind limits</p>
<p>4: Different types of limits</p>	<ul style="list-style-type: none"> • Students will learn the idea of a difference quotient. • Students will be able to compare and contrast the difference quotient to how they approximated "holes" in the prior graph. • Students will be able to represent instantaneous rates of change graphically and 	<p>CLEAR Calculus Lab 4 has students contrast finding a limit with approximating an instantaneous rate of change.</p>

	<p>symbolically.</p> <ul style="list-style-type: none"> Students will be able to approximate both under and over estimates of an instantaneous rates of change symbolically and algebraically. 	
5: Derivative at a point	<ul style="list-style-type: none"> Students will be able to approximate an instantaneous rate of change utilizing average rates of change Students will be able to combine graphical, symbolic, and algebraic approximations on one graph Students will utilize the limit process and Δx notation to define the instantaneous rate of change as a value lying within a specific error bound <p>LANDMARK: Find an approximation for instantaneous rates of change within some bound</p> <p>LANDMARK: Define instantaneous rates of change using a limit and formal definition</p>	<p>CLEAR Calculus Lab 5: Using the graph and equation of the bolt of a crossbow flying through the air to find the velocity of the bolt at a specific point in time.</p>

Break Down of Events (Findings)

In this section, I describe the learning trajectory that emerged in the course, including the learning activities and their rationale, and impact of the learning activities on students' understanding . See Appendix A for an overview of the trajectory.

Rationale for an introductory problem

Beginning the class, I needed a problem that could reveal what students know about average rates of change. Research on how students learn covariation lead me to believe that students would not have smooth covariational understanding when dealing with complex graphical situations. Thus I used a problem with three identifiable factors:

- Two functions, $f_1(x)$ and $f_2(x)$, are compared. $f_1(x)$ varies at a constant rate of change over the entire domain. $f_2(x)$ has a varying rate of change, but with the special condition that over any sub-interval of uniform length the average rate of change is also constant.

2. The length of the uniform sub-intervals for $f_2(x)$ are of a non-integer or uncommon fraction length, so as to make the problem more abstract.
3. $f_2(x)$'s constant average rate of change is less than $f_1(x)$'s rate of change, yet $f_2(x)$ reaches the end goal before $f_1(x)$.

Race Problem

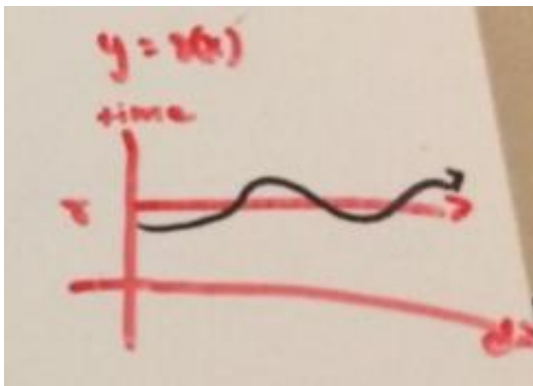
The race problem is brief in its presentation but satisfies all the aforementioned conditions.

Chort and Frey ran a marathon (26.2 miles). Chort ran at a perfectly uniform pace of eight-minutes-per-mile. Frey took exactly eight minutes and one second to complete each one-mile interval. This refers to all one-mile intervals, including, for example, the interval from 5.63 miles to 6.63 miles. Nevertheless, Frey finished ahead of Chort. Explain how.

This problem serves as an essential key to the course as a whole. It provides an opportunity for the instructor to accomplish the following goals: First, find out what and to what degree of complexity students perceive rates of change. Second, establish that class procedures will be student lead.

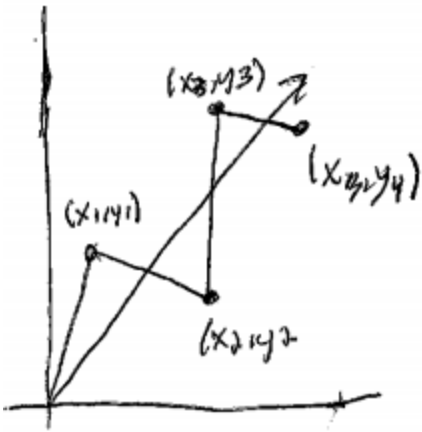
- Covariation

Initially all students responded to the bike problem by drawing some form of constant line for Chort with a wavy line overlapping it to represent Frey.



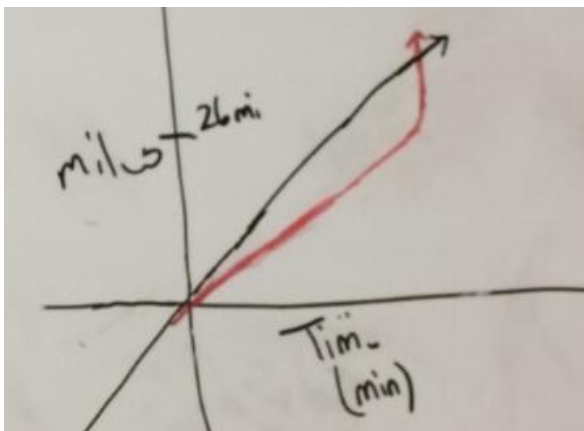
(Figure 1)

Any attempt to further clarify Chort's path was extremely difficult for students. At best we could identify singular discrete points such as in the photo below.



(Figure 2)

Note that students understood that Chort ran a continuous race, and thus the points were necessarily connected. However, students had no clear understanding of what was happening between said parts. I believe this was an example of continuous chunky covariation (Thompson and Carlson, 2017, p. 435). To solve this problem we had to reduce the noise caused by these multiple points and view Chort's path as one curve seen below.



(Figure 3)

Another note was that some students seemed to see a number line as a discrete set of "small" or "big" points, such as .1 or .9. I think such reasoning may be in direct correlation to chunky reasoning.

- Average Rates of Change

A key component to understanding the bike problem is understanding the meaning of an average rate of change. Students had no experience with this term and continually fell back upon the classic definition of average as a total summation divided by the quantity of its summands, as seen in Figure 1. By the end of the problem students had been exposed to average rates of change as a constant line between two points.

Reaction and New Rationale

I had not realized how much work it would take to capture the movement of Chort's race. Even after significant time spent on the problem, students would still slip back into describing average rates of change as an average value between two extremes. This outcome would be dealt with through an almost daily discussion of rates of change. With regards to covariation, students seemed firmly rooted in chunky continuous covariation. This outcome was planned to be dealt with in lab 2 which focused entirely on covariation between volume and height.

Going forward, I decided to implement a worksheet on limits so as to gain more initial insight into what my students knew about the key components of the derivative definition.

Intro Worksheet

Intro Worksheet presents two sequences of fractions, one with a decreasing denominator and the other with an increasing denominator. Students are asked to describe in their own words what is happening to these sequences as their respective denominators change. Students were also told that $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ and asked to rewrite the equation for slope using delta notation.

- Limit

Students had not been exposed to a limit before. When presented with the fractions approaching either 0 or infinity, they demonstrated Oerhtman's (2009) infinity as a number metaphor in that they saw the fractions approaching something either really big or really small (p. 417). For a few students this may have been due to their chunky understanding of the real number line.

- Delta Notation

This activity was the first presentation of delta notation. However, no explanation other than that $\Delta y = y_2 - y_1$ was given. Thus the activity served to introduce it to students, but it was in no way meaningful to them at this point and time.

Reaction and New Rationale

This worksheet was all new material and students answered exactly as I expected them to respond.

Lab 2

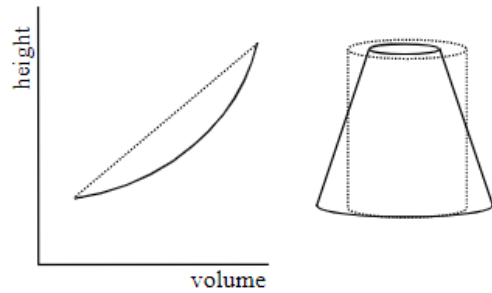
Lab 2 presents questions concerning bottles of various dimensions and how said dimensions might affect the rate at which they fill with water. The lab utilizes delta notation and asks specific questions relating constant and average rates of change. Below I present three important questions.

2. Suppose a height vs. volume graph is a straight line. Describe what bottle shapes could correspond to such a graph. (Be creative to think of all the possibilities!) A linear graph represents a *constant rate of change* between the two quantities, height and volume. Explain what this means in terms of amounts of change in height and volume.

(Figure 4)

3. The diagram to the right depicts a bottle that is wide at the bottom and narrow at the top (drawn with a solid line). The solid line in the graph shows the relationship of height vs. volume for this bottle.

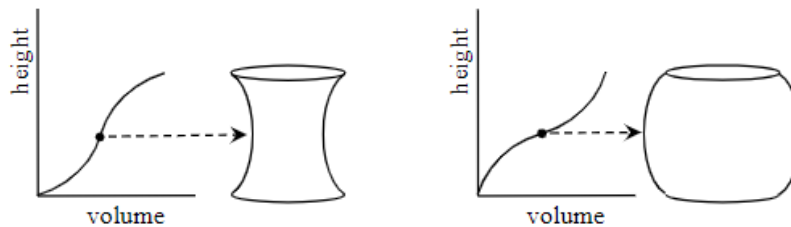
To think about the meaning of an average rate of change it is often helpful to introduce an auxiliary situation where the rate is constant. In this case, for the auxiliary situation we can imagine a cylindrical bottle (as drawn with a dotted line) and corresponding linear graph.



Use the auxiliary cylindrical bottle and graph to explain the meaning of the average rate of change of height with respect to volume for the original bottle that is wide at the bottom and narrow at the top.

(Figure 5)

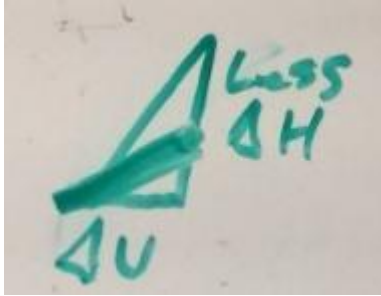
4. Inflection points correspond to points where the bottle changes from getting narrower to getting wider (or vice-versa). This is because an inflection point on the graph occurs when the graph changes from getting steeper to becoming less steep (or vice-versa). Explain what is happening at the inflection points for a bottle that is narrower in the middle using language about *amounts of change*.



(Figure 6)

- Covariation

Student responses to Figure 6 suggested that the bottle filling content supported them to change from Thompson's categories of continuous chunky to continuous smooth covariation (20017, p. 435). Proof of this was demonstrated in several ways. The first was in student actions. Students were continuously using their hands to try to illustrate how rates of change were smoothly changing on either side of the inflection point. I defined this landmark as rates of rates. During this time, they also continually made statements such as, "the slope is increasing but then... it goes down". Drawings of a figurative sliding slope bar were also made such as the one below.



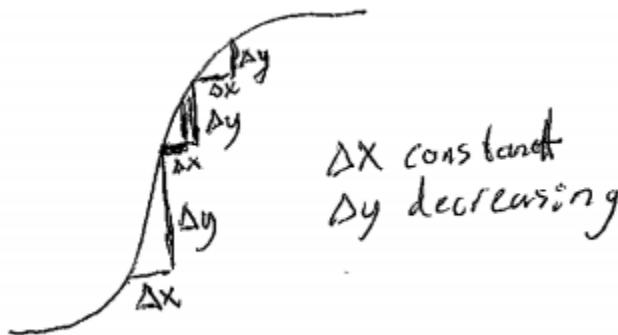
(Figure 7)

These actions were in stark contrast to the placing of arbitrary points and connecting them with lines performed during race activity. Here students were clearly imagining a form of tangent line continuously and smoothly sliding along the function's path.

- Delta Notation

Lab 2 was where the development of delta notation became a meaningful tool for students to express their thinking. For example, one student drew an almost square like coffee cup to answer the question 2. When asked to justify this choice, the student responded with $\Delta h = \Delta v$. Unfortunately, delta notation began to be over generalized by the end of the lab.

When working on problem 4, almost all students described the inflection point as where $\Delta h / \Delta v$ flipped from $\Delta h > \Delta v$ to $\Delta h < \Delta v$ or vice versa. Although false, it represents thinking similar to Thompson's (2017) pre-coordination of covariational thinking, but applied to the ratio $\Delta y / \Delta x$ (p. 435). Understanding rates of change approaching a point can be incredibly complex. To better understand this complexity, students repeatedly fixed some change in x (the denominator) and then analyzed the corresponding change in y (the numerator) over a series of sub intervals of width Δx . Thus statements such as $\Delta y < \Delta x$ would represent the following thinking.

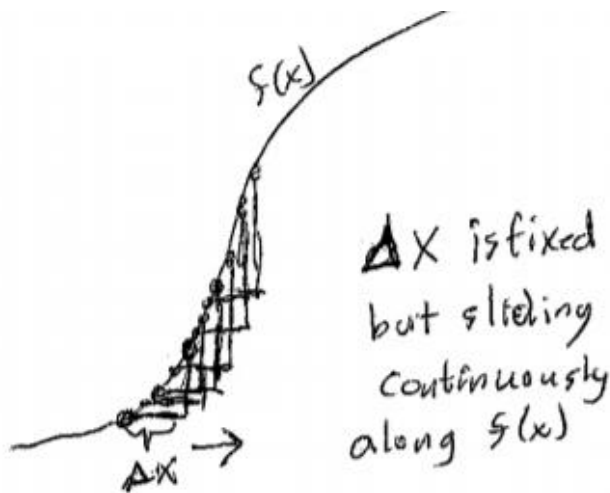


(Figure 8)

Such logic may have been reinforced by the fact that all examples involved a cup filling up with water rather than being drained, thus avoiding the problem of describing negative

slope, but regardless, it presented a powerful tool by which students could attempt to articulate their understanding of changing rates of change, i.e. the rates of rates landmark.

An interesting feature of this use of delta notation is that it neither confirmed nor denied the smooth covariational understanding demonstrated by the use of the sliding tangent bar in figure 7. When looking at figure 8, one sees a series of connected triangles where the hypotenuse of each is $\Delta y/\Delta x$. Whether or not students envisioned $\Delta y/\Delta x$ to be these triangles connected back to back like in Figure 8 or smoothly and continuously placed along each point of the number line like in the figure below is unclear.



(Figure 9)

- Average Rates of Change

Question 3 paired with conversations concerning question 2 were essential moments in understanding average rates of change. As mentioned, students would respond to question 2 by drawing coffee cups of equal height and width. Thus they saw a constant rate of change as only applying to ideal circumstances. By the end of question 3, over half of the students seemed to understand the average rate of change as the constant rate of change between two points as demonstrated by the following statement: "it's like, the straight line is the change for the skinny bottle and it's the same as... the other bottle, the average rate of change". Thus students saw that average rates of change applied constant rates of change to situations outside of their ideal coffee cup.

It is also interesting to note that this constant rate of change was seen by students to be a 2 step process, the process being rise (pause) over run. Thus for average rates of change I again saw a form of ratio pre-coordination.

Reaction and New Rationale

Throughout this lab, I was continually blown away by how much more powerful questions concerning filling water into cups were in comparison to questions concerning distance or velocity. This may have been due to the fact that younger students experience car trips passively as passengers and thus have a less concrete mental image of how that motion works over time.

Going forward, several factors needed to be addressed. Students had started to tackle the complexity of covariation both across $f(x)$ and $f'(x)$ but often resorted to hands or inaccurate inequalities to describe it. Continual effort would need to be made to further push this reasoning and make it more concrete. Delta notation had developed a dual meaning between its equation definition ($\Delta x = x_2 - x_1$) and as a way to describe variation in the change in one variable over some interval. Further work would need to be done in calculating the individual components so as to clarify student understanding mathematically.

The next activity I had planned was a 3 week summary to assess what the students had learned about rates of change. Specifically, it would check for any overlapping student definitions still needing to be addressed.

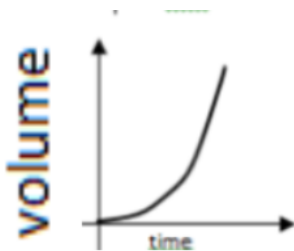
3 week summary

The 3 week summary activity is composed of three styles of questions.

1. What is the difference between a constant, average, and instantaneous rate of change? (**Answer in at least 3 complete sentences!!**)

Plot: Below are graphs modeling the volume in a cup over time. The cup starts out empty and is being filled at a **constant** rate. For each graph, **what is the shape of the cup and why?** (You may draw the cup if you can't describe it)

2.



PLOT: A car is resting at a stop light. The light turns green and then goes for several blocks until it is stopped by another redlight.

7. Make a distance v time graph modeling this problem.

8. Make a velocity v time graph modeling this problem.

(Figure 10)

- Covariation

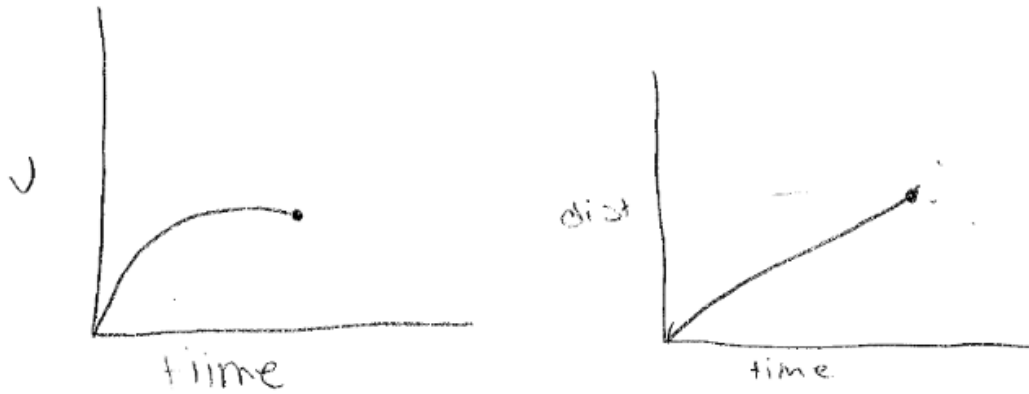
Question 2 was intended to be volume versus height with the height moving at a constant rate. Failure to label it as such inadvertently created a paradox, but students seemed to understand the intended question from their experience in lab 2 as shown in the figure below.



(Figure 11)

As seen, the student imagined a horizontal line moving up the flask at a constant rate with respect to time. Volume was seen to be varying at whatever rate would be necessary to guarantee the constant rise of the horizontal fill line. Furthermore, the student in Figure 11 clearly say Δv as varying smoothly and continuously. Thus the lack of clarity concerning the smooth versus chunky continuous variation in Δy , when given $\Delta y/\Delta x$, from lab 2 seems to be further clarified.

When asked to solve questions 7 and 8, several students answered perfectly. However, some demonstrated how an understanding of $f(x)$ may conflict with and define their vision of $f'(x)$. As noted by Nemirovsky and Rubin (1992), many students expect the function to share common features with its derivative, and thus often try to “match” the two graphs (as cited in understanding concepts of calc, Larsen). For example, in the figure below a student confused the fact that the car would stop farther away with the fact that velocity would be 0 at a stop. Thus the velocity graph decreased, but did not hit the x-axis.



(Figure 12)

- Average Rates of Change

In question 1 students defined the constant rate of change as a straight line where $\Delta y = \Delta x$, and only one student described average as the mean of variation. Several students even described the average rate of change as the rate of change between two points, but none drew the connection that their dashed line connecting the two points was, in fact, the constant rate of change between those two points.

Constant rate of change is when the change of y is constant with the value of x .
 average rate of change is when the change in the value of a quantity divided by the elapsed time.
 Instantaneous rate of change is expressed using limits.

(Figure 13)

A constant rate of change would be a linear equation ↗

An average rate would be the time it takes to get from one point to another, (dotted line),

An instantaneous rate of change would be immediately speeding up or slowing down,



(Figure 14)

Thus students seemed to be rooted in their image of the ideal coffee cup for constant rates of change and still finding pre-coordination language more personally meaningful for average rates of change.

It is interesting to note that I had not introduced the phrase “instantaneous” rate of change until this point. Thus I found that most students, similar to their initial stance on average rates of change, believed instantaneous to be quite literal. Instantaneous meant to never be constant. For example, a rocket exponential speeding towards space would be an object with instantaneous rates of change. The mention of limits in Figure 13 can be attributed to the student cleverly drawing a connection with the question posed and the intended topic for the following class.

Reaction and New Rationale

It was exciting to see that students had left behind the concept of average rates of change as some sort of mean in variation. However, their variational understanding of rates of change seemed to be firmly rooted in pre-coordinational understanding. Such conceptions were not a concern, because the derivative lab would be incorporated with technology so as to allow students the ability to “see” the components Δy and Δx varying continuously together.

Although more work could clearly be done on rates of change, I felt that it was important to start introducing the limit process in class. The limit process would introduce the idea of approximating acceptable values of a limit within some restricted interval. The derivative lab would replicate the same methods of approximation and would thus be a natural continuation of the limit lab.

Limits Lab

The Limits Lab presents students with complex rational functions which each have holes due to a problematic value of x in their denominator. The steps within the Limit Lab will be repeated for all examples from the Limit Lab until the formal definition of the derivative itself. These steps are as follows: What is an approximation? Is this approximation an overestimate or underestimate? What is the error? Explain why the error cannot be found exactly. What is a bound on the error? What are possible values within that bound? Upon completion of these questions, a table is presented, as seen in Figure 15, for students to organize their work. They are then asked to go a step further and find an approximation specifically within a bound of .001. Thus students are guided towards a natural algorithmic way to find an acceptable limit.

It should be noted that my students had not seen nor heard of functions with graphical holes in their mathematical careers and thus the lab required a simple introduction on such functions. A complex function was presented in Geogebra, such as the one in Figure 15. Students were then asked to decide what they believed the function value to be at the point of discontinuity. Since Geogebra does not place any indicator of such a graphical hole on the screen, students immediately assumed the graph to be continuous and were surprised for the answer to be undefined. From this short introduction students understood that the proceeding problems were to concern the behavior of a graph around some problematic x value affiliated with the causation of a 0 in the denominator.

Graph 1: The graph of $f(x) = \frac{\sqrt[3]{x+7}-2}{x-1}$ has a hole. Your task is to determine the location of this hole using approximation techniques

12. List three fairly decent pairs of underestimates and overestimates (you can include the one you computed above). For each pair, give a bound for the error and use this to determine a range of possible values for the actual y -value of the hole in a table with headers as shown.

Underestimate	Overestimate	Error Bound	Range of Possible Values

(Figure 15)

- Limit

At the start of the lab, students had only seen limits as the process of getting a number very large or very small. By the end of the lab, students described the limit as a value within some given error bound, which Oehrtman (2009) described as understanding through an approximation metaphor (p. 421). This definition was not initially meaningful to students but became meaningful after being forced to find limit approximations within error bounds made small enough to be inconvenient.

For example, one group found the task of finding x_1 and x_2 values such that $|f(x_1) - f(x_2)| < .001$ to be extremely complicated since it required symbolically representing decimals close to one. The group would fix one value to the left of 1, say $x_1 = .99$, and then arbitrarily change the decimal digits of the second number to the right of 1. Thus when they calculated $|f(.99) - f(1.1)|$, $|f(.99) - f(1.01)|$, $|f(.99) - f(1.11)|$, they were baffled that the bound had only gotten larger. Throughout this struggle, the group refused to accept a limit value outside of the given bounds of $.001$. Thus it could be seen that they understood the limit to be an approximation within some bound, even though they lacked the numeric literacy to effectively find it.

Reaction and New Rationale

The largest takeaway of the limit lab was how hard the process of approximating a limit was to start but how easy it became after only a few times entering the algorithm into Geogebra. By the end of the lab students had the approximation method for finding a limit adequately mastered, and the strongest remaining hurdle was the technical aspect of choosing the correct x values around c so as to be within the necessary error bound. Furthermore, students had left the notions of a limit being some really large or small value behind.

The lab was so successful that I felt that students were more than ready to begin the derivative lab which would take the methods of the limit lab and apply them to approximating rates of change. However, this paper would not be complete without an adequate explanation of the degree to which Geogebra played in the aforementioned progress of student understanding.

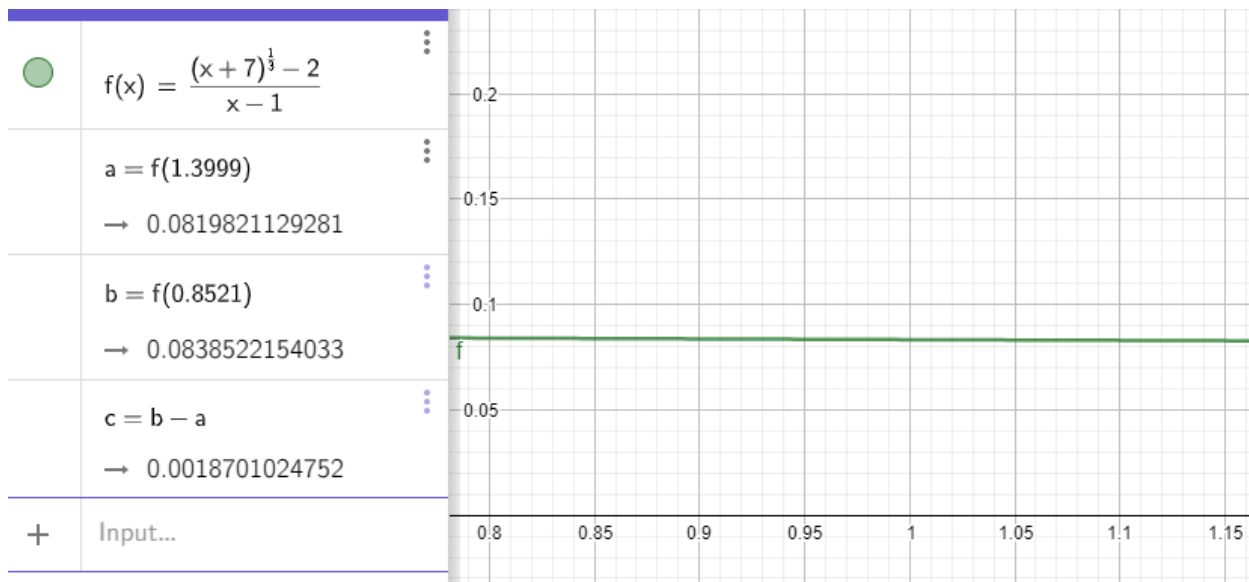
Geogebra

Geogebra is my graphing software of choice, as it provides several features I believe to be essential to this course. Some of these include a mobile app, the ability to “pinch” or “squeeze” your graph, and the ability to create pseudo algorithms by defining variables or functions prior to computation.

- Limit

As discussed, students in the limit lab developed an understanding of limits as an approximation within some error bound. This understanding could be described as very procedural in nature and only became truly meaningful through the use of Geogebra. The program allowed students to compute and graph complex functions that might have normally been out of the mathematical reach of the students involved.

For example, one student wrote a quick algorithm in which values to the left and right of $x=1$ could be entered into parenthesis and an error bound would automatically be generated (Figure 16). In the picture presented, the student had already found a strong limit estimate within the required bounds and was attempting to find maximally distant x values that would still satisfy the error bound in y of $.001$. This level of play would result in the student describing Geogebra as “quite fun”, and the same student would later physically write down a similar algorithm as justification for written homework. Furthermore, such play demonstrates how the use of Geogebra encourages the natural formalization of abstract mathematical concepts. Although the student discussed was never given instructions concerning arbitrary, sufficient, delta, or epsilon, his “play” was clearly an attempt to find a *sufficient* interval of x values for which the distance between them, *delta*, implied a range bound by $.001$, *an arbitrary epsilon*.



(Figure 16)

Geogebra also backed up research done by Oehrtman’s (2009) which identified the process of “zooming” (p. 405) or “proximity” (p. 417) as common metaphors through which students come to understand limits. Going forward, students would need to alternate from

calculating the values of holes to approximating slope values. Whenever grappling with either of these questions students would plot the function and repeatedly zoom in and out of the function over smaller and smaller intervals. I believe this to have been an essential part in them coming to understand the role of limits in the formal definition of a derivative and, this process relied entirely upon the ability of Geogebra to transform graphs into malleable objects to be successful.

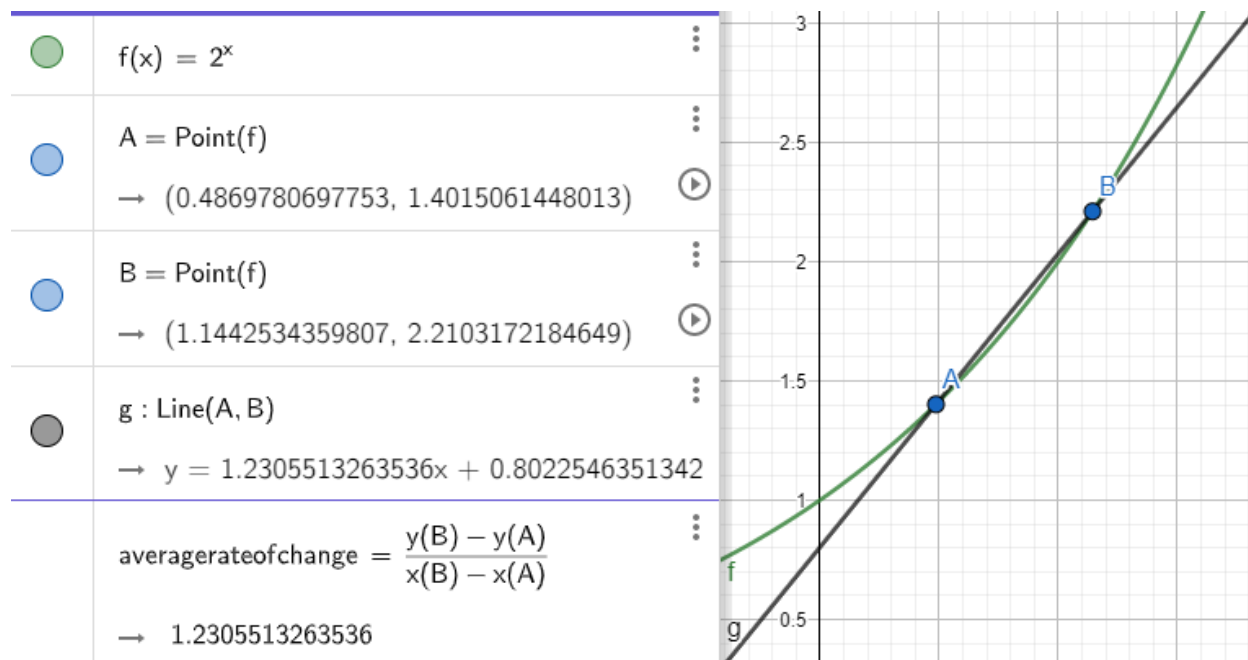
Derivative Lab

Similar to the Race Problem, the Derivative Lab begins with a brief problem

A bolt is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Accounting for wind resistance proportional to the speed of the bolt, its height above the ground is given by the equation $h(t) = 7350 - 245t - 7350e^{-t/25}$ meters (with t measured in seconds). Approximate the speed when $t = 2$ seconds accurate to within 0.1 m/s.

(Figure 17)

After several attempts and variations of an acceptable error bound are completed by students, I demonstrate how points and lines may be combined into moveable slopes.

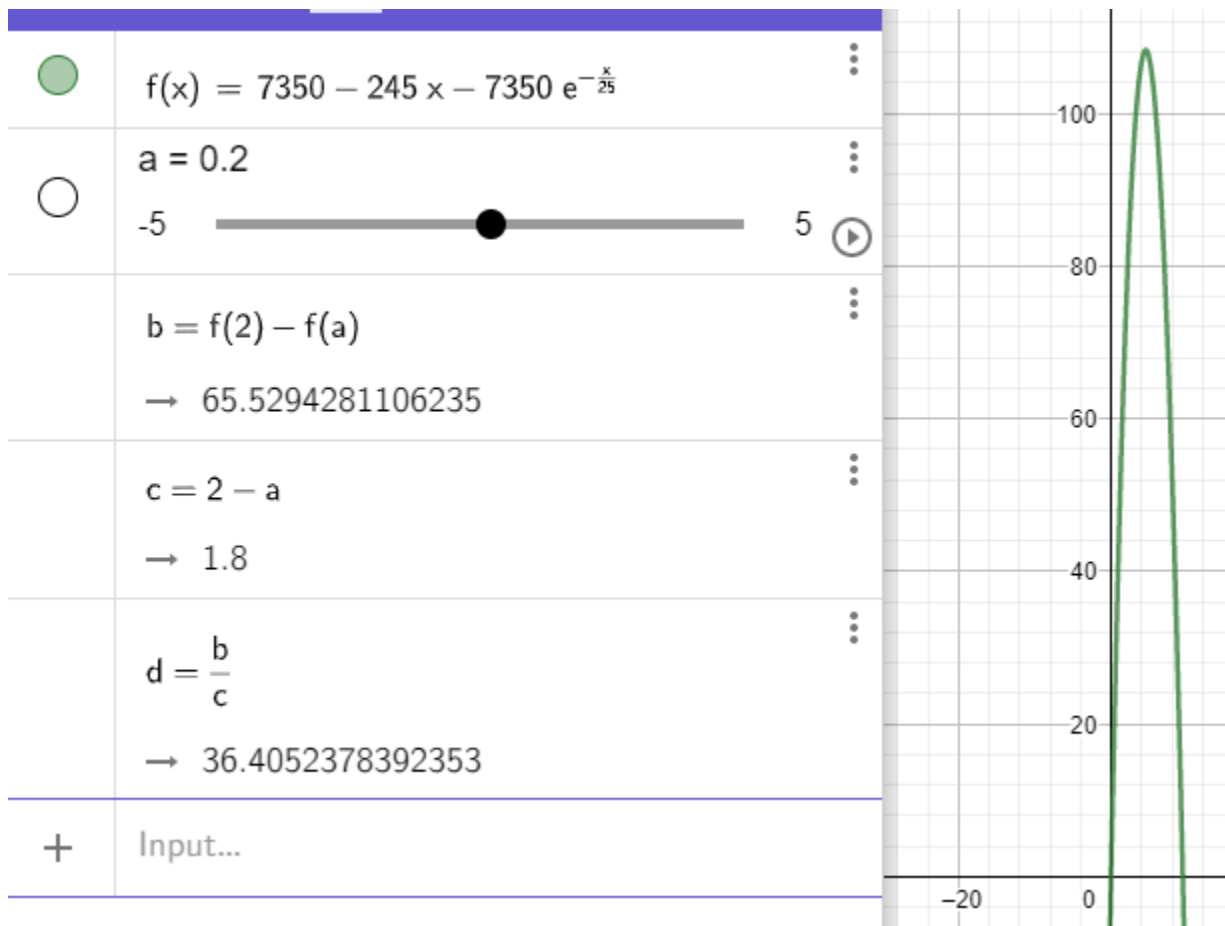


(Figure 18)

- Covariation

Prior to the Derivative Lab, students would deal with the ratio $\Delta y/\Delta x$ by fixing either Δy or Δx . I had assumed that students would naturally transition to seeing Δy and Δx varying smoothly together as they were continually required to find accurate approximations of

instantaneous rates of change. This transition was surprisingly hard to physically identify. For example, one student created an algorithm which incorporated a slider to more efficiently calculate the Δx and Δy pieces of the ratio $\Delta y/\Delta x$ (Figure 19). However, it is unclear whether the student had included the variable, d , simply for computational ease or as a result of recognition that Δy and Δx vary simultaneously together within $\Delta y/\Delta x$.



(Figure 19)

- Average Rates of Change

It was in this lab that students began to explicitly describe the average rate as a ratio of change in y and change in x defined over two points. Occasionally language such as “rise over run” was used, but such statements still referred to a ratio relating two points. The use of the process to coin the name was simply due to its familiarity to the students who had limited ways to describe the object at hand.

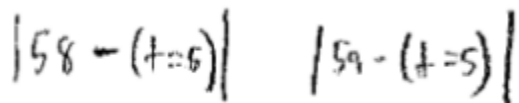
Similar to finding graphical holes, students began to search for instantaneous rates of change through an approximation within some upper and lower bound of slope caused by two points being cinched together. However, the differences between finding this slope and finding

limit values of graphical holes proved too subtle to remain in the long term memory of all students. Some students would go on to describe the instantaneous rate of change as being an acceptable slope within some bound, yet others would confuse finding a hole with finding an average rate of change when trying to solve a problem. For the latter, the approximation method of finding limits within an error bound had become more important than what the error bound was actually bounding. Thus students would flip between describing the rate of change between two points and the value of $f(x)$ around some specific point when trying to find the instantaneous rate of change.

- Delta Notation

It was also in this lab that students began to leave behind the use of delta to over generalize change in a variable and used it to describe distance explicitly instead. Initially, this led to some confusion as students did not bother to use absolute values in the calculations. Thus some would describe both a slope approaching 0 and an exponential negative slope as “getting smaller”. This problem was not resolved until the class took time to discuss the use of absolute value signs as a way of expressing magnitude. Post this discussion, several students began to incorporate absolute value signs into their delta notation.

Another significant outcome of the derivative lab was that students who had slipped into describing graphical holes would occasionally see the variable t as representing both t and Δt or vice versa. Some became so confused with the duplicity of these variables that function notation was dropped entirely, such as in the following example:



The image shows two handwritten mathematical expressions side-by-side. The first expression is $|58 - (t=6)|$ and the second is $|59 - (t=5)|$. The variable t is used in both, but the second expression uses $t=5$ instead of Δt , illustrating the confusion mentioned in the text.

(Figure 20)

Reaction and New Rationale

With the success of the limits lab, I had hoped that the derivative lab would be just as meaningful for students. Although there was initially much success, it was disheartening to see some students slip back to thinking about graphical holes as early as one day after the lesson. This confusion further amplified their confusion over when to use Δt rather than t . The number one solution to this problem would have been more practice, but there was not enough time left in the course for further examples or practice.

With the derivative lab complete, all students had seen all the necessary components of the formal definition of an instantaneous rate of change. Thus given the time restraints of the course, it was time to have students be exposed to said definition. Although there were clearly several misconceptions lingering in student's minds, I believed that attempting to define the formal definition in their own words might have lead to some powerful insight into how students learn about rates of change.

1st Exposure

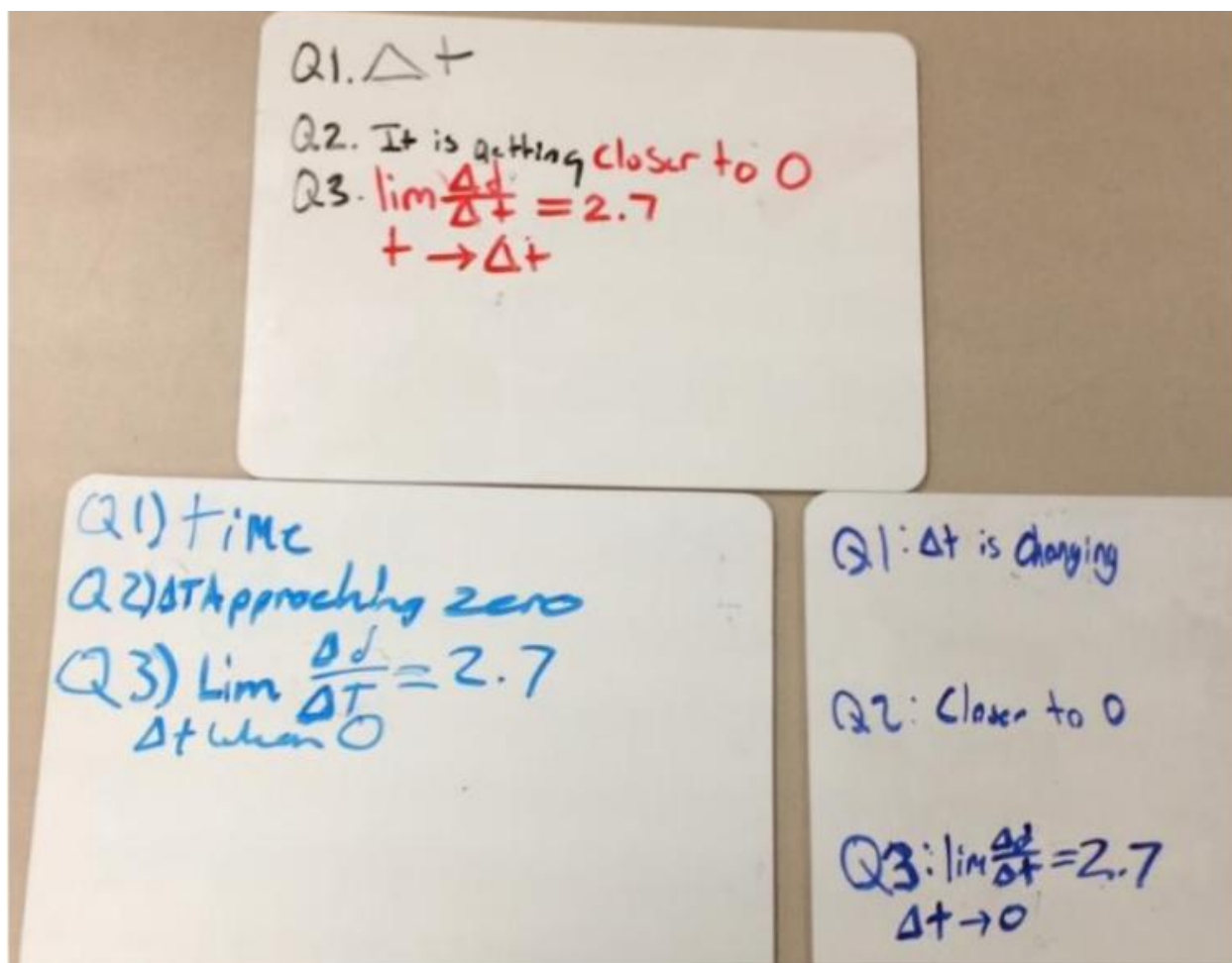
First exposure involves the placement of the semi-familiar continuous function $f(t)=2^t$ on the whiteboard with an accompanying graph and several marked, decreasing intervals. Students are then expected to find the average rate of change over each of these intervals (specifically $[2,3]$, $[2.1,3]$, $[2.001,3]$). After all measurements have been found, the following three questions are asked: What is the only part changing over time? Where is it going? How do we say this as a limit?

- Covariation

Post the introduction of limits students saw instantaneous rates of change as a composition of moving pieces. Whether it was time or the change in time dictating this movement varied for each student, and can be seen in Figure 21 below. Thus again, it was unclear if students had developed a sense of Δy and Δx varying simultaneously together.

- Delta Notation

During first exposure, some students still demonstrated a struggle differentiating Δt from t . However, all students clearly understood t and Δt being the malleable pieces used to find the instantaneous rate of change. The following picture shows student responses to the questions What is the only part changing over time? Where is it going? How do we say this as a limit?



(Figure 21)

Reaction and New Rationale

My greatest concern in the 1st exposure activity was that my use of average rates of change between the intervals [2,3], [2.1,3], and [2.001,3] would result in some sort of chunky understanding. I also regretted not having enough time to have them “discover” the limit notation on their own. One could imagine this same activity done over a more tedious number of intervals (say 10), and students asked to spend the following day creating an algorithm in Geogebra and their own formal notation for which to expedite the process.

After the 1st exposure, there was only one day of class left. Thus I felt it was time for students to complete the final assessment “instant formal definition”.

Instant Formal Definition

Formal definition serves as the cumulative assessment of the course. It has students define each part of the formal limit definition of the derivative, as shown in the figure below. The only clue on

the board was a picture with a tangent drawn at $x=2$ and another point $(2, f(2+\Delta x))$ with a secant line connecting them.

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

1. Using complete sentences, what does $f(x+\Delta x)$ represent?
2. Using complete sentences, what does $f(x+\Delta x) - f(x)$ represent?
3. Using complete sentences, what does Δx represent?
4. Using complete sentences, what does $\lim_{\Delta x \rightarrow 0}$ represent?
5. Using complete sentences, what does the fraction $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ represent?
6. Using complete sentences, define **constant** rate of change.
7. Using complete sentences, define **average** rate of change.
8. Using complete sentences, define **instantaneous** rates of change.
9. Sketch me a graph where the **instantaneous** and **average** rate of change are the same from $x = 0$ to $x = 2$

(Figure 22)

- Covariation

During the assessment, students described the formal definition as a composition of malleable pieces. The order and exactly which individual pieces varied changed from student to student. The following is an example of such a viewpoint from a student answering question 2.

*$f(x)$ is the regular function for y , but what ~~this~~
 Δx does is add a difference from ~~the~~ y' and y'' and
allows the function to not equal infinity.*

(Figure 23)

One could also see such behavior in this response for question 5.

The fraction represents the change in y over the change in x .

(Figure 24)

- Average Rates of Change

Originally, the goal of the course was to have students gain a deeper understanding of instantaneous rates of change. What I found was that there were essentially two camps of understanding. The first of these camps was defined by an understanding of the ratio within the formal definition as being the result of two points being drawn together and the instantaneous rate of change being at best described as a slope at a point. This can best be seen in the following responses to question 5 and 8.

Δ represents coordinates

Instant. rate of change is the slope at a single point

(Figure 25)

This camp was placed under average rates of change to signify that such students post the Derivative Lab tended to demonstrate an ability to replicate the procedure of approximating an instantaneous rate of change, but often could not describe what they were doing rigorously.

- Delta Notation

It was with delta notation that I saw the second “camp” of student understanding for the instantaneous rate of change. Students in this camp answered question 5 similarly but avoided using the word “slope” to answer question 9.

The fraction represents the change in y over the change in x .

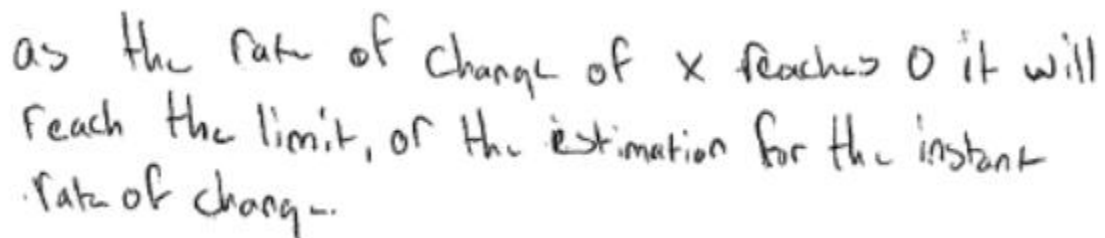
The constant rate of change at one point.

(Figure 26)

This camp was placed under delta notation for two reasons. The students in question routinely used delta notation accurately post the Derivative Lab and always described the limit to be a desired value made acceptable by being within some error bound, such as in Figure 27

- Limit

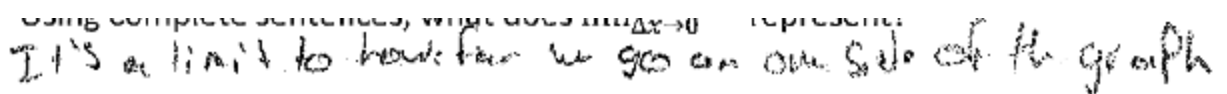
The assessment demonstrated that most of the students had a grasp of the limit as a tool to simplify the process of taking Δx and letting it approach 0. However, some still had traces of seeing the limit as a process for finding graphical holes. For example, the response to question 4 in the figure below is clearly discussing rates of change.



as the rate of change of x reaches 0 it will reach the limit, or the estimation for the instant rate of change.

(Figure 27)

In contrast, another answer to question 4 below seems to see the limit as a way to generate error bounds. What kind of errors those error bounds are bounding is unclear.

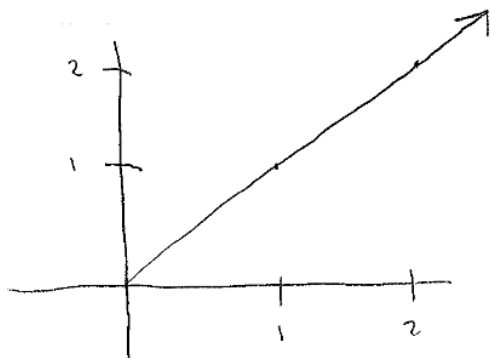


Using complete sentences, what does $\Delta x \rightarrow 0$ represent?
It's a limit to how far we go on one side of the graph

(Figure 28)

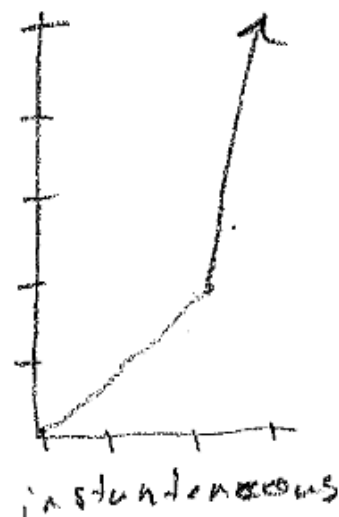
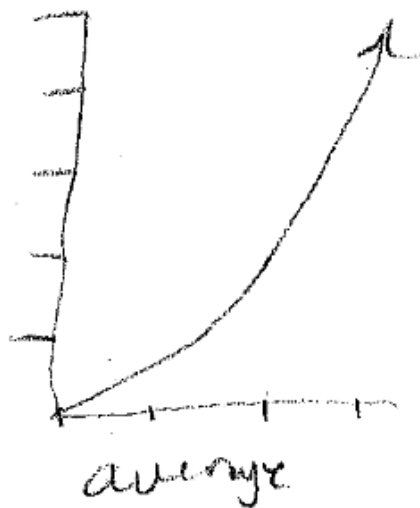
Bonus

Question 9 was special for the diversity in which students responded. Students in the limits as an approximation camp all gave purely linear graphs, as shown in Figure 29. This graph is the simplest answer, yet it could be argued to require more complex thinking. Drawing this graph implies a logical process from the definition of instantaneous rates of change as the constant rate of change at one point to the realization that there exists a simple graph satisfying this condition over an infinite domain.



(Figure 29)

Such responses were in contrast to those who saw the ratio within an instantaneous rate of change as the containment of some “rise over run” between coordinates. These students gave a semi-linear graph. Specifically, they correctly drew a linear graph over $[0, 2]$, but a graph in a different direction post $x=2$. It was as if a point was first placed satisfying the necessary conditions for “rise over run” and what happened after that point was arbitrary.



(Figure 28)

Summative Findings

My breakdown of class time events focused on four narratives, covariation, rates of change, limits, and delta notation. Covariation was the first of these narratives and, in many ways, was the most complex. As discussed in the literature review, Thompson found there to be many unique ways in which students understand variation and covariation. The situation only becomes more complex when students are asked to understand the formal definition of a derivative. The definition itself involves a

covariational relationship between the output of $\lim((f(x+h)-f(x))/h)$ and a length h . However, within this relationship lie other covariational relationships as well. $F(x+h)-f(x)$ involves covariation between the change in y and h , and one must not forget the covariational relationship between x and $f(x)$ for which the entire problem is built upon.

At the beginning of the course, students tended to demonstrate chunky continuous covariation, as shown in Figure 2. Classroom discussions concerning the behavior of various cups being filled with water seemed to break away from chunky continuous covariation, but it was never made clear if students generalized these findings to problems concerning distance and time. As we transitioned into discussing average rates of change, most students developed a pre-coordination understanding of $\Delta y/\Delta x$, in which either Δx or Δy were fixed and the other varied. The goal was to have students understand $\Delta y/\Delta x$ as value composed of two lengths varying smoothly together, but whether or not such formalization occurred was never made clear.

The second narrative was rates of change. Prior to this course, all participating students had completed a secondary algebra 2 course. This fact is significant because students in such a course will have practiced calculating the slope of a line and will have heard the phrase “average rate of change” used at least once. I was therefore quite surprised to see all students take the phrase quite literally as the mean of both high and low rates of change, as seen in Figure 1.

By the end of the second lab many students had advanced to understanding an average rate of change as a constant rate of change between two points. This reasoning was then advanced through the derivative lab to also be understood as the change in y divided by the change in x . An interesting finding was that some students found the metaphor of a limit as an approximation within some error bound meaningful enough to be extended to rates of change as well. By the end of the course, these students would describe the average rate of change was “a change in y divided by a change in x ” within some acceptable error bound.

The third narrative was limits. At the beginning of the course, students had neither seen nor heard the expression of limit in a math course. Thus their interpretation usually involved some number getting really large or really small. For example, to solve the limit of $1/x$ as x approaches infinity, most students simply evaluated the expression when x was 10000.

This interpretation of a limit as a number quickly changed as we dived into the limit lab. By working with graphical holes and asking students to find limit approximations within a narrow error bound, students came to see limits through approximation metaphors. Thus any choice of limit was a value which we, as a class, had deemed to be sufficiently close enough to be satisfactory. Geogebra was

essential to this process as it allowed students to quite literally play towards a solution. Towards the end of the course, some students became confused between finding graphical holes and approximating instantaneous rates of change, but such confusion did not alter the perception of finding a limit as an approximation process.

The final narrative was delta notation. Just as with limits, students had neither seen nor heard expressions concerning delta notation. Students began to utilize delta notation of their own accord around lab 2. Their use, however was different than originally intended. Students correctly understood Δx to be “change in x ” and Δy to be “change in y ”, but exactly how or which was changing was arbitrary. This is best seen in Figure 8 where Δx is fixed and $\Delta y < \Delta x$ translates to a decreasing change in y over multiple equivalent intervals.

It is unclear if some students ever entirely broke out of using delta notation ambiguously. For example, some students would correctly use dt as a malleable distance between points when looking for the instantaneous rate of change. However, others would exchange Δt and t arbitrarily, such as in Figure 21. These students might have understood the numerator of the derivative definition as a changing change in y , but the habit of fixing the denominator, Δt , would have been problematic to describing this change. Thus they might have used t as a way to attempt to express their recognition that the change in y corresponded to some non fixed change in x . The course ended with delta notation being a familiar tool for which students might better articulate their thoughts, but the rigor of use varied greatly from student to student.

These narratives culminated in students understanding instantaneous rates of change as either an estimation based upon an average rate of change defined over points being drawn together or upon an average rate of change defined over some Δx approaching 0 with the latter being the more rigorous understanding. It should be noted that there was never any evidence if either of these understandings included a recognition of the formal derivative as a multiplicative object rather than strictly a composition of smaller bits.

Implications for Future Teaching

My course explored alternate ways in which to teach students fundamental calculus concepts such as rates of change, limits, and the derivative definition. The approach took approximately five weeks and covered significantly less content than might have been covered by a more traditional calculus course of equivalent length. However, I would argue that the concepts learned by students through this approach, such as the approximation approach to a limit, made the course well worth repeating in the future. I had taught an extensive calculus one course in the past but in a more

traditional method. Although I was proud of what my students accomplished in that course, their engagement never was to a high enough degree to be labeled as “play” and their solution methods were oftentimes based in rote memorization, rather than problem solving. This stood in stark contrast to the key moments of this modified approach. In the new approach students were continually using interactive graphs both to solve problems and to explore potential problems of their own. They were forced to continually confront and analyze their preconceptions of rates of change, and most importantly, they learned to connect the pieces of an equation to actual math rather than simply plug and chug. The fact that I found this approach worthy of future use, however, does not mean that this first iteration was without flaws.

The first problem to address would be to tweak the questions and remove the accompanying diagrams to the questions in lab 2. In most cases, the questions and diagrams were meant to spark conversation but instead took the mystery out of the question by being too explicit. A great example of this can be seen in Figure 5. One may note students are asked to describe the behavior of the graph around a point using rates of change, yet justification is already given in both the “because” statement and graphs. Imagine the discussion this same question might yield if both of these were removed. It should be noted that although I would tweak lab 2, its inclusion is a necessity. Calculus courses tend to borrow many concepts from physics, such as velocity, to justify the use of derivatives. Although these ideas are certainly interesting, they made significantly less impact on student understanding than the volume versus height problems of lab 2. I would argue that volume was more meaningful since the students were young enough to have spent most of their lives traveling in vehicles passively and thus had a harder time seeing velocity as anything more formal than simply “fast” or “slow”. Volume versus height, in comparison, provides a simple, visual and verbal common ground for all.

The second problem was the student’s development of interpreting $\Delta y/\Delta x$ as some set of consecutive intervals in which the change in Δy was compared between each one (Figure 8). This form of thinking would dominate how many students interpreted graphs and it was never clear if they saw these intervals as overlapping (Figure 9) or lying consecutively (Figure 8). Since the end goal was to have students imagine a secant line being smoothly and continuously drawn along $f(x)$ towards one of its fixed endpoints, students were not initially analytically empowered by their interpretation of delta notation. However, I would argue that their mathematical identities were empowered to better discuss abstract notions with their peers. Thus lab 2 needs to be tweaked in such a way so as to make delta notation’s use more naturally rigorous for students. A simple way could be to include some of the following questions to the beginning of the lab: Where is the rate of change positive? Where is it

negative? Is it constant in these intervals? Where is it increasing? Where is decreasing? Can it be decreasing and still positive? If the average rate of change between two points is represented as $\Delta y/\Delta x$, how can we re-state our answers using Δy and Δx ?

The next change I would make to the course would be for students to spend several days attempting to find the instantaneous rate of change using only their own constructions of secant lines and algorithms in Geogebra. At the end of each day I would have students write a sample instruction manual for generating their solution. The use of properly defined “shortcut” notation would be strongly encouraged as well. I believe, given these extra days, that the class would have developed something very close to the formal definition of a derivative on their own, which would have been significantly more meaningful. As noted by Kuster et. al. (2018), tying student created notation to notation of the broader community is one way in which to bring mathematical formalization as a consequence of student work rather than a starting point (p. 23). Furthermore, such a student first approach would potentially allow for an opportunity to assess whether students saw the derivative, in its entirety, as a mathematical object. No such evidence was ever clearly shown in either the course or the final assessment. Although the final assessment could be tweaked to better illicit this kind of response, I would argue the outcome would be same without first addressing this issue of extra time.

One last change should be considered, as well. As previously discussed, the race problem was composed of three specific factors. However, these factors were generic in that the problem could be tweaked to any number of situations in which one object was varying constantly and the other was not. Since the students seemed to gain such significant insight from lab 2, it would be interesting to change the race problem to a problem concerning volume or height. One example could be two pools of equivalent total capacity being filled over time. Such an experiment could help verify my hypothesis that younger students do not find problems concerning physical motion to be personally meaningful.

Geogebra Throughout	Essential Instructional Final Sequence:	
	Activity	Time Weight (Weeks)
	Race Problem	1
	Intro Worksheet	Homework
	Lab 2	1.4
	3 Week Summary	Homework
	Limits Lab	1.4
	Derivative Lab	.6
	First Exposure	1.4
	Derivative Formal Definition	.2

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