Uniquely and 2-Uniquely Hamiltonian Graphs

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Presentation outline:

A) Preface

- Quick introduction to graph theory
- What is a Hamilton cycle?
  - Maximum uniquely Hamiltonian Graphs

B) A look at Maximum uniquely Hamiltonian graphs

- Verifying Hamiltonicity in Polynomial time

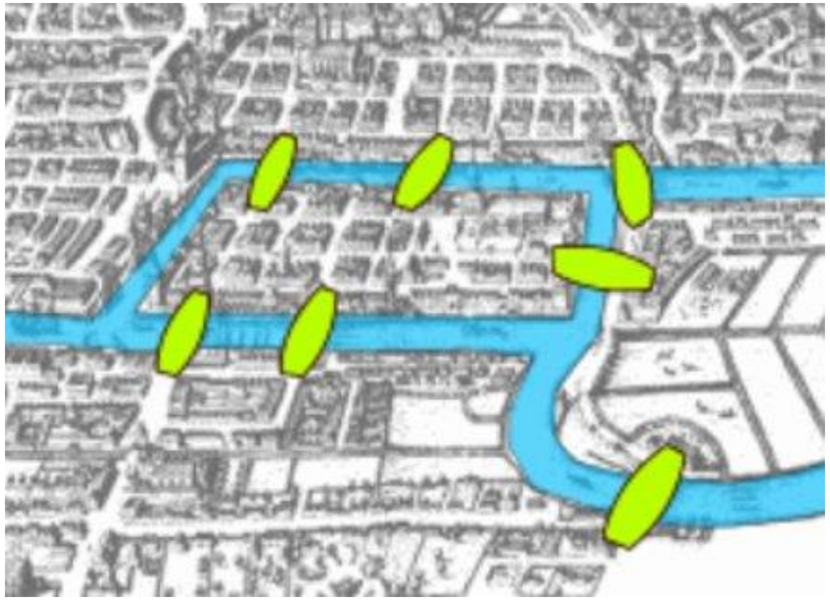
C) Extending the maximum uniquely Hamiltonian graph

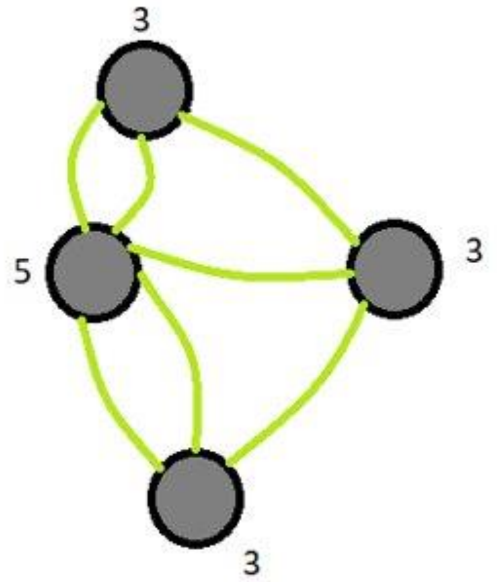
- Adding cycles to a max unique Ham graph
- Proof of maximal edge case for 2-unique Hamiltonian graphs

Preface:

1) Graph theory developed by Leonhard Euler (1707 – 1783)

- Konigsberg bridge problem (represented Konigsberg as a graph) (1736)
- Simple representation of groups of objects sharing some relationship
- Found that counting node degrees (number of edges incident on a node) determines the existence of an Euler tour.





Travelling Salesman Problem (1930)

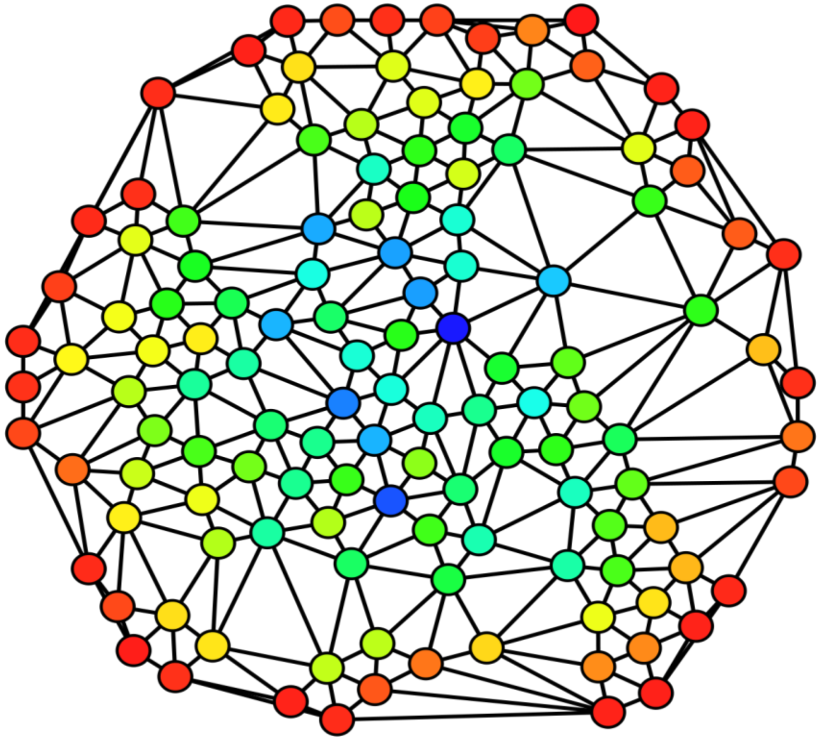
- Euler circuits are not efficient for a travelling salesman

1) Introducing Hamilton cycles

- Touch every node just once and return to the node from which you started.
- Analogous to Euler circuits, but concerned with nodes instead of edges.

2) Unfortunately for our salesman, Hamilton cycles are very hard to find

- NP-complete problem
- Arbitrary graphs hard to solve
- Easier for graphs with fewer cycles?



3) Maximum uniquely Hamiltonian graphs

- As many edges as we can fit into a graph and maintaining a single Hamilton cycle.

A look at maximum uniquely Hamiltonian graphs (MUHG's)

1) Sheehan , Entringer, and Barefoot's work

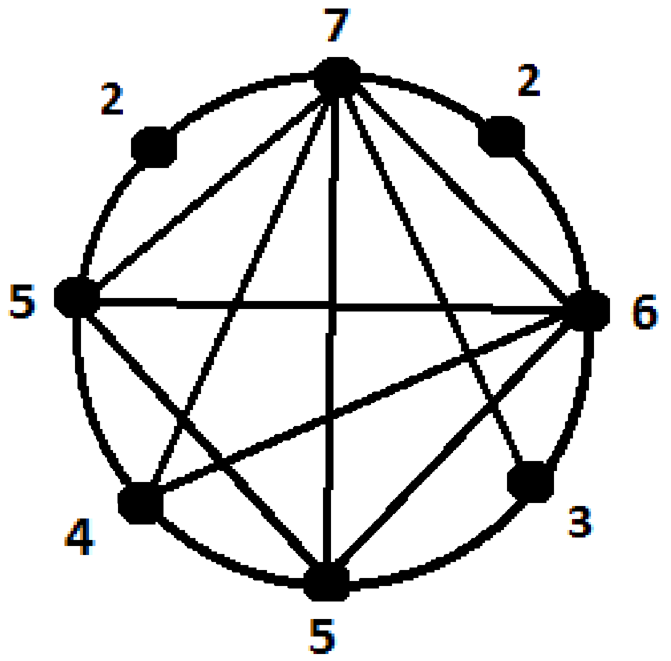
- Sheehan identified maximal edge case for graphs with a single cycle (1977)
- Entringer and Barefoot discovered family of MUHG's of size $2^{([n/2]-4)}$ (1981)
- The configuration of these graphs permits a polynomial computational solving-time

2) AAAlgorithm (AAAlg):

```
while (any_deg > 2){

   find(maxDegreeNode);

   if (maxDegreeNode isAdjacentTo 2 Deg2Nodes){
      delete(all_other_edges);
      update(degree_values);
   } end if;
} end loop;
```



Proof of correctness by induction:

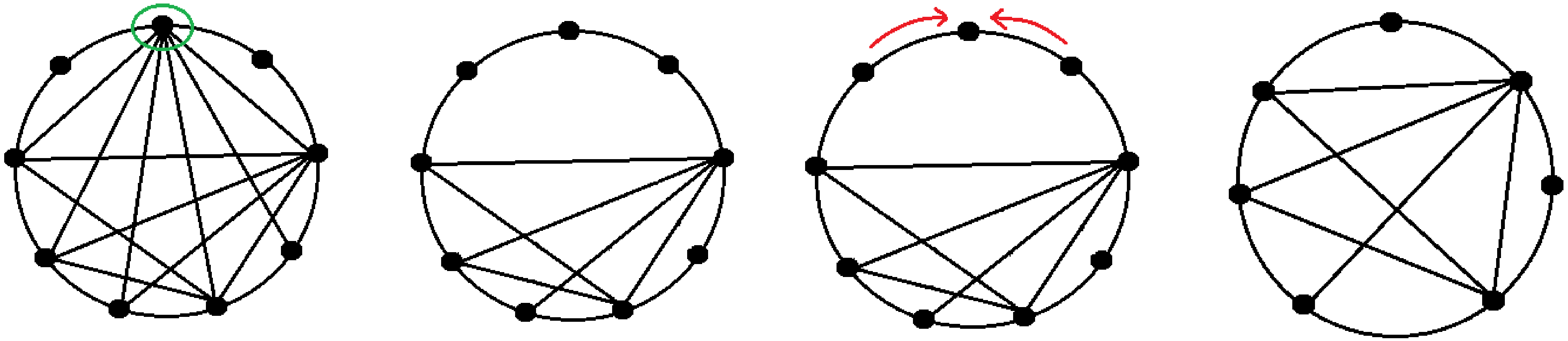
Take the 7-node base case graph from the previous slide, we've seen that our algorithm works for this graph.
It works similarly for the 8-node base case.

Now consider the n-2 case for graph G of order n > 8:

Step 1: Locate global max degree node

Step 2: remove all edges not connecting degree 2 nodes

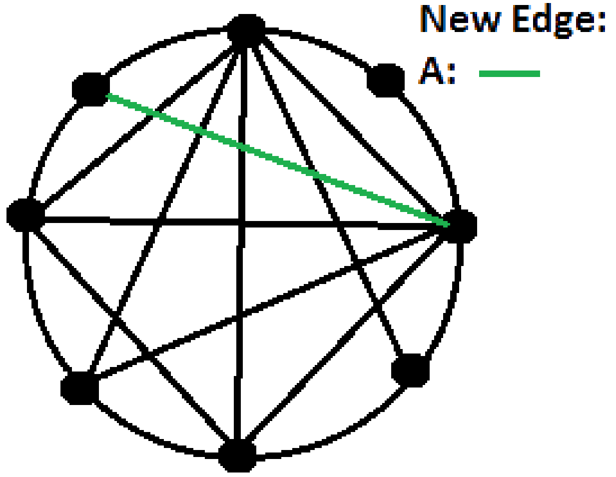
Step 3: Retract edges connecting previous global max node and nodes of degree 2.

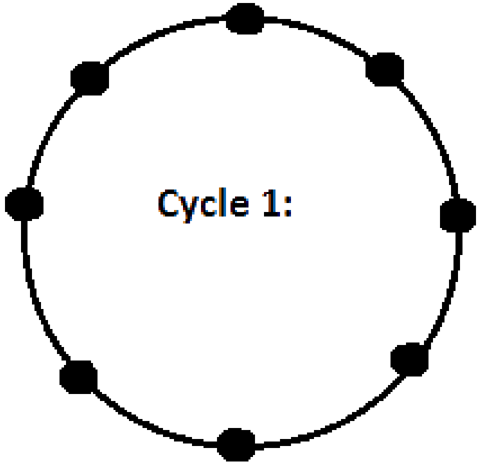


We have reduced our graph to the (n-2) MUHG, and the proof is complete.

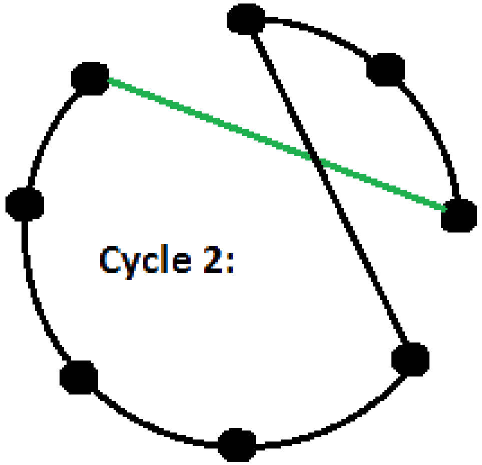
Extending the MUHG:

What about other graphs with few Hamilton cycles?
- Can we modify the MUHG to obtain a graph with 2 cycles?





Adding edge A (carefully) adds an additional cycle to our graph:



Actually: the new number of edges, N + 1, is extremal for 2 cycles where N is the number of edges in a MUHG: $N = ([n^2/4] + 1)$.

Proof for lower bound:

Take a look at the MUHG of size n = 8.

Now notice we can add one particular edge (edge A) to an extremal G_1 graph to produce a graph with exactly 2 Ham cycles.

Therefore: $\text{Max}(E(G_2)) \geq N + 1$

Proof for upper bound:

Take the graph G_2 with exactly 2 Ham cycles H_1 & H_2.

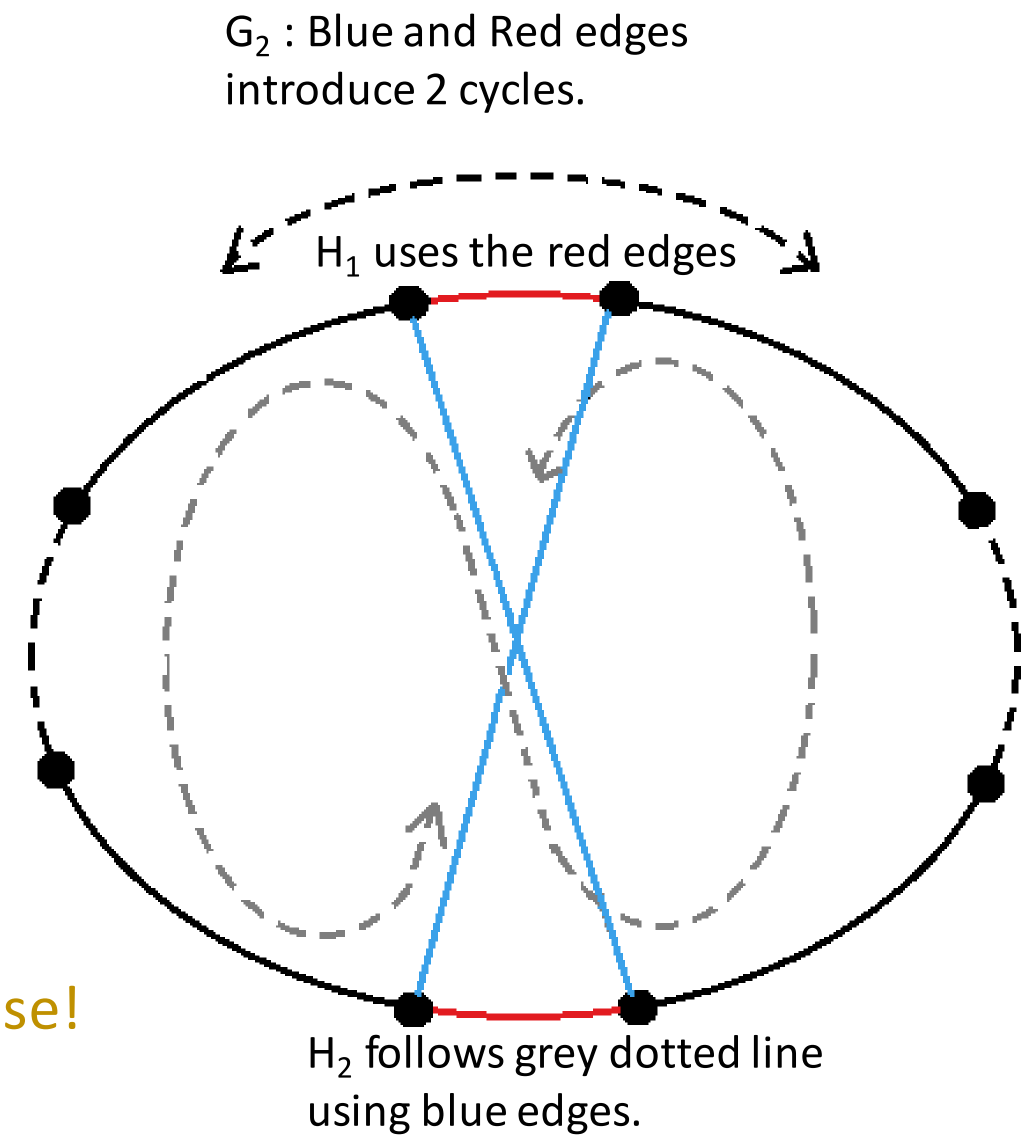
Remove an edge e from G_2 that's in H_1 but not H_2

We get: $E(G_2) \leq N + 1$.

We've proved both the upper bound: $E(G_2) \leq N + 1$,
as well as the lower bound: $\text{Max}(\ E(G_2)\ ) \geq N + 1$.
Therefore $E(G_2) = N + 1$ and our proof is complete.

An interesting discovery for graphs containing 2 cycles:

- Configuration for the n case carries up to n + 2 case!
- This fact lends these graphs to a polynomial time algorithm. But we won't cover that today.



Works cited and thanks!

- Sheehan, J. (1977), Graphs with exactly one hamiltonian circuit. J. Graph Theory, 1: 37-43. doi:10.1002/jgt.3190010110

-Barefoot, C.A. and Entringer, R.C. (1981), A census of maximum uniquely hamiltonian graphs. J. Graph Theory, 5:315-321. doi:10.1002/jgt.3190050313

THANKS!

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