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Derivation of the (Closed-Form) Particular Solution of the Poisson's Equation in 3D Using Oscillatory Radial Basis Function

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CLIMBING THE BRANCHES OF THE GRACEFUL TREE CONJECTURE

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Abstract. This paper presents new ways to look at proving the Graceful Tree Conjecture, which was first posed by Kotzig, Ringel, and Rosa in 1967. In this paper, we will define an adjacency diagram for a graph, and we will use this diagram to show that several classes of trees are graceful.

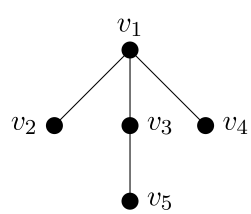
Keywords: *Graceful labeling, trees*

MSC (2010): Primary 05C78, Secondary 05C05

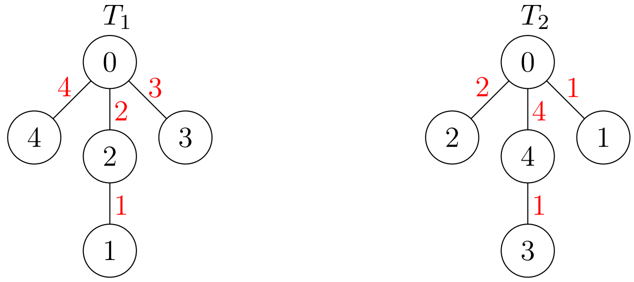
1. Introduction

A simple graph, G, is a pair, (V, E), where V is a set whose elements are referred to as *vertices*; and where E is a set consisting of 2-element subsets of V, whose elements are referred to as *edges*. We refer the reader to [1] for the definition of any unfamiliar terms. Furthermore, we will let $|V|$ and $|E|$ denote the cardinality of V and E, respectively. A *vertex labelling* of the graph G is an assignment of integers to the vertices of G. More formally, a vertex labelling is a function from V to $\mathbb{Z}$. Using the vertex labelling, we obtain an *induced edge labelling* by assigning to each edge $\{v_i, v_j\}$ the label $|f(v_i) - f(v_j)|$. When both the vertex labelling f_V and the induced edge labelling are one-to-one, f_V is said to be a *graceful* labelling. Moreover, a graph G is said to be *graceful* if there exists some vertex labelling of G that is graceful.

Example 1.1. Consider the graph



Consider, the following two vertex labellings T_1 and T_2 with the induced edge labelling shown in red.



Note that both T_1 and T_2 are bijections from V to $\{0, 1, 2, 3, 4\}$. Since the induced edge labelling from T_1 assigns distinct labels to every edge, T_1 is graceful. However, the induced edge labelling from T_2 assigns the edge label of 1 to two different edges, and thus does not produce a graceful labelling.

The graph in the example above is graceful despite the fact that the labelling T_2 is not graceful, since at least one graceful labelling of the graph exists (namely, T_1). To be able to state the graceful tree conjecture, we need to first define the type of graph known as a *tree*.

Definition 1.1. Let $G = (V, E)$ be a graph having vertex set V and edge set E. Then G is called a *tree* if G is a simple graph that is connected and does not contain any cycles.

From this point forward in this paper, we will assume G is a tree. Note that for any tree, $|E| = |V| - 1$ (see Corollary 1.5.3 in [1]). Hence, a tree is graceful if and only if there is an assignment of the integers $0, 1, 2, \ldots, |V| - 1$ to the vertices of the graph in such a way that the edges are labelled by $1, 2, \ldots, |V| - 1$ under the induced edge labelling. In response to a problem posed by Gerhard Ringel in 1963 in [3], Alexander Rosa made the famous Graceful Tree Conjecture in [4].

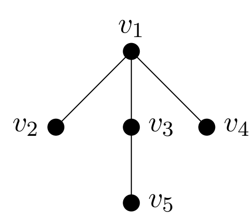
Conjecture 1.1 (Graceful Tree Conjecture)**.** *Every tree is graceful.*

In [3], the focus was on *β-valuations*, and the current formulation of the problem was introduced in 1972 in a paper by Solomon Golumb [2].

2. Adjacency Diagrams

Let $G = (V, E)$ where $V = \{v_1, v_2 \ldots, v_n\}$. We will further consider the vertices to be labelled in such a way that the vertex labelling is given by $f(v_i) = i - 1$. Then we can form the associated $n \times n$ *adjacency matrix* whose $(i, j)^{th}$ entry is the number of edges going from vertex v_i to v_j.

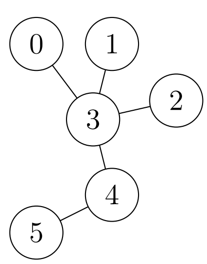
Example 2.1. The graph G is given below with its associated adjacency matrix A_G:



$$\Longleftrightarrow \quad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

When G is a tree, we note that the entries of the adjacency matrix will either be 0 or 1, since trees are simple graphs. In addition, since simple graphs have no loops and are undirected, A_G will be a symmetric matrix with all main diagonal entries being zero. Therefore, we will focus solely on the upper triangle of A_G. Furthermore, instead of writing down the zeros and ones in the adjacency matrix, we will just use a circle to denote the ones and omit the zeros. We will call the result of this process the *adjacency diagram.*

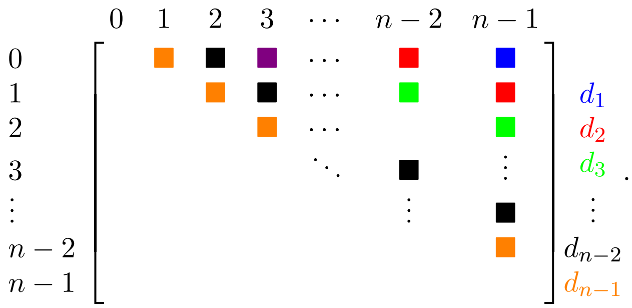
Example 2.2. The graph G is give below with its associated adjacency diagram :



$$\Longleftrightarrow \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} & & \bullet & & & \\ & & \bullet & & & \\ & & \bullet & & & \\ & & & \bullet & & \\ & & & & \bullet & \\ & & & & & \end{bmatrix} \end{array}$$

Notice that the adjacency diagram for a graph is just the upper triangle of the adjacency matrix for the graph with circles placed where the entries would typically be and the removal of the zeros. In the context of the Graceful Tree Conjecture, we only need to consider the adjacency diagrams.

Before stating our first proposition, we will notate the diagonals of the adjacency diagram with $d_1, d_2, \ldots, d_n$, as indicated in the following diagram:



Proposition 2.1. *Let G be a labelled graph with $n-1$ edges. This labelling is graceful if and only if the adjacency diagram has exactly one circle in each diagonal.*

Proof. Let G be a labelled graph with $n-1$ edges. Assume $G = \{v_1, \ldots, v_n\}$ with vertex labelling $f(v_i) = i-1$. Using the induced edge labelling, each edge $\{v_i, v_j\}$ of G is labelled by $|f(v_i) - f(v_j)| = |i-j|$. Note that each entry of the adjacency diagram of G corresponds to an edge of G. It follows that each edge represented as an entry of diagonal d_i in the adjacency diagram of G will have edge labelling $n-i$. For the graph to be graceful, the edges must be labelled with $1, 2, \ldots, n-1$. Hence, there must be exactly one circle in each of the diagonals of the adjacency diagram of G. ∎

We can use the previous proposition to count the number of gracefully labelled graphs with $n-1$ edges.

Corollary 2.1. *The number of graceful labelings of graphs with $n-1$ edges is $(n-1)!$.*

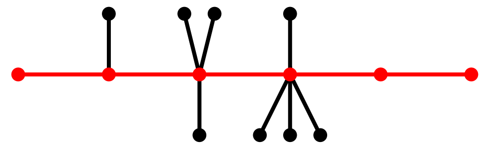
Proof. Let G be a graph with $n-1$ edges. From Proposition 2.1, the adjacency diagram of G must have exactly one circle in each of the $(n-1)$ diagonals. Moreover, diagonal d_i has i entries for $1 \leq i \leq n-1$. Hence, there are $(n-1)!$ possible arrangements of the circles. ∎

It should be noted that the above Corollary applies to all graphs, including all trees as well as disconnected graphs with $n-1$ edges that contain cycles. Moreover, the above Corollary does not say that there are $(n-1)!$ different graceful graphs with $n-1$ edges, since a single graph may have several graceful labellings. It is interesting to note that if f defines a graceful labelling of a graph G, then $g(v) = (n-1) - f(v)$ will define another graceful labelling of G.

3. Some Graceful Graphs

In this final section, we will show how Proposition 2.1 can be used to show that *caterpillar* graphs are graceful. A *caterpillar* graph is a tree in which all vertices are within distance 1 of a central path. Rosa [4] first noted that all caterpillar graphs are graceful, and he gave an example of how to define a vertex labelling for any caterpillar graph. The presentation below uses adjacency diagrams to better explain why this vertex labelling will always be graceful.

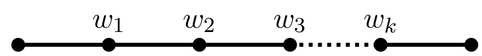
Example 3.1. The graph given below is a caterpillar graph whose central path is indicated in red.



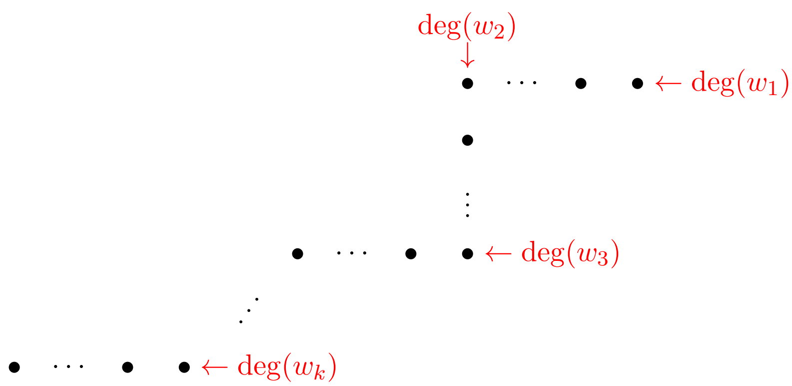
In the following proposition, we will use the method of adjacency diagrams to show that caterpillar graphs are graceful.

Proposition 3.1. *Every caterpillar graph is graceful.*

Proof. Let G be a caterpillar graph. If G is a path of length one, then labelling one vertex with 0 and the other vertex with 1 produces a graceful labelled. Assume the central path of G has length greater than one. Label the vertices of the central path having degree 2 by w_i as in the following diagram:



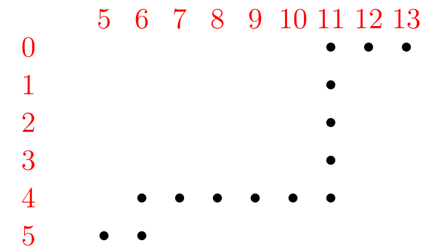
Construct the following adjacency diagram beginning in the upper right corner, where the number of circles in each row (or column) is given by $\deg(w_i)$ for $1 \leq i \leq k$:



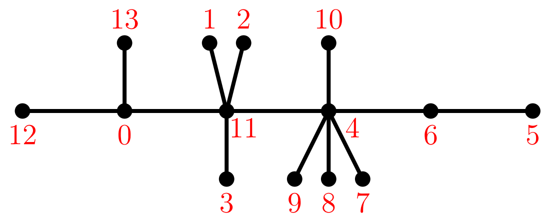
Note that the number of edges in G is given by $\left(\sum_{i=1}^{k} \deg(w_i)\right) - (k-1)$. Hence, the diagram constructed above corresponds to an adjacency diagram for G. Clearly two circles in the same row or column are on different diagonals of the adjacency diagram. In addition, any two circles that are not in the same row or column will have the higher circle to the right of the lower one. Thus the circles must be in different diagonals, and as a result, the labelling is graceful by Proposition 2.1. Therefore, the graph G is graceful. ∎

The adjacency diagram constructed in Proposition 3.1 might end with a vertical sequence of circles (as opposed to a horizontal sequence of circles). To end this section, we provide an example illustrating a graceful labelling of the caterpillar graph from Example 3.1.

Example 3.2. Considering the graph in Example 3.1, the construction given in the proof of Proposition 3.1 yields the following adjacency diagram:



This adjacency diagram corresponds to the following graceful labelling of the graph G:



4. Conclusion

The Graceful Tree Conjecture is still an open problem, but the use of adjacency diagrams and Proposition 2.1 developed in this paper may prove to be a useful tool to study this problem. This proposition can be used to prove that classes of graphs are graceful, as showcased in Proposition 3.1. It remains to show that every tree has an adjacency diagram satisfying Proposition 2.1.

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