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“As soon as the four sides are all equal, then the angles must be 90° each”. Children's misconceptions in geometry

Humphrey Uyouyo Atebe & Marc Schäfer

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"As soon as the four sides are all equal, then the angles must be 90º each". Children's misconceptions in geometry

Humphrey Uyouyo Atebe [1] and Marc Schäfer [2]
Education Department, Rhodes University, South Africa
[1] humphreyate@yahoo.com and [2] m.schafer@ru.ac.za

Abstract

This study describes Nigerian and South African students' conceptual understanding in high school geometry based on the van Hiele model of geometric thinking levels. The study further highlights students' misconceptions in school geometry. Concepts of triangles and quadrilaterals were investigated among 36 mathematics learners drawn from grades 10 through 12 who participated in this study. The tasks included identifying and naming shapes, sorting of shapes, stating the properties of shapes, defining of shapes and establishing class inclusions of shapes. The results indicated that many Nigerian and South African high school learners in Grade 10, 11 and 12 hold a number of misconceptions about geometric concepts of triangles and quadrilaterals. The use of imprecise terminology in describing many geometric shapes was common among the learners. The majority of learners in this study were at van Hiele level 0. Although many were able to distinguish between triangles and quadrilaterals, they lacked the appropriate vocabulary to distinguish among shapes in the same class. For many learners, the task of naming shapes was easier than giving a description of their properties. Also many students demonstrated a relatively better understanding of the concept of triangle than that of quadrilaterals. Knowledge of class inclusions of shape was absent among these learners. The results of this study were found to be consistent with those of earlier studies and recommendations are offered.

Keywords: van Hiele theory, Geometry, Thinking levels, Misconceptions, Understanding.

Introduction

The problem of the mismatch between teachers' classroom instruction and learners' cognitive levels of thinking in mathematics in general appears to be widespread and one that seems far from being solved (Peterson, 1998; Feza & Webb, 2005). Research evidence seems to indicate that instruction in geometry, specifically in Nigerian and South African high schools is inadequate in terms of providing learners with requisite skills needed to operate at a formal deductive thinking level required for most high school courses (van der Sandt & Nieuwoudt, 2003; Adedayo, 2000). According to Norwood and Carter (1996) and Mansfield and Happs (1996), many traditional teaching strategies do little to enhance teachers' understanding of their learners' mathematical thought. Van Hiele (1986) asserts that the inability of many teachers to match instruction with their pupils' levels of understanding in geometry more than anything else accounts for their failure to promote students' conceptual understanding in this subject.

An analysis of the Nigerian and South African mathematics curricula for high school learners reveals that school geometry is presented largely at the formal deductive and axiomatic thinking levels corresponding to van Hiele level 4 (Federal Republic of Nigeria (FRN), Ministry of Education (MoE), 1985; South Africa, Department of Education (DoE), 2003). Published research, however, suggests that all too often the elementary and junior high school experiences of these learners are insufficient to enable them operate with understanding at these high levels of geometric reasoning (Hoffer, 1981:14; Pegg, 1995). Students need prerequisite understandings about geometry before being rushed into formal axiomatic geometry. When this is not the case, students tend to imitate the action schemes of the teacher by memorizing, but without understanding, the concepts being developed (Hoffer, 1981; Clements & Battista, 1992). One direct consequence of this situation is that many high school learners hold several misconceptions about geometric concepts. It is against this backdrop of the mismatch between instruction and students’ levels of understanding that this study seeks to explore students’ misconceptions in geometry within the Nigerian and South African contexts. Specifically, this study aims to explore students’ misconceptions in high school geometry using a manipulative instrument.

Background and Significance

Results of the Third International Mathematics and Science Study-Repeat (TIMSS-R) conducted in 1999 indicated that of the 38 countries that participated, South African pupils recorded the poorest performance in mathematics compared to pupils from other countries (Brombacher, 2001; Howie, 2001). The average score of 275 points out of 800 points was well below the international average of 487 points (Howie, 2001). The TIMSS-R 1999 results further revealed that of the South African participants, pupils from the Eastern Cape province ranked 7th (out of 9) behind children from the other provinces. Howie’s (2001) analysis of TIMSS-R indicates that of the mathematics topics on which pupils were tested, South African pupils found geometry-related questions the most difficult. This revelation makes a case for re-appraisal of the South African pupils’ conception of geometric ideas.

Because Nigeria has never participated in TIMSS, very little seems to be known about the Nigerian children’s mathematical competencies in relation to children of other countries. However, in one of our other studies, Atebe and Schäfer (2008), the Nigerian students’ conception of geometry was found to be consistent with that of their South African counterparts with regard to the van Hiele levels of geometric understanding. Many of the Nigerian (53%) and South African (41%) grades 10, 11 and 12 learners were at van Hiele level 0. We thus recommended that further study is needed to explicate more rigorously the Nigerian and South African learners’ geometric thought using a manipulative instrument that could elucidate learners’ misconceptions in geometry – the focus of this paper.

This study makes a contribution to our understanding of children misconceptions in geometry that account for their continued weak performance in the subject. It is hoped that with this understanding of children’s misconceptions, teachers and curriculum designers would be able to appropriately structure instructions and learning programmes in geometry in ways that incorporates learners’ levels of geometric thinking in order to improve children’s conceptual understanding in geometry.

Theory Informing the Study

This study utilizes a theory of levels of thought in geometry called the van Hiele theory (van Hiele, 1986). The van Hiele theory was developed by two mathematics educators from the Netherlands – Pierre van Hiele and Dina van Hiele-Geldof, when they did research on thought and concept development in geometry among school children.

Resulting from their research, the van Hieles identified and proposed five sequential and hierarchical levels of understanding in geometry through which learners progress in their development of geometric ideas (Usiskin, 1982; Burger and Shaughnessy, 1986; Senk, 1989; Pegg, 1995). Although the van Hiele theory was developed in the late 1950s, currently, the theory is generally acclaimed as one of the best-known frameworks for studying teaching and learning processes in geometry (Battista, 2002). The van Hiele theory unravels and enables exploration into why many students experience difficulty in their geometry courses. The theory also offers a model of teaching that teachers could apply in order to promote their learners' levels of understanding in geometry.

The Van Hiele Levels of Geometric Thought

Two different numbering schemes are commonly used in the literature to describe the van Hiele levels: level 0 through to 4, and level 1 through to 5 (Senk, 1989). The van Hieles originally made use of the level 0 through to 4 numbering scheme. However, Hoffer (1981) and van Hiele's (1986, 1999) more recent writings made use of the level 1 through to 5 numbering system. This, according to Senk (1989, p.310) permits the 0 to be used for students who do not operate even at the van Hiele's "basic" level. In this study, all references made to research studies that used the 0 to 4 scheme were adapted to the 1 to 5 numbering scheme. The van Hiele levels can be described as follows:

Level 1: Visualization. At this level, the student recognizes a geometric shape by its appearance alone. A figure is perceived as a whole, recognizable by its visible form. Properties of a figure are not yet perceived by the student (Mayberry, 1983).

Level 2: Analysis. The student at this level is able to reason about a geometric shape in terms of its properties. The student can recognize and name the properties of a figure, but does not yet understand the relationships between these properties. The relationships between different figures are not yet understood (Hoffer, 1981).

Level 3: Informal deduction. At this level, the student can logically order the properties of figures and begins to perceive the relationships between these properties and between different figures. The student uses the properties that they already know to formulate definitions of simple geometric shapes, and class inclusions are understood (Mayberry, 1983; van Hiele, 1999). The role and importance of deduction, however, is not yet understood.

Level 4: Deduction. The learner can appreciate the role of deduction and can now prove theorems deductively. The meaning of necessary and sufficient conditions is understood and can establish inter-relationships among networks of theorems (Pegg, 1995).

Level 5: Rigour. The student can establish theorems in different axiomatic systems. The role and significance of indirect proofs is understood. Non-Euclidean geometries can be studied and different systems can be compared (Mayberry, 1983; Feza & Webb, 2005).

Clements and Battista (1992, p.429) argue that many school children exhibit thinking about geometric concepts more "primitive than, and probably prerequisite to, van Hiele's level 1". They therefore proposed the existence of level 0 which they called pre-recognition. In essence, some earlier van Hiele researchers like Usiskin (1982) and Senk (1989, p.318) had earlier proposed the existence of level 0 and used it to describe students who are "…unable to recognize common plane geometric figures…", but the literature seems to indicate that it was Clements and

Battista (1992, p.429) that first explicitly used the word “pre-recognition” to refer to this class of students. Students at this level can distinguish between curvilinear and rectilinear shapes, but not among shapes in the same class. A student at this level, for example, may differentiate between a circle and a rectangle, but not between a rhombus and a square.

What is important in the van Hiele theory in terms of pedagogy is the linguistic character of the levels. Each of the levels has its own linguistic symbols and network of relations. People reasoning at different levels speak different languages and the same term is interpreted differently. The mismatch between instruction and students’ cognitive levels in geometry is caused largely by teachers’ failure to deliver instruction to the pupils in a language that is appropriate to students’ thinking levels (van Hiele, 1986). Van Hiele emphasised the hierarchical nature of the levels. Thus a student cannot operate with understanding at level n without having mastered level n–1 (Usiskin, 1982). It is therefore asserted that teachers and curriculum developers should therefore consider this in their geometry instructional design and delivery.

Some critical comments about the van Hiele theory concerning class inclusion

Class inclusion is a subject of some controversy regarding the level at which it occurs among school children in the van Hiele hierarchy and is, therefore, accorded some special attention in this paper. In its original form, the van Hiele theory posits that students only come to grips with the knowledge of class inclusion of geometric shapes at level 3. There are, however, opposing views from empirical research that seem to suggest that with appropriate instructional tasks, students could understand and accept class inclusion even at van Hiele levels 1 and 2 (De Villiers, 1994; 1998). We intend here to briefly engage in this issue and clarify our position as it relates to this paper.

De Villiers (1994) distinguishes between two forms of classification of mathematical concepts (e.g. classification of the real numbers) with a major focus on the classification of quadrilaterals. These are partition classification and hierarchical classification. We found his classification forms relevant to this study because of their potential possibility to clarify the issue of class inclusion regarding the van Hiele theory.

By hierarchical classification is meant “the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts” (De Villiers, 1994, p.11). Partition classification, on the other hand, relates to a classification done such that “the various subsets of concepts are considered to be disjoint from one another” (*ibid*.). He contrasts the two forms of classification as shown in Figure 1. We have taken the liberty to reverse the arrowheads so as to better convey the idea of ‘subset’ implicit in the figure. That is, the representation, Squares → Rhombi, may be read as “the squares are subset/subclass of the rhombi”.

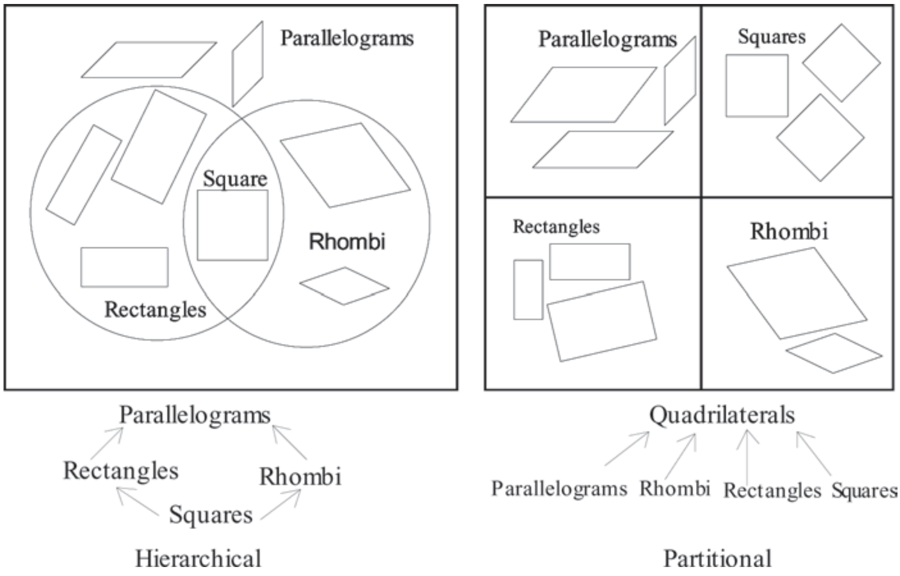


Figure 1: De Villiers' two Classification forms of quadrilaterals

As evident in Figure 1, in hierarchical classification, the rectangles and rhombi are clearly subsets of the parallelograms with the squares as the intersection between the rectangles and the rhombi. In contrast, in partition classification, the squares are neither (seen as) rectangles or rhombi, nor the rectangles and rhombi (seen as) parallelograms.

De Villiers (1994, p.12) further asserts that "the classification of any set of concepts does not take place independently of the process of defining". He substantiated his claim with the following illustration:

> For example, to hierarchically classify a parallelogram as a trapezium requires defining a trapezium as "a quadrilateral with *at least* one pair of opposite sides parallel". If on the other hand we want to exclude the parallelograms from the trapeziums we need to define a trapezium as "a quadrilateral with *only* one pair of opposite sides parallel". (De Villiers, 1994, p.12)

More importantly, De Villiers (1994; 1998) stresses the view that hierarchical and partitional definitions (and classification) are both mathematically correct. He, however, added that hierarchical definition is more economical (linguistically speaking) than partitional definition, but that students more generally prefer partitional definition (and classification). This empirical viewpoint brings us closer to the theoretical standpoint of the van Hiele theory concerning class inclusion, and raises two fundamental questions concerning class inclusion in the theory. These are: 1) If two students define a geometric shape (e.g. a rectangle) such that the one student offers a correct hierarchical definition (economical or not) and the other offers a correct partitional definition (economical or not), can both students be described as belonging to the same van Hiele level (in this case, level 3)? 2) How do we know for sure that a student who offers a partitional definition (and classification) of a geometric concept is capable of van Hiele level 3 reasoning (as claimed by De Villiers, 1994; 1998) in a given geometry task?

According to the van Hiele theory, definitions at level 3 are expected to be hierarchical. Hence, it would seem reasonable that only the student who offered a correct hierarchical definition could be said to be at level 3 on the van Hiele scale, which partly answers the first question. But this again raises yet another question, which is: What level then is the student who offered a correct partitional definition? De Villiers (1998) expresses the view that students can construct their own definitions at each van Hiele level and asserts that students' definitions at van Hiele levels 1 and 2 are typically partitional. The following excerpt represents his views.

> **Van Hiele [level] 1**: *Visual* definitions, e.g. a rectangle is a quad that looks like this (draws or identifies one) or describes it in terms of *visual* properties, e.g. all angles 90°, two long and two short sides.
>
> **Van Hiele [level] 2**: *Uneconomical* definitions, e.g. a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.
>
> **Van Hiele [level] 3**: *Correct, economical* definitions, e.g. a rectangle is a quadrilateral with an axis of symmetry through each pair of opposite sides.
>
> (De Villiers, 1998, p.253)

Clearly the first two definitions in the above excerpt indicate that "…students' definitions at these levels would tend to be partitional [as] they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long sides and two short sides)" (De Villiers, 1998, p.253). What the above excerpt has illustrated is that even on a single geometry task that is designed to explore students' knowledge of class inclusion, different students could be on different levels depending on their individual responses. In this study, we employed among others De Villiers' ideas (as illustrated in the above excerpt) to assign learners to various van Hiele levels on classification and defining shapes geometry tasks.

We now turn to the second question. As stated earlier, research has indicated that many students who exhibit appreciable competence in logical reasoning at level 3 still prefer to define/classify quadrilaterals partitionally (De Villiers, 1994). This empirical finding seems to pose a challenge to researchers to reliably describe students as having (or not having) knowledge of class inclusion of shapes according to the van Hiele model. De Villiers (1994; 1998) in acknowledgement of this problem suggests that researchers should carefully design and select their testing instrument, and where possible incorporate the use of dynamic software such as Sketchpad in their studies. As the sample for this study comprised learners from schools where the presence of computers is little felt, we could not apply De Villiers' suggestion, but instead, we incorporated informal interviews to further interrogate participants' knowledge of class inclusion of shapes based on their written responses.

In this study, Burger and Shaughnessy's (1986) descriptors of the van Hiele levels as well as van Hiele's (1986; 1999) own writings about the characteristics of the levels were used as the theoretical framework of analysis. De Villiers' ideas of hierarchical and partitional definitions (and classification) as discussed above were especially employed in our judgement of students' knowledge of class inclusions, and hence, the attainment of van Hiele level 3.

Method

This case study was undertaken to elicit selected students' levels of conceptual understanding in geometry within the premise of respecting and recognizing the uniqueness of each individual. Manipulatives in the form of picture (concept) cards of triangles and quadrilaterals were presented to each participant. A set of structured questionnaires that required the learners to carry out various operations (identifying, naming, classifying, defining) was then given to each of the participants. The learners were then asked to write down their responses as they worked on the various tasks. The questions were structured in a manner that the researchers could easily decode learners' understanding of and thought about the geometric concepts that were presented to them.

The method adopted in this study has a wide acceptability among researchers in the field of mathematics education who seek an understanding of children's thought about geometric concepts, as it has been used in many earlier studies (Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys & Liebov, 1997; Renne, 2004; Feza & Webb, 2005). In most of these studies, interview schedules, of whatever form (structured or unstructured) were used to tease out students' thought about geometric concepts while the students are engaged in the tasks involving the manipulatives (Burger & Shaughnessy, 1986; Renne, 2004; Feza & Webb, 2005).

In this study, the selected students were required to engage with the manipulative tasks before being interviewed. Burger and Shaughnessy's (1986) interview format was, however adapted into the question format used in this study.

Sample

A total of 36 Nigerian and South African high school mathematics learners were involved in this study. 18 of these learners were drawn from a state high school in Ojo Local Education District in Nigeria, while the remaining 18 learners were drawn from a Township high school in the Eastern Cape Province in South African. Both schools are representatives of those schools accessed by the majority of children in Nigeria and South Africa. The schools were chosen on the basis of their functionality and proximity.

Six learners of mixed mathematics abilities (high, average, low achievers) each were drawn from grades 10, 11 and 12 in each of the two schools. Selection was therefore purposive, as the respective mathematics teachers in each grade were requested to assist in the selection process using the students' academic records. The reason was for us to have a sample that is representative of each grade in terms of cognitive ability. It must be emphasised that the findings of this study are thus not generalisable, but pertain to this specific sample only. The study nevertheless provides a frame of reference on which to build a more comprehensive understanding of learners' geometric thinking in Nigeria and South Africa.

Instrument

The main instrument used in this study for data collection was a questionnaire together with a set of manipulatives consisting of triangles and quadrilaterals of the various kinds.

The Manipulatives

Van Hiele (1999) suggests that giving learners ample opportunity for playful exploration of hands-on manipulatives gives teachers a chance to observe and assess informally learners' understanding of and thinking about geometric shapes and their properties. This is, because a teacher's knowledge of what students are thinking is important for instructional design and implementation. Since this study explores students' understanding of geometric concepts, the use of hands-on manipulatives allows the learners to demonstrate what they know and think about these concepts. This is supported by Kilpatrick's (1978, p.191) assertion that "we learn by doing… and by thinking about what we do".

The manipulatives used in this study consisted of numbered concept cards of triangles and quadrilaterals. In all, there were 30 numbered cards comprising 10 triangular cards and 20 cards of various quadrilaterals. There is a growing concern that static manipulatives like the ones used in this study tend to steer learners' thinking towards partition definition and classification as they do not offer the learners the opportunity to dynamically transform a given shape into some more general or special case of the shape (De Villiers, 1994). De Villiers (*ibid*.), therefore, recommends (among other things) "an appropriate negotiation of *linguistic* meaning", i.e. clarity of language concerning the questions or the tasks that the learners are required to respond to in geometry (p.17). This precaution as suggested by de Villiers was taken into account in this study.

Cards Construction and Composition

Initially the manipulatives constructed and used by Feza and Webb (2005) were acquired on request for adoption, which they generously gave to us. But, because our intention in this study was to focus primarily on the geometric concepts of triangles and quadrilaterals, it became necessary to construct our own. The reason for this focus is that triangles are the key building blocks for most geometrical configurations, while according to French (2004), many children are familiar with various quadrilaterals from an early age. Feza and Webb's manipulatives, however, offered a useful insight into our own constructions.

The concept cards were made from cardboard cutouts. Straight edges, protractors and a pair of scissors were used for constructing the cards so as to guarantee accurate side-angle relation properties of the various shapes. The triangular shapes constructed included isosceles, equilateral, scalene, right-angled triangle, and several combinations of these. The quadrilaterals constructed included squares, rectangles, rhombuses, parallelograms, kites and trapeziums. There were at least two of each type of shapes differentiated by varying either the size, colour or the orientation of the number written on the card.

Data Collection

One set of questionnaire consisting of five distinct tasks was developed and used for data collection in this study. Each student was given the questionnaire together with the pack of concept cards numbered 1 to 30. Straightedges and protractors were given each student. The students were required to work through all five tasks contained in the questionnaires in accordance with detailed instructions for each task.

Task 1: Identifying and naming shapes. This task required the students to identify each shape by stating the correct name of the shapes. Each student was requested to justify his or her naming.

Task 2: Sorting of shapes. This task required the students to sort all 30 shapes into two groups – groups of triangles and quadrilaterals. The students were required to state the criterion for their grouping, and to state the general/common or collective name of the shapes in either group.

Task 3: Sorting by class inclusion of shapes. This task required the students to make a further sorting of the shapes in either group into smaller subgroups of shapes that are alike in some way. The students were requested to state how the shapes in each subgroup were alike. This tasks, thus explores students' knowledge of class inclusion or the lack of it. It was indicated to the students that a shape may belong to two or more subgroups. This hint was necessary so as to forestall the problem of partitional thinking inherent in this kind of task on the part of the students as explained earlier on.

Task 4: Defining shapes. This task required the students either to state a definition of a shape or list the defining properties of a shape. A sample question from this task is as follows:

What would you tell someone to look for in order to pick out all the parallelograms from among these shapes?

This question was repeated for rectangles, rhombuses, squares, trapeziums and isosceles triangles.

Task 5: Class inclusion of shapes. Students were required to state with justification whether a given shape belongs to a class of shapes with some more general properties. A sample question from this task is:

> *Is shape No. 23 a rectangle?*
> *How do you know?*

Shape No. 23 was a concept card of a square in this study. Similar questions were asked for other shapes.

Assignment of levels

Burger and Shaughnessy's (1986) point of view concerning the use of manipulatives for tasks such as the ones described above is that these tasks do not distinguish (or elicit) reasoning beyond van Hiele level 3, a point buttressed by De Villiers'(1998) partitional and hierarchical definition (and classification) explained above. Thus, the highest level assignable to any student in this study is level 3.

Results

Previous research studies have drawn some important conclusions which we considered relevant to this study. Mayberry (1983) and Senk (1989), for instance, found that students can be on different levels for different concepts. Renne's (2004) study revealed that although many students can easily identify many 2-dimensional shapes, many of these students do not have the right vocabulary to express distinguishing attributes and compare shapes in a systematic manner. As a result, students' responses to each of the five tasks described earlier were analyzed distinctly for the attainment of specific van Hiele levels.

Exploratory Analysis

Given the fact that this study was undertaken within the notion of respecting and recognizing the uniqueness of each individual we adopted an exploratory analytic method to enable us to elicit patterns in the students' responses to the various tasks. This method was also used by Mayberry (1983).

Identifying and naming shapes task: Students' responses to this task are presented in Table 1.

Table 1: Students who named shapes correctly and stated the correct reason

Shape No.	Name of shape	No. correctly naming shape		No. stating correct reason	
		Nigeria (n=18)	S.A (n=18)	Nigeria (n=18)	S.A (n=18)
1	Rhombus	8	12	0	1
2	Isosceles trapezium	10	14	0	0
3	Rectangle	16	17	2	6
4	Obtuse-angled scalene triangle	15	16	4	6
5	Rectangle	6	9	0	6
6	Square	16	18	0	3
7	Isosceles trapezium	12	13	0	1
8	Kite	10	10	2	0
9	Rhombus	4	8	0	2
10	Isosceles triangle	16	16	11	12
11	Parallelogram	14	12	1	3
12	Equilateral triangle	17	17	7	9
13	Rhombus	5	11	0	2
14	Isosceles triangle	17	14	5	9
15	Rectangle	17	18	3	9
16	Isosceles trapezium	11	13	0	0
17	Rectangle	15	16	1	6
18	Equilateral triangle	18	15	8	5
19	Rectangle	9	10	1	5
20	Right-angled trapezium	9	9	0	0
21	Right-angled isosceles triangle	17	16	7	7
22	Right-angled isosceles triangle	16	15	8	7
23	Square	12	17	1	4
24	Right-angled isosceles triangle	16	16	9	10
25	Parallelogram	12	11	1	6
26	Right-angled trapezium	5	6	0	0
27	Scalene triangle	13	13	9	9
28	Kite	10	10	2	1
29	Parallelogram	12	12	0	6
30	Right-angled scalene triangle	16	13	7	8

As evident in Table 1, there were a lot of inconsistencies regarding students' ability to correctly name the various shapes on this task. Lack of conceptual understanding in geometry became evident, as many students seemed able to recognize shapes only in some basic (or standard) orientations. As indicated in Table 1, three of the shapes in the pack of shapes used in this study were rhombuses (shape Nos. 1, 9 & 13). With regard to the number written on the card, shape No.1 was presented in the more basic orientation than shape Nos. 9 and 13 as illustrated in Figure 2.

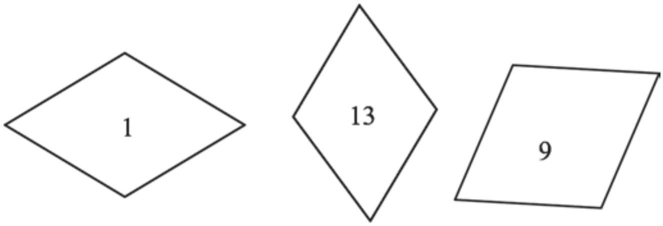


Figure 2: Rhombuses in different orientations

Although 8 learners (44%) from the Nigerian cohort were able to identify shape No.1 as a rhombus, only 4 (22%) of them, however, were successful in identifying shape No.9 as a rhombus. Similarly, 12 learners (67%) from the South African cohort were able to identify shape No.1 as a rhombus, but only 8 (44%) of them succeeded in recognizing shape No.9 as a rhombus. This lack of conceptual ability to recognize rhombuses in different orientations was also evident in students' identification and naming of other shapes as revealed in Table 1. Similar findings had earlier been reported by Mayberry (1983, p.64) where. in the U.S., some preservice elementary teachers "had difficulty in recognizing a square with a nonstandard orientation".

Table 1 also indicates that more students can name shapes than describe the properties of shapes. For instance, although 8 students from the Nigerian subsample were able to name shape No.1 as a rhombus, none of them could state the discerning properties of it. Also, only 1 out of the 12 South African learners who correctly stated the name of shape No.1 as a rhombus was able to state its properties. This seems to link up with the hierarchical property of the van Hiele levels explained above.

The use of imprecise properties for describing the shapes was common among many of the students. The majority of the students described the shapes entirely by the property of sides while neglecting angle properties of the shapes. For example, all the 8 Nigerian learners that correctly identified shape No.1 as a rhombus used only the side property to justify their naming; "It is a polygon having four equal sides", "It has 4 equal sides", "Four sides are equal", and so forth. Among the South African learners, only 1 out of the 12 that correctly identified shape No.1 as a rhombus made use of both sides and angle properties to justify his naming – "It is a quad with 4 equal sides, angles are not right angles", exemplifying De Villiers' partitional definition as clarified earlier on. The rest, like their Nigerian counterparts focused only on the property of sides; "it has 4 equal sides, stude (for skewed) square", "4 equal sides are equal and it is the same as square but its squde (for skewed)", "It has four equal sides", "It is like a square (a square has 4 equal sides) but skewed" and so forth.

There was no student (Nigerian and South African learners alike) that used more than one attribute of a shape in naming the shape. For instance, right-angled isosceles triangles (shape Nos. 21, 22 & 24) were either named as “isosceles triangle” or “right-angled triangle” by students who named them correctly, with the majority showing preference for the former name. In short, only 2 learners (1 Nigerian and 1 South African) named these shapes as “right-angled triangle”. About half of the learners simply referred to these shapes (and other different triangles) as simply “triangle”.

This manner of naming shapes by reference to a single attribute was absent with regard to quadrilaterals. None of the learners used such words as “right-angled trapezium” (shape Nos.20 & 26) or “isosceles trapezium” (shape Nos.2, 7 & 16) even when they used straightedges and protractors to establish these attributes. The learners simply called them “trapezium”.

During an initial on-site analysis of students’ responses, some of the inconsistencies that we noticed in the learners’ responses prompted us to interview them. These interviews helped to tease out more information about students’ misconceptions in geometry. Although all 36 learners involved in this study were interviewed, only a sample of 2, however, was selected for reporting in this narrative. This sample was chosen for the variety of responses the learners exhibited during the interviews, a variety that was representative of the entire study sample and best reflected the theme of this paper. In the identifying and naming shapes task, Vusumzi (pseudonym), a grade 12 learner from the South African subsample named a rhombus (shape No.1) as a “square” and gave such reason as “it has four equal sides”. The following interview took place:

Researcher: Do you mean that all shapes having four equal sides are squares?

Vusumzi: Yes.

Researcher: If a shape is a square, what other property would it have apart from four equal sides?

Vusumzi: All four angles measure 90° each.

Researcher: Did you measure the angles of shape No.1?

Vusumzi: No.

Researcher: Why?

Vusumsi: I know that as soon as the four sides are all equal, then the angles must be 90° each.

This line of reasoning was common among the majority of the learners, while many others made reference to a visual prototype as evident in the following interview that we held with Asisat (pseudonym), a grade 11 learner from the Nigerian subsample. Asisat had correctly named shape No.1 as a rhombus and shape No.6 as a square. She, however, stated that “it has 4 equal sides” as the only reason for naming both shapes as she did. She was interviewed as follows:

Researcher: Do you mean that if a shape has four equal sides, then it is a rhombus?

Asisat: No. The shape has to have four equal sides and look like a kite.

Researcher: You gave the same reason [four equal sides] for the rhombus and the square. How is a square different from a rhombus?

Asisat: A square is like a carpet, a rhombus is like a kite.

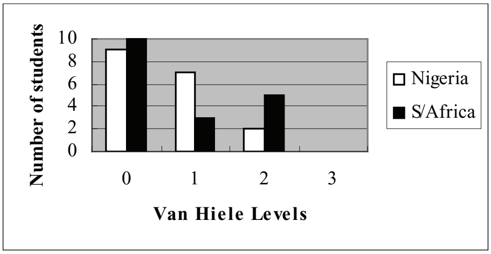
Researcher: Is there anything else that you can tell me about the properties of a square apart from having four equal sides?

Asisat:	[Prolonged silence, then shook her head slowly] No.
Researcher:	Ok. But is there anything that you can remember about the angles of a square and a rhombus?
Asisat:	Em …em… am not sure.
Researcher:	That's fine. Now tell me, how were you sure that the two shapes were not both either squares or rhombuses?
Asisat:	You see, how I used to know them is that the one that is like a carpet is the square, and that one [pointing at the rhombus] that is like a kite I know that it is a rhombus.

Clearly, Asisat, like many other learners, was not attending to the properties of the shapes. A shape was what it is called because it looked like some known shape or object – typical of van Hiele level 1 reasoning. With these interviews together with the students' written responses we were able to assign levels to the students using Burger and Shaughnessy's (1986) descriptors of the van Hiele Levels. Table 2 as well as its associated bar graph gives the number of students at each van Hiele levels for the task of identifying and naming shapes. This task could not distinguish reasoning beyond level 3 (see Burger & Shaugnessy, 1986: p. 43).

Table 2: Van Hiele levels of students on the task of identifying and naming shapes.

Van Hiele Levels	Number of students	
	Nigeria (n=18)	South Africa (n=18)
0	9	10
1	7	3
2	2	5
3	0	0



Sorting shapes task: Many students (25 out of 36 i.e. 69%) were able to sort the shapes into two groups – one of triangles and the other of quadrilaterals – using the property of sides. More South African learners (15) were successful on this task than Nigerian learners (10). Even though many students sorted the shapes into two groups of 3-sided shapes (triangles) and 4-sided shapes (quadrilaterals), only very few of them, 5 learners each from the Nigerian and South African subsamples, were able to use the correct terminology 'quadrilateral' for the group of 4-sided shapes. The results are summarized in Table 3.

Table 3: Number of students who successfully sorted shapes into groups of triangles and quadrilaterals.

	Nigeria (n=18)		South Africa (n=18)	
Number correctly sorting shapes into 3-sided and 4-sided shapes.	10		15	
No. sorting shapes by property of sides.	9		14	
No. stating the correct name of the group of shapes	Triangles	Quadrilaterals	Triangles	Quadrilaterals
	9	5	14	5

The sorting shapes task revealed some important misconceptions about geometric concepts among the students. 8 students (2 Nigerians and 6 South Africans) reasoned that all 4-sided shapes are called square. There were 4 other students (3 Nigerians and 1 South African) who reasoned that all 4-sided figures are called rectangle. And yet one other student (a Nigerian) thought that all 4-sided shapes are called parallelogram.

On the assignment of levels, learners that were either not able to (or were only partly able to) sort the shapes into groups of triangles and quadrilaterals were assigned van Hiele level 0. Those who were able to sort the shapes into distinct groups of triangles and quadrilaterals and were also able to state the correct criterion (number of sides) for sorting were assigned van Hiele level 1. Learners who successfully sorted the shapes into groups of triangles and quadrilaterals and made use of the right terminology – triangle for the 3-sided shapes and quadrilateral for the 4-sided shapes, were assigned van Hiele level 2. This task could not elicit reasoning beyond level 2. Table 4 summarizes these results.

Table 4: Students' van Hiele levels on sorting shapes task.

Van Hiele Levels	Number of students	
	Nigeria (n=18)	South Africa (n=18)
0	8	3
1	5	12
2	5	3

Mayberry's claim. Mayberry (1983) claims that students can be on different van Hiele levels on different concepts. By comparing individual student's van Hiele levels on the identification task and sorting shapes task, the data from this study supports Mayberry's claim, as students either lost, gained or remained on the same level on the two tasks. These results are presented in Table 5.

Table 5: Number of students who lost, gained or remained on the same van Hiele levels on two discrete tasks of identifying shapes and sorting shapes.

Country	No. Losing	No. Gaining	No. on the same level
Nigeria	4	3	11
South Africa	4	7	7

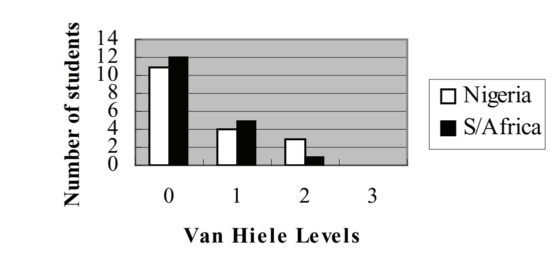
Two important new ideas about the van Hiele levels were revealed from our comparisons of students' levels across different tasks. First, no student lost or gained more than one level on the two tasks. Second, loss of levels was primarily from level 1 to level 0, as only 3 (8%) students (all South Africans) declined from level 2 to level 1. Maintenance of levels for the Nigerian learners was largely at level 0 (33%), while it was at level 1 (17%) for the South African subsample.

Sorting by class inclusion task. All the students sorted the shapes so as to prohibit class inclusions. Rectangles, squares and rhombuses, for example, were all excluded from the class of parallelograms by all the students the same way that squares were not perceived as rectangle (or rhombus). That is, students who attempted forming subclasses of shapes did so partitionally.

Right-angled isosceles triangles were either excluded from the class of right-angled triangles or that of isosceles triangles by the students (the learners focusing only on a single attribute). Even with prohibition of class inclusion, the majority of the learners could not sort the shapes into distinct classes of triangles and quadrilaterals, as many either omitted many members or included non-members of a class. Forming distinct classes of triangles (with class exclusion), however, appeared easier for many students than that of quadrilaterals. Again this task could not distinguish reasoning beyond level 3. Table 6 and its accompanying bar graph summarize these results with regard to the van Hiele levels.

Table 6: Students' van Hiele levels on sorting shapes by class inclusion task.

Van Hiele Levels	Number of students	
	Nigeria (n=18)	South Africa (n=18)
0	11	12
1	4	5
2	3	1
3	0	0



Defining shapes task. This task revealed a number of imprecise visual qualities that many learners used in describing the shapes. Reference to visual prototypes was common in learners' definitions of the shapes, "Rhombuses look like squares but if you look carefully it's sides are slanting/elevational and all equal", consistent with De Villiers' (1998) findings concerning students' partiotional definition at van Hiele level 1. Many students defined the shapes so as to prohibit class inclusions, while some others gave a litany of their properties in defining them. The concept of isosceles triangle appeared to be understood by many students. A number of misconceptions were also noticed in some students' definitions. Bulelwa (pseudonym), a grade 12 learner, for example, would tell someone to look for a shape that has "four unequal sides" in order to pick out all the rhombuses from a set of quadrilaterals. The evidence that Bulelwa's case is an issue of misconception came from our analysis of his responses to the identifying and naming shapes task. Bulelwa had named each of the isosceles trapeziums as rhombus and gave such reason as "it has four unequal sides". Table 7 summarizes these results.

Table 7: Students' van Hiele levels on defining shapes task.

Van Hiele Levels	Number of students	
	Nigeria (n=18)	South Africa (n=18)
0	12	10
1	5	5
2	1	3
3	0	0

Class inclusion task. The class exclusion that dominates the learners' reasoning about geometric concepts in the previous tasks became more evident in this task. For instance, only 1 South African learner perceived a square as belonging to the class of rectangles and that of rhombuses.

No learner from the Nigerian subsample perceived a square as belonging to either class of shape. Students' denial of a shape with the more specific properties as not belonging to the class of the one with the more general properties is usually by a listing of a few properties of the special case which the more inclusive shape does not have. For example, some of the students reasoned that a square is not a rectangle "because all the sides [of a square] are equal"; just as some others said a rhombus is not a parallelogram because "all four sides are equal". These results link up with those of De Villiers (1994) when he stated that students' classification of quadrilaterals is generally spontaneously partitional. The results of this task are presented in table 8.

Table 8: Students' responses to the class inclusion task.

Question posed	No. with correct response	
	Nigeria (n=18)	South Africa (n=18)
Is shape No. 23 a rectangle?	0	1
Is shape No. 17 a parallelogram?	0	2
Is shape No. 6 a rhombus?	0	1
Is shape No. 1 a parallelogram?	3	1
Is shape No. 30 a scalene triangle?	8	7

NB: A student was considered to have answered correctly if he/she responded in the affirmative and gave a correct reason to justify his/her answer.

In terms of the van Hiele levels, 11 students each from the Nigerian and South African subsamples were at level 0. These were students who could not even state a partitional definition even at the visual level (i.e. the "look like" type of definition). There were 7 students each from the Nigerian and South African participants at level 1 on this task.

Students' Misconceptions and Use of Imprecise Terminology

Clements and Battista (1992) make a list and bemoan students' misconceptions in geometry. Compounding this problem is students' use of imprecise geometrical terminology. Hoffer (1981) suggests that the right terminology should be introduced to the students early in their geometry courses.

Although some of students' use of imprecise terminology may be due to language problems (as may be revealed in various spelling errors), the majority of them are due to conceptual misunderstandings. Figure 3 is a list of some of the imprecise terminology (in the form of spelling problems) that many students used in describing various geometrical shapes in this study. Figure 4 represents some of the students' misconceptions about some geometric concepts in this study.

Equalateral triangle, equadilateral triangle, equadrilateral triangle, regular triangle –all for equilateral triangle.

Isosceless, Iscocelene, Isoscyle, Isocelist, Isoscilice – all for isosceles triangle.
Scarlene, scaline, scalelan, Irregular – all for scalene triangle
Robus, rombus – all for rhombus.
Squer, squre – for square
Parallelogramme – for parallelogram
Traipyzium – for trapezium

Figure 3: Some of students' imprecise terminology.

The use of imprecise terminology such as these by students should not be taken for granted on the basis of "these are mere spelling mistakes", because our analysis of students' written work revealed that students who made use of the correct terminology have better conceptual understanding of the geometric concepts tested in this study.

- Regular rectangle – It is a rectangle having four equal side.
- Small rectangle – It is a mall rectangle that has no equal side.
- A square is not a rectangle because it have the same size.
- A square is not a rhombus because it has all it shapes equal.
- As soon as the four sides [of a geometric figure] are all equal, then the angles must be 90^o each.
- If the opposite sides of a shape are equal, the shape is a rectangle.
- A rectangle is not a parallelogram because it is not a quadrilateral.
- All quadrilaterals have four shapes [not sides].
- A rectangle is not a parallelogram, because it is a straight shape.
- If a shape has 4 equal sides, then it is a square.
- A square is like a carpet, a rhombus is like a kite.
- A rhombus is not a parallelogram because a parallelogram has 2 pairs of opposite sides
  equal, not 4 sides equal.
- A rhombus is not a parallelogram why is because a rhombus look like a square and it is bended.

Figure 4: Some of students' misconceptions in geometry.

Teachers should pay attention to students' imprecise use of and misconceptions about geometric concepts and correct them early in their geometry course.

Conclusion

This study investigated Nigerian and South African Grades 10, 11 and 12 high school students' conceptual understanding in geometry within the notion of respecting and recognizing the uniqueness of each individual. The van Hiele theory of the levels of geometric thinking formed the theoretical framework, while Burger and Shaughnessy's (1986) characterization of the van Hiele levels together with De Villiers’ (1994; 1998) notions of hierarchical and partitional definition (and classification) was used as a basis for analysis.

The results of this study showed that many Nigerian and South African high school learners involved in this study hold a number of misconceptions about geometric concepts of triangles and quadrilaterals. Most of these students used several imprecise terminology in describing many geometric shapes. A number of these learners were at van Hiele level 0, as many were only able to distinguish between triangles and quadrilaterals, but lacked the requisite vocabulary to distinguish among shapes in the same class. These results were found to be consistent with those of Clements and Battista (1992).

The task of naming shapes was easier for many students than giving a description of their properties. This is consistent with the study of Renne (2004). The concept of triangle (particularly isosceles triangle) was better understood by many students than the concept of quadrilaterals (particularly rhombus). The few students who attempted to describe shapes by their properties did so by focusing only on a single attribute-characteristic of a level 2 reasoning. Knowledge of class inclusions of shapes was simply absent among these students. Earlier studies by Feza and Webb (2005) and Atebe and Schäfer (2008) found similar results.

Recommendations

We recommend that at the senior secondary phase, irrespective of grade levels, the properties of shapes and the relationships between the properties and among different shapes should be explored by the students whenever such shapes come up for discussion. Teachers should pay attention to students’ use of imprecise terminology and correct them early in their geometry courses. Tasks, such as the ones used in this study should be developed (or adapted) for use in geometry classroom instruction as they hold promises in enabling teachers to understand many of students’ misconceptions in geometry.

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