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Gray Codes in Music Theory

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GRAY CODES IN MUSIC THEORY

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An Abstract of the Thesis Presented
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In the branch of Western music theory called serialism, it is desirable to construct chord progressions that use each chord in a chosen set exactly once. We view this problem through the scope of the mathematical theory of *Gray codes*, the notion of ordering a finite set X so that adjacent elements are related by an element of some specified set R of involutions in the permutation group of X. Using some basic results from the theory of permutation groups we translate the problem of finding Gray codes into the problem of finding Hamiltonian paths and cycles in a Schreier coset graph of the permutation group generated by the involutions R. Having made this translation we can use known results about Hamiltonian paths in Schreier (and Cayley) graphs of groups to generate serialism-like chord progressions. We illustrate the method by examining two theorems from the literature on Hamiltonian paths, due to Conway, Sloane, and Wilks (*Graphs Combin.* 5 (1989), no. 4, 315–325), and to Eades and Hickey (*J. Assoc. Comput. Mach.* 31 (1984), no. 1, 19–29). We give proofs of these theorems that complement the published proofs by filling in some details and clarifying some potentially confusing points, and we then use the algorithms extracted from these proofs to produce chord progressions.

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CHAPTER 1

INTRODUCTION

This thesis will relate the search for Hamiltonian paths and cycles in graphs to the problem of producing chord progressions in Western music theory that cycle through a set of chords in an organized way. This problem is important in the branch of music theory called *serialism* - developed by the so-called "Second Viennese School", consisting of German composer Schönberg and his pupils - which focuses on composing music that uses each of the twelve chromatic tones before repeating any. (More information on serialism can be found in Schönberg's book [1].) We extend the ideas of serialism to chords by finding a chord progression that uses every chord in a chosen set exactly once. First, we will represent connections between chords and graphs, and then establish existence theorems and develop rigorous algorithms that produce Hamiltonian cycles in such graphs.

Serialism is an application of the theory of *Gray codes*, the notion of cycling through a finite set X with respect to a collection of involutions R of the elements of the set. An overview of Gray codes is given in [11]. We will draw a connection between the search for Gray codes and the existence of Hamiltonian paths in Cayley and Schreier graphs. Then we will prove two main theorems giving algorithms that produce Gray codes. The first main theorem comes from *Gray Codes for Reflection Groups* by Conway, Sloane, and Wilks [3], and the second main theorem comes from *Some Hamiltonian Paths and a Minimal Change Algorithm* by Eades and Hickey [6]. We will interpret both of these examples of Gray through the scope of serialism, providing chord progressions that use each chord in a chosen set exactly once.

The search for Gray codes of a set X with respect to a set R of involutions of X goes hand in hand with graph theory, as the vertices and edges of a graph may represent, respectively, the elements of X and their connections in R. In Conway, Sloane, and Wilks [3], the authors construct an algorithm that produces a Gray code for a finite *Coxeter* *group* G with respect to a system of *fundamental reflections* R (see [9] for this terminology). We will not need to introduce the specifics of Coxeter groups here, as we will observe in Theorem 7.1.1 that the main result of [3] remains valid, with the same proof, under a more general and simpler assumption on R.

In Eades and Hickey [6], the authors determine necessary and sufficient conditions for the existence of a Gray code for the set of k-element subsets of the set $\{1, 2, ..., n\}$ with respect to the set of involutions R that change one element of the subset by a difference of 1. An account of Eades and Hickey's work is given in Chapter 9. This Gray code is interpreted musically by assigning n notes to the numbers 1 to n, so that k-element subsets of $\{1, 2, ..., n\}$ represent k-note chords, and successive chords move by changing one note at a time.

The application of algebra to Western music theory is a relatively new area of mathematics, developed within the last century. Expanding the work of music theorist Hugo Riemann (not to be confused with the mathematician), the study of Neo-Riemannian music theory identifies relationships between chords that do not depend on any tonal center, demonstrating that operations on sets of chords generate a permutation group. In *A Graph-Theoretic Approach to Efficient Voice-Leading* [12], the authors Wixey and Sturman use the vertices and edges of graphs to represent chords and their connections, in order to show the "harmonic proximity" of chords. In *Musical Actions of Dihedral Groups* [4], the authors Crans, Fiore, and Satyendra completely characterize the dihedral structure of the TI and PLR groups (permutation groups of the set of 24 major and minor triads). The authors also provide a Gray code for these musical groups with respect to their generating sets. In Section 8.2 we will apply the theorem of Conway, Sloane, and Wilks, in the more general form proved in Theorem 7.1.1, to extend the Gray-code construction in [3] to larger permutation groups of larger sets of chords, including four-note chords called *seventh chords*.

Cannas and Andreatta [2] have generalized the PLR group to the set of major seventh, dominant seventh, minor seventh, fully diminished seventh and half diminished seventh chords, creating a much larger graph of chords and their connections than that of Crans, Fiore, and Satyendra. Cannas and Andreatta provide a Gray code for this set of seventh chords with respect to a certain set of permutations. However, the fact that there are only three distinct fully diminished seventh chords (as opposed to one for each of the 12 chromatic tones) complicates this set-up. A mathematical explanation for this complication is that the graph of seventh chords considered in [2] is not *regular* - the vertices corresponding to the fully diminished chords have higher degree than the other vertices - and so this graph cannot be a Cayley graph of any group. The construction that we give in Section 8.1 is different to that of Cannas and Andreatta as we exclude the fully diminished chords, and the graph corresponding to our set of chords and permutations is, by design, the Cayley graph of a group generated by involutions.

We will establish a general mathematical context in which to understand these works in the application of graph-theory to serialism by establishing a bijection between the set of Gray codes for a set X with respect to a set involutions $R \subseteq \mathrm{Perm}(X)$ on one hand, and on the other, the set of Hamiltonian paths in the Schreier graph of the permutation group generated by R with respect to the stabilizer group of an element of X and the set R. Our Gray code for seventh chords is an example of the applicability of this correspondence to music theory: we can apply the algorithm from Conway, Sloane, and Wilks [3] to a group of permutations of a set of seventh chords that is generated by involutions, and thus obtain a Gray code.

A potential application of our identification of Gray codes and Hamiltonian paths in Schreier graphs is discussed in Section 10.1. The Gray code for the k-element subsets of $\{1, ..., n\}$ studied in [6] corresponds to a Hamiltonian path in the Schreier graph for the subgroup $S_k \times S_{n-k}$ of the symmetric group S_n. This is an example of a *parabolic* subgroup in the Coxeter group S_n, and the results of [6] naturally raise the question of whether similar results hold for arbitrary parabolic subgroups of finite Coxeter groups. A full examination of this problem is beyond the scope of this thesis, but in Section 10.1, we formulate an explicit question on the existence of a Gray code for parabolic subgroups of the hyperoctahedral groups. Musically, such a Gray code is an extension of the k-subsets of $\{1, 2, ..., n\}$ Gray code, where k-note chords have each note of the chord in one of two instruments, and chords change by moving one note, switching the voice of the note designated by 1, or swapping the voices of two notes in a chord. We leave the full examination of this problem for future research.

Another potential extension of the work presented in this thesis is the study of subgroups of the permutation group of a set of chords that are not necessarily generated by involutions. For instance, in [8], Julian Hook studies *uniform triadic transformations*, a group of permutations of the set of 24 major and minor triads. The elements of this group that are of order 24 are particularly interesting, because they have the potential to cycle through the entire set of major and minor triads.

CHAPTER 2
GRAY CODES

Given a set of chords and a set of permutations of those chords, we wish to find a chord progression that uses each chord in the set where the movements between successive chords are the result of one of the specified permutations. Generalizing this problem, let X be a finite set and R a set of *involutions* in the permutation group of X - that is, R is a subset of Perm(X) such that each element of R has order 2. Then, we can search for an ordering of the elements of the set X with respect to the set of involutions R in the following way.

Definition 2.0.1. Let X be a finite set of cardinality a, and let R be a set of involutions in Perm(X). Then, a *Gray code* for (X, R) is an ordering of the elements of X,

$$C = (x_0, x_1, ..., x_{a-1}), \quad x_i \neq x_j \text{ for } i \neq j,$$

such that for each $0 \leq i < a-1$, there exists $r_i \in R$ with $r_i(x_i) = x_{i+1}$. The ordering C is a *cyclic Gray code* if there exists $r_{a-1} \in R$ such that $r_{a-1}(x_{a-1}) = x_0$.

Named after physicist Frank Gray, Gray codes have been used in many areas of computer science since the 1980's [11]. It should be noted that there is a more general definition of a Gray code, where the set R of permutations of the set X need not be involutions. We restrict to involutions because many of the groups found in music theory are those generated by order 2 chord operations.

Example 2.0.2. Let X be the set of n-digit binary numerals and let R be the set of involutions in Perm(X) that change a single digit. Then, a Gray code for (X, R) is called a *reflected binary Gray code*, and it is an ordering of the binary numerals such that successive numbers differ in a single digit. This is the situation that Gray originally studied.

Reflected binary Gray codes can be interpreted musically by assigning n distinct notes to each of the n digits. Then, an n-digit binary number represents the chord that contains the notes with a 1 in their respective digit, and a Gray code is a chord progression that uses all possible chord combinations of the n notes, such that movements between successive chords are the result of adding or removing a single note.

Example 2.0.3. Let X be the set of subsets of $\{1, 2, ..., n\}$ that have some fixed number $k \leq n$ of elements, and let $R \subseteq \mathrm{Perm}(X)$ be the set of involutions $\{(12), (23), ..., (n-1\ \ n)\}$, the adjacent transpositions. Then, a Gray code for (X, R) is called a *k-subsets of n Gray code*, an ordering of the k-element subsets of n such that successive subsets differ in only one element by a difference of 1.

This example can be interpreted in music theory by assigning notes to each element of $\{1, 2, ..., n\}$ so that k-element subsets of n are k-note chords, and a Gray code is a serialism-like chord progression that moves only one note at a time.

We will define a correspondence between the search for Gray codes and the search for Hamiltonian paths in Schreier graphs.

CHAPTER 3
HAMILTONIAN PATHS AND CYCLES

3.1 Graphs and Paths

In view of defining Hamiltonian paths, we begin by defining graphs and paths, and developing some of their properties. See [7], for instance, for more background on graph theory. A *finite graph* Γ is a nonempty finite set of *vertices*, $V(\Gamma)$, and a finite set of *edges*, $E(\Gamma)$, where each edge is a 2-element subset of $V(\Gamma)$. (Graphs of this kind are called *simple, undirected* graphs, and this definition prohibits multiple edges between two vertices and loops on single vertices.) Additionally, a graph Δ is a *subgraph* of a graph Γ if $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. A *spanning subgraph* of Γ is a subgraph Δ with $V(\Delta) = V(\Gamma)$. If U is a subset of $V(\Gamma)$, then the *subgraph of* Γ *induced by* U is the subgraph Δ with $V(\Delta) = U$ and $E(\Delta) = \big\{\{u, v,\} \in E(\Gamma) : u, v \in U\big\}$. Two vertices v and w are said to be *adjacent* in a graph Γ if $\{v, w\} \in E(\Gamma)$. Finally, given a finite graph Γ and a vertex $v \in V(\Gamma)$, the *degree* of v is the number of edges in $E(\Gamma)$ that contain v.

It is useful to represent graphs visually in 2-dimensional Euclidean space where vertices are represented by distinct points and edges are represented by curves that connect vertices. In this setting, vertices and edges can have labels to designate them. The vertices may be positioned anywhere in the space as long as the edge information remains the same.

The following definition demonstrates how to move around a graph.

Definition 3.1.1. A *path* in a graph is a sequence of vertices $(v_0, v_1, ..., v_n)$ such that $n \geq 1$ and $\{v_i, v_{i+1}\}$ is an edge for all $0 \leq i < n$. A *cycle* is a path $(v_0, v_1, ..., v_n)$ with $n \geq 3$ such that $\{v_0, v_n\}$ is an edge and $v_i \neq v_j$ for $i \neq j$.

A path in a graph is like moving around a map, where each intersection connects to those adjacent to it. In some cases, it is useful to interpret a path $(v_0, v_1, ..., v_n)$ as a sequence of edges $(e_0, e_1, ..., e_{n-1})$ where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i < n$. Sometimes a sequence of roads on a map is easier to follow than a sequence of intersections.

A graph Γ is said to be *connected* if for all vertices $v, w \in V(\Gamma)$, there is a path $(v_0, ..., v_n)$ such that $v = v_0$ and $w = v_n$. In other words, there is a path connecting any two vertices of Γ. The *length* of a path $(v_0, ..., v_n)$ is the integer $n - 1$, or equivalently the number of edges in the path. In addition, if there is a path between two vertices v and w in a graph, then the *distance* between v and w is the minimum length of a path $(v_0, ..., v_n)$ with $v = v_0$ and $w = v_n$. A graph is called *acyclic* if it contains no cycles.

For our purposes, graphs will be constructed with vertices and edges that represent chords and their connections, and a path in such a graph represents a chord progression (a sequence of chords).

The following lemmas will be useful in the proof of Theorem 7.1.1:

Lemma 3.1.2. Every acyclic finite graph contains a vertex of degree less than 2.

Proof. Assume to the contrary that Γ is an acyclic finite graph such that each vertex of Γ has degree greater than or equal to 2. Fix a vertex v_0 and an edge $\{v_0, v_1\}$; since the degree of v_1 is at least 2, there exists a vertex $v_2 \neq v_0$ that is adjacent to v_1. If the path (v_0, v_1, v_2) is a cycle, we have a contradiction; so v_0 is not adjacent to v_2. Since the degree of v_2 is also at least 2, there exists a vertex $v_3 \neq v_0, v_1$ that is adjacent to v_2. If the path (v_0, v_1, v_2, v_3) is a cycle, we have a contradiction. Otherwise, recursively, if $(v_0, ..., v_i)$ is a path in Γ with distinct vertices and edges $e_0, ..., e_{i-1}$, then there is an edge $e_i \neq e_{i-1}$ that adjoins v_i to some v_{i+1}, because $\deg(v_i) \geq 2$. If v_{i+1} is distinct from $v_0, ..., v_i$, continue in this way, but since $V(\Gamma)$ is finite, at some point $v_{i+1} = v_j$ for some $0 \leq j \leq i$. In which case, the path $(v_j, ..., v_i)$ is a cycle in Γ, a contradiction to Γ being acyclic. □

Lemma 3.1.3. The graph obtained by removing a vertex from an acyclic finite graph is also acyclic.

Proof. Let Δ be the graph obtained by removing a vertex from an acyclic finite graph Γ; then, each path in Δ is also a path in Γ, where no path is a cycle, so Δ cannot contain any cycles. □

The following definition provides a condition to consider two graphs equivalent.

Definition 3.1.4. Two graphs Γ and Δ are *isomorphic* if there exists a bijection $\varphi : V(\Gamma) \longrightarrow V(\Delta)$ such that $\{v, w\}$ is an edge in Γ if and only if $\{\varphi(v), \varphi(w)\}$ is an edge in Δ; such a map is called a *graph isomorphism*.

In other words, isomorphic graphs have the same structure, but with a different labelings of their vertices and edges.

Given a graph isomorphism $\varphi : V(\Gamma) \longrightarrow V(\Delta)$, if $(v_0, ..., v_n)$ is a path in Γ, then $\big(\varphi(v_0), ..., \varphi(v_n)\big)$ is a path in Δ.

3.2 Hamiltonian Paths and Cycles

The goal of this thesis is to relate the search for Hamiltonian paths in graphs to the search for Gray codes of a set X with respect to involutions $R \subseteq \mathrm{Perm}(X)$, so that we can find systematic chord progressions that use each chord in a chosen set exactly once.

Definition 3.2.1. A *Hamiltonian path* in a graph is a path that contains all the vertices exactly once, and a *Hamiltonian cycle* is a Hamiltonian path that is a cycle.

Given a graph Γ, a path $(v_0, ..., v_n)$ is thus a Hamiltonian path if $v_i \neq v_j$ for all $0 \leq i < j \leq n$, and for each $v \in V(\Gamma)$, $v = v_i$ for some i. If two graphs are isomorphic via a graph isomorphism $\varphi : V(\Gamma) \longrightarrow V(\Delta)$, then a Hamiltonian path in Γ will map to a Hamiltonian path in Δ. This idea will be used in the proof of our first main theorem, Theorem 7.1.1.

Lemma 3.2.2. If $(v_0, v_1, ..., v_n)$ is a Hamiltonian cycle in a graph Γ, then

$$R := (v_n, v_{n-1}, ..., v_0) \text{ and } T := (v_i, v_{i+1}, ..., v_{n-1}, v_n, v_0, v_1 ..., v_{i-1})$$

are Hamiltonian cycles in Γ for $2 \leq i \leq n$.

Proof. Since $(v_0, v_1, ..., v_n)$ is a Hamiltonian cycle in Γ, $\{v_n, v_0\}$ and $\{v_i, v_{i+1}\}$ are edges of Γ for all $1 \leq i < n$, meaning both R and T are cycles in Γ. Also, each vertex of $(v_0, v_1, ..., v_n)$ appears exactly once in both R and T; therefore, R and T are Hamiltonian cycles in Γ. $\square$

3.3 Bipartite Graphs

The problem of determining whether a given graph admits a Hamiltonian path or cycle can be quite difficult. Nonetheless, there are some useful necessary conditions that can be easily checked. In the proof of our second main theorem, Theorem 9.1.1, we search for a Hamiltonian path in a graph that has the following property.

Definition 3.3.1. A graph Γ is *bipartite* if there are subsets $A, B \subseteq V(\Gamma)$ such that $A \cap B = \emptyset$, $A \cup B = V(\Gamma)$, and each edge of Γ is of the form $\{a, b\}$ for $a \in A$ and $b \in B$. The sets A and B are called the *parts* of the bipartite graph Γ.

If a graph Γ is bipartite with parts A and B, then there are no edges in Γ that connect elements of the same part. Therefore, any path in Γ must alternate vertices in A and B. This observation leads to the following lemma.

Lemma 3.3.2. Let Γ be a bipartite graph with parts A and B. If Γ has a Hamiltonian path, then $\big||A| - |B|\big| \leq 1$.

Proof. Assume that $(v_0, v_1, ..., v_{n-1})$ is a Hamiltonian path in Γ. Without loss of generality, we can assume $v_0 \in A$. Then, since Γ has no edges that connect vertices of the same part, $v_i \in A$ for all the even indices i and $v_j \in B$ for all the odd indices j. Therefore, we see that $\big||A| - |B|\big| =$

$$\Big|\big|\{i \in \{0, 1, ..., n-1\} : i \text{ is even}\}\big| - \big|\{j \in \{0, 1, ..., n-1\} : j \text{ is odd}\}\big|\Big| \leq 1. \qquad \square$$

3.4 Combs and Graph Products

Another kind of graph that will be considered in the proof of Theorem 9.1.1 is the product of a comb with the graph K_2, the graph with two vertices and an edge connecting them.

Definition 3.4.1. A *comb* with *main path* M and *boundary points* u and v is a connected, acyclic, finite graph C such that each vertex is of maximum degree 3, and all the vertices of

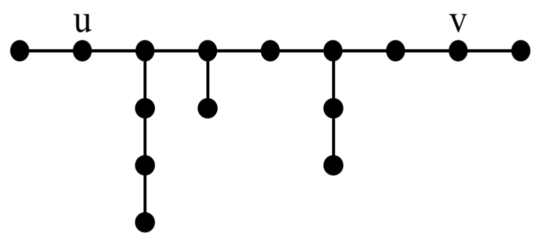

Figure 3.1: Comb with boundary points u and v.

degree 3 lie on the main path M, strictly between the boundary points u and v. For each vertex x on the main path between the boundary points u and v, the *tooth* at x is the longest path in C that intersects the main path only at x (and if the degree of x is 2, the tooth at x is considered trivial). If Γ is a finite graph, then a *spanning comb for* Γ is a subgraph C of Γ that is a comb and $V(C) = V(\Gamma)$.

An example of a comb with boundary points u and v is given in Figure 3.1. The main path of this comb is the horizontal string of vertices, and the teeth are the vertical paths.

The following defines the product of a finite graph with the graph K_2 (the graph with $V(K_2) = \{v_0, v_1\}$ and $E(K_2) = \big\{\{v_0, v_1\}\big\}$). It should be noted that there are more general notions of the product of two finite graphs, but it is not necessary for the proof of Theorem 9.1.1.

Definition 3.4.2. If Γ is a finite graph, the *product of* Γ *and* K_2 is the graph ΓK_2 with $V(\Gamma K_2) = \{(i, v) : i \in \{1, 2\}, v \in V(\Gamma)\}$ such that two vertices (i, u) and (j, v) are adjacent if either $i = j$ and $\{u, v\} \in E(\Gamma)$ or $i \neq j$ and $u = v$.

The introduction of combs and graph products of K_2 is advantageous because the product of any comb with K_2 contains a Hamiltonian path, as shown in the following lemma.

Lemma 3.4.3. If C is a comb with boundary points u and v and the distance between u and v is even, then CK_2 has a Hamiltonian path with endpoints $(2, u)$ and $(1, v)$.

Proof. Let $(u_0, u_1, ..., u_a, u, t_{0,0}, t_{1,0}, ..., t_{b,0}, v, v_0, v_1, ..., v_c)$ be the main path of the comb C where u and v form a pair of boundary points. Also, for each $0 \leq i \leq b$, let

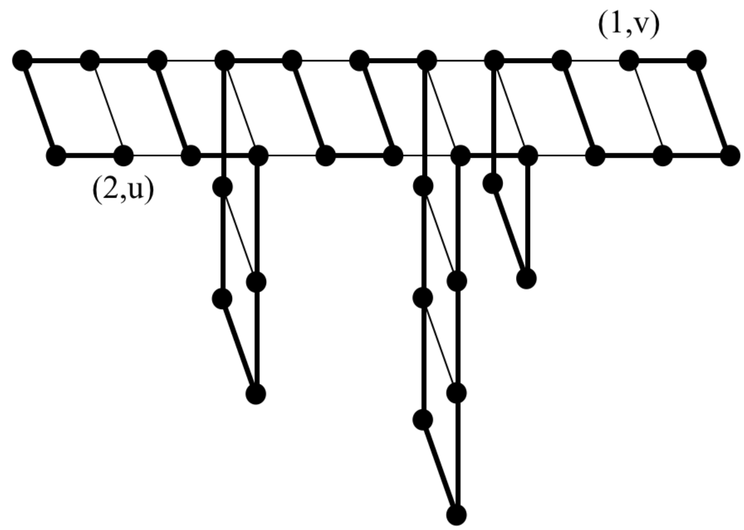

Figure 3.2: Hamiltonian path in the product of a comb and K_2.

$T_i = (t_{i,0}, t_{i,1}, ..., t_{i,k_i})$ be the tooth at $t_{i,0}$. Note that since the distance between u and v in C is even, b is even. Then, the following sequence of vertices is a Hamiltonian path in CK_2:

$$\begin{array}{cccccc}
(2,u), & (2,u_a), & (2,u_{a-1}), & ..., & (2,u_0), & \\
(1,u_0), & (1,u_1), & (1,u_2), & ..., & (1,u), & \\
(1,t_{0,0}), & (1,t_{0,1}), & (1,t_{0,2}), & ..., & (1,t_{0,k_0}), & \\
(2,t_{0,k_0}), & (2,t_{0,k_0-1}), & (2,t_{0,k_0-2}), & ..., & (2,t_{0,0}), & \\
(2,t_{1,0}), & (2,t_{1,1}), & (2,t_{1,2}), & ..., & (2,t_{1,k_0}), & \\
(1,t_{1,k_0}), & (1,t_{1,k_0-1}), & (1,t_{1,k_0-2}), & ..., & (1,t_{1,0}), & \\
..., & & & & & \\
(1,t_{b,0}), & (1,t_{b,1}), & (1,t_{b,2}), & ..., & (1,t_{b,k_b}), & \\
(2,t_{b,k_b}), & (2,t_{b,k_b-1}), & (2,t_{b,k_b-2}), & ..., & (2,t_{b,0}), & \\
 & (2,v), & (2,v_0), & (2,v_1), & ..., & (2,v_c) \\
 & (1,v_c), & (1,v_{c-1}), & (1,v_{c-2}), & ..., & (1,v)
\end{array}$$

□

An example of the Hamiltonian path defined in the proof of Lemma 3.4.3 for the product of a comb and the graph K_2 is given in Figure 3.2. This lemma is useful because if a graph Γ has a spanning subgraph that is isomorphic to the product of a comb and K_2, then Γ contains a Hamiltonian path.

CHAPTER 4
FINITE GROUPS GENERATED BY INVOLUTIONS

4.1 Finite Groups Generated by Involutions

We wish to understand the structure of groups that appear in the context of music theory, and in general, the groups associated with Gray codes. Many useful groups in Western music are generated by involutions. An *involution* is a group element of order 2, so then a *finite group generated by involutions* is a pair (G, R) where G is a finite group and $R \subseteq G$ is a set of involutions that generates G. These types of groups appear in the study of Gray codes as subgroups of the permutation group of a set X that are generated by order 2 permutations.

Example 4.1.1. A familiar example includes the permutation group S_n of the set of n integers $\{1, 2, ..., n\}$. It can be shown that for any n, the group S_n is generated by the adjacent transpositions $(12), (23), ..., (n-1\ \ n)$. These transpositions are each of order 2, making the pair $(S_n, \{(12), ..., (n-1\ \ n)\})$ a finite group generated by involutions. It should be noted that the set of adjacent transpositions is not the only set of involutions that generates S_n; another such generating set of involutions includes the transpositions $(12), (13), ..., (1\, n)$.

Example 4.1.2. An important example of a finite group generated by involutions that will be used later in this thesis is the group C_2^n, the product of n copies of the additive group of two elements $C_2 = \{0, 1\}$. As a set, C_2^n comprises the n-bit binary numbers, and the group can be generated by the set of involutions

$$R_2^n := \{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)\}.$$

Thus, the pair (C_2^n, R_2^n) is a finite group generated by involutions.

4.2 Finite Groups Generated by Two Involutions

This section considers the case when a finite group is generated by only two involutions. If G is a finite group generated by two involutions r and s, we want to show that G is determined up to isomorphism by the order of the element rs.

Definition 4.2.1. For $n \geq 3$, the *dihedral group of order 2n* is the automorphism group D_{2n} of the graph Γ_n with

$$V(\Gamma_n) = \{1, 2, ..., n\} \text{ and } E(\Gamma_n) = \big\{\{1,2\}, \{2,3\}, ..., \{n-1, n\}, \{n, 1\}\big\}.$$

The graph Γ_n is the graph with n vertices and edges that form a loop, demonstrating that the dihedral group of order $2n$ is the group of symmetries of a regular n-gon.

Lemma 4.2.2. For $n \geq 3$, the group D_{2n} has order $2n$ and is generated by two involutions.

Proof. Let $g \in D_{2n}$ be an automorphism of the graph Γ_n. Then, there are n options for $g(1)$, and once $g(1)$ is fixed, $g(2)$ must be adjacent to $g(1)$, leaving two options for $g(2)$. If both $g(1)$ and $g(2)$ are determined, since g is a graph isomorphism, there is only one option for each $g(3), g(4), ..., g(n)$. Therefore, g is completely determined by $g(1)$ and $g(2)$, with $2n$ possibilities for $g(1)$ and $g(2)$, meaning $|D_{2n}| = 2n$.

Now, we want to show that D_{2n} is generated by two involutions. Let $r \in D_{2n}$ have $r(1) = 1$ and $r(2) = n$, and let $t \in D_{2n}$ have $t(1) = 2$ and $t(2) = 3$. Then, r represents an order 2 reflection of Γ_n about the vertex 1 and t represents a single rotation of Γ_n. Thus, the subgroup $H = \langle t \rangle \subseteq D_{2n}$ is the group of n rotations of Γ_n. Also, since $r(1) = 1$ and the only rotation in H that fixes 1 is the identity, we see that r is not in H, so the cosets rH and H are distinct. Therefore, $|H \cup rH| = 2n = |D_{2n}|$, meaning

$$D_{2n} = H \cup rH = \{1, t, t^2, ..., t^{n-1}, r, rt, rt^2, ..., rt^{n-1}\}.$$

Next, notice that $rtr = t^{-1}$ as

$$rtr(1) = rt(1) = r(2) = t^{-1}(1) \text{ and } rtr(2) = rt(n) = r(1) = 1 = t^{-1}(2).$$

Hence, if we let $s = rt$, then $t = rs$ and $s^2 = (rtr)t = t^{-1}t = 1$, so D_{2n} is generated by the two involutions r and s. □

If $r, t \in D_{2n}$ are the automorphisms of Γ_n that represent the reflection about the vertex 1 and a single rotation, as in the proof of the lemma, then the relation $rtr = t^{-1}$ implies that $t^{-k}r = (rtr)^k r = rt^k$ for each $0 \leq k \leq n$. Therefore, we can completely determine the multiplication in D_{2n} for each $0 \leq k \leq n$ and $0 \leq l \leq n$ as follows:

$$t^k t^l = t^{k+l}$$

$$t^k(rt^l) = rt^{-k}t^l = rt^{l-k}$$

$$(rt^k)t^l = rt^{k+l}$$

$$(rt^k)(rt^l) = (t^{-k}r)(rt^l) = t^{l-k}$$

Theorem 4.2.3. For each $n \geq 1$, there is a unique group up to isomorphism of order $2n$ that is generated by two involutions.

Proof. If $n = 1$, then C_2 is the only group of order 2, and it is generated by two identical involutions. If $n = 2$, then any group G of order 4 generated by two involutions r and s is $\{1, r, s, rs\}$, so G is isomorphic to $C_2 \times C_2$.

Otherwise, when $n \geq 3$, we know that D_{2n} is a group of order $2n$ generated by two involutions, so we must show that it is unique. Let G be a group of order $2n$ that is generated by involutions r and s. Set $t = rs$, and let H be the cyclic subgroup of G generated by t. Note that $t^2 = (rs)^2 \neq 1$, because $rs \neq sr$ as the order of G is greater than 4. We claim that r is not in H. In fact, if $r = t^k$ for some $k \geq 0$, then $1 = t^{2k}$. Also, $s = r(rs) = t^{k+1}$, so $t^{2k} = 1 = t^{2k+2}$, but that implies $t^2 = 1$, a contradiction.

Therefore, rH and H are distinct left cosets of H in G. Next, we want to show that $H \cup rH$ is a subgroup of G. For each $k \geq 0$, we have $rt^k = r(rs)^k = (sr)^k r = t^{-k}r$, so multiplication in $H \cup rH$ is given by

$$t^k t^l = t^{k+l} \in H$$

$$t^k(rt^l) = rt^{-k}t^l = rt^{l-k} \in rH$$

$$(rt^k)t^l = rt^{k+l} \in rH$$

$$(rt^k)(rt^l) = (t^{-k}r)(rt^l) = t^{l-k} \in H,$$

where $(rt^k)^{-1} = rt^k \in rH$. Therefore, $H \cup rH$ is a subgroup of G containing r and s, and $G = \langle r, s\rangle$, so $H \cup rH = G$. Since G has the same multiplication table as D_{2n}, they are isomorphic. $\square$

This theorem tells us that if G is a finite group generated by two involutions r and s, then

$$G \cong \begin{cases} C_2 & \text{if } r = s \\ C_2 \times C_2 & \text{if order}(rs) = 2 \\ D_{2n} & \text{if order}(rs) = n > 2. \end{cases}$$

4.3 The PLR Group

In the context of music theory, we will be focusing on subgroups of the permutation group of a set of chords that are generated by self-invertible chord operations. These involutions are permutations of a set of chords (say Π, the set of 24 major and minor triads) that represent "smooth" ways to move between chords, creating chord progressions that have minimal change between chords, or "parsimonious voice-leading". One such group is called the *PLR Group*, a subgroup of the permutation group of Π, the set of major and minor chords. The set Π is the union of Π^+ and Π^-, respectively the sets of major and minor triads defined as sets of three element subsets of $\mathbb{Z}_{12}$ as follows.

$$\Pi^+ = \big\{\{x, x+4, x+7\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}_{12}\big\}$$

$$\Pi^- = \big\{\{x, x+3, x+7\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}_{12}\big\}$$

The PLR Group is generated by three order 2 permutations of the set Π defined below:

$$\begin{aligned}
P:&\quad \{x, x+4, x+7\} \leftrightarrow \{x, x+3, x+7\}\\
L:&\quad \{x, x+4, x+7\} \mapsto \{x-1, x+4, x+7\}\\
&\quad \{x, x+3, x+7\} \mapsto \{x, x+3, x+8\}\\
R:&\quad \{x, x+4, x+7\} \mapsto \{x, x+4, x+9\}\\
&\quad \{x, x+3, x+7\} \mapsto \{x-2, x+3, x+7\}
\end{aligned}$$

It can be checked that these operations on major and minor triads are well-defined order 2 permutations of Π, making the PLR Group a finite group generated by involutions. Each generator P, L, and R operators on a triad by moving only one note of the chord, exemplifying the use of parsimonious voice-leading. The PLR group will be discussed in detail in Section 7.3, and more information on the PLR Group is given in [4].

CHAPTER 5

COMMUTING, CAYLEY, AND SCHREIER GRAPHS

5.1 Commuting Graphs

Given a finite group G that is generated by a subset $R \subseteq G$, we define a graph that displays which elements of R commute with one another in G.

Definition 5.1.1. If G is a finite group with generating set R, then the *commuting graph of* G *with respect to* R is the graph $\mathrm{Com}(G, R)$ with $V(\mathrm{Com}(G, R)) = R$ and $E(\mathrm{Com}(G, R)) = \big\{\{r_i, r_j\} : r_i r_j \neq r_j r_i\big\}$.

Given a finite group G generated by involutions $r_1, ..., r_n$, we have that two generators r_i and r_j commute with one another if and only if $(r_i r_j)^2 = 1$. Therefore the commuting graph of G with respect to involutions R can be determined by identifying the order of the product of each pair of generators. That is, the involutions r_i and r_j are adjacent in $\mathrm{Com}(G, R)$ if the order of $r_i r_j$ is greater than 2.

Example 5.1.2. Let $G = S_4$, the permutation group of four element set $\{1, 2, 3, 4\}$, and let R be the following set of generating transpositions $\{(12), (23), (34)\}$. Then, the commuting graph of G with respect to R is the graph shown in Figure 5.1. This graph tells us that the elements (12) and (34) commute in G because they are not adjacent. To see the dependence of a commuting graph on the generating set, if we instead let $R = \{(12), (23), (34), (14)\}$, we obtain the commuting graph shown in Figure 5.2.

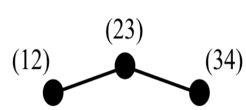

Figure 5.1: Commuting graph of S_4 with respect to $\{(12), (23), (34)\}$.

Example 5.1.3. Let $G = C_2^3$, the product of three copies of the additive group C_2, and let R be the set of involutions $R_2^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Then, the commuting graph of

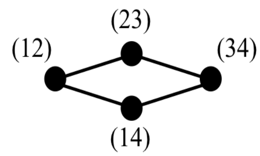

Figure 5.2: Commuting graph of S_4 with respect to $\{(12), (23), (34), (14)\}$.

G with respect to R is shown in Figure 5.3. This edgeless graph tells us that each of the generators commute with one another. In general, the commuting graph of C_2^n with respect to R_2^n (defined in Example 4.1.2) is edgeless, meaning that each of the generators commute with one another.

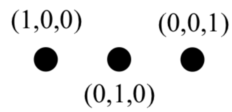

Figure 5.3: Commuting graph of C_2^3 with respect to R_2^3.

Example 5.1.4. Referring back to the PLR Group (Section 4.3), a subgroup of the permutation group of the set of major and minor triads, the commuting graph of the PLR Group with respect to the generating involutions P, L, and R is shown in Figure 5.4. This commuting graph demonstrates that none of the generators commute in the group.

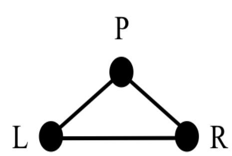

Figure 5.4: Commuting graph of the PLR group with respect to $\{P, L, R\}$.

5.2 Cayley Graphs

Given a finite group G that is generated by a set of involutions $R \subseteq G$, we wish to define a finite graph that represents how to move between the elements of G with respect to the generators in R. In this way, we obtain the following definition.

Definition 5.2.1. If (G, R) is a finite group generated by involutions, then the *Cayley graph of G with respect to R* is the graph $\mathrm{Cay}(G, R)$ with $V(\mathrm{Cay}(G, R)) = G$ and $E(\mathrm{Cay}(G, R)) = \big\{\{g, gr\} : g \in G, r \in R\big\}$

Given a Cayley graph, $\mathrm{Cay}(G, R)$, the edge $\{g, gr\}$ of the graph is labeled by the involution $r \in R$. A path in $\mathrm{Cay}(G, R)$ is a sequence of group elements $(g_0, g_1, ..., g_n)$ such that for each $0 \leq i < n$, there exists r_i with $g_i r_i = g_{i+1}$. In this way, a path in a Cayley graph is also a sequence of generators.

There is a more general definition of the Cayley graph of a group with respect to any generating set, and such a graph would be a directed graph. By restricting to finite groups generated by involutions, Cayley graphs are undirected graphs.

Example 5.2.2. Let $G = S_3$ and $R = \{(12), (23)\}$; then the the Cayley graph of G with respect to R is the graph shown in Figure 5.5. Notice that since R contains only the two involutions (12) and (23), at any vertex of $\mathrm{Cay}(G, R)$ there are only two edges, revealing a Hamiltonian cycle. More generally, if a finite group G is generated by two involutions r_1 and r_2, then $(1, r_1, r_1r_2, r_1r_2r_1, ..., (r_1r_2)^{m-1}r_1)$ is a Hamiltonian cycle for $\mathrm{Cay}(G, \{r_1, r_2\})$, where m is the order of r_1r_2.

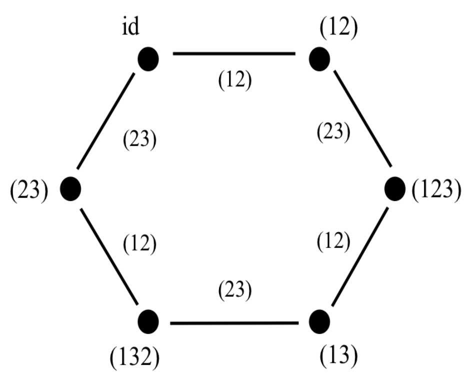

Figure 5.5: Cayley graph of S_3 with respect to $\{(12), (23)\}$.

Lemma 5.2.3. Let (G, R) be a finite group generated by involutions. Then, $\mathrm{Cay}(G, R)$ is connected.

Proof. To verify this claim, it is enough to show that there is a path connecting any vertex of $\text{Cay}(G, R)$ to the vertex that represents the identity of the group, because any two such paths may be concatenated to form a path between any two vertices. Thus, given any vertex labeled g of $\text{Com}(G, R)$, since R generates G, we have that $g = r_{x_1} r_{x_2} ... r_{x_k}$ for $1 \leq x_i \leq n$, meaning that the path using the edges $r_{x_1}, ..., r_{x_k}$ connects the identity vertex to the vertex labeled g. $\square$

Example 5.2.4. Let $G = C_2^n$ and $R = R_2^n$ (as defined in Example 4.1.2). Then, the Cayley graph of G with respect to R is the n-dimensional hypercube, because the vertices of $\text{Cay}(G, R)$ are the n-digit binary numbers and edges are formed between numbers that differ in only one digit. The Cayley graph of C_2^3 with respect to R_2^3 is given in Figure 5.6. Notice that a path in the n-dimensional hypercube graph is a sequence of binary numbers with minimal change between successive numbers, that is, only one digit changes at a time. Theorem 7.1.1 provides an algorithm that produces a binary reflected Gray code, an ordering of all the n-digit binary numbers such that adjacent numbers differ only in one digit [11]. The search for binary reflected Gray codes is equivalent to the search for Hamiltonian paths in the Cayley graph of C_2^n with respect to R_2^n.

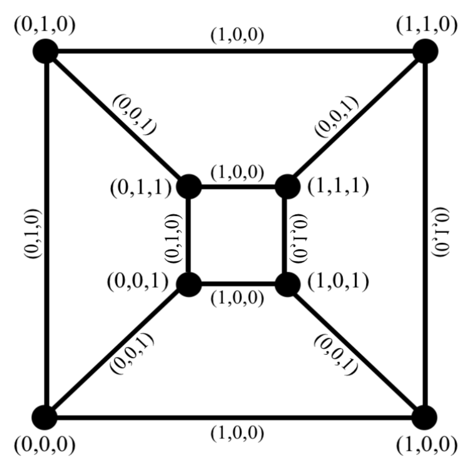

Figure 5.6: Cayley graph of C_2^3 with respect to R_2^3.

Example 5.2.5. Referring back to the PLR group (Section 4.3), the Cayley graph of the PLR group with respect to the set of generators $\{P, L, R\}$ is called the *Tonnetz* in Western music theory (meaning "tone network"). It has been shown in [4] that the PLR group acts simply transitively on the set of major and minor triads, allowing us to identify the elements of the PLR group with each of the 24 major and minor triads, producing the Tonnetz found in Figure 5.7. In this figure, uppercase letters denote major triads and lowercase letters denote minor triads. This graph is a handy compositional tool because it demonstrates how to move between chords with respect to the three simple operations.

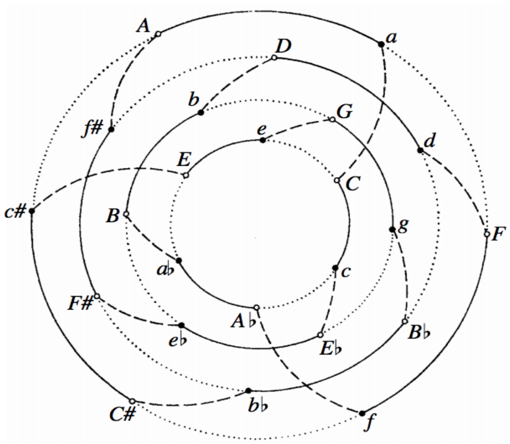

Figure 5.7: Cayley graph of the PLR group with respect to $\{P, L, R\}$ or Tonnetz [4]

5.3 Schreier Graphs

Given a finite group generated by involutions (G, R) and a subgroup $H \subseteq G$, we will define a graph that represents how to move between the right cosets of H in G with respect to the generators in R. In this way, we obtain the following definition.

Definition 5.3.1. If (G, R) is a finite group generated by involutions and H is a subgroup of G, then the *Schreier graph of G with respect to H and R* is the graph $\mathrm{Sch}(G, H, R)$ with $V(\mathrm{Sch}(G, H, R)) = H\backslash G$ and

$$E(\mathrm{Sch}(G, H, R)) = \big\{\{Hg, Hgr\} : Hg \in H\backslash G, r \in R\big\}.$$

Given a Schreier graph $\mathrm{Sch}(G, H, R)$, each edge $\{Hg, Hgr\}$ is labeled by the involution r. A path in $\mathrm{Sch}(G, H, R)$ is a sequence of right cosets $(Hg_0, Hg_1, ..., Hg_n)$ such that for each $0 \leq i < n$, there exists r_i with $Hg_ir_i = Hg_{i+1}$. Thus, a path in a Schreier graph can be expressed as a sequence of generators.

Similar to Cayley graphs, there is a more general definition of the Schreier graph of a group with respect to a chosen subgroup and generating set. The requirement that the generated set be a set of involutions allows us to work with undirected graphs.

If (G, R) is a finite group generated by involutions, then if $H \subseteq G$ is chosen to be the subgroup containing only the identity of G, then $\mathrm{Sch}(G, H, R)$ is canonically isomorphic to $\mathrm{Cay}(G, R)$. In this way, Cayley graphs are a special case of Schreier graphs.

Example 5.3.2. Let G be the symmetric group S_n and R be the set of adjacent transpositions $\{(12), (23), ..., (n-1\ n)\}$. For each $1 \leq k < n$, define

$$S_{k,n-k} := \Big\langle R - \{(k\ k+1)\} \Big\rangle,$$

the subgroup of S_n that permutes the first k numbers $\{1, 2, ..., k\}$ and the last $n-k$ numbers $\{n-k+1, n-k+2, ..., n\}$ separately. The Schreier graph $\mathrm{Sch}(G, H, R)$ will be the focus of Theorem 9.1.1.

When $n = 4$ and $k = 2$, the Schreier graph of S_4 with respect to the subgroup $S_{2,2}$ and $\{(12), (23), (34)\}$ is shown in Figure 5.8. Notice that this graph is bipartite with parts

$$A = \{H, H(234), H(132), H(13)(24)\},$$

$$B = \{H(23), H(1342)\}.$$

Since $|A| - |B| = 2 > 1$, from Lemma 3.3.2, we conclude that no Hamiltonian path exists in this graph.

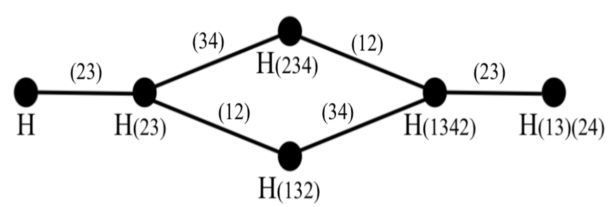

Figure 5.8: Schreier graph of S_4 with respect to the subgroup $S_{2,2}$ and $\{(12), (23), (34)\}$.

CHAPTER 6
CORRESPONDENCE BETWEEN GRAY CODES AND HAMILTONIAN PATHS

6.1 Simply Transitive Group Actions

In this chapter, we will relate the search for Gray codes and the search for Hamiltonian paths in Schreier and Cayley graphs. If X is a set and R is a set of involutions in $\mathrm{Perm}(X)$, then let $G \subseteq \mathrm{Perm}(X)$ be the subgroup generated by R. Then, define a group action of G on the set X in the usual way by the map $G \times X \longrightarrow X$ such that $gx \mapsto g(x)$.

Let G be a group that acts on a set X. Recall that the *orbit* of $x \in X$ is the set $Gx = \{gx \in X : g \in G\}$, and the *stabilizer* of $x \in X$ is the subgroup $G_x = \{g \in G : gx = x\}$ of G. The orbits in X are the equivalence classes of the equivalence relation on X such that $x \sim y$ if there exists $g \in G$ with $gx = y$. Using these notions, we have the following definition.

Definition 6.1.1. Let G be a group that acts on a set X. Then, the group action is *transitive* if $Gx = X$ for each $x \in X$, and the group action is *simply transitive* if it is transitive and $G_x = \{1\}$ for each $x \in X$.

Example 6.1.2. Let $X = \{0,1\}^n$, the n-digit binary numerals, and let $G = C_2^n$, the product of n copies of the additive group $\{0,1\}$. Then, G acts on X is the usual way, and the group action is simply transitive.

Example 6.1.3. Let $X = \Pi$, the set of major and minor triads, and let G be the PLR group. Then, it can be shown that G acts simply transitively on X [4]. As noted in Example 5.2.5, since this group action is simply transitive, we can associate each group element to one of the 24 major and minor triads. This property is generalized in the following section.

The following theorem is called the orbit-stabilizer theorem, drawing the connection between the orbit and stabilizer of each element of X.

Theorem 6.1.4. Let G be a group that acts on a set X. Then, for each $x \in X$, the map $f : G_x\backslash G \longrightarrow Gx$ defined $f(G_x g) = g^{-1}x$ is a bijection of sets.

Proof. The map f is well defined and injective because for each $g, g' \in G$,

$$G_x g = G_x g' \iff g' g^{-1} \in G_x \iff g' g^{-1} x = x \iff g^{-1} x = g'^{-1} x.$$

Also, f is surjective because for each $gx \in Gx$, we have $f(G_x g^{-1}) = gx$. Thus, f is a bijection. □

Corollary 6.1.5. Let G be a finite group that acts on a finite set X. Then, for each $x \in X$, $|G|/|G_x| = |Gx|$.

Proof. Since G and X are finite, from the orbit stabilizer theorem and Lagrange's theorem, we have $|G_x\backslash G| = |G|/|G_x| = |Gx|$. □

6.2 Correspondence Between Gray Codes and Hamiltonian Paths

The following theorem determines the connection between the search for Gray codes and the search for Hamiltonian paths in Schreier graphs.

Theorem 6.2.1. Let X be a finite set and $R \subseteq \mathrm{Perm}(X)$ a set of involutions. If the group G generated by R acts transitively on the set X, then for each $x \in X$, there is a bijection between the set of Gray codes for (X, R) and the set of Hamiltonian paths in $\mathrm{Sch}(G, G_x, R)$, where cyclic Gray codes correspond to Hamiltonian cycles.

Proof. To begin, define the graph Γ with $V(\Gamma) = X$ and $E(\Gamma) = \big\{\{x, rx\} : r \in R\big\}$. Then, Hamiltonian paths in Γ are exactly the Gray codes for (X, R), so we want to show that $\mathrm{Sch}(G, G_x, R)$ is isomorphic to Γ for each $x \in X$.

Let $x \in X$ and define a map $\varphi : V(\mathrm{Sch}(G, G_x, R)) \longrightarrow V(\Gamma)$ by $\varphi(G_x g) = g^{-1}x$. Then, the map φ is exactly the map f from the orbit-stabilizer theorem (Theorem 6.1.4), so it is a bijection.

To see that the map φ is a graph isomorphism, $\{G_x g, G_x gr\}$ is an edge of $\mathrm{Sch}(G, G_x, R)$ if and only if

$$\{\varphi(G_x g), \varphi(G_x gr)\} = \{g^{-1}x, (gr)^{-1}x\} = \{g^{-1}x, r(g^{-1}x)\} \in E(\Gamma).$$

Therefore, Γ is isomorphic to $\mathrm{Sch}(G, G_x, R)$, meaning there is a bijection between the Hamiltonian paths in $\mathrm{Sch}(G, G_x, R)$ and the Gray codes for (X, R), where cyclic Gray codes for (X, R) are in correspondence with Hamiltonian cycles in $\mathrm{Sch}(G, G_x, R)$. □

Corollary 6.2.2. Let X be a finite set and $R \subseteq \mathrm{Perm}(X)$ a set of involutions. If the group $G = \langle R \rangle$ acts simply transitively on the set X, then for each $x \in X$, there is a bijection between the set of Gray codes for (X, R) and the set of Hamiltonian paths in $\mathrm{Cay}(G, R)$, where cyclic Gray codes correspond to Hamiltonian cycles.

Proof. Since G acts simply transitively on X, the stabilizer of each $x \in X$ is the trivial subgroup of G, meaning $\mathrm{Sch}(G, G_x, R)$ is isomorphic to $\mathrm{Cay}(G, R)$. Thus, the proof of the corollary follows directly from Theorem 6.2.1. □

6.3 Examples of Gray Codes as Hamiltonian Paths

Having identified the correspondence between Gray codes and Hamiltonian paths in Schreier graphs, we can interpret the two examples of Gray codes given in Chapter 2 as Hamiltonian paths in respective Schreier graphs.

Example 6.3.1. Introduced in Example 2.0.2, a reflected binary Gray code is an ordering of the n-digit binary numerals in which successive numbers differ in only one digit. In this example, X is the set of n-digit binary numerals and $R \subseteq \mathrm{Perm}(X)$ is the set of involutions that change exactly one digit.

The subgroup G of $\mathrm{Perm}(X)$ generated by R is isomorphic to the group C_2^n, the product of n copies of the additive group $\{0,1\}$. As defined in Example 4.1.2, the set R is the set of generating involutions R_2^n. In this case, the group C_2^n acts simply transitively on the set X, meaning there is a correspondence between binary reflected Gray codes and Hamiltonian paths in $\mathrm{Cay}(C_2^n, R_2^n)$.

Theorem 7.1.1 provides an algorithm that constructs a Hamiltonian path in $\mathrm{Cay}(C_2^n, R_2^n)$ and equivalently a binary reflected Gray code.

Example 6.3.2. Introduced in Example 2.0.3, a k-subsets of n Gray code is the ordering of the k-element subsets of $\{1, 2, ..., n\}$ such that successive subsets differ in only one element by a difference of 1. In this example, we have

$$X = \{(a_1, a_2, ..., a_k) : 1 \leq a_1 < a_2 < ... < a_k \leq n\}$$

and $R \subseteq \mathrm{Perm}(X)$ is the set of involutions that swap only two consecutive integers.

The subgroup G of $\mathrm{Perm}(X)$ generated by R is isomorphic to the group S_n, where R is the set of adjacent transpositions $\{(12), (23), ..., (n-1\ n)\}$. The group G acts transitively on the set X but not simply transitively. Let $x \in X$ be the k-element set $\{1, 2, ..., k\}$. Then, the stabilizer of x is the set of elements of G that permute the numbers 1 to k and $k+1$ to n separately; that is, $G_x = S_{k,n-k}$, as defined in Example 5.3.2. Therefore, each k-subsets of n Gray code is a Hamiltonian path in the Schreier graph $\mathrm{Sch}(S_n, S_{k,n-k}, R)$.

CHAPTER 7

FIRST MAIN THEOREM

7.1 Statement and Proof of First Main Theorem

The following theorem provides a sufficient condition for the existence of a Hamiltonian cycle in the Cayley graph of a finite group with respect to a generating set of involutions, including an algorithm that produces such a cycle. This theorem is originally stated and proved in Conway, Sloane, and Wilks [3], but it has been adapted for our purposes.

Theorem 7.1.1. Let (G, R) be a finite group generated by involutions with $|R| \geq 2$. Then, if $\mathrm{Com}(G, R)$ is acyclic, there is a Hamiltonian cycle in $\mathrm{Cay}(G, R)$. [3]

Proof. We prove the theorem by induction on $n = |R|$, the number of involutions that generate the finite group G.

When $n = 2$ and $R = \{r_1, r_2\}$, then $(1, r_1, r_1r_2, r_1r_2r_1, ..., (r_1r_2)^{d-1}r_1)$ is a Hamiltonian cycle in $\mathrm{Cay}(G, R)$ where d is the order of r_1r_2.

Now, proceeding for $n \geq 3$, assume that there is a Hamiltonian cycle in the Cayley graph of any finite group with respect to $n - 1$ generating involutions with an acyclic commuting graph, and let (G, R) be a finite group generated by involutions with $|R| = n$ such that $\mathrm{Com}(G, R)$ is acyclic. By Lemma 3.1.2, there is a vertex of $\mathrm{Com}(G, R)$ of degree less than 2, so we can index the elements of R so that the vertex of the commuting graph that represents r_n has degree 0 or is adjacent only to the vertex r_{n-1}.

Let $H \subseteq G$ be the subgroup of G generated by $R' := \{r_1, ..., r_{n-1}\}$, and let Γ and Δ denote $\mathrm{Cay}(G, R)$ and $\mathrm{Cay}(H, R')$, respectively. Since G is the union of $m = [G : H]$ cosets of H, the graph Γ is partitioned into m disjoint subgraphs; let Δ_g be the subgraph of Γ induced by the left coset gH. Then, each Δ_g is isomorphic to Δ via the map $\Delta \longrightarrow \Delta_g$ defined $h \mapsto gh$.

Applying Lemma 3.1.3, since $\mathrm{Com}(H, R')$ is the result of removing a vertex from the acyclic graph $\mathrm{Com}(G, R)$, by the induction hypothesis, there is a Hamiltonian cycle

$B = (h_0, ..., h_{b-1})$ in Δ, where $b = |H|$. We can construct a Hamiltonian cycle in any subgraph Δ_g with edges identical to the edges contained in B by left multiplying B by some element of the coset gH. The goal is to string together Hamiltonian cycles in each of the subgraphs Δ_g via a recursive algorithm in order to construct a Hamiltonian cycle in Γ.

To define this algorithm, consider for each subset $X \subseteq G$ and $1 \leq i \leq n$ the set

$$\delta_{r_i}(X) = \{g \in G - X : g = xr_i, x \in X\},$$

the elements of $G - X$ that we can get to from the set X via right multiplication by the involution r_i.

Now, suppose we have a subset $X \subset G$ that is a union of left cosets of H, and we have a Hamiltonian cycle C in the subgraph of Γ induced by X, such that every edge label occurring in C except for the edge label r_n also occurs in the Hamiltonian cycle B in Δ.

Since X is a union of left cosets of H, we have that $\delta_{r_i}(X) = \emptyset$ for each $1 \leq i < n$. Here, if $\delta_{r_n}(X) = \emptyset$, then we are done, as the Cayley graph Γ is connected and then $X = G$. Otherwise, there exists $y \in G - X$ and $x \in X$ such that $y = xr_n$. Let Δ_y be the subgraph of Γ induced by the coset yH.

As noted in Lemma 3.2.2, given a Hamiltonian cycle, we can shift the cycle to begin at any element and reverse the order of the cycle as necessary. Therefore, if $c = |X|$, we can choose an ordering for the cycle $C = (x_0, ..., x_{c-1})$ such that $x_{c-1}r_\beta = x_0 = x$ where $r_\beta \neq r_{n-1}, r_n$ (which is possible as $n \geq 3$).

Next, the cycle C was chosen so that the edge r_β occurs in the cycle B in Δ. Thus, we can reorder $B = (h_0, ..., h_{b-1})$ such that $h_{b-1}r_\beta = h_0$. Therefore,

$$\begin{aligned} yh_0^{-1}B &= (yh_0^{-1}h_0, ..., yh_0^{-1}h_{b-1}) \\ &=: (y_0, ..., y_{b-1}) \end{aligned}$$

is a Hamiltonian cycle for Δ_y such that $y_{b-1}r_\beta = y_0 = y$.

Since $r_\beta \neq r_{n-1}$ and the vertex r_n of $\mathrm{Com}(G, R)$ is adjacent to at most the vertex r_{n-1}, the generators r_β and r_n commute. Thus, $x_{c-1}r_n = x_0r_\beta r_n = y_0r_\beta = y_{b-1}$, implying that the vertices x_{c-1} and y_{b-1} are adjacent in Γ.

Finally, we have

$$D = \{x_0, ..., x_{c-1}, y_{b-1}, y_{b-2}, ..., y_0\}$$

is a Hamiltonian cycle for the subgraph of Γ induced by $X \cup yH$ because each element of $X \cup yH$ is contained in D, and $y_0 = y$ and $x_0 = x$ are adjacent in Γ. Since $X \cup yH$ is a union of left cosets of H and every edge label occurring in D except r_n occurs in B, we have a well defined recursive algorithm.

Because G is finite, this algorithm can be repeated until Hamiltonian cycles for all the Δ_g's are concatenated to form a Hamiltonian cycle in Γ, completing the proof of the theorem. □

7.2 Application to Reflected Binary Gray Codes

The first application of the Theorem 7.1.1 is one that is a useful result in computer science. As introduced in Example 2.0.2, a binary reflected Gray code is an ordering of the binary numeral system in which successive numbers only differ in one digit. Gray codes are used in the error correction of digital communications like satellites and cable [11]. Additionally, reflected binary Gray codes can be interpreted musically by assigning n distinct notes to each of the n digits. Then, an n-digit binary number represents the chord that contains the notes with a 1 in their respective digit, and a Gray code is a chord progression that uses all possible chord combinations of the n notes, such that movements between successive chords are the result of adding or removing a single note.

In Example 6.3.1, we applied Theorem 6.2.1 to see that there is a bijection between reflected binary Gray codes and Hamiltonian paths in the Cayley graph of C_2^n with respect to the involutions R_2^n. And in Example 5.2.4, we found that the graph $\mathrm{Cay}(C_2^n, R_2^n)$ is the n-dimensional hypercube.

Theorem 7.1.1 applies in this case because, as noted in Example 5.1.3, the commuting graph of C_2^n with respect to R_2^n is edgeless, and hence acyclic. To see the algorithm in action, we will construct a Hamiltonian cycle in the 4-dimensional hypercube. The algorithm in the proof of Theorem 7.1.1 involves starting with two generating involutions, and adding the remaining generators one at a time. More explicitly, we will construct a Hamiltonian cycle in the 2-cube, use that to construct a Hamiltonian cycle in the 3-cube, and then obtain a Hamiltonian cycle in the 4-cube.

Let H be the subgroup of C_2^4 generated by the two involutions $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. Here, we are in the base case of the induction, so a Hamiltonian cycle for the subgraph induced by H is given by alternating generators. Moving to the 3-dimensional hypercube, if we left multiply the Hamiltonian cycle for the subgraph induced by H by the third generator $(0, 0, 1, 0)$, we obtain a Hamiltonian cycle for the subgraph induced by the only nontrivial left coset of H in C_2^3. These two cycles are shown in Figure 7.1. Then, we string the two cycles together as shown in Figure 7.2, resulting in a Hamiltonian cycle in the 3-dimensional hypercube.

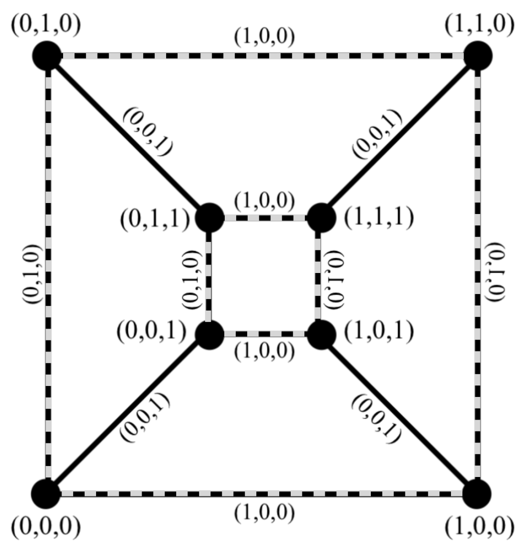

Figure 7.1: Constructing a Hamiltonian cycle in the 3-dimensional hypercube.

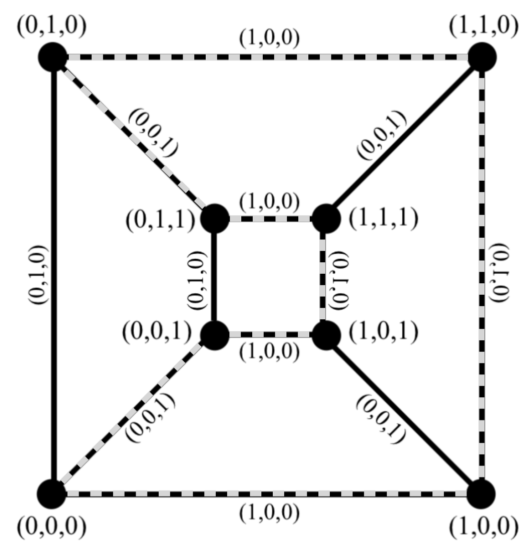

Figure 7.2: Hamiltonian cycle in the 3-dimensional hypercube.

Finally, we move to the 4-dimensional hypercube graph, or the Cayley graph of C_2^4 with respect to R_2^4. Figure 7.3 shows two isomorphic copies of a Hamiltonian cycle in the 3-dimensional hypercube, and Figure 7.4 shows how to string them together to obtain a Hamiltonian cycle in the 4-dimensional hypercube graph. The resulting Hamiltonian cycle represents a reflected binary Gray code in the 4-digit case. Notice that we can start the Gray code at any binary number and move in either direction around the cycle.

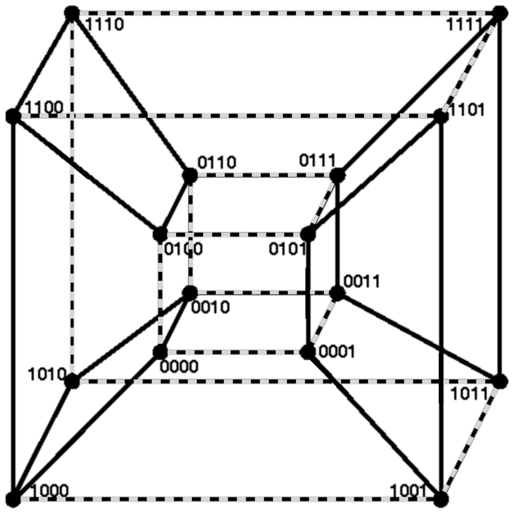

Figure 7.3: Constructing a Hamiltonian cycle in the 4-dimensional hypercube.

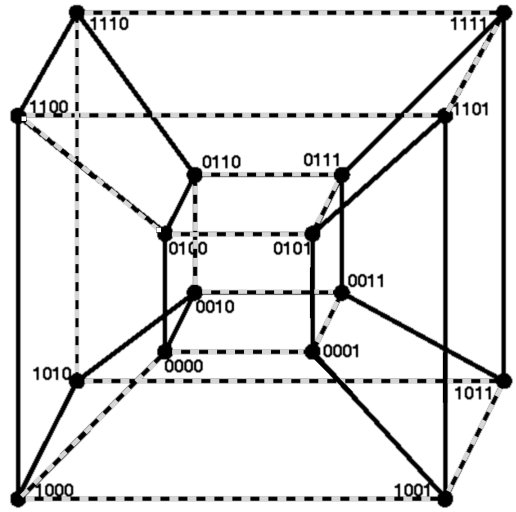

Figure 7.4: Hamiltonian cycle in the 4-dimensional hypercube.

7.3 Application to the PLR Group

Another application of Theorem 7.1.1 is useful in music theory. Given a group G of permutations on a set of chords X that is generated by a set of order two operations R, if the group G acts on the set X simply transitively, then each element of the group can be identified with a distinct chord in the set. Therefore, in this case, a Hamiltonian cycle in the Cayley graph of G with respect to X is a chord progression that uses each of the elements of X exactly once, the essence of serialism.

We have already noted in Example 5.2.5 that the PLR group acts simply transitively on the set of 24 major and minor triads Π; however, the commuting graph of the PLR group with respect to the generators P, L, and R contains a cycle, meaning the Theorem 7.1.1 does not immediately apply. Nonetheless, it can be shown that the PLR group is generated by the two elements L and R [4], where the commuting graph of the PLR group with respect to the set $\{L, R\}$ is in fact acyclic. Therefore, Theorem 7.1.1 applies to the Cayley graph of the PLR group with respect to the generators R and L. With only two generating involutions, we are in the base case of the induction argument. Therefore, if we alternately apply the operations L and R to the C major triad, we obtain the following sequence of triads, where uppercase letters represent major triads and lowercase letter represent minor ones.

$$\text{C, a, F, d, B}\flat\text{, g, E}\flat\text{, c, A}\flat\text{, f, D}\flat\text{, b}\flat\text{, G}\flat\text{, e}\flat\text{, B, g}\sharp\text{, E, c}\sharp\text{, A, f}\sharp\text{, D, b, G, e}$$

Interestingly enough, the first half of this chord progression is used in Beethoven's Ninth Symphony, demonstrating that the this application is not restricted to serialism.

The limitation of this example is that the PLR group is generated by only two involutions, meaning the real substance of the Theorem 7.1.1 is not fully applied. Thus, the next goal is to provide an example of a group G of permutations of a set of chords X that has the following properties:

1. The group G is generated by a set of involutions R such that $|R| > 2$ and $\mathrm{Com}(G, R)$ is acyclic.

2. The group G acts simply transitively on the set X.

CHAPTER 8

THE SBW GROUP

8.1 Permutations of Sets of Seventh Chords

In order to accomplish this goal, we will move to a larger set of chords: the set Ψ of all major seventh ($^{\Delta 7}$), dominant seventh (^{7}), minor seventh ($^{-7}$), and half diminished seventh ($^{\varnothing 7}$) chords. More specifically, the collection Ψ is the union of four set of four element subsets of Z_{12} defined as follows.

$$\Psi^{\Delta 7} = \big\{\{x, x+4, x+7, x+11\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}\big\}$$

$$\Psi^{7} \;\; = \big\{\{x, x+4, x+7, x+10\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}\big\}$$

$$\Psi^{-7} = \big\{\{x, x+3, x+7, x+10\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}\big\}$$

$$\Psi^{\varnothing 7} = \big\{\{x, x+3, x+6, x+10\} \subseteq \mathbb{Z}_{12} : x \in \mathbb{Z}\big\}$$

The size of this set of seventh chords is $|\Psi| = 12 \times 4 = 48$, as there are twelve notes and four types of seventh chords.

The group that we begin examining is a semidirect product that defines a group action on the set Ψ. Let S_4 act by group automorphisms on $\mathbb{Z}_{12}^4$, the product of four copies of $\mathbb{Z}_{12}$, by permuting coordinates. That is, for $\sigma \in S_4$ and $x \in \mathbb{Z}_{12}^4$, we define

$$\sigma(x_1, x_2, x_3, x_4) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}).$$

This action defines a semidirect product $S_4 \ltimes \mathbb{Z}_{12}^4$. Explicitly, for each (σ, x) and (w, y) in $S_4 \ltimes \mathbb{Z}_{12}^4$, we define

$$\big(\sigma, (x_1, x_2, x_3, x_4)\big)\big((w, (y_1, y_2, y_3, y_4)\big) := \big(\sigma w, (y_1 + x_{w(1)}, y_2 + x_{w(2)}, y_3 + x_{w(3)}, y_4 + x_{w(4)})\big).$$

In order to define a group action of $S_4 \ltimes \mathbb{Z}_{12}^4$ on the set Ψ, notice that any chord in the collection Ψ can be represented by the ordered pair $(i, r) \in \{1, 2, 3, 4\} \times \mathbb{Z}_{12}$, where r is the root of the chord and i represents the type of chord: 1 for major seventh, 2 for dominant seventh, 3 for minor seventh, and 4 for half diminished seventh. Therefore, using this representation of the set Ψ, for any $(\sigma, x) \in S_4 \ltimes \mathbb{Z}_{12}^4$ and $(i, r) \in \{1, 2, 3, 4\} \times Z_{12}$, define a group action

$$(\sigma, x)(i, r) = (\sigma(i), r + x_i).$$

Example 8.1.1. To see how this group action works in practice, consider the example where $(\sigma, x) = \big((12)(34), (7, 5, 11, 1)\big)$ in the group $S_4 \ltimes \mathbb{Z}_{12}^4$ and $(i, r) = (1, 0)$ in the set $\{1, 2, 3, 4\} \times Z_{12}$. Then, $\big((12)(34), (7, 5, 11, 1)\big)$ operates on the chord $(1, 0)$ - which represents C major seventh ($\mathrm{C}^{\Delta 7}$) - by transposing the root 0 up by the interval in the third entry of x and changing the chord type to the image of 3 under the permutation σ. In this case,

$$\big((12)(34), (7, 5, 11, 1)\big)(1, 0) = (2, 7),$$

meaning that the operation $\big((12)(34), (7, 5, 11, 1)\big)$ applied to the chord C major seventh ($\mathrm{C}^{\Delta 7}$) is the chord G dominant seventh (G^7).

Example 8.1.2. In order to demonstrate that the group action of $S_4 \ltimes \mathbb{Z}_{12}^4$ on the set Ψ is a significant chord permutation group, consider the following chord progression taken from part of the chorus of the jazz standard "Tune Up" by Miles Davis [5]:

$$\mathrm{E}^{-7},\ \mathrm{A}^7,\ \mathrm{D}^{\Delta 7},\ \mathrm{D}^{-7},\ \mathrm{G}^7,\ \mathrm{C}^{\Delta 7},\ \mathrm{C}^{-7},\ \mathrm{F}^7,\ \mathrm{B}\flat^{\Delta 7}.$$

Consecutive chords in this progression are the result of the element $\big((132), (0, 5, 5, 0)\big)$ of $S_4 \ltimes \mathbb{Z}_{12}^4$. That is, each minor seventh chord is transposed up an interval of 5 and changed to a dominant seventh chord; each dominant seventh chord is transposed up and interval of 5 and changed to a major seventh chord; and each major seventh chord is changed to its parallel minor seventh. As the jazz giant Thelonious Monk once said, "all musicians are subconsciously mathematicians" [10].

Notice that in this group action, given an element (σ, x) of $S_4 \ltimes \mathbb{Z}_{12}^4$ and $t \in \mathbb{Z}_{12}$, for each chord (i, r), if $(\sigma, x)(i, r) = (i', r')$, then $(\sigma, x)(i, r + t) = (i', r' + t)$. It is said that each element of $S_4 \ltimes \mathbb{Z}_{12}^4$ acts on chords of the same type the "same way". This result is analogous to the *uniformity condition* in Julian Hook's "Uniform Triadic Transformation" [8], where a group action is similarly defined on the set of major and minor triads Π using the semidirect product $C_2 \ltimes \mathbb{Z}_{12}^2$.

8.2 The SBW Group

The action of the group $S_4 \ltimes \mathbb{Z}_{12}^4$ on the set of seventh chords Ψ is transitive, but the group $S_4 \ltimes \mathbb{Z}_{12}^4$ is far too large to act simply transitively on Ψ. Therefore, we will define a subgroup of $S_4 \ltimes \mathbb{Z}_{12}^4$ that is generated by involutions and acts simply transitively on the set Ψ.

Named after Schönberg, Berg, and Webern of the Second Viennese School, define the *SBW group* to be the subgroup of $S_4 \ltimes \mathbb{Z}_{12}^4$ generated by the elements

$$\begin{aligned} S &:= \big((12)(34), (7, 5, 11, 1)\big), \\ B &:= \big((13)(24), (2, 10, 10, 2)\big), \\ W &:= \big((14)(23), (6, 0, 0, 6)\big). \end{aligned}$$

Routine calculations show that each of these generators has order 2 in $S_4 \ltimes \mathbb{Z}_{12}^4$, meaning the SBW group is a finite group generated by involutions. It remains to show that the SBW group acts simply transitively on the set of seventh chords Ψ, and that the commuting graph is acyclic.

Proposition 8.2.1. The SBW group acts simply transitively on the set Ψ.

Proof. In order to understand the group action of the SBW group on the set Ψ, we can identify the group $S_4 \ltimes \mathbb{Z}_{12}^4$ with a subgroup of the general linear group of degree 5 over the ring $\mathbb{Z}_{12}$, the group $\mathrm{GL}(5, \mathbb{Z}_{12})$ of invertible 5×5 matrices with coefficients in $\mathbb{Z}_{12}$. Then, the SBW group is isomorphic to a subgroup of $\mathrm{GL}(5, \mathbb{Z}_{12})$, so we can easily compute the order of the SBW group and the stabilizer of an element of Ψ using Sage.

Define a map $\varphi : S_4 \ltimes \mathbb{Z}_{12}^4 \longrightarrow \mathrm{GL}(5, \mathbb{Z}_{12})$ by

$$\big(\sigma, (x_1, x_2, x_3, x_4)\big) \mapsto \left[\begin{array}{c|c|c|c|c} \sigma\begin{bmatrix}1\\0\\0\\0\end{bmatrix} & \sigma\begin{bmatrix}0\\1\\0\\0\end{bmatrix} & \sigma\begin{bmatrix}0\\0\\1\\0\end{bmatrix} & \sigma\begin{bmatrix}0\\0\\0\\1\end{bmatrix} & \sigma\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Then, the map φ is a group homomorphism because

$$\begin{aligned} &\varphi\Big(\big(\sigma, (x_1, x_2, x_3, x_4)\big)\big(w, (y_1, y_2, y_3, y_4)\big)\Big) \\ &= \varphi\big(\sigma w, w^{-1}(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)\big) \\ &= \left[\begin{array}{c|c|c|c|c} \sigma w\begin{bmatrix}1\\0\\0\\0\end{bmatrix} & \sigma w\begin{bmatrix}0\\1\\0\\0\end{bmatrix} & \sigma w\begin{bmatrix}0\\0\\1\\0\end{bmatrix} & \sigma w\begin{bmatrix}0\\0\\0\\1\end{bmatrix} & \sigma\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \sigma w\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ &= \varphi\big(\sigma, (x_1, x_2, x_3, x_4)\big)\varphi\big(w, (y_1, y_2, y_3, y_4)\big) \end{aligned}$$

for each $(\sigma, x), (w, y) \in S_4 \ltimes \mathbb{Z}_{12}^4$. Also, the kernel of φ is clearly trivial, so φ is injective. Therefore, the group $S_4 \ltimes \mathbb{Z}_{12}^4$ is isomorphic to the image of φ in $\mathrm{GL}(5, \mathbb{Z}_{12})$.

Now, the SBW group is isomorphic to the subgroup G of $\mathrm{GL}(5, \mathbb{Z}_{12})$ generated by the matrices $\varphi(S), \varphi(B)$, and $\varphi(W)$. Using the Sage code in Appendix A, we find that the order of G is 48. Next, we want to compute the stabilizer of the element $(1, 0) \in \{1, 2, 3, 4\} \times \mathbb{Z}_{12}$. Recall that $(\sigma, x)(1, 0) = (\sigma(1), x_1)$, so (σ, x) is in the stabilizer of $(1, 0)$ if $\sigma(1) = 1$ and $x_1 = 0$; that is, the matrix $\varphi(\sigma, x) \in G$ has a 1 in the 1,1 coordinate and a 0 in the 1,5 coordinate. The Sage code in Appendix A counts the number of elements in G with these conditions and determines that the the stabilizer of $(1, 0)$ is trivial.

Since the SBW group has order 48 and the stabilizer of $(1, 0)$ has order 1, the orbit-stabilizer theorem (Theorem 6.1.4) guarantees that the orbit of $(1, 0)$ has size 48=$|\Psi|$.

Thus, there is only one orbit in the group action and the stabilizer of each element of Ψ is trivial, so the SBW group acts simply transitively on Ψ. $\square$

8.3 Application of Theorem 7.1.1 to the SBW Group

Since the SBW group acts simply transitively on the set of seventh chords Ψ, each Hamiltonian cycle in the Cayley graph of the SBW group with respect to the generating involutions is associated to a Gray code for $(\Psi, \{S, B, W\})$. We will apply Theorem 7.1.1 to the SBW group, but first we have to verify that the commuting graph of the SBW group with respect to the generating involutions is acyclic.

Lemma 8.3.1. The commuting graph of the SBW group with respect to $\{S, B, W\}$ is acyclic.

Proof. The commuting graph of the SBW group with respect to the involutions S, B, and W can be constructed by determining the order of each pair of generators. Routine calculations show that order$(SB) = 6$, order$(BW) = 4$, and order$(SW) = 2$. Therefore, the commuting graph of the SBW group with respect to $\{S, B, W\}$ is given in Figure 8.1. Since this graph has only one vertex of degree greater than 1, it is acyclic. $\square$

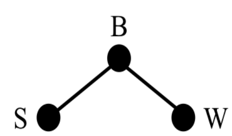

Figure 8.1: Commuting graph of the SBW Group with respect to $\{S, B, W\}$.

Therefore, since the SBW group acts simply transitively on the set Ψ and the commuting graph of the SBW group with respect to $\{S, B, W\}$ is acyclic, we can apply Theorem 7.1.1 to obtain a Gray code for $(\Psi, \{S, B, W\})$. If we choose the chord $C^{\Delta 7}$ to represent the identity element of the SBW group and implement the algorithm, we construct the following cyclic Gray code.

$$
\begin{tikzcd}
C^{\Delta 7} \arrow[r, dash, "W"] \arrow[d, dash, "B"] & F\sharp^{\varnothing 7} \arrow[d, dash, "B"] & F\sharp^{\Delta 7} \arrow[r, dash, "W"'] \arrow[d, dash, "B"] & C^{\varnothing 7} \arrow[d, dash, "B"] \\
D^{-7} \arrow[d, dash, "S"] & G\sharp^{7} \arrow[d, dash, "S"] & G\sharp^{-7} \arrow[d, dash, "S"] & D^{7} \arrow[d, dash, "S"] \\
C\sharp^{\varnothing 7} \arrow[d, dash, "B"] & C\sharp^{\Delta 7} \arrow[d, dash, "B"] & G^{\varnothing 7} \arrow[d, dash, "B"] & G^{\Delta 7} \arrow[d, dash, "B"] \\
E\flat^{7} \arrow[d, dash, "S"] & E\flat^{-7} \arrow[d, dash, "S"] & A^{7} \arrow[d, dash, "S"] & A^{-7} \arrow[d, dash, "S"] \\
A\flat^{\Delta 7} \arrow[d, dash, "B"] & D^{\varnothing 7} \arrow[d, dash, "B"] & D^{\Delta 7} \arrow[d, dash, "B"] & G\sharp^{\varnothing 7} \arrow[d, dash, "B"] \\
B\flat^{-7} \arrow[d, dash, "S"] & E^{7} \arrow[r, dash, "W"] & E^{-7} & B\flat^{7} \arrow[d, dash, "S"] \\
A^{\varnothing 7} \arrow[d, dash, "B"] & A^{\Delta 7} \arrow[r, dash, "W"] \arrow[d, dash, "B"] & D\sharp^{\varnothing 7} \arrow[d, dash, "B"] & E\flat^{\Delta 7} \arrow[d, dash, "B"] \\
B^{7} \arrow[d, dash, "S"] & B^{-7} \arrow[d, dash, "S"] & F^{7} \arrow[d, dash, "S"] & F^{-7} \arrow[d, dash, "S"] \\
E^{\Delta 7} \arrow[d, dash, "B"] & A\sharp^{\varnothing 7} \arrow[d, dash, "B"] & B\flat^{\Delta 7} \arrow[d, dash, "B"] & E^{\varnothing 7} \arrow[d, dash, "B"] \\
F\sharp^{-7} \arrow[d, dash, "S"] & C^{7} \arrow[d, dash, "S"] & C^{-7} \arrow[d, dash, "S"] & F\sharp^{7} \arrow[d, dash, "S"] \\
F^{\varnothing 7} \arrow[d, dash, "B"] & F^{\Delta 7} \arrow[d, dash, "B"] & B^{\varnothing 7} \arrow[d, dash, "B"] & B^{\Delta 7} \arrow[d, dash, "B"] \\
G^{7} \arrow[r, dash, "W"] & G^{-7} & C\sharp^{7} \arrow[r, dash, "W"] & C\sharp^{-7}
\end{tikzcd}
$$

By starting at any point in the cycle and moving in either direction, we have a serialism-like chord progression that uses each of the 48 seventh chords in the set Ψ exactly once. The chord progressions found in this cycle exemplify a shifting and occasionally nonexistent tonal center, deviating from the realm of functional harmony. This trend is found in contemporary classical and jazz music.

CHAPTER 9

SECOND MAIN THEOREM

9.1 Statement of the Second Main Theorem

Our second main theorem concerns the existence of k-element subsets of n Gray code - this theorem is stated and proved in Eades and Hickey [6]. Given the set of positive integers from 1 to n and some k such that $1 < k < n-1$, we wish to order the k-element subsets of $\{1, 2, ..., n\}$ such that successive subsets differ in only one element by a difference of 1. Recall from Example 6.3.2 that this is equivalent to finding a Hamiltonian path in the Schreier graph of the symmetric group S_n with respect to the subgroup $S_{k,n-k}$ and the set of generating adjacent transpositions.

To achieve this goal, define the graph $G_{n,k}$ such that

$$V(G_{n,k}) = \{(a_1, a_2, ..., a_k) : 1 \leq a_1 < a_2 < ... < a_k \leq n\},$$
$$E(G_{n,k}) = \Big\{\{(a_1, ..., a_k), (b_1, ..., b_k)\} : \Big(\sum_{i=1}^{k}(a_i - b_i)^2\Big)^{1/2} = 1\Big\}.$$

Notice that the vertex set of $G_{n,k}$ represents the set of all k-element subsets of $\{1, 2, ..., n\}$ and that two such k-tuples are adjacent in $G_{n,k}$ if they differ by a quantity of 1 in only one position. Therefore, the task at hand is to find a Hamiltonian path in the graph $G_{n,k}$. With this construction, we obtain the following theorem.

Theorem 9.1.1. If $n \geq 4$ and $1 < k < n-1$, then $G_{n,k}$ has a Hamiltonian path if and only if n is even and k is odd.

The necessity of n being even and k being odd is proved in Section 9.2, while the sufficiency is proved in Section 9.3.

9.2 Proof of Necessity

This section proves the necessity of n being even and k being odd in Theorem 9.1.1. Assume that for some $n \geq 4$ and $1 < k < n-1$, there is a Hamiltonian path in the graph

$G_{n,k}$. First notice that $G_{n,k}$ is bipartite with parts defined

$$E_{n,k} = \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : \sum_{i=1}^{k} a_i \text{ is even}\}$$

$$O_{n,k} = \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : \sum_{i=1}^{k} a_i \text{ is odd}\}.$$

Let $\eta_{n,k}$ and $\omega_{n,k}$ denote the size of the sets $E_{n,k}$ and $O_{n,k}$ respectively. Then, $G_{n,k}$ contains a Hamiltonian path only if $|\tau_{n,k}| = |\eta_{n,k} - \omega_{n,k}| \leq 1$, from Lemma 3.3.2.

Next, in order to determine the number $\tau_{n,k}$, we want to enumerate the vertices of $V(G_{n,k})$ whose elements sum to some number N. Consider the polynomial in x and y

$$f(x, y) = \prod_{r=1}^{n} (1 + xy^r).$$

Here, if the integer r is used in a vertex of $V(G_{n,k})$, it contributes r to the number N and 1 to the number k, so the number of vertices whose elements sum to N is given by the coefficient of $x^k y^N$ in $f(x, y)$. Therefore, the number $\tau_{n,k}$ is the coefficient of x^k in $f(x, -1)$.

If n is even, then $n = 2m$ for some integer m, so

$$\begin{aligned} \prod_{r=1}^{2m} (1 + x(-1)^r) &= (1 + x)^m (1 - x)^m \\ &= (1 - x^2)^m. \end{aligned}$$

Thus, using the binomial theorem,

$$\tau_{n,k} = \begin{cases} 0 & \text{if } k \text{ is odd and } n \text{ is even} \\ (-1)^{k/2} \binom{m}{k/2} & \text{if } k \text{ is even and } n \text{ is even} \end{cases}$$

Next, if n is odd, then $n = 2m + 1$ for some integer m, so

$$\begin{aligned} \prod_{r=1}^{2m+1} (1 + x(-1)^r) &= (1 + x)^m (1 - x)^{m+1} \\ &= (1 - x^2)^m (1 - x). \end{aligned}$$

Thus, using the binomial theorem,

$$\tau_{n,k} = \begin{cases} (-1)^{\frac{k+1}{2}} \binom{m}{\frac{k-1}{2}} & \text{if } k \text{ is odd and } n \text{ is odd} \\ (-1)^{k/2} \binom{m}{k/2} & \text{if } k \text{ is even and } n \text{ is odd} \end{cases}$$

Finally, since $n \geq 4$ and $1 < k < n-1$, the only case with $|\tau_{n,k}| \leq 1$ is when n is even and k is odd, proving the necessity of Theorem 9.1.1.

9.3 Proof of Sufficiency

In this section, we will prove the sufficiency of Theorem 9.1.1. That is, assuming n is even and k is odd, we want to show that $G_{n,k}$ contains a Hamiltonian path.

The proof uses induction on n. The following subsets of $V(G_{n,k})$ will be used in the proof.

$$\begin{aligned} A_{n,k} &= \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : a_1 = 1, a_2 = 2\} \\ B_{n,k} &= \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : a_1 \geq 3\} \\ C_{n,k} &= \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : a_1 = 1, a_2 \geq 3\} \\ D_{n,k} &= \{(a_1, a_2, ..., a_k) \in V(G_{n,k}) : a_1 = 2\} \end{aligned} \tag{9.3.1}$$

If we let $\hat{A}_{n,k}$, $\hat{B}_{n,k}$, $\hat{C}_{n,k}$, and $\hat{D}_{n,k}$ denote the subgraphs of $G_{n,k}$ induced by $A_{n,k}$, $B_{n,k}$, $C_{n,k}$, and $D_{n,k}$ respectively, then each of these subgraphs is isomorphic to some $G_{n',k'}$ as follows.

$$\begin{aligned} \hat{A}_{n,k} &\cong G_{n-2,k-2} \text{ via } (1, 2, a_3, a_4, ..., a_k) \mapsto (a_3 - 2, a_4 - 2..., a_k - 2), \\ \hat{B}_{n,k} &\cong G_{n-2,k} \text{ via } (a_1, a_2, ..., a_k) \mapsto (a_1 - 2, a_2 - 2, ..., a_k - 2), \\ \hat{C}_{n,k} &\cong \hat{D}_{n,k} \text{ via } (1, a_2, a_3, ..., a_k) \mapsto (2, a_2, a_3, ..., a_k), \\ \hat{D}_{n,k} &\cong G_{n-2,k-1} \text{ via } (2, a_2, a_3, ..., a_k) \mapsto (a_2 - 2, a_3 - 2, ..., a_k - 2). \end{aligned}$$

Notice that the requirement of k being odd means the induction hypothesis will only apply to the subgraphs $\hat{A}_{n,k}$ and $\hat{B}_{n,k}$, but not to $\hat{C}_{n,k}$ and $\hat{D}_{n,k}$. Thus, we must strengthen the inductive hypothesis with a Lemma.

Lemma 9.3.1. If n is even and $1 \leq k \leq n$, then

1. If k is odd, then $G_{n,k}$ has a Hamiltonian path with endpoints $(1, 2, ..., k)$ and $(n-k+1, n-k+2, ..., n)$

2. If k is even, then $G_{n,k}$ has a spanning comb with boundary points $(1, n-k+2, n-k+3, ..., n)$ and $(2, 3, ..., k+1)$.

Proof. The proof of the lemma is by induction on n. For $n = 2$, the statement is easily verified. Now, let n be an even integer and assume that the lemma holds for any even integer $m < n$. When $k = 1$ or $k = n$, the graph $G_{n,k}$ is just a single path of vertices, so the lemma holds. Thus, we can restrict to the case when $1 < k < n$.

If k is odd, then by the induction hypothesis, the subgraphs $\hat{A}_{n,k}$ and $\hat{B}_{n,k}$ have Hamiltonian paths with respective endpoints

$$x_A = (1, 2, ..., k) \;\;\&\;\; y_A = (1, 2, n-k+3, n-k+4, ..., n),$$
$$x_B = (3, 4, ..., k+2) \;\;\&\;\; y_B = (n-k+1, n-k+2, ..., n).$$

Additionally, the inductive hypothesis guarantees that the subgraph $\hat{C}_{n,k}$ has a spanning comb T with boundary points

$$x_C = (1, 3, n-k+3, n-k+4, ..., n) \;\;\&\;\; y_C = (1, 4, 5, ..., k+2).$$

Let J denote the subgraph of $G_{n,k}$ induced by $C_{n,k} \cup D_{n,k}$. Recall that $C_{n,k}$ is isomorphic to $D_{n,k}$ by the map $\varphi(1, a_2, ..., a_k) = (2, a_2, ..., a_k)$, and notice that for each $c \in V(C_{n,k})$, we have $\{c, \varphi(c)\} \in E(G_{n,k})$. Therefore, J contains a spanning subgraph that is isomorphic to the product TK_2.

Since the boundary points of the comb T are in the same part of the bipartite graph $G_{n,k}$ (as described in the proof of necessity), the distance between x_C and y_C is even, so using Lemma 3.4.3, the graph TK_2 has a Hamiltonian path with endpoints $(1, x_C)$ and $(2, y_C)$. Thus, J has a Hamiltonian path with endpoints x_C and $y_D = (2, 4, 5, ..., k+2)$.

Therefore, since y_A is adjacent to x_C and y_D is adjacent to x_B in $G_{n,k}$, we can concatenate the Hamiltonian paths for the subgraphs $\hat{A}_{n,k}$, J, and $\hat{B}_{n,k}$ to obtain a Hamiltonian path in $G_{n,k}$ with endpoints x_A and y_B.

Now, if k is even, then by the induction hypothesis, the subgraph $\hat{C}_{n,k}$ has a Hamiltonian path $(c_1, c_2, ..., c_m)$ with $c_m = (1, n-k+2, n-k+3, ..., n)$. From the isomorphism of $\hat{C}_{n,k}$ and $\hat{D}_{n,k}$, there is a Hamiltonian path $(d_1, d_2, ..., d_m)$ for $\hat{D}_{n,k}$ such that $d_1 = (2, 3, ..., k+1)$ and $\{c_i, d_i\} \in E(G_{n,k})$ for all $1 \leq i \leq m$.

Therefore, since m is even, the path

$$M = (d_1, c_1, c_2, d_2, ..., c_{m-1}, c_m, d_m)$$

is a Hamiltonian path for J. We want to construct a spanning comb for $G_{n,k}$ with M as part of the main path. To accomplish this, for any vertex x in the path M, we will define a path T_x that is the tooth at x.

If $x = c_i = (1, x_2, x_3, ..., x_k)$, then let r be the smallest index such that $x_r > r+1$, and if no such index exists let $r = k+1$. For any $1 \leq j < r$, define $t_j = (1, 2, ..., j, x_{j+1}, x_{j+2}, ..., x_k)$. Then, $t_1 = x$ and $\{t_j, t_{j-1}\} \in E(G_{n,k})$ for all $1 < j < r$, so let $T_x = (t_1, t_2, ..., t_{r-1})$ be the tooth at x.

If $x = d_i = (2, x_2, x_3, ..., x_k)$, then let $r = x_2 - 1$ and $t_j = (j+1, x_2, x_3, ..., x_k)$ for $1 \leq j < r$. Again, $t_1 = x$ and $\{t_j, t_{j-1}\} \in E(G_{n,k})$ for all $1 < j < r$, so let $T_x = (t_1, t_2, ..., t_{r-1})$ be the tooth at x.

Notice that tooth defined at c_m is trivial (because the associated r for c_m is 2), while the tooth defined at $d_m = (2, n-k+2, n-k+3, ..., n)$ is nontrivial. If T_{d_m} is the tooth at d_m, we can define T to be the comb in $G_{n,k}$ with main path (M, T_{d_m}) (the concatenation of M and T_{d_m}) and teeth T_x. We want to show that T is a spanning comb for $G_{n,k}$.

If $y = (1, 2, y_3, ..., y_k) \in A_{n,k}$ and r is the largest index with $y_r = r$, then y lies only on the tooth T_x where

$$x = (1, 3, 4, ..., r+1, y_{r+1}, y_{r+2}, ..., y_k).$$

And if $y = (y_1, y_2, ...y_k) \in B_{n,k}$ where $y_1 > 2$, then y lies only on the tooth T_x where

$$x = (2, y_2, y_3, ..., y_k).$$

Therefore, T is an acyclic spanning subgraph of $G_{n,k}$ of degree at most 3 such that all the degree 3 vertices of T lie on the main path strictly between the boundary points $(1, n-k+2, n-k+3, ..., n)$ and $(2, 3, ..., k+1)$. This completes the proof of the lemma. $\square$

The proof of sufficiency of Theorem 9.1.1 follows directly from the proof of this lemma.

9.4 Application of Theorem 9.1.1

In this section, we will carryout the algorithm provided in the proof of sufficiency of Theorem 9.1.1 in the case where $n = 6$ and $k = 3$, and then the case where $n = 8$ and $k = 3$.

Example 9.4.1. We will first apply Theorem 9.1.1 to the case where $n = 6$ and $k = 3$. Since n is even and k is odd, there exists a Hamiltonian path in the graph $G_{6,3}$. The algorithm begins by splitting $G_{6,3}$ into the subgraphs $\hat{A}_{6,3}$, $\hat{B}_{6,3}$, $\hat{C}_{6,3}$, and $\hat{D}_{6,3}$ (defined in Section 9.3). First, $\hat{A}_{6,3}$ and $\hat{B}_{6,3}$ have the respective trivial Hamiltonian paths

$$(\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\}) \text{ and } (\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}).$$

The graph $G_{6,3}$ with the paths for $\hat{A}_{6,3}$ and $\hat{B}_{6,3}$ outlined is shown in Figure 9.1.

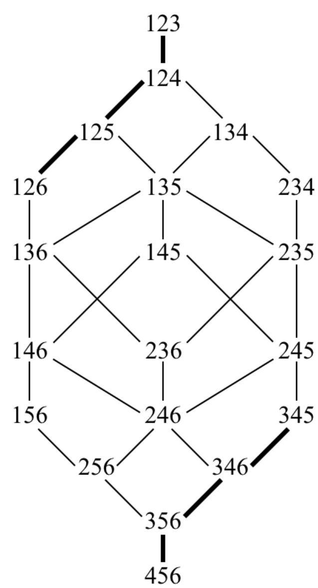

Figure 9.1: The graph $G_{6,3}$ with Hamiltonian paths for the subgraphs $\hat{A}_{6,3}$ and $\hat{B}_{6,3}$.

Now, we obtain isomorphic spanning combs for $\hat{C}_{6,3}$ and $\hat{D}_{6,3}$, and find a Hamiltonian path for J, the subgraph of $G_{6,3}$ induced by $C_{6,3} \cup D_{6,3}$. The Hamiltonian path for J described in the proof of Theorem 9.1.1 is shown in Figure 9.2.

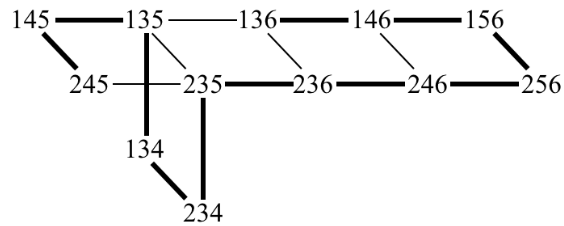

Figure 9.2: Hamiltonian path in the subgraph of $G_{6,3}$ induced by $C_{6,3} \cup D_{6,3}$.

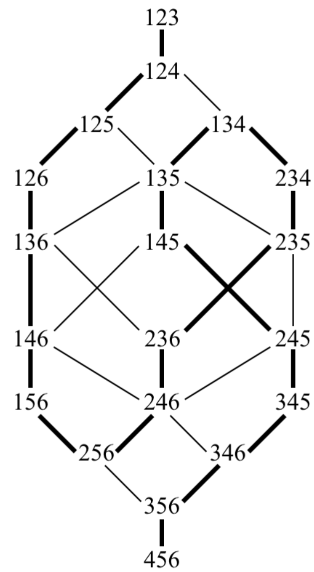

Figure 9.3: Hamiltonian path in the graph $G_{6,3}$.

Finally, we concatenate the Hamiltonian paths for $\hat{A}_{6,3}$, J, and $\hat{B}_{6,3}$ to obtain the Hamiltonian shown in Figure 9.3.

Example 9.4.2. The second application of Theorem 9.1.1 is in the case where $n = 8$ and $k = 3$. The subgraph $\hat{A}_{8,3}$ of $G_{8,3}$ is isomorphic to the graph $G_{6,1}$, so $\hat{A}_{8,3}$ has the trivial Hamiltonian path

$$(\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,2,7\}, \{1,2,8\}).$$

Since $\hat{B}_{8,3}$ is isomorphic to $G_{6,3}$, we can use the Hamiltonian path found in Example 9.4.1 to obtain the corresponding path in $\hat{B}_{8,3}$. The paths for $\hat{A}_{8,3}$ and $\hat{B}_{8,3}$ are shown in Figure 9.4.

Now, we construct isomorphic spanning combs for $\hat{C}_{8,3}$ and $\hat{D}_{8,3}$, and find a Hamiltonian path in the the subgraph J induced by $C_{8,3} \cup D_{8,3}$. The Hamiltonian path for J described in the proof of Theorem 9.1.1 is shown in Figure 9.5.

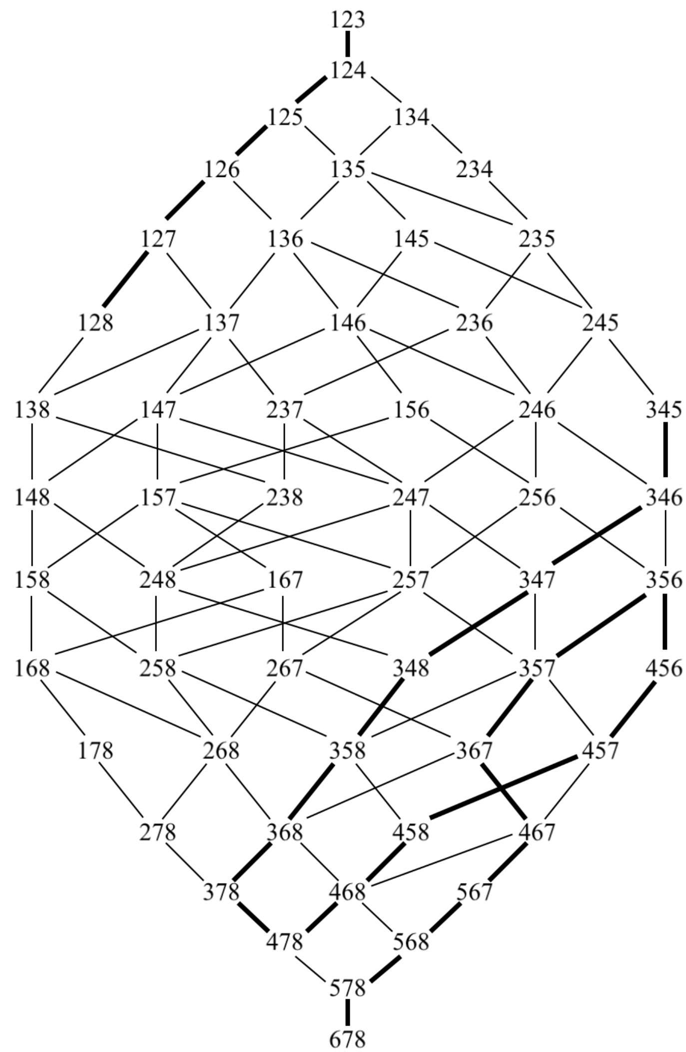

Figure 9.4: The graph $G_{8,3}$ with Hamiltonian paths for the subgraphs $\hat{A}_{8,3}$ and $\hat{B}_{8,3}$.

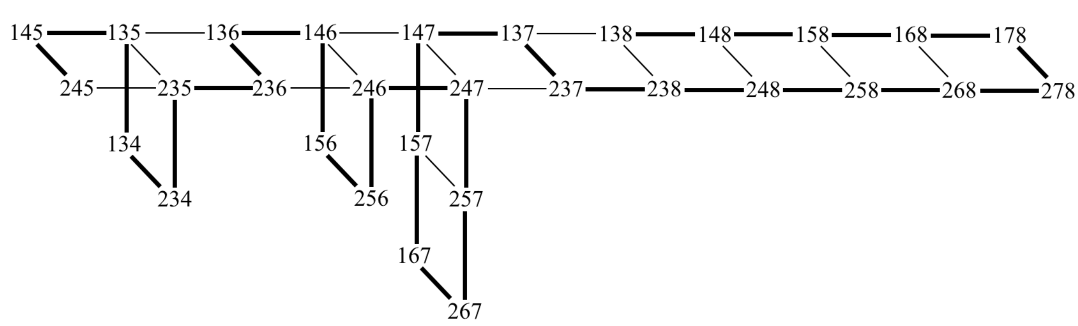

Figure 9.5: Hamiltonian path in the subgraph of $G_{8,3}$ induced by $C_{8,3} \cup D_{8,3}$.

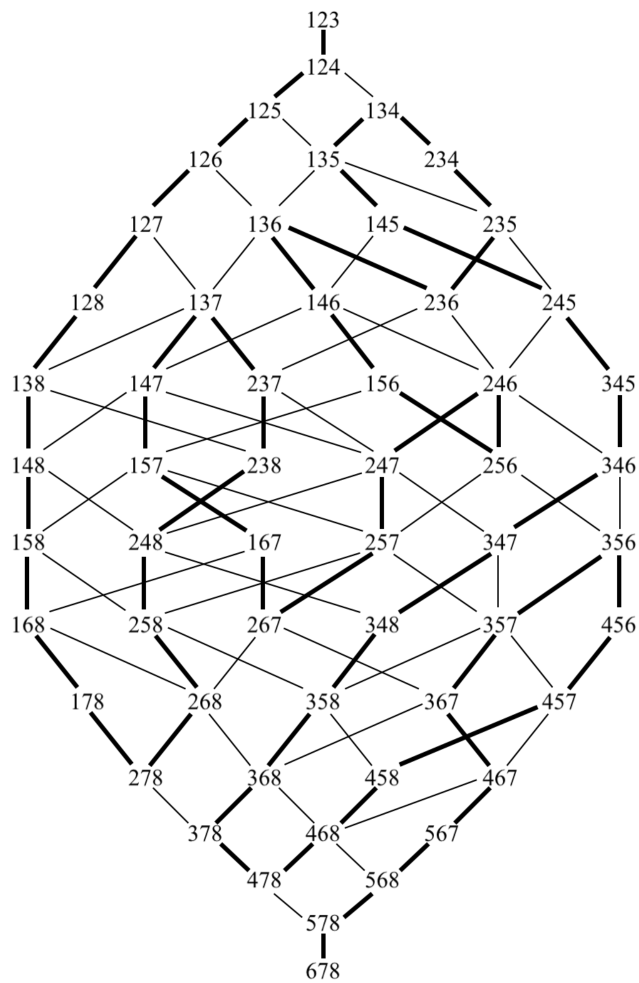

Figure 9.6: Hamiltonian path in the graph $G_{8,3}$.

Finally, we concatenate the Hamiltonian paths for $\hat{A}_{8,3}$, J, and $\hat{B}_{8,3}$ to obtain the Hamiltonian shown in Figure 9.6.

Example 9.4.3. As an application to music composition, a set of n notes can be assigned to the set $\{1, 2, ..., n\}$ so that the Gray code is a progression of k-note chords that uses each chord exactly once. Take $n = 6$ and $k = 3$, and respectively assign the six notes C, D, E, F, G, and A to the numbers 1, 2, 3, 4, 5, and 6. Then, a Hamiltonian path in the graph $G_{6,3}$ is a chord progression in the key C major that contains all of the three note chords such that chords change with minimal movement of notes. This example is particularly pleasing because most of the harmonies are consonant. If we instead used a larger set of notes (say all 12 chromatic tones), there would be many unpleasant dissonances.

CHAPTER 10

FURTHER WORK

We conclude this thesis by presenting two directions of future research that extend and generalize the work we have already done.

10.1 The Hyperoctahedral Group

We have shown that the k-elements of n Gray code corresponds to a Hamiltonian path in the Schreier graph of S_n with respect to the subgroup $S_{k,n-k}$ and the involutions $\{(12), (23), (n-1\ n)\}$. The subgroup $S_{k,n-k}$ is called a *parabolic* subgroup of S_n, and it is the result of removing one involution from the canonical generating set. The first main theorem (Theorem 7.1.1), which concerns the existence of a Hamiltonian path in a Schreier graph for the trivial subgroup, holds not just for S_n, but for a very general class of groups generated by involutions. In this context, the result of our second main theorem (Theorem 9.1.1) raises the question of whether similar results hold for other parabolic subgroups of other finite groups generated by involutions. In this section, we will set up an analogous problem for a parabolic subgroup of the hyperoctahedral group.

For each positive integer n, the *hyperoctahedral* group B_n is the set of signed permutations of $\{1, 2, ..., n\}$, and it is isomorphic to the semidirect product $S_n \ltimes C_2^n$, where S_n acts on C_2^n by permuting coordinates. The group B_n is generated by the set of involutions

$$R = \Big\{ \big(id, (0, ..., 0, 1)\big), \big((12), (0, ..., 0)\big), \big((23), (0, ..., 0)\big), ..., \big((n-1\ n), (0, ..., 0)\big) \Big\},$$

the set of adjacent transpositions and the involution that changes the last digit.

For each $k \in \{1, 2, ..., n\}$, consider the following set:

$$\mathcal{X}_{n,k} := \big\{ (X, f) : X \subseteq \{1, 2, ..., n\}, |X| = k, f : X \longrightarrow \{0, 1\} \big\}.$$

Define a group action of B_n on the set $\mathcal{X}_{n,k}$ as follows: for each $\big(\sigma, (b_1, ..., b_n)\big) \in B_n$ and $(X, f) \in \mathcal{X}_{n,k}$, we define $\big(\sigma, (b_1, ..., b_n)\big)(X, f) = (\sigma(X), g)$, where g is the function

$\sigma(X) \longrightarrow \{0,1\}$ defined $g(\sigma(x)) = f(x) + b_x$. It can be checked that this action is transitive. For each $1 \leq k \leq n$, consider the pair $(Y_k, h_k) \in \mathcal{X}_{n,k}$ such that $Y_k = \{1, 2, ..., k\}$ and $h_k(y) = 0$ for all $y \in Y_k$. Then, the stabilizer of (Y_k, h_k) in B_n is the parabolic subgroup P_k defined as follows:

$$P_k = \begin{cases} \left\langle R - \Big\{ \big((k\ k+1), (0, ..., 0) \big) \Big\} \right\rangle & \text{if } 1 \leq k \leq n-1 \\ \left\langle R - \Big\{ \big(id, (0, ..., 0, 1) \big) \Big\} \right\rangle = S_n & \text{if } k = n \end{cases}$$

Therefore, the orbit-stabilizer theorem tells us that the coset space $P_k \backslash B_n$ is in bijection with $\mathcal{X}_{n,k}$.

In this setting, a Hamiltonian path in the Schreier graph of B_n with respect to the subgroup P_k and involutions R is an ordering of the set $\mathcal{X}_{n,k}$ of signed k-element subsets of $\{1, 2, ..., n\}$ such that successive subsets differ by a single element, swap the signs of two elements, or change the sign of the element n. Musically, if n notes are assigned to the set $\{1, 2, ..., n\}$ and the signs represent two different instrument voices, then such a Gray code is a progression of k-note chords with each of the notes in one of two instruments, such that successive chords change by moving one note, swapping the instruments of two notes in a chord, or switching the instrument of the note designated by n.

In this situation, we pose the question: for which values of n and k does there exist a Hamiltonian path in the Schreier graph $\mathrm{Sch}(B_n, P_k, R)$?

10.2 Arbitrary Chord Permutations

A second potential extension of the work done in this thesis is the study of subgroups of the permutation group of a set of chords that are not necessarily generated by involutions. In this generalized setting, the graphs that represent chords and their connects are *directed* graphs, where edges are ordered pairs of vertices. That is, a permutation may send chord x to chord y, but not necessarily the converse, so directed edges are more appropriate. There are more general definitions of Gray codes and Schreier graphs that account for finite groups of this kind.

Not all of the groups that appear in music theory are generated by involutions. In [8], Hook studies the group of *uniform triadic transformations*, a permutation group of the set of 24 major and minor triads that is isomorphic to the semidirect product $C_2 \ltimes \mathbb{Z}_{12}^2$. This group acts on the set of major and minor triads analogously to the the group $S_4 \ltimes \mathbb{Z}_{12}^4$ defined in Section 8.1. Many subgroups of the group of uniform triadic transformations are not generated by involutions, but still carry musical significance. For example, if we take an element u of $C_2 \ltimes \mathbb{Z}_{12}^2$ that has order greater than 2, and repeated apply u to some triad, then we obtain a perfectly good chord progression. Although many groups found in music theory are generated by involutions, there is still work to be done.

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APPENDIX

SAGE CODE

The following code was used for computations described in Chapter 8. The code can be run by copying and pasting it into `https://sagecell.sagemath.org/` and pressing “Evaluate”.

```
R=Integers(12)
r=matrix(R,5, [[0,1,0,0,5],[1,0,0,0,7],[0,0,0,1,1],[0,0,1,0,11],[0,0,0,0,1]])
s=matrix(R,5, [[0,0,1,0,10],[0,0,0,1,2],[1,0,0,0,2],[0,1,0,0,10],[0,0,0,0,1]])
t=matrix(R,5, [[0,0,0,1,6],[0,0,1,0,0],[0,1,0,0,0],[1,0,0,0,6],[0,0,0,0,1]])

G = MatrixGroup([r,s,t])
print("Order of G:")
print(order(G))

genorders=(order(G.subgroup([r])),order(G.subgroup([s])),order(G.subgroup([t])))
print("Order of r,s,t:")
print(genorders)

e1=vector(R,[1,0,0,0,0])
L=[(g.matrix().column(0),g.matrix()[0,4]) for g in G]
I=L.count((e1,0))
print("Order of the stabiliser of (1,0):")
print(I)
```

BIOGRAPHY OF THE AUTHOR

Isaac Luke Vaccaro was born in New York City on August 30, 1998. He was raised in Kennebunk, Maine, where he graduated in 2016 from Kennebunk High School. He completed a Bachelor of Arts in Mathematics at the University of Maine in 2019, and was matriculated into the graduate school through the "Four Plus" mathematics program. After receiving his degree, Isaac will become a secondary school teacher in mathematics and music. Isaac Luke Vaccaro is a candidate for the Master of Arts degree in Mathematics from the University of Maine in May 2020.