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# The Mathematics of Chinese Checkers

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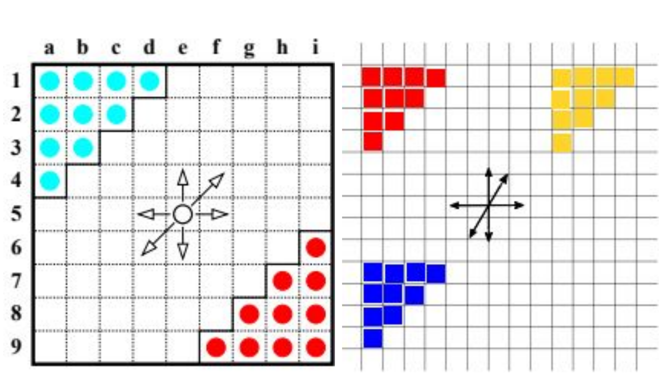
## Abstract

Our goal for this project was to expand and improve upon the findings of George I. Bell and Nicholas Fonseca, who have both written papers on optimization in Chinese Checkers. While their work focuses mainly on cooperative games between one, two, and three players, we have considered games for six players. While doing this, we have redefined the playing board in a more intuitive manner, while developing and proving its associated distance formula. As well, we have conjectured the shortest game for six players, and are working to generalize a formula for the number of moves required to finish a six player game as fast as possible. This could further incite research to generalize a lower bound for any number of players.


## Definitions

- Cell - Position in which a marble may be placed.
- Move - Changing the marble's position from one cell to another. This may be done in three ways:
  - Step - Moving a marble to an adjacent cell. An adjacent cell may be horizontal or diagonal, but not vertical.
  - Jump - Moving a marble by hopping over another marble of any army into an empty cell. This does not capture the other marble, and jumps are not compulsory.
  - Chain - More than one jump in a single turn by the same marble.
- Base - Starting position for an army of marbles.
- Goal - Opposing base, ending position for an army of marbles.
- Man - Single marble owned by a player. This is part of their army.
- Army - Set of ten men owned by a single player.
- Turn - Single move by a player in a round of turns.
- Round - All of the players take one turn.
- Movement Structure - A movement structure allows a player to move along a chain. There is one defined movement structure.
  - Ladder - Marbles that allow for a player to move as a chain.
  - Rung - A single marble that is part of a ladder.



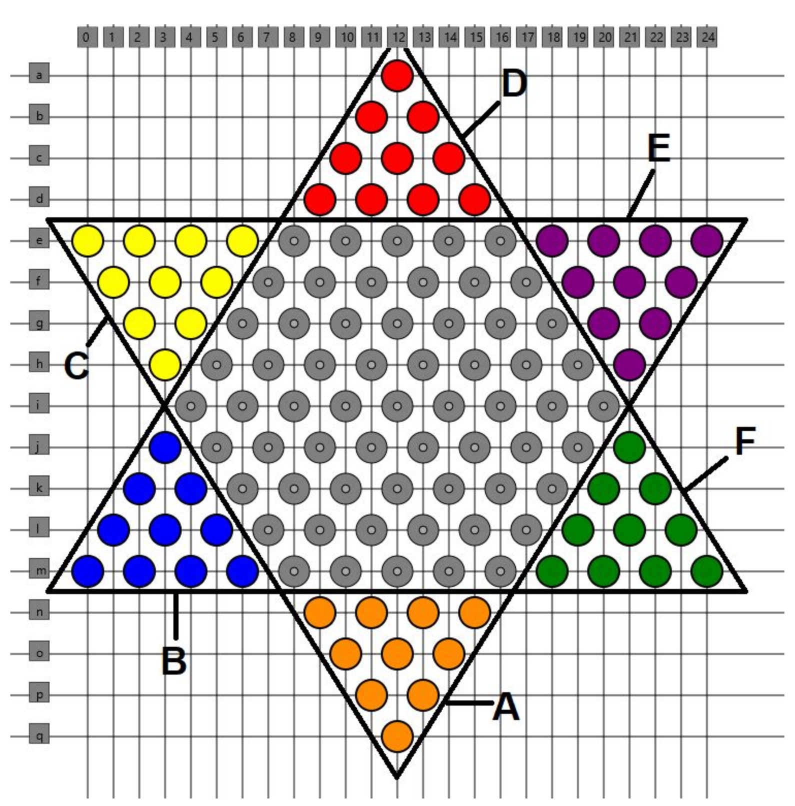



Figure 1 from George I. Bell, "The shortest game of chinese checkers and related problems," Integers 9.1 (2009), 2.
Figure 1 from Nicholas Fonseca, "Optimizing a Game of Chinese Checkers," (2015), 5.

## Graphical Representation

### Previous Board Design

- Cartesian Board
- Movement constricted
  - No movement along $y=-x$
- Chess notation
- More difficult for more players
- Semi-intuitive movement

### Our Board Design

- Cartesian Board overlaid playing board
- Holes in the board (e.g. b12 does not exist as a cell)
- Modified chess notation
- Any number of players
- Intuitive movement
- Identified player bases A-F

## Distance Formula

$$||x,y|| = \begin{cases} |y| + \dfrac{(|x|-|y|)}{2} & \text{if } |x| > |y| \\ |y| & \text{if } |x| \le |y| \end{cases}$$

As previous authors have provided a distance formula for their coordinate system, we had to develop our own.

*Proof.*

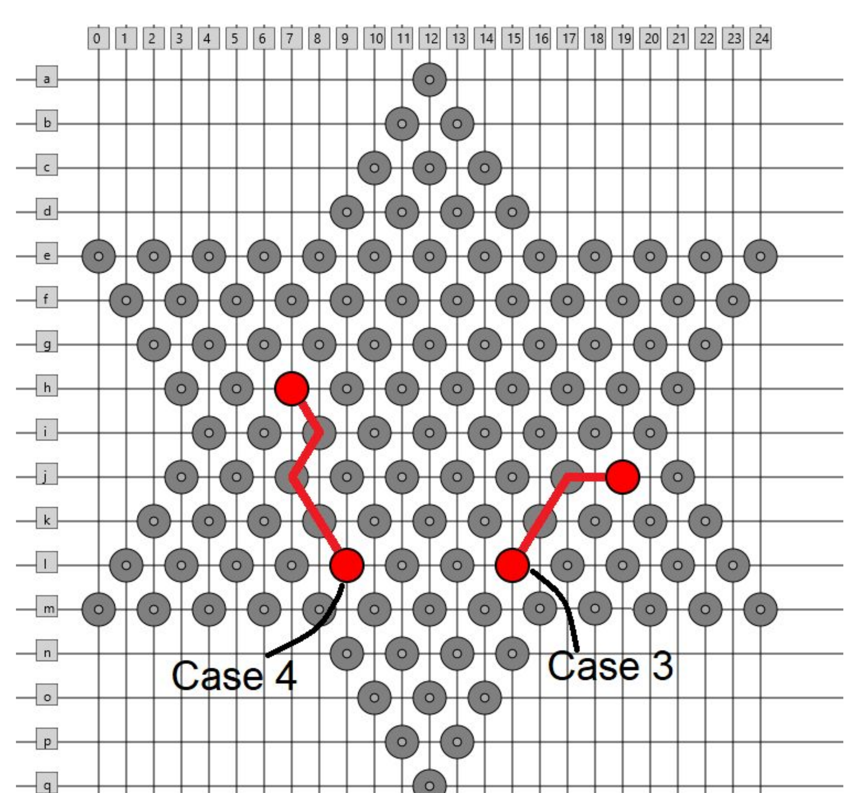
We know that the *x*-coordinates are spaced by 2 while the *y*-coordinates are spaced by 1. We define the *x*-distance to be $|x|$ and the *y*-distance to be $|y|$. Because the rules of Chinese Checkers allow a player to move directly horizontal and/or diagonally, we have four cases to prove:

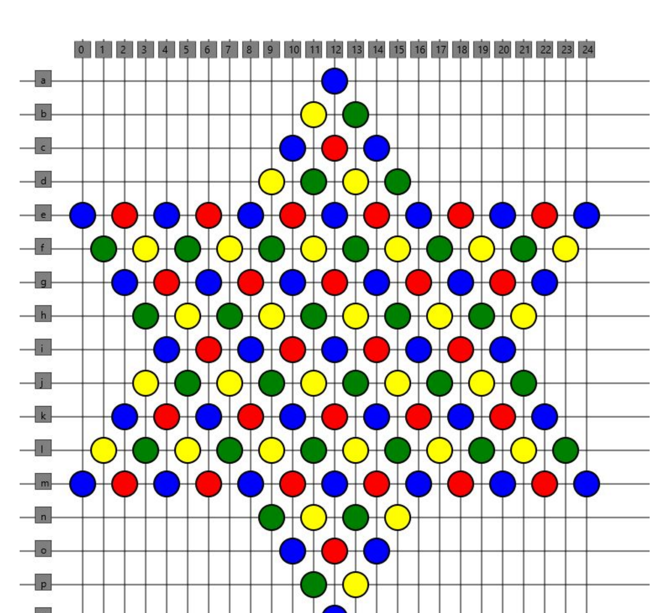
Case 1) Strictly horizontal movements: No vertical moves mean that $|y| = 0$ and $|x| > |y|$. Because the *x*-coordinates are spaced by 2, the number of horizontal steps, *n*, must be half of the *x*-coordinates, say $n = |x|/2$. We may then manipulate $n = |x|/2 = 0 + ½(|x| - 0) = |y| + ½(|x| - |y|)$.

Case 2) Diagonal movements along a single diagonal: Because the piece would be moving diagonally, the *x*-coordinate movement is spaced by 1 since it falls between a strict horizontal movement (of 2 coordinates). Hence, $|x| = |y|$. Thus, the number of steps would be $n = |y|$.

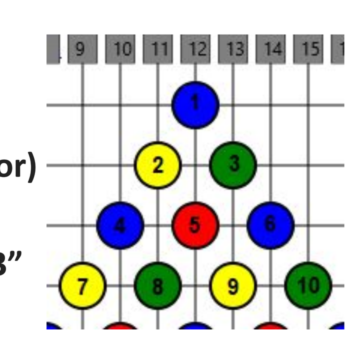
Case 3) Diagonal movements one a single diagonal with horizontal movements: This case can be broken up into two cases previously proven. Because the diagonal movement is on the same diagonal, we apply Case 2, allowing this movement to be described as $|y|$. As with Case 1, we can include horizontal movement as $½|x| = ½|x| - 0 = ½(|x| - |y|)$. Combining the vertical and horizontal cases gives $|y| + 1/2(|x| - |y|)$ again.

Case 4) Diagonal movements of different diagonals: Diagonal movement of different diagonals may be expressed as two distinct movements. Note that when considering a marble at (0, *y*) the first move is not necessary. First, the marble moves along a diagonal from (0,0), or on our graph, i12. This movement will place the marble along a vertical line to (*x*,*y*). This movement is shown to be $|x|$ by Case 2. Then, the marble may move in a zigzag pattern of one step diagonal movements. When the marble zigs, it will increase $|x|$ and $|y|$ by 1. When the marble zags, it will again increase $|y|$ by 1, but will return to its previous *x* position. Note that these zigzags do not need be strictly consecutive, but only that each zig needs an associated zag. This second motion may be defined as $|y| - |x|$, as we have already move the *x* distance. Thus, the full movement is $|x| + (|y| - |x|) = |y|$.



## Location Types



- Four distinct locations (as shown by color)
  - Cannot jump between locations
- Bell calls these locations "man types 0-3"
  - 0-Blue, 1-Green, 2-Yellow, 3-Red
- Bases and their respective goals have equivalent location types
  - Within base (see image): 1,4, and 6 are the same type; 2, 7 and 9 are the same type; 3, 8, and 10 are the same type
- Ladder Building Rules
  - Blue and Red may jump over Green and Yellow
  - Green and Yellow may jump over Blue and Red


## References

Bell, George I. "The shortest game of chinese checkers and related problems." Integers 9.1 (2009): 17-39.

Fonseca, Nicholas. "Optimizing a Game of Chinese Checkers." Honors Thesis (2015).


## Searching for the Lower Bound

### Absolute Lower Bound: $9p + 1$, where *p* is the number of players

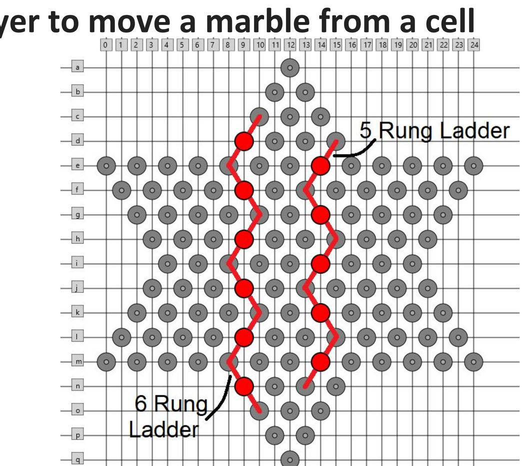
- Assume winning player moves first
- All non-winning players move 9 times
- This requires 9 rounds and 1 additional turn, thus, 9 turns for each player, and an additional turn for the winning player
- Impossible to put piece in goal on first move, so more moves necessary

### Realistic Lower Bound: $9p + c + 1$, where *c* is the number of moves it takes to build a successful chain

Successful chain - a chain that allows the winning player to move a marble from a cell in their starting base to a cell in the goal

Ladder options:

- 5 rung option (10 steps or 5 jumps)
  - Moves piece from row d to row n
  - Requires building 5 rungs
- 6 rung option (12 steps or 6 jumps)
  - Moves piece from row c to row o
  - 2 preexisting rungs from base and goal
  - Requires building 4 rungs




### 6-Player Game

Our formula for the 6-player game provides a realistic lower bound of 63, as c = 8. We have found an example of a 66 move solution, and we conjecture the existence of a 64 move solution. Our current solution follows the current structure:

- Assume Player A is winning player and Player B moves first, moving clockwise
- Players B and C build ladders for blue and red locations
  - Built using green and yellow locations as rungs
- Players E and F build ladders for green and yellow locations
  - Built using blue and red locations as rungs
- Player D empties in the order 4, 1, 6, 9, 2, 7, 5, 10, 3, 8.
  - This order follows the base groupings mentioned in Location Types
- Player A fills its goal in the order 1, 6, 4, 2, 7, 9, 5, 3, 8, 10.
  - The grouping order is the same as for Player D
- Player A empties their base in the order 5, 6, 1, 4, 7, 9, 2, 8, 3, 10.
  - This follows roughly the same grouping order as it fills its goal
  - Note: Location 5 is emptied first in order to fulfill a mandatory move
    - Least disruptive to jump a group of one piece

## *c* vs. *α*

Bell and Fonseca propose *α* as a constant for their lower bound equations, where *α* is the critical move. This move is the first move in which the winning player gets a man in the goal. After this point, the winning player is expected to get a man in the goal with each subsequent move. However, the value *α* requires supposing many correct moves. Thus, the value *c* is more useful, as it provides an easier standard for building ladders for games with more players, while making weaker assumptions. This is of course at the cost of some accuracy in low number player games.

### Previous Formulas

- Norm
  - Bell: $½(|x|+|y|+|x - y|)$
  - Fonseca: $max(|x|, |y|)$
- Lower Bound
  - Bell (two player game): $α + 2(s - 1)$, where *s* is the size of the army
  - Fonseca (three player game): 40

### Our Formulas

- Distance
  - If $|x| > |y|$
    - $½(|x|+|y|)$
  - If $|x| \le |y|$
    - $|y|$
- Lower Bound (for *p* players)
  - $9p + c + 1$

## Future Work

Thus, the challenge for find the lower bound for games with *p* players is in finding the value of *c* for each value of *p*. We have shown that with 6 players this was a relatively easy task. However, with games with fewer players, this becomes somewhat more challenging.