Spring 5-10-2020

Nonlocal Helmholtz Decompositions and Connections to Classical Counterparts

Andrew Haar
University of Nebraska - Lincoln, ahaar@huskers.unl.edu

Petronela Radu
University of Nebraska - Lincoln, pradu@unl.edu

Nonlocal Helmholtz Decompositions and Connections to Classical Counterparts

Andrew Haar[1]

Work supported by UNL UCARE award, Advisor Prof. Petronela Radu, and the UNL Department of Mathematics.

1: University of Nebraska-Lincoln

Introduction

Classical Continuum Mechanics

- Considers only local interactions; phenomena captured only at x
- Based on partial differential equations with differential operators
- Example: classical heat equation

$$u_t(x,t) = \Delta u(x,t)$$

where u is temperature, x is position, t is time and all other relative physical constants are scaled to one.

- Problems when partial derivatives appear in materials with discontinuities! (partially solved by the use of weak derivatives)

The Problem

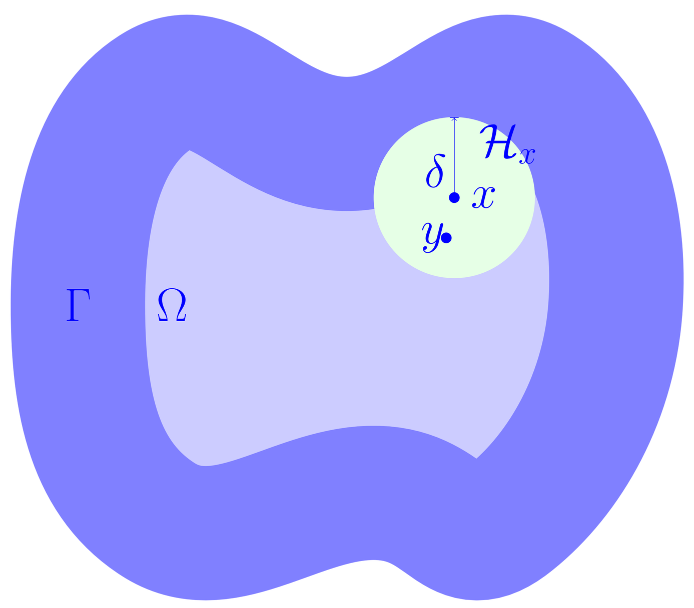
As mentioned in the last bullet point above, problems begin to arise with discontinuities especially when they evolve in time. Common examples of this happening include

- evolution of cracks (very bad; **two** derivatives on discontinuous functions) [3]
- phase-separation model (a binary discontinuous function, marking the two phases is approximated by a smooth profile)
- image processing (transition from white to black is discontinuous; "fixing" such images should not blur contour lines)

An Alternative to Classical Continuum Mechanics

- Peridynamics aims to more effectively model phenomena involving material discontinuities [4]
- Replaces differential operators with (weakly singular) integral operators, hence one can deal with spatial discontinuities
- Interactions occur through bonds on a horizon, $\mathcal{H}_x$, of radius δ, between all points x and its "neighboring" points, y, that are within the horizon.
- These interactions occur inside a body, Ω, with specified boundary conditions on the surrounding collar, Γ. [1, 2]

Below is a domain, Ω, with boundary conditions specified in the surrounding collar, Γ. For a point x in the domain, all points, y, within the specified horizon, δ interact with the point x.



Our Work

Definitions

We have defined three new operators that are analogous to the common local operators in vector calculus.

Definition We define the dot convolution operator $*.$. Suppose $\mathbf{f}, \mathbf{g}$ are vector valued functions. Then

$$\mathbf{f} *. \mathbf{g} := \int_{\mathbb{R}} \mathbf{f}(\tau) \cdot \mathbf{g}(t-\tau)\, \mathrm{d}\tau$$

Definition We define the cross convolution operator $*_\times$. To compute $\mathbf{f} *_\times \mathbf{g}$, compute a cross product, but instead of normal multiplication, use convolution, i.e.

$$\mathbf{f} *_\times \mathbf{g} := \begin{pmatrix} f_2 * g_3 - g_2 * f_3 \\ g_1 * f_3 - f_1 * g_3 \\ f_1 * g_2 - g_1 * f_2 \end{pmatrix}$$

We can now define our new operators based on these new types of convolution.

Nonlocal Gradient We define the nonlocal gradient of a scalar function u with respect to the kernel $\boldsymbol{\alpha}$ to be

$$\mathcal{G}_{\boldsymbol{\alpha}}[u] := \boldsymbol{\alpha} * u$$

Nonlocal Divergence We define the nonlocal divergence, with kernel $\boldsymbol{\alpha}$, of a vector function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^m$ to be

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}] := \boldsymbol{\alpha} *. \mathbf{v} = \alpha_1 * v_1 + \alpha_2 * v_2 + \alpha_3 * v_3$$

Nonlocal Curl We define the nonlocal curl with kernel $\boldsymbol{\alpha}$ of a function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^3$ to be

$$\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{v}] := \boldsymbol{\alpha} *_\times \mathbf{v}$$

Results

Nonlocal Helmholtz Decomposition Suppose $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^3$. Then there exist functions Φ and $\mathbf{A}$ such that

$$\mathbf{F}(\mathbf{x}) = -\mathcal{G}(\Phi) + \mathcal{C}(\mathbf{A})$$

In the local case, this says that we can decompose a vector function into a solenoidal and a curl-free component. In other words, a volume-preserving and an irrotational component. In addition to our main result, we have proven some propositions that begin to show the connections between our nonlocal operators.

Curl of the Gradient is 0 Consider a scalar function $f \in L^1(\mathbb{R})$. We have

$$\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[f]] = \mathbf{0}$$

Divergence of the Curl is 0 Consider a vector function $\mathbf{f} \in L^1(\mathbb{R}^3)$. We have

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] = 0$$

Remarks

As an intuition for defining our nonlocal operators in this way, we can consider the Dirac mass. Note that derivatives can be freely transferred between two functions in a convolution (i.e. $\partial f * g = f * \partial g$) and that the Dirac mass is the identity for convolution. It also, in a sense, allows us to look at these operators as they become local, because, at least intuitively, the Dirac mass represents the point at which the horizon of interaction is zero; that is, where we have a local operator.

So by letting $\alpha = \partial \delta_0$ we can transfer the derivative to the function we are operating on and we are left with the local counterparts of our operators. Using this, we can use our proof of the nonlocal Helmholtz decomposition to prove the local version, so we hope to see convergence of the operators. This does not, however, give any inkling of the rate of convergence.

Conjectures and Future Work

The source of many of our conjectures is seeing the connections between the nonlocal and the local operators.

(i) If the horizon of interaction is a ball of radius δ, then we hope to see convergence in some norm (likely the L^2 norm) at a rate of δ^2 for all three of our operators.

(ii) If $\mathcal{C}[\mathbf{f}] = \mathbf{0}$ then there exists some function h such that $\mathcal{G}[h] = \mathbf{f}$. This is the converse of the first identity above.

(iii) Investigations of other identities of the local operators. Some examples of this would be Green's identities and integration by parts.

(iv) We plan to begin doing numerical analysis of these operators to gain insight on their geometric interpretation. This could also possibly lead to conjectures and properties that are unique to the nonlocal operators.

These next steps are aimed at understanding the properties of the new operators we have defined. Once we have a better understanding of them, we can begin to apply them by creating models and potentially obtaining solutions to nonlocal boundary value problems set in the framework of these new operators.

References

[1] M. D'Elia, C. Flores, X. Li, P. Radu, and Y. Yu. Helmholtz-hodge decompositions in the nonlocal framework. well-posedness analysis and applications. *Journal of Peridynamics and Nonlocal Modeling*, 2020.

[2] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume–constrained problems, and nonlocal balance laws. *Mathematical Models and Methods in Applied Sciences*, 23(03):493–540, 2013.

[3] Y. D. Ha and F. Bobaru. Studies of dynamic crack propagation and crack branching with peridynamics. *International Journal of Fracture*, 162(1-2):229–244, 2010.

[4] S. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48:175–209, 2000.