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A note on the fine structure constant

Bilal Khan
University of Nebraska-Lincoln, bkhan2@unl.edu

Irshadullah Khan
Quaid-i-Azam University, Islamabad, Pakistan, irshadk2@gmail.com

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Bilal Khan*
Department of Computer Science and Sociology
University of Nebraska-Lincoln
Lincoln, Nebraska, USA
bkhan2@unl.edu

Irshadullah Khan†
Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
irshadk2@gmail.com

Abstract

We derive the numerical value of the fine structure constant α in purely number-theoretic terms, under the assumption that in a system of charges between two parallel conducting plates, the Casimir energy and the mutual Coulomb interaction energy agree.

In 1916, based on Michelson and Morley's precise measurement of the hydrogen atom spectrum [1], Arnold Sommerfeld extended the Bohr model to include elliptical orbits and relativistic dependence of mass on velocity, introducing the term α for the fine-structure constant [2]. While the first physical interpretation of α was as the ratio of the velocity of the electron in the first circular orbit of the relativistic Bohr atom to the speed of light in a vacuum, the physical interpretations of α have widened greatly since that time.

As a dimensionless constant which does not appear to be directly related to any mathematical constants, the fine-structure constant has long fascinated physicists. Arthur Eddington argued that the value could be "obtained by pure deduction" and he related it to the Eddington number, his estimate of the number of protons in the universe [3], conjecturing in 1929 that $\alpha = 1/137$ precisely [4]. Wolfgang Pauli reportedly collaborated with psychoanalyst Carl Jung on the possible significance of α^{-1} [5], and Max Born is said to have believed that if the value of α were different, "our task to disentangle the natural laws would be hopelessly difficult" [6]. By the 1940s, however, experimental values for α^{-1} deviated sufficiently from 137 to refute Eddington's argument [7]. Richard Feynman, one of the originators of the theory of quantum electrodynamics (QED), referred to the fine-structure constant in these terms:

> There is a most profound and beautiful question associated with the observed coupling constant, e – the amplitude for a real electron to emit or absorb a real photon. It is a simple number that has been experimentally determined to be close to 0.08542455. (My physicist friends won't recognize this number, because they like to remember it as the inverse of its square: about 137.03597 with about an uncertainty of about 2 in the last decimal place. It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it.) Immediately you would like to know where this number for a coupling comes from: is it related to pi or perhaps to the base of natural logarithms? Nobody knows. It's one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the "hand of God" wrote that number, and "we don't know how He pushed his pencil." We know what kind of a dance to do experimentally to measure this number very accurately, but we don't know what kind of dance to do on the computer to make this number come out, without putting it in secretly! [8]

In this note, we will argue that (modulo a clearly stated and reasonable hypothesis), the theoretical value of the inverse of the fine structure constant is given by the expression:

$$\alpha^{-1} \quad = \quad \frac{6}{\zeta(-3)\cdot\pi^2}\cdot\left(\frac{2}{5}(1+\sqrt{2}-2\sqrt{3})-\frac{2}{3}\pi-6\log 2+2\log(1+\sqrt{2})+12\log(1+\sqrt{3})-4\log(2+\sqrt{3})\right)$$

where $\zeta(-3) = 1/120$ based on the analytic continuation of the Riemann zeta function to negative values.

*See http://transdiscipline.com

†Visiting Faculty under the UNDP TOKTEN programme.; https://www.researchgate.net/profile/Irshadullah_Khan/research

Imagine two parallel, square conducting plates that are aligned so that they form opposite faces of a rectangular parallelopiped. The separation between the plates is $a > 0$. Each of the two plates has sides of length $\ell = na$, where n is a positive integer. For purposes of analysis, we mark out an $n \times n$ grid on each plate, forming n^2 subsquares, with each subsquare having sides of length a. Now each pair of corresponding marked subsquares (on the two respective plates) define opposite faces of a cube $\mathcal{C} \cong [0, a]^3$ having volume a^3. There are n^2 such cubes which, together, fill the space between the plates. The system is illustrated in Figure 1.

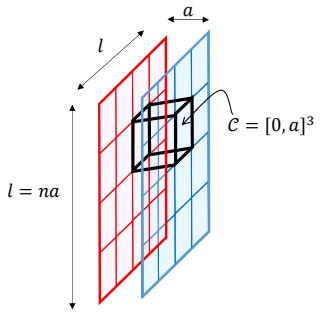

Figure 1: Two parallel plates.

We assume the charge on the electron is $-e$.

Imagine that *each* of the aforementioned n^2 cubes contains both one negative charge $(-e)$ and one positive charge (e). More specifically, suppose that each of the two charges has been uniformly distributed across the cube without mutual annihilation. In the Semi-Classical Model (SCM), smearing $-e$ across a cube ensures that each point (x, y, z) is associated with a charge density

$$\rho_-(x, y, z) = -e/a^3.$$

Similarly, smearing $+e$ across a cube ensures that each point (x', y', z') within it is associated with a charge density

$$\rho_+(x', y', z') = +e/a^3.$$

In the SCM, charges $-e$ and $+e$ are not associated with any spin.

The mutual Coulomb interaction energy E for the aforementioned system of charges between two plates is calculated as follows: For point charges, it can be shown that E is the sum of mutual interaction energies of distinct pairs of charges. In the system being discussed, each of the n^2 cubes has a charge $+e$ and a charge $-e$ smeared across it. Consider the positive charge distribution inside one particular cube, C_0. This can be paired with (i) the negative charge distribution inside the *same* cube C_0, or (ii) the *positive* charge distributed inside a *different* cube C_1; or (iii) the *negative* charge distribution inside a different cube C_1. The contributions of the pairings of type (ii) and (iii) mutually cancel within the calculation of E. The interaction energy E thus reduces to n^2 times U, where U is the mutual interaction energy of the positive and negative charge distributions inside a *single* cube. It is straightforward to see that

$$U = -\frac{1}{4\pi\epsilon_0} \cdot \left(\frac{e}{a^3}\right)^2 \cdot \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} \int_{x'=0}^{a} \int_{y'=0}^{a} \int_{z'=0}^{a} \frac{dx \cdot dy \cdot dz \cdot dx' \cdot dy' \cdot dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \tag{1}$$

where ϵ_0 is the absolute dielectric permittivity of classical vacuum. All variables in Equation (1) are in SI units, i.e. U is in Joules, a is in meters, e is in Coulombs, and ϵ_0 is in $C^2 m^{-1} J^{-1}$. Performing a change of variables $x = au$, $y = av$, $z = aw$, $x' = au'$, $y = av'$, and $z = aw'$, the mutual interaction energy can be re-expressed as follows

$$U = -\frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{a} \cdot \int_{u=0}^{1} \int_{v=0}^{1} \int_{w=0}^{1} \int_{u'=0}^{1} \int_{v'=0}^{1} \int_{w'=0}^{1} \frac{du \cdot dv \cdot dw \cdot du' \cdot dv' \cdot dw'}{\sqrt{(u-u')^2 + (v-v')^2 + (w-w')^2}} \tag{2}$$

The value of the 6-dimensional integral appearing in (2) is known as $\Delta_3(-1)$ in prior work on "box integrals"[1] by Bailey, Borwein, and Crandall [9]. Indeed, the closed form value of $\Delta_3(-1)$ is known:

$$\begin{aligned} \Delta_3(-1) &= \frac{2}{5}(1 + \sqrt{2} - 2\sqrt{3}) - \frac{2}{3}\pi - 6\log 2 + 2\log(1+\sqrt{2}) + 12\log(1+\sqrt{3}) - 4\log(2+\sqrt{3}) \\ &\approx 1.88231264438966016010560038868367587849\ldots \end{aligned} \tag{3}$$

It follows that

$$U = -\frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{a} \cdot \Delta_3(-1) \tag{4}$$

and so

$$E = -\frac{n^2}{a} \cdot \frac{1}{4\pi\epsilon_0} \cdot e^2 \cdot \Delta_3(-1) \tag{5}$$

[1] This cited work generalizes hypercube line picking and the derivation of Robbin's constant, and is not to be confused with the "box integrals" of particle physics.

Separately, we know that the Casimir energy due to quantum fluctuations of the electromagnetic ground state between two parallel plates of area A and separation a is (in the limit $A \to \infty$) given by

$$E_{\text{Casimir}} \quad = \quad -\hbar c \cdot \frac{\pi^2}{6} \cdot \zeta(-3) \cdot \frac{1}{a^3} \cdot A \tag{6}$$

where $\zeta(-3) = 1/120$ based on the analytic continuation of the Riemann zeta function to negative values of the argument. In the system being discussed $A = a^2 n^2$, and so, by substituting this into expression (6) we conclude that in our system

$$E_{\text{Casimir}} \quad = \quad -\frac{n^2}{a} \cdot \frac{\hbar c \pi^2}{720} \tag{7}$$

Let us *hypothesize* that in the limit of large n, the mutual interaction energy of positive and negative charge distributions in our system is equal to the Casimir energy. From this hypothesis that $E_{\text{Casimir}} = E$, it follows that

$$-\frac{n^2}{a} \cdot \frac{\hbar c \pi^2}{720} \quad = \quad -\frac{n^2}{a} \cdot \frac{1}{4\pi\epsilon_0} \cdot e^2 \cdot \Delta_3(-1) \tag{8}$$

wherein $n^2/a \neq 0$ cancels on both sides. From the equality (8), we may deduce the inverse of the fine structure constant directly:

$$\alpha^{-1} \quad \stackrel{\text{def}}{=} \quad \frac{4\pi\epsilon_0 \hbar c}{e^2} \tag{9}$$

$$= \quad \frac{6 \cdot \Delta_3(-1)}{\zeta(-3) \cdot \pi^2} \tag{10}$$

$$= \quad \frac{6}{\zeta(-3) \cdot \pi^2} \cdot \left(\frac{2}{5}(1 + \sqrt{2} - 2\sqrt{3}) - \frac{2}{3}\pi - 6\log 2 + 2\log(1 + \sqrt{2}) + 12\log(1 + \sqrt{3}) - 4\log(2 + \sqrt{3}) \right) \tag{11}$$

$$\approx \quad 137.31706448243941647438631497184952683 83\ldots \tag{12}$$

The 2018 CODATA [10] experimental value for α^{-1} is 137.035999084(21), which is $\approx 0.2\%$ lower than our closed form derivation in (12). Theoretical explanations of the 0.2% divergence will be given in a subsequent article (see acknowledgements section below for details).

Acknowledgements

The first author (Bilal Khan, Ph.D.) is trained as a mathematician (group theorist), and is the son and collaborator of the second author (Irshadullah Khan) who was a physicist and a former Ph.D. advisee of P. W. Higgs in 1965. The second author passed away on January 6, 2020, leaving behind a significant collection of hand-written unpublished manuscripts, notes, and diaries. The first author is attempting to synthesize the second author's writings, and this preprint is one outcome of that process. Given that the first author is not a physicist by training, he would like to appeal for assistance from the physics community.

The discovery of prior work by Bailey, Borwein, and Crandall was made by the first author in the process of developing this paper. The original manuscripts and notes by Irshadullah Khan contain independent lengthy derivations based on Equation (1), alongside numerical computations using the Maple software package. The second author's original approach (which might be of independent interest) was omitted from this presentation, in view of the availability of a closed form expression for $\Delta_3(-1)$.

The second author's manuscripts, notes, and diaries contain extensive treatment of the implications of the 0.2% discrepancy, e.g. *"Assuming the hypothesis is correct, it follows that the value presently assigned to the fine structure constant is incorrect. It requires a small correction which has the important consequence that the usually accepted mechanism for Lamb shift based on quantum electrodynamics is incomplete. There is a small residual Lamb shift which can not be accounted for without introducing new elements into our currently accepted picture of the hydrogen atom"* (Irshadullah Khan, 2009). The first author hopes that, with the help of the physics community, the second author's arguments regarding these assertions can be the subject of followup submissions to the ArXiv.

References

[1] A. A. Michelson and E. W. Morley. *Amer. J. Sci*, 34:427, 1887.

[2] Arnold Sommerfeld. *Atombau und Spektrallinien*. Braunschweig: Friedrich Vieweg und Sohn, 1919.

[3] A. S. Eddington. The constants of nature. *The World of Mathematics*, 2:1074–1093, 1956.

[4] Edmund Whittaker. Eddington's theory of the constants of nature. *The Mathematical Gazette*, 29(286):137–144, 1945.

[5] P. Varlaki, L. Nadai, and J. Bokor. Number archetypes and background control theory concerning the fine structure constant. *Acta Polytechnica Hungarica*, 5(2), 2008.

[6] A. I. Miller. *Deciphering the Cosmic Number: The Strange Friendship of Wolfgang Pauli and Carl Jung*. W.W. Norton & Co, 2009.

[7] H. Kragh. Magic number: A partial history of the fine-structure constant. *Archive for History of Exact Sciences*, 57 (5):395–431, 2003.

[8] Richard P. Feynman. *QED: The Strange Theory of Light and Matter*. Princeton University Press, 1985.

[9] D. Bailey, J. Borwein, and R. Crandall. Advances in the theory of box integrals. *Mathematics of Computation*, 79(271):1839–1866, 2010.

[10] Fundamental Physics Constants: inverse fine-structure constant. `https://physics.nist.gov/cgi-bin/cuu/Value?alphinv`. Accessed: 2020-04-29.