MOTIVIC INFORMATION

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Abstract. We introduce notions of information/entropy and information loss associated to exponentiable motivic measures. We show that they satisfy appropriate analogs to the Khinchin-type properties that characterize information loss in the context of measures on finite sets.

In memory of Paolo de Bartolomeis

1. Introduction

I was invited to contribute a paper to a volume of the Bulletin of the Italian Mathematical Society dedicated to the memory of Paolo de Bartolomeis. I met Paolo during my postdoc years at MIT, while he was visiting Gang Tian. Since that time, he has always been a nice and generous friend, and I regret the fact that we no longer had occasions to see each other in recent years: after the main focus of my own research shifted away from the area of differential geometry we no longer frequented the same conferences and the occasions to meet professionally became much more sporadic. I was deeply saddened by the news of his untimely death this year. In thinking about a possible contribution to this volume, I decided to avoid the typically more formal style of mathematical papers, which seemed to me a bit too dry for the occasion, and I settled instead for a more freely flowing collection of thoughts, somewhat speculative in nature, revolving around the ideas of entropy and information loss, revisited in the context of motivic measures.

1.1. **Entropy and information.** The relation between Entropy and Information is one of the fundamental ideas of contemporary science, introduced by Shannon in the first extensive mathematical account of the theory of information and communication, [27]. The Shannon entropy detects the information content of a probability measure and constrains the amount of information that can be transmitted on a channel, in terms of a bound on data compression. In the simplest case of a probability measure $P = (P_i)$ on a finite set of cardinality n, the Shannon entropy is given by

$$S(P) = -\sum_{i=1}^{n} P_i \log P_i. \tag{1.1}$$

There is an axiomatic characterization of the Shannon entropy given by the Khinchin axioms [15], reformulated in a more coincise way by Faddeev [9]: continuity with a maximum at equidistribution, additivity over subsystems $S(A \cup B) = S(A) + S(B|A)$, and expansibility (a compatibility for changing n by restriction to the faces of the simplex of probability measures) suffice to characterize $S(P)$ completely up to a multiplicative constant $C > 0$.

Recently, the axiomatic characterization of the Shannon entropy was reinterpreted in modern categorical terms in [2], [17], [21], [20]. In particular, we are interested here in the notion of information loss for morphisms of finite sets with probability measures and its axiomatic characterization discussed in [2], which we will review briefly in §4.1.

1.2. **Information loss in the Grothendieck ring of varieties.** Our goal in this paper is to propose an information theoretic point of view in the context of motivic measures, where we are interested in quantifying phenomena of "information loss", associated to morphisms of algebraic varieties. Motivic measures are meant here as ring homomorphism from the Grothendieck ring of varieties to various other rings (the integers in the case of the Euler characteristic, or a polynomial ring in the case of the Poincaré polynomial, etc.). In particular, the motivic Euler characteristic is the ring homomorphism of Gillet–Soulé [10] mapping the Grothendieck ring of varieties to the Grothendieck ring of Chow motives.

The structure of the Grothendieck ring of varieties is very subtle, with phenomena such as the existence of zero-divisors, including the Lefschetz motive, [5], [22], [24] only recently uncovered. Motivic measures can be seen as ways to probe the structure of the Grothendieck ring, by mapping it to various kinds of "Euler characteristic type" invariants.

Within this general framework we think it is interesting to consider possible notions of information associated to the evaluation of a motivic measure on a given variety or motive and information loss associated to morphisms.

2. Motivic Measures, Integration, and Motivic Information

2.1. **Hasse-Weil information function.** For a variety X over a finite field $\mathbb{F}_q$, the Hasse–Weil zeta function is given by the (exponential) generating function for the number of points of X over the field extensions $\mathbb{F}_{q^m}$,

$$Z(X,t) = \exp\left(\sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m \right).$$

For a variety X defined over $\mathbb{Z}$ with reductions X_p at the primes p, the associated L-function is defined as

$$L(X,s) = \prod_p Z(X_p, p^{-s}).$$

It is convenient to write the Hasse–Weil zeta function in the equivalent form

$$Z(X,t) = \prod_x (1 - t^{\deg(x)})^{-1},$$

where the product is over the set of closed points of X and $\deg(x) = [k(x) : \mathbb{F}_q]$ with $k(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$ at x. Indeed, by writing $\#X(\mathbb{F}_{q^m}) = \sum_{r|m} r\, a_r$ with $a_r = \#\{x : [k(x) : \mathbb{F}_q] = r\}$, one obtains

$$Z(X,t) = \prod_{r\geq 1}(1-t^r)^{-a_r}.$$

Equivalently, for $\alpha = \sum_i n_i x_i$ effective zero-cycles with $n_i \in \mathbb{Z}_{\geq 0}$ and x_i closed points of X, one can write

$$Z(X,t) = \sum_\alpha t^{\deg(\alpha)},$$

where $\deg(\alpha) = \sum_i n_i \deg(x_i)$.

It is natural, if one regards the Hasse-Weil zeta function as a motivic measure, as in [25], [26], to associate to it an information function of the form

$$H(X,t) := -\sum_\alpha t^{\deg(\alpha)} \log(t^{\deg(\alpha)}). \tag{2.1}$$

This expression occurs naturally if we write the Shannon entropy for a distribution of the form

$$P(\alpha) := \frac{t^{\deg(\alpha)}}{Z(X,t)} \tag{2.2}$$

over the set of degree zero effective cycles α in X, that is, the quantity $t^{\deg(\alpha)}/Z(X,t)$ is the relative weight assigned by the zeta function to a degree zero effective cycle α in X.

Definition 2.1. *For a variety X over a finite field $\mathbb{F}_q$, the local Hasse–Weil entropy is defined as the Shannon entropy of the distribution $P = (P(\alpha))$ of* (2.2) *on degree zero effective cycles,*

$$S(X) := -\sum_\alpha P(\alpha)\log(P(\alpha)) = \log Z(X,t) + Z(X,t)^{-1} H(X,t). \tag{2.3}$$

In the classical Shannon entropy case, for a product distribution PQ one has

$$S(PQ) = -\sum_i \sum_j P_i Q_j \log(P_i Q_j) = -\sum_i P_i \log(P_i) - \sum_j Q_j \log(Q_j),$$

that is, the usual additivity property for independent systems.

Thus, in the case of a variety X over $\mathbb{Z}$ one can consider the reductions X_p at the various primes, with the corresponding Hasse–Weil zeta functions, as independent systems and assign to X an information function of the form

$$H_{\mathbb{Z}}(X,s) := \sum_p Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}). \tag{2.4}$$

This corresponds to a distribution $P(\alpha) = \prod_p P(\alpha_p)$ with

$$P(\alpha_p) = \frac{p^{-s\deg(\alpha_p)}}{Z(X_p, p^{-s})}. \tag{2.5}$$

Definition 2.2. *For a variety X over $\mathbb{Z}$, the global Hasse–Weil entropy is the Shannon entropy of the distribution* (2.5),

$$S(X) := \sum_p Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) + \sum_p \log Z(X_p, p^{-s}) = H_{\mathbb{Z}}(X, s) + \log L(x, s). \tag{2.6}$$

In both (2.3) and (2.6) we see that the Shannon entropy consists of a term of the form $\log Z(X,t)$ or $\log L(X,s)$ and a term of the form $H(X,t)$ normalized by the zeta function. In fact, the Hasse–Weil entropy can be completely described in a simple form in terms of the logarithm of the arithmetic L-function.

Proposition 2.3. *The Hasse–Weil entropy* (2.6) *is given by*

$$S(X) = \log L(X,s) + s \sum_p \log(p) \sum_{m\geq 1} \#X_p(\mathbb{F}_{p^m}) p^{-sm}.$$

The latter term can be equivalently written as $s\frac{d}{ds}\log L(X,s)$, so that

$$S(X) = (1 - s\frac{d}{ds}) \log L(X,s). \tag{2.7}$$

Proof. The term $H(X_p, p^{-s})$ is simply

$$H(X_p, p^{-s}) = s\log(p) \sum_\alpha p^{-s\deg(\alpha)} \deg(\alpha) = s\log(p)(t\frac{d}{dt} Z(X_p, t))|_{t=p^{-s}}$$

$$Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) = s\log(p)(tZ(X_p,t)^{-1}\frac{d}{dt} Z(X_p,t))|_{t=p^{-s}}.$$

For a generating function $G(t) = \exp(\sum_r c_r \frac{t^r}{r})$ in exponential form, one has $t\frac{1}{G}\frac{dG}{dt} = t\frac{d\log G}{dt} = \sum_r c_r t^r$. This operation corresponds to passing to ghost components in the Witt ring, as we discuss below. Thus, we obtain

$$Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) = s\log(p) \sum_{m\geq 1} \#X_p(\mathbb{F}_{p^m}) p^{-sm}.$$

We have

$$\frac{d}{ds} L(X,s) = \frac{d}{ds}\prod_p Z(X_p, p^{-s}) = \sum_p \frac{d}{ds} Z(X_p, p^{-s}) \cdot \prod_{\ell\neq p} Z(X_\ell, \ell^{-s})$$

$$= \sum_p Z(X_p, p^{-s})^{-1} \frac{d}{ds} Z(X_p, p^{-s}) \cdot L(X,s) = L(X,s) \cdot \sum_p \frac{d}{ds} \log Z(X_p, p^{-s}).$$

This gives

$$\frac{d}{ds}\log L(X,s) = \sum_p \frac{d}{ds}\log Z(X_p, p^{-s}) = -\sum_p \log(p) t\frac{d}{dt}\log Z(X_p,t)|_{t=p^{-s}}.$$

Thus, we obtain

$$Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) = -s\frac{d}{ds}\log L(X,s)$$

Thus, we obtain the simpler expression for the Hasse–Weil entropy of the form (2.7). □

The explicit $\log(p)$ factors can be absorbed into a change of basis, using base p logarithm in the expression for the entropy local factor $H(X_p, p^{-s})$.

2.1.1. *Hasse-Weil entropy of a point.*

Example 2.4. *For $X_p = \mathrm{Spec}(\mathbb{F}_p)$ the Hasse-Weil entropy* (2.6) *is given by*

$$S(\mathrm{Spec}(\mathbb{F}_p)) = (1 - s\frac{d}{ds}) \log \zeta(s), \tag{2.8}$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. This is immediate from Proposition 2.3. It can also be seen by direct computation as follows. For $X_p = \mathrm{Spec}(\mathbb{F}_p)$ we have $Z(\mathrm{Spec}(\mathbb{F}_p), p^{-s}) = (1 - p^{-s})^{-1}$ and $L(X, s) = \zeta(s) = \prod_p (1 - p^{-s})^{-1}$. Thus we have

$$Z(\mathrm{Spec}(\mathbb{F}_p), p^{-s})^{-1} H(\mathrm{Spec}(\mathbb{F}_p), p^{-s}) = \frac{s \log(p) p^{-s}}{(1 - p^{-s})}.$$

Thus, in this case the first term in the Shannon entropy (2.6) is given by

$$\sum_p Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) = s \sum_p \frac{\log(p) p^{-s}}{(1 - p^{-s})} = s \sum_p \log(p) \sum_{k \geq 1} p^{-ks} = s \sum_n \Lambda(n) n^{-s}$$

where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log(p) & n = p^k, \ k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\sum_p Z(X_p, p^{-s})^{-1} H(X_p, p^{-s}) = -s \, \frac{\zeta'(s)}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta function. The second term in (2.6) is simply given by $\log L(X, s) = \log \zeta(s)$. Thus, the Hasse–Weil entropy in this case is given by $\log \zeta(s) - s(\log \zeta(s))'$. □

In Quantum Statistical Mechanics, given a system with partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$, the entropy can be computed as the function $S = \frac{\partial}{\partial T}(T \log Z)$, where $T = 1/\beta$ is the temperature parameter. This is the same as

$$S = (1 - \beta \frac{\partial}{\partial \beta}) \log Z(\beta),$$

expressed in terms of the inverse temperature β. Thus, we see that the computation of the Hasse–Weil entropy of a point given in Lemma 2.4 is exactly the thermodynamical entropy of a quantum statistical mechanical system that has the Riemann zeta function as partition function. It is well known that the Riemann zeta function admits an interpretation as partition function in Quantum Statistical Mechanics, either in terms of the simpler "Riemann gas" system of [13], [29], or in terms of the more refined Bost–Connes system [6] (see also [7]).

2.1.2. *Hasse–Weil entropy of affine spaces.*

Example 2.5. *For $X = \mathbb{A}^n$ the Hasse–Weil entropy is given by*

$$S(\mathbb{A}^n) = \log \zeta(s-n) + s \sum_p \log(p) \frac{p^{-(s-n)}}{1-p^{-(s-n)}} = (1 - s\frac{d}{ds}) \log \zeta(s-n). \tag{2.9}$$

Proof. For $X = \mathbb{A}^n$ we have $Z(X_{\mathbb{F}_q}, t) = (1-q^n t)^{-1}$ and $L(\mathbb{A}^n, s) = \prod_p (1-p^{-s+n})^{-1} = \zeta(s-n)$. Thus, the Hasse–Weil entropy is given by (2.9). □

Thus, the effect of passing from a point to an affine space $\mathbb{A}^n$ is simply a shift in the inverse temperature variable $\beta \mapsto \beta - n$ of the quantum statistical mechanical system, namely one obtains the entropy of a system with partition function $Z_n(\beta) = Z(\beta - n)$. As n grows large, this system captures the thermodynamical properties of the original systems at increasingly low temperatures, that is, for inverse temperatures $\beta > n$.

2.1.3. *Hasse–Weil entropy of projective spaces.*

Example 2.6. *For $X = \mathbb{P}^n$ the Hasse–Weil entropy is given by*

$$S(\mathbb{P}^n) = (1 - s\frac{d}{ds}) \prod_{m=0}^{n} \zeta(s-m). \tag{2.10}$$

Proof. For $X = \mathbb{P}^n$ we have

$$Z(\mathbb{P}^n_{\mathbb{F}_q}, t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}$$

hence the L-function is given by

$$L(\mathbb{P}^n, s) = \prod_{m=0}^{n} \zeta(s-m).$$

The expression (2.10) is then immediate from Proposition 2.3. □

The expression (2.10) also agrees with the thermodynamical entropy of a known quantum statistical mechanical system. Indeed, the GL_n generalizations of the Bost–Connes system considered in [28] (see also the "determinant part" considered in [8]) have partition function $Z(\beta) = \prod_{m=0}^{n} \zeta(\beta - m)$ and entropy (2.10).

2.2. **Exponentiable motivic measures and zeta functions.** The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$ of varieties over a field $\mathbb{K}$ is generated by isomorphism classes $[X]$ of varieties with the inclusion-exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for $Y \subset X$ a closed subvariety and with the product given by $[X] \cdot [Y] = [X \times Y]$, the class of the product over $\mathrm{Spec}(\mathbb{K})$. The Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line.

We follow the terminology used for instance in [26] and we call *motivic measure* any ring homomorphisms $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$, where R is a commutative ring.

When one interprets the classes $[X]$ in the Grothendieck ring as a universal Euler characteristic (see [3]) a motivic measure in the sense specified above is determined by (and in turn determines) an invariant of algebraic varieties that satisfies the two main properties of an Euler characteristic, namely inclusion-exclusion $\mu(X) = \mu(Y) + \mu(X \smallsetminus Y)$ and mulitiplicativity under products $\mu(X \times Y) = \mu(X)\mu(Y)$.

As shown in [14], [25], [26], to any motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ one can associate the Kapranov zeta function, which can be seen as a map $\zeta_\mu(\cdot, t) : K_0(\mathcal{V}_{\mathbb{K}}) \to W(R)$ with values in the big Witt ring $W(R)$ of R, and is defined as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu([S^n(X)])\, t^n, \tag{2.11}$$

where $S^n(X)$ is the n-fold symmetric product of X, given by the quotient $S^n(X) = X^n/S_n$ of the n-fold product by the action of the symmetric group S_n of permutations. This can be regarded as an exponentiated version of the original measure μ, by interpreting the terms $\mu([S^n(X)])$ as analogs of the terms $\mu(X)^n/n!$ in an exponential series, [25].

Here we view the left-hand-side of (2.11) as an element in $(1 + R[[t]])^*$ and we identify the big Witt ring $W(R)$, as an additive group, with $((1 + R[[t]])^*, \times)$ with the usual product of formal series, which is the addition $+_W$ of the Witt ring, while the product $\star$ of the Witt ring is uniquely determined by setting

$$(1 - at)^{-1} \star (1 - bt)^{-1} = (1 - abt)^{-1} \tag{2.12}$$

for all $a, b \in R$, see [1], [4]. In general, the zeta function (2.11) defines a group homomorphism $\zeta_\mu(\cdot, t) : K_0(\mathcal{V}_{\mathbb{K}}) \to W(R)$ but not necessarily a ring homomorphism.

A motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ is called *exponentiable* (see [26]) if the associated Kapranov zeta function $\zeta_\mu(\cdot, t) : K_0(\mathcal{V}_{\mathbb{K}}) \to W(R)$ is a ring homomorphism, that is, if the zeta function is itself a motivic measure.

The motivic measure given by the counting of points over finite fields is exponentiable, [25], and the Gillet–Soulé motivic measure of [10] (the motivic Euler characteristic) $\mu_{GS} : K_0(\mathcal{V}_{\mathbb{K}}) \to K_0(\mathrm{Chow}(\mathbb{K})_{\mathbb{Q}})$ is also exponentiable, [26]. Several motivic measures that factor through μ_{GS}, like the topological Euler characteristic, the Hodge and Poincaré polynomials, are also exponentiable (see [26]), while the Larsen–Lunts motivic measure [16] is not exponentiable, since as shown in Proposition 4.3 of [26] in the exponentiable case if the zeta functions of two varieties are rational then the zeta function of the product also is, while the Larsen–Lunts motivic measure provides an example where zeta functions of curves are rational but the zeta function of a product of two positive genus curves is not.

The exponentiable property of motivic measures is related to λ-ring structures. A λ-ring R is a commutative ring endowed with maps $\lambda^n : R \to R$ satisfying $\lambda^0(a) = 1$, $\lambda^1(a) = a$ and $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$, so that $\lambda_t(a) = \sum_n \lambda^n(a)t^n$ is a group homomorphism $\lambda_t : R \to W(R)$. Assume that R is a λ-ring such that the group homomorphism $\sigma_t : R \to W(R)$ given by $\sigma_t(a) = \lambda_{-t}(a)^{-1}$ (the opposite λ-structure) is a ring homomorphism. Then as shown in [25], [26], the exponentiable condition for a motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ can be phrased as the property that

$$\mu([S^n(X)]) = \sigma^n(\mu([X])), \tag{2.13}$$

where $\sigma_t(a) = \sum_n \sigma^n(a)t^n$.

In the following we will restrict our attention to motivic measures that are exponentiable.

2.3. **A motivic entropy function.** Given an exponentiable motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ and an associated motivic zeta function $\zeta_\mu(X,t)$, we consider an associated Shannon type entropy function, which generalizes the Hasse-Weil entropy described in the previous sections. By analogy to Definition 2.1 we expect an expression of the form

$$S_\mu(X) := \log \zeta_\mu(X,t) + \zeta_\mu(X,t)^{-1} H_\mu(X,t), \tag{2.14}$$

where we need to specify more precisely what the terms mean in the context of motivic zeta functions with values in the Witt ring $W(R)$. As in the Hasse–Weil case discussed above, we expect the term $\zeta_\mu(X,t)^{-1}H_\mu(X,t)$ to take the form of a logarithmic derivative. Thus, a candidate definition for a motivic entropy of an exponentiable motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ would be given by

$$S_\mu(X) := (1 - s\frac{d}{ds}) \log \zeta_\mu(X, \lambda^{-s}), \tag{2.15}$$

where λ is a parameter in $\mathbb{R}_+^*$ and the change of variables $t = \lambda^{-s}$ is meant to interpret the s variable as an inverse temperature thermodynamic parameter. This means interpreting the motivic zeta function $\zeta_\mu(X, \lambda^{-s})$ as a partition function and (2.15) as its thermodynamical entropy.

In terms of the t variable, this means defining the entropy function as

$$S_\mu(X) = (1 - t\log(t)\frac{d}{dt}) \log \zeta_\mu(X,t). \tag{2.16}$$

2.3.1. *Lambda ring structure and Adams operations.* The term $t\frac{d}{dt}\log \zeta_\mu(X,t)$ in (2.16) has a natural interpretation in terms of lambda ring structures and the associated

Adams operations. Indeed, one defines the n-th Adams operation $\Psi_n(a)$ on the λ-ring R as the n-th ghost component of the opposite λ-structure $\sigma_t(a)$, that is,

$$t\frac{d}{dt}\log\sigma_t(a) = \psi_t(a) = \sum_{n\geq 1}\Psi_n(a)t^n. \tag{2.17}$$

(Here we follow the sign convention as in [12] for $\Psi_n(a)$ rather than as in [25].) These are ring homomorphisms $\Psi_n : R \to R$, satisfying $\Psi_n \circ \Psi_m = \Psi_{nm}$.

Lemma 2.7. *Let R be a commutative ring with no $\mathbb{Z}$-torsion and with opposite λ-ring structure σ_t. The motivic entropy* (2.15) *of an exponentiable motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ is given by*
(2.18)

$$S_\mu(X) = (1-t\log(t)\frac{d}{dt})\log\sigma_t(\mu([X])) = \sum_{n\geq 1}\frac{\Psi_n(\mu([X]))}{n}t^n - \sum_{n\geq 1}\Psi_n(\mu([X]))\,t^n\log(t).$$

2.3.2. *Motivic entropy of the Euler characteristics.* As shown in [25], the Macdonald formula for the Euler characteristics of symmetric products

$$\sum_{n=0}^{\infty}\chi(S^n(X))t^n = (1-t)^{-\chi(X)} = \exp(\sum_{r>0}\chi(X)\frac{t^r}{r}) \tag{2.19}$$

implies that the motivic measure on $K_0(\mathcal{V}_{\mathbb{C}})$ given by the Euler characteristic can be exponentiated. We can also read directly the value of the associated entropy function from (2.19). We obtain the following.

Example 2.8. *The motivic entropy of the motivic measure $\chi : K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}$ given by the Euler characteristics is given by*

$$\begin{aligned} S_\chi(X) = &\ (1-t\log(t)\frac{d}{dt})\log(1-t)^{-\chi(X)} \\ = &\ \chi(X)\frac{S(t,1-t)}{(1-t)} \\ = &\ \chi(X)\,\zeta_\chi(\mathrm{Spec}(\mathbb{K}),t)\,S(t,1-t), \end{aligned} \tag{2.20}$$

where $S(t,1-t) = -t\log(t) - (1-t)\log(1-t)$ is the binary Shannon entropy function and $\zeta_\chi(\mathrm{Spec}(\mathbb{K}),t) = (1-t)^{-1}$ is the zeta function of a point.

We should regard the dependence of the entropy on the variable t as a thermodynamic parameter, namely after a change of variable $t = e^{-\beta}$ we can think of the zeta function

$$\sum_{n=0}^{\infty}\chi(S^n(X))e^{-n\beta}$$

as a partition function, where (at least in the case of non-negative Euler characteristics) the coefficient $\chi(S^n(X))$ represents the degeneracy of the n-th energy level. In

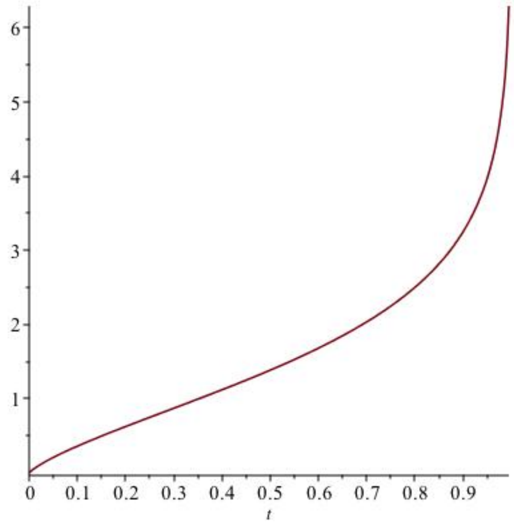

FIGURE 1. The motivic entropy of the Euler characteristic.

this view, the behavior of the function (2.19) with respect to t, shown in Figure 1 for a value $\chi(X) = 1$, corresponds near $t = 0$ (large $\beta \to \infty$) to the low temperature $T \to 0$ behavior of the system, while the behavior near $t = 1$ (near $\beta = 0$) corresponds to the high temperature $T \to \infty$ limit.

2.3.3. *Motivic entropy of Poincaré polynomials.* Similarly, the Mcdonald formula for the Poincaré polynomials,

$$\sum_{n=0}^{\infty} \mathcal{P}(S^n(X), z) t^n = \prod_{j=0}^{2n} (1 - z^j t)^{(-1)^{j+1} b_j(X)} = \exp(\sum_{r>0} \mathcal{P}(X, z^r) \frac{t^r}{r}), \tag{2.21}$$

used in [25] to show that the associated motivic measure is exponentiable, gives the value of the motivic entropy.

Example 2.9. *The motivic entropy of the motivic measure defined by the Poincaré polynomial is given by*

$$S_{\mathcal{P}}(X) = \sum_{j=0}^{2n} (-1)^j b_j(X) \tau(z^j) \left(S(z^j t, 1 - z^j t) + z^j t \log(z^j)\right), \tag{2.22}$$

where $\tau : \mathbb{Z}[z] \to W(\mathbb{Z}[z])$ *is the Teichmüller character to the Witt ring and* $S(u, 1 - u) = -u \log(u) - (1 - u) \log(1 - u)$ *is the binary Shannon entropy.*

Proof. We have

$$S_{\mathcal{P}}(X) = (1 - t \log(t) \frac{d}{dt}) \log \zeta_{\mathcal{P}}(X, t) = (1 - t \log(t) \frac{d}{dt}) \sum_{j=0}^{2n} (-1)^{j+1} b_j(X) \log(1 - z^j t)$$

$$= \sum_j (-1)^{j+1} b_j(X)(\log(1-z^jt) + \frac{z^jt\log(t)}{1-z^jt})$$

$$= \sum_j \frac{(-1)^j b_j(X)}{1-z^jt}(-(1-z^jt)\log(1-z^jt) - z^jt\log(z^jt) + z^jt\log(z^j))$$

$$= \sum_j \frac{(-1)^j b_j(X)}{1-z^jt}(S(z^jt, 1-z^jt) + z^jt\log(z^j)),$$

where $(1-z^jt)^{-1} = \tau(z^j)$ is the image in the Witt ring $W(\mathbb{Z}[z])$ of the element $z^j \in \mathbb{Z}[z]$ under the Teichmüller character $\tau : R \to W(R)$ mapping $R \ni a \mapsto \tau(a) = (1-at)^{-1} \in W(R)$. □

Note that the shift in the binary Shannon entropy $S(z^jt, 1-z^jt) + z^jt\log(z^j)$ is similar to the shift of the Shannon entropy one usually encounters in coding theory, where the q-ary Shannon entropy is defined as

$$S_q(\delta, 1-\delta) = S(\delta, 1-\delta) + \delta\log_q(q-1) = -\delta\log_q\delta - (1-\delta)\log_q(1-\delta) + \delta\log_q(q-1).$$

This is the form of the Shannon entropy that describes the asymptotic behavior of the volume of the Hamming balls (see for instance [30]).

3. Khinchin Properties of Motivic Entropy

The classical Shannon entropy is characterized in terms of the Khinchin axioms, [15]. It is natural to consider the question of what formal properties, analogous in some sense to the Khinchin characterization of entropy, are satisfied by the motivic version described above.

3.1. **Extensivity of motivic entropy.** The main property of the Shannon entropy is the extensivity property, namely its additive behavior on subsystems. The extensivity property is usually expressed as the relation

$$S(A \cup B) = S(A) + S(B|A) = S(B) + S(A|B).$$

We show here that the analogous property satisfied by the motivic entropy is the inclusion–exclusion property, where we think of subvarieties of a given ambient variety as subsystems and we identify the conditional entropy with the difference

$$S_\mu(B|A) = S_\mu(B) - S_\mu(A \cap B).$$

The case of additivity over independent subsystems then becomes just the scissor-congruence relation $[X] = [Y] + [X \smallsetminus Y]$ in the Grothendieck ring inherited by the entropy function S_μ.

Proposition 3.1. *The motivic entropy $S_\mu(X)$ of an exponentiable motivic measure $\mu : K_0(\mathcal{V}_\mathbb{K}) \to R$ satisfies*

- *Additivity over independent subsystems: for closed embeddings $Y \hookrightarrow X$*

$$S_\mu(X) = S_\mu(Y) + S_\mu(X \smallsetminus Y). \tag{3.1}$$

- *Extensivity over subsystems: inclusion–exclusion*

$$S_\mu(X_1 \cup X_2) = S_\mu(X_1) + S_\mu(X_2) - S_\mu(X_1 \cap X_2). \tag{3.2}$$

Proof. A motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ is a ring homomorphism. In particular, the Grothendieck group relations $[X] = [Y] + [X \smallsetminus Y]$ for closed embeddings $Y \hookrightarrow X$ imply that $\mu(X) = \mu(Y) + \mu(X \smallsetminus Y)$, which in turn implies the more general inclusion–exclusion property $\mu(X_1 \cup X_2) = \mu(X_1) + \mu(X_2) - \mu(X_1 \cap X_2)$.

The motivic zeta function $\zeta_\mu(X, t)$ in turn satisfies the relation

$$\zeta_\mu(X, t) = \zeta_\mu(Y, t)\zeta_\mu(X \smallsetminus Y, t) = \zeta_\mu(Y, t) +_W \zeta_\mu(X \smallsetminus Y, t), \tag{3.3}$$

where the addition $+_W$ in the Witt ring is the multiplication of power series. More generally, for $X = X_1 \cup X_2$, one has

$$\zeta_\mu(X, t) = \frac{\zeta_\mu(X_1, t)\zeta_\mu(X_2, t)}{\zeta_\mu(X_1 \cap X_2, t)} = \zeta_\mu(X_1, t) +_W \zeta_\mu(X_2, t) -_W \zeta_\mu(X_1 \cap X_2, t). \tag{3.4}$$

Thus, the motivic entropy satisfies (3.2). □

3.2. **Mutual motivic information.** In information theory the mutual information of two systems is defined as

$$\mathcal{I}(X, Y) = S(X) + S(Y) - S(X \cap Y),$$

or equivalently

$$\mathcal{I}(X, Y) = \sum_{x,y} P(x, y) \log \frac{P(x, y)}{P(x)P(y)}$$

$$= -\sum_x P(x) \log P(x) - \sum_y P(y) \log P(y) + \sum_{x,y} P(x, y) \log P(x, y),$$

which is the expression above. Thus, the mutual information is directly defined in terms of an inclusion-exclusion form, where one interprets $\mathcal{I}(X, Y)$ as the information of $X \cup Y$.

Thus, in our interpretation of the extensivity of the motivic entropy, given two subvarieties X, Y of some ambient variety, we can interpret as mutual information the quantity

$$\mathcal{I}_\mu(X, Y) = S_\mu(X \cup Y) = S_\mu(X) + S_\mu(Y) - S_\mu(X \cap Y).$$

3.3. **Zeros.** Another of the formal Khinchin properties of the Shannon entropy is the fact that it is stationary (and in fact maximal) at the uniform distribution and it is zero at the most non-uniform distributions $P = (P_i)$ where one of the $P_i = 1$ and all others are zero. We discuss here the meaning of the vanishing of the motivic entropy.

So far we have treated the motivic entropy function purely formally, without defining precisely in what ring of functions it is taking values. Because of the presence of the $\log(t)$ term, we cannot just view this function as an element of a power series ring $(1 + tR[[t]])^*$ or a Witt ring $W(R)$. It is better to think of $S_\mu(X)$ as an element of a ring $\mathcal{L}(R, t)$ of formal power series of logarithmic type, in the sense of [18].

We can describe the motivic entropy as follows.

Lemma 3.2. *The motivic entropy S_μ is the group homomorphism that fits in the commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{\sigma_t} & W(R) = (1 + tR[[t]])^* \\ \uparrow{\scriptstyle \mu} & \nearrow{\scriptstyle \zeta_\mu} & \downarrow{\scriptstyle \mathcal{L}} \\ K_0(\mathcal{V}_{\mathbb{K}}) & \xrightarrow{S_\mu} & \mathcal{L}(R, t) \end{array}$$

where μ is an exponentiable motivic measure, σ_t is the opposite λ-ring structure, $\mathcal{L}(R,t)$ is the ring of formal power series of logarithmic type, and $\mathcal{L} : W(R) \to \mathcal{L}(R,t)$ $\mathcal{L}(f) = (1 - t\log(t)\frac{d}{dt})\log(f)$ is a group homomorphism.

Proof. The fact that the composition $\sigma_t \circ \mu = \zeta_\mu$ is the motivic zeta function is the condition of exponentiability of the motivic measure μ, see [25], [26]. The map homomorphism $\mathcal{L}(f) = (1-t\log(t)\frac{d}{dt})\log(f)$ satisfies the logarithmic functional equation $\mathcal{L}(f +_W g) = \mathcal{L}(f \cdot g) = \mathcal{L}(f) + \mathcal{L}(g)$, hence it defines a group homomorphism $\mathcal{L} : W(R) \to \mathcal{L}(R,t)$. □

Lemma 3.3. *The kernel of the motivic entropy S_μ is the same as the kernel of the motivic measure ζ_μ.*

Proof. It suffices to show that the kernel of $\mathcal{L}$ is trivial. An element $f \in W(R) = (1 + tR[[t]])^*$ of the form $f(t) = \exp(\sum_{n\geq 1} \frac{a_n}{n} t^n)$ is in the Kernel of $\mathcal{L}$ if $\log(f) = t\log(t)\frac{d}{dt}\log(f)$, which is verified as an identity in $\mathcal{L}(R,t)$ only if $\log(f) = 0$, that is, if $f = 1$ is the additive unit of $W(R)$. Thus, a class $A = \sum_i n_i[X_i] \in K_0(\mathcal{V}_{\mathbb{K}})$ is in the kernel of S_μ iff it is in the kernel of the exponentiated motivic measure, $\zeta_\mu(A) = 1$. □

Thus, we can see the elements X in the kernel of the motivic measure as corresponding to the distributions with least information, or in other words they are the source of information loss in the motivic measure.

3.4. **Functoriality.** The remaining Khinchin axioms for the Shannon entropy are continuity over the simplex of measures $P = (P_i)$ and a consistence condition when viewing an n-dimensional simplex as a face of an $(n + 1)$-dimensional simplex,

$$S_{n+1}(P_1, \ldots, P_n, 0) = S_n(P_1, \ldots, P_n),$$

together with the symmetry of S under permutations of its arguments. We can view this requirement as a kind of functoriality requirement, when we consider the inclusion of faces as morphisms. Thus, the analogous property we require for the entropy function defined in the motivic setting is to satisfy a functoriality property induced by the funtoriality of Witt rings.

Lemma 3.4. *The motivic entropy is functorial. Namely, if* $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ *and* $\mu' : K_0(\mathcal{V}_{\mathbb{K}}) \to R'$ *are exponentiable motivic measures related by a (pre)-*λ*-ring homomorphism* $\phi : R \to R'$*, so that* $\mu' = \phi \circ \mu$*, then there exists a group homomorphism* $S : \mathcal{L}(R,t) \to \mathcal{L}(R',t)$ *such that* $S_{\mu'} = S(\phi) \circ S_{\mu}$*.*

Proof. The Witt rings are functorial, in the sense that a ring homomorphism $\phi : R \to R'$ induces a ring homomorphism $W(\phi) : W(R) \to W(R')$. A morphism of (pre)-λ-rings is a ring homomorphism $\phi : R \to R'$ for which one has a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\sigma_t} & \Lambda(R) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Lambda(\phi)} \\ R' & \xrightarrow{\sigma'_t} & \Lambda(R'), \end{array}$$

with $\Lambda(R) = (1+tR[[t]])^*$. The ghost map $gh : W(R) \to tR[[t]]$ is also functorial, and so is the ring of formal power series of logarithmic type. Thus, we obtain a diagram

$$\begin{array}{ccccccc} & \overset{\mu}{\nearrow} & R & \xrightarrow{\sigma_t} & \Lambda(R) & \xrightarrow{\mathcal{L}} & \mathcal{L}(R,t) \\ K_0(\mathcal{V}_{\mathbb{K}}) & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Lambda(\phi)} & & \downarrow{\scriptstyle S(\phi)} \\ & \underset{\mu'}{\searrow} & R' & \xrightarrow{\sigma'_t} & \Lambda(R') & \xrightarrow{\mathcal{L}} & \mathcal{L}(R',t). \end{array}$$

□

4. Motivic Entropy as Information Loss

The proposal discussed above for a notion of Entropy/Information in the setting of motivic measures is based on our initial observation that we can interpret the Hasse–Weil zeta function, when written in terms of effective zero-cycles, as a distribution as in (2.2) for which we formally compute the ordinary Shannon entropy. The resulting expression was then generalized in the form (2.15) for an arbitrary exponentiable motivic measure.

This proposal, however, has the drawback that it does not lend itself easily to a relative form, a motivic version of a Kullback–Leibler divergence, or better a measure of information loss associated to morphisms, which would provide a motivic analog of the characterization of information loss of [2].

We discuss here how one can modify the original proposal so as to accommodate a notion of information loss.

4.1. **Information loss on finite sets.** In the usual setting of probability measures on finite sets and classical information theory, given a morphism $f : (\Sigma, P) \to (\Sigma', Q)$, where Σ, Σ' are finite sets and P, Q are probability measures, one counts the information loss of f as a Kullback–Leibler divergence

$$\mathcal{I}(f) = S(P) - S(Q) = \sum_{s\in\Sigma} P_s \log \frac{Q_{f(s)}}{P_s} = \mathrm{KL}(P||Q). \tag{4.1}$$

The second equality follows by a simple calculation, see [2], using the assumption that morphisms are measure preserving, namely that

$$Q_j = \sum_{i\in f^{-1}(j)} P_i. \tag{4.2}$$

In our setting we will need to consider more general morphisms, which do not necessarily satisfy the condition (4.2), hence we will consider the Kullback–Leibler divergence $\mathrm{KL}(P||Q) = \sum_{s\in\Sigma} P_s \log \frac{Q_{f(s)}}{P_s}$ as our model of information loss, even when this does not necessarily agree with the difference $S(P) - S(Q)$.

The function $\mathcal{I}(f)$ of (4.1) satisfies an axiomatic characterization (up to a constant multiplicative factor), which follows from the Khinchin axioms of the Shannon entropy (reformulated as in [9]):

- Additivity under composition of morphisms: $\mathcal{I}(f \circ g) = \mathcal{I}(f) + \mathcal{I}(g)$;
- Additivity under direct sums: $\mathcal{I}(f \oplus g) = \mathcal{I}(f) + \mathcal{I}(g)$;
- Homogeneity under scaling: $\mathcal{I}(\lambda f) = \lambda\mathcal{I}(f)$, for $\lambda \in \mathbb{R}^*_+$.

The last two properties are replaced by the single additivity over convex combinations

$$\mathcal{I}(\lambda f \oplus (1-\lambda)g) = \lambda\mathcal{I}(f) + (1-\lambda)\mathcal{I}(g), \tag{4.3}$$

for $\lambda \in [0, 1]$, if the normalization of measures is preserved, see [2]. Additivity under composition plays the role of a functoriality property in the framework of [2].

4.2. **Sources of Information Loss.** We are interested here in a similar counting of information loss associated to motivic measures. As we discussed above, the kernel of an exponentiated motivic measure can be viewed as the amount of information contained in the Grothendieck ring of varieties that is lost when seen through the given motivic measure. It is also the kernel of the motivic entropy reflecting this interpretation as information loss.

If we want to make this idea of information loss in the motivic context more precise, we can identify two different possible sources of information loss:

- Ring homomorphisms $\phi : R \to R'$
- Morphisms of varieties $f : X \to Y$ (or correspondences of motives).

The first case corresponds to modifying the motivic measure $\mu : K_0(\mathcal{V}_\mathbb{K}) \to R$ by composition with a ring homomorphism $\phi : R \to R'$, while keeping the variety it is evaluated on unchanged, while the second case corresponds to maintaining the motivic measure unchanged while modifying the varieties through morphisms $f : X \to Y$ of algebraic varieties, for motivic measures defined on the Grothendieck ring of varieties

$K_0(\mathcal{V}_{\mathbb{K}})$, or correspondences $\alpha : h(X) \to h(Y)$ of Chow motives, for motivic measures on $K_0(\mathrm{Chow}(\mathbb{K}))$.

4.3. **Power structures.** In the next subsection we introduce an information loss function associated to a triple (ϕ, μ, μ') consisting of motivic measures $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ and $\mu' : K_0(\mathcal{V}_{\mathbb{K}}) \to R'$ and a ring homomorphism $\phi : R \to R'$.

In order to discuss an analog of the convex combination property (4.3) of information loss, we need to first recall the notion of a power structure, see [11].

Definition 4.1. *A* power structure *on a ring R is a map $(1+R[[t]])\times R \to 1+R[[t]]$, $(f(t), a) \mapsto f(t)^a$, with the properties that*

- $f(t)^0 = 1$, *for all* $f \in 1 + R[[t]]$,
- $f(t)^1 = f(t)$, *for all* $f \in 1 + R[[t]]$,
- $(f(t) \cdot g(t))^a = f(t)^a \cdot g(t)^a$, *for all* $f, g \in 1 + R[[t]]$, $a \in R$,
- $f(t)^{a+b} = f(t)^a \cdot f(t)^b$, *for all* $f \in 1 + R[[t]]$, $a, b \in R$,
- $f(t)^{ab} = (f(t)^a)^b$, *for all* $f \in 1 + R[[t]]$, $a, b \in R$.

Example 4.2. *As shown in* [11]*, there exists a power structure on the Grothendieck ring of varieties $K_0(\mathcal{V}_{\mathbb{C}})$ such that the universal motivic zeta function*

$$\zeta_{\mu_u}(X, t) = \sum_{n=0}^{\infty} [S^n(X)]\, t^n,$$

which is the exponentiation of $\mu_u = \mathrm{id} : K_0(\mathcal{V}_{\mathbb{C}}) \to K_0(\mathcal{V}_{\mathbb{C}})$, satisfies

$$(1-t)^{-[X]} = \zeta_{\mu_u}(X, t). \tag{4.4}$$

It is obtained by setting

$$f(t)^{[X]} := 1 + \sum_{k=1}^{\infty} \sum_{\sum i k_i = k} \left[(\prod_i X^{k_i} \smallsetminus \Delta) \times \prod_i X_i^{k_i} / \prod_i S_{k_i} \right] t^k,$$

for $f(t) = 1 + \sum_i [X_i]\, t^i$ with $[X_i] \in K_0(\mathcal{V}_{\mathbb{C}})$, see [11] *for more details.*

4.4. **Information loss from ring homomorphisms.** A measure of information loss associated to a ring homomorphism $\phi : R \to R'$ and a pair of given exponentiable motivic measures $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ and $\mu' : K_0(\mathcal{V}_{\mathbb{K}}) \to R'$ can be obtained simply by the difference of the motivic entropies

$$\mathcal{I}_X(\phi, \mu, \mu') = S_{\phi\circ\mu}(X) - S_{\mu'}(X) = (1 - t\log(t)\frac{d}{dt}) \log \frac{\zeta_{\phi\circ\mu}(X,t)}{\zeta_{\mu'}(X,t)}, \tag{4.5}$$

where $S_{\phi\circ\mu}(X) = S(\phi) \circ S_\mu(X)$ and $\zeta_{\phi\circ\mu}(X,t) = \Lambda(\phi)\zeta_\mu(X,t)$, by Lemma 3.4.

This measure of information loss satisfies an analog of the properties of information loss described in [2].

Lemma 4.3. *Let $\phi : R \to R'$ be a morphism of commutative rings and let $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ and $\mu' : K_0(\mathcal{V}_{\mathbb{K}}) \to R'$ be exponentiable motivic measures. Then the information loss function $\mathcal{I}_X(\phi, \mu, \mu')$ of (4.5) satisfies*

(1) *Additivity under composition $R \xrightarrow{\psi} R' \xrightarrow{\phi} R''$:*

$$\mathcal{I}_X(\phi \circ \psi, \mu, \mu'') = \mathcal{I}_X(\phi, \mu', \mu'') + S(\phi) \circ \mathcal{I}_X(\psi, \mu, \mu'). \tag{4.6}$$

(2) *Additivity under combination: for $\phi_1, \phi_2 : R \to R'$ ring homomorphisms, where the ring R' has a power structure,*

$$\mathcal{I}_X(\lambda\phi_1 + (1-\lambda)\phi_2, \mu, \mu') = \lambda\, \mathcal{I}_X(\phi_1, \mu, \mu') + (1-\lambda)\, \mathcal{I}_X(\phi_2, \mu, \mu'), \tag{4.7}$$

where

$$\mathcal{I}_X(\lambda\phi_1 + (1-\lambda)\phi_2, \mu, \mu') := (1 - t\log(t)\frac{d}{dt}) \log \frac{\zeta_{\phi_1\circ\mu}(X,t)^\lambda \cdot \zeta_{\phi_2\circ\mu}(X,t)^{1-\lambda}}{\zeta_{\mu'}(X,t)}. \tag{4.8}$$

Proof. For the composition $\phi \circ \psi : R \to R''$, by Lemma 3.4 we have

$$S_{(\phi\circ\psi)\circ\mu}(X) - S_{\mu''}(X) = S(\phi\circ\psi) \circ S_\mu(X) - S_{\mu''}(X)$$

$$= S(\phi) \circ S_{\psi\circ\mu}(X) - S(\phi) \circ S_{\mu'}(X) + S_{\phi\circ\mu'}(X) - S_{\mu''}(X)$$

$$= S(\phi)(S_{\psi\circ\mu}(X) - S_{\mu'}(X)) + S_{\phi\circ\mu'}(X) - S_{\mu''}(X),$$

hence we obtain (4.6).

For $\lambda \in R'$, consider the element

$$\zeta_{(\lambda\phi_1 + (1-\lambda)\phi_2)\circ\mu}(X,t) := \zeta_{\phi_1\circ\mu}(X,t)^\lambda \cdot \zeta_{\phi_2\circ\mu}(X,t)^{1-\lambda}, \tag{4.9}$$

where the product as power series is the addition in the Witt ring and the powers, for λ and $1-\lambda \in R'$, are determined by the power structure of R', so that (4.9) is clearly the analog of a convex combination in $W(R')$. We have

$$\mathcal{I}_X(\lambda\phi_1 + (1-\lambda)\phi_2, \mu, \mu') = (1 - t\log(t)\frac{d}{dt}) \log \frac{\zeta_{(\lambda\phi_1 + (1-\lambda)\phi_2)\circ\mu}(X,t)}{\zeta_{\mu'}(X,t)}$$

$$= (1 - t\log(t)\frac{d}{dt}) \log \frac{\zeta_{\phi_1\circ\mu}(X,t)^\lambda \cdot \zeta_{\phi_2\circ\mu}(X,t)^{1-\lambda}}{\zeta_{\mu'}(X,t)^\lambda \cdot \zeta_{\mu'}(X,t)^{1-\lambda}}$$

$$= \lambda(S_{\phi_1\circ\mu}(X) - S_{\mu'}(X)) + (1-\lambda)(S_{\phi_2\circ\mu}(X) - S_{\mu'}(X)),$$

so that we obtain (4.7). ∎

4.5. **Hasse–Weil information loss.** We then consider the question of how to construct an information loss function associated to morphisms of varieties. To this purpose we analyze again the case of the Hasse-Wil zeta function and the motivic measure given by the counting measure for varieties over finite fields.

As we have seen before, when we describe the Hasse-Weil zeta function as a generating function for effective 0-cycles, we can associate to it the distribution $P(\alpha) = t^{\deg(\alpha)}/Z(X,t)$, for $\alpha = \sum_i n_i x_i$ a 0-cycle on X, with $\deg(\alpha) = \sum_i n_i \deg(x_i)$.

Using the Kullback–Leibler divergence point of view on how to measure information loss, we aim at computing a relative entropy of the distribution $P = (P(\alpha))$ on 0-cycles on X and the corresponding distribution for 0-cycles on Y, by comparing them via the morphism $f : X \to Y$.

Cycles push forward under proper morphisms and pull back under flat morphisms. Thus, we can consider two different information loss functions for these two classes of morphisms.

4.5.1. *Hasse-Weil information loss for proper morphisms.* Given a proper morphism $f : X \to Y$ of algebraic varieties, for a subvariety $V \subset X$, one defines the pushforward $f_*(V)$ as zero if $\dim f(V) < \dim V$ and as $f_*(V) = \deg(V/f(V))\, f(V)$ if $\dim f(V) = \dim V$, where $\deg(V/f(V))$ is the degree $[\mathbb{K}(V) : \mathbb{K}(f(V))]$ of the finite field extension $\mathbb{K}(V)$ of $\mathbb{K}(f(V))$. The definition is then extended by linearity to combinations $\sum_i n_i V_i$. In particular, for a 0-cycle $\alpha = \sum_i n_i x_i$ in X, the pushforward under a proper morphism $f : X \to Y$ is given by

$$f_*(\alpha) = \sum_i n_i \deg(x_i/f(x_i)) \deg(f(x_i)), \tag{4.10}$$

where $\deg(x/f(x)) = [\mathbb{K}(x) : \mathbb{K}(f(x))]$.

Over the field of complex numbers the degree $\deg(x/f(x))$ represents geometrically the number of points of the fiber $\#f^{-1}(y)$ for $y = f(x)$ (counted with the appropriate multiplicity in the case of ramification). However, this is not necessarily the case in positive characteristics, where for example the map induced by $\mathbb{K}[t^p] \to \mathbb{K}[t]$ has degree p but is one-to-one on points.

Definition 4.4. *The Hasse–Weil information loss of a proper morphism $f : X \to Y$ is given by*

$$\mathcal{I}_{HW}(f_*) := \sum_{\alpha \in \mathcal{Z}^0_{\rm eff}(X)} P(\alpha) \log \frac{Q(f_*(\alpha))}{P(\alpha)}, \tag{4.11}$$

where $P(\alpha)$ is defined as in (2.2)*, $\mathcal{Z}^0_{\rm eff}(X)$ is the set of zero-dimensional effective cycles on X, and Q is the analogous distribution on Y,*

$$Q(\gamma) = \frac{t^{\deg(\gamma)}}{Z(Y,t)}, \quad \text{for } \gamma \in \mathcal{Z}^0_{\rm eff}(Y).$$

4.5.2. *Hasse-Weil information loss for flat morphisms.* Let $f : X \to Y$ be a flat morphism of relative dimension n. For an irreducible subvariety $V \subset Y$ the pullback $f^*(V)$ is defined as the $f^{-1}(V)$ and extended by linearity.

Definition 4.5. *The Hasse–Weil information loss of a flat morphism $f : X \to Y$ is given by*

$$\mathcal{I}_{HW}(f^*) := \sum_{\gamma \in \mathcal{Z}^0_{\rm eff}(Y)} Q(\gamma) \log \frac{P(f^*(\gamma))}{Q(\gamma)}. \tag{4.12}$$

4.6. **Proper morphisms.** The case of proper morphisms, defined in (4.11), is the one that most closely resembles the definition of information loss for finite sets that we recalled above from [2]. However, because of the behavior of degrees of cycles under pushfoward, it turns out that the information loss function $\mathcal{I}_{HW}(f_*)$ of Definition 4.4 is simply a logarithmic difference of zeta function.

Lemma 4.6. *The Hasse–Weil information loss* (4.11) *is given by*

$$\mathcal{I}_{HW}(f_*) = \log \frac{Z(X,t)}{Z(Y,t)}. \tag{4.13}$$

Proof. By proceeding as in our previous discussion of the Hasse-Weil entropy, we can equivalently write the expression (4.11) as

$$\mathcal{I}_{HW}(f_*) = \log \frac{Z(X,t)}{Z(Y,t)} - Z(X,t)^{-1} H(f_*,t), \tag{4.14}$$

where the term $H(f,t)$ is given by

$$H(f_*,t) = - \sum_{\alpha \in \mathcal{Z}^0_{\rm eff}(X)} t^{\deg(\alpha)} \log(t^{\deg(f_*(\alpha)) - \deg(\alpha)}). \tag{4.15}$$

We have $\deg(x) = [\mathbb{K}(x) : \mathbb{K}]$ and similarly $\deg(f(x)) = [\mathbb{K}(f(x)) : \mathbb{K}]$, hence these degrees are related by

$$\deg(x) = [\mathbb{K}(x) : \mathbb{K}] = [\mathbb{K}(x) : \mathbb{K}(f(x))] \cdot [\mathbb{K}(f(x)) : \mathbb{K}] = \deg(x/f(x)) \cdot \deg(f(x)),$$

hence $\deg(f_*(\alpha)) = \sum_i n_i d_f(x_i) \deg(f(x_i)) = \sum_i n_i \deg(x_i) = \deg(\alpha)$. Thus, the term $H(f_*,t)$ of (4.15) vanishes and one is left with (4.13). □

We check that this notion of information loss satisfies properties of additivity under composition and combination. In order to formulate the appropriate condition of additivity under combination, we consider a decomposition $X = X_1 \cup X_2$ as a disjoint union, and a corresponding decomposition $Y = Y_1 \cup Y_2$ with the property that $f_i = f|_{X_i} : X_i \to Y_i$. We write $f = f_1 \oplus f_2$ to refer to such data. We generalize this to weighted combinations $\lambda f_1 \oplus (1-\lambda) f_2$, by considering the distribution, for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathcal{Z}^0_{\rm eff}(X_i)$,

$$Q_\lambda(\alpha) = Q((\lambda f_1 \oplus (1-\lambda) f_2)_*(\alpha) := Q_1((f_1)_*(\alpha_1))^\lambda \cdot Q_2((f_2)_*(\alpha_2))^{1-\lambda}, \tag{4.16}$$

where for $\gamma_i \in \mathcal{Z}^0_{\rm eff}(Y_i)$, we have $Q_i(\gamma_i) := t^{\deg(\gamma_i)}/Z(Y_i,t)$. Similarly, we also consider the distribution $P_i(\alpha_i) := t^{\deg(\alpha_i)}/Z(X_i,t)$ and the distribution

$$P_\lambda(\alpha) = P_1(\alpha_1)^\lambda \cdot P_2(\alpha_2)^{1-\lambda}. \tag{4.17}$$

Proposition 4.7. *The Hasse–Weil information loss* (4.11) *satisfies additivity under composition*

$$\mathcal{I}_{HW}((g\circ f)_*) = \mathcal{I}_{HW}(f_*) + \mathcal{I}_{HW}(g_*)$$

and additivity under combination

$$\mathcal{I}_{HW}((\lambda f_1 \oplus (1-\lambda) f_2))_*) = \lambda \mathcal{I}_{HW}((f_1)_*) + (1-\lambda)\mathcal{I}_{HW}((f_2)_*).$$

Proof. Clearly the function $\mathcal{I}_{HW}(f_*)$ of (4.14) satisfies additivity under composition since

$$\mathcal{I}_{HW}((g\circ f)_*) = \log \frac{Z(X,t)}{Z(W,t)} = \log\frac{Z(X,t)}{Z(Y,t)} + \log\frac{Z(Y,t)}{Z(W,t)} = \mathcal{I}_{HW}(f_*) + \mathcal{I}_{HW}(g_*)$$

for proper morphisms $f : X \to Y$ and $g : Y \to W$.

For a decomposition $f_i : X_i \to Y_i$ and $f = f_1 \oplus f_2$ as above, we have

$$\mathcal{I}_{HW}((f_i)_*) = \log\frac{Z(X_i,t)}{Z(Y_i,t)}.$$

Since $Z(X,t) = Z(X_1,t)\cdot Z(X_2,t)$ and $Z(Y,t) = Z(Y_1,t)\cdot Z(Y_2,t)$, we have additivity

$$\mathcal{I}_{HW}(f_*) = \log\frac{Z(X,t)}{Z(Y,t)} = \log\frac{Z(X_1,t)}{Z(Y_1,t)} + \log\frac{Z(X_2,t)}{Z(Y_2,t)} = \mathcal{I}_{HW}((f_1)_*) + \mathcal{I}_{HW}((f_2)_*).$$

In the case of weighted combinations the information loss is computed by the Kullback-Leibler divergence

$$\sum_\alpha P_\lambda(\alpha) \log\frac{Q_\lambda(\alpha)}{P_\lambda(\alpha)}, \tag{4.18}$$

where

$$Q_\lambda((\alpha_1,\alpha_2)) = \frac{t^{\lambda \deg((f_1)_*(\alpha_1))}}{Z(Y_1,t)^\lambda} \cdot \frac{t^{(1-\lambda)\deg((f_2)_*(\alpha_2))}}{Z(Y_2,t)^{1-\lambda}}$$

Arguing as in Lemma 4.6 above, we see that this gives

$$\mathcal{I}_{HW}((\lambda f_1 \oplus (1-\lambda) f_2))_*) = \log\frac{Z(X_1,t)^\lambda \cdot Z(X_2,t)^{1-\lambda}}{Z(Y_1,t)^\lambda\cdot Z(Y_2,t)^{1-\lambda}},$$

which gives the additivity property. $\square$

4.7. **Finite surjective flat morphisms.** We consider then the case of flat morphisms and we focus on the simpler case of finite flat surjective morphisms $f : X \to Y$ of smooth quasi-projective varieties, with constant degree $\delta = \deg(f)$. In this case the pullback of effective zero-cycles is given by $f^*(\gamma) = \sum_i n_i \sum_{x_{i,j} \in f^{-1}(y_i)} x_{i,j}$, for $\gamma = \sum_i n_i y_i$ an effective zero-cycle in Y, with $\deg(f^*(\gamma)) = \deg(f) \cdot \deg(\gamma)$.

Lemma 4.8. *Let $f : X \to Y$ be a finite flat surjective morphism, with constant degree $\delta = \deg(f)$. Then the information loss function $\mathcal{I}_{HW}(f^*)$ of (4.12) is given by*

$$\mathcal{I}_{HW}(f^*) = \log \frac{Z(Y,t)}{Z(X,t)} + (\delta - 1)\, t \log(t) \frac{d}{dt} \log Z(Y,t). \tag{4.19}$$

Proof. We have

$$\sum_\gamma t^{\deg(\gamma)} Z(Y,t) \log \frac{Z(Y,t)}{t^{\deg(\gamma)}} \frac{t^{\deg(f^*(\gamma))}}{Z(X,t)}$$
$$= \log \frac{Z(Y,t)}{Z(X,t)} - \sum_\gamma t^{\deg(\gamma)} \log t^{(\deg(f)-1)\deg(\gamma)}.$$

As in §2.3 we see that this equals (4.19). ☐

We can use this description of the information loss function to give a more general definition for an arbitrary exponentiable motivic measure.

Definition 4.9. *Let $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ be an exponentiable motivic measure and let $f : X \to Y$ be a finite flat surjective morphism, with constant degree $\delta = \deg(f)$. The information loss is given by*

$$\mathcal{I}_\mu(f^*) := \log \frac{\zeta_\mu(Y,t)}{\zeta_\mu(X,t)} + (\delta - 1) t \log(t) \frac{d}{dt} \log \zeta_\mu(Y,t). \tag{4.20}$$

4.8. **Information loss of the Euler characteristics.** We consider again the example of the motivic measure given by the Euler characteristics.

Proposition 4.10. *For $\mathbb{K} = \mathbb{C}$ and $\chi : K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}$ the Euler characteristics, the information loss of a finite flat surjective morphism $f : X \to Y$ of degree $\delta = \deg(f)$ is given by*

$$\mathcal{I}_\chi(f^*) = S_\chi(Y) - S_\chi(X) + (\chi(f^{-1}(S)) - \delta \cdot \chi(S))\, \zeta_\chi(\mathrm{Spec}(\mathbb{K}), t)\, t \log(t) \tag{4.21}$$

where $S_\chi(X)$ is the motivic information of the Euler characteristics as in (2.20) and $S \subset Y$ is the locus such that f is étale over $Y \smallsetminus S$. If the morphism $f : X \to Y$ is étale, then $\mathcal{I}_\chi(f^*) = S_\chi(Y) - S_\chi(X)$.

Proof. By the Macdonald formula we have $\zeta_\chi(X,t) = (1-t)^{-\chi(X)}$. Thus, we obtain

$$\mathcal{I}_\chi(f^*) = \log \frac{(1-t)^{-\chi(Y)}}{(1-t)^{-\chi(X)}} + (\delta - 1) t \log(t) \frac{d}{dt} \log(1-t)^{-\chi(Y)}$$
$$= \frac{-1}{1-t} \left((\chi(Y) - \chi(X))(1-t) \log(1-t) - (\delta \cdot \chi(Y) - \chi(Y)) t \log(t) \right).$$

For a finite flat surjective morphism $f : X \to Y$ with degree $\delta = \deg(f)$, the Euler characteristics satisfies the Riemann–Hurwitz relation

$$\chi(X) = \delta \cdot \chi(Y) + \chi(f^{-1}(S)) - \delta \cdot \chi(S),$$

where f is étale over $Y \smallsetminus S$. Thus, we can write the above as

$$\mathcal{I}_\chi(f^*) = \frac{S(t,1-t)}{1-t}(\chi(Y)-\chi(X)) + (\chi(f^{-1}(S)) - \delta\cdot\chi(S))\frac{t\log(t)}{1-t}$$
$$= \zeta_\chi(\mathrm{Spec}(\mathbb{K}),t)\,\big((\chi(X)-\chi(Y))S(t,1-t) + (\chi(f^{-1}(S)) - \delta\cdot\chi(S))\,t\log(t)\big).$$

In the case where the morphism $f : X \to Y$ is étale, we have $\chi(X) = \delta \cdot \chi(Y)$ and we obtain simply the difference of the entropies

$$\mathcal{I}_\chi(f^*) = \zeta_\chi(\mathrm{Spec}(\mathbb{K}),t)\,(\chi(Y)-\chi(X))\,S(t,1-t) = S_\chi(Y) - S_\chi(X).$$

□

In the case of the Euler characteristics, the class of étale coverings appears to be the suitable class of morphisms for which the information loss function behaves as in the case of finite sets and agrees with the difference of entropies. However, this is not necessarily the case for arbitrary motivic measures. Indeed, unlike the case of Zariski locally trivial fibrations, in general if $f : X \to Y$ is an étale covering, the class $[X]$ in the Grothendieck ring does not necessarily factor as a multiple of the class $[Y]$. Indeed, by [16] in characteristic zero the quotient of the Grothendieck ring by imposing the relation $[X] = \delta \cdot [Y]$ for étale coverings of degree δ is isomorphic to $\mathbb{Z}$ via the Euler characteristics. Thus, one does not expect in general to have $\mathcal{I}_\mu(f^*) = S_\mu(Y) - S_\mu(X)$ for étale coverings for an arbitrary motivic measure μ.

4.9. **Additivity properties.** For a decomposition $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ with $f_i = f|_{X_i} : X_i \to Y_i$, and an exponentiable motivic measure $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to R$ where R has a power structure, we consider the information loss function

(4.22)
$$\mathcal{I}_\mu((\lambda f_1 \oplus (1-\lambda) f_2)^*) = \log \frac{\zeta_\mu(Y_1,t)^\lambda \cdot \zeta_\mu(Y_2,t)^{1-\lambda}}{\zeta_\mu(X_1,t)^\lambda \cdot \zeta_\mu(X_2,t)^{1-\lambda}} - (\deg(f)-1)t\log(t)\frac{d}{dt}\log(\zeta_\mu(Y_1,t)^\lambda \cdot \zeta_\mu(Y_2,t)^{1-\lambda}).$$

In the Hasse-Weil case, this corresponds to considering the distributions

$$P_\lambda(\gamma) = P_1(f_1^*(\gamma_1))^\lambda P_2(f_2^*(\gamma_2))^{1-\lambda} \quad \text{and} \quad Q_\lambda(\gamma) = Q_1(\gamma_1)^\lambda Q_2(\gamma_2)^{1-\lambda},$$

with $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_i \in \mathcal{Z}^0_{\mathrm{eff}}(Y_i)$ and computing the Kullback–Leibler divergence

$$\sum_\gamma Q_\lambda(\gamma) \log \frac{P_\lambda(\gamma)}{Q_\lambda(\gamma)}.$$

Since $\deg(f) = \deg(f_i)$ the information loss (4.22) satisfies the additivity property

$$\mathcal{I}_\mu((\lambda f_1 \oplus (1-\lambda) f_2)^*) = \lambda \mathcal{I}_\mu(f_1^*) + (1-\lambda)\mathcal{I}_\mu(f_2^*).$$

The question of additivity under composition of morphisms is more delicate, because of the observation mentioned at the end of the previous subsection on the behavior under étale coverings (and more generally under flat surjective morphisms of constant degree). A simple example where one recovers the behavior of information loss for finite sets is given by the following class of varieties and morphisms.

Example 4.11. *Given a variety Y over $\mathbb{K}$ consider the set of $X = Y \times S$ where S is a zero-dimensional variety of the form $S = \mathrm{Spec}(\oplus_{i=1}^N \mathbb{K})$, for some N. Let $\pi_S : X \to Y$ be the projection map $\pi_S(s, y) = y$. For this set of varieties and maps the information loss satisfies*

$$\mathcal{I}_\mu(\pi_S^*) = S_\mu(Y) - S_\mu(X). \tag{4.23}$$

In particular, $\mathcal{I}_\mu(\pi_S^*)$ satisfies both additivity under composition $\mathcal{I}_\mu((\pi_S \circ \pi_{S'})^*) = \mathcal{I}_\mu(\pi_S^*) + \mathcal{I}_\mu(\pi_{S'}^*)$ and additivity under combination (4.22).

Proof. For an exponentiable measure $\mu : K_0(\mathcal{V}_\mathbb{K}) \to R$, the zeta function of a product satisfies $\zeta_\mu(X, t) = \zeta(Y, t) \star_{W(R)} Z(S, t)$, where $\star_{W(R)}$ is the product in the Witt ring. Moreover, since S is a union of N copies of $\mathrm{Spec}(\mathbb{K})$ we have $\zeta_\mu(S, t) = (1-t)^{-N} = (1-t)^{-1} +_{W(R)} \cdots +_{W(R)} (1-t)^{-1}$. Thus, since $(1-t)^{-1}$ is the multiplicative unit of $W(R)$, we obtain

$$\begin{aligned} \zeta_\mu(X, t) &= \zeta(Y, t) \star_{W(R)} ((1-t)^{-1} +_{W(R)} \cdots +_{W(R)} (1-t)^{-1}) \\ &= \zeta(Y, t) +_{W(R)} \cdots +_{W(R)} \zeta(Y, t) = \zeta_\mu(Y, t)^N. \end{aligned}$$

Thus, we have

$$\mathcal{I}_\mu(\pi_S^*) = \log \frac{\zeta_\mu(Y, t)}{\zeta_\mu(X, t)} + (N-1) t \log(t) \frac{d}{dt} \log \zeta_\mu(Y, t)$$

$$= (1 - t\log(t)\frac{d}{dt}) \log \zeta_\mu(Y, t) - \log \zeta_\mu(X, t) + N t \log(t) \frac{d}{dt} \log \zeta_\mu(Y, t)$$

$$= (1 - t\log(t)\frac{d}{dt}) \log \zeta_\mu(Y, t) - (1 - t\log(t)\frac{d}{dt}) \log \zeta_\mu(X, t).$$

It is then clear that this difference satisfies the required additivity properties. □

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