J. Fluid Mech. (1991), *vol.* 222, *pp.* 207–213
Printed in Great Britain

Water-wave instability induced by a drift layer

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(Received 2 October 1989 and in revised form 4 June 1990)

A simple water-wave instability induced by a shear flow is re-examined, using a cubic equation first derived by Stern & Adam (1973) for a piecewise constant vorticity model. The instability criteria and the growth rate are computed. It is found that this mechanism is effective only if the surface drift velocity exceeds the minimum wave speed for capillary–gravity waves, and only if the drift-layer thickness lies within a band which depends on the wavelength and the drift velocity.

1. Introduction

Stern & Adam (1973) studied the generation of capillary–gravity waves by a shear current in the water. They derived a cubic equation for complex wave speeds using a step profile for the vorticity distribution. An instability was identified when $u^2/g\Delta$ exceeds a critical value, where u is the surface drift speed, g is the gravitational acceleration, and Δ is the thickness of the shear layer. The same equation was derived by Kawai (1977) but not analysed, and by Voronovich, Lobanov & Rybak (1980) as a limit of a more general problem with a vortex sheet at the bottom of the shear layer. In the present note, we investigate in more detail the dependence of this instability on the various parameters.

It is convenient to formulate the problem as one of steady motion in which the wave speed c is supposed real. Imaginary roots of the equation for c correspond to unstable disturbances growing like $e^{kc_i t}$, where $k = 2\pi/\lambda$ and $c_i = \operatorname{Im} c$. We follow closely the notation of Milinazzo & Saffman (1990), who studied the effect of a drift layer on the properties of finite amplitude capillary–gravity waves of permanent form on deep water.

The shape of the upper interface $H_1(x)$, the stream function $\Psi_1(x,y)$ in the drift layer, the shape of the lower interface between the drift layer and the irrotational flow $H_2(x)$, and the stream function in the irrotational fluid $\Psi_2(x,y)$ are

$$\left.\begin{aligned} H_1 &= c_1 \cos kx, \\ \Psi_1 &= \tfrac{1}{2}\Omega y^2 - (c-\Omega\Delta)\,y + (a_1 e^{ky} + b_1 e^{-ky})\cos kx, \\ H_2 &= -\Delta + e_1 \cos kx, \\ \Psi_2 &= -cy + d_1 e^{ky}\cos kx + \text{const}, \end{aligned}\right\} \quad (1.1)$$

where $\Omega = u/\Delta$ is the strength of the vorticity in the shear layer. The boundary conditions, that the interfaces are streamlines and that the pressure is continuous across them, are linearized and applied on $y = 0$ and $y = -\Delta$. After some algebra we obtain a cubic equation for $\theta = (c-\Omega\Delta)/c_0$,

$$\theta^3 + \theta^2(q + \tfrac{1}{2}\beta(1-e^{-2q/\beta})) + \theta(-1 + \beta q - \tfrac{1}{2}\beta^2(1-e^{-2q/\beta})) - q + \tfrac{1}{2}\beta(1-e^{-2q/\beta}) = 0, \quad (1.2)$$

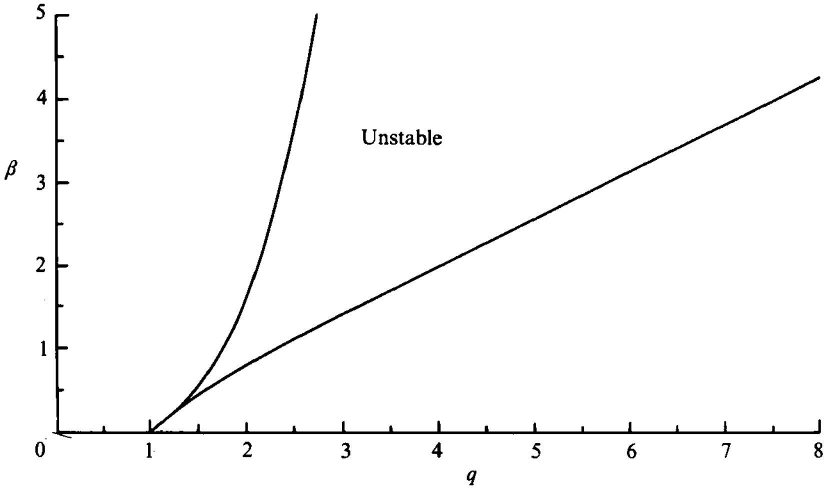

FIGURE 1. The wedge of instability in the (q, β)-plane.

where
$$q = \frac{\Omega \Delta}{c_0}, \quad \beta = \frac{\Omega}{2\pi f_0}, \quad c_0 = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda}\right)^{\frac{1}{2}}, \quad f_0 = \frac{c_0}{\lambda}. \tag{1.3}$$

The wave speed $c = (\theta + q)\, c_0$. Note that if $q = 0$, we obtain $c/c_0 = \pm 1$ and $c = 0$. If $\beta = 0, q > 0$, then $c/c_0 = q \pm 1$ and $c = 0$. Also, $q \rightarrow -q$ corresponds to $\theta \rightarrow -\theta$.

There is a range of values of q and β for which c is imaginary. This means that given Ω and Δ, the plane interface is unstable to disturbances of wavelength λ such that the values of q and β fall in the unstable region shown in figure 1. This region is calculated as follows.

Take
$$\left.\begin{aligned} a &= 1, \\ b &= \tfrac{1}{3}(q + \tfrac{1}{2}\beta(1 - \mathrm{e}^{-2q/\beta})), \\ \tilde{c} &= \tfrac{1}{3}(-1 + \beta q - \tfrac{1}{2}\beta^2(1 - \mathrm{e}^{-2q/\beta})), \\ d &= -q + \tfrac{1}{2}\beta(1 - \mathrm{e}^{-2q/\beta}), \end{aligned}\right\} \tag{1.4}$$
$$H = a\tilde{c} - b^2, \quad G = a^2 d - 3ab\tilde{c} + 2b^3. \tag{1.5}$$

Then
$$D \equiv D(q, \beta) = (G^2 + 4H^3)/a^2 \tag{1.6}$$

is the discriminant of the cubic. The roots are real if $D < 0$. If $D > 0$, there are two imaginary roots. When $D = 0$, the equation has two equal roots $-(-H)^{\frac{1}{2}} \operatorname{sgn} G$. Instability occurs for those values of λ for which
$$D\left(\frac{\Omega \Delta}{c_0(\lambda)}, \frac{\Omega \lambda}{2\pi c_0(\lambda)}\right) > 0. \tag{1.7}$$

The instability boundary in figure 1 is the locus $D(q, \beta) = 0$. The wedge of instability has asymptotes $q \sim (3\beta)^{\frac{1}{3}}$ and $q \sim 1.83\beta$. The wedge is exponentially thin near its tip, where it asymptotes the line $\beta = q - 1$.

The dimensional growth rate of the instability is
$$\sigma = 2\pi f_0 \theta_{\mathrm{i}} = \sqrt{3}\pi f_0 |Q + H/Q|, \tag{1.8}$$

where $\theta_i = \mathrm{Im}\,\theta$, $\theta_r = \mathrm{Re}\,\theta$, and

$$Q^3 = \tfrac{1}{2}(-G+(G^2+4H^3)^{\frac{1}{2}}). \tag{1.9}$$

The phase speed is

$$c_P = u+c_0\theta_r = u-bc_0-\tfrac{1}{2}c_0(Q-H/Q). \tag{1.10}$$

The transition from real to imaginary wave speeds is associated with the coalescence of two of the real speeds. It is found that it is the two smaller speeds which coincide, so the unstable waves will propagate somewhat slower than the propagating wave mode which has phase speed $u-bc_0+c_0(Q-H/Q)$.

2. Results

The wedge in figure 1 shows that there is no instability unless $q > 1$, i.e. $u > c_m$, where $c_m = (4gT)^{\frac{1}{4}}$ is the minimum phase speed and occurs when $\lambda = \lambda_m = 2\pi(T/g)^{\frac{1}{2}}$. Thus this mechanism will not produce waves until y exceeds a critical value.

Suppose now that u is fixed greater than c_m. For $g = 981$ cm/s² and $T =$ 72 dynes/cm, $c_m = 23$ cm/s and $\lambda_m = 1.7$ cm. For each value of λ, such that

$$u > c_0(\lambda) > c_m, \tag{2.1}$$

the value of $q = u/c_0$ is greater than 1, and hence there is instability if Δ is such that $\beta = u\lambda/(2\pi c_0 \Delta)$ lies in the unstable range shown in figure 1. The range of λ given by (2.1) is

$$\lambda_{max} = \lambda_m[(u/c_m)^2+((u/c_m)^4-1)^{\frac{1}{2}}] > \lambda > \lambda_m[(u/c_m)^2-((u/c_m)^4-1)^{\frac{1}{2}}] = \lambda_{min}. \tag{2.2}$$

We suppose that u/c_m is given and examine the range of λ in which unstable waves exist for a range of values of Δ/λ_m. In figure 2, we show for several values of u/c_m plots of the range of λ/λ_m *vs.* Δ/λ_m for which there is instability. It can be seen that there is no instability unless Δ exceeds a minimum Δ_{crit} which depends on u/c_m. Note that for a fixed λ/λ_m, instability is confined to a band of Δ/λ_m; consequently as the drift layer thickens an unstable wave restabilizes. Also shown in the figures is the point of maximum dimensionless growth rate σ_{max}/f_m, where $f_m = c_m/\lambda_m$ is the frequency of the minimum speed capillary–gravity wave, and the contour where the growth rate is half the maximum.

The dependence of λ_{crit}/λ_m, λ^+/λ_m, λ_{max}/λ_m, and λ_{min}/λ_m is shown in figure 3(*a*). The supercript + denotes values at the point of maximum growth rate, crit refers to values at the nose where $\Delta = \Delta_{crit}$. Figure 3(*b*) shows the variation of Δ^+/λ_m and Δ_{crit}/λ_m. Figure 3(*c*) shows the phase speeds and the growth rates. It can be shown that as $u/c_m \to 1$, $\lambda_{crit}/\lambda_m \sim 1-2(u/c_m-1)$, $\Delta_{crit}/\lambda_m \sim 1/2\pi(u/c_m-1)^{-1}$ and σ_{max}/f_m is exponentially small. The phase speed of the unstable modes approaches zero as $u/c_m \to 1$, and the phase speed of the stable mode at the critical point approaches $2c_m$. This is consistent with the idea that the drift layer convects the waves with a velocity u, and the instability can be thought of as a choking of the modes propagating against the drift layer.

We wish to point out that although a motivation for this work was the attempt to understand and explain simply some aspects of the generation of waves by wind, in particular the time for the appearance of relatively short waves and their phase speeds, when wind starts blowing over a flat calm, we have so far been unable to find concrete experimental data to support the prospect that this mechanism can play a primary role in wind-wave generation.

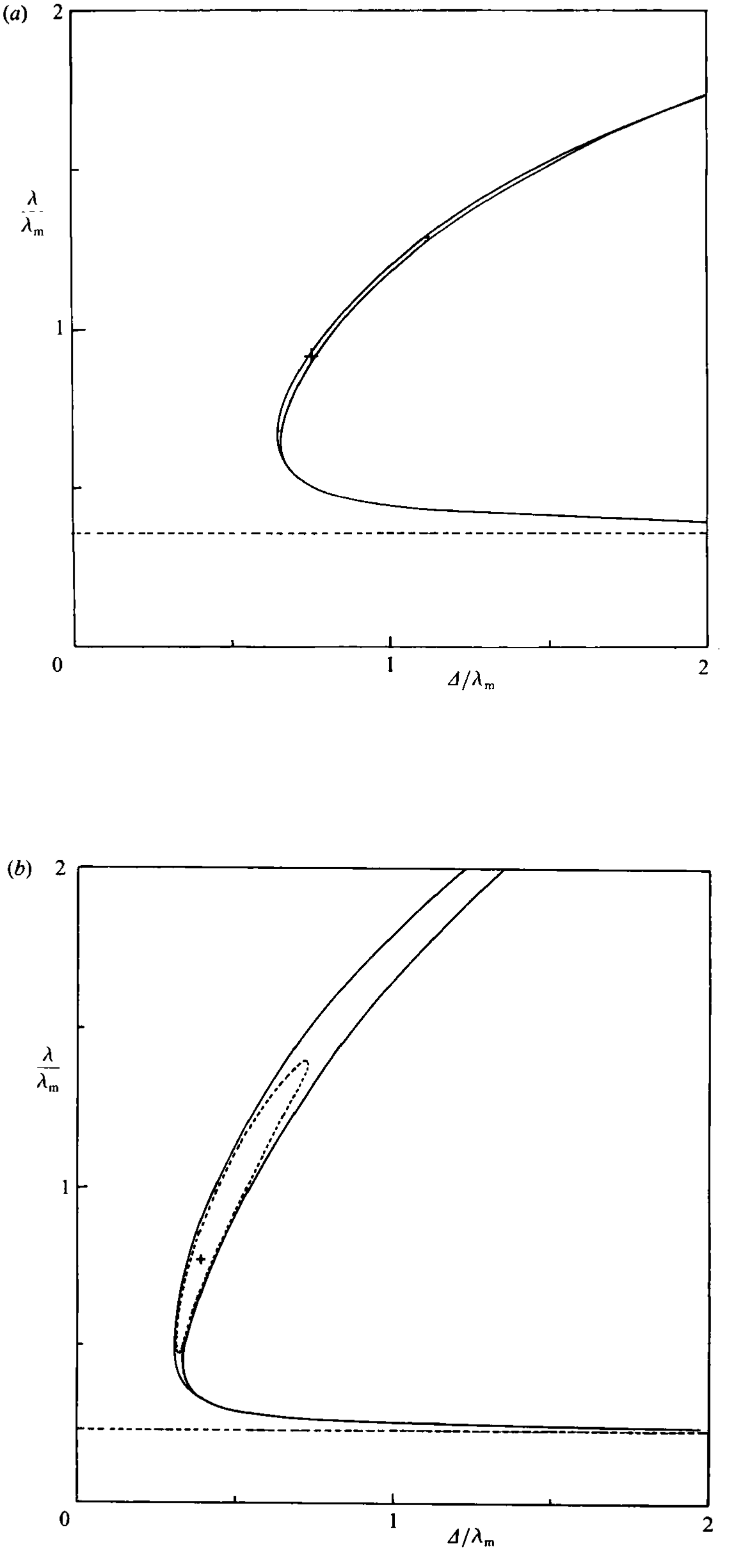

FIGURE 2 (a, b). For caption see facing page.

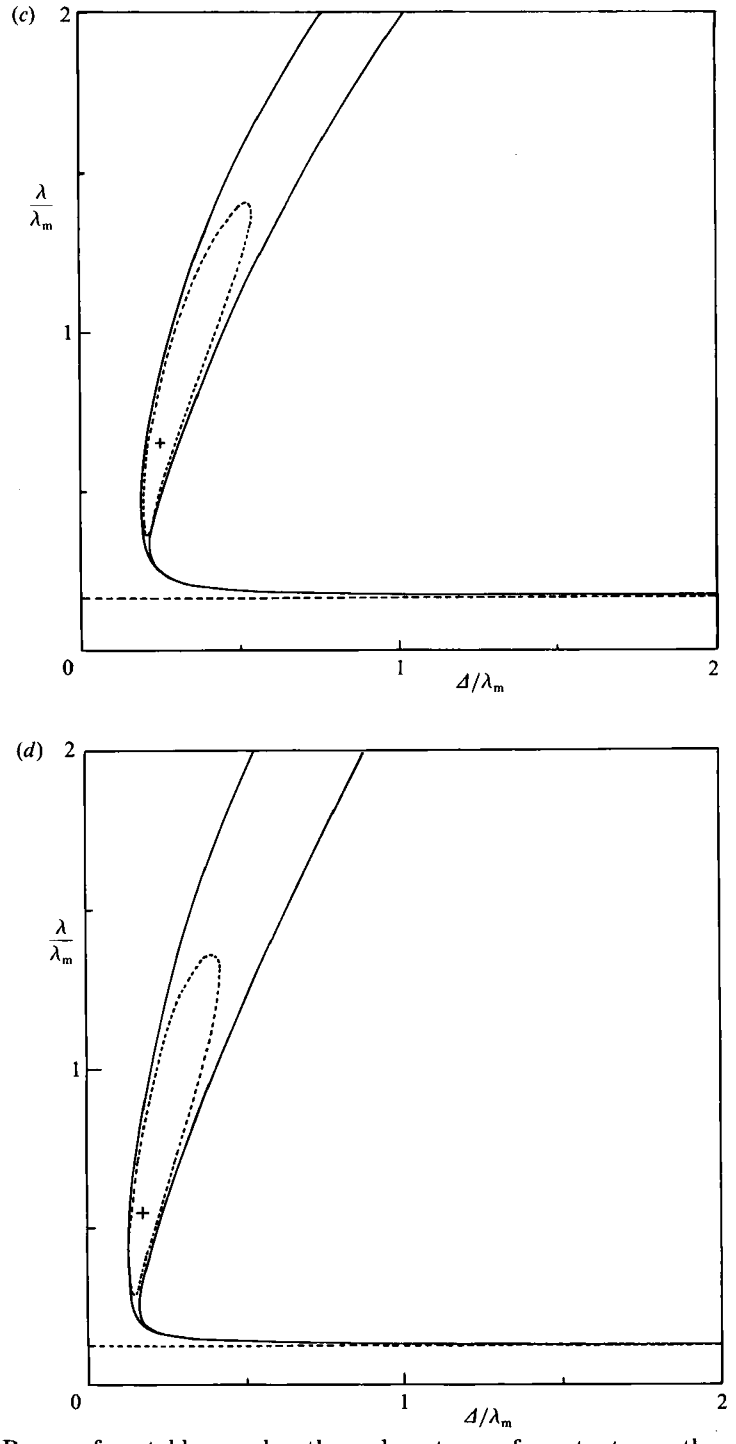

FIGURE 2. Range of unstable wavelengths and contours of constant growth rate *vs.* drift-layer thickness for varying drift velocity. (*a*) $u/c_m = 1.25$. (*b*) $u/c_m = 1.5$. (*c*) $u/c_m = 1.75$. (*d*) $u/c_m = 2.0$. +, location of the maximum growth rate. Interior contour gives locus of the growth rate of half the maximum.

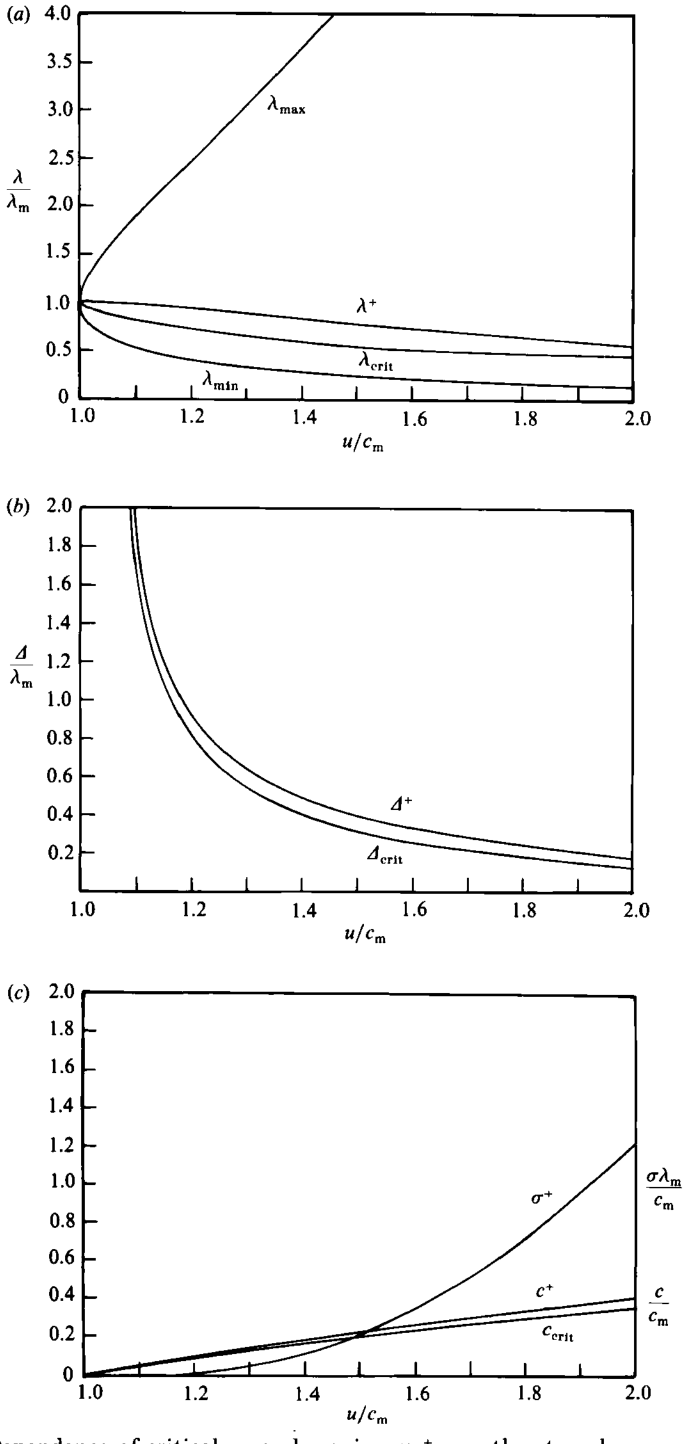

FIGURE 3. Dependence of critical $_{crit}$ and maximum $^+$ growth rate values on u/c_m. (*a*) Wavelength behaviour. (*b*) Drift-layer depth. (*c*) Maximum growth rate and phase speeds.

The authors wish to thank the referees for constructive criticisms and references to relevant previous work. This work is supported by Office of Naval Research under Contract nos. N00014-88-C-0356 and N00014-89-J-1164.

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