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On Quantum Effects of Vector Potentials and Generalizations of Functional Analysis

A Dissertation by
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Chapman University
Orange, CA

Schmid College of Science and Technology

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Computational and Data Sciences

May 2020

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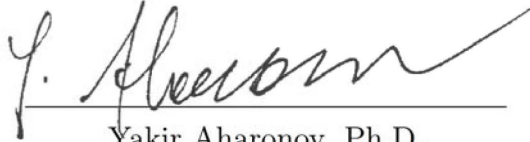
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May 2020

On Quantum Effects of Vector Potentials and Generalizations of
Functional Analysis

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LIST OF PUBLICATIONS

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- D. Alpay, I. L. Paiva, and D. C. Struppa, Positivity, rational Schur functions, Blaschke factors, and other related results in the Grassmann algebra, *Integr. Equ. Op. Th.* **91**, 8 (2019).
- I. L. Paiva, Y. Aharonov, J. Tollaksen, and M. Waegell, Magnetic forces in the absence of a classical magnetic field, *Phys. Rev. A* **101**, 042111 (2020).
- D. Alpay, I. L. Paiva, and D. C. Struppa, A general setting for functions of Fueter variables: differentiability, rational functions, Fock module and related topics, *Isr. J. Math.* **236**, 207 (2020).
- D. Alpay and I. L. Paiva, On the extension of positive definite kernels to topological algebras, [arXiv:2003.05632](https://arxiv.org/abs/2003.05632) (2020).
- I. L. Paiva, Y. Aharonov, J. Tollaksen, and M. Waegell, Complex-valued vector potential in a pre and post-selected system, (in preparation) (2020).

ABSTRACT

On Quantum Effects of Vector Potentials and Generalizations of Functional Analysis

by Ismael Lucas de Paiva

This is a dissertation in two parts. In the first one, the Aharonov-Bohm effect is investigated. It is shown that solenoids (or flux lines) can be seen as barriers for quantum charges. In particular, a charge can be trapped in a sector of a long cavity by two flux lines. Also, grids of flux lines can approximate the force associated with continuous two-dimensional distributions of magnetic fields. More, if it is assumed that the lines can be as close to each other as desirable, it is explained how the classical magnetic force can emerge from the Aharonov-Bohm effect. Continuing, the quantization of the source of the magnetic field, and not just of the degrees of freedom of the particle interacting with it, is considered. Special attention is given to the cases where the source has a relatively small spreading and is post-selected. As it will be discussed, in those cases, the weak value plays a role in the determination of the effective vector potential “experienced” by the particle. In the second part of this work, notions from functional analysis are extended to Banach algebras and completions of Grassmann algebras. A notion of analyticity is given to the functions of a single Banach algebra variable. This notion allows the introduction of holomorphic polynomials, power series, and rational functions. With that, the analogous of Hilbert spaces of power series are also considered. Finally, closures of Grassmann algebras with respect to the 1 and the 2-norms are explored. The analogous of the complex analysis in the open disk or a half-plane (usually referred to as Schur analysis) is presented in the 1-norm closure. Also, a Wiener-like algebra, interpolation problems, and a process known as the Schur algorithm are studied in this setting. Now, an inner product between two elements can be introduced in the 2-norm closure, revealing similarities between this space and the non-commutative Fock-Bargmann-Segal space. It is, then, defined a class of stochastic processes. To conclude, the derivatives of these processes are analyzed in an analogous of the space of stochastic distributions.

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LIST OF ABBREVIATIONS

| | |
|------|--------------------------------|
| AB | Aharonov-Bohm |
| BA | Banach algebra |
| BMFS | Banach module of Fueter series |
| CK | Cauchy-Kovalevskaya |
| DA | Drury-Arveson |
| FBS | Fock-Bargmann-Segal |
| FV | Fueter variables |
| GA | Grassmann algebra |
| HS | Hilbert space |
| NP | Nevanlinna-Pick |
| RF | Rational function |
| SG | Schur-Grassmann |
| WG | Wiener-Grassmann |
| WV | Weak value |

LIST OF SYMBOLS

| | |
|-----------------------------|--|
| \mathbb{N} | Natural numbers |
| \mathbb{N}_0 | Union of \mathbb{N} with $\{0\}$ |
| \mathbb{Z} | Integers |
| \mathbb{Z}_n | Integers module n |
| \mathbb{Z}_n^* | Set subtraction of \mathbb{Z}_n by $\{0\}$ |
| \mathbb{R} | Real numbers |
| \mathbb{C} | Complex numbers |
| \mathcal{A} | Banach algebra |
| Λ_n | Grassmann algebra with n generators |
| Λ | Union of Λ_n for every $n \in \mathbb{N}$ |
| $\bar{\Lambda}_{(p)}$ | Closure of Λ with respect to the p -norm |
| $ \cdot\rangle$ | Dirac ket |
| $\langle\cdot $ | Dirac bra |
| $\langle\cdot \cdot\rangle$ | Inner product between a bra and a ket |
| $\text{Tr}(\cdot)$ | Trace functional |
| Tr_s | Partial trace over a subspace s |
| $\langle\cdot\rangle$ | Expected value |
| $\langle\cdot\rangle_w$ | Weak value |

1 Introduction

This dissertation is divided into two parts. In the first part, it is studied some results involving a quantum particle in the presence of a magnetic field, but never in direct contact with it.

A better comprehension of electromagnetism was an essential factor in the revolution that started in physics during the past century. This revolution gave rise to two theories that completely shifted the understanding of the foundation of the physical world: the theory of relativity and the theory of quantum mechanics.

The former emerged from the studies of the apparent necessity of an absolute reference frame in the study of electromagnetism. While the equations of motion in Newtonian mechanics are invariant under Galilean transformation, Maxwell's equations, which are the basis of electromagnetism, are not. It was, then, conjectured that there existed a medium called *ether*, which was believed to be everywhere in the universe and would serve as an absolute system of reference where Maxwell's equations assumed their well-known forms. However, the Michelson-Morley experiment, which was an interference experiment meant to detect differences in the speed of light in systems moving with respect to the ether, did not return the expected results. It seemed, some thought, that the ether was an elastic solid. Lord Kelvin, for instance, thought that "we may simply deny the scholastic axiom that two portions of matter cannot jointly occupy the same space, and may assert, as an admissible hypothesis, that ether does occupy the same space as ponderable matter, and that ether is not displaced by ponderable bodies moving through space occupied by ether" [1].

In parallel with that, Voight [2], in 1887, and Lorentz [3], in 1904, considered a transformation that consisted in a way to change the system of coordinates for which Maxwell's equations are invariant. There is a very radical idea that is implied by this transformation: the speed of light is the same in *every* inertial system of reference. Moreover, because of that, the

coordinates of space and time are always mixed in a change of referential. Even though these ideas were, in a sense, already present in the works of Voight and Lorentz, they only got their due attention with one of the two seminal publications in 1905 by Albert Einstein [4], where he went straight to the crux of the problem. This new theory led to necessary corrections to the Newtonian mechanics since the Galilean transform had to be replaced by the new transform introduced by Voight and Lorentz, named *Lorentz transform* by Poincaré [5], who likely was unaware of Voight's work.

Quantum theory, on the other hand, can be traced back to 1900, when Plank, continuing his studies on the black body radiation, was led to the quantization of the energy exchange between a cavity and an electromagnetic field within it [6]. In 1905, in his other seminal article published that year, Einstein introduced the idea that electromagnetic radiation is *always* quantized to explain the photoelectric effect [7]. Inspired by these results, Bohr presented, in 1913, his model for a hydrogen atom [8], in which electrons had discrete levels of energy.

Later, in 1925, Heisenberg published a seminal article where he proposed a theory where the coordinates of position and the momentum of electrons in atoms were non-commuting physical quantities [9]. Following that, Dirac developed an algebraic formulation that provided a formal framework for the commutativity law [10]. Also, Born and Jordan further developed this theory in Ref. [11] and in a follow up article with Heisenberg [12]. The celebrated Heisenberg commutation relation was published in 1927 [13].

Moreover, in 1926, Pauli used Heisenberg's matrix mechanics to compute the spectrum and the Stark effect for a hydrogen atom [14]. However, the computation of the spectrum of more generic systems was overly complicated in this theory — at that time, physicists were not familiar with matrix algebra.

In parallel with Heisenberg's theory, a wave mechanics theory was also being developed.

In 1923, de Broglie used the idea that electromagnetic waves were quantized, together with Einstein's formula $E = mc^2$, which was also published in 1905 [15], to suggest that corpuscular objects, i.e., masses, should also have a wavelength associated with them [16]. Motivated by it, in 1926, Schrödinger developed his acclaimed equation to describe the dynamics of such waves [17], now called *wave functions*. Also, still in 1926, Born introduced the probabilistic interpretation for the wave function [12]. The theory emerging from these ideas was broadly diffused in the physics community since it required a mathematical knowledge that was more familiar to physicists: linear partial differential equations.

Finally, Schrödinger himself noticed that matrix and wave mechanics were mathematically equivalent [18]. His work was followed by further analysis from Dirac [19, 20], Jordan [21–23], and von Neumann [24].

It would still take some time until the first discussions about a quantum charge having its dynamics affected by a magnetic field in a region it never enters. In a talk to a physical society meeting in Danzig in 1939, Franz discussed that the interference of an electron traveling in a region with magnetic depends on the total magnetic flux enclosed by it [25]. However, it is not clear if it was discussed the possibility of having the electron still being affected by the field without directly interacting with it [26].

Later, in 1949, Ehrenberg and Siday, in the context of the theory of refraction of electron rays, concluded that a magnetic flux, even if isolated, would affect the dynamics of a charge encircling it [27]. Their result went unnoticed — perhaps because it did not seem a general effect. In fact, their practical problem was the development of a magnetic lens for a beta spectrometer, and they concluded that “[o]ne might therefore expect wave-optical phenomena to arise which are due to the presence of a magnetic field but not due to the magnetic field itself, i.e., which arise whilst the rays are in field-free regions of space.” Finally, unaware of the Ehrenberg and Siday's article, Aharonov and Bohm published their seminal work in

1959 [28]. Besides also introducing an electric version of it, they went straight to the crux of the problem, making the fundamental character of the result evident. With that, their work got traction since its publication. The effect, which is known as the Aharonov-Bohm (AB) effect, was observed experimentally [29–32], and it has been the subject of a vast study in the literature [32–53]. Also, a version of the effect for neutral particles with a magnetic moment, usually referred to as the Aharonov-Casher effect, was introduced [54]. Moreover, the effect still suscitates debates about its non-local nature [55–65].

Here, new consequences of the AB effect are presented. One of them, which can also be found in Refs. [66, 67], shows that solenoids (or flux lines) can be seen as barriers for quantum charges. In particular, two flux lines can trap a particle in a sector of a long cavity. Moreover, grids of flux lines can approximate the force associated with continuous two-dimensional distributions of magnetic fields. More, assuming the lines can be as close to each other as desirable, it is explained how the classical magnetic force can emerge from the AB effect.

Furthermore, as can also be seen in Ref. [68], the effect is considered in the presence of a quantized source of magnetic fields. When the source is post-selected, the *weak value*, as it will be studied, plays a role in the determination of the effective vector potential “experienced” by the particle.

In the second part of this dissertation, a variety of results from functional analysis, operator theory, and the theory of stochastic distributions is extended to a Banach algebra (BA) with an involution and completions of Grassmann algebras (GAs). A structure underlying most of this study is the concept of a Hilbert space (HS) of power series. HSs appeared as generalizations of finite-dimensional linear spaces. They were the result of a shift in the view of sequences — and, later, of functions — that arose in the study of linear integral equations. Instead of being seen as isolate objects, they started being analyzed as elements

of a space. Part of the origin of HSs can be associated with Hilbert's study of linear integral equations in a series of six papers published by him between 1904 and 1910, and composed his acclaimed book from 1912 [69].

In parallel to his work, Lebesgue introduced his notorious integral in 1904 [70]. The connection with HSs appeared in 1907 with Schmidt and Fréchet showing that the space of square summable functions, which is associated with a particular type of Lebesgue integration, had a geometry similar to HSs. Following that, Riesz showed that, in fact, there exists a one-to-one correspondence between HSs of sequences and Lebesgue square measurable functions, which allowed Fréchet and Fisher to show that these functions form a complete metric space. More, Fréchet introduced an integral formula for the action of continuous linear functional on elements of these metric spaces [71]. Also, Fréchet [72] and Riesz [73] introduced important properties of the dual of HSs with what is now known as the *Riesz representation theorem* and, in 1908, Schmidt proved a type of spectral theorem [74]. Moreover, existing results by Bessel, Fourier, and Parseval were translated into this new framework.

The first complete formal treatment of HSs was presented by von Neumann. He, then, together with Hilbert and Nordheim, applied this formalism to quantum mechanics in 1928 [75]. After that, other applications in physics (including in classical mechanics) and in other areas (like signal processing) were found [76, 77]. Finally, it should also be mentioned that the name *Hilbert space* was only introduced by von Neumann in 1930 [78].

The HSs of interest here are HSs of power series. In particular, it is studied the *Hardy space*, formally introduced by Riesz in 1923 [79], but named by him after Hardy because, in 1915, Hardy had worked on the functions he was considering [80]. Another HS of power series of interest is the *Fock-Bargmann-Segal space*, which was introduced by Fock in 1932 [81] as the completion of a direct sum of HSs in the study of identical quantum particles. Later, Bargmann [82], in 1961, and Segal [83], in 1963, presented a HS of power series isomorphic

to the Fock space. Finally, a space introduced by Drury [84], which had its importance highlighted by Arveson [85], often referred to as the *Drury-Arveson space*, is also considered in this work.

These and other HSs of power series were extended from a single complex variable to more general settings, like several complex variables, upper triangular operators, quaternions, and bi-complex numbers. They have been a source of new problems and methods [86–91]. Here, as already mentioned, some results on the extension of these spaces to BAs and completions of GAs are presented.

In the extension to BAs, which is a work originally presented in Ref. [92], a notion of analyticity is given to functions of a single variable in this setting, which, later, allows the introduction of the analogous of HSs of power series. Now, in the case of GAs, their closure with respect to the 1 and the 2-norms are considered. The 1-norm closure was studied in Ref. [93]. The convergence of power series and interpolation problems are investigated in this setting. The space of functions of a single complex variable with image in the closure of the Grassmann algebra with respect to the 2-closure, however, presents a structure similar to the non-commutative FBS space. A class of stochastic processes, which includes the analogous of the fractional Brownian motion, is defined and studied here, as done in Ref. [94].

This work is structured as follows. Chapter 2 briefly introduces some basic concepts from quantum mechanics and electromagnetism. This provides some tools for the presentation of the AB effect. Also, some results associated with it are reviewed.

Chapter 3 discusses a peculiar consequence of the AB effect: flux lines introduce a field-free force on quantum charges that can emulate continuous two-dimensional magnetic fields. This result is explored, leading to the obtainment of *topological bound states* and to the approximation of Landau levels. It is also discussed a semi-classical theory where grids of flux lines can be seen as the source of the classical magnetic force, at least for two-dimensional

fields.

Chapter 4 consider a case where the AB effect happens in the presence of quantized sources of magnetic fields. For that, a study on the quantization of sources is presented. Weak interactions between the source and the charge are of particular interest, and, because of it, the concepts of weak measurements and weak values are reviewed. Finally, it is shown how the effective vector potential “experienced” by the particle is complex-valued.

Chapter 5 starts the mathematical explorations of this work. A notion of derivative analogous to the one presented by Fueter for functions of a single quaternionic variable is extended to the case where the variable takes value in a BA that satisfies a minimum set of properties. With this result, holomorphic polynomials, power series, and rational functions are introduced. This, in turn, allows the study of Banach modules of holomorphic power series, the counterpart of HSs of power series in this context.

Finally, Chapter 6 considers various analysis problems in completions of GAs with respect to the 1 and the 2-norm. The analogous of the complex analysis in the open disk or a half-plane, often referred to as Schur analysis, is presented in the 1-norm closure. In particular, a Wiener-like algebra, which plays a role of the analogous of the Hardy space, is considered. Also, interpolation problems and a process called Schur algorithm are studied in this setting. In the 2-norm closure, it is noted that the inner product between two elements can be introduced, revealing similarities between this space and the non-commutative FBS space. It is, then, defined a class of stochastic processes. The derivatives of these processes are analyzed in a space analogous to the space of stochastic distributions.

It should be noted that, rather than including a chapter with final remarks, it was chosen to incorporate a discussion section at the end of every chapter that introduces a result that is a product of the studies for this dissertation.

2 Some Lessons from the Aharonov-Bohm Effect

As mentioned in the introduction, the Aharonov-Bohm (AB) effect is a vastly studied phenomenon, which has led to the development of many theoretical and experimental tools and numerous surprising consequences. This chapter presents an explanation of the effect and reviews some results that will be useful for the next two chapters. To start, a brief introduction to some basic concepts and definitions in quantum mechanics is given.

2.1 Preliminaries

A closed quantum system is represented by a vector, denoted by $|\psi\rangle$, in a complex Hilbert space (HS) \mathcal{H} . Its adjoint is denoted by $\langle\psi| = (|\psi\rangle)^\dagger$. Physical quantities that can be associated with the system and measured are given by self-adjoint operators acting on \mathcal{H} . They are often referred to as *observables*.

The spectral theorem assures that there exists an orthogonal basis $\{|o\rangle\}$ such that a self-adjoint operator O can be written as a diagonal operator

$$O = \int o|o\rangle\langle o|do. \quad (2.1)$$

The elements $|o\rangle$ are called the eigenvectors (or eigenstates) of O and o are the (real) eigenvalues associated with them

If \mathcal{H} admits a discrete basis, Eq. (2.1) becomes a sum of the type

$$O = \sum_{k=0}^{N-1} o_k|o_k\rangle\langle o_k|, \quad (2.2)$$

where N is the dimension of \mathcal{H} , which can be infinite. Also, the norm of the vectors of interest is always one, i.e.,

$$\langle \psi | \psi \rangle = 1, \quad (2.3)$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product in \mathcal{H} . The reason is that there exists a probabilistic interpretation of the state, given by the *Born rule*: if the observable O is measured and the state of the system is $|\psi\rangle$, the probability of finding the system in the state $|o_k\rangle$ is given by $|\langle o_k | \psi \rangle|^2$. Observe that the absolute value is necessary because $\langle o_k | \psi \rangle$, called the amplitude of probability, is, in general, a complex number. Moreover, with the choice associated with Eq. (2.3), the quantum states are defined up to a global phase. Then, two normal vectors in a complex HS represent the same quantum state if they differ by a global phase. Also, note that

$$\langle o_j | o_k \rangle = \delta_{jk}, \quad (2.4)$$

where δ_{jk} is the Kronecker delta. The change on the state that describes the system from $|\psi\rangle = \sum_k \psi_k |o_k\rangle$ to $|o_k\rangle$ is what is often referred to as the *collapse* of the state caused by the measurement of O . Furthermore, the average over repeated measurements of an observable O over a system in the state $|\psi\rangle$ results in the *expected value* $\langle O \rangle$ of O , which is given by

$$\langle O \rangle = \langle \psi | O | \psi \rangle = \sum_{j,k} o_k \bar{\psi}_j \psi_k \langle o_j | o_k \rangle = \sum_k o_k \bar{\psi}_k \psi_k. \quad (2.5)$$

Similar results hold in the case where the HS has a continuous basis. However, the relation between two elements of a basis is

$$\langle o | o' \rangle = \delta(o - o'), \quad (2.6)$$

where $\delta(o - o')$ is the Dirac delta. Also, the expected value of $\langle O \rangle$ becomes

$$\langle O \rangle = \langle \psi | O | \psi \rangle = \int \overline{o\psi(x)} \psi_k(x) dx. \quad (2.7)$$

The position (operator X) and the momentum (operator P) of a quantum particle are given by a vector in a continuous HS. Then, if the position of a system is described by the state $|\psi\rangle$, the *wave function* $\psi(x) = \langle x | \psi \rangle$ describes the probability amplitude of finding the particle at a certain position x . Observe that $\hat{\psi}(p) = \langle p | \psi \rangle$ can also be defined. In fact, $\hat{\psi}(p)$ is the Fourier transform of $\psi(x)$. Also, X and P satisfy the well-known canonical commutation relation

$$[X, P] \equiv XP - PX = i\hbar I, \quad (2.8)$$

where I is the identity operator.

The composition of quantum systems is given by the tensor product. Then, if a state $|\psi\rangle$ belongs to \mathcal{H}_1 and a state $|\varphi\rangle$ belongs to \mathcal{H}_2 , their joint state is

$$|\Theta\rangle = |\psi\rangle \otimes |\varphi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2. \quad (2.9)$$

A state that can be written in this form is called a *product* or a *separable* state. However, a general state of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is written as

$$|\Gamma\rangle = \iint \gamma(r, s) |r\rangle \otimes |s\rangle dr ds, \quad (2.10)$$

where $|r\rangle$ denotes a basis of \mathcal{H}_1 and $|s\rangle$ denotes a basis of \mathcal{H}_2 . If a state $|\Gamma\rangle$ cannot be written as a pure state, each of its parts is called an *entangled* or a *non-separable* state. In this case, the state associated with each system can be written with the definition of a *partial trace*,

i.e., the state of system 1 is a partial trace of $|\Gamma\rangle\langle\Gamma|$ over the system 2

$$\begin{aligned}
\rho_1 &\equiv \text{Tr}_2 |\Gamma\rangle\langle\Gamma| \\
&\equiv \iiint \overline{\gamma(r', s')} \gamma(r, s) |r\rangle\langle r'| \text{Tr}(|s\rangle\langle s'|) ds ds' dr dr' \\
&= \iiint \overline{\gamma(r', s')} \gamma(r, s) |r\rangle\langle r'| \langle s'|s\rangle ds ds' dr dr' \\
&= \iiint \overline{\gamma(r', s)} \gamma(r, s) |r\rangle\langle r'| ds dr dr'.
\end{aligned} \tag{2.11}$$

Similarly, the state of system 2 is

$$\rho_2 \equiv \text{Tr}_1 |\Gamma\rangle\langle\Gamma| = \iiint \overline{\gamma(r, s')} \gamma(r, s) |s\rangle\langle s'| ds ds' dr. \tag{2.12}$$

Observe that, in this case, the states are no longer vectors in \mathcal{H}_1 and \mathcal{H}_2 . They are operators, called *density matrices*, acting on those spaces. Although this is the most general form to represent a system, most of the time only vectors in HSs are considered here.

The dynamics of quantum systems is governed by Hamiltonians. The Schrödinger equation states that a Hamiltonian H is the generator of translations in time of the a state $|\psi(t_0)\rangle = |\psi\rangle$, i.e.,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle. \tag{2.13}$$

This means that there exists an unitary U such that

$$i\hbar \frac{\partial}{\partial t} U(t; t_0) |\psi\rangle = H U(t; t_0) |\psi\rangle \Rightarrow i\hbar \frac{\partial}{\partial t} U = H U. \tag{2.14}$$

In general, the Hamiltonian is a function of time and Eq. (2.14) has to be carefully evaluated. However, in cases where $H(t)$ commutes with $H(t')$ for every t and t' , Eq. (2.14) leads to

$$U(t; t_0) = e^{-\int_{t_0}^t H(t') dt' / \hbar}. \tag{2.15}$$

A common type of Hamiltonian is

$$H = \frac{1}{2m}P^2 + V(\vec{X}, t), \quad (2.16)$$

where V is an scalar potential, the canonical momentum \vec{P} is the “vector of operators” (P_x, P_y, P_z) , and $\vec{X} = (X, Y, Z)$. In this case, the Schrödinger equation for a state $|\psi(t)\rangle$ in the position basis is

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}, t) + V(\vec{x}, t)\psi(\vec{x}, t), \quad (2.17)$$

where it was used the fact that

$$\vec{P} = -i\hbar\vec{\nabla}. \quad (2.18)$$

Observe that, because systems are, in general, entangled, the dynamics of an individual system is not always unitary. Here, however, this matter is not looked into.

Besides the Schrödinger picture, there is another way to consider dynamics in quantum mechanics, which is given by the Heisenberg picture. In this perspective, even though quantum states still identifies a system, they do not evolve. What evolves are their properties, i.e., the observables. The dynamics of the observables are governed by the Heisenberg equation

$$\frac{d}{dt}O(t) = \frac{i}{\hbar}[H, O(t)] + \frac{\partial O}{\partial t}(t). \quad (2.19)$$

Most of the systems considered here are in the presence of an electromagnetic field. Because of it, instead of the Hamiltonian in Eq. (2.16), their dynamics are governed by

$$H = \frac{1}{2m} \left(P - q\vec{A}(\vec{X}, t) \right)^2 + qV_q(\vec{X}, t) + V(\vec{X}, t), \quad (2.20)$$

where \vec{A} is the vector potential and V_q is the scalar potential. These potentials are such that

the electric field is

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V_q \quad (2.21)$$

and the magnetic field is

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (2.22)$$

Because the potentials are defined as a way to mathematically “reconstruct” the fields \vec{E} and \vec{B} , it can be seen from Eqs. (2.21) and Eq. (2.22) that there is not a unique way to define \vec{A} and V_q . In fact, if Λ is a function of the position coordinates and time, the potentials

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \quad (2.23)$$

and

$$V'_q = V_q - \frac{\partial \Lambda}{\partial t} \quad (2.24)$$

are associated with the same electromagnetic field. The change from the potentials \vec{A} and V_q to \vec{A}' and V'_q is called a *gauge transformation*. Also, Λ is associated with a *gauge choice*.

To conclude this section, it should be noted that, in quantum mechanics, the *gauge invariance* implies that, given the unitary

$$U = e^{iq\Lambda(\vec{X},t)/\hbar}, \quad (2.25)$$

the Hamiltonian in Eq. (2.20) is physically equivalent to the Hamiltonian

$$H' = UHU^\dagger - q\frac{\partial \Lambda}{\partial t}(\vec{X},t), \quad (2.26)$$

as can be checked by direct computation and observing that Eq. (2.8) implies that

$$\left[\vec{P}, f(\vec{X}) \right] = i\hbar \vec{\nabla} \Lambda(\vec{X}). \quad (2.27)$$

With that set, the AB effect and its consequences can finally be studied.

2.2 Aharonov-Bohm Effect

In classical physics, the dynamics of a particle with charge q is only affected by a magnetic field which it directly interacts with it, i.e., if it travels in a region where the magnetic field is non-zero. However, in quantum mechanics, this is not always the case. If a charge encircles a region in space that contains a magnetic field, it accumulates a quantum phase that is proportional to the magnetic flux inside the region enclosed by its trajectory, regardless of whether there was any magnetic field on the trajectory of the particle. This is the (magnetic) AB effect, which was introduced in 1959 by Aharonov and Bohm [28].

The way this effect is usually presented is by considering a charge encircling a solenoid whose axis lies in the z axis, as represented by Fig. 2.1. For simplicity, the solenoid is taken to be infinitely thin — in which case it will be referred to as a flux line. Also, the particle is assumed to travel in the xy plane on the superposition

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_L\rangle + |\psi_R\rangle), \quad (2.28)$$

where $|\psi_L\rangle$ is a packet that passes to the left and $|\psi_R\rangle$ is a packet that passes to the right of the flux line. This superposition can be achieved, for instance, with the use of a double-slit. The Hamiltonian of the particle can be written as

$$H = \frac{1}{2m} \left(\vec{P} - q\vec{A}(\vec{X}) \right)^2, \quad (2.29)$$

where $\vec{P} = P_x\hat{x} + P_y\hat{y}$ and $\vec{X} = X\hat{x} + Y\hat{y}$.

To obtain the solution of this Hamiltonian, let the states $|\psi_L^0\rangle$ and $|\psi_R^0\rangle$ be the solutions of

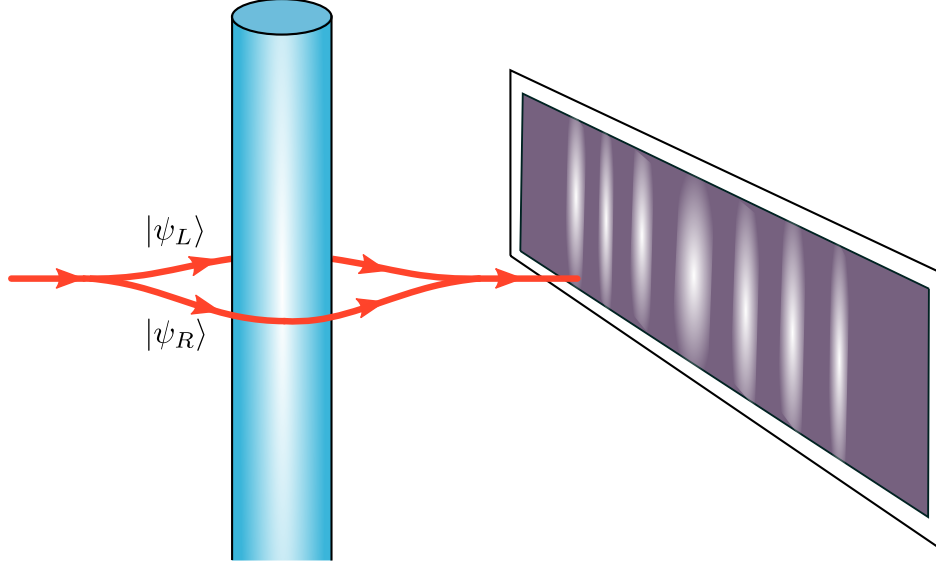


Figure 2.1: Illustration of an experiment that verifies the AB effect. A charge travels in a superposition of a state $|\psi_L\rangle$ traveling to the left and a state $|\psi_R\rangle$ traveling to the right of the solenoid (blue object), as represented by the red paths. This causes a shift in the interference pattern proportional to the magnetic flux inside the solenoid.

the case where there is no magnetic field in the flux lines, i.e.,

$$H_0 = \frac{1}{2m} P^2. \quad (2.30)$$

Then, following a procedure introduced by Dirac [95], it can be obtained that, after encircling the flux line,

$$|\psi_L\rangle = e^{iq \int_{\gamma_L} \vec{A} \cdot d\ell / \hbar} |\psi_L^0\rangle \quad (2.31)$$

and

$$|\psi_R\rangle = e^{iq \int_{\gamma_R} \vec{A} \cdot d\ell / \hbar} |\psi_L^0\rangle, \quad (2.32)$$

where γ_L and γ_R are, respectively, the trajectory of the center of $|\psi_L\rangle$ and $|\psi_R\rangle$. Then, because quantum states are equivalent up to a global phase,

$$\int_{\gamma_R} \vec{A} \cdot d\ell - \int_{\gamma_L} \vec{A} \cdot d\ell = \oint \vec{A} \cdot d\ell, \quad (2.33)$$

and the magnetic flux Φ_B inside the region enclosed by the charge is

$$\Phi_B = \oint \vec{A} \cdot d\ell, \quad (2.34)$$

the state of the system after it encircles the flux line is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\psi_L^0\rangle + e^{i\varphi_{AB}} |\psi_R^0\rangle \right), \quad (2.35)$$

where

$$\varphi_{AB} \equiv \frac{q\Phi_B}{\hbar} \pmod{2\pi} \quad (2.36)$$

is the quantum phase accumulated by the charge, usually called the AB phase. The state in Eq. (2.35) should be compared to

$$|\Psi^0\rangle = \frac{1}{\sqrt{2}} \left(|\psi_L^0\rangle + |\psi_R^0\rangle \right), \quad (2.37)$$

which is the final state of the system that undergoes a regular double-slit experiment. Then, if an interference pattern is obtained from a double-slit experiment without a flux line, this pattern will be shifted by an amount determined by the AB phase if the same experiment is performed with a flux line placed in between the slits.

Finally, observe that, since a phase has a periodicity of 2π , the AB phase contains just the modular part of $q\Phi_B/\hbar$. Thus, the magnetic flux

$$\Phi_0 \equiv \frac{2\pi\hbar}{q}, \quad (2.38)$$

called a *fluxon*, creates an equivalence relation for which two congruent magnetic fluxes modulo Φ_0 give rise to the same AB phase.

2.3 Modular Variables

Consider a particle in the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle + e^{i\alpha} |\psi_2\rangle \right), \quad (2.39)$$

where $\psi_1(x) \equiv \langle x|\psi_1\rangle$ and $\psi_2(x) \equiv \langle x|\psi_2\rangle$ are wave packets, i.e., wave functions localized in a certain interval, with the same profile, meaning that, up to a shift in their domain, they are represented by the same function. The question that will be addressed now concerns the detection of the relative phase α between ψ_1 and ψ_2 . It is known that this phase can be observed by interference experiments, i.e., the phase can be detected when there is an overlap between the two wave packets. Nevertheless, it can be asked whether there exists an observable capable of detecting α when ψ_1 and ψ_2 do not overlap.

In principle, it seems that such an operator does not exist. The reason is that it would have to be written as a linear combination of powers of the operators of position and momentum. However, for every $n, m \in \mathbb{N}_0$,

$$\begin{aligned} \langle X^n P^m \rangle &= \int \bar{\psi}(x) X^n P^m \psi(x) dx \\ &= \frac{1}{2} \left(\int \bar{\psi}_1(x) X^n P^m \psi_1(x) dx + \int \bar{\psi}_2(x) X^n P^m \psi_2(x) dx \right), \end{aligned} \quad (2.40)$$

where it was used the fact that, because ψ_1 and ψ_2 do not overlap,

$$e^{i\alpha} \int \bar{\psi}_1(x) X^n P^m \psi_2(x) dx = e^{-i\alpha} \int \bar{\psi}_2(x) X^n P^m \psi_1(x) dx = 0. \quad (2.41)$$

Therefore, $\langle X^n P^m \rangle$ does not depend on α for every $n, m \in \mathbb{N}_0$.

However, suppose that the center of the packets are separated by a distance L , i.e., $\psi_1(x) = \psi_2(x - L)$ or $\psi_1(x) = \psi_2(x + L)$. Also, consider the operator $e^{iPL/\hbar}$, which is the operator

of translations by L — more precisely, for any function f that admits a Fourier transform,

$$\begin{aligned}
e^{iPL/\hbar}f(x) &= e^{iPL/\hbar} \left(\frac{1}{\sqrt{2\pi\hbar}} \int \hat{f}(p)e^{ipx/\hbar} dp \right) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int \hat{f}(p)e^{ip(x+L)/\hbar} dp \\
&= f(x+L),
\end{aligned} \tag{2.42}$$

where \hat{f} is the Fourier transform of f . Then,

$$\begin{aligned}
\langle e^{iPL/\hbar} \rangle &= \int \bar{\psi}(x)e^{iPL/\hbar} dx \\
&= \frac{1}{2} \left(e^{i\alpha} \int \bar{\psi}_1(x)\psi_2(x+L) dx + e^{-i\alpha} \int \bar{\psi}_2(x)\psi_1(x+L) dx \right) \\
&= \frac{1}{2} e^\alpha
\end{aligned} \tag{2.43}$$

if $\psi_1(x) = \psi_2(x+L)$. Hence, this operator is capable of detecting the phase α . Moreover, while $e^{iPL/\hbar}$ is not Hermitian and, thus, it is not an observable, the operators

$$\cos\left(\frac{PL}{\hbar}\right) = \frac{e^{iPL/\hbar} + e^{-iPL/\hbar}}{2} \tag{2.44}$$

and

$$\sin\left(\frac{PL}{\hbar}\right) = \frac{e^{iPL/\hbar} - e^{-iPL/\hbar}}{2i} \tag{2.45}$$

are, and they can also detect α . In fact, $\langle \cos(PL/\hbar) \rangle = (\cos \alpha)/2$ and $\langle \sin(PL/\hbar) \rangle = (\sin \alpha)/2$. Then, it is possible, in principle, to construct observables capable of detecting the relative phase of two non-overlapping wave packets.

It should be noticed that only the value

$$P_{\text{mod}} \equiv P \bmod \frac{2\pi\hbar}{L} \tag{2.46}$$

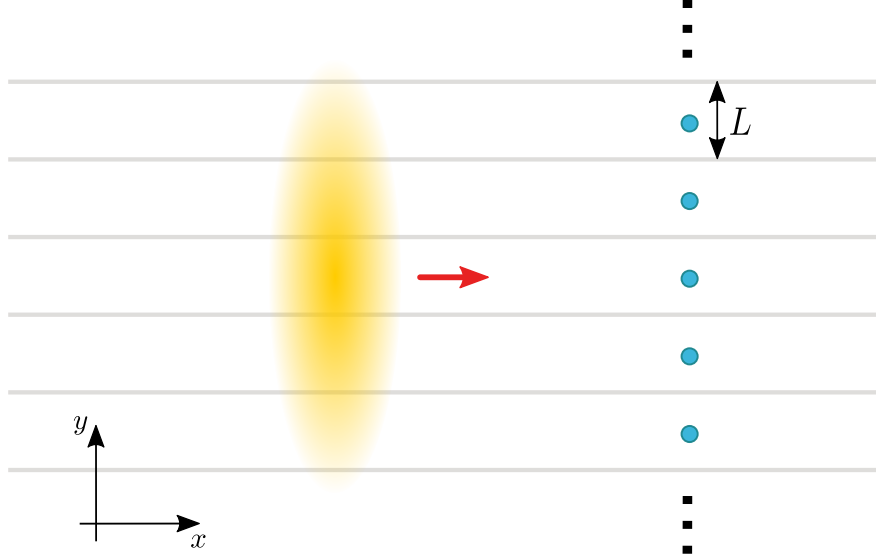


Figure 2.2: Quantum charge q (yellow cloud) sent through a lattice of flux lines (blue dots), each carrying a magnetic flux Φ_B . Low-energy particles are reflected. The gray lines are just horizontal lines that cross the lattice in the midpoint between two flux lines and are not part of the configuration of the system.

is relevant for $e^{iPL/\hbar}$ and any other operator derived from it. The operator P_{mod} is called the *modular momentum* of the charge and it is a type of *modular variable* — a concept introduced in 1969 by Aharonov, Pendleton and Petersen [96], and further developed after that [58, 97–100]. Modular variables of position and energy can be also defined [58].

Before proceeding, observe that, from Eq. (2.46), it can be introduced a Hermitian operator N such that

$$P = \frac{2\pi\hbar}{L}N + P_{\text{mod}}. \quad (2.47)$$

As an application of this concept, consider the scenario represented in Fig. 2.2 of a charge traveling towards a lattice, which coincides with the y axis, of impenetrable lines spaced by a length L . Then, the Hamiltonian of the particle can be written as

$$H = \frac{1}{2m} \left(P_x^2 + P_y^2 \right) + V(X, Y), \quad (2.48)$$

where V is a potential associated with lattice. Observe that $V(x, y) = 0$ for every $x \neq 0$. Moreover, it is periodic in y when $x = 0$. Using the Heisenberg equation of motion, it can, then, be observed that, for $x \neq 0$,

$$\frac{d}{dt}P_y = \frac{i}{\hbar} [H, P_y] = 0, \quad (2.49)$$

i.e., there is no change in the momentum P_y . The same can be said about P_x . Now, at $x = 0$, $[H, P_y] \neq 0$, which implies that P_y is, in general, not conserved. However,

$$\begin{aligned} \frac{d}{dt}e^{i(P_y)_{\text{mod}}L/\hbar} &= \frac{i}{\hbar} [H, e^{iP_yL/\hbar}] \\ &= \frac{i}{\hbar} [V(Y), e^{iP_yL/\hbar}] \\ &= \frac{i}{\hbar} (V(Y) - V(Y + L)) e^{i(P_y)_{\text{mod}}L/\hbar}, \end{aligned} \quad (2.50)$$

where $V(y) \equiv V(0, y)$. Because of the periodicity of $V(y)$, the term between parenthesis in the last part of Eq. (2.50) vanishes, i.e., there is no change in the modular momentum $(P_y)_{\text{mod}}$. Thus, from Eq. (2.47), it can be concluded that the momentum P_y of the particle can only change by multiples of $2\pi\hbar/L$ during its interaction with the lattice, i.e.,

$$\Delta p_y = \frac{2\pi n\hbar}{L}, \quad (2.51)$$

where $n \in \mathbb{Z}$.

To verify this result, observe that, if the lattice is sufficiently heavy and the initial wave length of the charge is λ , the total momentum $p = 2\pi\hbar/\lambda$ of the particle should also be unchanged. Then, while crossing the lattice, the charge is scattered by angles θ_n such that

$$\sin \theta_n = \frac{\Delta p_y}{p} = \frac{n\lambda}{L}, \quad (2.52)$$

which is the well-known expression for the discrete scattering angles in a diffraction experiment.

Now, suppose that each line on the lattice carries a magnetic flux Φ_B . Then, because of the AB effect, the angles in Eq. (2.52) are shifted by angles θ'_n such that

$$\sin \theta'_n = \frac{\Delta p_y}{p} = \left(n + \frac{q\Phi_B}{2\pi\hbar} \right) \frac{\lambda}{L}. \quad (2.53)$$

This implies that the change in the y direction of the momentum is

$$\Delta p_y = p \sin \theta'_n = \left(2\pi n + \frac{q\Phi_B}{\hbar} \right) \frac{\hbar}{L}, \quad (2.54)$$

i.e., the modular momentum in the y direction is not conserved, differently from the scenario without a magnetic flux on the lines. In fact, there is a change in modular momentum given by

$$(p_y)_{\text{mod}} = \frac{q\Phi_B}{L}. \quad (2.55)$$

It should be noticed that the value of the modular momentum in Eq. (2.55) refers to the kinematic momentum, i.e., the product of the mass of the particle by its velocity, in opposition to the canonical momentum, which is the canonical conjugated of the position. This distinction is important because, although both momenta coincide when the Hamiltonian is the one in Eq. (2.48), the two concepts diverge when the charge's dynamics is governed, for instance, by the Hamiltonian in Eq. (2.29).

2.4 Instantaneous Change in the Transverse Velocity

The discussion at the end of the previous section suggests that there is a sudden change in the velocity of charges when the line connecting two wave packets crosses a flux line carrying

a magnetic flux Φ_B . As shown in Ref. [57], this is, in fact, the case.

To see that, assume that a particle with charge q is prepared in the state

$$\Psi_0(x, y) = \frac{1}{\sqrt{2}} (\psi(x, y) + \psi(x + L_x, y + L_y)), \quad (2.56)$$

where ψ is a wave packet and $\vec{L} = L_x\hat{x} + L_y\hat{y}$ is the separation between the centers of the wave packets. Suppose the packets evolve in time without changing the distance $L = |\vec{L}|$ between them until the particle crosses a flux line placed at the origin of the coordinate system, meaning that the line that connects the centers of both wave packets crosses the flux line. Also, let the Hamiltonian that governs the dynamics of the charge be

$$H = \frac{1}{2m} \left[(P_x - qA_x(x, y))^2 + (P_y - qA_y(x, y))^2 \right], \quad (2.57)$$

where $\vec{A} = A_x\hat{x} + A_y\hat{y}$ is the vector potential associated with the flux line.

In this case, the modular velocity \vec{v}_{mod} of the particle can be defined, and it is such that

$$m\vec{v}_{\text{mod}} \equiv (\vec{P} - q\vec{A}) \pmod{\frac{2\pi\hbar}{L}}. \quad (2.58)$$

Furthermore, the modular velocity u_{mod} in the direction that connects the packets satisfies

$$mu_{\text{mod}} \equiv (\vec{P} - q\vec{A}) \cdot \frac{\vec{L}}{L} \pmod{\frac{2\pi\hbar}{L}}. \quad (2.59)$$

Also, preparing for the results that will be presented next, observe that, for every holomorphic

function f and unitary U ,

$$\begin{aligned}
f(U^\dagger \vec{P} \cdot \vec{L} U) &= \sum_{n=0}^{\infty} \left(U^\dagger \vec{P} \cdot \vec{L} U \right)^n f_n \\
&= \sum_{n=0}^{\infty} U^\dagger \left(\vec{P} \cdot \vec{L} \right)^n U f_n \\
&= U^\dagger f(\vec{P} \cdot \vec{L}) U.
\end{aligned} \tag{2.60}$$

Then, letting $f(\vec{P} \cdot \vec{L}) = e^{i\vec{P} \cdot \vec{L}/\hbar}$ and $U = e^{-iq \int_{\gamma(\vec{r})} \vec{A} \cdot d\vec{\ell}/\hbar}$, and noting that

$$e^{iq \int_{\gamma(\vec{r})} \vec{A} \cdot d\vec{\ell}/\hbar} \left(\vec{P} \cdot \vec{L} \right) e^{-iq \int_{\gamma(\vec{r})} \vec{A} \cdot d\vec{\ell}/\hbar} = \left(\vec{P} - q\vec{A}(\vec{r}) \right) \cdot \vec{L}/\hbar, \tag{2.61}$$

it can be concluded that

$$e^{i\left(\vec{P}-q\vec{A}(\vec{r})\right) \cdot \vec{L}/\hbar} = e^{iq \left(\int_{\gamma(\vec{r})} \vec{A} \cdot d\vec{\ell} - \int_{\gamma(\vec{r}+\vec{L})} \vec{A} \cdot d\vec{\ell} \right) / \hbar} e^{i\vec{P} \cdot \vec{L}/\hbar} = e^{iq \int_{\tau(\vec{r})} \vec{A} \cdot d\vec{\ell}/\hbar} e^{i\vec{P} \cdot \vec{L}/\hbar}, \tag{2.62}$$

where γ is an arbitrary path and τ is a segment of line with length L that goes from the center of packet 2 to the center of packet 1.

Now, suppose the flux line carries zero magnetic flux and denote the state of the charge at an instant of time t by

$$\Psi_0(\vec{r}, t) = \frac{1}{\sqrt{2}} (\Psi_1(\vec{r}, t) + \Psi_2(\vec{r}, t)). \tag{2.63}$$

In this case, there is no change in modular velocity. However, if the flux line carries a magnetic flux Φ_B , its state Ψ as a function of position and time can be written as

$$\Psi(\vec{r}, t) = \frac{1}{\sqrt{2}} \left(e^{iq \int_{\gamma_1} \vec{A} \cdot d\vec{\ell}/\hbar} \Psi_1(\vec{r}, t) + e^{iq \int_{\gamma_2} \vec{A} \cdot d\vec{\ell}/\hbar} \Psi_2(\vec{r}, t) \right), \tag{2.64}$$

where γ_1 refers to the trajectory of the center of packet 1 and γ_2 refers to the trajectory of

the center of packet 2. With help of Eq. (2.62), this implies that

$$\begin{aligned}
\langle e^{imu_{\text{mod}}L/\hbar} \rangle &= \iint \overline{\Psi(\vec{r}, t)} e^{iq \int_{\tau(\vec{r})} \vec{A} \cdot d\vec{\ell}/\hbar} e^{i\vec{P} \cdot \vec{L}/\hbar} \Psi(\vec{r}, t) dx dy \\
&= \frac{1}{2} e^{iq \oint_{\gamma} \vec{A} \cdot d\vec{\ell}/\hbar} \iint \overline{\Psi_1(\vec{r}, t)} \Psi_2(\vec{r} + \vec{L}, t) dx dy \\
&= \frac{1}{2} e^{iq \oint_{\gamma} \vec{A} \cdot d\vec{\ell}/\hbar} \\
&= \frac{1}{2} e^{iq\Phi_{\text{enc}}/\hbar},
\end{aligned} \tag{2.65}$$

where Φ_{enc} is the total magnetic flux enclosed by the closed path. In Eq. (2.65), the Stokes' theorem and the fact that

$$\int_{\gamma_1} \vec{A} \cdot d\vec{\ell} - \int_{\gamma_2} \vec{A} \cdot d\vec{\ell} - \int_{\tau(\vec{r})} \vec{A} \cdot d\vec{\ell} = \oint_{\gamma} \vec{A} \cdot d\vec{\ell} \tag{2.66}$$

were used. To see that Eq. (2.66) holds, it should be observe that, to be prepared in the superposition of packets given by Eq. (2.56), the charge must have started localized in a certain region. Then, γ_1 and γ_2 start at the same point.

Finally, because Eq. (2.65) implies that the modular velocity only changes with a variation in the magnetic flux inside the area “covered” by the packets, there is an instantaneous change in the modular velocity u_{mod} when the line that connects the center of the two packets crosses the flux line. Before and after that, there should be no change to the modular velocity.

It should be noticed that this result depends on the fact that the wave packets satisfies the property of being the translation of each other in a Cartesian system of coordinates. In a general case, the analysis of how the modular velocity changes is more complicated.

To conclude, the fact that the AB effect can be seen as a change in the distribution of the velocity of a charge suggests that there may be a force acting on the particle. In fact, forces associated with the AB effect were previously discussed in Refs. [56, 101–103]. This aspect will be further investigated in Chapter 3.

2.5 Nodal Lines

A quantum particle confined to a finite region of space behaves drastically different from a classical particle. In particular, its energy becomes quantized. In fact, consider the problem known as the *infinite well*, which was likely [104] introduced by Mott [105] in 1930, and is typically one of the first examples presented in an introductory course on quantum mechanics. It consists of a particle confined in a region $|x| < \ell/2$, $\ell > 0$, by the potential

$$V(x) = \begin{cases} 0 & \text{if } |x| < \ell/2 \\ \infty & \text{if } |x| \geq \ell/2 \end{cases}. \quad (2.67)$$

In this case, the solutions of the Schrödinger equation vanish outside the region with zero potential. Moreover, inside the well, the eigensolutions of the Schrödinger equation are

$$\vartheta_n(x) = \sqrt{\frac{2}{\ell}} \sin \left[n\pi \left(\frac{x}{\ell} + \frac{1}{2} \right) \right], \quad (2.68)$$

for every $n \in \mathbb{N}$. Moreover, their associated energy is

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2m\ell^2}. \quad (2.69)$$

The quantization of energy is a general result of the confinement of quantum particles that, then, holds not only for one-dimensional cases.

Here, following Refs. [47, 55], it is studied how the presence of a flux line affects the eigenstates of two-dimensional cavities. For that, consider a cavity with an impenetrable flux line on its interior. First, suppose there is no magnetic field traveling through the flux line. Then, the Hamiltonian of a free particle inside the cavity is given by Eq. (2.30), with the consideration that, besides being zero outside the cavity, the state that describes the particle

should also vanish at the flux line. Now, any eigenfunction ψ_0 of such a Hamiltonian, i.e.,

$$H_0\psi_0 = E\psi_0, \quad (2.70)$$

can be assumed to be a real function without loss of generality. Then, if the flux line carries a magnetic flux Φ_B , which is associated with a vector potential \vec{A} , the Hamiltonian of the cavity becomes the one given by Eq. (2.29). Moreover, if the wave function ψ of a charge is an eigenfunction of such a Hamiltonian, it satisfies

$$H\psi = E\psi. \quad (2.71)$$

With a similar approach used for Eqs. (2.31) and (2.32), it can be shown that there exists an eigenfunction ψ_0 of H_0 such that

$$\psi = \psi_0 e^{iq \int_{\gamma} \vec{A} \cdot d\vec{\ell} / \hbar}, \quad (2.72)$$

where γ denotes a path. In this case, Eq. (2.72) can be seen as generating a complex continuation rule of ψ_0 . In general, these extensions determine a Riemann surface [106, 107].

The rule in this case is

$$\psi(\vec{r}_{\gamma}) = \psi_0(\vec{r}) e^{iW_{\gamma}\varphi_{AB}}, \quad (2.73)$$

where the winding number of γ , W_{γ} , is such that [108]

$$\oint_{\gamma} \vec{A} \cdot d\vec{\ell} = W_{\gamma} \Phi_B. \quad (2.74)$$

Eq. (2.73) should be read as a relation between the end points of a closed path γ in real space that starts in \vec{r} and finishes in \vec{r}_{γ} in the Riemann surface.

In order to obtain a single-valued function, a sheet of such a manifold may be chosen. This

choice, however, introduces a discontinuity in the wave function in an arbitrary location. Nonetheless, it should be noticed that this discontinuity is only in the system's phase. This approach of making ψ single-valued can be intuitively seen as an imposition of a periodic boundary condition on ψ_0 . With this view, instead of solving Eq. (2.29) for ψ , one solves Eq. (2.30) for ψ_0 with the extra condition that, in polar coordinates (ρ, θ) ,

$$\psi_0(\rho, \theta + 2\pi) = e^{-i\varphi_{AB}}\psi_0(\rho, \theta). \quad (2.75)$$

This is physically relevant because it implies that the absolute value of the wave function and, also, its probability density are independent of the choice of sheet.

Now, assume the flux line inside the cavity carries a half-integer fluxon, also called a *semifluxon*, i.e., $\Phi_B = \Phi_0/2$. This particular value is interesting because it makes the extension rule given by Eq. (2.73) real since $\varphi_{AB} = \pi$ and, then, $e^{i\varphi_{AB}} = -1$. As a result, ψ can be taken to be real.

Furthermore, because ψ changes its sign in a path which encircles a semifluxon, it must be zero somewhere in the path, regardless of the shape of the cavity. Therefore, by the continuity of the surface determined by ψ (or the continuity of its probability distribution in case a sheet is chosen), there must exist a nodal curve, i.e., a curve where ψ vanishes, starting at the flux line and finishing at the boundary of the cavity — or the position of another flux line carrying a semifluxon [47, 55].

It should be noted that linear combinations of eigenstates also possess nodal curves. Moreover, nodal curves of eigenstates become straight lines according to the symmetry of the cavity.

To summarize and illustrate the main points of the discussion presented here, consider a circular cavity of unit radius with a semifluxon at its center. In this case, the ground state

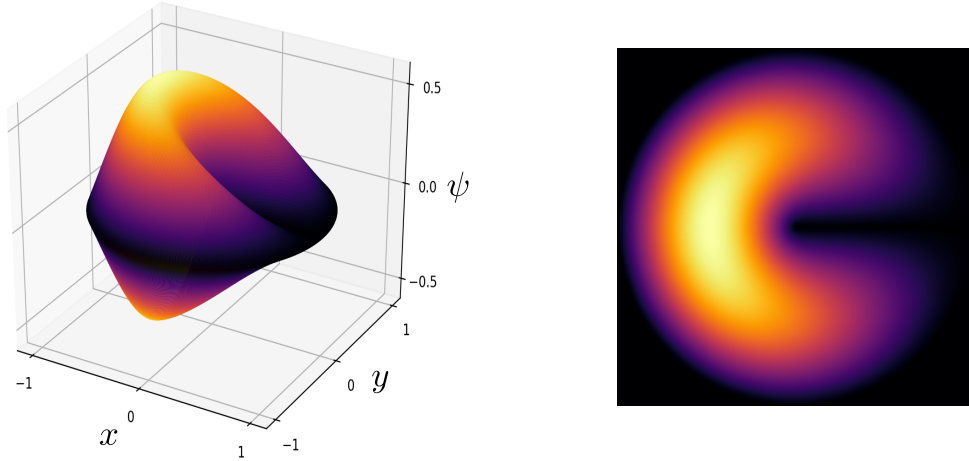


Figure 2.3: Representation of the real ground state of a circular cavity with a flux line carrying a semifluxon on its center. The three-dimensional plot on the left is the real Riemann surface associated with the “multivalued” wave function of Eq. (2.73) for $\varphi = \pi$. The color map defined on it associates colors to the magnitudes of ψ . This allows the drawing of the heat map of any arbitrarily chosen sheet of that surface, as represented on the right.

of the cavity when $\vec{A} = 0$ is

$$\psi_0(\vec{r}) = J_1(ar), \quad (2.76)$$

where r is the radial coordinate, J_1 is the Bessel function of the first kind and parameter 1, and $a \approx 3.83$ is the smallest positive zero of J_1 . Then, when the flux line is carrying a semifluxon, the real ground state can be expressed as

$$\psi(\vec{r}) = \psi_0(r) \cos(\theta/2), \quad (2.77)$$

where θ is the angular coordinate. Note that ψ is a “multivalued function”. On the left-hand side of Fig. 2.3, the Riemann surface associated with ψ is represented, while the heat map of any arbitrary choice of sheet is represented on the right-hand side. Observe the nodal line induced by the flux line.

To conclude, it is worth mentioning that nodal curves appear in a variety of topological

systems, not only in systems with flux lines. Recently, special attention has been dedicated to them, and they have been realized in multiple physical configurations [109–116].

2.6 Equivalence Principle

The AB effect has an impact that goes beyond the study of electromagnetic systems. As an example of it, it will be shown, following Ref. [117], that it requires an extension of the classical *equivalence principle* in quantum physics.

The equivalence principle states that inertial and gravitational mass are equivalent. However, another way of seeing it is by stating that pseudo-forces that arise in non-inertial frames can be canceled by other forces and, then, physical systems in the non-inertial frames behave as if they were in an inertial frame.

Now, consider a laboratory given by a narrow ring with an inner radius R_1 and an outer radius R_2 , as represented by Fig. 2.4. Also, assume the lab is rotating with angular velocity ω and, to simplify the discussion, let all particles inside the lab have the same ratio of charge over mass. Moreover, it should be noticed that the disc with radius R_1 is massive.

Because of its rotation, the lab experiences two pseudo-forces, namely the *centrifugal* and the *Coriolis* force. Then, a quantum physicist, Alice, and a classical physicist, Bob, can detect that the lab is a non-inertial frame. However, suppose that another physicist, Charlie, decides to trick Alice and Bob. For that, Charlie observes that the Coriolis force \vec{F}_C acting on an object with mass m and velocity (measured in the lab) \vec{v} can be written as $\vec{F}_C = m\vec{v} \times \vec{C}$, where \vec{C} is the field associated with \vec{F}_C . Moreover, \vec{C} satisfies

$$\vec{\nabla} \cdot \vec{C} = 0, \tag{2.78}$$

which implies that \vec{F}_C is given by a field that is the rotation of a vector potential \vec{V}_C . Also,

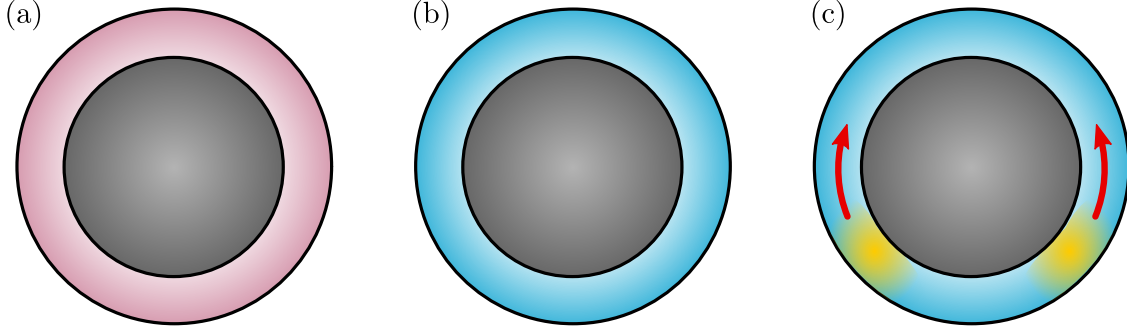


Figure 2.4: Representation of a laboratory given by a narrow ring. (a) A classical physicist, Bob, and a quantum physicist, Alice, can verify whether the laboratory (pink region) is spinning or not. (b) However, when an electromagnetic field (blue region) is added to cancel the non-inertial forces associated with the spinning inside the laboratory, Bob can no longer know he is not in an inertial frame. (c) Nonetheless, Alice can detect the spinning with a particle in a superposition (yellow clouds) encircling the disk internal to the laboratory (gray area), where the forces were not canceled by the electromagnetic field, which causes a shift in the interference pattern proportional to the angular velocity of the disk.

if \vec{F}_c is the centrifugal force,

$$\vec{\nabla} \times \vec{F}_c = 0, \quad (2.79)$$

i.e., \vec{F}_c can be written as the gradient of a scalar potential. Therefore, there exists a parallel between the centrifugal and the Coriolis forces with electromagnetic forces.

Charlie, then, decides to apply an electromagnetic field *inside* the lab in order to remove the pseudo-forces. Because of this, Bob is not able to detect that the lab is a non-inertial frame. In fact, any experiment he performs with classical particles will give results that correspond to measurements made in inertial systems.

Nevertheless, because of the AB effect, Alice can conduct an interference experiment with a quantum particle and detect that the lab is rotating. The reason is that, although Charlie canceled the pseudo-forces inside the lab, these pseudo-forces were not canceled in the interior of the disk with radius R_1 . Then, if a quantum particle encircles the disk in a superposition of packets traveling in the clockwise and counterclockwise directions, the interference pattern

detected by Alice will be shifted by a phase proportional to

$$m \oint \vec{V}_C \cdot d\vec{\ell} = m \int_{D_1} \vec{C} \cdot d\vec{S} = 2\pi m R_1^2 \omega, \quad (2.80)$$

where D_1 is the disk with radius R_1 .

The scenario considered here shows that the equivalence principle needs to be modified in quantum physics, and this is an example of how the implications of the AB effect can reach areas that go beyond the dynamics of quantum charges.

3 Magnetic Forces from the Aharonov-Bohm Effect

As discussed in the previous chapter, a force can be associated with the Aharonov-Bohm (AB) effect, even though it does not result from a local interaction between the charge and the magnetic field. In this chapter, the main results from Refs. [66, 67] are presented, showing that such a force corresponds, in a certain sense, to a magnetic force.

Specifically, it is discussed how an infinite lattice of flux lines acts as an energy barrier for quantum charges and that two flux lines can be used to confine low-energy charges in a sector of a long cavity. More generally, it is shown that grids of flux lines can emulate continuous two-dimensional magnetic fields. This allows the obtaining of the Landau levels in a uniform grid of flux lines. This result also suggests a semi-classical model where the force associated with a grid of flux lines can be seen as quantum counterparts of forces from classical continuous two-dimensional magnetic fields.

3.1 Continuous Magnetic Fields in Classical Physics

In this section, a simple example of a magnetic field as an energy barrier in classical physics is presented. This example will be further explored in the context of quantum mechanics in the next two sections.

Consider a (penetrable) wall parallel to the y axis with a uniform magnetic field $\vec{B} = B\hat{z}$ and a point particle with mass m and charge q traveling in the x direction, as represented in Fig. 3.1. Moreover, let $\Phi_B = BLw$ be the magnetic flux associated with any arbitrary region with length L and width w . When the particle is inside the wall, a force $\vec{F} = \vec{v} \times \vec{B}$, where \vec{v} is the particle's velocity, acts on it. Then, using Newton's second law, it can be

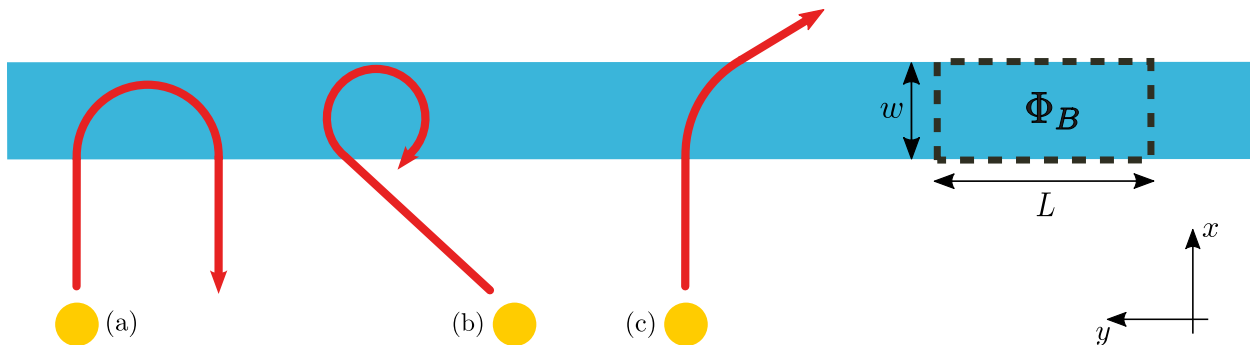


Figure 3.1: Classical charges (yellow dots) traveling towards a uniform magnetic field wall (blue region) given by $\vec{B} = B\hat{z}$ and such that any rectangular region with length L and width w encloses a flux $\Phi_B = BLw$. Their trajectories are represented in red. In (a) and (b), low-energy particles cannot cross the wall, while a high-energy particle is scattered by it in (c). In general, if a particle passes through the wall, its transverse kinematic momentum is changed by $-q\Phi_B/L$.

checked that the charge is deflected by the field into a circular arc of radius

$$R = \frac{mvv}{|q\phi_B|}, \quad (3.1)$$

where

$$\phi_B = \frac{\Phi_B}{L} \quad (3.2)$$

is the magnetic flux per unit of transverse length. Then, if the speed $v = \|\vec{v}\|$ of the particle is such that $R < w$, i.e., if the magnitude of the particle's kinematic momentum $p_x = mv$ is such that

$$p_x < |q\phi_B|, \quad (3.3)$$

the particle is reflected by the wall.

Note that, here, and everywhere in the present chapter, there is no bound on the z component of the charge's kinematic momentum (or on the energy associated with it), i.e., it can be arbitrarily large and remain conserved without affecting the results.

Now, if the width w of the wall is reduced, while the magnitude of the field B is simultaneously increased in a way that ϕ_B remains constant, the maximum speed v for which a given charge is reflected by the wall remains unchanged. This continues to hold even in the limit $w \rightarrow 0$, i.e., when the magnetic field wall becomes widthless. Therefore, the barrier is characterized entirely by the magnetic flux per unit of length ϕ_B associated with the wall.

Furthermore, Eq. (3.1) is valid even if the particle is incident at an arbitrary angle in the xy plane, as represented in Fig. 3.1(c). Then, it can be checked with a simple geometric analysis that, in general, the wall reflects all those charges if $2R < w$, i.e., if their planar kinematic momentum $mv = \sqrt{p_x^2 + p_y^2}$ is such that

$$\sqrt{p_x^2 + p_y^2} < \frac{|q\phi_B|}{2}. \quad (3.4)$$

Finally, if the charge crosses the wall in a time interval Δt , the change in transverse kinematic momentum is

$$\Delta p_y = -qB \int_{\Delta t} v_x(t) dt = -qBw = -q\phi_B, \quad (3.5)$$

i.e., the change in the transverse kinematic momentum is independent of the angle of incidence and the velocity of the particle. It can also be verified that, when the charge is reflected, there is no change in the magnitude of its transverse kinematic momentum after the interaction with the wall: the angle of incidence equals the angle of reflection.

3.2 Continuous magnetic fields in quantum mechanics

In this section, the direct quantum analog of the result from the previous section will be investigated. For simplicity, only the infinite widthless magnetic wall will be considered.

First, the reference frame is such that the wall is placed on the y -axis, and the particle starts

on the left-hand side. A possible choice of vector potential for this scenario is

$$\vec{A} = \phi_B \Theta(x) \hat{y}, \quad (3.6)$$

where Θ is the Heaviside step function, i.e.,

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}. \quad (3.7)$$

In this gauge, the Hamiltonian of the system can be written as

$$H = \frac{1}{2m} \left[P_x^2 + (P_y - q\phi_B \Theta(X))^2 \right], \quad (3.8)$$

which implies that the initial average energy of the charge can be computed as

$$\langle E \rangle_i = \frac{1}{2m} \left(\langle P_x^2 \rangle_i + \langle P_y^2 \rangle_i \right) \quad (3.9)$$

and its final average energy, in case it crosses the lattice, is

$$\begin{aligned} \langle E \rangle_f &= \frac{1}{2m} \left(\langle P_x^2 \rangle_f + \langle P_y^2 \rangle_f + q^2 \phi_B^2 - 2q\phi_B \langle P_y \rangle_f \right) \\ &= \frac{1}{2m} \left(\langle P_x^2 \rangle_f + \langle P_y^2 \rangle_i + q^2 \phi_B^2 - 2q\phi_B \langle P_y \rangle_i \right), \end{aligned} \quad (3.10)$$

where it was used the fact that the canonical transverse momentum is conserved, i.e., $\langle P_y \rangle_i = \langle P_y \rangle_f$ and $\langle P_y^2 \rangle_i = \langle P_y^2 \rangle_f$, which follows from $[H, P_y] = 0$.

To avoid possible confusion due to the asymmetry in \vec{A} , note that, if the particle had started on the right-hand side, its initial kinematic momentum would be given by $mv_y = P_y - q\phi_B$. Then, if the particle started with zero transverse kinematic momentum, for instance, its initial canonical transverse momentum would be such that $\langle P_y \rangle_i = q\phi_B$. Again, the canonical

momentum would be conserved, which implies that the particle's final transverse kinematic momentum after it crossed the wall would be, on average, $q\phi_B$. This verifies the symmetry of rotation of the xy plane by 180° , which justifies the restriction to particles starting on the left-hand side of the wall.

With the requirement of change in the average transverse kinematic momentum, it is possible to conclude that a charge cannot completely cross the wall whenever $\langle E \rangle_i < \langle E \rangle_{f,\min}$. To be precise, higher-energy components of the wave packet associated with the charge pass through the wall, while lower-energy components are reflected. Generally, the probability of finding the particle goes to zero in the far-field on the side opposite to the one it started.

Then, if the particle is incident with average (canonical and kinematic) momentum $\langle P_x \rangle_i$ in the x axis and $\langle P_y \rangle_i$ in the y axis, its minimum average energy after crossing the wall is obtained when $\langle P_x \rangle_f \rightarrow 0$, and it cannot completely pass it if

$$\langle P_x^2 \rangle_i < q^2 \phi_B^2 - 2q\phi_B \langle P_y \rangle_i. \quad (3.11)$$

It follows that, if the particle is perpendicularly incident, i.e., $\langle P_y \rangle_i = 0$, the above condition becomes

$$\langle P_x \rangle_i < |q\phi_B|, \quad (3.12)$$

where it was used the fact that $\langle P_x^2 \rangle \geq \langle P_x \rangle^2$. Observe that Eq. (3.12) is analogous to Eq. (3.3). Moreover, if $\langle P_y \rangle_i \neq 0$, Eq. (3.11) only admits solution if its right-hand side is positive, i.e., if $|\langle P_y \rangle_i| < |q\phi_B|/2$, then, the condition for a charge to be at least partially reflected by the wall, regardless of the angle of incidence, is

$$\sqrt{\langle P_x \rangle_i^2 + \langle P_y \rangle_i^2} < \frac{|q\phi_B|}{2}, \quad (3.13)$$

which is analogous to Eq. (3.4).

3.3 Emulating Continuous Magnetic Fields with Distributions of Flux Lines

In this section, the widthless magnetic wall is replaced by flux lines — or infinitely thin solenoids. In two dimensions, these lines are point objects. Without loss of generality, the magnetic flux on each line is assumed to be positive. This can always be achieved with a rotation of the referential system. Also, recall that the influence of each flux line in the dynamics of quantum charges is invariant under the addition of a fluxon, i.e., the magnetic flux Φ_0 defined in Eq. (2.38). Because of this, the magnetic fluxes can be taken to be limited to the interval $[0, \Phi_0)$.

The continuous magnetic wall is replaced with a lattice of flux lines with spacing L , with each line carrying a magnetic flux Φ_B , as represented in Fig. 2.2. As discussed in the previous chapter, an incident charge represented by a plane wave has the modular part of its transverse kinematic momentum changed according to Eq. (2.55). What is most important for the present analysis is the fact that this implies that there is a minimum change in the transverse kinematic momentum of the charge, which is given by $|\Delta p_y|_{\min} = |q\phi_B|$ if $\Phi_B \leq \Phi_0/2$, and $2\pi\hbar/L - q\phi_B$ if $\Phi_B > \Phi_0/2$. Observe that the latter corresponds to the minimum deflection if $-\Phi_0/2 < \Phi_B < 0$. Because of this extra symmetry, hereby, only magnetic fluxes in the interval $(0, \Phi_0/2]$ are considered.

Let the charge start with average kinematic momentum $p_i = p_x^{(i)}\hat{x} + p_y^{(i)}\hat{y}$. Then, if $p_y^{(i)} = 0$, the particle acquires a transverse kinematic momentum of at least $-q\phi_B$. Therefore, if $p_x^{(i)} < |q\phi_B|$ (Eq. (3.3)) is satisfied, the charge is reflected. Moreover, if $p_y^{(i)} \neq 0$, its transverse kinematic momentum can, in principle, decrease in magnitude after it crosses the lattice. However, if $|p_y^{(i)}| < |\Delta p_y|_{\min}/2$, the magnitude of the final transverse kinematic momentum of the particle cannot be smaller than $\left| |p_y^{(i)}| - |\Delta p_y|_{\min} \right|$, which is still greater than $|p_y^{(i)}|$. In other words, if Eq. (3.4) is satisfied, the charge must bounce off the lattice of

flux lines. In conclusion, the lattice constitutes an energy barrier similar to the continuous magnetic wall. Interestingly, Eq. (3.4) does not depend explicitly on \hbar . However, because Φ_B is upper-bounded by $\Phi_0 = 2\pi\hbar/q$, for any fixed lattice spacing L , $\phi_B = \Phi_B/L \rightarrow 0$ when $\hbar \rightarrow 0$. However, L can be adjusted so that $L \rightarrow 0$ and ϕ_B is constant in that limit. This shows that, in specially designed configurations, consequences of the AB effect can still hold in the classical limit.

3.4 Topological Bound States

In this section, it is shown that flux lines can be used to create bound states. However, to continue the parallel between discrete and continuous distribution of fields, the cases with continuous magnetic fields are considered first.

In classical physics, it was studied in Section 3.1 that widthless magnetic walls behave like mirrors for low-energy charges. If the wall is placed inside a long rectangular cavity, as represented in Fig. 3.2(a), it continues to behave like a mirror. Furthermore, if a low-energy charge is placed between two of those walls, independently of how close they are from each other, it remains trapped in that region, as shown in Fig. 3.2(b).

A similar result holds in a quantum treatment of the problem. An argument close to the one presented in Section 3.2 can be given. For that, consider a quantum charge traveling towards a magnetic wall, as represented in Fig. 3.2(c). Choosing a gauge for which the vector potential inside the cavity is written as Eq. (3.6), it is possible to conclude that there should be an increase in the particle's average kinematic energy associated with the y direction after it completely crosses the wall. Then, if its initial average energy is sufficiently small, the charge is at least partially reflected.

Now, a quantum charge can also be trapped by two walls of magnetic field, as illustrated

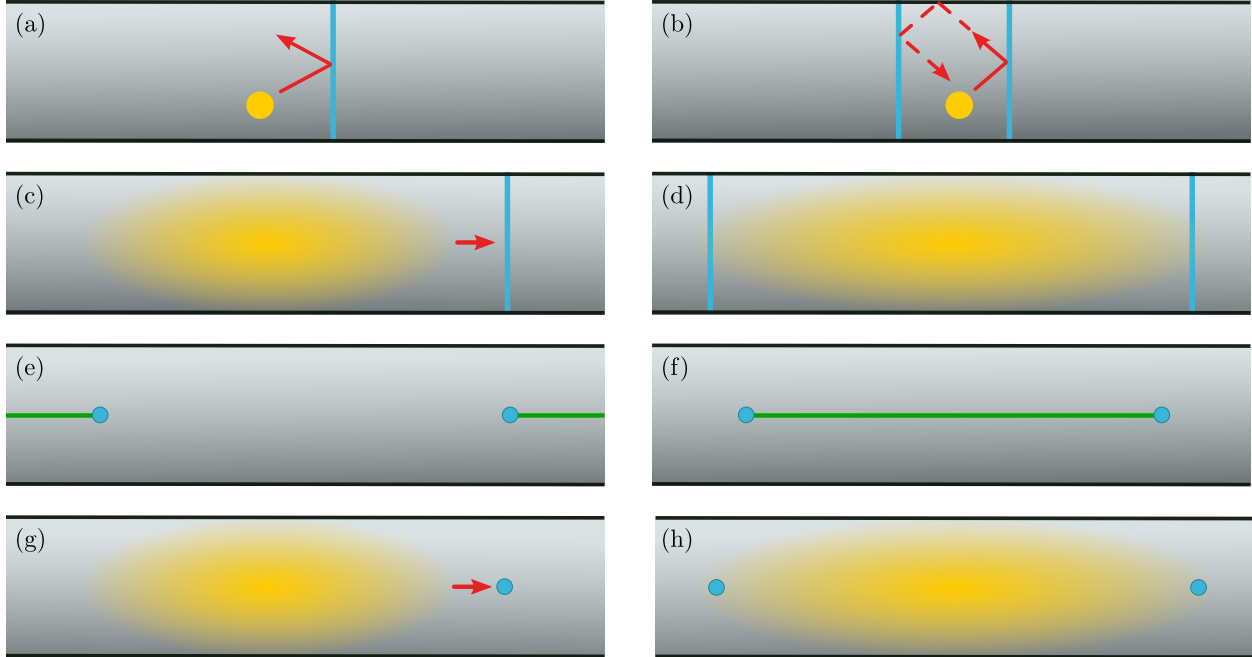


Figure 3.2: Representation of particles inside a cavity. A Classical particle (yellow dot) can be (a) reflected trapped and (b) trapped by magnetic walls (blue lines). The red curves represent the particle's trajectory. For a quantum charge q (yellow cloud) inside a cavity, the magnetic walls also (c) work as an energy barrier, and (d) can trap the particle. This effect persists in the quantum case even if the wall is replaced by a flux line (blue dot), as shown in (g) and (h). In particular, if the flux line carries a semifluxon, nodal lines are induced in the wave functions of charges, as illustrated by the green lines in (e) and (f).

in Fig. 3.2(d). As discussed in Section 2.5, the energy of a quantum particle confined to a cavity becomes quantized. There, the case of an infinite well was given as an example. The rectangular cavity of interest here is a two-dimensional version of that problem. In fact, because the Schrödinger equation becomes separable, the solution to this problem is similar to the one-dimensional case. The overall eigenstates are products of the eigenstates of each dimension and their energies are the sum of the energy associated with each direction.

While these ideas did not play a role in the discussion of the scenario represented in Fig. 3.2(c), they are fundamental in the case where a particle is confined by two magnetic walls inside a long rectangular cavity. In fact, letting L be the width of the cavity and D be the distance between the walls, the minimum average energy of a particle that starts in the

region between the walls must have at least an average energy of

$$\langle E \rangle_{i,\min} = \frac{\pi^2 \hbar^2}{2mL^2} + \frac{\pi^2 \hbar^2}{2mD^2}, \quad (3.14)$$

i.e., the minimum energy of a particle inside a two-dimensional box with lengths D and L . Moreover, the amount of average energy necessary for a charge to completely cross one of the walls is greater than or equal to

$$\langle E \rangle_{f,\min} = \frac{\pi^2 \hbar^2}{2mL^2} + \frac{q^2 \phi_B^2}{2m}. \quad (3.15)$$

Hence, if $|q\phi_B| > \pi\hbar/D$, i.e., if the separation D between the walls is such that

$$D > \frac{\pi\hbar}{|q\phi_B|}, \quad (3.16)$$

it is possible to have a low energy particle that stays (at least partially) trapped in the region between the two walls. Observe that, in general, the particle will have a non-trivial dynamics inside the region between the walls. In any case, the result just presented implies that the walls create bound states inside the cavity. Moreover, different from the classical case, there is a minimum distance between the walls required for the possibility of having a particle confined by them. However, in the classical limit, i.e., when $\hbar \rightarrow 0$, the value of this minimum distance goes to zero.

Finally, the case where the magnetic walls are replaced by flux lines, which is the main focus of this section, is now considered. Once again, consider a rectangular cavity with width L equipped with two flux lines placed with a distance D from each other on the long symmetry axis of the cavity, as shown in Fig. 3.2(e). Then, the following question can be asked: do the flux lines work similarly to the walls of magnetic field just considered? Or, more specifically, do they create bound states inside the cavity?

To approach this problem, first consider flux lines carrying a semifluxon each, i.e., a magnetic flux given by $\Phi_B = \Phi_0/2$. This simplifies the analysis because, as studied in Section 2.5, semifluxons induce nodal lines in the wave function of charges. In this particular configuration, the symmetry of the problem implies that the nodal lines associated with the flux lines for eigenstates of the cavity should be as illustrated by the green lines in Figs. 3.2(e) or 3.2(f). To see that, note that the Hamiltonian of the cavity remains physically invariant if the cavity is rotated by 180° about its long axis of symmetry. The difference between the two configurations (the original and the rotated one) is the direction of the semifluxon. Then, the flux lines differ by an entire fluxon Φ_0 , which does not affect any physical system.

A charge whose wave function is a packet moving towards the right-hand side, initially confined in the sector delimited by the two flux lines, and given by a product state $\psi = \psi_x\psi_y$, with

$$\psi_y = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi y}{L}\right), \quad (3.17)$$

has a minimum average energy $\langle E \rangle_{i,\min}$ given by Eq. (3.14).

In this setup, the eigenstates that contribute to the eigendecomposition of the particle must be the ones whose nodal lines are in the configuration shown in Fig. 3.2(f). Thus, a charge that completely escapes the region between the flux lines will present a nodal line at the center of the y direction. Then, the state with minimum average energy after a particle completely escapes the flux lines corresponds to the first excited state of the y direction, i.e.,

$$\langle E \rangle_{f,\min} = \frac{2\pi^2\hbar^2}{mL^2}. \quad (3.18)$$

As a consequence, if the charge starts with average energy smaller than this value, which may be the case if

$$\frac{\pi^2\hbar^2}{2mL^2} + \frac{\pi^2\hbar^2}{2mD^2} < \frac{2\pi^2\hbar^2}{mL^2} \Rightarrow D > \frac{1}{\sqrt{3}}L, \quad (3.19)$$

it cannot completely leave the region. Even if parts of its original wave function associated with higher energies move past the flux lines, parts of it bounce back. This remaining part remains trapped in the region between the flux lines and, therefore, is given by a linear combination of eigenstates with nodal lines outside the region between the flux lines. Thus, there exist eigenstates with low energy associated with the configuration presented in Fig. 3.2(f) that are mostly confined in sector delimited by the flux lines.

This result is valid even if the cavity is infinitely long, i.e., if an open wave-guide is considered. Moreover, even though the confinement of these systems depends on the geometry of the cavity (it must be relatively narrow, for instance), they would not exist if it was not for the AB effect. Because of this, they are called *topological bound states*.

Also, the continuity of the probability density associated with the wave function implies that the topological bound states have tails outside the region between the flux lines. However, position measurements of topologically bounded charges must reveal, with substantial probability, that they are in the region between the solenoids, and the charge should not be found in the far (outer) field.

Now, what happens if each flux line carries a magnetic flux Φ_B different from a semifluxon? To answer this question, consider the cavity of Fig. 3.2(g) with a single flux line carrying a magnetic flux Φ_B placed, say, at the origin of a frame of reference whose x axis is parallel to the long symmetry axis of the cavity. Assume the cavity has a long extension to the left and the right of the flux line. Like in the case with semifluxons, let the charge start in a separable state $\psi = \psi_x \psi_y$, where ψ_y is given by Eq. (3.17) and ψ_x is a state with low average energy in the direction of motion.

If $\Phi_B = 0$, ψ returns to its initial shape after the packet has completely passed the flux line. Now, if $\Phi_B = \epsilon \ll 1$, then ψ is slightly disturbed as the charge passes it, since it must present a phase discontinuity after the charge completely encircles the flux line. To treat the

problem, consider a gauge for which the vector potential \vec{A} associated with the flux line is given by

$$\vec{A}(x, y) = \Phi_B \Theta(x) \delta(y) \hat{y}. \quad (3.20)$$

Then, after crossing the flux line, ψ turns into ψ' , which must have a phase discontinuity at $y = 0$. Since the magnetic flux is considered to be sufficiently small, $\psi' \approx \psi$ and, while $\psi'_y \approx \psi_y$, ψ'_y has a phase discontinuity of ϵ at $y = 0$. To see that this phase discontinuity implies that the flux line creates an energy barrier, observe that the change in the average energy associated with the y direction can be approximated as

$$\begin{aligned} \Delta \langle E \rangle &\approx \frac{1}{2m} \lim_{\gamma \rightarrow 0} \int_{-\gamma}^{\gamma} \psi_y^* (P_y - q\epsilon\delta(y))^2 \psi_y dy \\ &= \frac{1}{2m} \epsilon^2 q^2 \lim_{\gamma \rightarrow 0} \int_{-\gamma}^{\gamma} \delta(y)^2 |\psi_y|^2 dy. \end{aligned} \quad (3.21)$$

Because ψ_y was already prepared with the minimum possible energy in the y direction, this change of energy necessarily implies an increase in the average energy associated with that direction. As a result, as long as the initial average energy of ψ_x is smaller than a certain threshold, the charge is at least partially reflected by the flux line.

The problem, now, becomes the quantification of the minimum amount of extra energy associated with ψ'_y after the particle crosses the flux line. For that, consider, once more, the lattice of flux lines represented in Fig. 2.2 and let the initial state of the charge be $\psi = \psi_x \psi_y$, where ψ_y is given by Eq. (3.17), which has lines of zeros at $y = nL + L/2$, evenly spaced between the fluxes of the lattice, as represented by the gray lines in Fig. 2.2. For the incoming particle, the dynamics will be unchanged if infinite cavity walls (not magnetic) are added along the nodal lines, to the left of the lattice. Now, when the particle passes the flux lines, these walls end, and the dynamics goes back to the case of free diffraction, where Eq. (2.55) applies. Moreover, adding in the cavity walls on the right-hand side of the lattice to the initial Hamiltonian can only increase the minimum energy associated with the

y direction. In other words, the free case gives a lower bound on the energy for the stacked cavities case. Hence, it is possible to conclude that the energy increase associated with ψ_y after the charge crosses the flux line corresponds to at least the amount $q^2\phi_B^2/2m$. Thus, low-energy particles cannot cross the flux line.

With this in mind, consider a cavity with two flux lines separated by a distance D . If a charge starts in the region between the fluxes, the minimum amount of average energy it can have is given by Eq. (3.14). After crossing the flux line, the charge's minimum amount of average energy is expressed in Eq. (3.15). As with the case of two magnetic walls inside a cavity, it can be concluded that, if Eq. (3.16) is satisfied, there exist topological bound states in the sector of the cavity delimited by the flux lines.

Finally, observe that, if each wall has a magnetic flux greater than a semifluxon (i.e., $\Phi_0/2$), then it should be replaced by multiple flux lines, instead of a single one. In any case, the results discussed here should still hold.

3.5 Landau Levels

So far, in this chapter, the region with magnetic field was taken to be widthless. But this was done merely for convenience. In fact, as it will be shown in this section, the effects of two-dimensional magnetic fields can be emulated by two-dimensional grids of flux lines. In particular, approximations of the Landau levels will be obtained in the presence of a uniform two-dimensional grid of flux lines.

In classical physics, a classical charge inside a region with a uniform magnetic field travels in a circular motion. In quantum mechanics, these circular orbits are also present in the dynamics of charges. However, they are quantized, and each level of energy is called a Landau level. To see that, consider a particle in a uniform magnetic field $\vec{B} = B\hat{z}$. Then,

in the gauge known as the Landau gauge, for which the vector potential is $\vec{A} = Bx$, the Hamiltonian of the particle is given by

$$H = \frac{1}{2m} \left[P_x^2 + (P_y - qBX)^2 \right]^2. \quad (3.22)$$

Because $[H, P_y] = 0$, the transverse momentum can be replaced by its eigenstate $\hbar k_y$. With that, the Hamiltonian is simplified to

$$H = \frac{1}{2m} \left[P_x^2 + (\hbar k_y - qBX)^2 \right]^2, \quad (3.23)$$

which can be identified as the Hamiltonian of the one-dimensional *harmonic oscillator*. Harmonic oscillators play a fundamental role in the study of systems near their equilibrium. Also, they correspond to a one-dimensional projection of two-dimensional circular motions. The fact that Eq. (3.23) is the Hamiltonian of a harmonic oscillator indicates that the particle travels on a circular orbit, although the asymmetry of the chosen gauge does not seem to corroborate with this conclusion at first glance. However, a more careful analysis with, for instance, the choice of the symmetric gauge ($\vec{A} = (-y\hat{x} + x\hat{y})B/2$) can reveal the circular characteristic of the solution.

The idea, now, is to show that an approximation of the Landau levels can be obtained with the use of flux lines. To see that, consider a region with constant magnetic field $\vec{B} = B\hat{z}$. Also, let this region be divided into squares with length L , each with a flux $\Phi_B = BL^2 < \Phi_0/2$ — or a magnetic flux per unit of transverse length $\phi_B = B/L$. Now, replace each square by a flux line with magnetic flux Φ_B . Then, it is still possible to obtain the Landau levels with this two-dimensional square grid of flux lines with spacing L . In the

singular gauge, the Hamiltonian of a charge can be written as

$$H = \frac{1}{2m} \left[P_x^2 + \left(P_y - q \sum_{n,s \in \mathbb{Z}} A_{ns}(X, Y) \right)^2 \right], \quad (3.24)$$

where $A_{ns}(X, Y) = \Phi_B \Theta(X - nL) \delta(Y - sL)$ is the vector potential associated with each flux line. The solutions of this Hamiltonian cannot be easily found. However, the Hamiltonian can be simplified by using the fact that, for each vertical layer, the average effect of the flux lines is a change of $q\phi_B$ in vertical momentum. Hence, Eq. (3.24) can be approximated as

$$H = \frac{1}{2m} \left[P_x^2 + \left(P_y - q\phi_B \sum_{n \in \mathbb{Z}} \Theta(X - nL) \right)^2 \right]. \quad (3.25)$$

Now, because the new expression for the Hamiltonian commutes with the canonical transverse momentum P_y , it is possible to replace this operator by its eigenvalue $\hbar k_y$. Then,

$$H = \frac{1}{2m} \left[P_x^2 + \left(\hbar k_y - q\phi_B \sum_{n \in \mathbb{Z}} \Theta(X - nL) \right)^2 \right]. \quad (3.26)$$

One can easily see that the Hamiltonian in Eq. (3.26) is formally an approximation of the one-dimensional harmonic oscillator. In fact, if $L \ll 1$, the term $\phi_B \sum_{n \in \mathbb{Z}} \Theta(X - nL)$ can be approximated as BX . This shows that, indeed, the Landau levels can be recovered with the use of a two-dimensional grid of flux lines.

3.6 Semi-Classical Toy Model

Recall that the magnetic flux Φ_B associated with a flux line vanishes in the classical limit. However, this does not necessarily imply that ϕ_B also vanishes. Indeed, as it will be discussed in this section, it is possible to take the distance L between the flux lines to zero in the

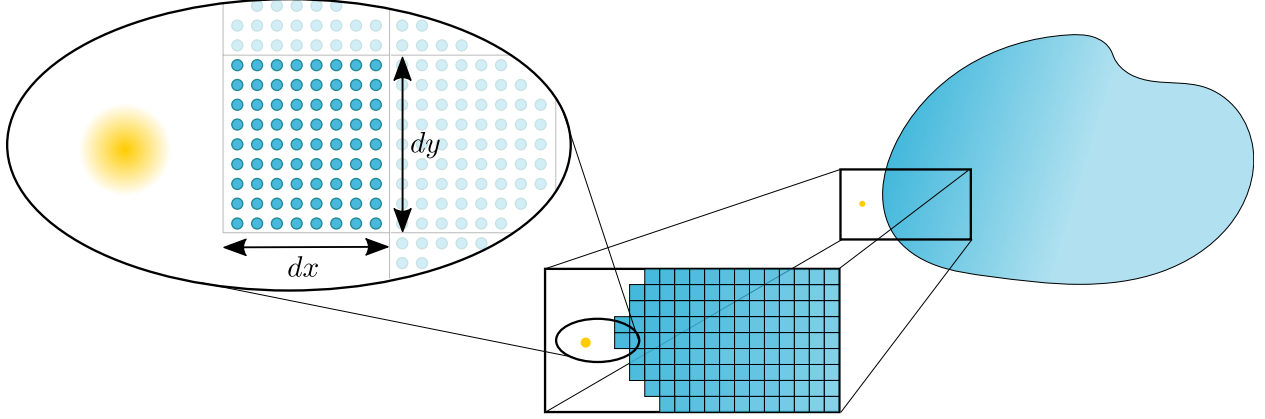


Figure 3.3: Schematic representation of a semi-classical theory where a region with an arbitrary continuous distribution of magnetic field in the z direction that does not depend on the z coordinate (blue region) is replaced by a discrete distribution of flux lines (blue dots). The region can be split into infinitesimal areas, each with constant magnetic fields, as represented in the rectangular zoomed-in cut-away section. These infinitesimal areas can be replaced by a grid of flux lines, as shown in the elliptical zoomed-in cut-away section. The charge (yellow object) is assumed to have a spread much smaller than the infinitesimal areas.

classical limit in a way that keeps ϕ_B constant. In this case, the minimum deflection does not vanish. It seems, then, that the AB effect generates a classical force.

In fact, under certain seemingly reasonable assumptions, the classical magnetic force in an arbitrary continuous field $\vec{B}(x, y) = B_z(x, y)\hat{z}$ can be seen as arising from the topological AB force. To show this, the magnetic field has to be first broken up into differential rectangles of area $dx \cdot dy$, each with flux $d\Phi_B = B_z(x, y)dxdy$, as illustrated in Fig. 3.3. Then, the uniform magnetic field B_z in the differential region $dx \cdot dy$ is replaced by an $M \times N$ grid of lines with fluxes $d\Phi_B/MN$.

Now, consider a quantum charge q spread over a region much smaller than $dx \cdot dy$ (but large enough to be diffracted by some of the flux lines) and incident on one of the infinitesimal cells with average velocity

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}. \quad (3.27)$$

Also, assume that the AB force acts on the wave function of the particle when it is crossing

the lattice, which is consistent with the discussion in Section 2.4. Then, neglecting the terms with \hbar in Eq. (2.54), which vanish in the classical limit, the change in kinematic momentum per vertical layer of N flux lines in the y direction amounts to $-qB_z(x, y)dx/M$. Similarly, the change in kinematic momentum in the x direction per horizontal layer of M flux lines is $qB_z(x, y)dy/N$. Then, if the center of the charge spread crossed $n_1 \leq N$ vertical layers and $n_2 \leq M$ horizontal layers while passing over the $dx \cdot dy$ infinitesimal cell, the total change in kinematic momentum can be approximated as

$$\begin{aligned} d\vec{p} &= dp_x \hat{x} + dp_y \hat{y} \\ &\approx qB_z(x, y) \frac{n_1}{M} dy \hat{x} - qB_z(x, y) \frac{n_2}{N} dx \hat{y}. \end{aligned} \quad (3.28)$$

Noticing that the particle's average velocity is kept approximately constant in each infinitesimal cell, i.e., $v_x \approx (n_2/N)(dx/dt)$ and $v_y \approx (n_1/M)(dy/dt)$, where dt is the amount of time the center of the charge's distribution remains in the cell, a simple application of the chain rule gives

$$\vec{F} = \frac{dp_x}{dt} \hat{x} + \frac{dp_y}{dt} \hat{y} = qB_z v_y \hat{x} - qB_z v_x \hat{y}, \quad (3.29)$$

which correspond to

$$\vec{F} = q\vec{v} \times \vec{B}. \quad (3.30)$$

Finally, taking the classical limit where the particle spread reduces to zero, \vec{v} becomes the classical velocity, and \hbar goes to zero, the classical force \vec{F} experienced by a point charge q in a magnetic field \vec{B} arises purely from the topological AB force.

3.7 Discussion

The results discussed in this chapter show a certain equivalence between the quantum and the classical treatment of the dynamics of charges in the presence of continuous magnetic

fields. This, on itself, is a manifestation of how special magnetic fields are. To evidence it, it is interesting to compare the magnetic wall with a usual potential barrier. First, recall that, even though the energy barrier associated with the magnetic walls is, in general, dependent on the angle of incidence, there exists a value ($|q\phi_B|/2$) for which any low-energy charge is reflected by the wall. For simplicity, take that value as the value of the energy barrier imposed by the wall. In that case, if the wall was replaced by a scalar potential given by

$$V(x) = \frac{q\phi_B}{2}\delta(x), \quad (3.31)$$

or even by

$$V(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x \geq w \\ \frac{|q\phi_B|}{2} & \text{if } 0 \leq x < w \end{cases}, \quad (3.32)$$

the potential wall would behave like a mirror for low-energy charges for particles only in the classical treatment of the problem. For quantum systems, those scalar barriers are not enough to reflect low-energy particles. This is because of a phenomenon known as *quantum tunneling*, discovered in 1927 by Hund [118].

The idea of this phenomenon is that, even though the particle does not have enough energy to overcome the barrier, slowly it starts “leaking” on the other side. Therefore, the particle can be found in the far-field. The potential barriers only reflect low-energy particles in the limit where $w \rightarrow \infty$, i.e., when the potential is uniformly applied everywhere on the opposite side of incidence of the particle.

Besides this perspective on magnetic fields, the results presented in this chapter also show that the AB effect enables the construction of energy barriers with lattices of flux lines. For magnetic fluxes between zero and $\Phi_0/2 = \pi\hbar/q$, these lattices and thin continuous magnetic walls behave alike. Their similarity, however, vanishes outside that interval because of the periodicity discussed in Section 3.2. Nonetheless, this is not a significant limitation of our

results. In fact, if a magnetic wall has a flux $\Phi_B > \Phi_0/2$ associated to any L , there is a length $L' < L$ such that the magnetic flux Φ'_B associated with a region of that length is $\Phi'_B < \Phi_0$. Thus, in general, any widthless uniform magnetic wall with ϕ_B can be replaced by a lattice of flux lines with $\Phi'_B = \phi_B L' \leq \Phi_0/2 = \pi\hbar/q$, where L' is the spacing of the lattice.

Additionally, it was discussed how these results could be extended to the case where the magnetic walls have some width, and the lattices are two-dimensional grids of flux lines. In particular, in Section 3.5, it was shown how approximations of Landau levels could be obtained with distributions of flux lines.

It is important to note that, before this work, Andrei Shelankov studied wave functions with finite width in the presence of flux lines using paraxial analysis and obtained some of the results presented here [119, 120]. However, this work not only uses a more straightforward approach to generalize Shelankov's results, but also introduces new ideas, like the topological bound states, for instance.

Finally, the basic scheme for a semi-classical theory presented in Section 3.6 suggests that the AB effect in quantum mechanics may be the fundamental source of the classical magnetic force. Further developments of this semi-classical model where the spread of the particle is introduced as a free parameter are necessary to shed more light on this quantum-to-classical transition.

4 Complex Vector Potentials in Pre and Post-Selected Systems

Up until now, the AB effect has been considered in scenarios where the source of the electromagnetic field is not quantized. This chapter, however, presents the results from Ref. [68], which studies quantum charges encircling a quantized magnetic flux. It is shown that the state of the particle after it surrounds a flux with small uncertainty is approximately the final state of a particle enclosing a region with the average magnetic flux. Furthermore, it is proved that, if a post-selection of the magnetic flux is considered, the magnetic vector potential is, in general, complex-valued.

4.1 Weak Measurements and Weak Values

As already mentioned, if a property O of a system in a state $|\psi\rangle$ is measured, the system will be found in an eigenstate $|o\rangle$ of O with probability given by

$$p(o) = |\langle o|\psi\rangle|^2. \quad (4.1)$$

However, for a measurement to happen, a measurement device has to interact with the system of interest, and the result is, then, read from the device (which is also called the *pointer*). Thus, the measurement of the system is made indirectly.

The model of how the interaction between the device and the system takes place was introduced by von Neumann [24]. He considered the following interaction Hamiltonian

$$H_{\text{int}}(t) = h(t)O \otimes P, \quad (4.2)$$

where O is the observable of interest (that acts in the Hilbert space (HS) \mathcal{H}_s of the system), P is the momentum operator that acts in the HS \mathcal{H}_d of the device, and h is a function of time that is non-zero from t_0 to t_1 , the interval of time where the measurement takes place.

The interest here is on interactions that happen fast enough when compared to the evolution of the system and the apparatus. In this case, H_{int} is the effective Hamiltonian during the measurement. These are called *impulsive* measurements. Hence, the unitary evolution is

$$U(t; t_0) = e^{-i \int_{t_0}^t H_{\text{int}}(t') dt' / \hbar}. \quad (4.3)$$

To see how the device ends up with information about the system, let the initial state of the device be

$$|\xi\rangle = \int \xi(x) |x\rangle dx. \quad (4.4)$$

Moreover, let the initial state of the system be

$$|\psi\rangle = \int \psi(o) |o\rangle do, \quad (4.5)$$

where $|o\rangle$ are eigenstates of the observable O . Then, writing $|\Psi(t_0)\rangle = |\xi\rangle \otimes |\psi\rangle$,

$$\begin{aligned} |\Psi(t)\rangle &= \iint \psi(o) \xi(x) U(t; t_0) [|o\rangle \otimes |x\rangle] do dx \\ &= \iint \psi(o) \xi(x) |o\rangle \otimes e^{-i \int_{t_0}^t h(t') o P dt' / \hbar} |x\rangle do dx \end{aligned} \quad (4.6)$$

and, in particular,

$$\begin{aligned} |\Psi(t_1)\rangle &= \iint \psi(o) \xi(x) |o\rangle \otimes |x + go\rangle do dx \\ &= \iint \psi(o) \xi(x - go) |o\rangle \otimes |x\rangle do dx, \end{aligned} \quad (4.7)$$

where $g \equiv \int_{t_0}^{t_1} h(t) dt$.

More generally, in the Heisenberg picture,

$$I \otimes X(t) = U^\dagger(t) (I \otimes X(t_0) U(t; t_0)) = I \otimes X(t_0) - \int_{t_0}^t h(t') dt' O(t_0) \otimes I. \quad (4.8)$$

Hence,

$$\langle X(t_1) \rangle = \langle X(t_0) \rangle - g \langle O(t_0) \rangle, \quad (4.9)$$

where it was used that

$$[X, f(P)] = i\hbar \frac{\partial f}{\partial P}(P), \quad (4.10)$$

which follows from Eq. (2.8). Eq. (4.9) shows that, in the average over many trials of the experiment, the measurement device is shifted by an amount that is proportional to the average of the observable of the measured system. While Eq. (4.9) also holds for density matrices, for simplicity, only pure initial states are considered hereby.

It can be observed that the constant g is associated with how well separated the eigenvalues of the observable will be in the reading of the device, i.e., how well the device can resolve the observable O . If g is large enough, the measurement is *sharp*, and the experimentalist has high confidence about the state of the system of interest after the measurement is complete. However, if g is small enough, the measurement will not provide enough information for the experimentalist to know with high certainty the state of the system of interest.

It turns out that, while a large g can highly disturb the system, the case of a sufficiently small g keeps the system relatively intact. In fact, from Eq. (4.7), it holds that the state of the system after the interaction with the device is

$$\begin{aligned} \rho_s &= \text{Tr}_d |\Psi(t_1)\rangle \langle \Psi(t_1)| \\ &= \iiint \psi(o) \overline{\psi(o')} \xi(x) \overline{\xi(x')} \langle x' + go' | x + go \rangle |o\rangle \langle o'| do do' dx dx' \\ &= \iiint \psi(o) \overline{\psi(o')} \xi(x) \overline{\xi(x - g(o - o'))} |o\rangle \langle o'| do do' dx. \end{aligned} \quad (4.11)$$

Now, assuming ξ is analytic,

$$\xi(x - g(o - o')) \approx \xi(x) - g(o - o') \frac{d\xi}{dx}(x) \quad (4.12)$$

as long as $|g(o - o')| \ll 1$ for every o and o' such that $\psi(o) \neq 0$ and $\psi(o') \neq 0$. In this case,

$$\begin{aligned} \rho_s &\approx \iiint \psi(o) \overline{\psi(o')} \xi(x) \overline{\xi(x)} |o\rangle \langle o'| do do' dx \\ &= \iint \psi(o) \overline{\psi(o')} |o\rangle \langle o'| do do' \\ &= |\psi\rangle \langle \psi|. \end{aligned} \quad (4.13)$$

This means that the final state is almost a product state if g is much smaller than the inverse of the spreading of the measured system. In particular, the measured system remains approximately unchanged. This is called a *weak measurement*. With that, the case where the system is highly disrupted is, then, called a *strong measurement*.

Furthermore, in a weak measurement,

$$|g(o - o')| \ll \hbar \Rightarrow |g(o - \langle O \rangle)| \ll \hbar \Rightarrow e^{-ig(o - \langle O \rangle) \otimes P/\hbar} \approx I_d \quad (4.14)$$

for every o and o' such that $\psi(o) \neq 0$ and $\psi(o') \neq 0$, and the final state of the joint system is such that

$$\begin{aligned} |\Psi(t_1)\rangle &= \iint \psi(o) \xi(x) |o\rangle \otimes e^{-igoP/\hbar} |x\rangle do dx \\ &= \iint \psi(o) \xi(x) |o\rangle \otimes e^{-ig\langle O \rangle P/\hbar} e^{-ig(o - \langle O \rangle) P/\hbar} |x\rangle do dx \\ &\approx \iint \psi(o) \xi(x) |o\rangle \otimes e^{-ig\langle O \rangle P/\hbar} |x\rangle do dx \\ &= |\psi\rangle \otimes \int \xi(x - g\langle O \rangle) |x\rangle dx, \end{aligned} \quad (4.15)$$

i.e., the measurement device is shifted by an amount proportional to $\langle O \rangle$.

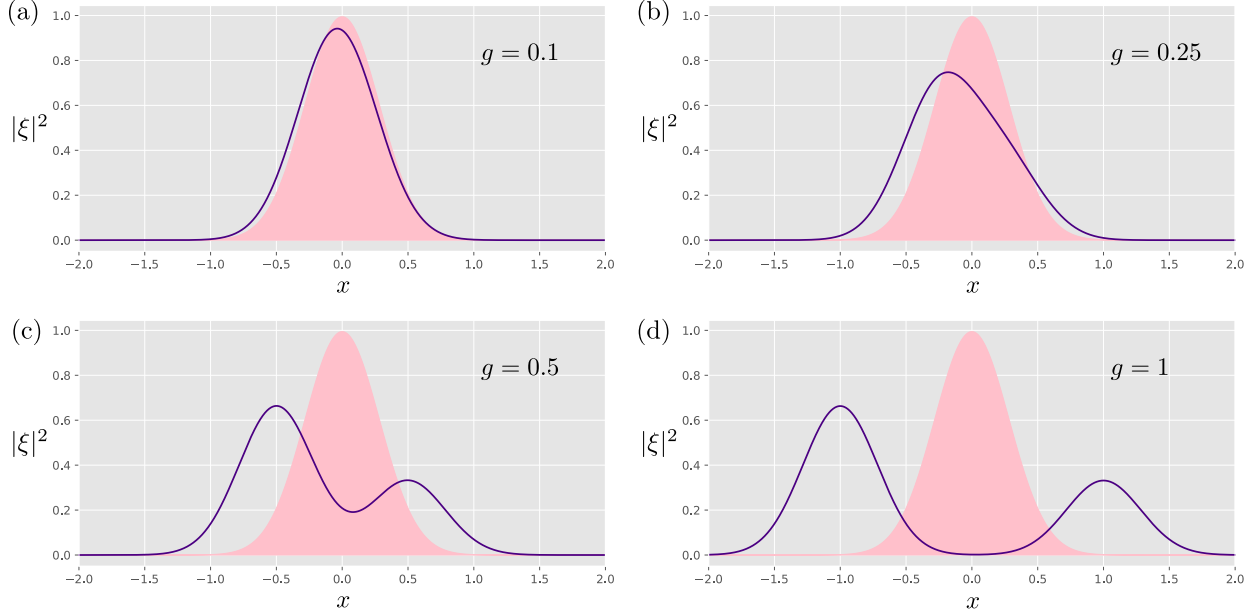


Figure 4.1: Effect of measurements in the measurement device. The initial distribution of the measurement device (pink area) is a Gaussian with zero mean and a standard deviation of $\sigma = 0.4$. The purple curve represents its distribution after the measurement is complete. It should be noticed that weak measurements (small g) almost does not disturb the device, while strong measurements (large g) drastically changes its distribution.

As an example, consider the case where the initial state of the device is a Gaussian centered at the origin, i.e.,

$$\xi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]. \quad (4.16)$$

Also, suppose the system of interest is a qubit, i.e., a quantum system associated with a complex HS \mathcal{H}_s with dimension 2, in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \quad (4.17)$$

where $|0\rangle$ and $|1\rangle$ are an orthonormal basis of \mathcal{H}_s . Moreover, consider the measurement of an observable σ_z such that $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$. Thus, Eq. (4.7) becomes

$$|\Psi(t_1)\rangle = \frac{1}{\sqrt{3}}|0\rangle \otimes \left(\int \xi(x-g)|x\rangle dx\right) + \sqrt{\frac{2}{3}}|1\rangle \otimes \left(\int \xi(x+g)|x\rangle dx\right). \quad (4.18)$$

Since the final state of the device ρ_d can be computed as $\rho_d = \text{Tr}_s |\Psi(t_1)\rangle\langle\Psi(t_1)|$, the final probability distribution of it is

$$|\xi|^2(x) = \langle x|\rho_d|x\rangle = \frac{1}{6\sigma^2\pi} e^{-\left(\frac{x-g}{\sigma}\right)^2} + \frac{1}{3\sigma^2\pi} e^{-\left(\frac{x+g}{\sigma}\right)^2}. \quad (4.19)$$

This state is compared with the initial state of the device (the square of Eq. (4.16)) in Fig. 4.1 for various choices of g and $\sigma = 0.4$. It should be noticed that, for weak measurements, as a result of the measured system remaining approximately undisturbed, the probability distribution of the pointer is also kept almost unchanged. This implies that little information can be extracted from a single measurement since the device does not have enough resolution to give a precision of how much its average was shifted. However, observe that repetitions of the measurement effectively decrease the variance associated with the pointer, allowing the shift in the average to be detectable.

Now, weak measurements are often considered in scenarios where the system of interest is post-selected. This means that, after the interaction with the device, the measured system continues its evolution until it is strongly measured and “collapses” in a state

$$|\varphi\rangle = \int \varphi(o)|o\rangle do, \quad (4.20)$$

which is assumed to be non-orthogonal to $|\psi\rangle$, i.e., $\langle\varphi|\psi\rangle \neq 0$. When the post-selection is made, the small entanglement between the system and the device that conducted the weak

measurement is destroyed. In this case, the final state of the device is

$$\begin{aligned}
(\langle\varphi| \otimes I_p)|\Psi(t_1)\rangle &= \iint \psi(o)\overline{\varphi(o)}\xi(x)e^{-ig\langle O\rangle P/\hbar}e^{-ig(o-\langle O\rangle)P/\hbar}|x\rangle do dx \\
&\approx \iint \psi(o)\overline{\varphi(o)}\xi(x)e^{-ig\langle O\rangle P/\hbar}\left(I_d - i\frac{g(o-\langle O\rangle)P}{\hbar}\right)|x\rangle do dx \\
&= \left(\int \psi(o)\overline{\varphi(o)} do\right) \int \xi(x)e^{-ig\langle O\rangle P/\hbar}\left(I_d + i\frac{g\langle O\rangle P}{\hbar}\right)|x\rangle dx \\
&\quad - \frac{g}{\hbar}\langle\varphi|\psi\rangle \int \xi(x)e^{-ig\langle O\rangle P/\hbar}\left(\int \frac{o\psi(o)\overline{\varphi(o)}}{\langle\varphi|\psi\rangle} do\right)P|x\rangle dx.
\end{aligned} \tag{4.21}$$

Observe that

$$\langle O\rangle_w \equiv \int \frac{o\psi(o)\overline{\varphi(o)}}{\langle\varphi|\psi\rangle} do = \frac{\langle\varphi|O|\psi\rangle}{\langle\varphi|\psi\rangle} \tag{4.22}$$

depends only on the measured system. This quantity, which was introduced in 1988 by Aharonov, Albert, and Vaidman in Ref. [121], is called the *weak value* (WV) of the operator O . It should be noticed that, differently from the expected value $\langle O\rangle$, the WV is, in general, a complex number.

Now, assuming $|g(\langle O\rangle_w - \langle O\rangle)| \ll \hbar$ and recognizing

$$\langle\varphi|\psi\rangle = \int \psi(o)\overline{\varphi(o)} do, \tag{4.23}$$

it holds that

$$\begin{aligned}
(\langle\varphi| \otimes I_p)|\Psi(t_1)\rangle &\approx \langle\varphi|\psi\rangle \int \xi(x)e^{-ig\langle O\rangle P/\hbar}\left(I_d - i\frac{g(\langle O\rangle_w - \langle O\rangle)P}{\hbar}\right)|x\rangle dx \\
&\approx \langle\varphi|\psi\rangle \int \xi(x)e^{-ig\langle O\rangle_w P/\hbar}|x\rangle dx \\
&= \langle\varphi|\psi\rangle \int \xi(x - g\text{Re}(\langle O\rangle_w))e^{g\text{Im}(\langle O\rangle_w)P/\hbar}|x\rangle dx.
\end{aligned} \tag{4.24}$$

Therefore, the measurement device is purely affected by the WV.

As an example, consider again the case where the initial states of the device and the system of

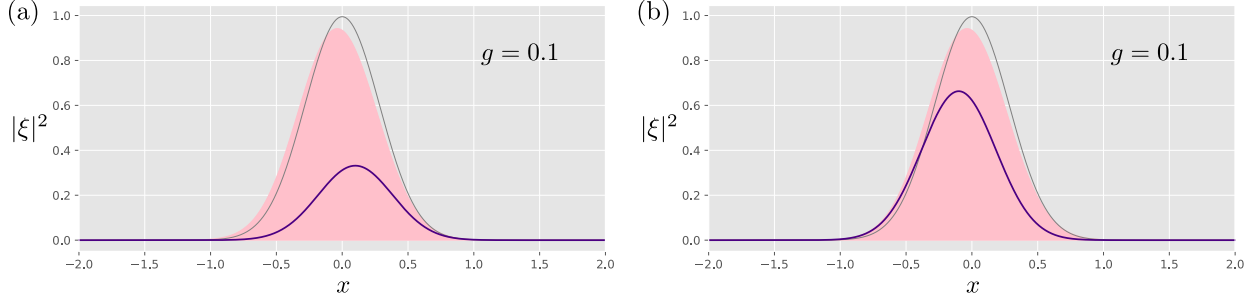


Figure 4.2: Effect of post-selections in weak measurements. The initial distribution of the measurement device (gray curve) is a Gaussian with zero mean and a standard deviation of $\sigma = 0.4$. The weak measurement slightly changes the distribution of the device (pink area). However, when considering post-selections of the measured system, the device's distribution is affected. The purple curves show the (unnormalized) probability distributions after two different (and complementary) post-selections are made.

interest are given by Eqs. (4.16) and (4.17), respectively. Then, if the qubit is post-selected in the state $|0\rangle$, Eq. (4.18) implies that the (unnormalized) final probability distribution of the device is

$$|\xi|^2(x) = \frac{1}{6\sigma^2\pi} e^{-\left(\frac{x-g}{\sigma}\right)^2}. \quad (4.25)$$

This situation is represented in Fig. 4.2(a). Moreover, if the post-selection is made on the state $|1\rangle$, the final (unnormalized) distribution of the device is

$$|\xi|^2(x) = \frac{1}{3\sigma^2\pi} e^{-\left(\frac{x+g}{\sigma}\right)^2}, \quad (4.26)$$

which is illustrated by Fig. 4.2(b).

As it can be observed, the distribution of the device is, in general, significantly changed when considering weak measurements with post-selections. In fact, it can be shown from the definition in Eq. (4.22) that the WV can assume any complex value. When it has an imaginary part or its value goes outside the range of eigenvalues of the operator O , the WV is said to be *anomalous*.

To conclude, it should be mentioned that WVs arose from the study of the two-state vector

formalism of quantum mechanics, introduced by Aharonov, Bergmann, and Lebowitz [122] in 1964. This formalism considers that the time asymmetry existing in the dynamics of quantum systems due to the “collapse” of the wave function is the result of an incomplete setting of boundary conditions. Specifically, it considers that the state that can be attributed to a system in a certain instant of time does not depend only on how the system was prepared (the pre-selection), but also on the state it will be measured in the future (the post-selection). The idea is, then, that the post-selection travels backward in time and combines with the pre-selection, which travels forward in time, to define the “reality” of the system at the present [123–125].

Also, the research on WVs has become a rich area of study [126–142]. In particular, it has uncovered many surprising effects in quantum theory [58, 143–147], besides shedding some light in the understanding of problems that were already known [148–152], like the interaction-free measurements [148, 153]. Also, anomalous WVs are proof of *contextuality* [154, 155]. Finally, WVs inspired new type of quantum experiments [156–163].

4.2 Quantization of the Sources of Electromagnetic Fields

Classical or quantum treatments of electromagnetic systems are characterized for being gauge invariant. It can be shown, however, that, when the source of electromagnetism is quantized, a more careful analysis of the problem is required. This section presents the model introduced in Ref. [164] and further studied in Ref. [165], which includes the quantization of the sources of the field. Moreover, it discusses the delicate aspects of gauge transformations in this context.

Consider an infinitely long cylindrical shell with radius a , rotational inertia I_c , and uniform charge density σ rotating around its long axis of symmetry (which coincides with the z axis) with angular velocity $\dot{\eta}$, in a similar manner as the solenoid represented in Fig. 2.1. Such a

cylinder generates a uniform magnetic field on its interior given by

$$\vec{B} = \mu_0 \sigma \eta a \hat{z}. \quad (4.27)$$

Outside of it, the magnetic field vanishes. Also, a uniformly charged wire is added in the z axis to cancel the electric field on the exterior of the cylinder. With that, observe that this cylinder behaves like a solenoid. Finally, consider a charge q with mass μ traveling outside the cylinder in the xy plane, and let the coordinates of such a charge be given by (r, θ) .

First, a classical treatment of the problem is given. For that, the joint system of the cylinder and the charge has its dynamics described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} I_d \dot{\theta}^2 + \frac{1}{2} I_c \dot{\eta}^2 + q\tau \dot{\eta} \dot{\theta}, \quad (4.28)$$

where $I_q = \mu r^2$ is the moment of inertia of the charge and τ is a constant associated with the interaction between the cylinder and the particle. From the Lagrangian, the canonical momentum with respect to each coordinate variable can be found. The canonical radial momentum of the particle is

$$p_r \equiv \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r}. \quad (4.29)$$

Also, the canonical angular momentum of the charge is

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_q \dot{\theta} + q\tau \dot{\eta}. \quad (4.30)$$

Furthermore, the canonical angular momentum of the cylinder is

$$p_\eta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = I_c \dot{\eta} + q\tau \dot{\theta}. \quad (4.31)$$

Solving for $\dot{\eta}$ in Eq. (4.31) and replacing it in Eq. (4.30), it holds that

$$\dot{\theta} = \frac{p_\theta}{I'_q} - \frac{q\tau p_\eta}{I'_q I_c}, \quad (4.32)$$

where $I'_q = I_q - q^2\tau^2/I_c$. Moreover,

$$\dot{\eta} = \frac{p_\eta}{I_c} - \frac{q\tau p_\theta}{I'_q I_c}. \quad (4.33)$$

The Hamiltonian of the system can, now, be derived with the Legendre transform. In fact, neglecting terms with τ^2/I_c^2 ,

$$H = p_r \dot{r} + p_\theta \dot{\theta} + p_\eta \dot{\eta} - \mathcal{L} \approx \frac{1}{2\mu} p_r^2 + \frac{1}{2I'_q} \left(p_\theta - \frac{q\tau}{I_c} p_\eta \right)^2 + \frac{1}{2I_c} p_\eta^2. \quad (4.34)$$

Now, take the limit $I_c \gg 1$ such that $\dot{\eta} \approx \tau/I_c$ is a finite constant $K/2\pi$. Then, $I'_q \approx I_q$ and

$$H \approx \frac{1}{2\mu} p_r^2 + \frac{1}{2I_q} \left(p_\theta - \frac{qK}{2\pi} p_\eta \right)^2 + \frac{1}{2I_c} p_\eta^2. \quad (4.35)$$

With that, the system can be quantized. The Hamiltonian associated with it is

$$H = \frac{1}{2\mu} P_r^2 + \frac{1}{2I_q} \left(P_\theta - \frac{qK}{2\pi} P_\eta \right)^2 + \frac{1}{2I_c} P_\eta^2, \quad (4.36)$$

where P_r , P_θ and P_η are Hermitian operators canonical conjugated of the observable coordinates R , Θ and E , respectively. Observe that each operator P acts on a different HS. Furthermore, the ‘‘operator vector potential’’ associated with the cylinder is

$$\vec{A} = \frac{K}{2\pi} R^{-1} P_\eta \hat{\theta}, \quad (4.37)$$

i.e., it is given in the Coulomb gauge.

Now, there is a subtlety involved in gauge transformations that arises when the transformation is a function of non-commuting observables. To see that, consider the case where the total time derivative of the function $f = -q\tau\dot{\eta}\theta$ is added to the Lagrangian in Eq. (4.28). The new Lagrangian is, then,

$$\mathcal{L}' = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}I_d\dot{\theta}^2 + \frac{1}{2}I_c\dot{\eta}^2 - q\tau\dot{\eta}\theta, \quad (4.38)$$

and it depends on the angular acceleration $\ddot{\eta}$ of the cylinder. A rigorous derivation of the Hamiltonian associated with this Lagrangian, and its respective quantization is presented in Ref. [165]. Here, it should just be noticed that this gauge change corresponds to the application of the unitary transformation

$$U = e^{-iqK\Theta P_\eta/2\pi\hbar} \quad (4.39)$$

to H in Eq. (4.36). Neglecting terms with $q^2K^2I_c/4\pi^2$ and using the canonical commutation relation between Θ and P_Θ ,

$$U \left(P_\theta - \frac{qK}{2\pi} P_\eta \right) \approx P_\theta - i \frac{qK}{2\pi\hbar} P_\theta \Theta P_\eta \Rightarrow U \left(P_\theta - \frac{qK}{2\pi} P_\eta \right)^2 U^\dagger \approx P_\theta^2. \quad (4.40)$$

Moreover, this operation translates the coordinate η in such a way that it becomes $\eta' = \eta - qK\theta p_\eta/2\pi$. Thus, similarly to what was done in Eq. (2.26), the new Hamiltonian is

$$H' = U H U = \frac{1}{2\mu} P_r^2 + \frac{1}{I_q} P_\theta^2 + \frac{1}{2I_c} P_{\eta'}^2 \quad (4.41)$$

since $P_{\eta'} = -i\hbar\partial/\partial\eta'$ is equivalent to P_η . Observe that, now, the coordinate of the cylinder is η' . It can, then, be verified that

$$[E', P_\theta] = -i \frac{qK}{2\pi\hbar} I, \quad (4.42)$$

where I is the identity operator that acts in the HS space associated with the coordinates η' and θ . This means that the usual canonical commutations (which, in particular, consider that the cylinder's and the charge's observables commute with each other) do not hold anymore. As stated in Ref. [165], this has a physical interpretation: a measurement of E' introduces an angular acceleration in the cylinder, which produces an electric field outside of it that changes P_θ .

Therefore, although the Hamiltonian of any system can be quantized in an arbitrary gauge, the quantization process is more delicate when the sources of the electromagnetic field are involved, and the Coulomb gauge seems to be the most appropriate choice.

4.3 Complex Vector Potentials

In the problem studied in the previous section, assume the charge travels in a circular trajectory with radius r and the inner evolution of the cylinder, i.e., $P_\eta^2/2I_c$ is negligible. In this case, the “operator vector potential” associated with the cylinder in the Coulomb gauge is

$$\vec{A}(r) = \frac{K}{2\pi r} P_\eta \hat{\theta}. \quad (4.43)$$

and the Hamiltonian in Eq. (4.36) can be written as

$$H = \frac{1}{2I_q} \left(P_\theta - \frac{qK}{2\pi} P_\eta \right)^2. \quad (4.44)$$

Also, denote by $|m\rangle$ the eigenstates of P_η such that

$$P_\eta |m\rangle = m\hbar |m\rangle \quad (4.45)$$

for every $m \in \mathbb{Z}$. Now, if the charge encloses the cylinder in the superposition

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} (|\psi_L\rangle + |\psi_R\rangle), \quad (4.46)$$

where $|\psi_L\rangle$ represents a packet that passes to the left of the cylinder and $|\psi_R\rangle$, a packet that travels to the right of it, as in Fig. 2.1, the state of the joint system after the particle encircles the cylinder can be written as

$$|\Lambda\rangle = \frac{1}{\sqrt{2}} \left(|\psi_L\rangle + e^{iqKm} |\psi_R\rangle \right) \otimes |m\rangle. \quad (4.47)$$

Thus, measurements of the position of the particle after the two wave packets recombine, closing the loop around the cylinder, will reveal the shift in the interference pattern characteristic of the AB effect. In this case, the effective vector potential has the standard classical form

$$\vec{A}(r) = \frac{Km\hbar}{2\pi r} \hat{\theta}. \quad (4.48)$$

However, in general, the angular momentum of the cylinder can be prepared in a superposition of eigenstates of P_η , i.e., in a state

$$|\xi\rangle = \sum_{j \in \mathbb{Z}} c_j |m_j\rangle, \quad (4.49)$$

where the constants c_j are such that $\sum_{j \in \mathbb{Z}} |c_j|^2 = 1$. Then, after the charge finishes the loop, the state of the joint system is

$$|\Lambda\rangle = \sum_{j \in \mathbb{Z}} \frac{c_j e^{-iKqm_j/2}}{\sqrt{2}} (|\psi_L\rangle + e^{iKqm_j} |\psi_R\rangle) \otimes |m_j\rangle. \quad (4.50)$$

In other words, the charge and the angular momentum of the cylinder become entangled.

Then, averaging over the states of the cylinder, the final state of the charge after it closes the loop around the cylinder is

$$\begin{aligned}
\rho &= \text{Tr}_c (|\Lambda\rangle\langle\Lambda|) \\
&= \frac{1}{2} \left[|\psi_L\rangle\langle\psi_L| + |\psi_R\rangle\langle\psi_R| + \left(\sum_{j \in \mathbb{Z}} |c_j|^2 e^{iKqm_j} \right) |\psi_R\rangle\langle\psi_L| \right. \\
&\quad \left. + \left(\sum_{j \in \mathbb{Z}} |c_j|^2 e^{-iKqm_j} \right) |\psi_L\rangle\langle\psi_R| \right].
\end{aligned} \tag{4.51}$$

Because of the entanglement, the interference between $|\psi_L\rangle$ and $|\psi_R\rangle$ is, in general, destroyed.

Now, a particular scenario of interest is the case where the charge interacts weakly with the vector potential \vec{A} , i.e., when $|Kq(m_j - \langle P_\eta \rangle)| \ll \hbar$ for every $j \in \mathbb{Z}$ such that $c_j \neq 0$. In this case,

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} |c_j|^2 e^{iKqm_j} &= e^{iKq\langle P_\eta \rangle/\hbar} \sum_{j \in \mathbb{Z}} |c_j|^2 e^{iKq(m_j \hbar - \langle P_\eta \rangle)/\hbar} \\
&\approx e^{iKq\langle P_\eta \rangle/\hbar}.
\end{aligned} \tag{4.52}$$

Hence, the final state of the charge is

$$|\psi_1\rangle \approx \frac{1}{\sqrt{2}} \left(|\psi_L\rangle + e^{iKq\langle P_\eta \rangle/\hbar} |\psi_R\rangle \right) \tag{4.53}$$

and the effective vector potential in the Coulomb gauge is

$$\vec{A}(r) = \frac{K}{2\pi r} \langle P_\eta \rangle \hat{\theta}. \tag{4.54}$$

As a result, under the assumptions that $|Kq(m_j - \langle P_\eta \rangle)| \ll \hbar$ for every $j \in \mathbb{Z}$ such that $c_j \neq 0$, the quantum treatment of the cylinder effectively produces the usual classical field expected by the average over the quantum states of the cylinder.

Now, suppose that, after the charge encircles it, the cylinder is post-selected in a state

$$|\varphi\rangle = \sum_{j \in \mathbb{Z}} d_j |m_j\rangle \quad (4.55)$$

such that $\langle \varphi | \xi \rangle \neq 0$. From Eq. (4.50), it holds that the final state of the charge after the post-selection of the cylinder is

$$|\psi_2\rangle = \frac{C}{\sqrt{2}} \sum_{j \in \mathbb{Z}} c_j \bar{d}_j \left(e^{-iKqm_j/2} |\psi_L\rangle + e^{iKqm_j/2} |\psi_R\rangle \right), \quad (4.56)$$

where $C = 1/\sqrt{\sum_{j \in \mathbb{Z}} |c_j \bar{d}_j|^2}$ is a normalization constant. If, in addition to letting $|Kq(m_j - \langle P_\eta \rangle)| \ll \hbar$ for every $j \in \mathbb{Z}$ such that $c_j, d_j \neq 0$, it is also assumed that $|Kq(\langle P_\eta \rangle_w - \langle P_\eta \rangle)/2| \ll \hbar$, where $\langle P_\eta \rangle_w$ is the WV of P_η , it holds that

$$\begin{aligned} \sum_{j \in \mathbb{Z}} c_j \bar{d}_j e^{\pm iKqm_j/2} &\approx e^{\pm iKq\langle P_\eta \rangle/2\hbar} \sum_{j \in \mathbb{Z}} c_j \bar{d}_j \left[1 \pm i \frac{Kq}{2} \left(m_j - \frac{\langle P_\eta \rangle}{\hbar} \right) \right] \\ &= e^{\pm iKq\langle P_\eta \rangle/2\hbar} \sum_{j \in \mathbb{Z}} c_j \bar{d}_j \left[1 \pm i \frac{Kq}{2\hbar} (\langle P_\eta \rangle_w - \langle P_\eta \rangle) \right] \\ &\approx e^{\pm iKq\langle P_\eta \rangle_w/2\hbar} \sum_{j \in \mathbb{Z}} c_j \bar{d}_j. \end{aligned} \quad (4.57)$$

Then, the final state of the charge after the post-selection of the angular momentum of the cylinder is given by

$$|\psi_2\rangle = \frac{D}{\sqrt{2}} \left(|\psi_L\rangle + e^{iKq\langle P_\eta \rangle_w/\hbar} |\psi_R\rangle \right), \quad (4.58)$$

where D is a normalization constant, which is necessary because, as mentioned in Section 4.1, $\langle P_\eta \rangle_w$ is, in general, a complex number. Also, the effective vector potential in this case is

$$\vec{A}(r) = \frac{K}{2\pi r} \langle P_\eta \rangle_w \hat{\theta}. \quad (4.59)$$

It can, then, be concluded that in a system with pre and post-selection, the effective electromagnetic vector potential is, in general, complex-valued. However, it should be noticed that the assumed conditions imply that the imaginary part of the vector potential, if it exists, is relatively small.

Another interesting effect that can be observed is the continuous change of probability of finding the charge on the left or the right arm of the loop. In fact, assuming the loop is a circle centered at the cylinder, the “probability” of finding the particle on the left and the right arm after each wave packet traveled an angle θ can be computed as

$$p_L(\theta) = \frac{e^{a\theta}}{e^{a\theta} + e^{-a\theta}} \quad (4.60)$$

and

$$p_R(\theta) = \frac{e^{-a\theta}}{e^{a\theta} + e^{-a\theta}}, \quad (4.61)$$

where $a = K\text{Im}(\langle P_\eta \rangle_w) / \pi\hbar$ is a real number. By probability here, it is meant the square of the magnitude of the amplitude. In other words, the amplitude of the charge’s state on each arm changes continuously in magnitude while it encircles the cylinder. This counterintuitive behavior results from the fact that, before the post-selection, there is a continuous increase in the entanglement between the charge and the cylinder while the particle encircles the cylinder — and, of course, this assumes that the charge is not observed in between the pre and post-selection of the cylinder’s angular momentum. Nevertheless, weak measurements of the position of the charge should reveal this surprising behavior.

4.4 Discussion

The results presented in this section shows that the vector potential can, in general, assume a complex value. If that is the case, the charge seems to behave in a somewhat counterintuitive way in the AB scenario, with a constant change in the magnitude of the amplitude on each

arm.

It should be noted that, in the limits considered in Sections 4.2 and 4.3, the charged cylinder can be taken to be a macroscopic object with the state of uncertain angular momentum given by Eq. (4.49). Also, if its rotational inertia I_c is sufficiently high, it is possible to know both the cylinder's position and angular velocity with a relatively high degree of certainty (since its angular velocity is given by the ratio P_η/I_c). As a result, such a cylinder qualifies as a classical object.

Now, it is also possible to post-select the cylinder in a state $|\varphi\rangle$ such that the vector potential in Eq. (4.59) is complex-valued. In this scenario, the complex vector potential is also a classical object.

This seems to suggest a need for the generalization of the correspondence principle, which refers to the connection between classical and quantum systems. The Ehrenfest theorem is often seen as a mathematical basis for this principle. However, it accounts only for real expectation values playing a role in classical physics. In fact, post-selections of quantum systems, and the WVs associated with them, are typically not considered in the study of the quantum-to-classical transition.

As shown here, nonetheless, even for macroscopic objects, complex WVs provide a more detailed physical picture than the usual real expectation values. Furthermore, the present results suggest that classical quantities (in the sense of quantities associated with macroscopic objects) are, in general, complex-valued. Also, because complex WVs are proof of contextuality, this work suggests that contextuality may play a role in the quantum-to-classical transition.

Finally, it should be noted that, even though the example considered here presents a scenario where the vector potential is complex, the same general derivation should hold for other physical quantities of macroscopic objects, like position and momentum.

5 Fueter Variables on Banach Algebras

In this chapter, the main results from Ref. [92] are presented. Specifically, a notion of analyticity introduced by Fueter for functions of a quaternionic variable [166] is extended to the functions of a single variable that takes value in a Banach algebra (BA) \mathcal{A} .

For that, Fueter variables (FVs) are defined in Section 5.3. Continuing, the referred notion of derivative of \mathcal{A} -valued functions is presented, followed by Fueter expansions and the Gleason problem, the theory of rational functions (RFs), and spaces of Fueter series, which include the Fock-Bargmann-Segal and the Drury-Arveson spaces. Before that, a short discussion on the extension of the notion of derivatives is presented.

5.1 Real Derivatives and Their Extension to Other Algebras

In real analysis, the derivative of a function f of a real variable x at a point $x_0 \in \mathbb{R}$ is defined as the limit

$$\frac{df}{dx}(x_0) \equiv \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R} \quad (5.1)$$

whenever it exists. A clear necessary, but not sufficient, condition for this limit to exist in a certain open set $\Omega_{\mathbb{R}} \subset \mathbb{R}$ is that the function belongs to the set $C^0(\Omega_{\mathbb{R}})$ of continuous functions in $\Omega_{\mathbb{R}}$. Moreover, if the derivative of the function exists for every point in $\Omega_{\mathbb{R}}$ and it is continuous, the function is said to belong to a subset $C^1(\Omega_{\mathbb{R}})$ of $C^0(\Omega_{\mathbb{R}})$.

Because the derivative of a function is itself a function, the same reasoning applied to f can be iterated to df/dx and higher order derivatives. Then, in general, a function belongs to a set $C^k(\Omega_{\mathbb{R}})$, for some $k \in \mathbb{N}$, if it admits at least k continuous derivatives. As an example of a function that belongs to $C^k(\mathbb{R})$ but does not belong $C^{k+1}(\mathbb{R})$, consider the function

$f(x) = |x|^{k+1}$ for $k = 2n$, where $n \in \mathbb{N}$. Its k -th derivative, given by

$$\frac{d^k f}{dx^k}(x) = (k+1)! |x|, \quad (5.2)$$

is continuous but not differentiable at $x = 0$.

If a function admits infinite continuous derivatives, i.e., if it belongs to $C^\infty(\Omega_{\mathbb{R}})$, it is called a *smooth* function. Moreover, if a $C^\infty(\Omega_{\mathbb{R}})$ function can be written as a convergent power series in a neighborhood of $x_0 \in \Omega_{\mathbb{R}}$, the function is said to be *analytic* at x_0 .

These notions can be extended to functions f of a complex variable z . The direct analogous of Eq. (5.1) for the definition of the derivative of f at a point $z_0 \in \mathbb{C}$ is

$$\frac{df}{dz}(z_0) \equiv \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}. \quad (5.3)$$

Writing $z = x + iy$, $z_0 = x_0 + iy_0$, and

$$f(z) = u(x, y) + iv(x, y), \quad (5.4)$$

where $x, y, x_0, y_0 \in \mathbb{R}$, u and v are real functions, and i is the imaginary complex unit, it can be shown that the limit in Eq. (5.3) exists if and only if u and v can be derived with respect to each variable at least once at (x_0, y_0) and

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{cases}, \quad (5.5)$$

which are known as the *Cauchy-Riemann equations*. Complex functions that are differentiable at a point $z_0 \in \Omega_{\mathbb{C}} \subset \mathbb{C}$ are called *holomorphic* functions at z_0 . Also, a complex function f is said to be analytic at a point $z_0 \in \mathbb{C} \in \Omega_{\mathbb{C}}$ if it can be written as a convergent

power series in a neighborhood of $z_0 \in \Omega_{\mathbb{C}}$. A remarkable theorem in complex analysis states a function is holomorphic if and only if it is analytic [167].

Although they are called Cauchy-Riemann equations, the expressions in Eq. (5.5) first appeared in d'Alembert's study of resistance of fluids in 1752 [168]. After Euler had already connected those equations to analytic functions [167], Cauchy, in 1814, used the system in Eq. (5.5) to create his theory of functions. Later, in 1851, Riemann studied this theory in his Ph.D. dissertation [169].

While the extension of Eq. (5.1) to the complex case works, it fails for arbitrary algebras. For instance, consider the case of the quaternions. Because of the lack of commutativity, the extension of Eq. (5.1) can be written either as

$$\lim_{q \rightarrow q_0} (f(q) - f(q_0)) (q - q_0)^{-1} \in \mathbb{H} \quad (5.6)$$

or

$$\lim_{q \rightarrow q_0} (q - q_0)^{-1} (f(q) - f(q_0)) \in \mathbb{H}, \quad (5.7)$$

where, now, f is a function of the quaternionic variable q and $q_0 \in \mathbb{H}$. However, in either of these cases, the derivative would only exist for linear functions, i.e., if $a_1, a_2, b_1, b_2 \in \mathbb{H}$,

$$f(q) = a_1 q + b_1 \quad (5.8)$$

or

$$f(q) = q a_2 + b_2, \quad (5.9)$$

respectively [170]. Therefore, if one wants to build a non-trivial analysis on the quaternions, other definitions of derivatives have to be considered.

In Ref. [166], Fueter studied quaternionic-valued functions and introduced non-commuting

hypercomplex variables that allowed power series expansion of hypercomplex functions. Such variables are now known as FVs. They are defined in a more general context in Section 5.3, right after the introduction of the BAs that are considered here.

5.2 The Algebra

The central object of study in most of this chapter are functions

$$f : \mathcal{A} \rightarrow \mathcal{A}, \quad (5.10)$$

where \mathcal{A} is an arbitrary BA over the field \mathbb{K} of the real or the complex numbers. First, however, some definitions and notation are introduced.

Let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathcal{A} \setminus \mathbb{K}$ be the generators of \mathcal{A} that differ from the identity element of \mathbb{K} , i.e., from $\mathbf{a}_0 \equiv 1$. For simplicity, the number n of generators is assumed to be finite, but it could, in principle, be taken to be infinity for all computations presented here. However, a more careful analysis would be required because, for instance, the operator defined in Eq. (5.19) would become a differential operator on infinitely many variables.

In some algebras, all “directions” are given by the elements \mathbf{a}_k , $k \in \mathbb{Z}_{n+1} \equiv \{0, 1, \dots, n\}$. If that is the case, there exist coefficients $c_{jkl} \in \mathbb{K}$, $j, k, l \in \mathbb{Z}_{n+1}$ such that

$$\mathbf{a}_j \mathbf{a}_k = \sum_{l=0}^n c_{jkl} \mathbf{a}_l \quad (5.11)$$

for every $j, k \in \mathbb{Z}_{n+1}^* \equiv \{1, \dots, n\}$. As an example, consider the quaternions. Denoting the imaginary units of the quaternions by i , j and k , the product of two elements (say, ij) is related to the third one (k in this case).

In other algebras, on the other hand, the product $\mathbf{a}_j \mathbf{a}_k$ might result in a new “direction.”

An example of such algebras is the real ternary algebra, which is generated by the number 1 and an element e that is not real (nor complex). However, e^2 gives a new direction in such an algebra. The quaternions themselves can be seen as another example when the elements 1, i , and j are considered their generators. In this case, ij results in a new direction of the algebra.

Then, in order to include algebras \mathcal{A} for which Eq. (5.11) does not necessarily hold, consider a set of t -uples \mathfrak{J} associated with all linearly independent directions of the algebra. Specifically, if, for a certain j and k , the product $\mathbf{a}_j\mathbf{a}_k$ results in a new direction, then (j, k) might be an element of \mathfrak{J} and $\mathbf{a}_{(j,k)} \equiv \mathbf{a}_j\mathbf{a}_k$. Note that, in some algebras, there might be multiple ways to build \mathfrak{J} . For instance, if $\mathbf{a}_j\mathbf{a}_k = -\mathbf{a}_k\mathbf{a}_j$, as is the case of Clifford algebras, either (j, k) or (k, j) should be part of \mathfrak{J} , but not both. In such case, if $(j, k) \in \mathfrak{J}$, $\mathbf{a}_k\mathbf{a}_j = -\mathbf{a}_{(j,k)}$. Moreover, the direction $\mathbf{a}_0 = 1$ is not included in \mathfrak{J} .

For simplicity of notation, instead of referring to the set \mathfrak{J} directly, a map from it into \mathbb{Z}_{m+1}^* , where m is the cardinality of \mathfrak{J} , is used. Hence, instead of considering indexes that take value in \mathfrak{J} , they will take value in \mathbb{Z}_{m+1}^* . Moreover, to avoid confusion, we also denote the “independent directions” of \mathcal{A} by e_k instead of \mathbf{a}_k . For instance, if the quaternions are generated by $\mathbf{a}_0 = 1$, $\mathbf{a}_1 = i$, and $\mathbf{a}_2 = j$, then, in the notation that will be used hereby, one must consider $e_0 = \mathbf{a}_0$, $e_1 = \mathbf{a}_1$, $e_2 = \mathbf{a}_2$, and $e_3 = \mathbf{a}_1\mathbf{a}_2$.

Therefore, with those definitions, the algebra \mathcal{A} can be identified with the space \mathbb{K}^{m+1} in the following way

$$\mathbb{K}^{m+1} \simeq \mathcal{A} = \left\{ a = \sum_{k=0}^m a_k e_k \mid a_0, a_1, \dots, a_m \in \mathbb{K} \right\}. \quad (5.12)$$

In particular, Eq. (5.11) can be rewritten as

$$e_j e_k = \sum_{\ell=0}^m (\chi_\ell)_{jk} e_\ell, \quad (5.13)$$

where the matrices χ_ℓ belong to $\mathbb{K}^{(m+1) \times (m+1)}$ for every $\ell \in \mathbb{K}_{m+1}$. The collection of these matrices will be called the *characteristic operators of the algebra* \mathcal{A} since they encode all properties of the product of the algebra. Moreover, each matrix χ_ℓ can be seen as a metric tensor-like object associated with the ℓ -th direction of \mathcal{A} . In fact, observe that, for any $a, b \in \mathcal{A}$, writing

$$ab = \sum_{\ell=0}^m c_\ell e_\ell, \quad (5.14)$$

Eq. (5.13) leads to

$$c_\ell = \sum_{j,k=1}^m a_j (\chi_\ell)_{jk} b_k \equiv a \chi_\ell b \quad (5.15)$$

for every $\ell \in \mathbb{K}_{m+1}$.

Now, the BAs \mathcal{A} of interest in this work are endowed with an involution, denoted by \dagger , which has the following properties:

- $\forall a, b \in \mathcal{A}, (ab)^\dagger = b^\dagger a^\dagger$;
- $\forall k \in \mathbb{K} \subset \mathcal{A}, kk^\dagger = k^\dagger k = |k|^2$.

Observe that, in general, aa^\dagger is not a real (nor a complex) number. In particular, $aa^\dagger = a^\dagger a$ does not always hold.

Continuing, because \mathcal{A} is a BA, it has a norm N for which

$$N(ab) \leq N(a)N(b), \quad \forall a, b \in \mathcal{A}. \quad (5.16)$$

Such a norm may or may not be induced by the involution \dagger . Nevertheless, it is assumed

that $N(a) = N(a^\dagger)$ for every $a \in \mathcal{A}$.

Note that because $\mathbb{K} \subset \mathcal{A}$, N is also a norm in \mathbb{K} . Therefore, without loss of generality, it can be presumed that $N(1) = 1$. Then, generally, $N(k) = |k|$, where $|k|$ denotes the usual norm of k in \mathbb{C} , which coincides with the norm of a real number if $\mathbb{K} = \mathbb{R}$. Moreover, it follows that

$$N(ka) = |k|N(a) \tag{5.17}$$

for every $a \in \mathcal{A}$.

Now, introduced basic notations and definitions in the algebra \mathcal{A} , it is possible to use Eq. (5.12) to rewrite the functions f of the type given by Eq. (5.10) as

$$f : \mathbb{K}^{m+1} \rightarrow \mathcal{A}. \tag{5.18}$$

In particular, most of the focus here is on the subset of functions of the type given by Eq. (5.18) that are \mathbb{K} -analytic, i.e., functions that are real analytic if $\mathbb{K} = \mathbb{R}$ or complex holomorphic if $\mathbb{K} = \mathbb{C}$.

5.3 A General Principle

The ideas introduced by Fueter for the quaternions [166] were adapted by Malonek for Clifford algebras [171, 172] and by other authors in other settings [173, 174]. These ideas can be also extended to the BAs considered here. To start, it should be noticed that, because of the identification between \mathbb{K}^{m+1} and \mathcal{A} presented in Eq. (5.12), it is possible to identify f in Eq. (5.10) as an \mathcal{A} -valued function of $m + 1$ \mathbb{K} -valued variables v_k , $k \in \mathbb{Z}_{m+1}$.

Now, consider functions f of class C^1 in an open set $\Omega \subset \mathbb{K}^{m+1}$. Also, assume they lie in

the kernel of the Cauchy-Fueter (CF) operator, defined as

$$D \equiv D_0 + \sum_{k=1}^m e_k D_k, \quad (5.19)$$

where

$$D_j \equiv \frac{\partial}{\partial v_j} \quad (5.20)$$

for every $j \in \mathbb{Z}_{m+1}$, i.e., they satisfy

$$Df = 0. \quad (5.21)$$

Such functions are called left D -hyperholomorphic or left monogenic [171]. The word left is used because one can also consider functions f that belongs to the kernel of D when it acts on f from the right. Here, because functions that satisfy Eq. (5.21) are the focus, the word left will be omitted, and the functions are just called D -hyperholomorphic or monogenic. However, observe that every result obtained from now on can be easily adapted for right D -hyperholomorphic functions.

Then, denoting $v = (v_0, v_1, \dots, v_m) \in \mathbb{K}^{m+1}$, it holds that

$$\begin{aligned} \frac{df}{dt}(tv) &= \sum_{k=0}^m v_k \frac{\partial f}{\partial v_k}(tv) \\ &= \sum_{k=1}^m v_k \frac{\partial f}{\partial v_k}(tv) - v_0 \left(\sum_{k=1}^m e_k \frac{\partial f}{\partial v_k}(tv) \right) \\ &= \sum_{k=1}^m (v_k - e_k v_0) \frac{\partial f}{\partial v_k}(tv) \end{aligned} \quad (5.22)$$

and, then,

$$f(v) - f(0) = \int_0^1 \frac{d}{dt} f(tv) dt = \sum_{k=1}^m (v_k - e_k v_0) \int_0^1 \frac{\partial f}{\partial v_k}(tv) dt. \quad (5.23)$$

The functions

$$\zeta_k(v) = v_k - e_k v_0, \quad k \in \mathbb{Z}_{m+1}^*, \quad (5.24)$$

are called the FVs or total regular variables [175] associated with D .

More generally, for a fixed $w \in \Omega$,

$$\begin{aligned} \frac{df}{dt}(tv + (1-t)w) &= \sum_{k=0}^m (v_k - w_k) \frac{\partial f}{\partial v_k}(tv + (1-t)w) \\ &= \sum_{k=1}^m (\zeta_k(v) - \zeta_k(w)) \frac{\partial f}{\partial v_k}(tv + (1-t)w) \end{aligned} \quad (5.25)$$

holds for every $v \in \Omega$. Therefore,

$$f(v) - f(w) = \int_0^1 \frac{d}{dt} f(tv) dt = \sum_{k=1}^m (\zeta_k(v) - \zeta_k(w)) \int_0^1 \frac{\partial f}{\partial v_k}(tv + (1-t)w) dt. \quad (5.26)$$

Following Malonek's approach for the Clifford algebra in Ref. [171], consider now $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m)$ and let \mathcal{H}^m be the set of all such vectors. Then, there is a one-to-one correspondence between \mathbb{K}^{m+1} and \mathcal{H}^m , namely

$$\mathbb{K}^{m+1} \simeq \mathcal{H}^m = \{\zeta = (\zeta_1, \dots, \zeta_m) \mid \zeta_k = v_k - e_k v_0; v_0, v_k \in \mathbb{K}\}. \quad (5.27)$$

Because of this correspondence, the notations $f(v)$ and $f(\zeta)$ are used interchangeably. Moreover, $\zeta(w)$ will often be denoted as $\xi \in \mathcal{H}^m$.

Observe that, although $\mathcal{H}^m \subset \mathcal{A}^m$, \mathcal{H}^m is not a submodule of \mathcal{A}^m . In fact, $c\zeta$ only belongs to \mathcal{H}^m for an arbitrary $\zeta \in \mathcal{H}^m$ if and only if $c \in \mathbb{K}$. As a consequence, the product of two D -hyperholomorphic functions is not always D -hyperholomorphic. For instance, if

$j, k \in \mathbb{Z}_{m+1}^*$, it holds that

$$\zeta_j \zeta_k = v_j v_k - e_j v_0 v_k - e_k v_0 v_j + e_j e_k v_0^2 \quad (5.28)$$

and, then,

$$D(\zeta_j \zeta_k) = [e_j, e_k] v_0^2, \quad (5.29)$$

where $[e_j, e_k] \equiv e_j e_k - e_k e_j$ is the commutator of e_j and e_k . Because, in general, $[e_j, e_k] \neq 0$, the right-hand side of Eq. (5.29) does not always vanish.

However, it follows from Eq. (5.29) that

$$D(\zeta_j \zeta_k + \zeta_k \zeta_j) = 0. \quad (5.30)$$

In general, as it will be seen in the next section, symmetrized products of FVs and, in particular, powers of a single FV are D -hyperholomorphic.

Finally, the operator

$$(\mathcal{R}_k(\xi)f)(\zeta) \equiv (\mathcal{R}_k(w)f)(v) \equiv \int_0^1 \frac{\partial f}{\partial v_k}(tv + (1-t)w)dt, \quad (5.31)$$

where $w \in \Omega$ is a fixed element, is called the *backward-shift operator centered at w* (or at ξ).

Using this definition, Eq. (5.26) can be rewritten as

$$f(v) - f(w) = \sum_{k=1}^m (\zeta_k(v) - \zeta_k(w))(\mathcal{R}_k(w)f)(v). \quad (5.32)$$

Furthermore, observe that

$$(\mathcal{R}_k(w)f)(w) = \frac{\partial f}{\partial v_k}(w). \quad (5.33)$$

Then, if f belongs to $C^2(\Omega)$, $\mathcal{R}_k(w)f(v)$ is D -hyperholomorphic. Generally, if f belongs to

$C^p(\Omega)$ for some $p \in \mathbb{N}$, the process given by Eq. (5.29) can be iterated p times, generating the so-called *Fueter polynomials*, which are the subject of study of Section 5.5. In particular, functions of class $C^\infty(\Omega)$ give origin to *Fueter series*, studied in Section 5.6.

5.4 Hyperholomorphicity of Functions from \mathbb{K}^{m+1} into \mathcal{A}

Following similar arguments of Ref. [171], it is now shown that functions of FVs admit a notion of a derivative that, in a sense, generalizes the real and the complex cases defined in Eqs. (5.1) and (5.3).

First, to set the framework, endow \mathcal{A}^m with the Hermitian form

$$[\zeta, \xi]_{\mathcal{A}} \equiv \sum_{k=1}^m \xi_k^\dagger \zeta_k, \quad (5.34)$$

which is, in general, \mathcal{A} -valued. Moreover, defining

$$\mathfrak{h}_k = (0, \dots, 0, 1, 0 \dots, 0) \in \mathcal{H}^m, \quad k \in \mathbb{Z}_{m+1}^*, \quad (5.35)$$

and

$$\mathfrak{h}_0 = -(e_1, e_2, \dots, e_m) \in \mathcal{H}^m, \quad (5.36)$$

it follows that an arbitrary element $\zeta \in \mathcal{H}^m$ can be written as

$$\zeta = \sum_{k=0}^m v_k \mathfrak{h}_k. \quad (5.37)$$

Furthermore,

$$[\zeta, \mathfrak{h}_k]_{\mathcal{A}} = \zeta_k \quad (5.38)$$

for every $k \in \mathbb{Z}_{m+1}^*$, and

$$[\zeta, \mathfrak{h}_0]_{\mathcal{A}} = -v_0 \sum_{k=1}^m e_k^2 - \sum_{k=1}^m v_k e_k. \quad (5.39)$$

Clearly, the set $\{\mathfrak{h}_k\}_{k \in \mathbb{Z}_{m+1}^*}$ is a canonical basis for \mathcal{A}^m . Then, motivated by it, the norm in \mathcal{A}^m is defined as

$$\|\zeta\|_{\mathcal{A}^m} \equiv \left(\sum_{k=1}^m N([\zeta, \mathfrak{h}_k]_{\mathcal{A}})^2 \right)^{1/2}. \quad (5.40)$$

Now, let $\mathcal{L}(\mathcal{A}^m, \mathcal{A})$ be the set of all left \mathcal{A} -linear operators from \mathcal{A}^m into \mathcal{A} . Then, if $L \in \mathcal{L}$, it holds that

$$L(au + bv) = aL(u) + bL(v) \quad (5.41)$$

for every $a, b \in \mathcal{A}$ and $u, v \in \mathcal{A}^m$. In particular, it follows that $L \in \mathcal{L}(\mathcal{H}^m, \mathcal{A})$ is \mathcal{A} -linear from the left if

$$L(a\zeta + b\xi) = aL(\zeta) + bL(\xi) \quad (5.42)$$

and, moreover, there exist constants $A_k \in \mathcal{A}$, $k \in \mathbb{Z}_{m+1}^*$, such that L is uniquely characterized by

$$L(\zeta) = \zeta_1 A_1 + \cdots + \zeta_m A_m. \quad (5.43)$$

With that set, it is possible to introduce a generalization of real differentiation and complex holomorphicity in the present setting. An \mathcal{A} -valued function f is said to be (left) hyperholomorphic at a point $\xi \in \Omega \subset \mathcal{H}^m$ if there exists a (left) \mathcal{A} -linear map $L_\xi \in \mathcal{L}(\mathcal{H}^m, \mathcal{A})$ such that

$$\lim_{\Delta\zeta \rightarrow 0} \frac{N(f(\xi + \Delta\zeta) - f(\xi) - L_\xi(\Delta\zeta))}{\|\Delta\zeta\|_{\mathcal{A}^m}} = 0, \quad (5.44)$$

in which case L_ξ is the (left) derivative of f at ξ . The function is also said to be hyperholomorphic in the open set Ω if there exists an L_ξ in Eq. (5.44) for every $\xi \in \Omega$. This notion

of derivative is called a *Fréchet derivative*.

Observe that, if f is hyperholomorphic, then its derivative at ξ , i.e., the linear map L_ξ in Eq. (5.44) is unique. In fact, Eqs. (5.43) and (5.44) imply that

$$f(\xi + \Delta\zeta) - f(\xi) = \Delta\zeta_1 A_1 + \cdots + \Delta\zeta_m A_m + o(\|\Delta\zeta\|_{\mathcal{A}^m}), \quad (5.45)$$

where

$$\lim_{\Delta\zeta \rightarrow 0} \frac{o(\|\Delta\zeta\|_{\mathcal{A}^m})}{\|\Delta\zeta\|_{\mathcal{A}^m}} = 0. \quad (5.46)$$

Then, assuming that f is a \mathbb{K} -analytic function, i.e.,

$$f(\zeta + \Delta\zeta) - f(\zeta) = \Delta f(\zeta) = \Delta v_0 \frac{\partial f}{\partial v_0} + \Delta v_1 \frac{\partial f}{\partial v_1} + \cdots + \Delta v_m \frac{\partial f}{\partial v_m} + o(|\Delta v|), \quad (5.47)$$

with

$$\lim_{\Delta v \rightarrow 0} \frac{o(|\Delta v|)}{|\Delta v|} = 0, \quad (5.48)$$

it is possible to show that f is hyperholomorphic if and only if it is D -hyperholomorphic. In fact, defining $v_0 \equiv \zeta_0$ and writing

$$v_k = \zeta_0 e_k + \zeta_k \quad k \in \mathbb{Z}_{m+1}^*, \quad (5.49)$$

Eq. (5.47) leads to

$$\begin{aligned} \Delta f(\zeta) &= \Delta\zeta_0 \left(D_0 f + \sum_{k=1}^m e_k D_k f \right) + \sum_{k=1}^m \Delta\zeta_k D_k f + o(\|\Delta\zeta\|_{\mathcal{A}^m}) \\ &= \Delta\zeta_0 D f + [\Delta\zeta, \nabla_v] f + o(\|\Delta\zeta\|_{\mathcal{A}^m}), \end{aligned} \quad (5.50)$$

where $\nabla_v \equiv \sum_{k=1}^m D_k \mathfrak{h}_k$. Comparing Eq. (5.50) with Eq. (5.43), one concludes that f is hyperholomorphic if and only if $Df = 0$, i.e., if and only if f is D -hyperholomorphic.

5.5 Fueter Polynomials

It is already established that FVs are hyperholomorphic. In general, however, their pointwise product is not. Nevertheless, as it will be shown now, these variables generate hyperholomorphic polynomials, called Fueter polynomials, with the *symmetrized product*.

The symmetrized product of N elements a_1, \dots, a_N of an algebra \mathcal{A} is defined as

$$a_1 \times a_2 \times \cdots \times a_N = \frac{1}{N!} \sum_{\sigma \in S_N} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(N)}, \quad (5.51)$$

where the sum is over the set S_N of all permutations of N indexes. Observe that, in general, the product \times is non-associative, i.e.,

$$(a \times b) \times c \neq a \times (b \times c). \quad (5.52)$$

Moreover, if \mathcal{A} is commutative, the symmetrized product reduces to the regular product. Furthermore, as pointed out by Malonek in Ref. [172], for every $k \in \mathbb{N}$,

$$(a_1 + \cdots + a_N)^k = \sum_{\substack{\alpha \in \mathbb{N}_0^N; \\ |\alpha|=k}} \frac{k!}{\alpha!} a^\alpha, \quad (5.53)$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_N!$, and $a^\alpha = a_1^{\alpha_1} \times \cdots \times a_N^{\alpha_N}$.

With those definitions, the same method used in Refs. [166, 173, 176] can be applied to show that, if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$, the symmetrized product of the terms $\zeta_k^{\alpha_k}$, $k \in \mathbb{Z}_{m+1}^*$, simply denoted by ζ^α , is hyperholomorphic. To start, rewrite ζ^α as

$$\zeta^\alpha = \frac{1}{|\alpha|!} \sum_{\sigma \in S_{|\alpha|}} \sum_{u=1}^m \sum_{\substack{k \in \mathbb{Z}_{|\alpha|+1}^* \\ \sigma(k)=u}} \zeta_{\sigma(1)} \cdots \zeta_{\sigma(k-1)} \zeta_u \zeta_{\sigma(k+1)} \cdots \zeta_{\sigma(|\alpha|)}, \quad (5.54)$$

where $|\alpha|$ denotes the sum of the components of α . Then,

$$\begin{aligned}
D(\zeta^\alpha) &= \frac{1}{|\alpha|!} \sum_{\sigma \in S_{|\alpha|}} \sum_{u=1}^m \sum_{\substack{k \in \mathbb{Z}_{|\alpha|+1}^* \\ \sigma(k)=u}} e_u \zeta_{\sigma(1)} \cdots \zeta_{\sigma(k-1)} \zeta_{\sigma(k+1)} \cdots \zeta_{\sigma(|\alpha|)} - \\
&\quad - \frac{1}{|\alpha|!} \sum_{\sigma \in S_{|\alpha|}} \sum_{u=1}^m \sum_{\substack{k \in \mathbb{Z}_{|\alpha|+1}^* \\ \sigma(k)=u}} \zeta_{\sigma(1)} \cdots \zeta_{\sigma(k-1)} e_u \zeta_{\sigma(k+1)} \cdots \zeta_{\sigma(|\alpha|)}.
\end{aligned} \tag{5.55}$$

Thus, because the sums in the previous equations over terms of the type

$$\zeta_u \zeta_{\sigma(1)} \cdots \zeta_{\sigma(k-1)} \zeta_{\sigma(k+1)} \cdots \zeta_{\sigma(|\alpha|)} \tag{5.56}$$

equals the sums over

$$\zeta_{\sigma(1)} \cdots \zeta_{\sigma(k-1)} \zeta_u \zeta_{\sigma(k+1)} \cdots \zeta_{\sigma(|\alpha|)}, \tag{5.57}$$

it can be shown that

$$v_0 D(\zeta^\alpha) = 0. \tag{5.58}$$

Finally, because v_0 can be taken to be non-zero, Eq. (5.58) implies that $D(\zeta^\alpha) = 0$, i.e., ζ^α is hyperholomorphic.

As a consequence, polynomials of FVs with the symmetrized product, i.e., linear combinations of ζ^α with different α 's (and \mathcal{A} -valued coefficients on the right-hand side of the powers) are also hyperholomorphic. These polynomials are referred to as *Fueter polynomials*.

To conclude, observe that, in a sense, Fueter polynomials appear naturally as an expansion of functions that admit many derivatives from repeated iterations of the backward-shift operator. To see that, assume that f is at least of class $C^2(\Omega)$ and observe that $\mathcal{R}_k(\xi)\mathcal{R}_j(\xi) = \mathcal{R}_j(\xi)\mathcal{R}_k(\xi)$ for every $j, k \in \mathbb{Z}_{m+1}^*$. Then, the application of Eq. (5.32) to $(\mathcal{R}_k(\xi)f)(\zeta)$ leads

to

$$\begin{aligned}
f(\zeta) &= f(\xi) + \sum_{k=1}^m (\zeta_k - \xi_k) \frac{\partial f}{\partial v_k}(\xi) \\
&\quad + \frac{1}{2} \sum_{j,k=1}^m [(\zeta_k - \xi_k)(\zeta_j - \xi_j) + (\zeta_j - \xi_j)(\zeta_k - \xi_k)] (\mathcal{R}_k(\xi) \mathcal{R}_j(\xi) f)(\zeta) \\
&= \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha| < 2}} (\zeta - \xi)^\alpha \frac{\partial^{|\alpha|} f}{\partial v^\alpha}(\xi) + \sum_{j,k=1}^m [(\zeta_k - \xi_k) \times (\zeta_j - \xi_j)] (\mathcal{R}_k(\xi) \mathcal{R}_j(\xi) f)(\zeta) \quad (5.59) \\
&= \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha| < 2}} (\zeta - \xi)^\alpha \frac{\partial^{|\alpha|} f}{\partial v^\alpha}(\xi) + \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha| = 2}} (\zeta - \xi)^\alpha (\mathcal{R}(\xi)^\alpha f)(\zeta).
\end{aligned}$$

More generally, recalling that $\zeta(w) = \xi$ and defining elements $\iota_k \in \mathbb{N}_0^m$ as

$$(\iota_k)_j \equiv \delta_{jk}, \quad j, k \in \mathbb{Z}_{m+1}^*, \quad (5.60)$$

observe that

$$\begin{aligned}
\mathcal{R}_k(\xi)(\zeta - \xi)^\alpha &= \int_0^1 \frac{\partial(\zeta - \xi)^\alpha}{\partial v_k} (tv + (1-t)w) dt \\
&= \int_0^1 \alpha_k (\zeta(tv + (1-t)w) - \xi)^{\alpha - \iota_k} dt \\
&= \int_0^1 \alpha_k t^{|\alpha| - 1} (\zeta - \xi)^{\alpha - \iota_k}(v) dt \\
&= \frac{\alpha_k}{|\alpha|} (\zeta - \xi)^{\alpha - \iota_k}
\end{aligned} \quad (5.61)$$

for every α such that $\alpha_k \neq 0$. Also, $\mathcal{R}_k(\xi)(\zeta - \xi)^\alpha = 0$ if $\alpha_k = 0$. Observe that Eq. (5.61) justifies the name backward-shift operator given to $\mathcal{R}_k(\xi)$.

As a consequence, every function f of class $C^p(\Omega)$ can be written as

$$f(\zeta) = \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha| < p}} \frac{(\zeta - \xi)^\alpha}{|\alpha|!} \frac{\partial^{|\alpha|} f}{\partial v^\alpha}(\xi) + \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha| = p}} (\zeta - \xi)^\alpha (\mathcal{R}(\xi)^\alpha f)(\zeta). \quad (5.62)$$

5.6 Fueter Series and the Gleason Problem

This section focuses on the study of hyperholomorphic functions, i.e., functions which are \mathbb{K} -analytic and belong to the kernel of the CF operator, given by Eq. (5.19). After that, the solution to a problem often referred to as the Gleason problem is studied.

To start, it is studied the convergence of infinite iterations of $\mathcal{R}(\xi)$ on a \mathbb{K} -analytic function f , i.e., the result of Eq. (5.62) in the limit $p \rightarrow \infty$. Such a process leads, at least formally, to the *Fueter series*

$$f(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} (\zeta - \xi)^\alpha f_\alpha(\xi), \quad (5.63)$$

where

$$f_\alpha(\xi) \equiv \frac{1}{|\alpha|!} \frac{\partial^{|\alpha|} f}{\partial v^\alpha}(\xi). \quad (5.64)$$

Then, it needs to be shown that the series in Eq. (5.63) converges in an open neighborhood of ξ .

To do so, observe that, if a function f that takes value in \mathbb{K}^{m+1} is \mathbb{K} -analytic in a neighborhood $\Omega(w)$ of a point w , then the power series

$$f(v) = \sum_{\alpha \in \mathbb{N}_0^{m+1}} (v - w)^\alpha F_\alpha(w), \quad (5.65)$$

where

$$F_\alpha(w) \equiv \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial v^\alpha}(w), \quad (5.66)$$

converges for every $v \in \Omega(w)$. In other words, there exist $K, r_1, \dots, r_m > 0$ such that

$$N(F_\alpha) \leq \frac{K}{r_0^{\alpha_0} \dots r_m^{\alpha_m}} \equiv \frac{K}{r^\alpha}. \quad (5.67)$$

Observe that the coefficients f_α and $F_{(0,\alpha)}$ can have their norm compared. In fact, using Eqs. (5.64) and (5.66), it can be derived that

$$N(f_\alpha) \leq N(F_{(0,\alpha)}) \quad (5.68)$$

for every $\alpha \in \mathbb{N}_0^m$.

Now, letting $\sigma_1, \dots, \sigma_m > 0$ be such that $\sigma_k < r_k$ for all $k \in \mathbb{Z}_{m+1}^*$ and, again, denoting by ξ the element in \mathcal{H}^m that corresponds to the point $w \in \mathbb{K}^{m+1}$, it is possible to define, in a similar way done in Ref. [172], the set

$$\mathcal{U}_\xi(\sigma) = \{\zeta \in \mathcal{H}^m \mid N(\zeta_k - \xi_k) \leq \sigma_k, k \in \mathbb{Z}_{m+1}^*\} \subset \mathcal{H}^m. \quad (5.69)$$

If a function f given by Eq. (5.65) is hyperholomorphic in a neighborhood of $w \in \mathbb{K}^{m+1}$, the Fueter series given by Eq. (5.63) converges absolutely for every $\zeta \in \mathcal{U}_\xi(\sigma)$. To prove that, first observe that, while either Eq. (5.63) and

$$f(\zeta) = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ |\alpha|=k}} (\zeta - \xi)^\alpha f_\alpha(\xi) \quad (5.70)$$

are, at least formally, possible representations of power series of functions of FVs, they generally have different domains of convergence. This is the case, for instance, if \mathcal{A} is a Clifford algebra [177]. Nevertheless, in $\mathcal{U}_\xi(\sigma)$, the two expressions, if individually convergent, coincide. In fact,

$$N((\zeta - \xi)^\alpha) \leq \prod_{k=1}^m N(\zeta_k - \xi_k)^{\alpha_k} \leq \sigma^\alpha < r^\alpha \quad (5.71)$$

and, as a consequence,

$$N \left(\sum_{\alpha \in \mathbb{N}_0^m} (\zeta - \xi)^\alpha f_\alpha(\xi) \right) \leq \sum_{\alpha \in \mathbb{N}_0^m} \sigma^\alpha N(f_\alpha(\xi)) \leq K \sum_{\alpha \in \mathbb{N}_0^m} \left(\frac{\sigma}{r} \right)^\alpha < \infty, \quad (5.72)$$

where Eqs. (5.67) and (5.68) were used. Hence, Eq. (5.63) converges absolutely in the domain $\mathcal{U}_\xi(\sigma)$.

With the convergence of Fueter series of \mathbb{K} -analytic functions proved, the Gleason problem can be considered. First, the Cauchy-Kovalevskaya (CK) product, also known as convolution, centered at $\xi \in \mathcal{H}^m$ is defined as

$$(\zeta - \xi)^\alpha a \odot_\xi (\zeta - \xi)^\beta b = (\zeta - \xi)^{\alpha+\beta} ab, \quad (5.73)$$

where $\alpha, \beta \in \mathbb{N}_0^m$ and $a, b \in \mathcal{A}$. This product can be extended to power series. In fact, remembering that $\xi = \zeta(w)$, let f be given by Eq. (5.63) and

$$g(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} (\zeta - \xi)^\alpha g_\alpha. \quad (5.74)$$

Setting $v_0 = w_0$, power series in $v_k - w_k$, $k \in \mathbb{Z}_{m+1}^*$, for each function with coefficients in \mathcal{A} are obtained:

$$f(w_0, v_1, \dots, v_m) = \sum_{\alpha \in \mathbb{N}_0^m} (v - w)^\alpha f_\alpha \quad (5.75)$$

and

$$g(w_0, v_1, \dots, v_m) = \sum_{\alpha \in \mathbb{N}_0^m} (v - w)^\alpha g_\alpha. \quad (5.76)$$

Observe that, in those equations, the coefficients f_α and g_α commute with the variables.

Then,

$$f(w_0, v_1, \dots, v_m)g(w_0, v_1, \dots, v_m) = \sum_{\alpha \in \mathbb{N}_0^m} (v - w)^\alpha h_\alpha, \quad (5.77)$$

where

$$h_\alpha \equiv \sum_{\substack{\gamma \in \mathbb{N}_0^m; \\ \alpha - \gamma \geq 0}} f_{\alpha - \gamma} g_\gamma. \quad (5.78)$$

With that, the CK product of two power series is defined as

$$(f \odot_\xi g)(\zeta) \equiv \sum_{\alpha \in \mathbb{N}_0^m} (\zeta - \xi)^\alpha h_\alpha. \quad (5.79)$$

Moreover, if $f(w)$ is invertible, there exist coefficients $d_\alpha \in \mathcal{A}$ for every $\alpha \in \mathbb{N}_0^m$ such that, in a neighborhood of w in $\Omega \subset \mathbb{K}^m$, we have

$$(f(w_0, v_1, \dots, v_n))^{-1} = \sum_{\alpha \in \mathbb{N}_0^m} (v - w)^\alpha d_\alpha. \quad (5.80)$$

Then, the CK inverse of f can be defined as

$$f^{-\odot_\xi}(\zeta) \equiv \sum_{\alpha \in \mathbb{N}_0^m} (\zeta - \xi)^\alpha d_\alpha. \quad (5.81)$$

An important fact about the CK product that should be noted is that it depends on the center ξ . For instance, consider, in the quaternionic setting, where $e_1 = i$, $e_2 = j$, and $e_3 = k$, the polynomials $p_1(\zeta) = \zeta_1 k$ and $p_2(\zeta) = \zeta_1 i$. Then, if $\xi = 0$, $P = p_1 \odot_0 p_2 = \zeta_1^2 j$. However, if $\xi = i$, because $p_1(\zeta) = (\zeta_1 - i)k - j$ and $p_2(\zeta) = (\zeta_1 - i)i - 1$, the result of the CK product is

$$Q(\zeta) = (p_1 \odot_i p_2)(\zeta) = (\zeta_1 - i)^2 j + j = \zeta_1^2 j - 2\zeta_1 k \quad (5.82)$$

since ζ_1 and i commute. Thus, $P \neq Q$, i.e., the CK products of p_1 and p_2 centered at the origin and centered at i are different.

Observe that it is also possible to define the right-hand side CK product as

$$(f \odot_{\xi}^R g)(z) \equiv \sum_{\alpha \in \mathbb{N}_0^m} h_{\alpha}(\zeta - \xi)^{\alpha}, \quad (5.83)$$

where

$$f(\zeta) = \sum_{\alpha \in \mathbb{N}^m} f_{\alpha}(\zeta - \xi)^{\alpha}, \quad g(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} g_{\alpha}(\zeta - \xi)^{\alpha} \quad (5.84)$$

and the coefficients h_{α} are, once again, given by Eq. (5.78).

With that set, the Gleason problem can be finally presented. The problem was introduced by Gleason in his studies of maximum ideals of a commutative BA [178]. He considered the algebra of holomorphic functions in the open unit ball in \mathbb{C}^n that can be continuously extended to the boundary. Then, he proved that, if this algebra is finitely generated, the maximum ideals of functions that are zero at the origin is generated by the coordinate functions.

The problem of verifying if these ideals in algebras of holomorphic (and hyperholomorphic) functions are generated by the coordinate functions became known as the Gleason problem, and it was vastly studied in different frameworks. A version of this problem in the present setting can be stated as a search for functions g_1, \dots, g_m such that a given hyperholomorphic function f with domain $\mathcal{U}_{\xi}(\sigma)$ can be written as

$$f(\zeta) - f(\xi) = \sum_{k=1}^m (\zeta_k - \xi_k) \odot_{\xi} g_k(\zeta) \quad (5.85)$$

for every $\zeta \in \mathcal{U}_{\xi}(\sigma)$.

Observe that, in principle, Eq. (5.32) is a solution to this problem if the CK product is replaced by the pointwise product. The problem, however, is not formulated in terms of the pointwise product because it has the disadvantage of, in general, not being hyperholomorphic.

Nevertheless, Eq. (5.32) can be seen as a good starting point in the obtaining of a solution to the problem. In fact, for every ζ such that $v_0 = w_0$, the CK product coincides with the pointwise product. As a result, the following equation also holds

$$f(\zeta) - f(\xi) = \sum_{k=1}^m (\zeta_k - \xi_k) \odot_{\xi} (\mathcal{R}_k(\xi)f)(\zeta). \quad (5.86)$$

However, not all solutions to the Gleason problem are given by Eq. (5.86). To see that, let \mathcal{G} denote the space of functions $f \in \mathcal{G}$ for which there exist $g_1, g_2, \dots, g_m \in \mathcal{G}$ that solve the Gleason problem. The space \mathcal{G} is said to be *resolvent invariant*. Moreover, let \mathcal{R} be the space of $\mathcal{R}_k(\xi)$ -invariant functions, called *backward-shift invariant*, i.e., the space for which $g_k = \mathcal{R}_k(\xi)f$. Then, it can be proved that $\mathcal{R} \subsetneq \mathcal{G}$.

First, it is necessary to characterize the elements of \mathcal{G} . For that, it will be assumed that \mathcal{G} is finite-dimensional. Also, let $f \in \mathcal{G}$ and G be a matrix-valued hyperholomorphic function whose columns form a basis for \mathcal{G} . Then, by definition, there exist a constant column matrix η with entries in \mathcal{A} such that $f = G\eta$. Moreover, there exist functions $g_1, \dots, g_m \in \mathcal{G}$ such that

$$f(\zeta) - f(\xi) = \sum_{k=1}^m (\zeta_k - \xi_k) \odot_{\xi} g_k(\zeta). \quad (5.87)$$

Finally, for every $k \in \mathbb{Z}_{m+1}^*$, there exist constant matrices A_k with entries in \mathcal{A} such that $g_k = GA_k\eta$. Hence, Eq. (5.87) can be rewritten as

$$(G(\zeta) - G(\xi))\eta = G(\zeta) \odot_{\xi} \sum_{k=1}^m (\zeta_k - \xi_k) A_k \eta, \quad (5.88)$$

which implies that G is given by

$$G(\zeta) = G(\xi) \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_{\xi}}. \quad (5.89)$$

Now, letting $f = G\eta$, where η is a constant column matrix with entries in \mathcal{A} , it holds that

$$\begin{aligned}
f(\zeta) - f(\xi) &= G(\xi) \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_{\xi}} \eta - G(\xi)\eta \\
&= G(\zeta) \odot_{\xi} \left(I - \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right) \right) \eta \\
&= \sum_{k=1}^m (\zeta_k - \xi_k) \odot_{\xi} G A_k \eta,
\end{aligned} \tag{5.90}$$

i.e., there exist functions $g_k = G A_k \eta \in \mathcal{G}$ which solves the Gleason problem for f .

Therefore, a function f belongs to a finite-dimensional resolvent invariant space \mathcal{G} if and only if it can be spanned by the columns of a matrix-valued function of the type given by Eq. (5.89).

Finally, to see that solutions given by Eq. (5.86) are only a subset of \mathcal{G} , observe that, because a function f in \mathcal{R} also belongs to \mathcal{G} , f can be written as $f = G\eta$, where G is of the form given by Eq. (5.89) and η is a constant column matrix with entries in \mathcal{A} . Moreover, since $g_k = G A_k \eta \in \mathcal{R}_k(\xi)f \in \mathcal{G}$, as just seen, it holds that $\mathcal{R}_k(\xi)G = G A_k$. Furthermore, because the operators $\mathcal{R}_k(\xi)$ commute, $\mathcal{R}_k(\xi)\mathcal{R}_j(\xi)G = \mathcal{R}_j(\xi)\mathcal{R}_k(\xi)G$. Then, $A_k A_j = A_j A_k$ for every $j, k \in \mathbb{Z}_{m+1}^*$.

This shows that functions in \mathcal{R} are a particular type of functions in \mathcal{G} for which all matrices A_k commute. In fact, every function in \mathcal{G} for which the matrices A_k commute belong to \mathcal{R} . To prove it, let $f \in \mathcal{G}$ be such that all matrices A_k in Eq. (5.89) commute. Let, moreover, η be a constant column matrix with entries in \mathcal{A} such that $f = G\eta$. Then, observe that the commutativity of the matrices A_k allows Eq. (5.89) to be rewritten as

$$G(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} \frac{|\alpha|!}{\alpha!} (\zeta - \xi)^{\alpha} G(\xi) A^{\alpha}, \tag{5.91}$$

where $A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_m^{\alpha_m}$. Moreover, using Eq. (5.61),

$$\begin{aligned}
(\mathcal{R}_k(\xi)f)(\zeta) &= \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ \alpha \geq \iota_k}} \frac{(|\alpha| - 1)!}{(\alpha - \iota_k)!} (\zeta - \xi)^{\alpha - \iota_k} G(\xi) A^\alpha \eta \\
&= \sum_{\substack{\alpha \in \mathbb{N}_0^m; \\ \alpha \geq \iota_k}} \frac{(|\alpha| - 1)!}{(\alpha - \iota_k)!} (\zeta - \xi)^{\alpha - \iota_k} G(\xi) A^{\alpha - \iota_k} A_k \eta \\
&= G(\zeta) A_k \eta \\
&= g_k(\zeta).
\end{aligned} \tag{5.92}$$

In other words, a function belongs to \mathcal{R} if and only if it is an element of \mathcal{G} for which all matrices A_k in Eq. (5.89) commute.

Before concluding, note that, if $f \in \mathcal{G}$, there exists a constant column matrix η with entries in \mathcal{A} such that $f = G\eta$, where G is given by Eq. (5.89). Moreover, f admits solutions to the Gleason problem, which are given by $g_k = GA_k\eta$. Then,

$$\begin{aligned}
f(\zeta) &= f(\xi) + G(\zeta) \odot_\xi \sum_{k=1}^m (\zeta_k - \xi_k) A_k \eta \\
&= f(\xi) + G(\xi) \odot_\xi \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_\xi} \odot_\xi \sum_{k=1}^m (\zeta_k - \xi_k) A_k \eta.
\end{aligned} \tag{5.93}$$

This equation characterizes a *hyperholomorphic RF*, the topic of the next section.

5.7 Hyperholomorphic Rational Functions

RFs, i.e., quotients of polynomials, of FVs are studied now. These functions play an important role in many areas, from mathematical analysis to signal processing. For instance, because polynomials can be easily evaluated, RFs are widely used for approximation and interpolation of functions in numerical analysis. Also, in the complex setting, these functions

appear in the study of linear systems as the z-transform (or the Laplace transform) of the impulse response of time-invariant systems is a RF.

The focus here is on RFs which are analytic in a neighborhood of a fixed $\xi \in \mathcal{H}^m$. From the theory of linear systems, it is known that such RFs can be written as

$$R(\zeta) = D + C \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_{\xi}} \odot_{\xi} \left(\sum_{k=1}^m (\zeta_k - \xi_k) B_k \right), \quad (5.94)$$

where A_k, B_k, C and D , for every $k \in \mathbb{Z}_{m+1}^*$, are matrices of appropriate sizes with entries in \mathcal{A} . Eq. (5.94) is called a *realization* of R , a notion that originated with the theory of linear systems [179].

While Eq. (5.94) is clearly a ratio, with respect to the CK product, of Fueter polynomials, it is still necessary to prove the converse, i.e., to show that every RF analytic at ξ admits a realization of the type given by Eq. (5.94). For that, observe that the inverse of a function that is invertible at ξ and whose realization is given by Eq. (5.94), with D invertible, admits a realization of the form

$$R(\zeta)^{-\odot_{\xi}} = D^{-1} - D^{-1} C \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k^{\square} \right)^{-\odot_{\xi}} \odot_{\xi} \left(\sum_{k=1}^m (\zeta_k - \xi_k) B_k \right) D^{-1}, \quad (5.95)$$

where

$$A_k^{\square} = A_k - B_k D^{-1} C. \quad (5.96)$$

This can be proved by writing

$$\begin{aligned} \left(\sum_{k=1}^m (\zeta_k - \xi_k) B_k \right) D^{-1} C &= \sum_{k=1}^m (\zeta_k - \xi_k) (A_k - A_k^{\square}) \\ &= \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k^{\square} \right) - \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right) \end{aligned} \quad (5.97)$$

and verifying that, indeed,

$$R(\zeta) \odot_{\xi} R(\zeta)^{-1} = I. \quad (5.98)$$

Now, if

$$R_u(\zeta) = D_u + C_u \odot_{\xi} \left(I_{N_u} - \sum_{k=1}^m (\zeta_k - \xi_k) (A_u)_k \right)^{-\odot_{\xi}} \odot_{\xi} \left(\sum_{k=1}^m (\zeta_k - \xi_k) (B_u)_k \right), \quad (5.99)$$

for $u = 1, 2$, gives two realizations of RFs with compatible sizes, then, for $k \in \mathbb{Z}_{m+1}^*$, it can be verified by direct computation that their product admits a realization given by Eq. (5.94)

with

$$\begin{aligned} A_k &= \begin{pmatrix} (A_1)_k & (B_1)_k C_2 \\ 0 & (A_2)_k \end{pmatrix}, & B_k &= \begin{pmatrix} (B_1)_k D_2 \\ (B_2)_k \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & D_1 C_2 \end{pmatrix}, & D &= D_1 D_2. \end{aligned} \quad (5.100)$$

Moreover, their sum admits the realization

$$\begin{aligned} A_k &= \begin{pmatrix} (A_1)_k & 0 \\ 0 & (A_2)_k \end{pmatrix}, & B_k &= \begin{pmatrix} (B_1)_k \\ (B_2)_k \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & C_2 \end{pmatrix}, & D &= D_1 + D_2. \end{aligned} \quad (5.101)$$

Thus, to show that any Fueter polynomial admits a realization, it suffices to prove that constant terms and terms of the form $(\zeta_k - \xi_k)M$, for every $k \in \mathbb{Z}_{m+1}^*$, admit realizations.

However, this is clear because

- a constant term corresponds to the realization where $A_j = B = C = 0$, for every $j \in \mathbb{Z}_{m+1}^*$;
- the function $(\zeta_k - \xi_k)M$ corresponds to $C = M$, $A_j = D = 0$, $B_j = \delta_{jk} I_N$, for every

$$j \in \mathbb{Z}_{m+1}^*.$$

Therefore, because every realization is a RF, as discussed, it can be said that a function R of FVs analytic at ξ is rational if and only if it admits a realization, which is given by Eq. (5.94).

Finally, to conclude, it can be shown that a function R of FVs analytic at the origin is rational if and only if there exists a finite-dimensional resolvent-invariant space \mathcal{G} such that the Gleason problem is solvable for every $f = R\eta \in \mathcal{G}$, where η is a constant column matrix with entries in \mathcal{A} . In fact, one direction of this statement was already proven in the discussion of Eq. (5.90), where it was shown that every f in a finite-dimensional \mathcal{G} admits a realization. For the converse, assume that R is given by Eq. (5.94) and that there exists η such that $f = R\eta$. Then,

$$f(\zeta) - f(\xi) = C \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_{\xi}} \odot_{\xi} \left(\sum_{k=1}^m (\zeta_k - \xi_k) B_k \right) \eta. \quad (5.102)$$

As a result,

$$g_k = C \odot_{\xi} \left(I - \sum_{k=1}^m (\zeta_k - \xi_k) A_k \right)^{-\odot_{\xi}} B_k \eta \quad (5.103)$$

solves the Gleason problem in a space \mathcal{G} generated by the columns of a function given by Eq. (5.89).

5.8 Banach Modules of Fueter Series

A fundamental aspect of functional analysis and operator theory is the study of positive kernels and their associated reproducing kernel Hilbert spaces (HSs) of power series. However, because the present work considers power series in BAs, the constructions that typically lead to HSs in many settings give rise to Banach modules of Fueter series (BMFS). These

modules are studied now.

For convenience, only Fueter series centered at the origin are considered hereby. However, one can easily reproduce the results that are presented here for Fueter series centered at a different point ξ . Moreover, because of this choice, the CK product is simply denoted by \odot (instead of \odot_0) and the backward-shift operator, by \mathcal{R}_k (instead of $\mathcal{R}_k(0)$).

To set the framework, let $\mathbf{c} = (c_\alpha)_{\alpha \in \mathbb{N}_0^m}$ be a family of non-null real numbers. Also, let $\mathring{\mathcal{W}}(\mathbf{c})$ denote the module of functions $f(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} \zeta^\alpha f_\alpha$ with coefficients $f_\alpha \in \mathcal{A}$ such that

$$\|f\| \equiv \left(\sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| (N(f_\alpha))^2 \right)^{1/2} < \infty. \quad (5.104)$$

Observe that Eq. (5.104) defines a norm in $\mathring{\mathcal{W}}(\mathbf{c})$. In fact, for every $a \in \mathcal{A}$,

$$\|af\| = \left(\sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| (N(af_\alpha))^2 \right)^{1/2} \leq N(a) \|f\|. \quad (5.105)$$

However, if a is restricted to \mathbb{K} , i.e., if $a \in \mathbb{K} \subset \mathcal{A}$,

$$\|af\| = |a| \|f\|, \quad (5.106)$$

where Eq. (5.17) was used. Moreover,

$$\begin{aligned} \|f\| = 0 &\Rightarrow \left(\sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| (N(f_\alpha))^2 \right)^{1/2} = 0 \\ &\Rightarrow f_\alpha = 0, \forall \alpha \in \mathbb{N}_0^m \\ &\Rightarrow f = 0. \end{aligned} \quad (5.107)$$

Finally, if $g(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} \zeta^\alpha g_\alpha$ belongs to $\mathring{\mathcal{W}}(\mathbf{c})$,

$$\begin{aligned}
\|f + g\|^2 &= \sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| (N(f_\alpha + g_\alpha))^2 \\
&\leq \sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| [N(f_\alpha) + N(g_\alpha)]^2 \\
&\leq \sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| \left[N(f_\alpha)^2 + 2N(f_\alpha)N(g_\alpha) + N(g_\alpha)^2 \right] \\
&\leq \|f\|^2 + 2\|f\| \cdot \|g\| + \|g\|^2 \\
&\leq (\|f\| + \|g\|)^2.
\end{aligned} \tag{5.108}$$

Now, let $f(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} \zeta^\alpha f_\alpha$ and $g(\zeta) = \sum_{\alpha \in \mathbb{N}_0^m} \zeta^\alpha g_\alpha$ be two elements of $\mathring{\mathcal{W}}(\mathbf{c})$. Then, the Hermitian form

$$[f, g] \equiv \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha g_\alpha^\dagger f_\alpha \tag{5.109}$$

converges in \mathcal{A} . In fact, using the Cauchy-Schwartz inequality,

$$\begin{aligned}
N([f, g]) &\leq \sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| N(f_\alpha) N(g_\alpha) \\
&\leq \left(\sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| N(f_\alpha)^2 \right)^{1/2} \left(\sum_{\alpha \in \mathbb{N}_0^m} |c_\alpha| N(g_\alpha)^2 \right)^{1/2} \\
&\leq \|f\| \cdot \|g\|.
\end{aligned} \tag{5.110}$$

In cases where the algebra \mathcal{A} does not have zero divisors, the results presented by Paschke in Ref. [180] are, then, valid in the context of the present section because the form defined in Eq. (5.109) satisfies the conditions to be what is called an \mathcal{A} -valued inner product. However, the goal here is the study of a more generic scenario, where \mathcal{A} is allowed to have zero divisors.

With the Hermitian form defined in Eq. (5.109), $\mathring{\mathcal{W}}(\mathbf{c})$ admits the reproducing kernel

$$K_{\mathbf{c}}(\zeta, \xi) \equiv \sum_{\alpha \in \mathbb{N}_0^m} \frac{\zeta^\alpha (\xi^\alpha)^\dagger}{c_\alpha} \quad (5.111)$$

in an open neighborhood of the origin in \mathcal{H}^m defined by

$$\Omega(\mathbf{c}) = \left\{ \zeta \in \mathcal{H}^m \left| \sum_{\alpha \in \mathbb{N}_0^m} \frac{(N(\zeta))^{2\alpha}}{|c_\alpha|} < \infty \right. \right\} \quad (5.112)$$

since

$$[f(\cdot), K_{\mathbf{c}}(\cdot, \xi)b] = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha \left(b^\dagger \frac{\xi^\alpha}{c_\alpha} \right) f_\alpha = b^\dagger f(\xi). \quad (5.113)$$

Now, it is important to note that, *a priori*, the module $\mathring{\mathcal{W}}(\mathbf{c})$ is not complete. However, by a general theorem on metric spaces, it has a completion to a Banach space, which is unique up to a metric space isometry – see, e.g., pages 54-58 of Ref. [181]. To see that, consider the vector module of Cauchy sequences in $\mathring{\mathcal{W}}(\mathbf{c})$, and say that two of such sequences, say (f_n) and (g_n) , are equivalent if

$$\lim_{n \rightarrow \infty} \|f_n - g_n\| = 0. \quad (5.114)$$

It can be checked that this is indeed an equivalence relation. The quotient module associated with it is denoted by \mathcal{CS} . Like in the complex case, the formula

$$\|\tilde{f}\| = \lim_{n \rightarrow \infty} \|f_n\| \quad (5.115)$$

where (f_n) belongs to the equivalence class \tilde{f} , does not depend on the given representative in the equivalence class, and defines a norm on \mathcal{CS} . Then, it follows from Eqs. (5.109) and

(5.113) that, for $f \in \mathcal{CS}$, the limit

$$b^\dagger f(w) \equiv \lim_{n \rightarrow \infty} b^\dagger f_n(w) \quad (5.116)$$

exists in the topology of \mathcal{A} and is independent of the given representative of the equivalence class.

Finally, the representation of \mathcal{CS} as a module of functions is denoted by $\mathcal{W}(\mathbf{c})$. It can be verified that the space $\mathcal{W}(\mathbf{c})$ endowed with the norm given by Eq. (5.115) is a Banach module, in which $\overset{\circ}{\mathcal{W}}(\mathbf{c})$ is naturally embedded in a dense way. Moreover, Eq. (5.113) also holds in $\mathcal{W}(\mathbf{c})$.

If for a certain algebra the form given by Eq. (5.109) is such that $[f, f] \geq 0$, i.e., it is a positive real number, for every f in $\mathcal{W}(\mathbf{c})$, then $\mathcal{W}(\mathbf{c})$ is a Hilbert module. If, on the other hand, $[f, f]$ is a real number for every $f \in \mathcal{W}(\mathbf{c})$ but it is not always positive for every f , then $\mathcal{W}(\mathbf{c})$ is a Potryagin or a Krein module. For example, the Banach module of power series associated with modules with positive coefficients c_α when the algebra \mathcal{A} is the quaternionic algebra is a HS. In the case of split-quaternions, however, the analogous module is a Pontryagin module.

This discussion can be extended by writing

$$[f, g] = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha g_\alpha^\dagger f_\alpha = \sum_{\ell=0}^m \left(\sum_{\alpha \in \mathbb{N}_0^m} c_\alpha g_\alpha^\dagger \chi_{\ell} f_\alpha \right) e_\ell, \quad (5.117)$$

where the characteristic operators of the algebra \mathcal{A} , defined in Eq. (5.15), have been used. Then, the inner summation can be identified with \mathbb{K} -valued inner product structures in

certain spaces (denoted by $\mathcal{K}_\ell(\mathbf{c})$) with the following definition

$$\langle f, g \rangle_{\mathcal{K}_\ell(\mathbf{c})} \equiv \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha g_\alpha^\dagger \chi_\ell f_\alpha. \quad (5.118)$$

Observe that each $\mathcal{K}_\ell(\mathbf{c})$ constitutes, in general, a Hilbert or a Pontryagin space. What determines if $\mathcal{K}_\ell(\mathbf{c})$ ends up being a Hilbert or a Pontryagin space is the coefficients c_α together with the characteristic operator χ_ℓ . For more information on Pontryagin or Krein spaces, see, e.g., Refs. [182–184].

An important characteristic of $\mathcal{W}(\mathbf{c})$ when it is endowed with the Hermitian form given by Eq. (5.109) is that, if $g \in \mathcal{W}(\mathbf{c})$ is such that

$$[f, g] = 0 \quad (5.119)$$

for every $f \in \mathcal{W}(\mathbf{c})$, then $g = 0$. In fact, for every $\alpha \in \mathbb{N}_0^m$,

$$[\zeta^\alpha, g] = c_\alpha g_\alpha^\dagger. \quad (5.120)$$

Because $c_\alpha \in \mathbb{R}$ is assumed to be non-zero, Eq. (5.119) implies that $g_\alpha = 0$ for every α , i.e., $g = 0$. Similarly, it can be shown that, if f is fixed and Eq. (5.119) holds for every $g \in \mathcal{W}(\mathbf{c})$, then $f = 0$.

Now that the module $\mathcal{W}(\mathbf{c})$ is introduced, the rest of this section focus on the study of operators that act on it. First, let O be an operator in $\mathcal{W}(\mathbf{c})$ and assume that it admits an adjoint, i.e., there exists an operator A characterized by

$$[Af, g] \equiv [f, Og] \quad (5.121)$$

for every $f, g \in \mathcal{W}(\mathbf{c})$ for which the term on the right-hand side converges. Then, A is

unique. In fact, suppose O admitted two adjoints, say A_1 and A_2 . In this case,

$$[A_1f - A_2f, g] = [A_1f, g] - [A_2f, g] = [f, Og] - [f, Og] = 0. \quad (5.122)$$

Therefore, following the discussion of Eq. (5.119), it can be concluded that

$$A_1f - A_2f = 0 \quad (5.123)$$

for every $f \in \mathcal{W}(\mathbf{c})$, which implies that $A_1 = A_2$. Because the adjoint of an operator O is unique, it will be denoted by $A = O^*$.

To address the question of whether an operator admits an adjoint in the first place, suppose O is bounded in $\mathcal{K}_\ell(\mathbf{c})$ for every $\ell \in \mathbb{Z}^{m+1}$. Then, it is a result from complex analysis that, in each $\mathcal{K}_\ell(\mathbf{c})$, O admits an adjoint, i.e., there exists an operator A_ℓ such that

$$\langle Of, g \rangle_{\mathcal{K}_\ell(\mathbf{c})} = \langle f, A_\ell g \rangle_{\mathcal{K}_\ell(\mathbf{c})}. \quad (5.124)$$

To show that the existence of such adjoints in every $\mathcal{K}_\ell(\mathbf{c})$ implies the existence of an adjoint in $\mathcal{W}(\mathbf{c})$, observe that

$$\langle Of, g \rangle_{\mathcal{K}_\ell(\mathbf{c})} = [\chi_\ell Of, g] \quad (5.125)$$

and

$$\langle f, A_\ell g \rangle_{\mathcal{K}_\ell(\mathbf{c})} = [\chi_\ell f, A_\ell g]. \quad (5.126)$$

These equations, together with Eq. (5.124), lead to

$$[\chi_\ell Of, g] = [f, \chi_\ell^* A_\ell g], \quad (5.127)$$

which, in turn, implies that

$$[Of, g] = \left[f, \left(\sum_{k=0}^m e_{\ell}^{\dagger} \chi_{\ell}^* A_{\ell} \right) g \right]. \quad (5.128)$$

Then, using the definition of the adjoint operator, it can be concluded that

$$O^* = \sum_{k=0}^m e_{\ell}^{\dagger} \chi_{\ell}^* A_{\ell}. \quad (5.129)$$

Therefore, it is proved that, if the operator O is bounded in every $\mathcal{K}_{\ell}(\mathbf{c})$, it admits an adjoint in $\mathcal{W}(\mathbf{c})$.

Before present specific examples of BMFS, the multiplication operator, which plays an important role in those modules, will be defined. For that, let \mathcal{M}_{ζ_k} , for every $k \in \mathbb{Z}_{m+1}^*$, denote the CK multiplication operator by ζ_k , i.e., if f belongs to $\mathcal{W}(\mathbf{c})$, then

$$\mathcal{M}_{\zeta_k} f \equiv \zeta_k \odot f. \quad (5.130)$$

More generally, if $\mathcal{W}(\mathbf{c})$ and $\mathcal{W}(\mathbf{d})$ are two BMFS, a multiplier \mathcal{M}_s can be defined

$$\begin{aligned} \mathcal{M}_s : \mathcal{W}(\mathbf{c}) &\rightarrow \mathcal{W}(\mathbf{d}) \\ \mathcal{M}_s f &\mapsto s \odot f, \end{aligned} \quad (5.131)$$

where $f \in \mathcal{W}(\mathbf{c})$ and $s \odot f \in \mathcal{W}(\mathbf{d})$.

5.8.1 Fock-Bargmann-Segal Module

The Fock-Bargmann-Segal (FBS) module is the particular Banach module of Fueter series $\mathcal{W}(\mathbf{c})$ for which

$$c_{\alpha} = \alpha!. \quad (5.132)$$

Its reproducing kernel is, then, given by

$$K_{\mathbf{c}}(\zeta, \xi) = \sum_{\alpha \in \mathbb{N}_0^m} \frac{1}{\alpha!} \zeta^\alpha (\xi^\alpha)^\dagger \quad (5.133)$$

and it is endowed with the Hermitian form

$$[f, g] = \sum_{\alpha \in \mathbb{N}_0^m} \alpha! g_\alpha^\dagger f_\alpha. \quad (5.134)$$

One of the results that most characterizes the FBS module, at least in the algebras where it was already studied, is the fact that the adjoint of the multiplication operator is the derivative operator on it. In the present framework, if the derivative operator ∂_k is taken to be

$$\partial_k f \equiv \sum_{\alpha \in \mathbb{N}_0^m} \alpha_k \zeta^{\alpha - \iota_k} f_\alpha \quad (5.135)$$

for every $k \in \mathbb{Z}_{m+1}^*$, this result also holds, i.e.,

$$[\partial_k f, g] = [f, \mathcal{M}_{\zeta_k} g] \quad (5.136)$$

for every $f, g \in \mathcal{W}(\mathbf{c})$. To prove that, let $\alpha, \beta \in \mathbb{N}_0^m$. Then,

$$\begin{aligned} [\partial_k \zeta^\alpha, \zeta^\beta] &= [\alpha_k \zeta^{\alpha - \iota_k}, \zeta^\beta] \\ &= (\alpha - \iota_k)! \alpha_k \delta_{\alpha - \iota_k, \beta} \\ &= \alpha! \delta_{\alpha, \beta + \iota_k} \\ &= [\zeta^\alpha, \zeta^{\beta + \iota_k}]. \end{aligned} \quad (5.137)$$

5.8.2 Drury-Arveson Module

The Drury-Arveson (DA) module, also known as the symmetric Fock module, is the particular case of a Banach module of Fueter series which has its coefficients c_α given by

$$c_\alpha = \frac{\alpha!}{|\alpha|!}, \quad (5.138)$$

i.e., the module $\mathcal{W}(\mathbf{c})$ with reproducing kernel

$$K_{\mathbf{c}}(\zeta, \xi) = \sum_{\alpha \in \mathbb{N}_0^m} \frac{|\alpha|!}{\alpha!} \zeta^\alpha (\xi^\alpha)^\dagger \quad (5.139)$$

and Hermitian form

$$[f, g] = \sum_{\alpha \in \mathbb{N}_0^m} \frac{\alpha!}{|\alpha|!} g_\alpha^\dagger f_\alpha. \quad (5.140)$$

Because contractions are important players in the study of DA modules, the idea of a contractive operator needs to be introduced in the present framework. First, observe that, in \mathcal{A} , a “natural” notion of non-negativity is given by

$$aa^\dagger \succeq 0, \quad (5.141)$$

for every $a \in \mathcal{A}$. Note that such a definition does not always imply real non-negativity — even if aa^\dagger is a real number. For example, aa^\dagger can be a negative real number in the setting of the split-quaternions.

With Eq. (5.141), an operator O in $\mathcal{W}(\mathbf{c})$ is said to be a contraction if

$$[Of, Of] \preceq [f, f] \quad (5.142)$$

for every $f \in \mathcal{W}(\mathbf{c})$.

One of the common characteristics of the DA modules already studied in the literature is the fact that the multiplication operators are a contraction from the module into itself. This is also the case here since, for every $f \in \mathcal{W}(\mathfrak{c})$, it holds that

$$\begin{aligned}
[\mathcal{M}_{\zeta_k} f, \mathcal{M}_{\zeta_k} f] &= \sum_{\alpha \in \mathbb{N}_0^m} [\zeta^{\alpha + \iota_k} f_\alpha, \zeta^{\alpha + \iota_k} f_\alpha] \\
&= \sum_{\alpha \in \mathbb{N}_0^m} \frac{(\alpha + \iota_k)!}{(|\alpha| + 1)!} f_\alpha^\dagger f_\alpha \\
&= \sum_{\alpha \in \mathbb{N}_0^m} \frac{\alpha_k + 1}{|\alpha| + 1} (c_\alpha f_\alpha^\dagger f_\alpha).
\end{aligned} \tag{5.143}$$

Then, letting

$$d_\alpha = \sqrt{\frac{c_\alpha (|\alpha| - \alpha_k)}{|\alpha| + 1}}, \tag{5.144}$$

it can be concluded that

$$\begin{aligned}
[f, f] - [\mathcal{M}_{\zeta_k} f, \mathcal{M}_{\zeta_k} f] &= \sum_{\alpha \in \mathbb{N}_0^m} \frac{|\alpha| - \alpha_k}{|\alpha| + 1} c_\alpha f_\alpha^\dagger f_\alpha \\
&= \sum_{\alpha \in \mathbb{N}_0^m} (d_\alpha f_\alpha)^\dagger (d_\alpha f_\alpha) \\
&\succeq 0.
\end{aligned} \tag{5.145}$$

Another important characteristic of DA modules is the fact that the multiplication and the backward-shift operators are adjoints of each other. Again, this holds in the present setting, as can be seen in a proof that is similar to the one presented for the quaternions in Ref. [185]. In fact, for every $f, g \in \mathcal{W}(\mathfrak{c})$ and $\alpha, \beta \in \mathbb{N}_0^m$ such that $\alpha_k \geq 1$,

$$[\mathcal{R}_k \zeta^\alpha, \zeta^\beta] = \left[\frac{\alpha_k}{|\alpha|} \zeta^{\alpha - \iota_k}, \zeta^\beta \right] = \frac{(\beta + \iota_k)!}{(|\beta| + 1)!} \delta_{\alpha - \iota_k, \beta} = [\zeta^\alpha, \zeta^{\beta + \iota_k}] = [\zeta^\alpha, \mathcal{M}_{\zeta_k} \zeta^\beta] \tag{5.146}$$

for every $k \in \mathbb{Z}_{m+1}^*$. Finally, if β is such that $\beta_k = 0$,

$$\left[\mathcal{R}_k \zeta^\alpha, \zeta^\beta \right] = 0 = \left[\zeta^\alpha, \zeta^{\beta+\iota_k} \right] = \left[\zeta^\alpha, \mathcal{M}_{\zeta_k} \zeta^\beta \right]. \quad (5.147)$$

Before concluding, an important tool for interpolations in complex DA spaces, namely the *Blaschke factor*, is introduced. For every ξ such that $\|\xi\|_{\mathcal{A}^m} < 1$, where the norm in \mathcal{A}^m is defined in Eq. (5.40), the Blaschke factor B_ξ is defined as

$$B_\xi \equiv (1 - \xi\xi^*)^{1/2} \odot (1 - \zeta\xi^*)^{-\odot} \odot (\zeta - \xi)(I - \xi^*\xi)^{-1/2}, \quad (5.148)$$

where

$$\xi^* \equiv \begin{pmatrix} \xi_1^\dagger \\ \xi_2^\dagger \\ \vdots \\ \xi_m^\dagger \end{pmatrix} \quad (5.149)$$

is the transpose of $\xi = (\xi_1 \ \xi_2 \ \cdots \ \xi_m)$.

Now, let C be the operator of evaluation at the origin, i.e., $Cf = f(0)$ for every $f \in \mathcal{W}(\mathbf{c})$.

Thus, it holds that

$$I - \sum_{k \in \mathbb{Z}_{m+1}^*} \mathcal{M}_{\zeta_k} \mathcal{M}_{\zeta_k}^* = C^* C. \quad (5.150)$$

With that, the following identity holds in the DA module $\mathcal{W}(\mathbf{c})$:

$$I - \mathcal{B}_\xi \mathcal{B}_\xi^* = (1 - \xi\xi^*)^{1/2} \left(I - \mathcal{M}_\zeta \mathcal{M}_\zeta^* \right)^{-1} C^* C \left(I - \mathcal{M}_\zeta \mathcal{M}_\zeta^* \right)^{-*} (1 - \xi\xi^*)^{1/2}, \quad (5.151)$$

where \mathcal{B}_ξ is the operator of multiplication by the Blaschke factor. The proof follows the one presented in Refs. [186] and [185]. First, observe that the operators $I - \mathcal{M}_\xi \mathcal{M}_\xi^*$ and $I - \mathcal{M}_\xi^* \mathcal{M}_\xi$ are self-adjoint. Moreover, because $\|\xi\|_{\mathcal{A}^m} < 1$, they are also contractive. As

a result, their square root (and the inverse of their square root) exist. Then, defining the Halmos extension of $-\mathcal{M}_\xi$, as done in Ref. [187],

$$\mathcal{H} \equiv \begin{pmatrix} (I - \mathcal{M}_\xi \mathcal{M}_\xi^*)^{-1/2} & -\mathcal{M}_\xi (I - \mathcal{M}_\xi^* \mathcal{M}_\xi)^{-1/2} \\ -\mathcal{M}_\xi^* (I - \mathcal{M}_\xi \mathcal{M}_\xi^*)^{-1/2} & (I - \mathcal{M}_\xi^* \mathcal{M}_\xi)^{-1/2} \end{pmatrix} \quad (5.152)$$

and setting

$$J = \begin{pmatrix} I_{\mathcal{W}(\mathbf{c})} & 0 \\ 0 & -I_{\mathcal{W}(\mathbf{c})^m} \end{pmatrix}, \quad (5.153)$$

it holds that

$$\mathcal{H}J\mathcal{H}^* = \mathcal{H}^*J\mathcal{H} = J. \quad (5.154)$$

Then, using Eq. (5.150),

$$\begin{aligned} C^*C &= I - \mathcal{M}_\zeta \mathcal{M}_\zeta^* \\ &= (I \quad \mathcal{M}_\zeta) J \begin{pmatrix} I \\ \mathcal{M}_\zeta \end{pmatrix} \\ &= (I \quad \mathcal{M}_\zeta) \mathcal{H}J\mathcal{H}^* \begin{pmatrix} I \\ \mathcal{M}_\zeta \end{pmatrix} \\ &= (\mathcal{X}_1 \quad \mathcal{X}_2) J \begin{pmatrix} \mathcal{X}_1^* \\ \mathcal{X}_2^* \end{pmatrix} \\ &= \mathcal{X}_1 \mathcal{X}_1^* - \mathcal{X}_2 \mathcal{X}_2^*, \end{aligned} \quad (5.155)$$

where

$$\mathcal{X}_1 = (I - \mathcal{M}_\zeta \mathcal{M}_\xi^*) (I - \mathcal{M}_\xi \mathcal{M}_\xi^*)^{-1/2} \quad (5.156)$$

and

$$\mathcal{X}_2 = (\mathcal{M}_\zeta - \mathcal{M}_\xi) (I - \mathcal{M}_\xi^* \mathcal{M}_\xi)^{-1/2}. \quad (5.157)$$

5.9 Discussion

This chapter started with a short explanation of the problem of extending the notion of derivatives to functions beyond functions of a complex variable. Nevertheless, following an approach started by Fueter for functions of a quaternionic variable and generalized by Malonek for functions of Clifford algebras, a derivative of functions of a single variable that takes values in a BA \mathcal{A} was constructed. This notion required the introduction of “special directions” in \mathcal{A} : the FVs. With those variables, hyperholomorphic polynomials and series were constructed with the use of the symmetrized and the CK product.

An aspect of the CK product introduced here in a way that seems to never been done in previous works and, then, requires further investigation is the fact that this product was defined at an arbitrary center $\xi \in \mathcal{H}^m$. Since the coefficients of the product of polynomials and power series are defined from the center of this product, it can be said that the point ξ determines the origin of the extension of the domain of functions from \mathbb{K} to \mathcal{A} . Then, because, in general, different centers are associated with different functions, the extension associated with them generates different mathematical structures. It would be interesting to better understand these structures and to look for possible relations between them.

Besides that, some applications of Fueter series were also considered. Basic definitions and results on the theory of RFs and realizations and BMFS were presented. These subjects are fundamental players in the study of functional analysis, operator theory, and signal processing. Because of it, in the complex setting, they are well-developed fields. Then, it would be interesting to verify if some form of the known results from complex analysis holds in the setting presented here.

Moreover, the spaces $\mathcal{K}_\ell(\mathbf{c})$ and their inner product were introduced in Section 5.8. It seems that a variety of characteristics of BMFS can be studied through them. This is a direction that should be further explored. Because the spaces $\mathcal{K}_\ell(\mathbf{c})$ take value in \mathbb{K} , they might

allow the “transference” of many results from the real and from the complex settings to the framework introduced here.

Also, some examples of BMFS were given, namely the FBS and the DA modules. Again, the two modules were already extensively studied in other settings, which means that there exists a large number of results whose validity in the present framework needs to be verified. For instance, in the complex case, the Blaschke factor and the correspondent counterpart of Eq. (5.151) allow the study of interpolation problems in the DA space [186]. In other settings, this line of research leads to unexpected conclusions. For instance, in the quaternionic case, even the simplest interpolation problem of finding all functions f in the quaternionic DA module such that $f(a) = 0$ involves an infinite number of interpolation conditions of the form [185]

$$(f \odot \zeta^\alpha)(a) = 0 \tag{5.158}$$

when α runs through \mathbb{N}_0^3 . Similar problems also arise in the present setting and will be addressed elsewhere.

Finally, it should be noticed that, although Fueter’s approach was extended here, this is not the only possible notion that can be generalized from the complex case. For instance, in 2007, Gentili and Struppa introduced the idea of *slice* derivatives in the quaternionic setting [188]. The advantage of their approach is the fact that the resultant power series does not correspond to power series of special variables. Instead, they consist of powers of a variable $q \in \mathbb{H}$. Similar ideas may hold in the setting presented here if more restrictions are imposed on it. This is another interesting research topic to be pursued in future works.

6 Analysis in Generalized Grassmann Algebras

This chapter presents the main results on analysis in closures of the Grassman algebra introduced in Refs. [93, 94]. The Grassmann algebra (GA) was first formalized by Grassmann in 1844 in a work that presented a formal framework for aspects of Cayley and Sylvester’s theory of multivectors [189]. Analysis in this algebra only started in 1899 with Cartan showing that the exterior algebra can be represented by it if the idea of derivatives by its generators is introduced [190]. The first application of this algebra in physics appeared in a work published by Martin in 1959, where the elements of the algebra were used in the study of “classical versions” of physical functions for fermions and the obtainment of their quantization through path integrals [191, 192]. Seven years later, this idea would be used by Schwinger to extend his quantum field theory to fermions [193]. Moreover, in 1979, Berezin independently started an extensive study of what is now known as *supermathematics* [194, 195].

The chapter is divided into two parts. In the first one, which goes from Section 6.2 to Section 6.9, the closure of the GA with respect to the 1-norm is studied. In this setting, many important ideas from functional analysis are introduced, like the one-step extension of Toeplitz matrices, rational functions (RFs), Schur functions, reproducing kernels, and interpolation problems. In the second part, which starts in Section 6.10 and ends in Section 6.12, a class of stochastic processes is introduced for functions whose image lies in the closure of the GA with respect to the 2-norm. Also, a framework is introduced where the derivative of the stochastic variables can be studied as continuous functions. Finally, a discussion of the chapter and an indication of future research directions are presented in the last section. Before that, in the next section, basic results, definitions, and notations are introduced.

6.1 Grassmann Algebra

The GA Λ is defined as the unital algebra over a field \mathbb{K} generated by 1 and a countable set of elements i_n , which do not belong to \mathbb{K} , are linearly independent over \mathbb{K} , and satisfy

$$i_n i_m + i_m i_n = 0, \quad (6.1)$$

where $n, m = 1, 2, \dots$. While Eq. (6.1) holds for a variety of hypercomplex numbers [196], the particularity of the generators of Λ is that Eq. (6.1) also holds for $n = m$, i.e.,

$$i_n^2 = 0. \quad (6.2)$$

Then, Λ has divisors of zero. Since \mathbb{K} is often taken to be the field of the complex numbers, it is assumed here that $\mathbb{K} = \mathbb{C}$.

The GA plays a fundamental role in supersymmetry and in quantum field theory, where it allows the construction of path integrals for fermions [197]. Because of its importance to supersymmetry, an element of Λ is commonly referred to as a *supernumber*.

In this work, if the GA has N generators i_n , it will be denoted by Λ_N . Moreover, the union of the algebras Λ_N will be denoted by Λ , i.e., $\Lambda \equiv \cup_{N \in \mathbb{N}} \Lambda_N$, as illustrated in Fig. 6.1. After this introductory section, closures of Λ , where the algebra has an infinite number of generators, will be the main focus of the chapter.

To set the framework, \mathfrak{J} is defined to be the set of t -uples $(a_1, \dots, a_t) \in \mathbb{N}^t$, where t runs through \mathbb{N} and $a_1 < a_2 < \dots < a_t$. Whith that, $i_\alpha \equiv i_{a_1} \cdots i_{a_t}$ for any $\alpha = (a_1, \dots, a_t) \in \mathfrak{J}$. Then, an element $z \in \Lambda$ can be written as the finite sum

$$z = z_0 + \sum_{\alpha \in \mathfrak{J}} z_\alpha i_\alpha, \quad (6.3)$$

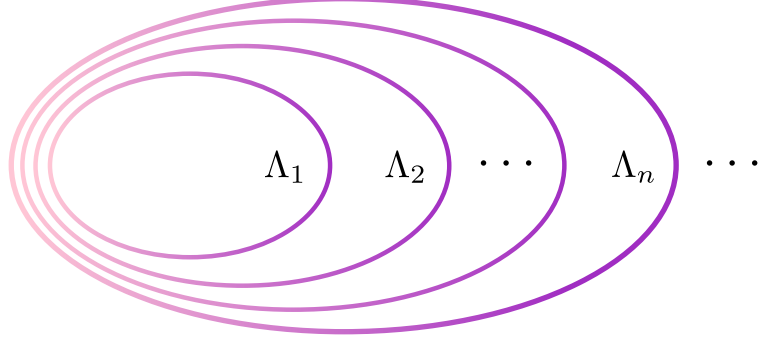


Figure 6.1: Schematic representation of the increasing family of Grassmann algebras.

where the coefficients z_0 and z_{a_1, \dots, a_k} are complex numbers.

The term that does not contain any Grassmann generator, z_0 , is called the *body* of the number, and it is often denoted by z_B . Meanwhile, $z_S = z - z_B$ is said to be the *soul* of the number [198]. As it will be discussed later in this section, it is possible to give a meaning to the sum in Eq. (6.3) when it has an infinite number of terms.

Also, setting $i_0 \equiv 1$, the set \mathfrak{J} can be extended by defining $\mathfrak{J}_0 \equiv \{0\} \cup \mathfrak{J}$. In this way, a supernumber can be written in the more compact form

$$z = \sum_{\alpha \in \mathfrak{J}_0} z_\alpha i_\alpha. \quad (6.4)$$

Furthermore, if $z = \sum_{\alpha \in \mathfrak{J}_0} z_\alpha i_\alpha$ and $w = \sum_{\beta \in \mathfrak{J}_0} w_\beta i_\beta$, their product can be written as

$$zw = \sum_{\alpha, \beta \in \mathfrak{J}_0} z_\alpha w_\beta i_\alpha i_\beta. \quad (6.5)$$

Now, let $\alpha, \beta \in \mathfrak{J}$ and note that $i_\alpha i_\beta = 0$ when i_α and i_β have a common factor i_u , with $u \in \mathbb{N}$. Moreover, when $i_\alpha i_\beta$ does not vanish, it might still not be an element of the set $\{i_\alpha : \alpha \in \mathfrak{J}\}$, since permutations might be necessary to obtain an element of that set. However, because these permutations only introduce powers of negative one, there exists a

uniquely defined $\gamma \in \mathfrak{J}$ such that

$$i_\alpha i_\beta = (-1)^{\sigma(\alpha, \beta)} i_\gamma, \quad (6.6)$$

where $\sigma(\alpha, \beta)$ is the number of permutations necessary to “build” an element $\gamma \in \mathfrak{J}$ from α and β . If such a relation holds, it is defined that

$$\alpha \vee \beta \equiv \gamma \quad (6.7)$$

and, therefore, $i_\alpha i_\beta = (-1)^{\sigma(\alpha, \beta)} i_{\alpha \vee \beta}$.

To include the possibility of $i_\alpha i_\beta = 0$ in the definition of the operation \vee , it is defined that $\alpha \vee \beta = \emptyset$ if there exists no $\gamma \in \mathfrak{J}_0$ such that Eq. (6.7) is satisfied. Then, the product of two supernumbers is written as

$$i_\alpha i_\beta = (-1)^{\sigma(\alpha, \beta)} \sum_{\gamma \in \mathfrak{J}_0} \delta_{\alpha \vee \beta, \gamma} i_\gamma, \quad (6.8)$$

where $\delta_{\alpha \vee \beta, \gamma}$ is the Kronecker delta.

It should be noted that Λ is a \mathbb{Z}_2 -graded algebra. The elements that commute with each other are of the form

$$z = z_0 + \sum_{\substack{\alpha \in \mathfrak{J} \\ \tau(\alpha) \equiv 0 \pmod{2}}} z_\alpha i_\alpha, \quad (6.9)$$

where $\tau(\alpha)$ is the number of elements of α . Those supernumbers are called *even supernumbers* and their set is denoted by Λ_{even} . It is easy to verify that they commute with *every* element of Λ and that they form a commutative subalgebra.

On the other hand, the elements that anticommute with each other are of the type

$$z = \sum_{\substack{\alpha \in \mathcal{J} \\ \tau(\alpha) \equiv 1 \pmod{2}}} z_\alpha i_\alpha. \quad (6.10)$$

They are known as *odd supernumbers* and do not form a subalgebra. In fact, it is an immediate result that the product of two odd supernumbers is an even supernumber. The set of odd supernumbers is denoted by Λ_{odd} .

Observe that, if $v \in \Lambda_{odd} \subset \Lambda$,

$$v^2 = \frac{1}{2} \sum_{\substack{\alpha, \beta \in \mathcal{J} \\ \tau(\alpha), \tau(\beta) \equiv 1 \pmod{2}}} v_\alpha v_\beta (i_\alpha i_\beta + i_\beta i_\alpha) = 0, \quad (6.11)$$

i.e., the square of odd supernumbers vanish.

Also, let $N \in \mathbb{N}$ and consider $N + 1$ elements $z_n \in \Lambda_N$ such that $z_{nB} = 0$ for every $n \in 1, \dots, N + 1$. Then, it can be checked by direct computation that

$$\prod_{n=1}^{N+1} z_n = 0 \quad (6.12)$$

and, in particular,

$$z_S^{N+1} = 0 \quad (6.13)$$

for every $z = z_B + z_S \in \Lambda_N$. This result can be also given for a $z \in \Lambda$ such that $z_B = 0$. In this case, there exists $n = n(z)$ such that $z \in \Lambda_{n(z)-1}$ and, then, $z^{n(z)} = 0$.

An important characteristic of the GA is that many of the properties of its elements depend solely on the properties of its complex body. For instance, a supernumber in Λ is invertible if and only if its body is different from zero. In fact, if $z = z_B + z_S \in \Lambda$, it holds that

$z_S^{n(z)} = 0$. Then, if $z_B \neq 0$, the inverse of z is given by

$$z^{-1} = z_B^{-1} \sum_{k=0}^{n(z)} \left(-\frac{z_S}{z_B} \right)^k. \quad (6.14)$$

Conversely, assuming z is invertible, let its inverse be $w = w_B + w_S \in \Lambda$. Then,

$$zw = 1 \Rightarrow z_B w_B = 1 \Rightarrow z_B \neq 0. \quad (6.15)$$

It is common to endow Λ with an involution \dagger for which the conjugation of a supernumber z given by Eq.(6.3) is

$$z^\dagger \equiv \overline{z_0} + \sum_{\alpha \in \mathfrak{J}} (-1)^{\pi(\alpha)} \overline{z_\alpha} i_\alpha, \quad (6.16)$$

where $\pi(\alpha) = \tau(\alpha)(\tau(\alpha) - 1)/2$. Observe that \dagger can be characterized by the complex conjugation of the coefficients z_α , $i_n^\dagger = i_n$, and $(zw)^\dagger = w^\dagger z^\dagger$.

Of course, this choice for an involution is arbitrary. In Ref. [94], other examples of involutions in Λ are presented.

If $z \in \Lambda$ is such that $z^\dagger = z$, it is said to be a *real supernumber*, or superreal; moreover, if $z^\dagger = -z$, it is said to be an *imaginary supernumber* [198]. Note that a real supernumber generally does not belong to \mathbb{R} . For instance, $i_1 + i_2$ is a real supernumber.

With an involution defined, it is also possible to introduce the idea of a *non-negative* and a *non-positive supernumber*. This is done by defining $z \in \Lambda$ as a non-negative (resp. non-positive) supernumber if there exists $w \in \Lambda$ such that $z = ww^\dagger$ (resp. $z = -ww^\dagger$), and writing $z \succeq 0$ (resp. $z \preceq 0$). Observe that, with such a definition, only real supernumbers can be classified as non-negative or non-positive.

Furthermore, if one requires invertibility from z , then w is invertible and z is called a *positive* (resp. *negative*) supernumber, or superpositive (resp. supernegative), a property that is

denoted by $z \succ 0$ (resp. $z \prec 0$).

It is, then, clear that:

- $z \succ 0 \Leftrightarrow z_B > 0$;
- $z \prec 0 \Leftrightarrow z_B < 0$;
- $z \succeq 0 \Rightarrow z_B \geq 0$;
- $z \preceq 0 \Rightarrow z_B \leq 0$.

With that, it can be shown that every positive supernumber in Λ has a square root, meaning the analytic extension of the real square root to $\mathbb{C} \setminus (-\infty, 0]$. To see that, let $z = z_B + z_S \in \Lambda$ be a positive supernumber, which means that $z_B > 0$. Thus,

$$z_B \left(1 + \frac{z_S}{z_B} \right) \quad (6.17)$$

and

$$\sqrt{z} = \sqrt{z_B} \sqrt{1 + \frac{z_S}{z_B}} = \sqrt{z_B} \left[1 - \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{2k}{k} \left(-\frac{z_S}{4z_B} \right)^{k+1} \right]. \quad (6.18)$$

Observe that the last sum converges because it has a finite number of non-zero elements.

Now, because the involution defined in the GA induces a modulus that is given either by $|z|^2 = z^\dagger z$ or $|z|^2 = z z^\dagger$, which, in general, is not a real number, the p -norm of a supernumber is introduced. For this purpose, let $p \geq 1$ be a real number and, then, define the p -norm of a supernumber $z \in \Lambda$ as

$$\|z\|_p = \left(\sum_{\alpha \in \mathfrak{I}_0} |z_\alpha|^p \right)^{1/p}, \quad (6.19)$$

where $|\cdot|$ is the usual modulus of a complex number.

It can be checked that, if $p = 1$,

$$\|zw\|_1 \leq \|z\|_1 \|w\|_1. \quad (6.20)$$

Hence, Λ endowed with the 1-norm is a Banach algebra (BA). However, that is not the case if $p > 1$, and the p -norm of a product is not necessarily bounded by the p norm of each of its factors. In particular, when considering completions $\overline{\Lambda}_{(p)}$ of Λ with respect to a p -norm, the resultant set $\overline{\Lambda}_{(p)}$ might not be an algebra.

This chapter will deal with this problem when $\overline{\Lambda}_{(2)}$ is introduced in Section 6.10. Before that, starting in the next section, the completion of Λ with respect to the 1-norm is studied.

6.2 1-Norm Completion of the Grassmann Algebra

From now until Section 6.9, it is studied the completion of Λ with respect to the 1-norm, which is the particular case of the p -norm of a supernumber z defined in Eq. (6.19) with $p = 1$, i.e.,

$$\|z\|_1 = \sum_{\alpha \in \mathcal{J}_0} |z_\alpha|, \quad (6.21)$$

where $|\cdot|$ is the usual modulus of a complex number. When this completion, which is denoted by $\overline{\Lambda}_{(1)}$, is endowed with the 1-norm, it becomes a complex vector space and has a BA structure.

It should be mentioned that some aspects of $\overline{\Lambda}_{(1)}$ were already considered in the literature [199–204]. The present work, however, follows a different approach.

Like in Λ , an element $z \in \overline{\Lambda}_{(1)}$ is invertible if and only if $z_B \neq 0$. To prove it, first assume $z \in \overline{\Lambda}_{(1)_{even}}$, which is a commutative BA. Then, let φ be a homomorphism between $\overline{\Lambda}_{(1)_{even}}$ and \mathbb{C} .

It can be shown that $\varphi(z) = z_B$. In fact, this follows from

$$\varphi(i_\alpha)^2 = \varphi(i_\alpha^2) = \varphi(0) = 0 \quad (6.22)$$

and

$$\varphi(1) = 1. \quad (6.23)$$

Therefore, by Gelfand's theorem on invertibility in commutative BAs (see, e.g., Ref. [205]), z is invertible if and only if $\varphi(z_B) \neq 0$.

Now, in case z is not assumed to be an even supernumber, observe that, if z is invertible in $\overline{\Lambda}_{(1)}$, there exists $w \in \overline{\Lambda}_{(1)}$ such that $zw = wz = 1$. In particular, $z_B w_B = 1$, which shows that $z_B \neq 0$. Conversely, without loss of generality, let $z = 1 + u + v \in \overline{\Lambda}_{(1)}$, where $u + v$ is the soul of z , with u being its even part and v being the odd one. As already discussed, $1 + u$ is invertible. Then,

$$z = (1 + u) \left[1 + (1 + u)^{-1} v \right] \quad (6.24)$$

is invertible if and only if $q = 1 + (1 + u)^{-1} v$ is invertible. Note that $q_S \in \overline{\Lambda}_{(1)_{odd}}$. Thus, to complete the proof, it is left to show that an element z such that $z_S \in \overline{\Lambda}_{(1)_{odd}}$ is invertible if $z_B \neq 0$. To do so, let $z = 1 + v$, where $v \in \overline{\Lambda}_{(1)_{odd}}$. Observing that the calculation shown in Eq. (6.11) is still valid in closures of Λ , it can be used that $v^2 = 0$ and, then, it follows that $w = 1 - v$ is the inverse of z . This finishes the proof that invertibility in $\overline{\Lambda}_{(1)}$ can be reduced to invertibility of the body of the number.

With this result, it is possible to show that, while the soul of any supernumber in Λ is nilpotent, the soul z_S of any element of $\overline{\Lambda}_{(1)}$ is quasi-nilpotent. In fact, the spectral radius formula asserts that

$$\lim_{n \rightarrow \infty} \|z_S^n\|_1^{1/n} = \sup \{|x| \mid x \in \rho(z_S)\}, \quad (6.25)$$

where $\rho(z_S)$ is the spectral radius of z_S , which can be determined by finding the values of $\lambda \in \mathbb{C}$ for which $z_S - \lambda$ is non-invertible. But, as just seen, a supernumber is non-invertible if and only if its body is non-invertible. Then,

$$\rho(z_S) = 0, \tag{6.26}$$

which implies that z_S is quasi-nilpotent.

Now, let $\lambda \in \mathbb{C}$ be a complex variable and the complex power series

$$f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \quad c_n \in \mathbb{C}, \tag{6.27}$$

be analytic in a neighborhood of the origin. Then, it follows directly from Eq. (6.26) that $f(z_S)$ converges in $\overline{\Lambda}_{(1)}$.

More generally, if $f(\lambda)$ is a complex analytic function for $\lambda \in \Omega$, then $f(z)$ converges in $\overline{\Lambda}_{(1)}$ for every z such that $z_B \in \Omega$. It can be seen by writing

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_B)}{n!} z_S^n. \tag{6.28}$$

It should be noted that Eq. (6.28) often appears in the literature as a formal power series — see, e.g., Ref. [198]. Here, however, a meaning is given to it as an infinite sum, in the sense that it converges in $\overline{\Lambda}_{(1)}$. Still on this discussion, a study of reproducing kernels associated with this type of extension in topological algebras was recently introduced [206].

Finally, observe that because of the results just presented, it is possible to conclude that many analytic functions can be extended to $\overline{\Lambda}_{(1)}$. For example, if the square root of a number is taken to be the analytic extension of the real square root to $\mathbb{C} \setminus (-\infty, 0]$, then a positive supernumber z admits a square root.

6.3 Matrix Algebra and Extension of Toeplitz Matrices

In this section, *Toeplitz matrices* taking values in $\overline{\Lambda}_{(1)}$ are studied. A Toeplitz matrix is a self-adjoint matrix with all elements of its main diagonal equal to each other. The aim here is to solve an extension problem for such matrices, namely how to obtain a Toeplitz matrix with a bigger size from a smaller one. The counterpart of this problem in the complex domain appears as an important player in many areas, like signal and image processing, system control, and in predictions of stationary processes of second order. Moreover, the center of a one-step extension in the classical theory is related to the concept of maximum entropy, and it is directly associated with the best estimation of parameters in stochastic processes, making it relevant in the solution to the Yule-Walker equations [207].

Before introducing Toeplitz matrices in the GA, some basic definitions and results in the theory of matrices with entries in $\overline{\Lambda}_{(1)}$ are studied. To begin, the conjugation \dagger presented in Section 6.1 can be extended to matrices M over the module $\overline{\Lambda}_{(1)}^{p \times q}$. This is done in the following way

$$M^* = (m_{kj}^\dagger). \quad (6.29)$$

In particular,

$$(ML)^* = L^*M^* \quad (6.30)$$

for matrices M and L of appropriate sizes.

Also, the norm of a matrix $M = (m_{jk}) \in \overline{\Lambda}_{(1)}^{p \times q}$ is defined as

$$\|M\|_1 \equiv \sum_{j,k} \|m_{jk}\|_1. \quad (6.31)$$

Now, in the algebra of complex matrices, if a matrix $M \in \mathbb{C}^{p \times p}$ is regular, meaning that all its main minor matrices are invertible, then M can be factorized as $M = LDU$, where D

is a diagonal matrix composed by invertible supernumbers, and L and U are, respectively, lower and upper-triangular matrices with main diagonals composed by ones. To see that a similar result holds in this setting, let $M = (m_{jk}) \in \overline{\Lambda}_{(1)}^{p \times p}$ be a matrix such that M_B is regular. Because of the regularity of M_B , it can be assumed that $(m_{kk})_B > 0$ for every $k \in \{1, 2, \dots, p\}$, which implies that m_{kk} is invertible. Thus,

$$\begin{aligned} M &= \begin{pmatrix} m_{11} & B \\ C & E \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ Cm_{11}^{-1} & I_{p-1} \end{pmatrix} \begin{pmatrix} m_{11} & 0 \\ 0 & E - Cm_{11}^{-1}B \end{pmatrix} \begin{pmatrix} 1 & m_{11}^{-1}B \\ 0 & I_{p-1} \end{pmatrix}, \end{aligned} \quad (6.32)$$

where B , C , and E are block matrices. By successively repeating this process, it can, then, be concluded that, if M is a matrix such that M_B is regular, $M = LDU$.

Also, an invertible matrix $M \in \overline{\Lambda}_{(1)}^{p \times p}$ has an invertible body. The result that was just proved, however, enables the proof of the converse, i.e., if M_B is invertible, M is invertible. In fact, if M_B is invertible, the body A_B of $A \equiv MM_B^*$ is regular. Then, as it was just discussed, $A = LDU$. Because any lower or upper-triangular matrix with main diagonal composed by ones is invertible and D , which is composed of positive supernumbers, is also invertible, it can be concluded that A is invertible. Then, M admits an inverse given by $M^{-1} = (M_B)^{-*} A^{-1}$. Therefore, any matrix $M \in \overline{\Lambda}_{(1)}^{p \times p}$ is invertible if and only if its body is invertible.

Now, preparing for the extension of positivity to matrices of supernumbers, let $M \in \overline{\Lambda}_{(1)}^{p \times p}$ and $c, d \in \overline{\Lambda}_{(1)}^{p \times 1}$. Then, it can be shown that

$$d^* M c = 0 \quad (6.33)$$

for every c and d if and only if

$$c^* M c = 0 \tag{6.34}$$

for every c . First, it is clear that, if the condition given by Eq. (6.33) holds, Eq. (6.34) also holds. To prove the converse, observe that, by the polarization identity,

$$d^* M c = \frac{1}{4} \sum_{k=0}^3 \left(c^* + (-i)^k d^* \right) M \left(c + i^k d \right). \tag{6.35}$$

Because, now, Eq. (6.34) is assumed to hold, each term of the sum in Eq. (6.35) vanishes since they are of the type $a^* M a$, where $a \in \overline{\Lambda}_{(1)}^{p \times 1}$.

Finally, it can be also shown that $M = 0$ whenever the condition given by either Eq. (6.33) or Eq. (6.34) holds. In fact, observe that, for the particular case where the j -th components of d and c are $d_j = \delta_{jr}$ and $c_j = \delta_{js}$, for some $r, s \in \{1, \dots, p\}$, Eq. (6.33) becomes just $m_{rs} = 0$.

With that, a matrix $M \in \overline{\Lambda}_{(1)}^{p \times p}$ is said to be a *non-negative supermatrix* (or simply super non-negative) if

$$c^* M c \succeq 0 \tag{6.36}$$

for every $c \in \overline{\Lambda}_{(1)}^{p \times 1}$. Similarly, M is defined to be *superpositive* if

$$c^* M c \succ 0 \tag{6.37}$$

for every $c \in \overline{\Lambda}_{(1)}^{p \times 1}$ such that c_B is not the null element.

With this definition, it follows that, if a supermatrix $M \in \overline{\Lambda}_{(1)}^{p \times p}$ is non-negative, then it is self-adjoint and its body is non-negative. To see that, starting by assuming that $M \in \overline{\Lambda}_{(1)}^{p \times p}$ is non-negative. Thus, Eq. (6.36) holds, which implies that $c^* M c$ is a real supernumber for

every $c \in \overline{\Lambda}_{(1)}^{p \times 1}$, i.e.,

$$c^* M c = c^* M^* c \Rightarrow c^* (M - M^*) c = 0. \quad (6.38)$$

Then, M is self-adjoint, i.e., $M^* = M$. Moreover, Eq. (6.34) implies that

$$c_B^* M_B c_B \geq 0, \quad (6.39)$$

for every $c \in \overline{\Lambda}_{(1)}^{p \times 1}$, which means that the body of M is non-negative.

Before introducing the problem of extension of Toeplitz matrices, one more result is presented. If $M \in \overline{\Lambda}_{(1)}^{p \times p}$, the following statements are equivalent:

- (i) M is superpositive;
- (ii) M is self-adjoint and M_B is positive;
- (iii) $M = LDU$, where D is a superpositive diagonal matrix, L is a lower triangular matrix with main diagonal composed by ones, and $U = L^*$;
- (iv) $M = LL^* = UU^*$, where $L, U \in \overline{\Lambda}_{(1)}^{p \times p}$, and L and U are, respectively, lower and upper-triangular matrices and their main diagonals are composed by ones.

To prove this statement, start by observing that (ii) follows directly from the definition of superpositivity. Then, (i) implies (ii). Also, it follows from the discussion of Eq. (6.32) that (ii) implies (iii).

To show that (iii) implies (iv), observe that the matrix D in $M = LDU$ is composed of positive supernumbers, which admit a positive square root. Then, D also has a superpositive diagonal square root. Denoting it by S , it holds that $D = S^2$ and

$$M = LDU = LS^2D = (LS)(LS)^* = L'L'^*. \quad (6.40)$$

In a similar way, one can start from a decomposition $M = UDL$, which is also valid, and conclude that $M = U'U'^*$.

Finally, the proof that (iv) implies (i) follows from the fact that, for every $c \in \overline{\Lambda}_{(1)}^{p \times 1}$ such that c_B is not the null element,

$$c^* M c = c^* L L^* c = (L^* c)^* (L^* c) \succ 0 \quad (6.41)$$

or, similarly, $c^* U U^* c \succ 0$.

The extension of Toeplitz matrices can be now studied. The problem can be stated as: given a superpositive Toeplitz matrix

$$T_N = \begin{pmatrix} r_0 & r_1 & \cdots & r_N \\ r_1^\dagger & r_0 & \cdots & r_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_N^\dagger & r_{N-1}^\dagger & \cdots & r_0 \end{pmatrix}, \quad (6.42)$$

where $r_0, \dots, r_N \in \overline{\Lambda}_{(1)}$ and $r_0^\dagger = r_0$, what are the conditions for a superpositive Toeplitz extension T_{N+1} be created from it? More specifically, what are the conditions that must be satisfied by a supernumber $r_{N+1} \in \overline{\Lambda}_{(1)}$ to assure that

$$T_{N+1} = \begin{pmatrix} T_N & b_{N+1} \\ b_{N+1}^* & r_0 \end{pmatrix}, \quad (6.43)$$

where b_{N+1} is a column matrix with coordinate $b_{N+1} = (r_{N+1}, r_N, \dots, r_1) \equiv (r_{N+1}, b_N)$, is superpositive?

To start solving this problem, observe that T_{N+1} can be written as

$$T_{N+1} = \begin{pmatrix} 1 & b_{N+1}^* T_N^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} r_0 - b_{N+1}^* T_N^{-1} b_{N+1} & 0 \\ 0 & T_N \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_N^{-1} b_{N+1} & I \end{pmatrix}. \quad (6.44)$$

Then, as already studied, T_{N+1} is a superpositive matrix if $r_0 - b_{N+1}^* T_N^{-1} b_{N+1}$ is a positive supernumber. Moreover, writing

$$\begin{aligned} T_N &= \begin{pmatrix} r_0 & a_N \\ a_N^* & T_{N-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & a_N T_{N-1}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} r_0 - a_N T_{N-1}^{-1} a_N^* & 0 \\ 0 & T_{N-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_{N-1}^{-1} a_N^* & I \end{pmatrix}, \end{aligned} \quad (6.45)$$

it can be concluded that

$$r_0 - b_{N+1}^* T_N^{-1} b_{N+1} \succ 0 \Leftrightarrow (r_{N+1} - c_N)^\dagger \alpha (r_{N+1} - c_N) \prec r_0 - b_N^* T_N^{-1} b_N, \quad (6.46)$$

where $\alpha = \left(r_0 - a_N T_{N-1}^{-1} a_N^* \right)^{-1}$ and $c_N = a_N T_{N-1}^{-1} b_N$. Furthermore, defining real supernumbers ξ such $\xi^\dagger \xi \equiv r_0 - b_N^* T_N^{-1} b_N$, Eq. (6.46) can be rewritten as

$$(r_{N+1} - c_N)^\dagger \alpha (r_{N+1} - c_N) \prec \xi^\dagger \xi. \quad (6.47)$$

This is, then, the solution to the problem of extension of Toeplitz matrices.

Observe that

$$\alpha^{1/2} (r_{N+1} - c_N) \prec \xi \Rightarrow r_{N+1} = c_N + \alpha^{-1/2} \eta \xi, \quad (6.48)$$

where η is a supernumber such that $\eta \prec 1$. Eq. (6.48) can be seen as a natural definition of a disk with center in c_N , left radius $\alpha^{-1/2}$, and right radius ξ . Such object is called a

superdisk.

Note that the “geometry” induced by the definition of positivity in the GA is different from the one induced by the 1-norm. In fact, elements in the superdisk, in general, do not have a bounded norm. For instance, consider

$$a = \frac{1}{2}(1 + \lambda i_1) \quad \lambda \in \mathbb{C}. \quad (6.49)$$

Even though it is clearly inside the superdisk $zz^\dagger < 1$, its norm is $\|a\|_1 = (1 + |\lambda|)/2$, which can be arbitrarily large.

6.4 Realization Theory and Rational Functions

Denote by $\Gamma^{p \times q}(\Omega)$ the space of power series whose coefficients take value in $\overline{\Lambda}_{(1)}^{p \times q}$ that converge on a neighborhood Ω of the origin in $\overline{\Lambda}_{(1)}$. Then, the main goal of this section is to define and study realization theory and RFs of power series $F \in \Gamma^{p \times q}(\Omega)$, i.e.,

$$F(z) = \sum_{n=0}^{\infty} z^n f_n, \quad (6.50)$$

where the variable z varies in Ω . The approach followed here is similar to the one used in the quaternionic setting (see, e.g., [86, 208–210]) and, also, to the Fueter variables on BAs presented in Section 5.7. An important distinction is the fact that, here, the focus is on functions that are also invertible at the origin.

To start, the Cauchy-Kovalevskaya (CK) product is introduced in this setting. For that, let F and G be two power series of the type given by Eq. (6.50). Then, the CK product is defined as

$$F \star G(z) \equiv \sum_{n \in \mathbb{Z}} z^n \left(\sum_{u \in \mathbb{Z}} f_u g_{n-u} \right). \quad (6.51)$$

The product is denoted by \star , and not by \odot , so it can be differentiated from the CK product defined in the previous chapter.

Note that such a product reduces to the pointwise product in the case of $z = \lambda \in \mathbb{C}$. Moreover, for $F(z)$ invertible, the star and the regular products are related by

$$\begin{aligned}
F \star G(z) &= \sum_{n \in \mathbb{Z}} z^n F(z) g_n \\
&= F(z) \sum_{n \in \mathbb{Z}} F(z)^{-1} z^n F(z) g_n \\
&= F(z) G \left(F(z)^{-1} z F(z) \right).
\end{aligned} \tag{6.52}$$

Now, $F \in \Gamma^{p \times q}(\Omega)$ is said to admit a realization if it can be represented in the form

$$F(z) = D + zC \star (I_N - zA)^{-\star} B, \tag{6.53}$$

where $D \equiv F(0)$, A, B, C are supermatrices of appropriate sizes, and $(I_N - zA)^{-\star} \equiv \sum_{n=0}^{\infty} z^n A^n$.

If the matrix D in Eq. (6.53) is invertible, it can be verified by direct computation that

$$F(z)^{-\star} = D^{-1} - zD^{-1}C \star (I - zA^\times)^{-\star} BD^{-1} \tag{6.54}$$

is a realization of $F^{-\star}$, with

$$A^\times = A - BD^{-1}C. \tag{6.55}$$

Moreover, it can be checked that, if

$$F_j(z) = D_j + zC_j \star (I_{N_j} - zA_j)^{-\star} B_j, \tag{6.56}$$

$j = 1, 2$, correspond to two realizations of compatible sizes, a realization of $F_1(z) \star F_2(z)$ is given by

$$\begin{aligned} A &= \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & D_1 C_2 \end{pmatrix}, \quad D = D_1 D_2 \end{aligned} \tag{6.57}$$

and a realization of $F_1 + F_2$ is given by

$$\begin{aligned} A &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \\ C &= \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad D = D_1 + D_2. \end{aligned} \tag{6.58}$$

Now, defining a k -th order polynomial in $\Gamma^{p \times q}(\Omega)$ as a finite sum of the form

$$M(z) = M_0 + zM_1 + \cdots + z^k M_k, \tag{6.59}$$

where $M_0, \dots, M_k \in \overline{\Lambda}_{(1)}^{p \times q}$, it is clear that any polynomial in z admits a realization.

It is also of interest to define a pair (C, A) as an *observable* pair of matrices if

$$\bigcap_{u=0}^{\infty} \ker CA^u = \{0\}. \tag{6.60}$$

Moreover, a pair (A, B) is said to be *controllable* if

$$\xi(I - \lambda A)^{-1} B \equiv 0 \Rightarrow \xi = 0, \tag{6.61}$$

where $\lambda \in \mathbb{R}$. Furthermore, a realization is called *minimal* if the pair (C, A) is observable and the pair (A, B) is controllable.

With that, it can be shown that two minimal realizations

$$F(z) = D_j + zC_j \star (I_{N_j} - zA_j)^{-\star} B_j, \quad j = 1, 2, \quad (6.62)$$

of a function F are similar. To do so, first observe that $D_1 = D_2$. Then, it is possible to write

$$\begin{aligned} \frac{F(x) - F(y)}{x - y} &= C_1(I_{N_1} - xA_1)^{-1}(I_{N_1} - yA_1)^{-1}B_1 \\ &= C_2(I_{N_2} - xA_2)^{-1}(I_{N_2} - yA_2)^{-1}B_2, \end{aligned} \quad (6.63)$$

where $x, y \in \mathbb{R}$ and the understanding that the left-hand side is equal to $F'(x)$ if $x = y$.

Moreover, define the operators

$$U \left((I_{N_1} - yA_1)^{-1} B_1 \xi \right) \equiv (I_{N_2} - yA_2)^{-1} B_2 \xi, \quad (6.64)$$

and

$$V \left((I_{N_2} - yA_2)^{-1} B_2 \xi \right) \equiv (I_{N_1} - yA_1)^{-1} B_1 \xi, \quad (6.65)$$

where $\xi \in \overline{\Lambda}_{(1)}^q$. The fact that the pairs (C_j, A_j) are observable assures that such operators are well defined. From the above definitions, and because the pairs (A_j, B_j) are controllable, it holds that

$$UV = \mathbb{I}_{N_2}, \quad (6.66)$$

and

$$VU = \mathbb{I}_{N_1}, \quad (6.67)$$

where \mathbb{I}_N denotes the $N \times N$ identity operator.

Now, defining elements e_j , $j = 1, \dots, N$, of modules $\overline{\Lambda}_{(1)}^N$ with components given by

$$(e_j)_k = \delta_{jk}, \quad (6.68)$$

it can be written that

$$(I_{N_1} - yA_1)^{-1}B_1\xi = \sum_{j=1}^{N_1} e_j\alpha_j, \quad (6.69)$$

where $\alpha_j \in \bar{\Lambda}_{(1)}$, and

$$U \left(\sum_{j=1}^{N_1} e_j\alpha_j \right) = \sum_{j=1}^{N_1} U(e_j)\alpha_j = \sum_{j=1}^{N_1} \left(\sum_{k=1}^{N_2} u_{kj}e_j \right) \alpha_j. \quad (6.70)$$

Hence, U can be represented by a $N_2 \times N_1$ matrix $\tilde{U} = (u_{kj})$. Similarly, V can be represented by a $N_1 \times N_2$ matrix \tilde{V} . To conclude the proof, because (A_1, B_1) is controllable, $N_2 \leq N_1$. However, because (A_2, B_2) is also controllable, $N_1 \leq N_2$. Therefore, $N_1 = N_2$ and two minimal representations are similar.

Presented some definitions and results on realizations, RFs can be discussed. As mentioned in Section 5.7, a RF $F \in \Gamma^{p \times q}(\Omega)$ is a quotient of polynomials. In particular, a RF $F \in \Gamma^{p \times r}$ can be written as a finite product of the type

$$F(z) = M_1(z) \star M_2(z)^{-\star} \star M_3(z), \quad (6.71)$$

where $M_1 \in \Gamma^{p \times q}$, $M_2 \in \Gamma^{q \times q}$, and $M_3 \in \Gamma^{q \times r}$ are polynomials. Then, it is clear that every realization is a RF. Therefore, because every polynomial admits a realization and the inverse of a realization also admits a realization, the concepts of RFs and realizations are equivalent.

Finally, it can be checked by direct computation that, because every RF admits a realization, and vice versa, they are characterized by the Taylor expansion

$$F(z) = D + \sum_{n=1}^{\infty} z^n CA^{n-1}B, \quad (6.72)$$

where the matrices A , B , C , and D are the ones defined in Eq. (6.53).

6.5 Rational Schur-Grassmann functions

The purpose of this section is to start the study of Schur analysis and related topics in $\overline{\Lambda}_{(1)}$, a subject that will continue to be explored until Section 6.9. To put the study into perspective, a brief review of it in the complex setting is presented.

Schur analysis is a part of function theory in the open unit disk \mathbb{D} or a half-plane. It is a rich and vastly developed field with numerous applications, which include, but are not limited to, signal processing [211], fast algorithms [212], and linear systems [213]. It originated with Schur in 1979 [214, 215], although this area can be traced back to Stieltjes's work in 1894 [216]. A collection of original papers on the topic can be found in Ref. [217]. The Hardy space and Blaschke factors are important players in this domain [218, 219], as well as the Wiener algebra and RFs.

The Hardy space of the unit disk \mathbf{H}_2 is the Hilbert space (HS) of power series $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$, $\lambda \in \mathbb{C}$, such that $\|f\|_{\mathbf{H}_2}^2 \equiv \sum_{n=0}^{\infty} |a_n|^2 < \infty$. From the signal processing point of view, it can be interpreted as the space of frequencies (z -transforms) of finite energy discrete signals. Such an interpretation motivates various interpolation problems in the Hardy space and related spaces. The Nevanlinna-Pick and the Carathéodory-Fejér problems are two examples of it. The former consists on, given $\lambda_1, \dots, \lambda_N$ in \mathbb{D} and complex numbers $\omega_1, \dots, \omega_N$, describing the set of all Hardy functions f such that $f(\lambda_j) = \omega_j$ for $j = 1, \dots, N$. The latter, on the other hand, refers to the problem of fixing the first N derivatives of a function at a given point. Moreover, for $\lambda = 0$, the Carathéodory-Fejér problem becomes trivial for Hardy functions — since the coefficients a_n of f are known. Nevertheless, this is a problem of central importance in the class of functions with a positive real part in the open unit disk, which is related to the theory of extension of Toeplitz matrices and has applications on the prediction theory of second-order stationary processes.

For both the Nevanlinna-Pick and the Carathéodory-Fejér problems, however, additional

metric constraints are imposed to f , such as requiring that it should take contractive values in the open unit disk. Functions satisfying such conditions are called Schur functions, and they are the transfer functions of dissipative systems [187, 220–224].

Schur functions can be characterized in a number of equivalent ways, including the just mentioned contractivity in the Hardy space. In the rational setting, which is our starting point here, a Schur function S can be defined as a matrix-valued RF which is analytic at infinity and has minimal realization given by

$$S(\lambda) = D + C(\lambda I - A)^{-1}B \quad (6.73)$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \leq \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \quad (6.74)$$

for some (not uniquely defined) $H < 0$. This characterization results from the positive real lemma, also known as Kalman-Yakubovich-Popov theorem [225–227], and its generalization [228]. For the description of all H satisfying Eq. (6.74), see Ref. [226, 227].

It should be noticed that Eq. (6.73) presents a different expression for a realization than the one studied in the previous section, which can be written as

$$S(\lambda) = H + \lambda G(I - \lambda T)^{-1}F, \quad (6.75)$$

which, in general, is not analytic at infinity. However, both expressions are equivalent if the matrix A is invertible. In fact, let $H = D - CA^{-1}B$, $G = -CA^{-1}$, $T = A$, and $F = AB$.

Then, Eq. (6.75) can be rewritten as

$$\begin{aligned}
S(\lambda) &= D - CA^{-1}B - \lambda CA^{-1}(I - \lambda A)^{-1}AB \\
&= D - CA^{-1}B + CA^{-1}(\lambda A - I)^{-1}(I + \lambda A - I)B \\
&= D + CA^{-1}(\lambda A - I)^{-1}B \\
&= D + C(\lambda I - A)^{-1}B.
\end{aligned} \tag{6.76}$$

With that, Eq. (6.73) with matrices conditioned to Eq. (6.74) is taken as the primary definition of a rational Schur-Grassmann (SG) function. To be precise, a $\overline{\Lambda}_{(1)}^{p \times q}$ -valued RF with realization given by

$$S(z) = D + C \star (zI_N - A)^{-\star} B \tag{6.77}$$

will be called a SG function if there exists a Hermitian strictly negative matrix $H \in \overline{\Lambda}_{(1)}^{N \times N}$, i.e., $H \prec 0$, such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \preceq \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix}. \tag{6.78}$$

Now, let S be a $\overline{\Lambda}_{(1)}^{p \times q}$ -valued RF of the type given by Eq. (6.77). Then, it can be verified that S is a SG function if and only if its body part S_B is a Schur function. This allows the direct translation to SG functions of a number of properties of Schur functions. Two examples of that can be given:

- Complex rational Schur functions $S(\lambda)$ can be characterized by $S(\lambda)S(\lambda)^* \preceq I$ for $|\lambda| \leq 1$. Then, it can be checked that rational SG functions $S(z)$ are such that $S(z)S(z)^* \preceq I$ for $zz^\dagger \preceq 1$.

- Complex rational Schur functions $S(\lambda)$ can be characterized by the fact that the kernel

$$\sum_{n=0}^{\infty} \lambda^n (I - S(\lambda)S(\omega)^*) \bar{\omega}^n \quad (6.79)$$

is positive definite. Also, rational SG functions $S(z)$ are such that the kernel

$$\sum_{n=0}^{\infty} z^n (I - S(z)S(w)^*) (w^\dagger)^n \quad (6.80)$$

is superpositive.

However, not every result from complex analysis holds in the present setting by a simple “reduction to the body” study of SG functions. As it will be seen in more details later, interpolation problems are an example of that.

Now, consider the Hermitian form

$$[F, G] \equiv \sum_{n=0}^{\infty} g_n^* f_n, \quad (6.81)$$

where

$$F = \sum_{n=1}^{\infty} z^n f_n \quad (6.82)$$

and

$$G = \sum_{n=1}^{\infty} z^n g_n \quad (6.83)$$

are such that $[F, F]$ and $[G, G]$ converge in $\bar{\Lambda}_{(1)}^{q \times q}$. Note that the restriction of Eq. (6.81) to its body corresponds to the matrix-valued Hermitian form associated to the Hardy space of $\mathbb{C}^{p \times q}$ -valued functions analytic in the open unit disk.

With this definition, another result that follows from the analysis of complex Schur functions

states that a RF S is a SG function if and only if

$$[M_S F, M_S F] \preceq [F, F], \quad (6.84)$$

where M_S is the CK multiplication operator, defined by $M_S F \equiv S \star F$.

Finally, let S be a $\overline{\Lambda}_{(1)}^{p \times q}$ -valued RF given by

$$S(z) = s_0 + z s_1 + \cdots, \quad (6.85)$$

where $s_0, s_1, \dots \in \overline{\Lambda}_{(1)}^{p \times q}$. Moreover, let L_N denote the lower triangular matrix

$$L_N = \begin{pmatrix} s_0 & 0 & \cdots & 0 & 0 \\ s_1 & s_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_N & s_{N-1} & \cdots & s_1 & s_0 \end{pmatrix}. \quad (6.86)$$

Then, it follows from complex analysis that S is a SG function if and only if $L_N^* L_N \preceq I$ for every $N \in \mathbb{N}$.

6.6 Wiener-Grassmann algebra

In this section, it is introduced the Wiener algebra associated with the GA $\overline{\Lambda}_{(1)}$, or simply the Wiener-Grassmann (WG) algebra. In the complex case, the characterization of boundary values of Hardy functions can be a quite challenging problem. Then, it is often desirable to consider functions in \mathcal{W}_+^p , which is a subset of the Wiener algebra

$$\mathcal{W}^p = \left\{ f(e^{it}) = \sum_{n \in \mathbb{Z}} e^{int} f_n \mid t \in \mathbb{R}, f_n \in \mathbb{C}^{p \times p}, \sum_{n \in \mathbb{Z}} \|f_n\| < \infty \right\}, \quad (6.87)$$

where $\|\cdot\|$ is a complex matrix norm. \mathcal{W}^p is endowed with the pointwise multiplication of functions.

Some facts about \mathcal{W}^p are relevant for the present discussion. For instance, a function $f \in \mathcal{W}^p$ is said to be strictly positive if $f(e^{it}) > 0$ for every real t . Moreover, the Wiener-Lévy theorem assures that f has an inverse in \mathcal{W}^p if and only if $f(e^{it}) \neq 0$ for every real t , i.e., invertibility in the algebra is equivalent to pointwise invertibility.

Two important subalgebras of \mathcal{W}^p are the aforementioned \mathcal{W}_+^p , composed by functions f of the type

$$f(e^{it}) = \sum_{n=0}^{\infty} e^{int} f_n, \quad (6.88)$$

and \mathcal{W}_-^p , which contains functions of the type

$$f(e^{it}) = \sum_{n=-\infty}^0 e^{int} f_n. \quad (6.89)$$

There are elements in \mathcal{W}_+^p (resp. \mathcal{W}_-^p) that have inverse in \mathcal{W}_+^p (resp. \mathcal{W}_-^p). They are denoted by f_+ (resp. f_-). Furthermore, some functions in \mathcal{W}^p can be factorized as $f = f_+ f_-$. An important theorem states that $f \in \mathcal{W}^p$ is strictly positive if and only if it has such a factorization and it is characterized by $f_- = f_+^*$, where f_+^* denotes the adjoint of f_+ .

Now, some of these results and definitions are introduced in the present setting. To start, the WG algebra is defined as

$$\mathcal{W}_G^p = \left\{ f(z) = \sum_{n \in \mathbb{Z}} z^n f_n \left| z \in \overline{\Lambda}_{(1)}, f_n \in \overline{\Lambda}_{(1)}^{p \times p}, \|[f, f]\|_1 < \infty \right. \right\}, \quad (6.90)$$

where, the Hermitian form $[\cdot, \cdot]$ is defined as

$$[f, g] = \sum_{n \in \mathbb{Z}} g_n^* f_n. \quad (6.91)$$

This algebra is endowed with the CK product introduced in Eq. (6.51), i.e.,

$$f \star g(z) \equiv \sum_{n \in \mathbb{Z}} z^n \left(\sum_{u \in \mathbb{Z}} f_u g_{n-u} \right), \quad (6.92)$$

for any two functions f and g in \mathcal{W}_G^p .

An import subalgebra of \mathcal{W}_G^p for this work is the set \mathcal{W}_{BP}^p , which will be called the Wiener-Bochner-Phillips algebra and is defined by

$$\mathcal{W}_{BP}^p = \left\{ f(t) = \sum_{n \in \mathbb{Z}} e^{int} f_n \mid t \in \mathbb{R}, f_n \in \overline{\Lambda}_{(1)}^{p \times p}, \sum_{n \in \mathbb{Z}} \|f_n\|_1 < \infty \right\}, \quad (6.93)$$

where $i \in \mathbb{C}$ is the imaginary unit. Note that the CK product is reduced to the pointwise product in \mathcal{W}_{BP}^p .

With this definition, it can be shown that a function $f \in \mathcal{W}_{BP}^p$ has an inverse in \mathcal{W}_{BP}^p if and only if $f(t)$ has an inverse in $\overline{\Lambda}_{(1)}$, for every t . First, it is clear that $f(t)$ is invertible for every t if f is invertible in \mathcal{W}_{BP}^p . Moreover, the converse is just an adapted version of the original result presented by Bochner and Phillips in Ref. [229, Theorem 1] to the case where the coefficients are elements of $\overline{\Lambda}_{(1)}^{p \times p}$ instead of a generic noncommutative ring.

Now, let $f \in \mathcal{W}_G^p$ and $f_{BP} \equiv f(e^{it}) \in \mathcal{W}_{BP}^p$, where $t \in \mathbb{R}$. Then, the following statements are equivalent:

- (i) f is invertible in \mathcal{W}_G^p ;
- (ii) f_{BP} is invertible in \mathcal{W}_{BP}^p ;
- (iii) the body of f_{BP} is invertible in the classical Wiener algebra \mathcal{W}^p ;
- (iv) $(f_{BP})_B(t) \neq 0$ for every t .

To see that (i) is equivalent to (ii), observe that, if (i) holds, there exists a function $g \in \mathcal{W}_G^p$ such that $f \star g(z) = 1$ for every z . In particular, this holds for points $z = e^{it}$, i.e., $f_{BP}g_{BP}(t) = 1$ for every t , where $g_{BP} \equiv g(e^{it})$. Conversely, if (ii) is true, there exists

$$g_{BP} = \sum_{n \in \mathbb{Z}} e^{int} g_n \in \mathcal{W}_{BP}^p \quad (6.94)$$

such that $f_{BP}g_{BP} = 1$. Then, it holds that

$$\sum_{u \in \mathbb{Z}} f_u g_{n-u} = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}. \quad (6.95)$$

Finally, consider the “extension” of g_{BP} to \mathcal{W}_G

$$g = \sum_{n \in \mathbb{Z}} z^n g_n, \quad (6.96)$$

where $z \in \bar{\Lambda}_{(1)}$ such that $z_B \neq 0$. It follows that $f \star g = 1$.

Moreover, to prove that (ii) is equivalent to (iii), observe that $f_{BP}g_{BP} = 1 \Rightarrow (f_{BP})_B (g_{BP})_B = 1$ and $(f_{BP})_B, (g_{BP})_B \in \mathcal{W}^p$. For the converse, assume that (iii) holds, which implies that f_{BP} has an inverse in $\bar{\Lambda}_{(1)}$. In this case, as already discussed, f_{BP} is invertible in \mathcal{W}_{BP}^p .

Finally, (iii) being equivalent to (iv) is just a restatement of the classical Wiener-Lévy theorem.

Before concluding this section, in a similar way done in the complex setting, the subsets

$$\mathcal{W}_{G+}^p = \{f \in \mathcal{W}_G^p \mid f_n = 0, n < 0\} \quad (6.97)$$

and

$$\mathcal{W}_{G-}^p = \{f \in \mathcal{W}_G^p \mid f_n = 0, n > 0\} \quad (6.98)$$

are introduced.

With that, a weak condition for the invertibility of a function in \mathcal{W}_{G+}^p is presented: a function $f \in \mathcal{W}_{G+}^p$ has an inverse in \mathcal{W}_{G+}^p if and only if $[f(z)]_B \neq 0$ for every z such that $\|z\|_1 \leq 1$.

To show that, assume that f is invertible in \mathcal{W}_{G+}^p . Then, there exists $g \in \mathcal{W}_{G+}^p$ such that

$$\left(\sum_{m=0}^{\infty} z^m f_m \right) \star \left(\sum_{n=0}^{\infty} z^n g_n \right) = 1 \quad (6.99)$$

for every $z \in \overline{\Lambda}_{(1)}$. In particular, restricting this equation to its body,

$$\left(\sum_{m=0}^{\infty} z_B^m (f_m)_B \right) \left(\sum_{n=0}^{\infty} z_B^n (g_n)_B \right) = 1, \quad (6.100)$$

i.e., $[f(z)]_B \neq 0$. Conversely, let $f \in \mathcal{W}_{G+}^p$ and $[f(z)]_B \neq 0$ for every z such that $\|z\|_1 \leq 1$. In particular, for $z = \xi \in \mathbb{C}$ with $|\xi| \leq 1$, $f_B(\xi)$ is an invertible element of \mathcal{W}_+^p . Hence, as already proved, f is invertible in \mathcal{W}_G^p . Therefore, if $g \in \mathcal{W}_G^p$ is its inverse,

$$\left(\sum_{m=0}^{\infty} z^m f_m \right) \star \left(\sum_{n \in \mathbb{Z}} z^n g_n \right) = 1. \quad (6.101)$$

Furthermore, using, again, the fact that $f_B(\xi)$ is invertible in \mathcal{W}_+^p , it can be concluded that $(g_n)_B = 0$ for every $n < 0$.

It should be noticed that a similar result to the one just presented holds for the invertibility in \mathcal{W}_{G-}^p . Moreover, a strong version of it, if it exists in the present setting, should characterize functions in \mathcal{W}_{G+}^p that have inverse in \mathcal{W}_{G+}^p , as it is done in complex analysis.

6.7 Reproducing Kernel Banach Modules and Interpolation

A module is said to admit a reproducing kernel if there exists a positive definite function $K(z, w)$ and a Hermitian form $[\cdot, \cdot]$ such that every function in such a set can be pointwise evaluated as

$$f(z) = [f(\cdot), K(\cdot, z)]. \quad (6.102)$$

In the complex case, this Hermitian form on a space of power series $f = \sum_{n=0}^{\infty} z^n a_n$, with $a_n \in \mathbb{C}$ and $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, is a map into \mathbb{C} and coincides with the usual inner product

$$[f, g] = \sum_{n=0}^{\infty} \bar{b}_n a_n, \quad (6.103)$$

where $g = \sum_{n=0}^{\infty} z^n b_n$, with $b_n \in \mathbb{C}$. Note that, with respect to this product, such a space of power series is a complex BA. Moreover, a similar definition could be given for matrix-valued coefficient a_n of f .

Now, when replacing the complex coefficients in Eq. (6.102) by elements of $\bar{\Lambda}_{(1)}$, it becomes clear that one type of power series of interest is the one defined by the Wiener algebra \mathcal{W}_{G+}^p studied in the previous section.

Since a Hermitian form of the type given by Eq. (6.103) is necessary, besides the CK product, this set is also endowed with the form defined in Eq. (6.81). Observe that, if

$$K(z, w) = \sum_{n=0}^{\infty} z^n (w^\dagger)^n, \quad (6.104)$$

the point evaluation of a function $f \in \mathcal{W}_{G+}$ is

$$f(z) = [f(\cdot), K(\cdot, z)]. \quad (6.105)$$

Moreover, the notion of orthogonality in $\mathcal{W}_{G_+}^p$ is defined according to the Hermitian form in Eq. (6.81), i.e., $f \in \mathcal{W}_{G_+}^p$ is said to be perpendicular to $g \in \mathcal{W}_{G_+}^p$ if $[f, g] = 0$.

Solutions to interpolation problems are the main concern of this section. For that, the introduction of the notion of linear independence is necessary. Since $\mathcal{W}_{G_+}^p$ is a module (and not a linear space) such a notion is, in general, delicate. However, consider the set $G = \{f_1, \dots, f_N\} \subset \mathcal{W}_{G_+}^p$ such that $(f_1)_B, \dots, (f_N)_B$ are linearly independent. Then, it is possible to show that G is a basis for $\text{span}(G)$. Indeed, it is only needed to prove that, given coefficients $c_j \in \overline{\Lambda}_{(1)}$, if

$$f_1 c_1 + \dots + f_N c_N = 0, \quad (6.106)$$

then $c_1 = c_2 = \dots = c_N = 0$. For this purpose, assume that f_1, \dots, f_N are such that their body are linearly independent, and Eq. (6.106) is satisfied by coefficients $c_j \neq 0$. Then, observe that the restriction of Eq. (6.106) to its body implies that the body of every c_j must be null. Moreover, writing

$$c_j = \sum_{\alpha \in \mathfrak{J}} c_{j\alpha} i_\alpha, \quad (6.107)$$

let m_j be the minimum value of $\tau(\alpha)$ for which $c_{j\alpha} \neq 0$. Also, let $m = \min\{m_1, m_2, \dots, m_N\}$. As a result, for every α such that $\tau(\alpha) = m$, Eq. (6.106) implies that

$$f_1 c_{1\alpha} + \dots + f_N c_{N\alpha} = 0. \quad (6.108)$$

Since at least one $c_{j\alpha}$ is not null, Eq. (6.108) contradicts the fact that f_1, \dots, f_N are linearly independent. Therefore, Eq. (6.106) can only be satisfied if $c_1 = c_2 = \dots = c_N = 0$.

Now, it is introduced the function

$$\Theta(z) \equiv I_p - (1 - z)C \star (I_q - zA)^{-\star} P^{-1} (I_q - A)^{-\star} C^* J, \quad (6.109)$$

where $(C, A) \in \overline{\Lambda}_{(1)}^{p \times q} \times \overline{\Lambda}_{(1)}^{q \times q}$ is an observable pair, $J \in \overline{\Lambda}_{(1)}^{p \times p}$ is a signature matrix, and $P \in \overline{\Lambda}_{(1)}^{q \times q}$ is an invertible self-adjoint matrix. This function is a fundamental object for the results presented here.

Furthermore, the CK product can be also defined for power series with the variable z placed on the right-hand side of the coefficients as

$$\left(\sum_{n=0}^{\infty} f_n z^n \right) \star_r \left(\sum_{n=0}^{\infty} g_n z^n \right) \equiv \sum_{n=0}^{\infty} \left(\sum_{u=0}^n f_u g_{n-u} \right) z^n. \quad (6.110)$$

As a result, the expression

$$\sum_{n=0}^{\infty} z^n (J - \Theta(z)J\Theta(w)^*) (w^\dagger)^n = C \star (I_q - zA)^{-\star} P^{-1} [(I_q - wA)^*]^{-\star_r} \star_r C^* \quad (6.111)$$

holds if and only if the Stein equation

$$P - A^*PA = C^*JC \quad (6.112)$$

holds. To verify it, consider $z = \lambda$ and $w = \omega$ in \mathbb{C} . Writing $\alpha(\lambda) = C(I_m - \lambda A)^{-1}P^{-1}(I_m - A)^{-\star}$ and $\beta = (I_q - A)^*P(I_q - A)$, observe that

$$\begin{aligned} J - \Theta(\lambda)J\Theta(\omega)^* &= J - \left[I_p - (1 - \lambda)C(I_q - \lambda A)^{-1}P^{-1}(I_q - A)^{-\star}C^*J \right] J \times \\ &\quad \times \left[I_p - (1 - \bar{\omega})JC(I_q - A)^{-1}P^{-1} [(I_q - \omega A)^*]^{-1} C^* \right] \\ &= \alpha(\lambda) [\beta - \lambda\beta\bar{\omega} + (1 - \lambda)(1 - \bar{\omega})(P - A^*PA - C^*JC)] \alpha(\omega)^*, \end{aligned} \quad (6.113)$$

which proves the result for $z = \lambda$. For an arbitrary $z \in \overline{\Lambda}_{(1)}$, the above calculation follows in a similar way. Because of this result and for reasons that will become clear later, the Stein equation is assumed to hold hereby.

The goal of the remaining of this section is the solution to the following interpolation problem:

Given an observable pair $(C, A) \in \overline{\Lambda}_{(1)}^{p \times q} \times \overline{\Lambda}_{(1)}^{q \times q}$ such that $C^*C = P - A^*PA$, find every f that satisfies

$$(C^* \star F)(A^*) = X. \quad (6.114)$$

First, it can be checked by direct computation that the function

$$F_{min} = C \star (I_q - zA)^{-\star} P^{-1} X = \sum_{n=0}^{\infty} z^n C A^n P^{-1} X \quad (6.115)$$

is a particular solution to Eq. (6.114). As a result, if F is another solution, then $G = F - F_{min}$ satisfies the homogeneous problem, i.e.,

$$(C^* \star G)(A^*) = 0. \quad (6.116)$$

Now, writing $G = \sum_{n=0}^{\infty} z^n g_n$, Eq. (6.116) is equivalent to

$$\sum_{n=0}^{\infty} (A^*)^n C^* g_n = 0, \quad (6.117)$$

which, in turn, is equivalent to

$$[G, C \star (I - zA)^{-\star} \xi] = 0, \quad (6.118)$$

for every ξ in \mathcal{W}_{G+}^p . In other words, G is orthogonal to $C \star (I - zA)^{-\star} \xi$.

Continuing, let $\mathcal{H}(\Theta)$ be the set of functions F of the type

$$F = C \star (I - zA)^{-\star} \xi. \quad (6.119)$$

Such a space is associated with the reproducing kernel

$$K_{\mathcal{H}(\Theta)}(z, w) = C \star (I_q - zA)^{-\star} P^{-1} [(I_q - wA)^*]^{-\star r} \star_r C^*, \quad (6.120)$$

with $P - A^*PA = C^*C$. Then, using Eq. (6.111),

$$K_{\mathcal{H}(\Theta)}(z, w) = \sum_{n=0}^{\infty} z^n [I - \Theta(z)\Theta(w)^*] (w^\dagger)^n \xi. \quad (6.121)$$

Finally, observe that $\mathcal{W}_{G_+}^p$ can be decomposed as the direct sum

$$\mathcal{W}_{G_+}^p = \Theta \mathcal{W}_{G_+}^p \oplus \mathcal{H}(\Theta). \quad (6.122)$$

In fact, using Eq. (6.121), the kernel K defined in Eq. (6.104) satisfies

$$K(z, w)\xi = \sum_{n=0}^{\infty} z^n I (w^\dagger)^n \xi = \sum_{n=0}^{\infty} z^n [\Theta(z)\Theta(w)^*] (w^\dagger)^n \xi + K_{\mathcal{H}(\Theta)}(z, w)\xi \quad (6.123)$$

for every $\xi \in \overline{\Lambda}_{(1)}^{p \times 1}$. The two terms on the right-hand side are orthogonal to each other. The term $\sum_{n=0}^{\infty} z^n [\Theta(z)\Theta(w)^*] (w^\dagger)^n \xi$ belongs to $\Theta \mathcal{W}_{G_+}^p$ and the term $K_{\mathcal{H}(\Theta)}(z, w)\xi$ belongs to $\mathcal{H}(\Theta)$. By Eq. (6.118), the two sets are orthogonal to each other. Therefore, Eq. (6.122) holds.

As a result, all solutions to Eq. (6.114) can be written as

$$F = F_{min} + \Theta \star H, \quad (6.124)$$

where H is an arbitrary element of \mathcal{W}_{G_+} .

6.8 Nevanlinna-Pick Interpolation

In Section 6.5, the original version of the Nevanlinna-Pick (NP) interpolation was briefly introduced in the complex setting. Now, a version of it in $\bar{\Lambda}_{(1)}$ is studied. Given N points z_k in the open unit superdisk and N values s_k in $\bar{\Lambda}_{(1)}$, the NP problem consists of characterizing all SG functions S satisfying

$$S(z_k) = s_k \quad (6.125)$$

and such that the Pick matrix P is superpositive, i.e.,

$$P = (p_{jk}) \equiv (p_k(z_j; s_j)) \succeq 0, \quad (6.126)$$

where $p_k(z; s)$ is defined as

$$p_k(z; s) \equiv (1 - ss_k^\dagger) \star (1 - zz_k^\dagger)^{-\star} \quad (6.127)$$

for every $k \in \{1, \dots, N\}$.

Following the solution to the complex case [187, 230], consider the function Θ defined in Eq. (6.109) with

$$A \equiv \begin{pmatrix} z_1^\dagger & & \\ & \ddots & \\ & & z_N^\dagger \end{pmatrix}, \quad C \equiv \begin{pmatrix} 1 & \cdots & 1 \\ s_1^\dagger & \cdots & s_N^\dagger \end{pmatrix}, \quad \text{and } J \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.128)$$

Note that $P - A^*PA = C^*JC$.

As a result, $\left(\begin{array}{cc} 1 & -s_k \end{array} \right) \star \Theta(z) \Big|_{z=z_k}$ equals

$$\left(\begin{array}{cc} 1 & -s_k \end{array} \right) - (1-z) \left(\begin{array}{cc} 1 & -s_k \end{array} \right) C \star (I_N - zA)^{-\star} P^{-1} (I_N - A)^{-\star} C^* J \Big|_{z=z_k}. \quad (6.129)$$

Moreover, because

$$\left(\begin{array}{cc} 1 & -s_k \end{array} \right) C = \left(\begin{array}{cccc} 1 - s_k s_1^\dagger & \cdots & 1 - s_k s_N^\dagger & \end{array} \right), \quad (6.130)$$

it holds that

$$\left(\begin{array}{cc} 1 & -s_k \end{array} \right) C \star (I_N - zA)^{-\star} \Big|_{z=z_k} = P^{[k]}, \quad (6.131)$$

where $P^{[k]}$ denotes the k -th row of P . Hence, if $I_N^{[k]}$ denotes the k -th row of I_N ,

$$\begin{aligned} \left(\begin{array}{cc} 1 & -s_k \end{array} \right) \star \Theta(z) \Big|_{z=z_k} &= \left(\begin{array}{cc} 1 & -s_k \end{array} \right) - (1-z_k) I_N^{[k]} (I_N - A)^{-\star} C^* J \\ &= \left(\begin{array}{cc} 1 & -s_k \end{array} \right) - (1-z_k) (1-z_k)^{-1} I_N^{[k]} C^* J \\ &= 0 \end{aligned} \quad (6.132)$$

since $I_N^{[k]} C^* J = \left(\begin{array}{cc} 1 & -s_k \end{array} \right)$. Therefore,

$$\left(\begin{array}{cc} 1 & -s_k \end{array} \right) \star \Theta(z) \Big|_{z=z_k} = 0 \quad (6.133)$$

for every $k \in \{1, \dots, N\}$.

Now, let σ be any SG function and consider the term

$$\left(\begin{array}{cc} 1 & -S(z) \end{array} \right) \star \Theta(z) \star \left(\begin{array}{c} \sigma(z) \\ 1 \end{array} \right). \quad (6.134)$$

Note that such a product is null at $z = z_k$. Also, writing

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \quad (6.135)$$

it holds that

$$s_k = (a \star \sigma(z) + b(z)) \star (c \star \sigma(z) + d(z))^{-\star} \Big|_{z=z_k}, \quad (6.136)$$

provided that the factor $c \star \sigma(z_k) + d(z_k)$ is CK invertible. In fact, that is the case since only the body of the factor needs to be invertible, and a theorem from complex analysis assures that it is. Observe that Eq. (6.136) allows to write the function S as

$$S \equiv T_{\Theta}(\sigma). \quad (6.137)$$

Then, for S to be a solution to the NP problem, it just needs to be shown that, if σ in Eq. (6.137) is a SG function, S is also a SG function. But this is, in fact, the case because, since

$$\Theta^* J \Theta \preceq J \Rightarrow \begin{pmatrix} \sigma^\dagger & 1 \end{pmatrix} \star \Theta^* J \Theta \star \begin{pmatrix} \sigma \\ 1 \end{pmatrix} \preceq \sigma^\dagger \sigma - 1, \quad (6.138)$$

it holds that

$$\begin{aligned} & \begin{pmatrix} (a \star \sigma + b)^\dagger & -(c \star \sigma + d)^\dagger \end{pmatrix} \begin{pmatrix} a \star \sigma + b \\ c \star \sigma + d \end{pmatrix} \preceq 0 \\ & (a \star \sigma + b)^\dagger (a \star \sigma + b) \preceq (c \star \sigma + d)^\dagger (c \star \sigma + d) \\ & [(a \star \sigma + b) \star (c \star \sigma + d)^{-\star}]^\dagger [(a \star \sigma + b) \star (c \star \sigma + d)^{-\star}] \preceq 1 \\ & S^\dagger S \preceq 1, \end{aligned} \quad (6.139)$$

which characterizes a SG function.

Finally, to conclude the solution, it just needs to be shown that all solutions are of the type given by Eq. (6.137), i.e., if a SG function S is a solution to the NP problem, then there exists a SG function σ such that S is given by Eq. (6.137). To see that this is the case, observe that there always exists a function σ given by

$$\sigma = T_{\Theta}^{-1}(S), \quad (6.140)$$

even if it, in principle, does not belong to the set of SG functions. However, because the restriction to the body corresponds to the NP interpolation problem in the complex setting (with Pick matrix $P_B > 0$), it can be stated that σ_B is a Schur function. Therefore, σ is a SG function, and all solutions to the NP problem in $\bar{\Lambda}_{(1)}$ are given by Eq. (6.137).

6.9 Schur Algorithm

In Ref. [214], using Schwarz' lemma, Schur associated to a Schur function $s(\lambda)$ a sequence, finite or infinite, of Schur functions $s_0, s_1 \dots$ with the recursion

$$\begin{aligned} s_0(\lambda) &= s(\lambda) \\ s_{n+1}(\lambda) &= \frac{s_n(\lambda) - s_n(0)}{\lambda(1 - \overline{s_n(\lambda)s_n(0)})}, \end{aligned} \quad (6.141)$$

for $n = 0, 1, \dots$. Such a recursion ends at a rank N if $|s_N(0)| = 1$, and this happens if and only if s is a finite Blaschke product. The numbers $\rho_n = s_n(0)$, $n = 0, 1, \dots$ are called the Schur coefficients of s . They lead to a continued fraction expansion of s , and prove more appropriate than the Taylor series of s to solve various approximation problems [187, 212].

Now, for $\rho \in \mathbb{D}$, let

$$\Theta_{\rho}(\lambda) \equiv \frac{1}{\sqrt{1 - |\rho|^2}} \begin{pmatrix} 1 & \rho \\ \bar{\rho} & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}. \quad (6.142)$$

Then, expressing s_n in terms of s_{n+1} , Eq. (6.141) can be rewritten as

$$s_n(\lambda) = T_{\Theta_{\rho_n}(\lambda)}(s_{n+1}(\lambda)), \quad (6.143)$$

with $\rho_n = s_n(0)$. The matrix-function $\Theta_{\rho_n}(\lambda)$ is, then, J -inner in the open unit disk. By the latter, it is meant that $\Theta_{\rho_n}(\lambda)$ is J -contractive in the open unit disk, i.e.,

$$\Theta_{\rho_n}(\lambda)J\Theta_{\rho_n}(1/\bar{\lambda})^* \leq J \quad (6.144)$$

for $\lambda \in \mathbb{D}$, and $\Theta_{\rho_n}(\lambda)$ is J -unitary (or symplectic) at the boundary of the unit disk, i.e.,

$$\Theta_{\rho_n}(\lambda)J\Theta_{\rho_n}(1/\bar{\lambda})^* = J \quad (6.145)$$

for $\lambda \in \partial\mathbb{D}$.

Observe that $\Theta_{\rho_n}(\lambda)$ remains J -inner if it is multiplied by a J -unitary constant, say X_n , on the right. Thus, Eq. (6.143) can be rewritten as

$$s_n(\lambda) = T_{\Theta_{\rho_n}(\lambda)X_n}(T_{X_n^{-1}}(s_{n+1}(\lambda))). \quad (6.146)$$

Since X_n is J -unitary, the function $T_{X_n^{-1}}(s_{n+1}(\lambda))$ is still a Schur function. This fact was used in Ref. [231, §3] to develop the Schur algorithm in the matrix-valued case — see, in particular, Eq. (4.13) in that paper.

A particular choice of X_n leads to

$$M_n \equiv \Theta_{\rho_n}(\lambda)X_n = I_2 - (1 - \lambda) \frac{\begin{pmatrix} 1 \\ \overline{\rho_n} \end{pmatrix} \begin{pmatrix} 1 & -\rho_n \end{pmatrix}}{1 - |\rho_n|^2}, \quad (6.147)$$

which allows the Schur algorithm to be rewritten as

$$\begin{aligned}\sigma_0(\lambda) &= s(\lambda) \\ \sigma_{n+1}(\lambda) &= T_{M_n(\lambda)^{-1}}(\sigma_n(\lambda))\end{aligned}\tag{6.148}$$

for $n = 0, 1, \dots$. This recursion with the counterpart of Eq. (6.147) is how the Schur algorithm is defined in $\bar{\Lambda}_{(1)}$. More precisely, using Eq. (6.109) with

$$A = 0, \quad C = \begin{pmatrix} 1 \\ \rho_n^\dagger \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\tag{6.149}$$

the Stein equation becomes

$$P = 1 - \rho_n \rho_n^\dagger.\tag{6.150}$$

Moreover, if

$$\begin{cases} a(z) \equiv 1 - (1 - z)(1 - \rho_n \rho_n^\dagger)^{-1}, \\ b(z) \equiv (1 - z)(1 - \rho_n \rho_n^\dagger)^{-1} \rho_n, \\ c(z) \equiv -(1 - z) \rho_n^\dagger (1 - \rho_n \rho_n^\dagger)^{-1}, \\ d(z) \equiv 1 + (1 - z) \rho_n^\dagger (1 - \rho_n \rho_n^\dagger)^{-1} \rho_n, \end{cases}\tag{6.151}$$

a version of M_n in $\bar{\Lambda}_{(1)}$ can be written as

$$M_n(z) = I - (1 - z) \begin{pmatrix} 1 \\ \rho_n^\dagger \end{pmatrix} (1 - \rho_n \rho_n^\dagger)^{-1} \begin{pmatrix} 1 & -\rho_n \end{pmatrix} = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.\tag{6.152}$$

Finally, the Schur algorithm for SG functions can be stated: If S is a SG function. Then, the recursion

$$\begin{aligned}\sigma_0(z) &= S(z) \\ \sigma_{n+1}(z) &= T_{M_n(z)^{-1}}(\sigma_n(z)), \quad \text{with } \rho_n = \sigma_n(0)\end{aligned}\tag{6.153}$$

defines a family of SG functions, provided that $P = 1 - \rho_n \rho_n^\dagger \succ 0$.

6.10 Fock-Bargmann-Segal Space

The complex Fock-Bargmann-Segal (FBS) space associated with ℓ_2 , i.e., the HS with reproducing kernel [232]

$$e^{\langle z, w \rangle_{\ell_2}} = \sum_{\alpha \in \ell} \frac{z^\alpha \bar{w}^\alpha}{\alpha!} \quad (6.154)$$

corresponds to function theory in a (countably) infinite number of commuting complex variables. In Eq. (6.154), the several complex variables notation was used and, then, $z = (z_1, z_2, \dots) \in \ell_2(\mathbb{N})$, where ℓ is the family of sequences

$$\alpha = (\alpha_1, \alpha_2, \dots), \quad \alpha_j \in \mathbb{N}_0 \quad (6.155)$$

for which at most a finite number of α_j 's are different from 0. Also, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots$ and $\alpha! = \alpha_1! \alpha_2! \dots$

In the noncommutative setting, the commuting variables are replaced by noncommuting ones and, then, a different set of indexes $\tilde{\ell}$ is needed. An element $\alpha \in \tilde{\ell}$ is given by

$$\alpha = ((\alpha_1, n_1), (\alpha_2, n_2), \dots, (\alpha_m, n_m)), \quad (6.156)$$

where $\alpha_u \in \mathbb{N}$ for every $u \in \{1, \dots, m\}$ and $n_v \neq n_{v+1}$ for every $v \in \{1, \dots, m-1\}$.

Moreover, the new kernel is written as

$$\sum_{\alpha \in \tilde{\ell}} z^\alpha \bar{w}^\alpha, \quad (6.157)$$

where $z^\alpha = z_{n_1}^{\alpha_1} z_{n_2}^{\alpha_2} \dots$ and $\bar{w}^\alpha = \dots \bar{w}_{n_2}^{-\alpha_2} \bar{w}_{n_1}^{-\alpha_1}$.

Note that the left concatenation gives $\tilde{\ell}$ a *monoid* structure. Furthermore, it defines a partial order as follows: for $\alpha, \beta \in \tilde{\ell}$, it is said that $\beta \leq \alpha$ if there is $\gamma \in \tilde{\ell}$ such that $\alpha = \beta\gamma$. There exists, then, a parallel between the indexes in $\tilde{\ell}$ and the ones in \mathfrak{J}_0 . Motivated by it and envisioning the construction of the counterpart of the FBS space in the GA, the inner product

$$\langle z, w \rangle = \sum_{\alpha \in \mathfrak{J}_0} z_\alpha \bar{w}_\alpha \quad (6.158)$$

is defined for elements of Λ . Observe that

$$\langle w, z \rangle = \langle z^\dagger, w^\dagger \rangle. \quad (6.159)$$

Also, the inner product defined in Eq. (6.158) induces the 2-norm. Then, it can be extended from Λ to $\bar{\Lambda}_{(2)}$. Moreover, $\bar{\Lambda}_{(2)}$ endowed with the inner product defined in Eq. (6.158) and the 2-norm is a HS. For simplicity of notation, such a HS is simply denoted by $\bar{\Lambda}_{(2)}$. By analogy with the noncommutative setting, as already discussed, and also noting the definition in Ref. [233], $\bar{\Lambda}_{(2)}$ is called the FBS space.

This section, then, starts a study, which will last until Section 6.12, of $\bar{\Lambda}_{(2)}$. This study considers functions $f : \mathcal{I} \rightarrow \bar{\Lambda}_{(2)}$, where \mathcal{I} is the domain of f , usually \mathbb{C} or \mathbb{R} . Whenever it is written that $f \in \bar{\Lambda}_{(2)}$, it is meant that $f(x) \in \bar{\Lambda}_{(2)}$ for every $x \in \mathcal{I}$.

As in the examples of FBS spaces in different settings, it is possible to define the left multiplication operation M_f in $\bar{\Lambda}_{(2)}$. However, the multiplication might not be a law of composition in $\bar{\Lambda}_{(2)}$. If that is the case, M_f is unbounded for an arbitrary $f \in \bar{\Lambda}_{(2)}$. One can, then, use two different approaches. In one of them, which will be considered in the next section, such operators can be studied in the space of stochastic distributions \mathfrak{S}_{-1} , where the product between two functions defines a new element of the space. In the other approach, M_f is restricted to functions $f \in \bar{\Lambda}_{(1)} \subset \bar{\Lambda}_{(2)}$ Eq. (6.191) assures that M_f is bounded in this case,

i.e., if $f \in \overline{\Lambda}_{(1)} \subset \overline{\Lambda}_{(2)}$ and $g \in \overline{\Lambda}_{(2)}$,

$$M_f g = fg \in \overline{\Lambda}_{(2)}. \quad (6.160)$$

For now, to avoid convergence problems, restrict M_f to $f \in \overline{\Lambda}_{(1)}$. Thus, if $f = \sum_{\alpha \in \mathfrak{J}_0} f_\alpha i_\alpha$,

$$M_f = \sum_{\alpha \in \mathfrak{J}_0} f_\alpha M_{i_\alpha}. \quad (6.161)$$

Therefore, the analysis can be focused on multiplication by elements of \mathfrak{J}_0 .

First, observe that M_1 is just the identity operator. Then, to get an expression for a general M_{i_α} for $\alpha \in \mathfrak{J}$, observe that Eq. (6.8) and

$$\langle i_\alpha, i_\beta \rangle = \delta_{\alpha, \beta} \quad (6.162)$$

lead to

$$\langle M_{i_\alpha} i_\beta, i_\gamma \rangle = (-1)^{\sigma(\alpha, \beta)} \sum_{\eta \in \mathfrak{J}} \delta_{\alpha \vee \beta, \eta} \langle i_\eta, i_\gamma \rangle = (-1)^{\sigma(\alpha, \beta)} \delta_{\alpha \vee \beta, \gamma}. \quad (6.163)$$

It can be also noted that the adjoint $M_{i_\alpha}^*$ of M_{i_α} , which, by analogy with the complex case, can be taken to be the left derivative with respect to i_α , is given by

$$\langle M_{i_\alpha}^* i_\beta, i_\gamma \rangle = \langle i_\beta, M_{i_\alpha} i_\gamma \rangle = (-1)^{\sigma(\alpha, \gamma)} \delta_{\beta, \alpha \vee \gamma}. \quad (6.164)$$

Also, because

$$M_{i_\alpha} = M_{i_{a_1}} M_{i_{a_2}} \dots M_{i_{a_{\tau\alpha}}} \quad (6.165)$$

and, as a consequence,

$$M_{i_\alpha}^* = M_{i_{a_{\tau\alpha}}}^* M_{i_{a_{\tau-1}}}^* \dots M_{i_{a_1}}^*, \quad (6.166)$$

close attention can be paid to left derivatives with respect to single generators. Thus,

$$M_{i_n}^* i_\alpha = \begin{cases} 0, & \text{if } a_k \neq n, \forall a_k \in \{1, \dots, |\alpha|\} \\ (-1)^{k-1} i_{a_1} i_{a_2} \cdots i_{a_{k-1}} i_{a_{k+1}} \cdots i_{a_{|\alpha|}}, & \text{if } \exists k; a_k = n \end{cases} \quad (6.167)$$

Hence, the left derivative constructed here corresponds to the one that is defined in superanalysis [194, 195, 198, 200].

Also, the Berezin integral is a concept widely used in superanalysis and supersymmetry [194, 195, 198, 200]. It coincides with the left derivative and, then, can be defined in terms of $M_{i_n}^*$:

$$\int di_n f \equiv M_{i_n}^* f. \quad (6.168)$$

More generally, if $|\alpha| < \infty$,

$$\int di_\alpha f \equiv \int di_{a_{|\alpha|}} \cdots \int i_{a_1} f = M_{i_{a_{|\alpha|}}}^* \cdots M_{i_{a_1}}^* f. \quad (6.169)$$

Furthermore, if f is generated by i_{a_1}, \dots, i_{a_N} and $i_\alpha = i_{a_1} \dots i_{a_N}$, the Berezin integral

$$\int di_\alpha f = f_{1,2,\dots,N} \quad (6.170)$$

reduces to

$$\int di_\alpha f = \langle M_f 1, i_\alpha \rangle, \quad (6.171)$$

which has some resemblance to a residue.

Finally, it is also worth considering the self-adjoint operator

$$T_f = M_f + M_f^*. \quad (6.172)$$

Then, if $f, g \in \overline{\Lambda}_{(2)}$ with $f_B = g_B = 0$, $M_f^*1 = f_B$ and $M_g^*1 = g_B$. As a result, it holds that

$$\langle T_f 1, T_g 1 \rangle = \langle f + f_B, g + g_B \rangle = \langle f, g \rangle. \quad (6.173)$$

The operator T_f is used later when stochastic processes associated with the FBS space are defined.

6.11 Topological Algebra Associated with $\overline{\Lambda}_{(2)}$

Let \mathcal{S} denote the space of Schwartz functions, also known as the space of test functions. Also, let \mathcal{S}' be its dual, known as the space of tempered distributions. The Gel'fand triple $(\mathcal{S}, \mathbf{L}_2(\mathbb{R}, dx), \mathcal{S}')$ plays an important role in analysis [234].

Moreover, Gel'fand triples are also defined in Hida's white noise space theory [235, 236] and in its noncommutative counterpart [237, 238]. They are used in the solutions of stochastic differential equations and the study of stochastic processes and their derivatives. In this section, Gel'fand triples are defined in $\overline{\Lambda}_{(2)}$ to solve similar problems. Most of the results presented here find a parallel to the ones presented in Refs. [239, 240].

One of the common reasons for the introduction of Gel'fand triples in the first place is the fact that, in a HS, products that are not a law of composition are often considered. As an example, in the white noise space, the Wick product is not a law of composition. Because of that, one embeds the white noise space into a space of stochastic distributions, where the product is a law of composition. This embedding can be made in different ways.

Furthermore, even if the convergence of the product is not a problem, the introduction of the space of stochastic distributions might still be necessary. In fact, in the study of stochastic processes, such spaces are necessary for the analysis of their derivatives.

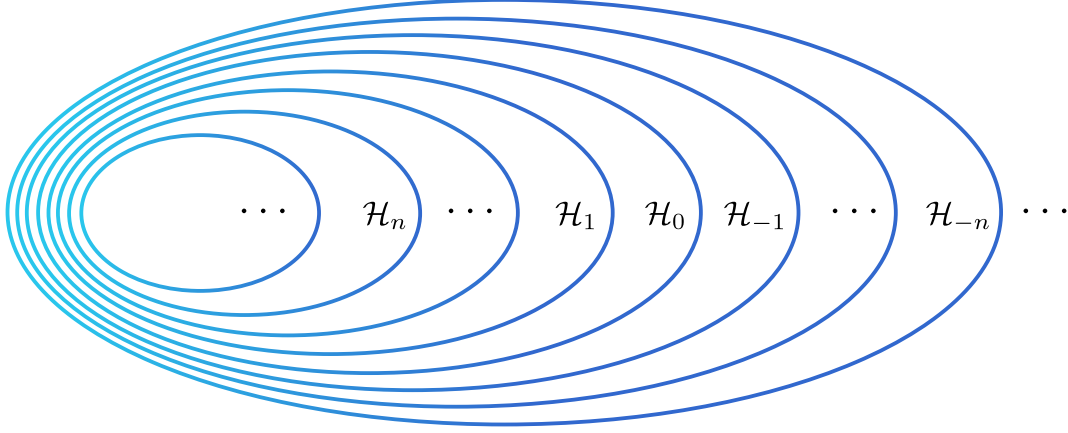


Figure 6.2: Schematic representation of a decreasing family of Hilbert spaces with increasing norm, where n is a positive integer.

In this section, the analogous of the space of stochastic distributions is introduced. Before that, a few facts from the complex case as well as from the theory of perfect spaces and strong algebras are briefly reviewed. More information on those spaces can be found in Ref. [241, 242].

Consider a decreasing family of HSs $(\mathcal{H}_p, \|\cdot\|_{\mathcal{H}_p})_{p \in \mathbb{Z}}$ with increasing norms, as represented in Fig. 6.2. The intersection $\mathcal{F} = \bigcap_{p=0}^{\infty} \mathcal{H}_p$ is a Fréchet space, which is a perfect space, meaning that compactness is equivalent to being bounded and compact. This is the fact, in particular, when, for every p , there exists $q > p$ such that the injection map from \mathcal{H}_q into \mathcal{H}_p is compact. An important instance is when this injection is nuclear and, then, \mathcal{H}'_p is identified with \mathcal{H}_{-p} . The dual $\mathcal{F}' = \bigcup_{p=0}^{\infty} \mathcal{H}_{-p}$, the space \mathcal{F} and \mathcal{H}_0 form a Gel'fand triple.

The space \mathcal{F}' is endowed with the strong topology, which is defined in terms of the bounded sets of \mathcal{F} . Then, \mathcal{F}' is locally convex, and the strong topology coincides with the inductive limit topology — see Ref. [239, Section 3] for a discussion on the subject.

As a consequence, the analysis is, *a priori*, done in a larger space of distributions, i.e., continuous functionals on the space \mathcal{F}' , which is a (non-metrizable) inductive limit of HSs. However, it is *in fact* done locally in a HS. There are two reasons why this is the case. The

first one is the fact that the space of distributions, as already mentioned, is the dual of a perfect space. The second reason is the algebraic structure of \mathfrak{S}_{-1} and, in particular, the Våge inequality, which is presented in Eq. (6.191).

Two of the main results related to such spaces that are relevant here and that illustrate why the analysis is locally done in a HS are:

- A set is (weakly or strongly) compact in \mathcal{F}' if and only if it is compact in one of the spaces \mathcal{H}_{-p} in the corresponding norm.
- Weak and strong convergence of sequences in a perfect space \mathcal{F}' are equivalent. Moreover, a sequence converges (weakly or strongly) if and only if it converges in one of the spaces \mathcal{H}_{-p} in the corresponding norm.

Before introducing definitions and results in the GA, it should be mentioned that a topological algebra is defined to be an algebra where the product is separately continuous in each variable. It is not a trivial fact, but it can be shown that a strong algebra is a topological algebra — see Ref. [243, IV.26, Theorem 2] and the discussion in Ref. [240, pp. 215-216].

Now, let $\{c_\alpha\}$ be a sequence of positive real numbers such that

$$\sum_{\alpha \in \mathfrak{I}_0} c_\alpha^{-2d} < \infty, \quad (6.174)$$

where $d \in \mathbb{N}$, and

$$c_\alpha c_\beta \leq c_\gamma \quad (6.175)$$

if $\alpha \vee \beta = \gamma$. Then, for every $p \in \mathbb{Z}$, a HS $\mathcal{H}_{-p}(c_\alpha)$, often simply denoted by \mathcal{H}_{-p} , is defined as

$$\mathcal{H}_{-p}(c_\alpha) = \left\{ f = \sum_{\alpha \in \mathfrak{I}_0} f_\alpha i_\alpha \in \overline{\Lambda}_{(2)} \mid \sum_{\alpha \in \mathfrak{I}_0} |f_\alpha|^2 c_\alpha^{-2p} < \infty \right\}. \quad (6.176)$$

It should be noticed that $\mathcal{H}_{-q}(c_\alpha) \subseteq \mathcal{H}_{-p}(c_\alpha)$ if $p \geq q$, as illustrated in Fig. 6.2.

The norm $\|f\|_{\mathcal{H}_{-p}}$ of $f \in \mathcal{H}_{-p}$ is defined as

$$\|f\|_{\mathcal{H}_{-p}} \equiv \sum_{\alpha \in \mathfrak{J}_0} |f_\alpha|^2 c_\alpha^{-2p}. \quad (6.177)$$

For simplicity, hereby, it will be assumed that

$$c_{\alpha \vee \beta} \equiv c_\alpha c_\beta. \quad (6.178)$$

In this case, $c_0 = 1$. In fact, if $c_0 > 1$, the inequality in Eq. (6.175) does not hold in general since $c_0 c_\alpha > c_\alpha = c_{0 \vee \alpha}$. Moreover, if $c_0 < 1$, the condition in Eq. (6.174) is not satisfied, as can be observed from the fact that $c_0^{-2d} < c_\alpha^{-2d}$ for every $\alpha \in \mathfrak{J}$, which implies that the sum in Eq. (6.174) diverges.

Now, let $f \in \mathcal{H}_{-p}$ with $c_\alpha > 1$ if $\alpha \neq 0$ and $c_0 = 1$. Then, because $\lim_{p \rightarrow \infty} c_\alpha^{-2p} = 0$ for every $\alpha \neq 0$,

$$\begin{aligned} \lim_{p \rightarrow \infty} \|f\|_{\mathcal{H}_{-p}} &= \lim_{p \rightarrow \infty} \sum_{\alpha \in \mathfrak{J}_0} |f_\alpha|^2 c_\alpha^{-2p} \\ &= \sum_{\alpha \in \mathfrak{J}_0} |f_\alpha|^2 \lim_{p \rightarrow \infty} c_\alpha^{-2p} \\ &= |f_0|^2. \end{aligned} \quad (6.179)$$

The next result introduces a Våge-like inequality, which is the analogous of a result due to Våge [244] and allows the analysis of stochastic processes to be done locally in a HS. For

that, let $f \in \mathcal{H}_{-q}$ and $g \in \mathcal{H}_{-p}$. Hence, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|fg\|_{\mathcal{H}_{-p}}^2 &= \sum_{\gamma \in \mathcal{I}_0} |(fg)_\gamma|^2 c_\gamma^{-2p} \\
&= \sum_{\gamma \in \mathcal{I}_0} \left| \sum_{\alpha \vee \beta = \gamma} (-1)^{\sigma(\alpha, \beta)} f_\alpha g_\beta \right|^2 c_\gamma^{-2p} \\
&\leq \sum_{\gamma \in \mathcal{I}_0} \left(\sum_{\substack{\alpha \vee \beta = \gamma \\ \alpha' \vee \beta' = \gamma}} |f_\alpha| |g_\beta| |f_{\alpha'}| |g_{\beta'}| \right) c_\gamma^{-2p} \\
&\leq \sum_{\gamma \in \mathcal{I}_0} \left(\sum_{\substack{\alpha \vee \beta = \gamma \\ \alpha' \vee \beta' = \gamma}} |f_\alpha| c_\alpha^{-p} |g_\beta| c_\beta^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} |g_{\beta'}| c_{\beta'}^{-p} \right) \\
&\leq \sum_{\alpha, \alpha' \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} \left(\sum_{\substack{\gamma \in \mathcal{I}_0; \exists \beta, \beta' \\ \alpha \vee \beta = \gamma \\ \alpha' \vee \beta' = \gamma}} |g_\beta| c_\beta^{-p} |g_{\beta'}| c_{\beta'}^{-p} \right) \tag{6.180} \\
&\leq \sum_{\alpha, \alpha' \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} \left(\sum_{\substack{\gamma \in \mathcal{I}_0; \exists \beta \\ \alpha \vee \beta = \gamma}} |g_\beta|^2 c_\beta^{-2p} \right)^{1/2} \left(\sum_{\substack{\gamma \in \mathcal{I}_0; \exists \beta' \\ \alpha' \vee \beta' = \gamma}} |g_{\beta'}|^2 c_{\beta'}^{-2p} \right)^{1/2} \\
&\leq \sum_{\alpha, \alpha' \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} \left(\sum_{\beta \in \mathcal{I}_0} |g_\beta|^2 c_\beta^{-2p} \right)^{1/2} \left(\sum_{\beta' \in \mathcal{I}_0} |g_{\beta'}|^2 c_{\beta'}^{-2p} \right)^{1/2} \\
&\leq \left(\sum_{\alpha \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} \right)^2 \|g\|_{\mathcal{H}_{-p}}^2 \\
&\leq \left(\sum_{\alpha \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-q} c_\alpha^{q-p} \right)^2 \|g\|_{\mathcal{H}_{-p}}^2 \\
&\leq \left(\sum_{\alpha \in \mathcal{I}_0} c_\alpha^{-2(p-q)} \right) \|f\|_{\mathcal{H}_{-q}}^2 \|g\|_{\mathcal{H}_{-p}}^2.
\end{aligned}$$

Now, it only remains to be shown that there exist coefficients c_α such that

$$\sum_{\alpha \in \mathfrak{J}_0} c_\alpha^{-2(p-q)} < \infty. \quad (6.181)$$

For that, let

$$c_\alpha = e^{\sum_{k=1}^n \varphi(a_k)} \quad (6.182)$$

for every $\alpha \in \mathfrak{J}_0$, where φ is such that $\varphi(n)$ form a monotonically increasing sequence for $n \in \mathbb{N}$ and $\varphi(0) = 0$. Then, Eqs. (6.174), (6.175) and (6.178) hold. Moreover, if $d = p - q > 0$,

$$\sum_{\alpha \in \mathfrak{J}_0} c_\alpha^{2(q-p)} = 1 + \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathfrak{J}; \\ |\alpha|=n}} e^{-2d \sum_{k=1}^n \varphi(a_k)}. \quad (6.183)$$

Finally, for Eq. (6.181) to hold, choose φ such that, for every $k \in \mathbb{N}$,

$$\varphi(a_k) \leq \xi a_k, \quad (6.184)$$

where ξ is a positive real number that will be appropriately chosen. Then,

$$\sum_{\substack{\alpha \in \mathfrak{J}; \\ \tau(\alpha)=1}} e^{-2d\varphi(a_1)} = \sum_{a_1=1}^{\infty} e^{-2d\varphi(a_1)} \leq \sum_{a_1=1}^{\infty} e^{-2d\xi a_1} = \frac{1}{e^{2d\xi} - 1}. \quad (6.185)$$

if it converges. Then, choosing

$$\xi > \ln 2^{1/2d}, \quad (6.186)$$

it holds that

$$\sum_{\substack{\alpha \in \mathfrak{J}; \\ \tau(\alpha)=1}} e^{-2d\varphi(a_1)} < 1. \quad (6.187)$$

Also,

$$\sum_{\substack{\alpha \in \mathcal{J}; \\ \tau(\alpha)=2}} e^{-2d(\varphi(a_1)+\varphi(a_2))} \leq \left(\sum_{a_1=1}^{\infty} e^{-2d\varphi(a_1)} \right) \left(\sum_{a_2=1}^{\infty} e^{-2d\varphi(a_2)} \right) \leq \left(\frac{1}{e^{2d\xi} - 1} \right)^2 \quad (6.188)$$

and, in general,

$$\begin{aligned} \sum_{\substack{\alpha \in \mathcal{J}; \\ \tau(\alpha)=n}} e^{-2d(\varphi(a_1)+\dots+\varphi(a_n))} &\leq \left(\sum_{a_1=1}^{\infty} e^{-2d\varphi(a_1)} \right) \dots \left(\sum_{a_n=1}^{\infty} e^{-2d\varphi(a_n)} \right) \\ &\leq \left(\frac{1}{e^{2d\xi} - 1} \right)^n. \end{aligned} \quad (6.189)$$

Therefore,

$$\sum_{\alpha \in \mathcal{J}_0} c_{\alpha}^{-2d} \leq 1 + \sum_{n=1}^{\infty} \left(\frac{1}{e^{2d\xi} - 1} \right)^n = 1 + \frac{1}{e^{2d\xi} - 2}. \quad (6.190)$$

The Våge-like inequality can, then, be written as

$$\|fg\|_{\mathcal{H}_{-p}} \leq C_{p-q} \|f\|_{\mathcal{H}_{-q}} \|g\|_{\mathcal{H}_{-p}}, \quad (6.191)$$

where C_{p-q} is a positive constant.

With that, it can be also shown that, if $f \in \mathcal{H}_{-p}$ and $g \in \mathcal{H}_{-q}$, with $p > q$, then

$$\|fg\|_{\mathcal{H}_{-p}} \leq C_{p-q} \|f\|_{\mathcal{H}_{-p}} \|g\|_{\mathcal{H}_{-q}}. \quad (6.192)$$

Now, it is introduced the space

$$\mathfrak{S}_1 = \bigcap_{p \in \mathbb{Z}} \mathcal{H}_p \quad (6.193)$$

and its topological dual

$$\mathfrak{S}_{-1} = \bigcup_{p \in \mathbb{Z}} \mathcal{H}_{-p}, \quad (6.194)$$

which are respectively the analogous of the space of test functions and the space of tempered distributions in the complex case.

It is possible to show that the space \mathfrak{S}_{-1} endowed with the pointwise product is a strong algebra. To see that, start by endowing \mathfrak{S}_{-1} with the inductive topology. Eq. (6.191) implies that the product is separately continuous in each \mathcal{H}_{-p} , which is equivalent to continuity in the inductive topology. Furthermore, \mathfrak{S}_{-1} inherits the associativity of the pointwise product from Λ . This space has, then, a BA structure. Therefore, \mathfrak{S}_{-1} can be seen as the inductive limit of Banach spaces, which makes it a strong algebra [240]. Observe that the inductive topology is equivalent to the strong topology.

It should be noticed that, if the strong convolution algebra associated to \mathfrak{J}_0 endowed with the convolution \vee is considered, a strong algebra that is closely related to \mathfrak{S}_{-1} is obtained. Although those two algebras are not isomorphic as a ring, they are isomorphic as locally convex topological vector spaces. As a consequence, it follows from Theorem 3.7 of Ref. [239] that \mathfrak{S}_{-1} is nuclear and, hence, perfect.

Now, let $n \in \mathbb{N}$ and $f \in \mathcal{H}_{-p} \subseteq \mathcal{H}_{-p-2}$. Then, it follows that

$$\|f\|_{\mathcal{H}_{-p-2}} \leq \|f\|_{\mathcal{H}_{-p}}. \quad (6.195)$$

Then, using Eq. (6.191),

$$\begin{aligned} \|f^n\|_{\mathcal{H}_{-p-2}} &\leq C_2 \|f\|_{\mathcal{H}_{-p}} \|f^{n-1}\|_{\mathcal{H}_{-p-2}} \\ &\leq C_2^2 \|f\|_{\mathcal{H}_{-p}}^2 \|f^{n-2}\|_{\mathcal{H}_{-p-2}} \\ &\leq C_2^{n-1} \|f\|_{\mathcal{H}_{-p}}^n. \end{aligned} \quad (6.196)$$

With that, consider the power series

$$F(\lambda) = \sum_{n \in \mathbb{N}_0} \alpha_n \lambda^n, \quad (6.197)$$

where $\alpha_n, \lambda \in \mathbb{C}$, and assume it is convergent absolutely in an open disk with radius R , i.e.,

$$\sum_{n \in \mathbb{N}_0} |\alpha_n \lambda^n| = \sum_{n \in \mathbb{N}_0} |\alpha_n| |\lambda^n| < \infty. \quad (6.198)$$

Now, observe that Eq. (6.196) allows the study of the convergence of $F(f)$ in \mathcal{H}_{-p} . Specifically,

$$\begin{aligned} \sum_{n \in \mathbb{N}_0} \|\alpha_n f^n\|_{\mathcal{H}_{-p-2}} &= \sum_{n \in \mathbb{N}_0} |\alpha_n|^2 \|f^n\|_{\mathcal{H}_{-p-2}} \\ &\leq \alpha_0 + C_2^{-1} \sum_{n \in \mathbb{N}} |\alpha_n|^2 \left(C_2 \|f\|_{\mathcal{H}_{-p}} \right)^n. \end{aligned} \quad (6.199)$$

Thus, $F(f)$ converges absolutely in \mathcal{H}_{-p-2} if

$$C_2 \|f\|_{\mathcal{H}_{-p}} < R \Rightarrow \|f\|_{\mathcal{H}_{-p}} < \frac{R}{C_2}. \quad (6.200)$$

In conclusion $F(f)$ converges in \mathcal{H}_{-p-2} if

A corollary of this result is that, $F(\lambda)$ given by Eq. (6.197) converges in \mathfrak{S}_{-1} for $f \in \mathfrak{S}_{-1}$ if the body of f satisfies Eq. (6.200). In fact, if $f \in \mathfrak{S}_{-1}$, there exists an integer q_0 such that $f \in \mathcal{H}_{-q}$ for every $q \geq q_0$. By the result just presented, the converge of $F(f)$ requires that $\|f\|_{\mathcal{H}_{-q}} < R/C_2$, what does not hold in general. However, because of Eq. (6.179), this condition can be rewritten as

$$|f_0|^2 < \frac{R}{C_2}. \quad (6.201)$$

Finally, it can be shown that $f \in \mathfrak{S}_{-1}$ is invertible if and only if its body f_0 is invertible.

To prove it, observe that, if g is the inverse of f and its body is given by g_0 , then

$$fg = 1 \Rightarrow f_0g_0 = 1 \Rightarrow f_0 \neq 0. \quad (6.202)$$

Conversely, if $f_0 \neq 0$, it can be assumed, without loss of generality, that $f_0 = 1$. Then, as just studied,

$$F(f) = \sum_{n \in \mathbb{N}_0} (1 - f)^n \quad (6.203)$$

converges if the body of $1 - f$ is smaller than C_2^{-1} . However, $(1 - f)_B = 0$. Therefore, $g = F(f) \in \mathfrak{S}_{-1}$. Furthermore, remembering that $F(x) = x^{-1}$ in the complex case, g is the inverse of f .

6.12 Stochastic Processes and Their Derivatives

In this section, a close counterpart of the noncommutative white noise space theory [237, 238] is presented for the study of stochastic processes in $\overline{\Lambda}_{(2)}$ as well as their derivatives.

First, a brief review of some aspects of stochastic processes is presented. More details can be found in Refs. [233, 237, 245]. The study of Gaussian stochastic processes can be made through the analysis of positive-definite kernels since there is a one-to-one correspondence between the two notions [246, 247]. In fact, the kernel coincides with the covariance of the stochastic process.

In the framework that is the basis for the model presented here, the processes are associated with positive-definite kernels of the form

$$K_\sigma(t, s) = \int_{\mathbb{R}} \frac{(e^{iut} - 1)(e^{-ius} - 1)}{u^2} d\sigma(u), \quad (6.204)$$

where σ is absolutely increasing continuous with respect to the Lebesgue measure, $d\sigma(u) =$

$m(u)du$, such that the Stieltjes integral converges, i.e.,

$$\int_{\mathbb{R}} \frac{m(u)du}{u^2 + 1} < \infty. \quad (6.205)$$

The reason for such choice is the fact that integrals of the type given by Eq. (6.204) correspond to correlation functions of zero-mean Gaussian processes with stationary increments [248, 249]. An important example of such processes is the fractional Brownian motion, for which $d\sigma(u) = |u|^{1-2H} du$, where $1 < H < 2$. In this case, the correlation function $K_{\sigma}(t, s)$ becomes

$$K(t, s) = \gamma_H \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad (6.206)$$

where γ_H depends only on H . If $H \neq 1/2$,

$$\gamma_H = \frac{\cos(\pi H)\Gamma(2 - 2H)}{(1 - 2H)H}, \quad (6.207)$$

where Γ is the Euler's Gamma function. Moreover, by continuity, $\gamma_{1/2} = \pi$.

Also, it is introduced the following operator S_m in $\mathbf{L}_2(\mathbb{R})$:

$$\widehat{S_m f}(u) = \sqrt{m(u)}\widehat{f}(u), \quad (6.208)$$

where \widehat{f} is the Fourier transform of f . Note that S_m is, in general, unbounded. Its domain is

$$\text{dom } S_m = \left\{ f \in \mathbf{L}_2(\mathbb{R}) \mid \int_{\mathbb{R}} m(u)|\widehat{f}(u)|^2 du < \infty \right\}, \quad (6.209)$$

which contains $\mathbf{1}_{[0,t]}$. Defining

$$f_m(t) = S_m \mathbf{1}_{[0,t]} \quad (6.210)$$

and using Plancherel's equality, which assures that

$$\langle f_m(t), f_m(s) \rangle_{\mathbf{L}_2(\mathbb{R})} = \frac{1}{2\pi} \left\langle \widehat{f}_m(t), \widehat{f}_m(s) \right\rangle_{\mathbf{L}_2(\mathbb{R})}, \quad (6.211)$$

it holds that

$$\begin{aligned} \langle f_m(t), f_m(s) \rangle_{\mathbf{L}_2(\mathbb{R})} &= \frac{1}{2\pi} \left\langle \sqrt{m(u)} \widehat{\mathbf{1}}_{[0,t]}, \sqrt{m(u)} \widehat{\mathbf{1}}_{[0,s]} \right\rangle_{\mathbf{L}_2(\mathbb{R})} \\ &= \frac{1}{2\pi} \left\langle m(u) \frac{e^{-iut} - 1}{u}, \frac{e^{-ius} - 1}{u} \right\rangle_{\mathbf{L}_2(\mathbb{R})} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(e^{iut} - 1)(e^{-ius} - 1)}{u^2} m(u) du. \end{aligned} \quad (6.212)$$

For the stochastic processes of interest here, a random variable is associated with the functions $f_m(t)$. This is done with the introduction of the creation operator ℓ_h , with $h \in \mathbf{L}_2(\mathbb{R})$, defined by

$$\ell_h(f) \equiv h \otimes f, \quad f \in \Gamma(\mathbf{L}_2(\mathbb{R})), \quad (6.213)$$

where $\Gamma(\mathbf{L}_2(\mathbb{R}))$ denotes the FBS space associated with $\mathbf{L}_2(\mathbb{R})$. Finally, letting $T_h = \ell_h + \ell_h^*$, a random variable $X_m(t)$ is defined as

$$X_m(t) \equiv T_{f_m(t)}. \quad (6.214)$$

Observe that the expected value of a random variable $X_m(t)$ can be defined by

$$E(X_m(t)) = \left\langle \Omega, T_{f_m(t)}(\Omega) \right\rangle_{\Gamma}, \quad (6.215)$$

where Ω is the vacuum state of Γ . Moreover, as expected,

$$E(X_m(t)X_m(s)) = \left\langle T_{f_m(t)}(\Omega), T_{f_m(s)}(\Omega) \right\rangle_{\Gamma} = \langle f_m(t), f_m(s) \rangle_{\mathbf{L}_2(\mathbb{R})} = K_{\sigma}(t, s), \quad (6.216)$$

where $K_\sigma(t, s)$ is given by Eq. (6.204) with $d\sigma(u) = m(u)du$, as already discussed.

Finally, for the processes of interest here, it is assumed that m satisfies

$$m(u) \leq \begin{cases} K|u|^{-b} & |u| \leq 1, \\ K|u|^{2N} & |u| > 1, \end{cases} \quad (6.217)$$

where $b < 2$, $N \in \mathbb{N}_0$, and K is a positive real constant. Then, letting ξ_n denote the Hermite functions,

$$|S_m \xi_n(t)| \leq D_1 n^{\frac{N+1}{2}} + D_2, \quad (6.218)$$

and

$$|S_m \xi_n(t) - S_m \xi_n(s)| \leq |t - s| \left(D_3 n^{\frac{N+2}{2}} + D_4 \right), \quad (6.219)$$

where D_1, D_2, D_3 , and D_4 are non-negative functions independent of n . The proof for this result are the Proposition 3.7 and the Lemma 3.8 of Ref. [250].

Now, in the GA, the operator T_f in Eq. (6.214) is replaced by the operator defined in Eq. (6.172) with

$$\begin{aligned} f_m(t) &= \sum_{n \in \mathbb{N}} \left\langle S_m \mathbf{1}_{[0,t]}, \xi_n \right\rangle_{\mathbf{L}_2(\mathbb{R})} i_n \\ &= \sum_{n \in \mathbb{N}} \left\langle \mathbf{1}_{[0,t]}, S_m \xi_n \right\rangle_{\mathbf{L}_2(\mathbb{R})} i_n \\ &= \sum_{n \in \mathbb{N}} \left(\int_0^t (S_m \xi_n)(u) du \right) i_n. \end{aligned} \quad (6.220)$$

where ξ_n denotes the Hermite functions. Hence,

$$X_m(t) = \sum_{n \in \mathbb{N}} \left(\int_0^t (S_m \xi_n)(u) du \right) T_{i_n}. \quad (6.221)$$

Moreover, the expected value function E can be defined in a similar manner as defined in

the well-known cases, i.e.,

$$E(X_m(t)) = \langle 1, X_m(t)1 \rangle_{\bar{\Lambda}_{(2)}}, \quad (6.222)$$

where 1 is the vacuum state in the FBS space introduced in Section 6.10. Note that $E(X_m(t)) = 0$.

Furthermore, using Eq. (6.173), it can be observed that the covariance, which gives the kernel $K(t, s)$, satisfies

$$E(X_m(t)X_m(s)) = K(t, s) = \langle X_m(t)1, X_m(s)1 \rangle_{\bar{\Lambda}_{(2)}} = \langle f_m(t), f_m(s) \rangle_{\bar{\Lambda}_{(2)}}. \quad (6.223)$$

Hence,

$$K(t, s) = \sum_{n \in \mathbb{N}} \left(\int_0^t (S_m \xi_n)(u) du \right) \left(\int_0^s (S_m \xi_n)(u') du' \right), \quad (6.224)$$

which is equivalent to the kernel defined in Eq. (6.204).

The goal here is to show that the framework just introduced allows the study of the derivatives of stochastic variables as continuous functions. However, a series of results has to be presented before that. To start, observe that the operator M_f is bounded from \mathcal{H}_p into \mathcal{H}_{-p} for every $f \in \mathfrak{S}_{-1}$ with $\|f\|_{\mathcal{H}_{-p}} < \infty$. In fact, if $g \in \mathcal{H}_p$, then, using Eq. (6.192), it holds that

$$\|M_f g\|_{\mathcal{H}_{-p}} = \|fg\|_{\mathcal{H}_{-p}} \leq C_{2p} \|f\|_{\mathcal{H}_{-p}} \|g\|_{\mathcal{H}_p}. \quad (6.225)$$

Also, for every $f \in \mathcal{H}_{-q}$, the operator M_f^* is bounded from \mathcal{H}_p into \mathcal{H}_{-p} , where $q < p$.

Again, letting $g \in \mathcal{H}_p$, this result can be seen from

$$\begin{aligned}
\|M_f^* g\|_{\mathcal{H}_{-p}}^2 &\leq \|M_f^* g\|_{\mathcal{H}_p}^2 = \sum_{\gamma \in \mathcal{I}_0} \left| \sum_{\alpha \vee \gamma = \beta} (-1)^{\sigma(\alpha, \gamma)} f_\alpha g_\beta \right|^2 c_\gamma^{2p} \\
&\leq \sum_{\gamma \in \mathcal{I}_0} \left(\sum_{\alpha \vee \gamma = \beta} |f_\alpha| |g_\beta| \right)^2 c_\gamma^{2p} \\
&\leq \sum_{\alpha, \alpha' \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} \sum_{\substack{\gamma \in \mathcal{I}_0; \exists \beta, \beta' \\ \alpha \vee \gamma = \beta \\ \alpha' \vee \gamma = \beta'}} |g_\beta| c_\beta^p |g_{\beta'}| c_{\beta'}^p \\
&\leq \sum_{\alpha, \alpha' \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-p} |f_{\alpha'}| c_{\alpha'}^{-p} \|g\|_{\mathcal{H}_p}^2 \\
&\leq \left(\sum_{\alpha \in \mathcal{I}_0} |f_\alpha| c_\alpha^{-q} c_\alpha^{-(p-q)} \right)^2 \|g\|_{\mathcal{H}_p}^2 \\
&\leq C_{p-q}^2 \|f\|_{\mathcal{H}_{-q}}^2 \|g\|_{\mathcal{H}_p}^2.
\end{aligned} \tag{6.226}$$

It should be also noted that, for every $f \in \mathfrak{S}_{-1}$, it is a consequence of the last two results presented that the operator $T_f = M_f + M_f^*$ is bounded from \mathfrak{S}_1 into \mathfrak{S}_{-1} . Moreover, there exist $p > q$ such that

$$\|T_f g\|_{-p} \leq 2C_1 \|f\|_{-q} \|g\|_p. \tag{6.227}$$

In fact, observe that, if $f \in \mathcal{H}_{-q}$ and $g \in \mathcal{H}_p$, it holds that

$$\begin{aligned}
\|T_f g\|_{-p} &\leq \|M_f g\|_{-p} + \|M_f^* g\|_{-p} \\
&\leq C_{p-q} \|f\|_{-q} \|g\|_{-p} + C_{p-q} \|f\|_{-q} \|g\|_p \\
&\leq 2C_{p-q} \|f\|_{-q} \|g\|_p \\
&\leq 2C_1 \|f\|_{-q} \|g\|_p,
\end{aligned} \tag{6.228}$$

where Eq. (6.191) was used.

Now, it can be finally shown that the derivatives of the stochastic variables considered here are constant in \mathfrak{S}_{-1} . This fact is stated as follows: Let m be a positive measurable function that satisfies Eqs. (6.217) and (6.205) with $d\sigma(t) = m(t)dt$. Then, for every $g \in \mathfrak{S}_1$, the function $t \mapsto X_m(t)f$ is strongly continuous in \mathfrak{S}_{-1} and there exists a continuous operator $W_m(t)$ from \mathfrak{S}_1 into \mathfrak{S}_{-1} such that

$$\frac{d}{dt}X_m(t)g = W_m(t)g. \quad (6.229)$$

Finally, the function $t \mapsto W_m(t)g$ is continuous from $[a, b]$ into \mathfrak{S}_{-1} .

To prove this result, observe that, since

$$f_m(t) = \sum_{n \in \mathbb{N}} f_n^m(t) i_n, \quad (6.230)$$

with

$$f_n^m = \int_0^t S_m \xi_n(u) du, \quad (6.231)$$

Eq. (6.218) leads to

$$\left\| \frac{d}{dt} f_m(t) \right\|_{\mathcal{H}_{-p}}^2 = \sum_{n \in \mathbb{N}} |S_m \xi_n(t)|^2 c_n^{-2p} \leq \sum_{n \in \mathbb{N}} (D_1 n^{\frac{N+1}{2}} + D_2)^2 c_n^{-2p}. \quad (6.232)$$

Then, for every suitable choice of coefficients c_n , there exists a positive integer p_0 such that, for every $p \geq p_0$, $df_m(t)/dt \in \mathcal{H}_{-p}$. Moreover, for $s = t + h$, where $h \neq 0$ is a real number, there exists p_1 such that,

$$K_p = \sum_{n \in \mathbb{N}} (D_3 n^{\frac{N+2}{2}} + D_4)^2 c_n^{-2p} < \infty, \quad \forall p \geq p_1 \quad (6.233)$$

and, using Eq. (6.219), the \mathcal{H}_{-p} norm of the difference of derivatives satisfies

$$\sum_{n \in \mathbb{N}} |S_m \xi_n(s) - S_m \xi_n(t)|^2 c_n^{-2p} \leq K_{p_1} |h|^2. \quad (6.234)$$

Therefore, $W_m = T_{df_m/dt}$ is a continuous operator from \mathfrak{S}_1 into \mathfrak{S}_{-1} . To see that Eq. (6.229) holds, observe that, for a real number $h \neq 0$ and $g \in \mathcal{H}_p$,

$$\left(\frac{X_m(t+h) - X_m(t)}{h} - W_m(t) \right) g = \sum_{n \in \mathbb{N}} \frac{\int_t^{t+h} (S_m \xi(u) - S_m \xi(t)) du}{h} M_{i_n} g = X_{\Delta(t,h)} g, \quad (6.235)$$

with

$$\Delta(t,h) = \sum_{n \in \mathbb{N}} \frac{\int_t^{t+h} (S_m \xi(u) - S_m \xi(t)) du}{h} i_n. \quad (6.236)$$

Thus, there exists $p > q \geq p_1$ such that, using Eq. (6.227),

$$\left\| X_{\Delta(t,h)} g \right\|_{\mathcal{H}_{-p}} \leq 2C_1 \|\Delta(t,h)\|_{\mathcal{H}_{-q}} \|g\|_{\mathcal{H}_p} \leq \left(2C_1 K_{p_1} \|g\|_{\mathcal{H}_p} \right) |h|^2. \quad (6.237)$$

This shows that $W_m(t) = T_{df_m/dt}(t) \equiv dX_m(t)/dt$ is a continuous operator from \mathfrak{S}_1 into \mathfrak{S}_{-1} . Finally, since a sequence converges in \mathfrak{S}_{-1} , i.e., a perfect space, if and only if it converges in one of the spaces \mathcal{H}_{-p} (as stated earlier in the section), it can be said that the function $t \mapsto W_m(t)g$ is continuous.

Now that the treatment of derivatives is formalized, the next step concerns, for instance, the development of the counterpart of Ito or Malliavin stochastic calculus. The first step in this direction is the introduction of stochastic integrals. This is done in the final result of this dissertation. The stochastic calculus itself will be developed in future works.

Let $t \mapsto Y(t)$, with $t \in [a, b]$, be a continuous \mathfrak{S}_{-1} -valued function in the strong topology of \mathfrak{S}_{-1} . If $g \in \mathfrak{S}_1$, there exists a positive integer p , which depends on g , such that the Pettis

integral

$$\int_a^b Y(t)W_m(t)g dt \tag{6.238}$$

can be computed as a limit of Riemann sums and converges in \mathcal{H}_{-p} .

The proof is similar to the one presented in Ref. [250]. Since the function $t \mapsto W_m(t)g$ is continuous from $[a, b]$ to \mathfrak{S}_{-1} and the product is jointly continuous in \mathfrak{S}_{-1} , the map $t \mapsto Y(t)W_m(t)g$ is also continuous and its image is, therefore, compact in \mathfrak{S}_{-1} . Now, recall that a set is (weakly or strongly) compact in \mathfrak{S}_{-1} , i.e., a perfect space, if and only if it is compact in one of the spaces \mathcal{H}_{-p} (as stated earlier in the section). Then, there exists p such that the image of $t \mapsto Y(t)W_m(t)g$ is in \mathcal{H}_{-p} . This function is also continuous with respect to the topology of \mathcal{H}_{-p} . Then the integral in Eq. (6.238) can be computed in \mathcal{H}_{-p} as a Riemann sum.

To conclude, it should be noticed that, as already discussed, the random variables introduced here have the same expected value and the same covariance of the variables associated with free stochastic processes [233]. Despite that, observe that the product in the integral in Eq. (6.238) is, ultimately, the product of i_α 's, which contrasts with the product of Hermite functions in free stochastic processes. This shows that the processes induced by them are drastically different and should be further explored.

6.13 Discussion

In this chapter, analysis on two different closures of the GA were studied, namely the closure with respect to the 1-norm, $\overline{\Lambda}_{(1)}$, and the closure with respect to the 2-norm, $\overline{\Lambda}_{(2)}$.

In the study of $\overline{\Lambda}_{(1)}$, the notion of positivity and the CK product for power series of a variable in $\overline{\Lambda}_{(1)}$ were introduced. Such ideas were used to study the counterpart of classical problems, such as the one-step extension problem for Toeplitz matrices and the Wiener algebra, and

to begin the development of Schur analysis in this setting.

The extension of these notions from a single complex variable to more general settings, like several complex variables, upper-triangular operators, quaternionic analysis, and bi-complex numbers, has been a source of new problems and methods [86–91]. Each of these settings has some interpretation in terms of signal processing theory and linear systems. For example, time-varying systems correspond to upper-triangular operators, and systems indexed by several indices correspond to function theory in the unit ball of \mathbb{C}^N , also known as the unit polydisk. Furthermore, in all those settings, there exists a natural counterpart of the Hardy space. In the case of the upper-triangular operators, it is the space of Hilbert-Schmidt upper-triangular operators. As to the case of function theory in the unit ball, it is the Drury-Arveson space [84], which differs from the classical Hardy space when the dimension N is greater than 1. In the setting introduced here, the counterpart of the Hardy space is a Wiener-type algebra.

Multiple research directions emerge from the results that were presented in this chapter. For instance, the problem of the one-step extension of Toeplitz matrices in complex analysis, as mentioned in Section 6.3, is associated with stochastic processes and, in particular, with the Yule-Walker equations. Then, it can be questioned if there is a version of those equations in $\overline{\Lambda}_{(1)}$. If there is, are their solution also connected in the same way to the problem of extension of Toeplitz matrices? Moreover, what are the stochastic processes that arise from this problem? Do they have any similarity with the class of processes introduced in Section 6.12?

Furthermore, only a few foundational tools from functional analysis were presented here. There is still a lot to be uncovered — from results that can be translated from other settings to results that fail to hold in $\overline{\Lambda}_{(1)}$.

In the last sections of the chapter, counterparts of notions from the theory of complex analysis

and stochastic processes were developed in $\overline{\Lambda}_{(2)}$. In particular, given the definitions already presented, the Gel'fand triple

$$\mathfrak{S}_1 \subset \overline{\Lambda}_{(2)} \subset \mathfrak{S}_{-1} \tag{6.239}$$

was considered. It should be mentioned that the space \mathfrak{S}_{-1} has an algebra structure of the type that was first introduced by Kondratiev in the setting of Hida's white noise space theory [235], and studied in a more generalized framework in Ref. [240]. There are many parallels (and differences) between the results presented here and other works where the complex numbers were replaced by the commutative algebra of Kondratiev stochastic distributions [235, 251–254]. Here, the main objective was the development of a framework that allows the introduction and analysis of stochastic processes and their derivatives.

The stochastic processes that were used as a basis for the one presented here have the concept of *freeness* — in opposition to independence — associated with their random variables. Also, the distributions associated with them are semi-circles — in opposition to Gaussians. A possible research direction is, then, the investigation of the questions: What is the independence-like concept associated with the random variables defined in this chapter? What are the distributions associated with the stochastic processes generated by them? Moreover, as already mentioned in the previous section, the counterpart of Ito and Malliavin stochastic calculus in the present setting has still to be developed.

Furthermore, another interesting direction is to look into generalizations of the class of processes that was considered here. In Section 6.12, the special cases for which $d\sigma(t) = m(t)dt$ were considered. For a general σ , the operator S_m defined by Eq. (6.208) cannot be introduced. However, it is possible to prove that there exists a continuous positive operator A from the Schwartz space \mathcal{S} into its dual \mathcal{S}' such that

$$\int_{\mathbb{R}} |\widehat{f}(u)|^2 d\sigma(u) = \langle Af, f \rangle_{\mathcal{S}', \mathcal{S}}. \tag{6.240}$$

Since \mathcal{S} is nuclear, the operator A can be factorized via a HS [255]. An explicit construction of A in the form $A = Q_\sigma^* Q_\sigma$, where Q_σ is continuous from \mathcal{S} into $\mathbf{L}_2(\mathbb{R})$, is given in Ref. [256]. It can, then, be investigated whether these results lead to a generalization of the processes presented here.

Finally, due to the importance of Grassmann numbers in quantum field theory and the physical motivation of the complex counterpart of what was introduced here, it can be asked what the applications (if there is any) in real-life problems of the work displayed in this chapter are.

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