1970

Mathematical Modelling of a First Order Transducer

Vijaykumar S. Monie

MATHEMATICAL MODELLING OF A

FIRST ORDER TRANSDUCER

BY

VIJAYKUMAR S. MONIE

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Electrical Engineering, South
Dakota State University

1970

MATHEMATICAL MODELLING OF A

FIRST ORDER TRANSDUCER

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser Date

Head, Electrical Engineering Department Date

2661-11

ACKNOWLEDGEMENTS

The author wishes to record his sincere appreciation to Dr. F. C. Fitchen for making it possible to execute his graduate studies at this school and his deep sense of gratitude to Dr. A. J. Kurtenbach for the excellent advice and guidance received, making this study a most rewarding experience.

V.S.M.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

LIST OF TABLES

CHAPTER ONE

INTRODUCTION

This chapter lays the ground work for this research paper. The problem is posed immediately, enabling the reader to gain a better appreciation of the material that follows. The next article explains the thought process that motivated this problem. This is followed by reviewing the available literature in the field and on which the author has relied quite extensively. The chapter concludes by explaining the more common terminology which will be used frequently throughout the dissertation.

1-1. <u>The Problem:</u>

Given a configuration of linear, passive network elements, termed filter, how faithful a reproduction of the input process is the process at the output of a first order transducer? Or, in other words, how much information is lost in a filter? This problem arises due to the necessity of using a filter in a system --- either for reasons of convenience or by force of circumstances. The paper attempts to model a transducer mathematically and to express the input and output processes statistically. In order to do so, a meaningful measure of error should be chosen and defined. The error measure chosen is the Mean-Integral Square Error (MISER). In other words, the average integral value of the square of the difference between the signal and the pattern[@] is considered to be the important characteristic measuring the quality of the filter, viz.,

$$\epsilon^2 = E\left\{\int_0^T \left[w(t) - x(t)\right]^2\right\}dt \tag{1.1-1}$$

The signal is assumed to be present since antiquity. A very good approximation of the situation is the existence of the signal for a long period of time with respect to the time constants of the filter. The problem then lends itself to what is known as steady state analysis.

This work is restricted to analyzing the pattern given a random Stationary Gaussian Markoff (SGM) process. An expression to calculate the MISER will be developed and a method to computerize the same will be indicated. However, actual numerical computations will be deferred. The expression for MISER will establish a relationship between the location of a pole and/or zero and the magnitude of MISER. The use of the SGM process as the input, enables one to describe output process in a concise manner. It also enables the results of this study to be compared with the results of others in related fields who have used the SGM process as a signal source.

@ Signal is defined as the input process to a transducer and is, usually, inconvenient to measure. Pattern is the output of a transducer conveniently observed and measured.

1-2. Motivation:

The problem stated in the preceding section was primarily motivated by the thought of giving the design engineer and the systems engineer a more gainful insight into systems using transducers and having a random signal source. The analysis of the pattern, given the statistics of the signal, enables one to compute the MISER --- a 'goodness' measuring criterion, and should be welcome tools at one's disposal.

One could conceive a mechanical device excited by several sources. During actual tests one would like to read various measuring instruments, measuring the continuously varying temperature, pressure, power output etc. This could be done by sampling various transducers in the individual circuits at any desired time. Under such conditions, when the input process is on for a considerable length of time before readings are taken, one essentially has a random process approximating the assumptions made in section 1-1. The device could be a major sub-assembly of a space craft, a large ship, an aircraft etc. Admittedly, the signal which has been assumed for this study, namely the SGM process, is hypothetical in nature. Nevertheless, the techniques used in this study could readily be adapted to a case where a random signal with known statistics is used.

There has been little or no direct work done in this area. It is considered important that the design or systems engineer be aware of the error introduced by a transducer in a system. The ready accessibility of a high speed digital computer allows for using good approximating methods for solving integral equations which would, otherwise, be rather laborious to solve. These reasons provided added impetus to undertake this study.

1-3. Review of Literature:

There are several books which treat linear networks in varying depths from the statistical viewpoint. An excellent, though elementary, treatment may be found in Cooper and McGillem's Methods of Signal and System Analysis [1]. This book deals with statistical principles and gives a precise treatment of random signals in linear systems. References [2] and [3] give a more sophisticated approach to linear networks and are thoroughly readable. However, none of these books have dealt directly with the problem of obtaining the MISER for a linear, passive network.

In the paper, "Homogeneous Wiener - Hopf Integral Function", D. C. Youla [4] solves the integral equation which inevitably has to be dealt with in approaching problems of this nature. J. Capon [5] in his paper, approximates the time domain solution of the homogeneous integral equation dealt with in [4].

1-4. Basic Definitions:

The following terms are used freely in this work and are defined here to avoid possible ambiguities.

(1) Correlation Function: It is defined as the expected value of the product of two random variables obtained by time sampling two random functions. This time sampling of two random functions, either periodically or otherwise, generates a set of points in space defining a function. When the two random functions come from the same random process it is called the Autocorrelation function.

$$R_{xx}(t,u) = E\left\{x(t)\ x(u)\right\} \qquad (1.4\text{-}4)$$

The suffixes indicate, in order, the time at which the random process is sampled. In the case of a stationary process, the autocorrelation function is only a function of the time difference [3].

$$R_{xx}(t,\ t+\tau) = E\left\{x(t)\ x(t+\tau)\right\}$$

or

$$R_{xx}(\tau) = E\left\{x(t)\ x(t+\tau)\right\} \qquad (1.4\text{-}2)$$

It is possible to consider the joint statistics of two different random processes $\{x(t)\}$# and $\{y(t)\}$. The cross-correlation function is then defined as

$$R_{xy}(t,u) = E\left\{x(t)\ y(u)\right\} \qquad (1.4\text{-}3)$$

A random process will be denoted by $\{\ \}$.

Again, if the processes are jointly stationary, the cross-correlation function is a function of only a single variable, τ .

$$R_{xy}(\tau) = E\{x(t+\tau)\ y(t)\} \qquad (1.4\text{-}4)$$

The order of the suffixes is important, indicating the time of sampling of the processes in that order.

(2) Completeness: It can be shown that a continuous function x(t) can be approximated by a discrete set of random functions [6] by

$$x_a(t) = \sum_{i=1}^{N} x_i\ \phi_i(t) \qquad 0 \leq t \leq T,$$

in which the N coefficients, x_i, depend only on the function x(t) to be represented but not on time and the N functions of time, $\phi_i(t)$, are specified independently of x(t). The error term is

$$\epsilon_N = \| x(t) - \sum_{i=1}^{N} x_i\ \phi_i(t) \|^{2\ @} \qquad (1.4\text{-}5)$$

@ $\|x\|$, is referred to as the norm or 'length' of the function x.

It is a short hand notation for

$$\|x\|^2 = \int_0^T |x(t)|^2\ dt$$

Observe that if x(t) is regarded as the voltage across a 1-Ohm resistor then $\|x\|^2$, the square of the norm, is the energy dissipated in the resistor in the time interval $[0,T]$.

If the set of $\emptyset_i(t)$ is such that

$$\lim_{N \to \infty} \epsilon_N = 0 \tag{1.4-6}$$

for any function x(t) such that

$$\int_0^T | x(t) |^2 dt < \infty \tag{1.4-7}$$

we say that the set of $\emptyset_i(t)$ is complete on $[0, T]$. Eq. (1.4-6) implies that any function of finite energy can be represented without error in terms of $\emptyset_i$'s. Note that when the $\emptyset_i$'s are a complete set we have "equality" between x(t) and

$$x_a(t) = \sum_{i=1}^{\infty} x_i \emptyset_i(t)$$

in the sense that there is no energy in the error signal $x(t) - x_a(t)$.

(3) Linearity: Assume that the responses to two different inputs $w_1(t)$ and $w_2(t)$ are $x_1(t)$ and $x_2(t)$, respectively. Let c_1 and c_2 denote two constants. A system is linear if the response to $w(t) = c_1 w_1(t) + c_2 w_2(t)$ is $x(t) = c_1 x_1(t) + c_2 x_2(t)$ for all values of w_1, w_2, c_1 and c_2.

(4) Nonanticipative: A system is nonanticipative (causal) if the present output does not depend upon future values of the input.

(5) Periodic and Non-Periodic Processes: A wide-sense stationary random process with sample functions x(t) is said to be

periodic with period T if its correlation function $R(\tau)$ is periodic with period T, i.e., if

$$R(\tau + T) = R(\tau), \quad \forall \tau \qquad (1.4\text{-}8)$$

If all the sample functions of a random process are periodic, or even if all except a set which occurs with probability zero are periodic, the process is periodic in the sense defined above [3]. It can be shown that if x(t) is periodic as defined above, then it can be expanded into a Fourier series [2]:

$$x(t) = \sum_{-\infty}^{\infty} x_n e^{jn\omega_o t}, \qquad (1.4\text{-}9)$$

where $\omega_o = 2\pi / T$

and the coefficients x_n given by

$$x_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_o t}\, dt, \qquad n = 1,2,3,\ldots\ldots\ldots \qquad (1.4\text{-}10)$$

are uncorrelated (and orthogonal) random variables such that,

$$E\{x_n\} = E\{x(t)\} \qquad n=0$$

$$= 0 \qquad n \neq 0 \qquad (1.4\text{-}11)$$

and $$E\{x_n x_m^*\} \overset{@}{=} 0 \qquad m \neq n$$

$$= b_n \qquad m = n \qquad (1.4\text{-}12)$$

@ The asterisk sign, *, when used as a superscript, will denote a conjugate of a function. For example, $x_m^* = (-x_m)$.

where $$b_n = \frac{1}{T} \int_0^T R(\tau)\, e^{-jn\omega_o \tau}\, d\tau \qquad (1.4\text{-}13)$$

If the time functions of the nth and mth terms, $n \neq m$ in (1.4-10) are orthogonal and also if (1.4-12) is true, then the series possesses a double orthogonality.

If $R(\tau)$ is not periodic, then the random variables (x_n; $n = 1,2,\ldots\ldots$) defined by Eq. (1.4-10) where T is a given constant, are no longer orthogonal and the series expansion (1.4-9) is not true for all t. However, it can be shown that (1.4-9) holds for $0 < t < T$ [3] . A non-periodic random process cannot be written as a Fourier series with uncorrelated random coefficients. The Karhunen-Loēve theorem, elaborated in article 2-3, enables one to give an approximate expansion of x(t) into a generalized Fourier series with uncorrelated coefficients.

(6) Stationarity:

(i) Strict Sense: A random process is stationary in the strict sense if its statistics are not affected by a shift in the time origin. This means that the two processes x(t) and $x(t + \epsilon)$ have the same statistics for any ϵ .

Two random processes $\{x(t)\}$ and $\{y(t)\}$ are jointly stationary if the joint statistics of x(t), y(t) are the same as the joint statistics of $x(t + \epsilon)$, $y(t + \epsilon)$ for any ϵ [2] .

(ii) Wide Sense: A random process $\{x(t)\}$ is stationary in the wide sense, if its expected value is a constant and its autocorrelation function depends only on the time difference, τ :

$$E\{x(t)\} = m_x = \text{constant}$$

$$E\{x(t)\ x(t+\tau)\} = R(\tau) \tag{1.4-14}$$

Two processes are jointly stationary in the wide sense if each satisfies (1.4-14) and their cross-correlation depends only on the time difference, τ :

$$E\{x(t+\tau)\ y(t)\} = R_{xy}(\tau) \tag{1.4-15}$$

1-5. Summary:

This chapter has laid down the ground work for further study of the problem. The problem ----- what is the nature of the pattern, given the signal? how is it affected by circuit configuration? how much information is lost in the transducer? ----- was initially posed. A 'goodness' measuring criterion was chosen. This was the mean-integral square error (MISER). It was followed by the motivating thoughts. Next, the literature was reviewed. Finally, common terminology, to be used frequently in this work, were defined and explained.

The following chapter will attempt to analyze the signal and pattern statistically. A general representation of the transducer will be used to achieve the goal. An expression for MISER will be developed in general terms.

CHAPTER TWO

GENERAL PRINCIPLES

This chapter deals with the analysis of the input, output processes in general terms and develops the concepts necessary to appreciate this study. Starting with a discussion of the assumptions made and their reasons, it proceeds to attempt a statistical analysis of the output process. The general transducer is assumed to have a simple pole and a simple zero. The homogeneous Fredholm equation, which arises when the Karhunen-Loēve theorem is used to express a non-periodic random process as a trigonometric Fourier series with uncorrelated random coefficients; is solved and the properties of the resultant eigenvalues and eigenfunctions are enumerated. Lastly, an expression for the MISER is developed.

2-1: Underlying and Simplifying Assumptions:

The first order transducer under consideration is studied by analyzing four resistance and capacitance configurations. In order to achieve the objective, namely to become aware of the nature of the output processes, the following assumptions have been made:

(a) The circuit configurations are linear, deterministic, non-anticipatory and physically realizable. Consequently, the following two conditions suggest themselves:

(i) The impulse response of the network, h(t), is zero for t less than zero, and

(ii)

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

so that the system is stable.

Alternative criteria for a physically realizable system are expounded upon in [7].

(b) The input process is a random Stationary Gaussian Markoff (SGM) process. This implies, among other things, that:

i) The output of the linear system is also a Gaussian process;

ii) If a process is stationary in the wide sense, it is also stationary in the strict sense;

iii) As a consequence of the Markoff process, the autocorrelation function of the input process must be an exponential function [2]. The autocorrelation function of the input process used in this study is $R_{ww}(\tau) = e^{-|\tau|}$

iv) For stationary processes, the autocorrelation function and the power spectral density form a Fourier transform pair.

A concise treatment of the Gaussian process may be found in [3] and [6].

2-2. Modelling of a General First Order Transducer:

Consider a first order transducer represented in figure 2-1. Let the input be a SGM process. Therefore, its autocorrelation function and power spectral density are given by [1] :

$$R_w(\tau) = B\, e^{-\alpha|\tau|} \tag{2.2-1}$$

$$S_w(\omega) = \frac{2\alpha B}{\alpha^2+\omega^2} \tag{2.2-2}$$

Let the transfer function of the system be represented, in general, by

$$H(s) = A\,\frac{c_1 s + e_1}{c_2 s + e_2} \tag{2.2-3}$$

Replacing s by $j\omega$ [8] , we have

$$H(j\omega) = A\,\frac{c_1 j\omega + e_1}{c_2 j\omega + e_2} \tag{2.2-4}$$

The output power spectral density is related to the input spectral density and the transfer function [6]

$$S_x(\omega) = |H(j\omega)|^2\, S_w(\omega) \tag{2.2-5}$$

$$|H(j\omega)|^2 = H(j\omega)\, H^*(j\omega)$$

$$= A^2\left(\frac{c_1^2\omega^2 + e_1^2}{c_2^2\omega^2 + e_2^2}\right)$$

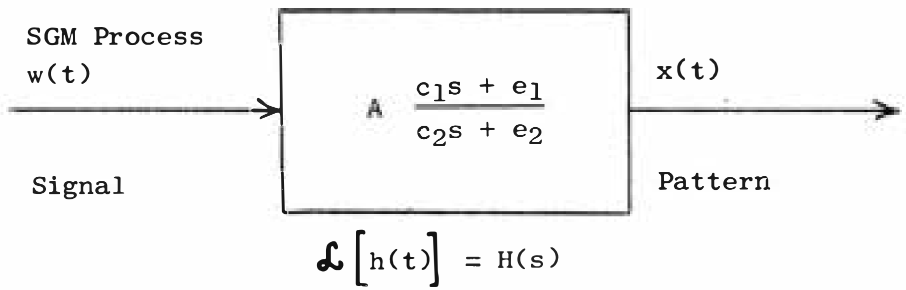


Figure 2-1: Block diagram of a general transducer

In the Laplace domain,

$$S_x(s) = A^2 B \cdot \frac{2\alpha}{\alpha^2 - s^2} \cdot \frac{e_1^2 - c_1^2 s^2}{e_2^2 - c_2^2 s^2}$$

By the technique of partial fractions and a few elementary algebraic manipulations, the following equation is arrived at

$$S_x(s) = A^2 \alpha B \left[\left\{ \frac{1}{\alpha} \left(\frac{e_1^2 - c_1^2 \alpha^2}{e_2^2 - c_2^2 \alpha^2} \right) \right\} \frac{1}{s + \alpha} \right.$$

$$+ \left\{ \frac{1}{c_2 e_2} \left(\frac{c_2^2 e_1^2 - c_1^2 e_2^2}{c_2^2 \alpha^2 - e_2^2} \right) \right\} \frac{1}{s + e_2/c_2}$$

$$+ \left\{ \frac{1}{\alpha} \left(\frac{e_1^2 - c_1^2 \alpha^2}{e_2^2 - c_2^2 \alpha^2} \right) \right\} \frac{1}{-s + \alpha}$$

$$\left. + \left\{ \frac{1}{c_2 e_2} \left(\frac{c_2^2 e_1^2 - c_1^2 e_2^2}{c_2^2 \alpha^2 - e_2^2} \right) \right\} \frac{1}{-s + e_2/c_2} \right] \quad (2.2\text{-}6)$$

Once again using $s=j\omega$ to our advantage and recognizing the fact that the correlation function is the inverse Fourier transform of the spectral density, it can be shown by rather straight forward manipulations that

$$R_x(\tau) = \frac{A^2 \alpha B}{(e_2^2 - c_2^2 \alpha^2)} \left[\frac{1}{\alpha} (e_1^2 - c_1^2 \alpha^2) e^{-\alpha|\tau|} \right.$$

$$\left. - \frac{1}{e_2 c_2} (e_1^2 c_2^2 - c_1^2 e_2^2) e^{-|\tau| e_2/c_2} \right] \quad (2.2\text{-}7)$$

$$R_x(0) = \frac{A^2 \alpha B}{(e_2^2 - c_2^2 \alpha^2)} \left[\frac{1}{\alpha} (e_1^2 - c_1^2 \alpha^2) \right.$$

$$\left. - \frac{1}{c_2 e_2} (c_2^2 e_1^2 - c_1^2 e_2^2) \right] \quad (2.2\text{-}8)$$

Equations (2.2-7) and (2.2-8) give the autocorrelation function of the output process. Notice that for the assumed stationary input, (2.2-7) is only a function of the time interval, τ. There is no simple physical interpretation of the significance of the autocorrelation function in the sense that a given value means a particular thing.

Among the properties of an autocorrelation function are [1] :

(i) The mean square value of the random process can always be obtained by setting $\tau = 0$.

(ii) The autocorrelation function is an even function of τ.

(iii) The largest value of the autocorrelation function always occurs at $\tau = 0$. There may be other values of τ for which it is just as large, but it cannot be larger.

The cross-correlation function is obtained for each case individually. However, it is worth noting the properties which follow [1] :

(i) The quantities $R_{xw}(0)$ and $R_{wx}(0)$ have no particular physical significance and do not represent mean square values. However, $R_{xw}(0) = R_{wx}(0)$.

(ii) Cross-correlation functions are not generally even functions of τ .

(iii) A type of symmetry exists as indicated by $R_{xw}(\tau) = R_{wx}(-\tau)$,

(iv) The cross-correlation function does not necessarily have its maximum value at $\tau = 0$. It can be shown, however, that [3] ;

$$|R_{wx}(\tau)| \leq \left[R_w(0)\ R_x(0)\right]^{\frac{1}{2}} \quad , \quad \forall\ \tau$$

2-3. <u>Solution of a Homogeneous Integral Equation</u>:

A non-periodic random process cannot be written as a trigonometric Fourier series with uncorrelated random coefficients [3] . However, it turns out that if the term, Fourier series, is extended ---as it often is ---to include any series of orthogonal functions $\phi_i(t)$ with coefficients properly determined, then non-periodic processes do have a Fourier series expansion with uncorrelated coefficients.

An expansion for x(t) on an interval a to b of the form

$$x(t) = \sum_{i=1}^{\infty} x_i \, \phi_i(t) \, \sigma_i \quad , \quad a \leq t \leq b \qquad (2.3\text{-}1)$$

where

$$\int_a^b \phi_i(t) \, \phi_j^*(t) \, dt = 1 \qquad \text{if } i = j$$

$$= 0 \qquad \text{if } i \neq j \qquad (2.3\text{-}2a)$$

$$E(x_i \, x_j^*) = 1 \qquad \text{if } i = j$$

$$= 0 \qquad \text{if } i \neq j \qquad (2.3\text{-}2b)$$

and the σ_i are real or complex numbers will be called an orthonormal expansion of the random process on the given interval. The equality in (2.3-1) is to mean precisely that for every t, $a \leq t \leq b$

$$x_t = \underset{N \to \infty}{\text{l.i.m.}^{@}} \sum_{i=1}^{N} \sigma_i x_i \phi_i(t)$$

@ l.i.m. denotes limit in the mean. Suppose that $E(|x_n|^2) < \infty$ for all n and that $E(|x|^2) < \infty$. The sequence of random variables x_n is said to converge in the mean to the random variable x if

$$\lim_{n \to \infty} E(|x_n - x|^2) = 0$$

in which case the abbreviation

$$\underset{n \to \infty}{\text{l.i.m.}} \; x_n = x$$

is used [3].

To obtain the functions $\phi_i(t)$ and the numbers $|\sigma_i|$, one needs to solve [3]

$$\int_a^b R(t, u)\, \phi_i(u)\, du = |\sigma_i|^2\, \phi_i(t) \qquad (2.3\text{-}3)$$

In the language of integral equation, the numbers $|\sigma_i|^2$ must be the eigenvalues and the functions $\phi_i(t)$ must be the eigenfunctions of the homogeneous Fredholm equation

$$\int_a^b R(t,u)\, \phi_i(u)\, du = \mu_i\, \phi_i(t) \qquad (2.3\text{-}4)$$

$$\text{for } a \leq t \leq b;\ i = 1,2,3,\ldots\ldots\ldots,n.$$

Conversely, one can construct an orthogonal expansion valid over any given interval $a \leq t \leq b$ for a random process with a continuous correlation function by using the σ_i 's and $\phi_i(t)$'s of Eq. (2.3-1), the positive square roots of eigenvalues and eigenfunctions of (2.3-4).

The preceding paragraphs form the essence of the Karhunen-Loēve theorem. The expansion given by (2.3-1) is very useful in certain theoretical problems; practically, its usefulness is severely limited by two facts: procedures for finding solutions of integral equations of the form of (2.3-4) are not known in general, and the decomposition of the signal or its power with respect to a set of orthonormal functions (the $\phi_i(t)$) which are not sines and cosines has no simple engineering interpretation.

A sufficient condition that the eigenvalues μ_i be discrete and that the $\phi_i(t)$'s form a complete set is that the (real) kernel R(t,u) be symmetric and positive definite on $(a \leq t,u \leq b)$. Usually, R(t,u) is positive semi-definite and such that at most there may be only a finite number of negative (real) eigenvalues; all other eigenvalues are real and positive. If the μ_i's remain distinct (as well as discrete), Eq. (2.3-4) holds as well for non-symmetric kernels. It may be noted that symmetry is a sufficient condition, not a necessary one; there are non-symmetric kernels where the $\phi_i(t)$'s form a complete orthonormal set [9]. Another useful, sufficient condition that the $\phi_i(t)$'s form a complete set is that the kernel R(t,u) be the Fourier transform of a spectral density, i.e., that $R(t,u) = R(t-u) = F^{-1}\{S(\omega)\}$ [7], [9].

Having made general remarks about the eigenvalues, the following observations can be made for the particular cases covered by this study. The input process is assumed to be a SGM process having a correlation function,

$$R_{ww}(u,\tau) = R_{ww}(u-\tau) = e^{-|u-\tau|}$$

This is a symmetric non-degenerate kernel.

As will be shown in the following chapter, the correlation function of the output process is also stationary, being a function of the time difference, (t-u). Specifically, the following will hold [9]:

(i) The kernel of the output process, $R_{xx}(t,u)$, is a non-degenerate symmetric kernel.

(ii) The homogeneous equation,

$$\int_a^b R(t,u)\, \phi_i(u)\, du = \mu_i\, \phi_i(t)$$

possesses a finite positive number, r, of linearly independent solutions $\phi_1(t)$, $\phi_2(t)$,.........., $\phi_r(t)$.

(iii) Every continuous symmetric kernel that does not vanish identically, possesses eigenvalues and eigenfunctions; their number is denumerably infinite if and only if the kernel is nondegenerate. All eigenvalues of a real symmetric kernel are real.

(iv). The sum of the reciprocals of the squares of the eigenvalues converges.

A brief discussion of the properties of eigenvalues of the homogeneous integral equation may also be found in [3] .

As already shown in section 2-2, the spectral density of the output process is

$$S_x(s) = A^2\, B\, \frac{2\alpha}{\alpha^2 - s^2}\, \frac{e_1{}^2 - c_1{}^2 s^2}{e_2{}^2 - c_2{}^2 s^2}$$

$S_x(s)$, a function of s, can be represented as a ratio of $N(s^2)$ to $D(s^2)$; $N(s^2)$ and $D(s^2)$ being polynomials of degrees m and n in s^2 respectively, viz.,

$$N(s^2) = \sum_{k=0}^{m} a_{2k} s^{2k}$$

$$D(s^2) = \sum_{k=0}^{n} b_{2k}\, s^{2k} , \qquad b_0 \neq 0$$

Using the notations as defined by D. C. Youla in his paper [4] ,

$$N(s^2) = -2\,\alpha\, A^2 B c_1{}^2 s^2 + 2\,\alpha\, A^2 B\, e_1{}^2$$

$$D(s^2) = c_2{}^2 s^4 - s^2(e_2{}^2 + c_2{}^2\,\alpha^2) + e_2{}^2\,\alpha^2$$

Let $K = 2\alpha A^2 B$, therefore,

$$D(s^2) - \lambda_i N(s^2) = c_2{}^2 s^4 - s^2(e_2{}^2 + c_2{}^2\alpha^2) + e_2{}^2\alpha^2 + K\,\lambda_i c_1{}^2 s^2 - K\,\lambda_i e_1{}^2 = 0 \qquad (2.3\text{-}5)$$

Solving (2.3-5) for s^2 gives:

$$s^2 = -\frac{1}{2c_2{}^2}\left[-(e_2{}^2 + c_2{}^2\,\alpha^2) + K\,\lambda_i c_1{}^2\right] \pm \left\{\left[\frac{1}{4c_2{}^4}\left[-(e_2{}^2 + c_2{}^2\,\alpha^2) + K\,\lambda_i c_1{}^2\right]^2\right] - \frac{1}{c_2{}^2}\left[\alpha^2 e_2{}^2 - \lambda_i K\, e_1{}^2\right]\right\}^{\frac{1}{2}} \qquad (2.3\text{-}6)$$

Letting $g(\lambda_i) = \frac{1}{2c_2^2}\left[(e_2^2 + c_2^2\alpha^2) - K\lambda_i C_1^2\right]$

and $f(\lambda_i) = -\frac{1}{c_2^2}\left[\alpha^2 e_2^2 - \lambda_i K e_1^2\right]$

reduces (2.3-6) to:

$$s^2 = g(\lambda_i) \pm \left[g^2(\lambda_i) + f(\lambda_i)\right]^{\frac{1}{2}} \qquad (2.3\text{-}7)$$

The solutions to which are

$$s_1 = +\left\{g(\lambda_i) + \left[g^2(\lambda_i) + f(\lambda_i)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$s_2 = -\left\{g(\lambda_i) + \left[g^2(\lambda_i) + f(\lambda_i)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$s_3 = +\left\{g(\lambda_i) - \left[g^2(\lambda_i) + f(\lambda_i)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$s_4 = -\left\{g(\lambda_i) - \left[g^2(\lambda_i) + f(\lambda_i)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \qquad (2.3\text{-}8)$$

The set of Eqs. (2.3-8) along with the ones preceding it have been computerized. The search technique used to determine the eigenvalues is shown in Appendix D. The program is flexible enough to enable one to determine the eigenvalues for all the cases under study. This point will be elaborated in Chapter Four. The fifty eigenvalues obtained by using the program are arranged in ascending order. The eigenvalues definitely depend upon the location of the zero; taking on larger values as the zero moves away from the origin. λ_i, as used here, is the inverse of the μ_i introduced earlier in this article. The eigenvalues, λ_i, obtained by solving (2.3-4) are seen to increase in value rather rapidly --- the rate of rise definitely being linked to the position of the zero, e_1. It may be pointed out that the eigenvalues obtained from solving Eq. (2.3-5) are the same as those from Eq. (2.3-4) solved in [4]. The increase in the values of the eigenvalues are consistent with Capon's solution [5] of the homogeneous Fredholm equation (2.3-4). It is also observed that the sum of the reciprocals of the squares of the eigenvalues converges.

The solution to Eq. (2.3-4) has two distinct parts --- eigenvalues and eigenfunctions. The method used to determine the eigenvalues has been indicated in the preceding paragraphs. Youla [4] and Capon [5] have solved the Fredholm homogeneous equation in the Laplace and time domain, respectively. However, Capon's method has a major drawback, viz., the eigenfunctions are grossly in error for the first few values of the index, i, which, incidentally, are rather important and substantial in value. Consequently, they cannot be used in computing MISER since their usage leads to completely erroneous results. Youla gives the eigenfunctions in the Laplace domain, reproduced below as:

$$\phi_i(x) \overset{@}{\doteqdot} \frac{P(s, \lambda_i)\, D^+(s)}{D(s^2) - \lambda_i N(s^2)} \tag{2.3-9}$$

where $S_x(s) \equiv \dfrac{N(s^2)}{D(s^2)}$

$$D(s^2) = D^+(s)\, D^-(s)$$

$$D^-(s) = D^+(-s)$$

$$P(s) = \sum_{k=0}^{n-1} p_k\, s^k$$

The p_k's being determined from

$$\sum_{k=0}^{n-1} \left[1 \mp (-1)^k x_r\right] \omega_r^{\,k} \, (\lambda_i)\, p_k = 0 \tag{2.3-10}$$

with the x_r defined as

$$x_r = e^{-\omega_r T} \frac{D^-(\omega_r)}{D^+(\omega_r)}$$

$$r = 1, 2, 3, \ldots\ldots\ldots, n.$$

$$i = 1, 2, 3, \ldots\ldots, \ldots\ldots, \ldots.$$

Note that the $\phi_i(x)$ defined by (2.3-9) are orthogonal but not orthonormal over $0 \leq x \leq T$. An expression for the normalizing constant can be derived [10] , but its complexity increases as the

---

@ The notation $\doteqdot$ will be used to denote the Laplace and Fourier transform, as the case may be. The pertinent domain will be explicit from the context.

order of $D(s^2)$ in s^2 increases beyond one. Eq. (2.3-9) will be used to compute MISER in the following chapters. The correctness of (2.3-9) was tested by analyzing a short circuited filter. This short circuited filter was also analyzed by using Capon's method. The two are tabulated in Table C.1, Appendix C.

A possible objection to the use of Youla's expression may be that the determination of the inverse Laplace transform, in certain cases, could be exceedingly difficult. Numerical techniques may overrule this objection to a certain extent. However, it must be emphasized that analytical solution is to be preferred, as then the normalizing constant could be calculated to a great degree of accuracy. Numerical techniques employed to obtain the inverse of Laplace transforms would be adequate enough to give an approximate value of the normalizing constant. This aspect will be dealt with in greater detail in article 4-2.

2-4. A General Expression for MISER:

Consider the block diagram illustrated in Figure 2-2. Let w(t) be the signal source to the first order transducer, and let x(t) be the pattern process at the output of the transducer. The signal source is assumed to be present since antiquity. Recall that a very good approximation to this situation is the presence of the signal for a considerable length of time compared to the time constant

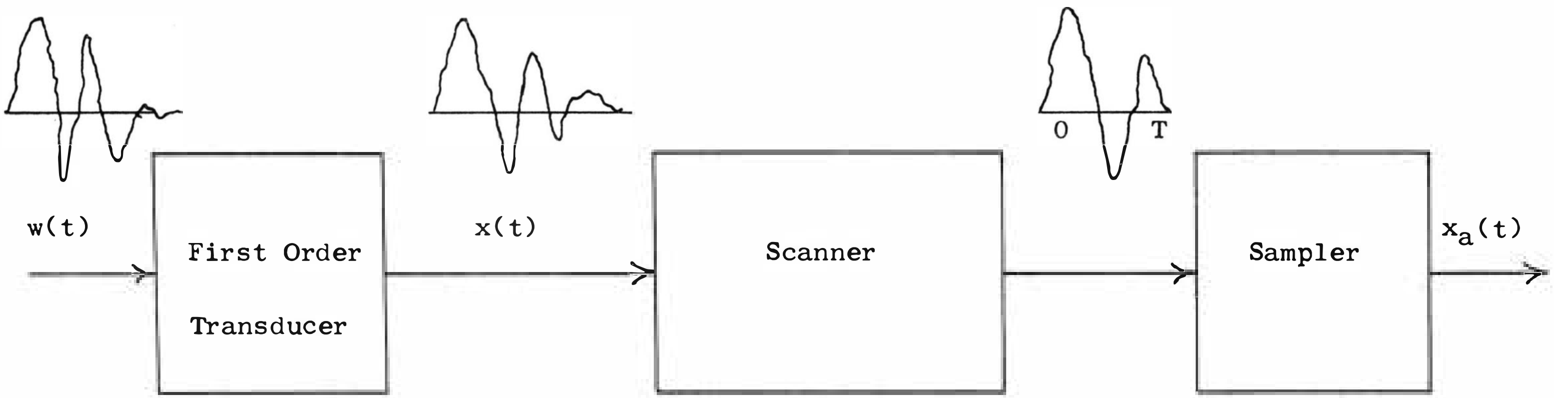


Figure 2-2: Block diagram of the system under study

of the transducer. Thus, a random process present since antiquity will be available at the output of the transducer. The transducer pattern, $x(t)$, is then fed into a scanner which continually scans the input during the time interval $[0,T]$. The output of the scanner is then sampled by a sampler, The scanner may be a step switch which will reproduce the input to the scanner during the interval $[0,T]$. The sampler could be an analog-to-digital converter or just an ordinary commutator with the arm rotating at a constant speed. The system illustrated in Figure 2-2 is representative of a part of the data transmission process. The signal process could be an electro-cardiogram (ECG), for example, or the weather conditions during a certain time of a year etc. In this case $x(t)$ will differ from $w(t)$ by the amount of information lost in the transducer. The scanner will reproduce $x(t)$ within a desired interval $[0,T]$. This continuous process is sampled and its approximations, $x_a(t)$, will be transmitted.

The set of discrete coefficients, $x(t)$, would represent to an observer the output of the transducers. One can then approximate $x(t)$ by a set of discrete coefficients over the interval $[0,T]$ as

$$x_a(t) = \sum_{i=1}^{n} x_i \emptyset_i(t) , \qquad 0 \leq t \leq T. \qquad (2.4\text{-}1)$$

in which the n coefficients of x_i depend only on the function $x(t)$ to be represented but not on time and the n functions of time, $\emptyset_i(t)$, are specified independently of x(t), [6]. The $\phi_i(t)$'s are picked such that they are orthonormal over the interval $[0,T]$. Under these conditions, it may be noted, that the best approximation in the least squares sense is obtained when x_i is taken as the projection of x(t) on $\phi_i(t)$ in the interval $[0,T]$, [11].

The MISER is

$$\epsilon^2 = E\left\{\int_0^T \left[w(t) - x_a(t)\right]^2 dt\right\} \tag{2.4-2}$$

$$= E\int_0^T \left[w(t) - x(t) + x(t) - x_a(t)\right]^2 dt$$

$$= E\int_0^T \left[w(t) - x(t)\right]^2 dt + 2E\int_0^T \left[w(t) - x(t)\right]\left[x(t) - x_a(t)\right] dt$$

$$+ E\int_0^T \left[x(t) - x_a(t)\right]^2 dt \tag{2.4-3}$$

Taking the expectation within the integral sign (See Appendix A), expanding the terms and performing some rather simple integrations, we obtain:

$$E\int_0^T \left[w(t) - x(t)\right]^2 dt = R_{ww}(0) - 2R_{wx}(0) + R_{xx}(0)$$

$$E\int_0^T \left[w(t) - x(t)\right]\left[x(t) - x_a(t)\right] dt = R_{wx}(0) - R_{xx}(0)$$

$$-E\left\{\sum_{i=1}^{n} x_i w_i\right\} + E\left\{\sum_{i=1}^{n} x_i^2\right\}$$

$$E\int_0^T \left[x(t) - x_a(t)\right]^2 dt = R_{xx}(0) - E\left\{\sum_{i=1}^{n} x_i^2\right\}$$

To obtain these one is required to interchange the integral and the summation signs [12]. Also, the property of orthonormality of the $\emptyset_i(t)$'s was invoked.

Adding the preceding three equations we have, after simplification;

$$\epsilon^2 = R_{ww}(0) - 2E\left\{\sum_{i=1}^{n} x_i w_i\right\} + E\left\{\sum_{i=1}^{n} x_i^2\right\} \qquad (2.4\text{-}4)$$

It can readily be shown that

$$E\left\{x_i^2\right\} = \int_0^T\int_0^T R_{xx}(t,u)\, \emptyset_i(t)\, \emptyset_i(u)\, dt\, du$$

and

$$E\left\{x_i w_i\right\} = \int_0^T\int_0^T R_{xw}(t,u)\, \emptyset_i(t)\, \emptyset_i(u)\, dt\, du$$

by using advantageously the fact that the output process is related to the input process through a convolution integral. In other words,

$$x(t) = \int_{-\infty}^{t} w(\tau)\, h(t-\tau)\, d\tau \qquad t \in [-\infty, T]$$

$$= \int_{-\infty}^{T} h(t-\tau) w(\tau)\, d\tau \qquad t > T$$

or

$$x(t) = \int_0^{t} w(t-\tau)\, h(\tau)\, d\tau \qquad t \in [-\infty, T]$$

$$= \int_{t-T}^{t} w(t-\tau)\, h(\tau)\, d\tau \qquad t > T$$

The $\emptyset_i(t)$'s expressed by the Eqs. (2.3-9) are valid over the range $0 \leq t \leq T$ only. Also, although the signal is assumed present since antiquity, it is being sampled in real time; the limits of integration are $[0,T]$. Consequently, (2.4-4) reduces to

$$\epsilon^2 = R_{ww}(0) - 2 \sum_{i=1}^{n} \int_0^T \int_0^T R_{xw}(t,u)\, \phi_i(t)\, \phi_i(u)\, dt\, du$$

$$+ \sum_{i=1}^{n} \int_0^T \int_0^T R_{xx}(t,u)\, \phi_i(t)\, \phi_i(u)\, dt\, du \qquad (2.4\text{-}5)$$

Being able to calculate each term in the above equation enables one to calculate the MISER --- a positive number less than unity. Observe that the signal source will be sensed in voltage units and the sampled output will also be identified in the same units.

2-5. Summary:

This chapter dealt with the basic principles of the techniques to be used later in this study. It was pointed out that the study is restricted to physically realizable, linear, deterministic, non-anticipatory systems. As a consequence of this, it was shown that the impulse response of the network must be finite. The signal is an assumed SGM process. The pattern was then statistically analyzed and certain properties of the correlation functions, considered important, were emphasized. The Fredholm homogeneous integral equation was then solved and the nature of the resultant eigenvalues and eigenfunctions was studied. Finally, a general formula for the

MISER was developed. The use of the Karhunen-Loēve theorem implied that the error obtained would be minimal.

CHAPTER THREE

STATISTICAL ANALYSIS

This chapter analyzes four different configurations of resistances and capacitances. The input in each case is the same, namely, the Stationary Gaussian - Markoff process. The output process, in each case is statistically analyzed; the correlation functions are determined and carefully studied. Salient features are noted and commented upon. The chapter concludes with a comparison of the results in each case.

Consider as an input the SGM process with the autocorrelation function as

$$R_{ww}(u - t) = e^{-|u - t|}, \qquad (3.1\text{-}1)$$

$$t, u \in (-\infty, T).$$

The cross-correlation function is defined as the expected value of the product of the random variables obtained by time sampling the input and output random processes. Noting that the pattern is related to the signal through the impulsive response of the filter, it may readily be shown that

$$R_{xw}(t,u) = E\left\{x(t)\, w(u)\right\} = \int_{-\infty}^{t} R_{xx}(u-\tau)\, h(t-\tau)\, d\tau \qquad (3.1\text{-}2)$$

The auto-correlation function of the pattern, defined as the expected value of the product of two random variables obtained by time sampling the pattern at different instants, is

$$R_{xx}(t,u) = E\left\{x(t)\,x(u)\right\}$$

$$= E\left\{\int_{-\infty}^{t} w(\alpha)\,h(t-\alpha)\,d\alpha \int_{-\infty}^{u} w(\beta)\,h(u-\beta)\,d\beta\right\}$$

$$= \int_{-\infty}^{t}\int_{-\infty}^{u} R_{ww}(\alpha,\beta)\,h(t-\alpha)\,h(u-\beta)\,d\alpha\,d\beta\ ^{@}$$

$$\alpha \in (-\infty,\ t);\quad \beta \in (-\infty,u) \qquad (3.1\text{-}3)$$

These equations will be used in the analysis of the cases which follow and, therefore, are derived before we can begin the analysis.

3-1a. Analysis of Case(A):

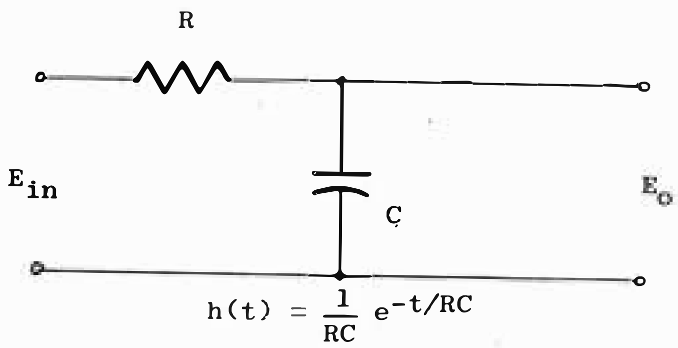
Consider a first order transducer represented by the R - C filter illustrated in figure 3.1(a). The impulse response of this low-pass filter is

$$h(t) \triangleq \frac{1}{RC}\,e^{-t/RC}$$

$$= \sigma\,e^{-\sigma t} \qquad (3.1\text{-}4)$$

where σ is the pole of the transfer function. Note that in the practical circuit under consideration, no pole can exist at origin.

@ See Appendices A and B



(With reference to Fig. 2-1; $A = 1$; $c_1 = 0$, $e_1 = 1/RC$, $c_2 = 1$, $e_2 = 1/RC$.)

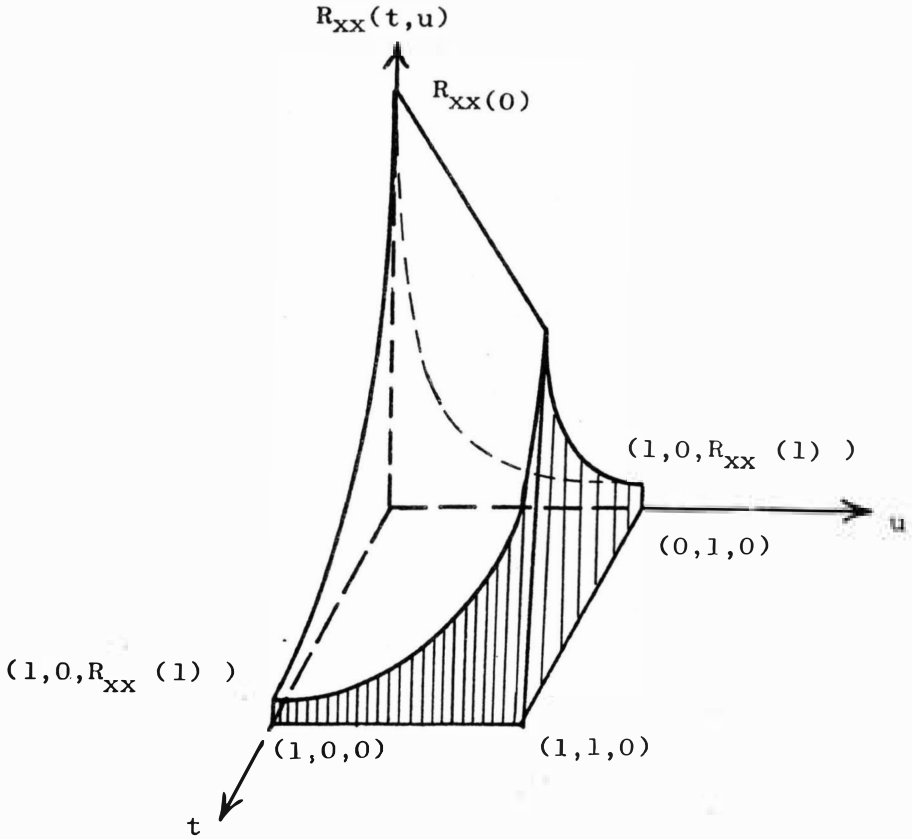


Figure 3-1. Case(A): a) Low pass filter and b) Pattern autocorrelation function.

3-1b. Correlation Functions:

The cross-correlation function is found by using Eq. (3.1-2).

Therefore,

$$R_{xw}(t,u) = \int_{-\infty}^{t} e^{-|u-\tau|} \sigma e^{-\sigma(t-\tau)} d\tau$$

$$= \int_{-\infty}^{o} e^{-|u-\tau|} \sigma e^{-\sigma(t-\tau)} d\tau$$

$$+ \int_{o}^{t} e^{-|u-\tau|} \sigma e^{-\sigma(t-\tau)} d\tau , \quad (u,t > 0)$$

Let $u > t > 0$, then

$$R_{xw}(t,u) = \sigma \int_{-\infty}^{o} e^{-(u-\tau)} e^{-\sigma(t-\tau)} d\tau$$

$$+ \sigma \int_{o}^{t} e^{-(u-\tau)} e^{-\sigma(t-\tau)} d\tau$$

Evaluation of the integrals gives

$$R_{xw}(t,u) = \left(\frac{\sigma}{\sigma + 1} \right) e^{(t-u)} \qquad (3.1\text{-}5)$$

Let $t > u > 0$; then

$$R_{xw}(t,u) = \int_{-\infty}^{o} e^{-(u-\tau)} \sigma e^{-\sigma(t-\tau)} d\tau$$

$$+ \int_{o}^{u} e^{-(u-\tau)} \sigma e^{-\sigma(t-\tau)} d\tau$$

$$+ \int_{u}^{t} e^{(u-\tau)} \sigma e^{-\sigma(t-\tau)} d\tau$$

Upon solving these integrals, one obtains

$$R_{xw}(t,u) = \left(\frac{\sigma}{\sigma - 1}\right) e^{-(t-u)} - \left(\frac{2\sigma}{\sigma^2 - 1}\right) e^{-\sigma(t-u)} \qquad (3.1\text{-}6)$$

The auto-correlation function of the pattern is calculated by using Eq. (3.1-3). Substituting appropriate values of the impulse responses, one gets

$$R_{xx}(t,u) = \sigma^2 \int_{-\infty}^{t} \int_{-\infty}^{u} e^{-|\alpha-\beta|} e^{-\sigma(t-\alpha)} e^{-\sigma(u-\beta)} d\alpha \, d\beta$$

Let $u > t > 0$, $\alpha \in (-\infty, t)$, $\beta \in (-\infty, u)$, then

$$R_{xx}(t,u) = \int_{-\infty}^{t} \sigma^2 e^{-\sigma(t-\alpha)} d\alpha \left\{ \int_{-\infty}^{\alpha} e^{-(\alpha-\beta)} e^{-\sigma(u-\beta)} d\beta \right.$$

$$\left. + \int_{\alpha}^{u} e^{(\alpha-\beta)} e^{-\sigma(u-\beta)} d\beta \right\}$$

Evaluation of these integrals by recognizing that t,u are parameters as far as α and β are concerned, yields

$$R_{xx}(t,u) = \left(\frac{\sigma^2}{\sigma^2 - 1}\right) \left[e^{(t-u)} - \frac{1}{\sigma} e^{\sigma(t-u)} \right] \qquad (3.1\text{-}7)$$

Let $t > u > 0$, $\alpha \in (-\infty, t)$, and $\beta \in (-\infty, u)$, then

$$R_{xx}(t,u) = \int_{-\infty}^{u} \sigma^2 e^{-\sigma(u-\beta)} d\beta \Big\{ \int_{-\infty}^{\beta} e^{(\alpha-\beta)} e^{-\sigma(t-\alpha)} d\alpha$$

$$+ \int_{\beta}^{t} e^{-(\alpha-\beta)} e^{-\sigma(t-\alpha)} d\alpha \Big\}$$

Proceeding on similar lines, we have

$$R_{xx}(t,u) = \left(\frac{\sigma^2}{\sigma^2 - 1}\right) \left[e^{-(t-u)} - \frac{1}{\sigma} e^{-\sigma(t-u)} \right] \qquad (3.1\text{-}8)$$

Eqs. (3.1-7) and (3.1-8) may be rewritten in a concise manner as

$$R_{xx}(t,u) = \frac{\sigma^2}{\sigma^2 - 1} \left[e^{-|t-u|} - \frac{1}{\sigma} e^{-\sigma|t-u|} \right], \quad t,u > 0 \qquad (3.1\text{-}9)$$

A superficial study of Eqs. (3.1-5), (3.1-6) and (3.1-9) indicates what may have been anticipated --- that the output process is also stationary in the wide sense, being solely dependent upon the time difference, (t-u). Furthermore, the cross-correlation function is not an even function --- consistent with the general properties stated in article 2-2.

Representing (t-u) by Δ, Eq. (3.1-9) becomes

$$R_{xx}(\Delta) = \left(\frac{\sigma^2}{\sigma^2 - 1}\right) \left[e^{-|\Delta|} - \frac{1}{\sigma} e^{-\sigma|\Delta|} \right] \qquad (3.1\text{-}9a)$$

having a maximum value of

$$R_{xx}(0) = \frac{\sigma}{\sigma + 1} \qquad (3.1\text{-}10)$$

at the origin. An interesting case is when $\sigma \longrightarrow 1$. Under this condition,

$$R_{xx}(0) = \frac{1}{2}$$

and for any other $\Delta \neq 0$, (3.1-9a) becomes a so-called indeterminate expression [13] . This may be solved by taking the derivatives of the numerator and denominator an equal number of times with respect to σ, till a finite or an infinite answer, in the limit as $\sigma \rightarrow 1$, is obtained. This approach leads to

$$R_{xx}(\Delta) = e^{-|\Delta|} \left(\frac{1 + |\Delta|}{2} \right) \qquad (3.1\text{-}11)$$

The value of the auto-correlation function at other locations of the pole, σ, may be obtained by direct substitution in (3.1-9).

The general nature of $R_{xx}(t,u)$ is shown in figure 3.1(b). The figure illustrates the general variation of $R_{xx}(t,u)$ for an assumed σ. The peak is $R_{xx}(0)$. The profile on the $[R_{xx}(t,u);\ u]$ plane is obtained by evaluating $R_{xx}(0,u)$. Similarly, the profile on the $[R_{xx}(t,u);\ t]$ plane is obtained by evaluating $R_{xx}(t,0)$. The nature of the auto-correlation function in space and the profiles on t = 0, u = 0 planes are seen to decay exponentially. Notice also, that the auto-correlation function is symmetrical about a plane perpendicular to the (t,u) plane and equi-angular (45°) to the other two planes.

This is not surprising --- it is merely a confirmation of the properties stated in article 2-2.

Eq. (3.1-9a) was analyzed for the variation of the autocorrelation function with changes in the location of the pole, σ. Figure 3-2 shows graphically this variation. Observe that the autocorrelation function approaches a value of 0.6 asymptotically. The time difference, $|t-u|$, was held constant at 0.5.

3-2a. Analysis of Case(B):

The first order transducer represented by the C - R filter in figure 3-3(a) is a high pass filter having an impulse response of

$$h(t) = \delta(t) - \frac{1}{RC} e^{-t/RC}$$

$$= \delta(t) - \sigma e^{-\sigma t} \qquad (3.2\text{-}1)$$

where σ is the pole of the transfer function. Once again, a pole at the origin is physically impossible. Observe that the location of the zero, γ, is at the origin and is independent of the location of the pole, σ. Therefore, the pole and the zero can never be coincident.

3-2b. Correlation Functions:

Using Eq. (3.1-2), the cross-correlation function is

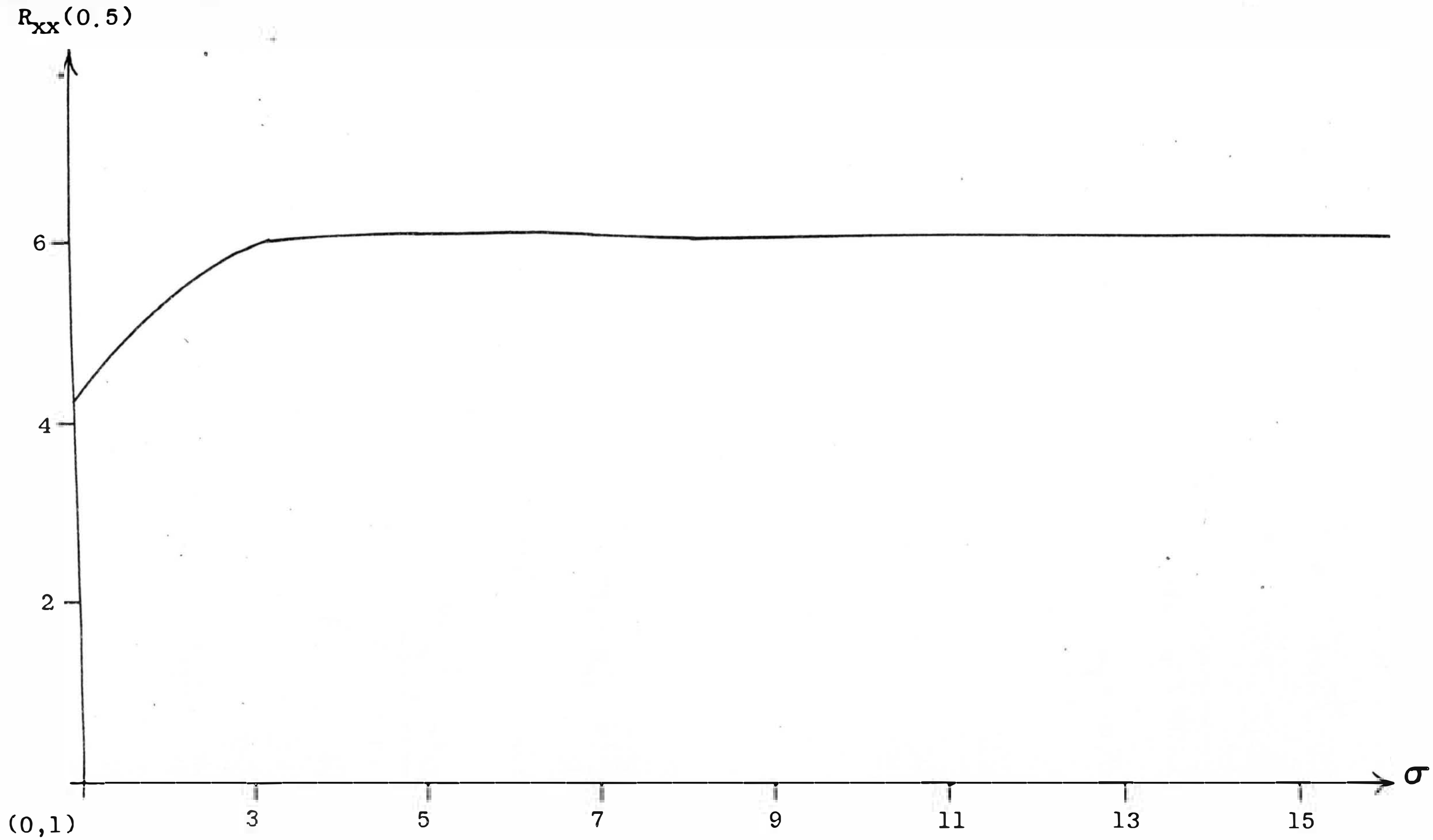
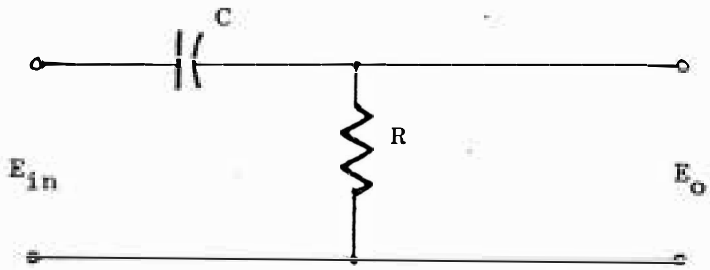


Figure 3-2. Variation of the pattern autocorrelation function, case (A).



$h(t) = \delta(t) - (1/RC)\, e^{-t/RC}$

(With reference to Fig. 2-1; $A = 1$, $c_1 = 1$, $e_1 = 0$, $c_2 = 1$, and $e_2 = 1/RC$.)

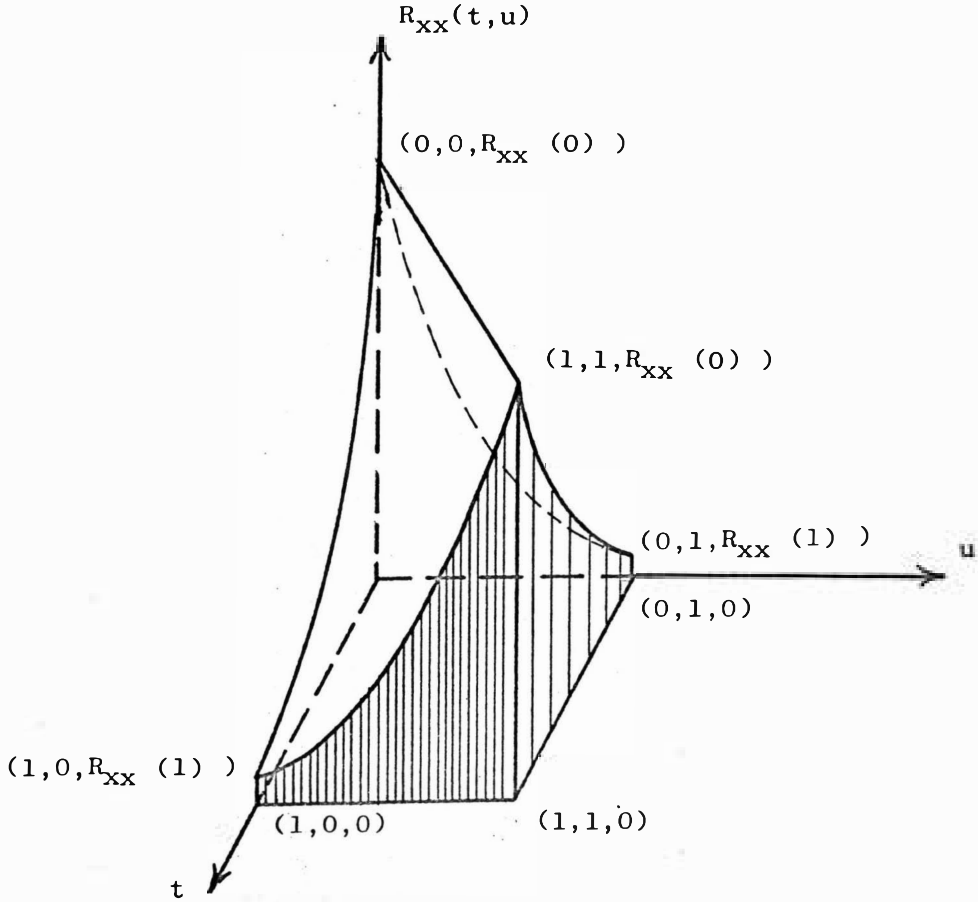


Figure 3-3. Case(B): a) High pass filter and b) Pattern autocorrelation function.

$$R_{xw}(t,u) = \int_{-\infty}^{t} e^{-|u-\tau|} \left[\delta(t-\tau) - \sigma e^{-\sigma(t-\tau)} \right] d\tau$$

$$u,\ t > 0$$

Let $u > t > 0$, then

$$R_{xw}(t,u) = \int_{-\infty}^{0} e^{-(u-\tau)} \delta(t-\tau)\, d\tau + \int_{0}^{t} e^{-(u-\tau)} \delta(t-\tau)\, d\tau$$

$$- \int_{-\infty}^{0} e^{-(u-\tau)} \sigma e^{-\sigma(t-\tau)}\, d\tau - \int_{0}^{t} e^{-(u-\tau)} \sigma e^{-\sigma(t-\tau)}\, d\tau$$

Solution of the integrals gives

$$R_{xw}(t,u) = \left(\frac{1}{\sigma + 1} \right) e^{(t-u)} \tag{3.2-2}$$

Let $t > u > 0$, then

$$R_{xw}(t,u) = \int_{-\infty}^{0} e^{-(u-\tau)} \left[\delta(t-\tau) - \sigma e^{-\sigma(t-\tau)} \right] d\tau$$

$$+ \int_{0}^{u} e^{-(u-\tau)} \left[\delta(t-\tau) - \sigma e^{-\sigma(t-\tau)} \right] d\tau$$

$$+ \int_{u}^{t} e^{(u-\tau)} \left[\delta(t-\tau) - \sigma e^{-\sigma(t-\tau)} \right] d\tau$$

Evaluation of these integrals yields

$$R_{xw}(t,u) = \left(\frac{2\sigma}{\sigma^2 - 1} \right) e^{-\sigma(t-u)} - \left(\frac{1}{\sigma - 1} \right) e^{-(t-u)} \tag{3.2-3}$$

The auto-correlation function of the pattern is calculated by using Eq. (3.1-3). Substituting appropriate values of the impulse responses, one gets

$$R_{xx}(t,u) = \int_{-\infty}^{t} \int_{-\infty}^{u} e^{-|\alpha - \beta|} \left[\delta(t-\alpha) - \sigma e^{-\sigma(t-\alpha)} \right] \times \left[\delta(u-\beta) - \sigma e^{-\sigma(u-\beta)} \right] d\alpha \; d\beta$$

Letting $u > t > 0$, $\alpha \in (-\infty, t)$ and $\beta \in (-\infty, u)$; we obtain

$$R_{xx}(t,u) = \int_{-\infty}^{t} \left[\delta(t-\alpha) - \sigma e^{-\sigma(t-\alpha)} \right] \left\{ \int_{-\infty}^{\alpha} e^{-(\alpha-\beta)} \left[\delta(u-\beta) - \sigma e^{-\sigma(u-\beta)} \right] d\beta + \int_{\alpha}^{u} e^{(\alpha-\beta)} \left[\delta(u-\beta) - \sigma e^{-\sigma(u-\beta)} \right] d\beta \right\} d\alpha$$

Evaluation of these integrals yields

$$R_{xx}(t,u) = \left(\frac{\sigma}{\sigma^2 - 1} \right) e^{\sigma(t-u)} - \left(\frac{1}{\sigma^2 - 1} \right) e^{(t-u)} \qquad (3.2\text{-}4)$$

Let $t > u > 0$, $\alpha \in (-\infty, t)$ and $\beta \in (-\infty, u)$, then

$$R_{xx}(t,u) = \int_{-\infty}^{u} \left\{ \delta(u-\beta) - \sigma e^{-\sigma(u-\beta)} \right\} \left[\int_{-\infty}^{\beta} e^{(\alpha-\beta)} \left\{ \delta(t-\alpha) - \sigma e^{-\sigma(t-\alpha)} \right\} d\alpha + \int_{\beta}^{t} e^{-(\alpha-\beta)} \left\{ \delta(t-\alpha) - \sigma e^{-\sigma(t-\alpha)} \right\} d\alpha \right] d\beta$$

Proceeding on similar lines, we have

$$R_{xx}(t,u) = \frac{\sigma}{\sigma^2 - 1} e^{-\sigma(t-u)} - \frac{1}{\sigma^2 - 1} e^{-(t-u)} \qquad (3.2\text{-}5)$$

Equations (3.2-4) and (3.2-5) may be rewritten in a concise manner as

$$R_{xx}(t,u) = \left(\frac{\sigma}{\sigma^2 - 1}\right)\left(e^{-\sigma|t-u|} - \frac{1}{\sigma} e^{-|t-u|}\right) \qquad (3.2\text{-}6)$$

$$t, u > 0$$

The general remarks made on page 39 , also hold here. Representing (t-u) by Δ, Eq. (3.2-6) becomes

$$R_{xx}(\Delta) = \left(\frac{\sigma}{\sigma^2 - 1}\right)\left(e^{-\sigma|\Delta|} - \frac{1}{\sigma} e^{-|\Delta|}\right) \qquad (3.2\text{-}6a)$$

having a maximum value of

$$R_{xx}(0) = \frac{1}{\sigma + 1} \qquad (3.2\text{-}7)$$

at the origin. As in the previous case consider the situation as $\sigma \longrightarrow 1$. Under this condition

$$R_{xx}(0) = \frac{1}{2}$$

and for any other $\Delta \neq 0$, (3.2-6a) becomes a so-called indeterminate expression [13] . Using the technique advocated in case(A), we have

$$R_{xx}(\Delta) = e^{-|\Delta|}\left[\frac{1-|\Delta|}{2}\right] \qquad (3.2\text{-}8)$$

The auto-correlation function may be determined at any other location of a pole by direct substitution in (3.2-6). The general nature of $R_{xx}(t,u)$ is shown in figure 3.3(b). Observations made in connection with figure 3.1(b) are also valid for this case.

Eq. (3.2-6a) was studied for the variation of the autocorrelation function with changes in the position of the pole, σ. Figure 3-4 illustrates this variation. Note that the auto-correlation function approaches zero value as the pole is moved away from origin. Absolute value of (t-u) was held constant at 0.5.

3-3a. Analysis of Case(C):

Consider the integrator circuit represented by the network in figure 3.5(a). The impulse response of this integrator is

$$h(t) = \frac{\sigma}{\gamma}\left[\delta(t) + \sigma k e^{-\sigma t}\right]$$

where

$$\sigma = \frac{R_2\gamma}{R_1 + R_2} = \frac{1}{1+k}\cdot\gamma$$

$$\gamma = \frac{1}{R_2 C}$$

k = dimensionless ratio of R_1 to R_2

σ, and γ are, respectively, the pole and zero of the transfer function. Once again, note that neither the pole nor the zero can be

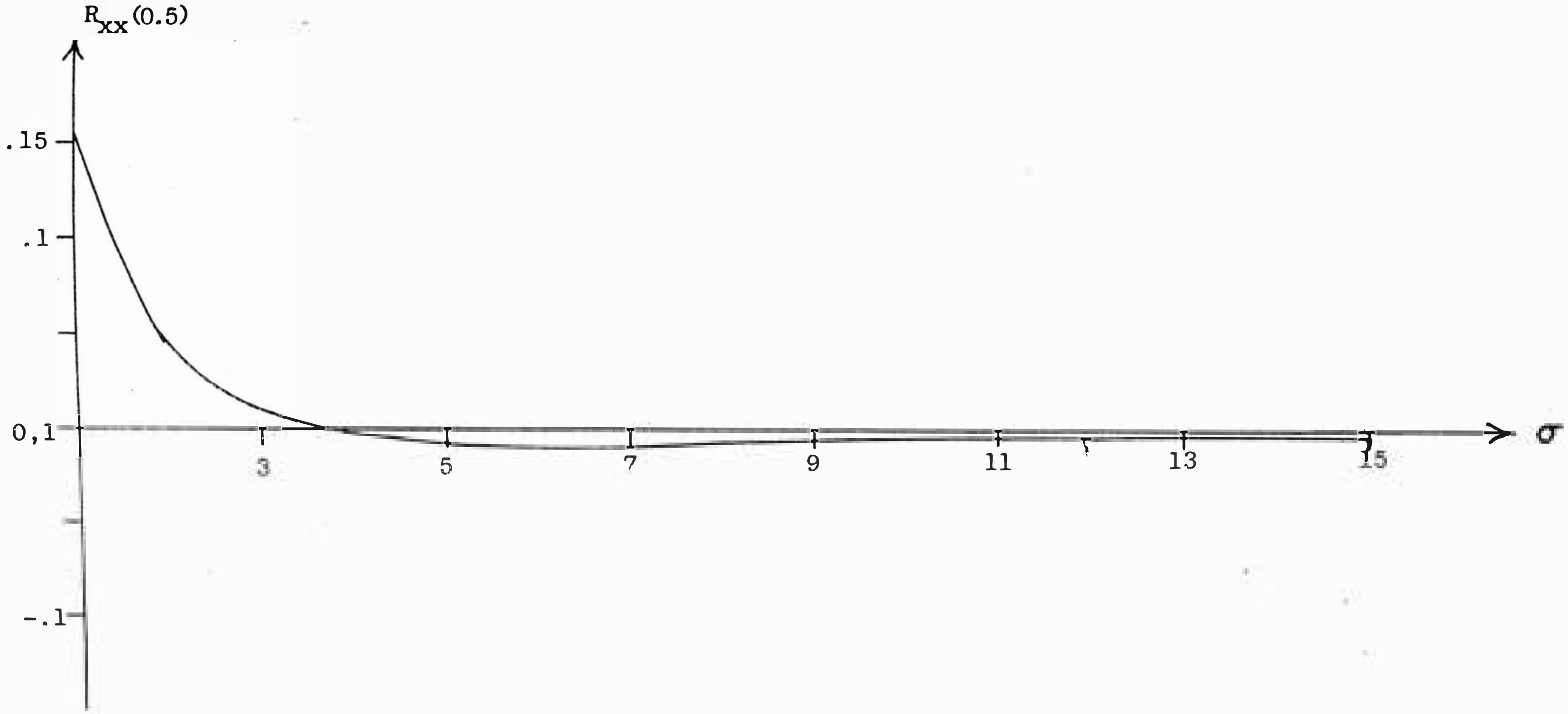
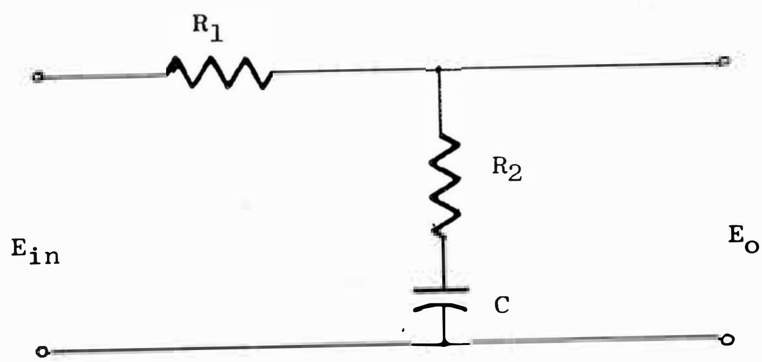


Figure 3-4. Variation of the pattern autocorrelation function, case(B).



(With reference to Fig. 2-1: $A = R_2/(R_1 + R_2)$; $c_1 = 1$; $e_1 = 1/R_2C$; $c_2 = 1$; $e_2 = 1/(R_1 + R_2)C$)

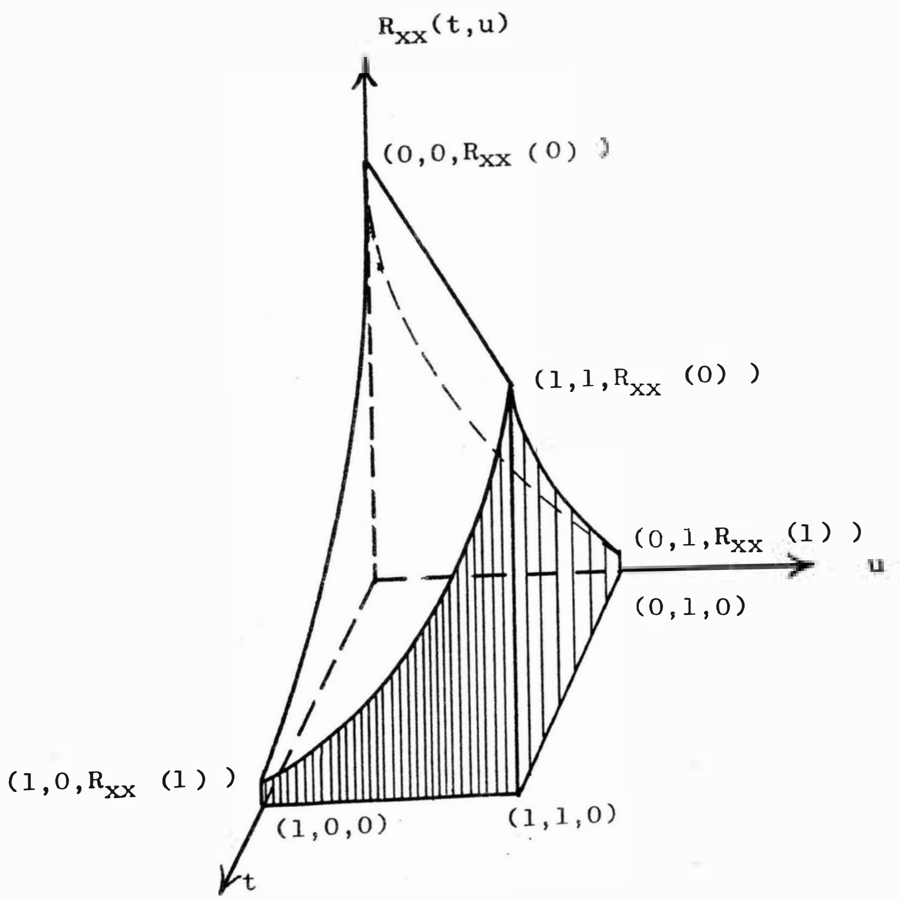


Figure 3-5. Case(C): a) Integrator circuit and b) Pattern autocorrelation function.

located at the origin. It is also interesting to note that $k \geq 0$, and therefore the location of the pole would be determined completely by the value of k and the location of the zero, γ. In other words, the pole will lie within the zero; both being coincident only when $k = 0$.

3-3b. Correlation Function:

Using Eq. (3.1-2), the cross-correlation function is

$$R_{xw}(t,u) = \int_{-\infty}^{t} e^{-|u-\tau|} \frac{\sigma}{\gamma} \left[\delta(t-\tau) + \sigma k e^{-\sigma(t-\tau)} \right] d\tau, \qquad t,u > 0.$$

Let $u > t > 0$, then

$$\begin{aligned} R_{xw}(t,u) = & \int_{-\infty}^{0} e^{-(u-\tau)} \frac{\sigma}{\gamma} \delta(t-\tau)\, d\tau \\ & + \int_{0}^{t} e^{-(u-\tau)} \frac{\sigma}{\gamma} \delta(t-\tau)\, d\tau \\ & + \frac{\sigma k}{\gamma} \left[\int_{-\infty}^{0} e^{-(u-\tau)} . \sigma . e^{-\sigma(t-\tau)}\, d\tau \right. \\ & \left. + \int_{0}^{t} e^{-(u-\tau)} . \sigma . e^{-\sigma(t-\tau)}\, d\tau \right] \end{aligned}$$

Evaluation of these integrals gives

$$R_{xw}(t,u) = \frac{\sigma}{\gamma} e^{(t-u)} + \frac{\sigma k}{\gamma} \left(\frac{\sigma}{\sigma + 1} \right) e^{(t-u)} \qquad (3.3\text{-}2)$$

Let $t > u > 0$, then

$$R_{xw}(t,u) = \frac{\sigma}{\gamma} \int_{-\infty}^{0} e^{-(u-\tau)} \left[\delta(t-\tau) + k\sigma e^{-\sigma(t-\tau)}\right] d\tau$$

$$+ \frac{\sigma}{\gamma} \int_{0}^{u} e^{-(u-\tau)} \left[\delta(t-\tau) + k\sigma e^{-\sigma(t-\tau)}\right] d\tau$$

$$+ \frac{\sigma}{\gamma} \int_{u}^{t} e^{+(u-\tau)} \left[\delta(t-\tau) + k\sigma e^{-\sigma(t-\tau)}\right] d\tau$$

Upon solving these integrals, one gets

$$R_{xw}(t,u) = \frac{\sigma}{\gamma} \left(\frac{\sigma(k+1)-1}{\sigma-1}\right) e^{-(t-u)} - \frac{k\sigma}{\gamma}\left(\frac{2\sigma}{\sigma^2-1}\right) e^{-\sigma(t-u)} \tag{3.3-3}$$

The auto-correlation function of the pattern is calculated by using Eq. (3.1-3). Substituting appropriate values of the impulse responses, one gets

$$R_{xx}(t,u) = \int_{-\infty}^{t} \int_{-\infty}^{u} e^{-|\alpha-\beta|}\left[\left\{\delta(t-\alpha) + k\sigma e^{-\sigma(t-\alpha)}\right\} \frac{\sigma}{\gamma}\right]$$

$$\times \left[\frac{\sigma}{\gamma} \left\{\delta(u-\beta) + k\sigma e^{-\sigma(u-\beta)}\right\}\right] d\alpha . d\beta$$

Letting $u > t > 0$; $\alpha \in (-\infty, t)$, and $\beta \in (-\infty, u)$ we obtain

$$R_{xx}(t,u) = \int_{-\infty}^{t} \left[\frac{\sigma}{\gamma} \left\{ \delta(t-\alpha) + k\sigma e^{-\sigma(t-\alpha)} \right\} \right] \times$$

$$\left[\int_{-\infty}^{\alpha} e^{-(\alpha-\beta)} \frac{\sigma}{\gamma} \left\{ \delta(u-\beta) + k\sigma e^{-\sigma(u-\beta)} \right\} d\beta \right.$$

$$\left. + \int_{\alpha}^{u} e^{(\alpha-\beta)} \frac{\sigma}{\gamma} \left\{ \delta(u-\beta) + k\sigma e^{-\sigma(u-\beta)} \right\} d\beta \right] d\alpha$$

Evaluation of these integrals yields

$$R_{xx}(t,u) = \xi e^{(t-u)} - \Omega e^{\sigma(t-u)} \qquad (3.3\text{-}4)$$

Letting $t > u > 0$; $\alpha \in (-\infty, t)$ and $\beta \in (-\infty, u)$ we obtain

$$R_{xx}(t,u) = \int_{-\infty}^{u} \left[\frac{\sigma}{\gamma} \left\{ \delta(u-\beta) + k\sigma e^{-\sigma(u-\beta)} \right\} \right] \times$$

$$\left[\int_{-\infty}^{\beta} e^{(\alpha-\beta)} \frac{\sigma}{\gamma} \left\{ \delta(t-\alpha) + k\sigma e^{-\sigma(t-\alpha)} \right\} d\alpha \right.$$

$$\left. + \int_{\beta}^{t} e^{-(\alpha-\beta)} \frac{\sigma}{\gamma} \left\{ \delta(t-\alpha) + k\sigma e^{-\sigma(t-\alpha)} \right\} d\alpha \right] d\beta$$

Proceeding on similar lines we have

$$R_{xx}(t,u) = \xi e^{-(t-u)} - \Omega e^{-\sigma(t-u)} \qquad (3.3\text{-}5)$$

where $\Omega = \left(\dfrac{k\sigma^2}{\gamma^2} \right) \left(\dfrac{2\sigma}{\sigma^2 - 1} \right) \left(\dfrac{2 + k}{2} \right)$

and $$\xi = \frac{\sigma^2}{\gamma^2}\left\{1 + k \cdot \frac{2\sigma^2}{\sigma^2 - 1} + k^2 \cdot \frac{\sigma^2}{\sigma^2 - 1}\right\}$$

Equations (3.3-4) and (3.3-5) may be rewritten in a concise manner as

$$R_{xx}(t,u) = \xi\, e^{-|t-u|} - \Omega\, e^{-\sigma|t-u|} \qquad (3.3\text{-}6)$$

$$t,u > 0$$

The general comments made on page 39, also hold here. Representing (t-u) by Δ, Eq. (3.3-6) becomes

$$R_{xx}(t,u) = \xi\, e^{-|\Delta|} - \Omega\, e^{-\sigma|\Delta|} \qquad (3.3\text{-}6a)$$

having a maximum value of

$$R_{xx}(0) = (\xi - \Omega) = \frac{\sigma^2}{\gamma^2}\left\{1 + \frac{2k\sigma}{\sigma+1} + \frac{k^2\sigma}{\sigma+1}\right\} \qquad (3.3\text{-}7)$$

at the origin. As in the previous cases consider $\sigma \to 1$. Under this condition

$$R_{xx}(0) = \frac{1}{\gamma^2}\left(\frac{2 + 2k + k^2}{2}\right) \qquad (3.3\text{-}7a)$$

Note that γ and k can take on only non-zero, positive values. Therefore, $R_{xx}(0)$ depends upon the position of the zero, γ, of the network --- in fact, $R_{xx}(0)$ varies inversely as the square of γ for a given k. For any other $\Delta \neq 0$, (3.3-6a) becomes a so-called indeterminate expression [13]. Using the technique advocated in case(A), we have

$$R_{xx}(\Delta) = \frac{e^{-|\Delta|}}{2\gamma^2}\left[2 + 2k(1 + |\Delta|) + k^2(1 - |\Delta|)\right] \quad (3.3\text{-}8)$$

The auto-correlation function may be determined at any other location of the pole by direct substitution in (3.3-6). The general nature of $R_{xx}(t,u)$ is shown in Figure 3.5(b). Observations made in connection with Figure 3.1(b) are, once again, valid.

Eq. (3.3-6a) was analyzed for the variation of the auto-correlation function with changes in the location of the pole and the zero. The three dimensional variation is illustrated in Figure 3-6. Recall that in this case the pole is located within the zero and in the limit, the two may be coincident. As a result the plane described by Eq. (3.3-6a) is of the nature shown in Figure 3-6. Figure 3-7 shows the variation of the variance of the pattern with respect to the changes in the location of the pole. A study of Eq. (3.3-7) results in Figure 3-7. Pole was held at γ = 25 for this study.

3-4a. Analysis of Case(D):

The first order transducer represented in Figure 3-8(a), is a differentiator circuit having an impulse response of

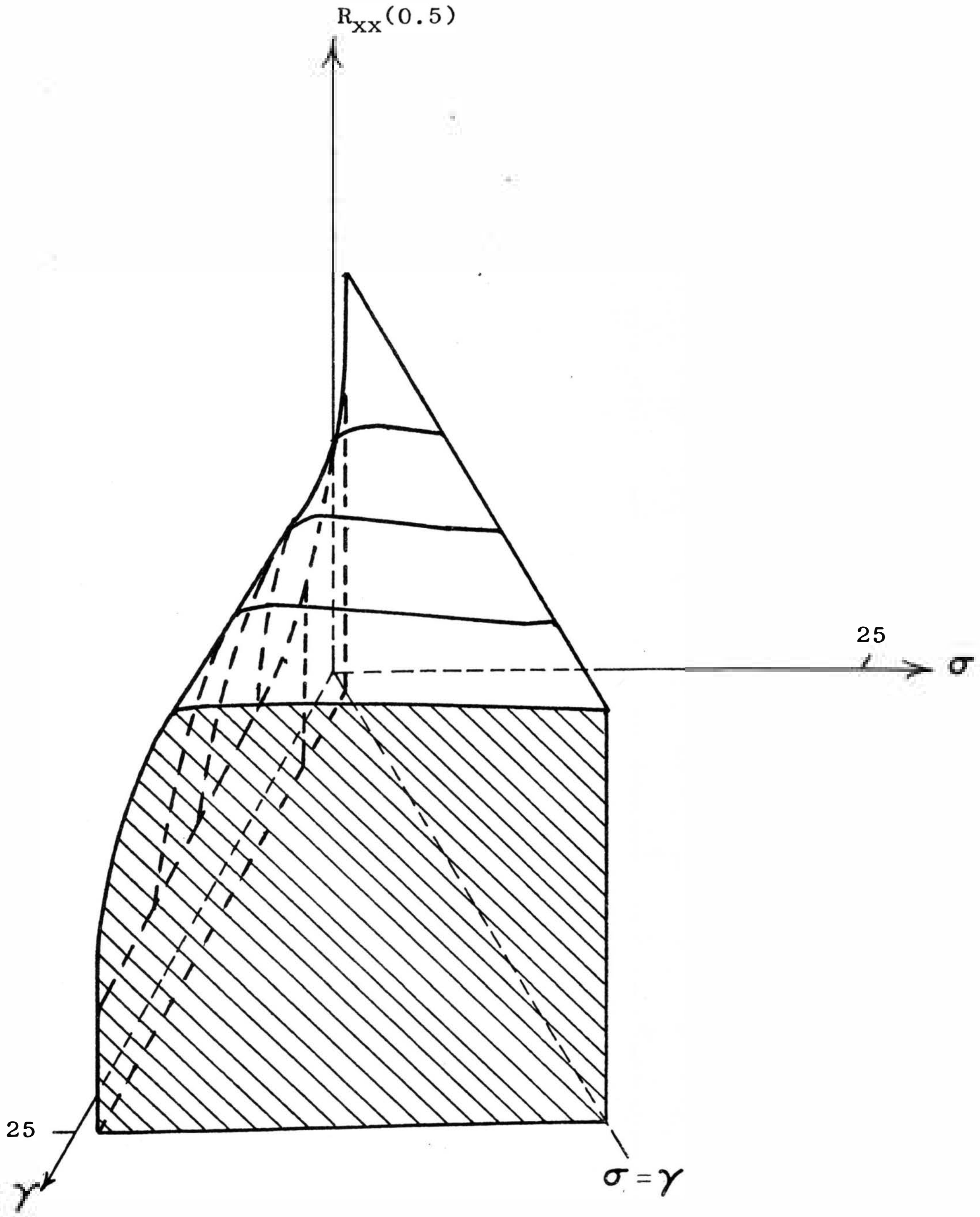


Figure 3-6. Variation of the pattern autocorrelation function, case(C).

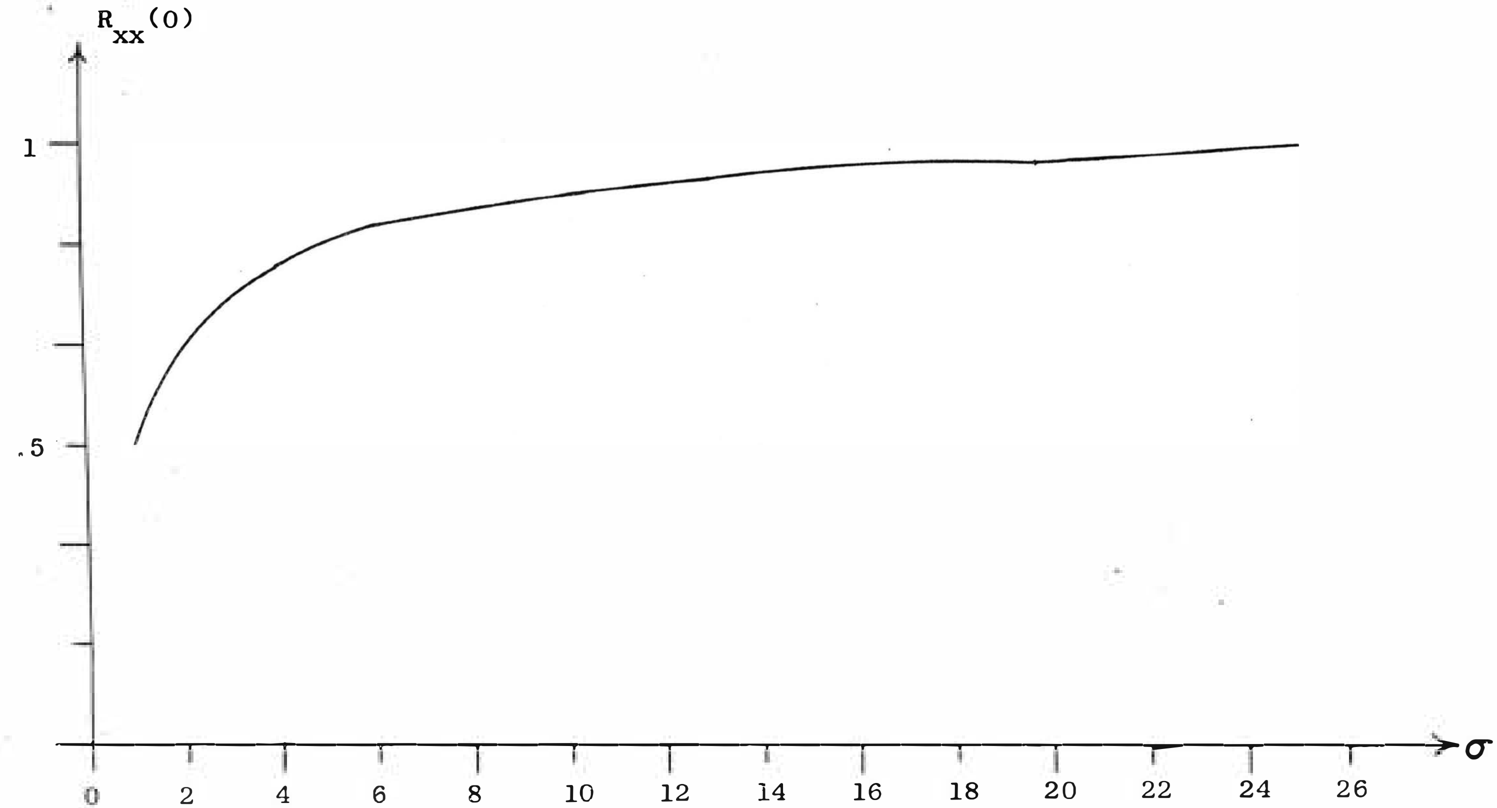
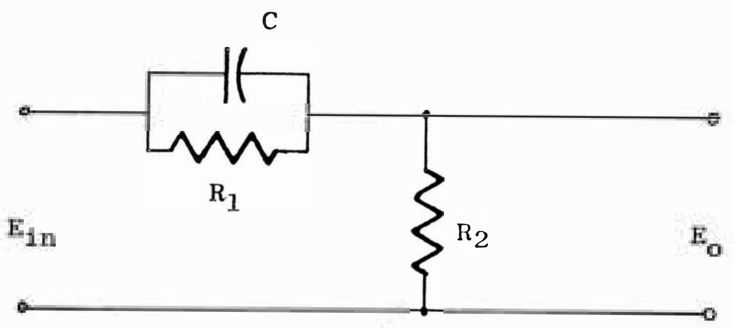


Figure 3-7. Case(C): Variance of the pattern



(With reference to Fig. 2-1: $A = 1$, $c_1 = 1$, $e_1 = 1/R_1C$; $c_2 = 1$; $e_2 = (R_1 + R_2)/R_1R_2C$)

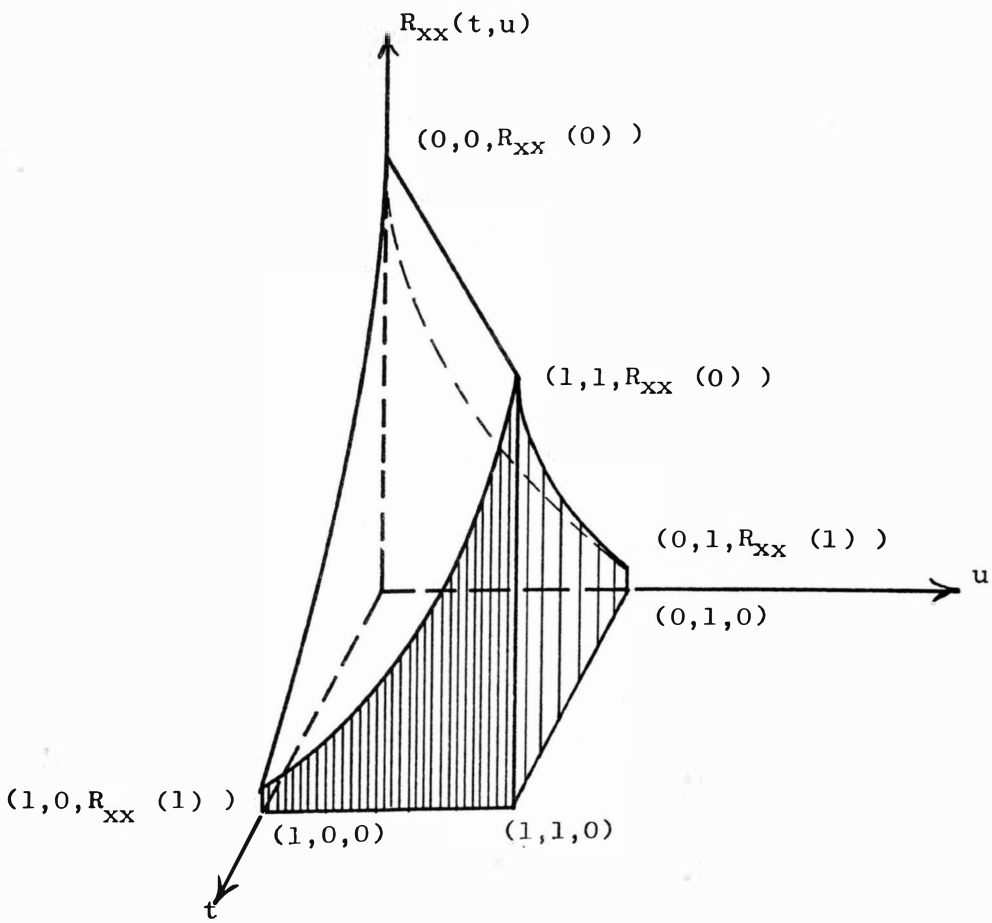


Figure 3-8. Case(D): a) Differentiator circuit and b) Pattern autocorrelation function.

$$h(t) = \delta(t) - \gamma e^{-\sigma t} \tag{3.4-1}$$

where $\sigma = \left(\frac{R_1+R_2}{R_1}\right)\gamma$

and $\gamma = \frac{1}{R_2 C}$

The parameters σ and γ are, respectively, the pole and the zero of the transfer function. Since no zero can exist at the origin, the location of the pole is not physically possible. In this case $\sigma \geq \gamma$, the equality holding only when R_2 assumes zero value. The pole, σ, will, consequently, lie outside the zero, γ. The location of the pole and zero will coincide when the output is short circuited.

3-4b. <u>Correlation Function:</u>

Using Eq. (3.1-2), the cross-correlation function is

$$R_{xw}(t,u) = \int_{-\infty}^{t} e^{-|u-\tau|}\left\{ \delta(t-\tau) - \gamma\, e^{-\sigma(t-\tau)}\right\} d\tau, \qquad t,u>0.$$

Let $u>t>0$, then

$$R_{xw}(t,u) = \int_{-\infty}^{o} e^{-(u-\tau)}\ \delta(t-\tau)\, d\tau + \int_{o}^{t} e^{-(u-\tau)}\ \delta(t-\tau)\, d\tau$$

$$- \gamma \int_{-\infty}^{o} e^{-(u-\tau)}\, e^{-\sigma(t-\tau)}\, d\tau - \gamma \int_{o}^{t} e^{-(u-\tau)}\, e^{-\sigma(t-\tau)}\, d\tau$$

Evaluation of these integrals gives

$$R_{xw}(t,u) = e^{(t-u)} - \frac{\gamma}{\sigma}\left(\frac{\sigma}{\sigma+1}\right) e^{(t-u)} \tag{3.4-2}$$

Let $t > u > 0$, then

$$R_{xw}(t,u) = \int_{-\infty}^{0} e^{-(u-\tau)} \left[\delta(t-\tau) - \gamma e^{-\sigma(t-\tau)}\right] d\tau$$

$$+ \int_{0}^{u} e^{-(u-\tau)} \left[\delta(t-\tau) - \gamma e^{-\sigma(t-\tau)}\right] d\tau$$

$$+ \int_{u}^{t} e^{+(u-\tau)} \left[\delta(t-\tau) - \gamma e^{-\sigma(t-\tau)}\right] d\tau$$

Upon solving these integrals, one obtains

$$R_{xw}(t,u) = \left(\frac{2\gamma}{\sigma^2 - 1}\right) e^{-\sigma(t-u)} + \left(1 - \frac{\gamma}{\sigma - 1}\right) e^{-(t-u)} \tag{3.4-3}$$

The auto-correlation function of the pattern is calculated by using Eq. (3.1-3). Substituting appropriate values of the impulse responses, one gets

$$R_{xx}(t,u) = \int_{-\infty}^{t}\int_{-\infty}^{u} e^{-|\alpha-\beta|}\left[\delta(t-\alpha) - \gamma e^{-\sigma(t-\alpha)}\right] \times$$

$$\left[\delta(u-\beta) - \gamma e^{-\sigma(u-\beta)}\right] d\alpha\, d\beta$$

Letting $u > t > 0$, $\alpha \in (-\infty, t)$, and $\beta \in (-\infty, u)$, we have

$$R_{xx}(t,u) = \int_{-\infty}^{t} \left[\delta(t-\alpha) - \gamma e^{-\sigma(t-\alpha)}\right] \times$$

$$\int_{-\infty}^{\alpha} e^{-(\alpha-\beta)} \left[\delta(u-\beta) - \gamma e^{-\sigma(u-\beta)}\right] d\beta$$

$$+ \int_{\alpha}^{u} e^{(\alpha-\beta)} \left[\delta(u-\beta) - \gamma e^{-\sigma(u-\beta)}\right] d\beta \quad d\alpha$$

Solution of these integrals yields

$$R_{xx}(t,u) = \eta e^{\sigma(t-u)} + \zeta e^{(t-u)} \qquad (3.4\text{-}4)$$

Letting $t > u > 0$, $\alpha \in (-\infty, t)$ and $\beta \in (-\infty, u)$, we have

$$R_{xx}(t,u) = \int_{-\infty}^{u} \left[\delta(u-\beta) - \gamma e^{-\sigma(u-\beta)}\right] \times$$

$$\left\{ \int_{-\infty}^{\beta} e^{(\alpha-\beta)} \left[\delta(t-\alpha) - \gamma e^{-\sigma(t-\alpha)}\right] d\alpha \right.$$

$$\left. + \int_{\beta}^{t} e^{-(\alpha-\beta)} \left[\delta(t-\alpha) - \gamma e^{-\sigma(t-\alpha)}\right] d\alpha \right\} d\beta$$

Proceeding on similar lines, we have

$$R_{xx}(t,u) = \eta e^{-\sigma(t-u)} + \zeta e^{-(t-u)} \qquad (3.4\text{-}5)$$

where $\eta = \left(\dfrac{2\gamma}{\sigma^2 - 1}\right)\left(1 - \dfrac{\gamma}{2\sigma}\right)$

and $\zeta = \left(1 - \dfrac{2\gamma\sigma}{\sigma^2 - 1} + \dfrac{\gamma^2}{\sigma^2 - 1}\right)$

Equations (3.4-4) and (3.4-5) may be rewritten in a concise manner as

$$R_{xx}(t,u) = \eta\, e^{-\sigma|t-u|} + \zeta\, e^{-|t-u|}, \qquad t,u > 0. \tag{3.4-6}$$

The general comments made on page 39, also apply here. Representing (t-u) by Δ, Eq. (3.4-6) becomes

$$R_{xx}(\Delta) = \eta\, e^{-\sigma|\Delta|} + \zeta\, e^{-|\Delta|} \tag{3.4-6a}$$

having a maximum value of

$$R_{xx}(0) = (\eta+\zeta) = \frac{\gamma^2 + \sigma(1-2\gamma) + \sigma^2}{\sigma\,(\sigma+1)} \tag{3.4-7}$$

at the origin. As in the previous cases, consider $\sigma \longrightarrow 1$. Under this condition

$$R_{xx}(0) = \frac{\gamma^2 - 2\gamma + 2}{2} \tag{3.4-7a}$$

Note that since γ can only take on non-zero real, positive values; the auto-correlation function for t=u will always be positive. For any other $\Delta \neq 0$, (3.4-6a) becomes a so-called indeterminate expression [13]. Using the technique advocated in case(A), we have

$$R_{xx}(\Delta) = e^{-|\Delta|}\left(\frac{\gamma^2 - 2\gamma(|\Delta|+1) + 2}{2}\right) \tag{3.4-8}$$

The auto-correlation function may be determined at any other location of the pole by direct substitution in (3.4-6). The general nature of $R_{xx}(t,u)$ is shown in Figure 3.8(b). Observations made in connection with Figure 3.1(b) are, once again, valid.

Analysis of Eq. (3.4-6a) by assuming $|\Delta| = 0.5$; results in Figure 3-9 which illustrates a plane described by moving the pole and the zero away from the origin. As already noted the pole should lie outside the zero and, therefore, the plane occupies one-half of the (γ, σ) plane. Variation of the variance with respect to the changes in the location of the pole is illustrated in Figure 3-10. It is a graphical representation of Eq. (3.4-7). Zero was held at $\gamma = 1$ for the study.

3-5. Comments

Four different configurations of resistors and capacitors have been analyzed in the preceding articles. Studying the results closely, the following general comments can be made:

(i) With the signal an assumed SGM process, the pattern in each case is stationary in the wide sense. By virtue of the fact that the signal has Gaussian characteristics, the pattern is also stationary in the strict sense and will have a Gaussian density function. Consequently, the pattern is fully specified in the statistical sense if the mean and the variance are known.

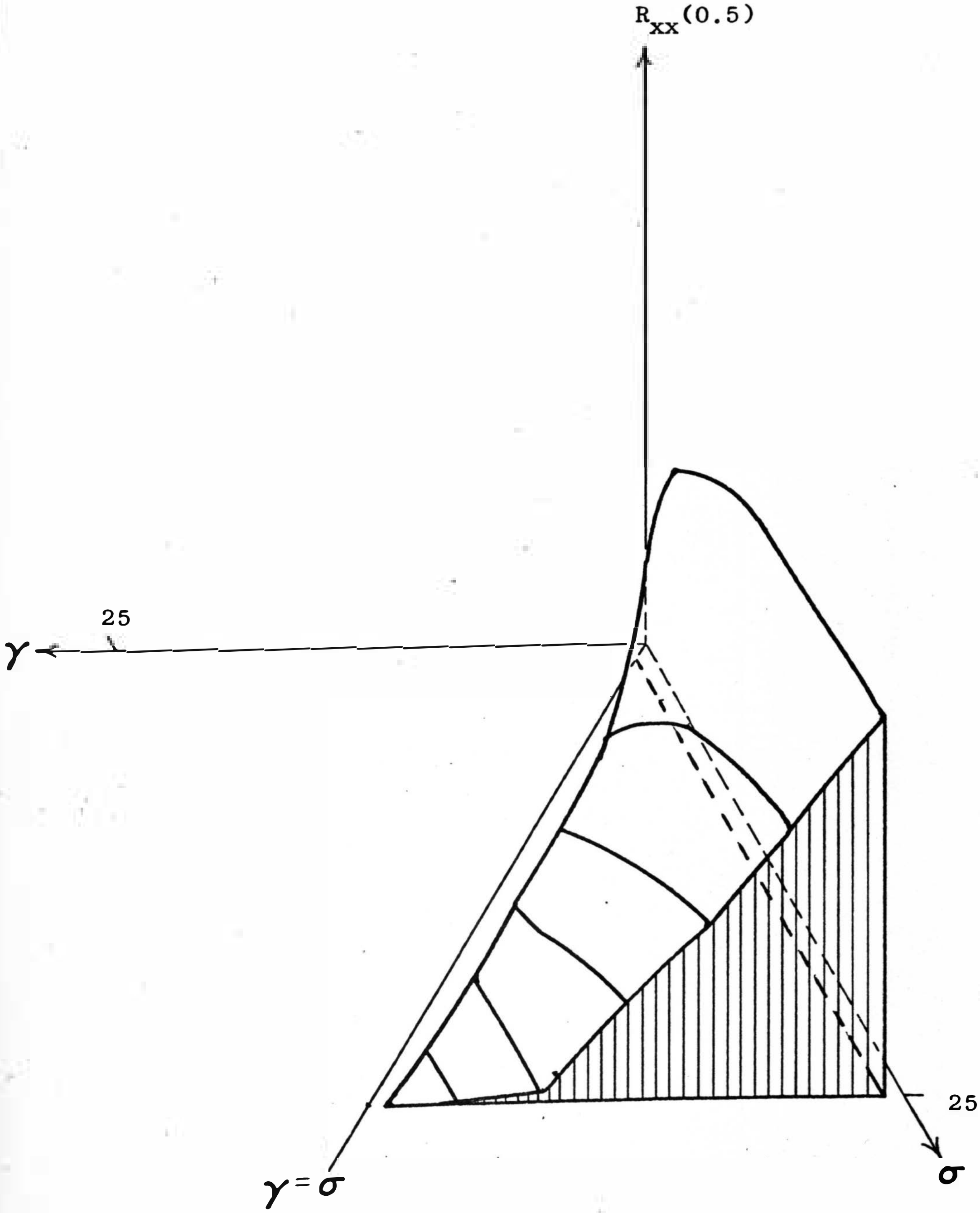


Figure 3-9. Variation of the pattern autocorrelation function, case(D).

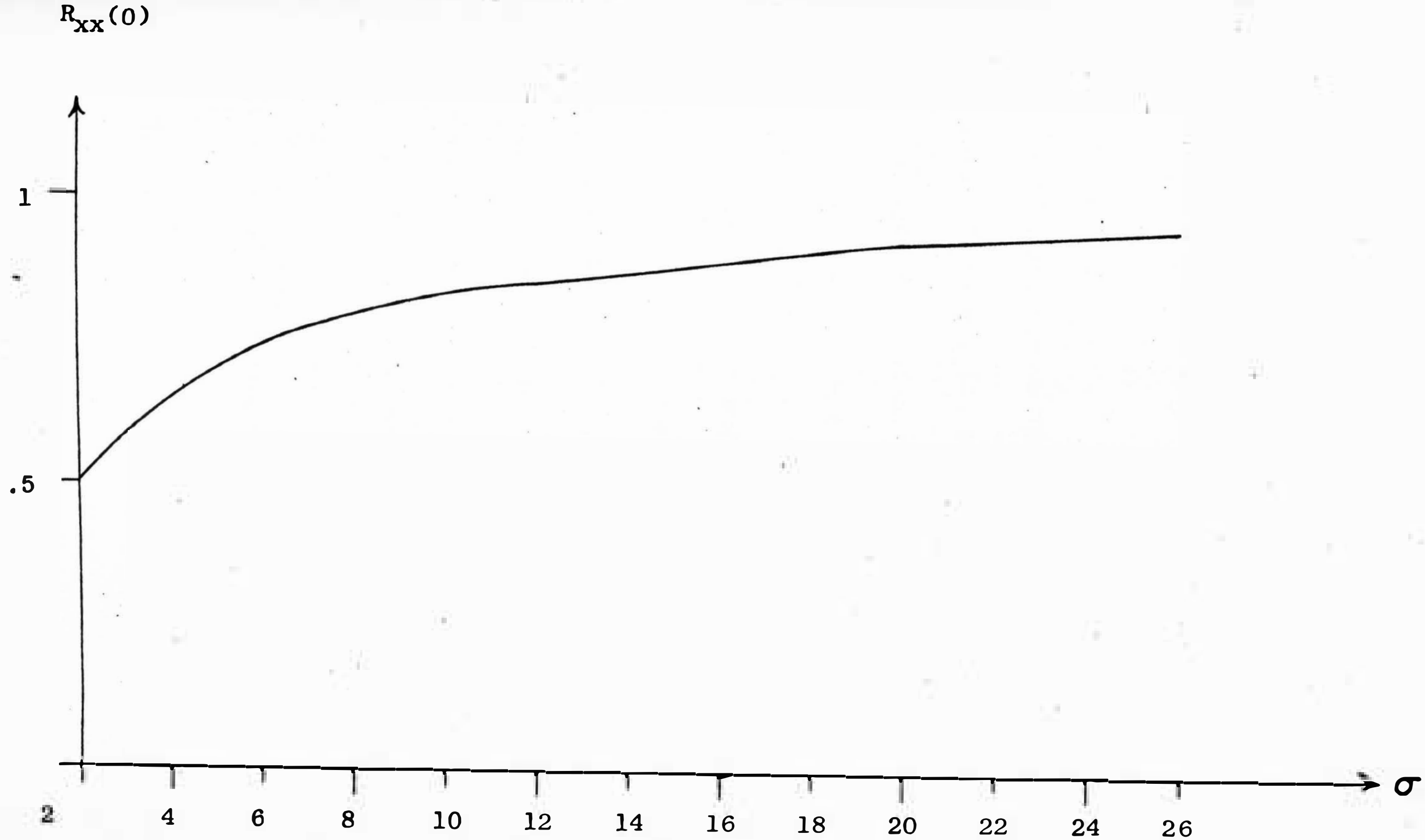


Figure 3-10. Case(D): Variance of the pattern

(ii) $R_{xx}(0)$ is dependent upon the location of the pole in the cases (A) and (B); and also upon the transfer function zero in the cases (C) and (D). Thus, the maximum value of the output auto-correlation function is dependent upon the transfer function pole and zero.

(iii) The auto-correlation function of the pattern, in each case, decays exponentially. Though they are similar in nature; the maximum value, rate of decay etc. depend upon the circuit configuration and values of the network parameters.

(iv) $\sigma \to 1$, in each case, makes an interesting study. The cross-correlation functions become 'indeterminate' in the limit, but may be evaluated by L'Hospital's rule. The cross-correlation function has not been evaluated as $\sigma \to 1$, since it has no physical significance and does not add, substantially, to the knowledge of the pattern. The auto-correlation function has been evaluated in the limit as $\sigma \to 1$, since it gives one an idea of the maximum value and also serves as a check on the calculations.

Though it was noted in article 2-2 that the auto-correlation function is an even function, this property was not invoked in our analysis. This property of evenness was proved for each case separately and was used advantageously to verify the computations.

3-6. Summary:

We have analyzed the patterns for four different configurations of resistances and capacitances with the same assumed SGM signal. The auto-correlation function of the pattern was calculated and $R_{xx}(0)$ computed in each case. The general nature of the pattern auto-correlation function was studied and appropriate comments were made. A definite relation between the location of the transfer function poles and zeros on one hand; and the maximum value, nature of the auto-correlation function on the other, was established. Table 3-6.1 summarizes the results. A graphical representation of the auto-correlation functions derived in each case was illustrated in Figures 3-2, 3-4, 3-6, 3-7, 3-9, and 3-10. Reasons for the type of curves obtained were briefly discussed. It was noted that in cases (C) and (D) the equations for the auto-correlation functions described planes with the movement in the location of a pole and a zero.

The next chapter will deal with the actual calculations of the MISER. The eigenfunctions will be determined in the Laplace domain. As will be shown, the time domain solution may be approximated by numerical techniques. However, analytical techniques are to be preferred.

CASES / DESCRIPTION	D
PATTERN AUTOCORRE FUNCTION $R_{xx}(\Delta)$; $\sigma \neq 1$	$\eta e^{-\sigma\vert\Delta\vert} + \zeta e^{-\vert\Delta\vert}$
VARIANCE OF THE PATTERN; $\sigma \neq 1$ $R_{xx}(0)$	$\dfrac{\gamma^2+\sigma(1-2\gamma)+\sigma^2}{\sigma(\sigma+1)}$
IN THE LIMIT AS $\sigma -$ a) $R_{xx}(\Delta)$	$\dfrac{e^{-\vert\Delta\vert}}{2}(\gamma^2-2\gamma(1+\vert\Delta\vert)+2)$
b) $R_{xx}(0)$	$(\gamma^2-2\gamma+2)/2$
THE POLE, $\sigma \equiv$	$(R_1+R_2)/R_1 R_2 C$
THE ZERO, $\gamma \equiv$	$1/R_2 C$
THE CONSTANTS ARE	$\eta = \left(\dfrac{2\gamma}{\sigma^2-1}\right)\left(1-\dfrac{\gamma}{2\sigma}\right)$ $\zeta = 1-\dfrac{2\sigma\gamma}{\sigma^2-1}+\dfrac{\gamma^2}{\sigma^2-1}$

CHAPTER FOUR

COMPUTATIONS OF MISER

With this chapter, we come to the crux of the problem, namely to determine how much information is lost in a filter. We start by calculating the power spectral density for each case. The method is explained, but actual calculations are left out as no difficulties are anticipated. The results and other relevant information are compiled in tabular form for ease of comparison. Youla's method [4] is used for finding the eigenfunctions in the Laplace domain. Time domain solution of the eigenfunctions, perhaps the most crucial single problem in this study, will be detailed. MISER calculations finally emerge, though actual numerical calculations will be deferred.

4-1. Preliminary Calculations for MISER:

The power spectral density of a stationary random process $\{x(t)\}$ is defined as the Fourier transform of the correlation function. Denoting this density function $S_x(\omega)$, we have

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau \qquad (4.1\text{-}1)$$

The spectral density of the transducer output process may be determined by [1]

$$S_x(\omega) = |\, H(j\omega)\, |^2 \quad S_w(\omega) \qquad (4.1\text{-}2)$$

where $\{w(t)\}$ and $\{x(t)\}$ are the signal and pattern processes respectively. Knowing the statistics of the input process and the impulse response of a transducer, Eq. (4.1-2) can be used. By the substitution $s = j\omega$, Eq. (4.1-2) may be expressed as a ratio of polynomials in s^2, viz.,

$$S_x(s) = \frac{N(s^2)}{D(s^2)} .$$

$N(s^2)$ and $D(s^2)$ are polynomials of degrees m and n in s^2, viz.,

$$N(s^2) = \sum_{k=0}^{m} a_{2k} s^{2k} \qquad (4.1\text{-}3)$$

$$D(s^2) = \sum_{k=0}^{n} b_{2k} s^{2k}, \qquad b_0 \neq 0.$$

In order that the spectral density be finite, it is required that the degree of $D(s^2)$ must exceed that of $N(s^2)$ by at least two and $D(s^2)$ must have no roots on the imaginary s axis [4].

Since the autocorrelation functions of the pattern have been found to be linear combinations of the form

$$R_x(\tau) = Me^{-N|\tau|} \qquad (4.1\text{-}4)$$

it may be helpful to determine the spectral density of $R_x(\tau)$ in Eq. (4.1-4). By definition, we have

$$S_x(\omega) = \mathcal{F}\left\{R_x(\tau)\right\}$$

$$= \int_{-\infty}^{0} M\, e^{N\tau}\, e^{-j\omega\tau}\, d\tau + \int_{0}^{\infty} M\, e^{-N\tau}\, e^{-j\omega\tau}\, d\tau$$

Solution of these integrals yields

$$S_x(\omega) = \frac{2\,MN}{N^2 + \omega^2} \qquad (4.1\text{-}5)$$

Table 4.1-1 gives the spectral density of the pattern for each case. The autocorrelation functions listed in Table 3.6-1 have been used to obtain these results. Table 4.1-1 also gives $N(s^2)$, $D(s^2)$ and $D^+(s)$ in each case. These are required to determine the eigenfunctions.

4-2a. Determination of Eigenfunctions:

Recall that the eigenfunctions, in the Laplace domain, are given by Eq. (2.3-9) which is rewritten here as

$$\phi_i(t) \doteqdot \frac{P(s, \lambda_i)\, D^+(s)}{D(s^2) - \lambda_i\, N(s^2)} \qquad (4.2\text{-}1)$$

$D^+(s)$, $D(s^2)$ and $N(s^2)$ for each case have been tabulated in Table 4.1-1. Observe that the polynomial $P(s)$ is arbitrary and of degree $(n-1)$. This arbitrary polynomial in s figures in the operational

	EIGENFUNCTIONS	
DESCRIPTION		D
SPECTRAL DENS s-DOMAIN	$2\Omega\sigma)$	$\frac{s^2(-2\eta\sigma-2\zeta)+(2\eta\sigma+2\zeta\sigma^2)}{s^4-(\sigma^2+1)s^2+\sigma^2}$
$N(s^2)=a_0+a_2s^2$		$a_2=-2\eta\sigma-2\zeta$ $a_0=2\eta\sigma-2\zeta\sigma^2$
$D(s^2)=b_0+b_2s$	→	→
m		1
n		2
$D^+(s)$	→	→
$D^-(s)$	σ →	→
THE POLE, σ ≡		$(R_1+R_2)/R_1$
THE ZERO, γ ≡		$1/R_2C$
⊖ FOR EXPLAN		

bilateral Laplace transform involving $\emptyset_i(t)$ of Eq. (2.3-4), [4]

Notationwise,

$$P(s) = \sum_{k=0}^{n-1} p_k\ s^k \tag{4.2-2}$$

In the cases under study, P(s) is a first order polynomial in s. It may be worth noting that $P(s, \lambda_i)$ is an arbitrary polynomial in s, with the p_k's being a function of λ_i. This concept is explained in sub-section 4-2c.

4-2b. Procedure for Determining the Eigenvalues:

At the risk of being repetitious, some of the equations of Chapter Two are rewritten below. With reference to Figure 2-1, the spectral density of the pattern is

$$S_x(s) = A^2\ B \left(\frac{2\ \alpha}{\alpha^2 - s^2}\right)\left(\frac{e_1{}^2 - c_1{}^2\ s^2}{e_2{}^2 - c_2{}^2\ s^2}\right) \tag{4.2-3}$$

It was also shown that

$$D(s^2) - \lambda_i\ N(s^2) = c_2{}^2\ s^4 - s^2(e_2{}^2 + c_2{}^2\alpha^2) + e_2{}^2\ \alpha^2 + K\lambda_i c_1{}^2 s^2 - K\ \lambda_i\ e_1{}^2 \tag{4.2-4}$$

The solutions of $D(s^2) - \lambda_i\ N(s^2) = 0$ were found to be

$$s_1, s_2 = \pm \left[g(\lambda_i) + \left\{ g^2(\lambda_i) + f(\lambda_i) \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (4.2\text{-}5)$$

$$s_3, s_4 = \pm \left[g(\lambda_i) - \left\{ g^2(\lambda_i) + f(\lambda_i) \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (4.2\text{-}6)$$

Solutions of $D(s^2) - \lambda_i N(s^2) = 0$, will be represented by $\pm\, \omega_1(\lambda_i)$ and $\pm\, \omega_2(\lambda_i)$. Note that the roots are arranged so that $0 \leq \text{Re}\,\omega_1 \leq \text{Re}\,\omega_2$.

The set of Eqs. (4.2-5) and 4.2-6) have been used to locate the eigenvalues, λ_i. The search technique employed in the computer solution is shown in Appendix D, where the program is reproduced. A typical computer print-out is also incorporated. Table 4.2-1 gives the values of the various constants which would replace the constants in the program to study a particular case.

Since the signal process is an assumed SGM process, the kernel $R_{ww}(t,u)$ is non-degenerate and hence should have an infinite number of eigenvalues. The eigenvalues are ordered in an ascending order and increase rather rapidly in value. Consequently, inverse of the eigenvalues approach zero quickly enough to consider a finite number of eigenvalues without being in gross error. The program in Appendix D will calculate the first fifty eigenvalues. Once the eigenvalues, λ_i, have been determined it will be a simple matter to calculate $\omega_r(\lambda_i)$. Table 4.2-2 illustrates the changes that will

Table 4.2-1: Values of the constants required to study a particular case. (This has reference to program given in Appendix D.)

Description \ Cases	A	B	C	D
Replace these constants (i) A1	0	1	1	1
(ii) A2	1	1	1	1
(iii) B1	1/RC	0	$1/R_2C$	$1/R_1C$
(iv) B2	1/RC	1/RC	$R_2/(R_1+R_2)R_2C$	$(R_1+R_2)/R_1R_2C$
(v) Alpha	1	1	1	1
(vi) A	1	1	$R_2/(R_1+R_2)$	1
(vii) B	1	1	1	1
(viii) $K=2\alpha A^2B$	2	2	$\frac{2R_2^2}{(R_1+R_2)^2} = \frac{2}{(1+k)^2}$	2

Table 4.2-2: Solutions of $D(s^2) - \lambda_i N(s^2) = 0$ for all the cases

Description \ Cases	A	B	C	D
$N(s^2)$	$2\sigma^2$	$-2s^2$	$(2\Omega\sigma - 2\xi)s^2 + (2\sigma\xi - 2\Omega\sigma)$	$(-2\eta\sigma - 2\zeta)s^2 + (2\eta\sigma + \zeta\sigma^2)$
$D(s^2)$	← $s^4 - (\sigma^2 + 1)s^2 + \sigma^2$ →			
In eqs. (4.2-5) and (4.2-6) replace $g(\lambda_i) =$	$\frac{\sigma^2 + 1}{2}$	$\frac{1}{2}(\sigma^2 + 1 - 2\lambda_i)$	$\frac{1}{2}\left(\sigma^2 + 1 - \frac{2\gamma^2\lambda_i}{(1+k)^2}\right)$	$\frac{1}{2}(\sigma^2 + 1 - 2\lambda_i)$
In eqs. (4.2-5) and (4.2-6) replace $f(\lambda_i) =$	$\sigma^2(2\lambda_i - 1)$	$-\sigma^2$	$\frac{2\lambda_i\gamma^2}{(1+k)^2} - \sigma^2$	$(2\lambda_i\gamma^2 - \sigma^2)$
The pole, $\sigma =$	$1/RC$	$1/RC$	$R_2/(R_1+R_2)$	$(R_1+R_2)/R_1$
The zero, $\gamma =$	—	—	$1/R_2C$	$1/R_2C$

be required, for the different cases, in Eqs. (4.2-5) and 4.2-6).

With the help of Table 4.2-2, $\omega_r(\lambda_i)$ may be calculated.

4-2c. <u>Determination of $P(s, \lambda_i)$:</u>

Recall that

$$P(s) = P(s, \lambda_i) = \sum_{k=0}^{n-1} p_k s^k \qquad (4.2\text{-}7)$$

where the p_k's are given by

$$\sum_{k=0}^{n-1} \left[1 \pm (-1)^k x_r\right] \omega_r^k(\lambda_i)\, p_k = 0 \qquad (4.2\text{-}8)$$

and

$$x_r \equiv e^{-\omega_r(\lambda_i)T} \frac{D^- \omega_r(\lambda_i)}{D^+ \omega_r(\lambda_i)} \qquad (4.2\text{-}9)$$

It is thus evident that $P(s, \lambda_i)$ will be dependent upon the order of $D(s^2)$. Actual values of $P(s, \lambda_i)$ are, of course, dependent upon the solutions of $D(s^2) - \lambda_i N(s^2) = 0$, viz., $\pm \omega_r(\lambda_i)$. In the cases under study, the order of $D(s^2)$ in s^2 is two. Eq. (4.2-8) then reduces to

$$(1 \mp x_r)\, p_o + (1 \pm x_r)\, \omega_r(\lambda_i)\, p_1 = 0 \qquad (4.2\text{-}10)$$

For a non-trivial solution we must have

$$p_1 = -\left(\frac{1 \mp x_r}{1 \pm x_r}\right)\left(\frac{1}{\omega_r(\lambda_i)}\right) p_o \qquad (4.2\text{-}11)$$

Substitution of this result into Eq. (4.2-7) gives

$$P(s,\lambda_i) = p_0 \left[1 - \frac{1 \mp x_r}{1 \pm x_r} \; \frac{1}{\omega_r(\lambda_i)} \; s \right] , \qquad (4.2\text{-}12)$$

$$r = 1,2.; \; i = 1,2,3,\ldots\ldots,\ldots\ldots,\ldots$$

This leads to

$$P(s,\lambda_i) = \begin{cases} p_0 \left[1 - \dfrac{1 - x_1}{1 + x_1} \; \dfrac{s}{\omega_1(\lambda_i)} \right] \\ \\ p_0 \left[1 - \dfrac{1 + x_1}{1 - x_1} \; \dfrac{s}{\omega_1(\lambda_i)} \right] \end{cases} , \; r = 1. \qquad (4.2\text{-}13)$$

$$\text{and } P(s,\lambda_i) = \begin{cases} p_0 \left[1 - \dfrac{1 - x_2}{1 + x_2} \; \dfrac{s}{\omega_2(\lambda_i)} \right] \\ \\ p_0 \left[1 - \dfrac{1 + x_2}{1 - x_2} \; \dfrac{s}{\omega_2(\lambda_i)} \right] \end{cases} , \; r = 2. \qquad (4.2\text{-}14)$$

$$x_1 = e^{-\omega_1(\lambda_i)} \left(\frac{\omega_1^2(\lambda_i) - (\sigma + 1)\,\omega_1(\lambda_i) + \sigma}{\omega_1^2(\lambda_i) + (\sigma + 1)\,\omega_1(\lambda_i) + \sigma} \right)$$

$$x_2 = e^{-\omega_2(\lambda_i)} \left(\frac{\omega_2^{\,2}(\lambda_i) - (\sigma + 1)\,\omega_2(\lambda_i) + \sigma}{\omega_2^{\,2}(\lambda_i) + (\sigma + 1)\,\omega_2(\lambda_i) + \sigma} \right)$$

Observe that all the terms can now be evaluated and hence $P(s, \lambda_i)$ can be calculated. With this we are in a position to compute the eigenfunctions. Before proceeding, however, it may be advisable to note a few interesting points:

(i) The constant term, p_0, will be eliminated while normalizing the eigenfunctions.

(ii) In the solution of $P(s, \lambda_i)$, the coefficient of s is a dimensionless number for a given r and i.

(iii) As a consequence of this, $P(s, \lambda_i)\, D^+(s)$ is a third degree polynomial in s and $D(s^2) - \lambda_i\, N(s^2)$ will be of the same nature for each case. It would, therefore, seem possible to obtain the inverse Laplace transform analytically.

4-2d. A General Time Domain Eigenfunction:

It was indicated in subsection a of this article that

$$\phi_i(t) \doteqdot \frac{P(s, \lambda_i)\, D^+(s)}{D(s^2) - \lambda_i\, N(s^2)}$$

Let the right hand side of this transform pair be represented by

$$\frac{P(s, \lambda_i)\, D^+(s)}{D(s^2) - \lambda_i\, N(s^2)} = \frac{A_3\, s^3 + A_2\, s^2 + A_1\, S + A_0}{s^4 + (\beta^2 + \tau^2)\, s^2 + \beta^2 \tau^2}$$

A good table of Laplace transforms [15] gives the transform of this s - domain polynomial as

$$\phi_i(t) = \frac{A_0 - A_2\beta^2}{\beta(-\beta^2 + \tau^2)} \sin(\beta t + \psi_1)\sqrt{1 + \psi_1^2}$$

$$+ \frac{A_0 - A_2\tau^2}{\tau(\beta^2 - \tau^2)} \sin(\tau t + \psi_2)\sqrt{1 + \psi_2^2} \qquad (4.2\text{-}16)$$

where $$\psi_1 = \tan^{-1} \frac{\beta(A_1 - A_3\beta^2)}{A_0 - A_2\beta^2} \qquad (4.2\text{-}17)$$

and $$\psi_2 = \tan^{-1} \frac{\tau(A_1 - A_3\tau^2)}{A_0 - A_2\tau^2} \qquad (4.2\text{-}18)$$

Observe that $(\beta^2 + \tau^2)$ is the coefficient of s^2 and $\beta^2\tau^2$ is a constant term. Knowing the actual values, β and τ may be determined. Table 4.2-3 makes a comparative study of the component terms required, in each case, to obtain the eigenfunctions. As already pointed out in sub-section c of this article, $P(s, \lambda_i)$ and $D(s^2)$ have similar forms in all the four cases. Eq. (4.2-10) may, therefore, be rewritten as

$$P(s, \lambda_i) = p_0(1 - P_z s)$$

P_z representing the coefficient of s in Eq. (4.2-12). Therefore,

TABLE 4.2-3 (†). [REFERENCE — EQS. (4.2-16) AND (4.2-19)]

DESCRIPTION		D
$D(s^2) - \lambda_i N(s^2)$	$+ s^2$	$s^4 + \{\lambda_i (2\eta\sigma + 2\zeta) - (\sigma^2 + 1)\} s^2 + \{\sigma^2 - \lambda_i (2\eta\sigma + 2\zeta\sigma^2)\}$
COEFFICIENT OF s^2;		$\{\lambda_i (2\eta\sigma + 2\zeta) - (\sigma^2 + 1)\}$
CONSTANT TERM;		$\{\sigma^2 - \lambda_i (2\eta\sigma + 2\zeta\sigma^2)\}$
THE POLE, $\sigma \equiv$		$(R_1 + R_2) / R_1 R_2 C$
THE ZERO, $\gamma \equiv$		$1 / R_2 C$
THE CONSTANTS		$\eta = \left(\frac{2\gamma}{\sigma^2 - 1}\right)\left(1 - \frac{\gamma}{2\sigma}\right)$ $\zeta = 1 - \frac{2\sigma\gamma}{\sigma^2 - 1} + \frac{\gamma^2}{\sigma^2 - 1}$

$$P(s,\lambda_i)\, D^{+}(s) = -P_z p_0 s^3 + \left[p_0 - P_z p_0(\sigma + 1)\right] s^2$$

$$+\left[p_0(\sigma + 1) - P_z p_0\right] s + p_0 \sigma \qquad (4.2\text{-}19)$$

In terms of the parameters defined in Eqs. (4.2-16), (4.2-17) and (4.2-18); we have

$$A_3 = - P_z\, p_0 \qquad (4.2\text{-}20)$$

$$A_2 = p_0 - P_z\, p_0\, (\sigma + 1) \qquad (4.2\text{-}21)$$

$$A_1 = p_0(\sigma + 1) - P_z\, p_0 \qquad (4.2\text{-}22)$$

$$A_0 = p_0\, \sigma \qquad (4.2\text{-}23)$$

where P_z is replaced by

$$\left.\begin{aligned} P_{1i} &= \frac{1 - x_1}{1 + x_2}\ \frac{1}{\omega_1(\lambda_i)} \\ \bar{P}_{1i} &= \frac{1 + x_1}{1 - x_1}\ \frac{1}{\omega_1(\lambda_i)} \end{aligned}\right\} \qquad (4.2\text{-}24)$$

$$\left.\begin{aligned} P_{2i} &= \frac{1 - x_2}{1 + x_2}\ \frac{1}{\omega_2(\lambda_i)} \\ \bar{P}_{2i} &= \frac{1 + x_2}{1 - x_2}\ \frac{1}{\omega_2(\lambda_i)} \end{aligned}\right\} \qquad (4.2\text{-}25)$$

The x_r have already been defined in Eq. (4.2-15). The other parameters viz., β and τ, may be identified with the help of Table 4.2-3.

Observe that in Eqs. (4.2-16), (4.2-17) and (4.2-18) the parameters are independent of time, t, and are constants for a given r and i. Rewriting Eq. (4.2-16) as

$$\phi_i(t) = Y_1 \sin(\beta t + \psi_1) + Y_2 \sin(\tau t + \psi_2)$$

where $$Y_1 = \frac{A_0 - A_2\beta^2}{\beta(\tau^2 - \beta^2)} \sqrt{1 + \psi_1^2} \qquad (4.2\text{-}26)$$

and $$Y_2 = \frac{A_0 - A_2\tau^2}{\tau(\beta^2 - \tau^2)} \sqrt{1 + \psi_2^2} \qquad (4.2\text{-}27)$$

Note that the $\phi_i(t)$ in Eq. (4.2-16) are orthogonal but not orthonormal. The next article proceeds to determine the normalizing constant.

4-3. Determination of the Normalizing Constant:

It can be shown [10] that the square of the normalizing constant is

$$C_i^2 = \int_0^1 \phi_i^2(t)\, dt \qquad (4.3\text{-}1)$$

where, $\phi_i(t) = Y_1 \sin(\beta t + \psi_1) + Y_2 \sin(\tau t + \psi_2)$

Therefore,

$$c_i^2 = \int_0^1 \left[Y_1 \sin(\beta t + \psi_1) + Y_2 \sin(\tau t + \psi_2)\right]^2 dt$$

Expansion and direct substitution yields

$$c_i^2 = \frac{Y_1^2}{2\beta}\left[\beta - \cos(\beta + 2\psi_1)\sin\beta\right]$$

$$+ \frac{Y_1 Y_2}{\beta - \tau}\left[2\cos\left(\frac{\beta - \tau + 2\psi_1 - 2\psi_2}{2}\right)\sin\left(\frac{\beta - \tau}{2}\right)\right]$$

$$- \frac{Y_1 Y_2}{\beta + \tau}\left[2\cos\left(\frac{\beta + \tau + 2\psi_1 + 2\psi_2}{2}\right)\sin\left(\frac{\beta + \tau}{2}\right)\right]$$

$$+ \frac{Y_2^2}{2\tau}\left[\tau - \cos(\tau + 2\psi_2)\sin\tau\right] \qquad (4.3\text{-}2)$$

The c_i^2 given by Eq. (4.3-2) will be dependent upon r and λ_i as the Y's are functions of λ_i. The other parameters may be computed with the help of Table 4.2-3.

4-4. MISER Calculations:

One is now in a position to compute the Mean Integral Square Error. Eq. (2.4-5), which is the equation for the MISER, is reproduced below as

$$\epsilon^2 = R_{ww}(0) + \frac{1}{C_i^2} \sum_{i=1}^{n} \int_0^1 \int_0^1 [R_{xx}(t,u) - 2\, R_{xw}(t,u)]\, \phi_i(t)\, \phi_i(u)\, dt\, du \tag{4.4-1}$$

The variance of the signal process is obviously 1. The pattern auto-correlation functions are detailed in Table 3.6-1. The cross-correlation functions have been derived in Chapter Three. Knowing the eigenfunctions derived in article 4-2; all the component terms can be determined and the MISER computed with the help of a high speed digital computer. Appendix E gives a flow chart which may be of assistance in programming a problem of interest.

4-5. Summary:

This chapter has given a definite direction to help calculate the MISER. All the seemingly uncorrelated articles of the preceding chapters were called upon to shape the final analytical solution. Though the expressions which have been derived seem formidable they would lend themselves to computer solutions without much effort. It must be remembered that by the nature of the solution, a family of eigenfunctions will be generated.

CHAPTER FIVE

CONCLUSIONS

Four different configurations of resistances and capacitances representing a filter were analyzed using the mean-integral square error as the 'goodness' measuring criterion. With the signal process an assumed SGM process, the patterns were found to be stationary in the wide sense. The pattern autocorrelation functions were seen to be identical in nature --- decaying exponentially for given pole and zero locations.

In all the four cases, it was shown that the location of a pole at the origin is a physical impossibility. Moreover, in case(C) the pole is always located within that of the zero or in the limit may coincide; while in case(D), the pole must lie outside the location of the zero. The variation of the pattern autocorrelation function for a given time interval, $|t-u|$, with respect to changes in location of the poles and zeros made an interesting study. Figures 3.2 through 3.10 illustrate the variations. Case(A) has no finite zero, case(B) has a zero at the origin with the pole location remaining independent; cases (C) and (D) have poles and zeros with the location of the poles depending upon the position of the zero. The pattern autocorrelation functions, therefore, describe a plane with the movement of the poles and zeros.

Chapter Four dealt with the heart of the problem viz., giving an expression to determine loss of information in a filter. Eq. (4.4-1) expresses the MISER valid for the cases under consideration. Though actual numerical computations were avoided due to lack of facilities; experience indicates that the MISER would be very small --- close to zero. The MISER would increase in value as the pole is moved away from the origin in all the cases. The use of the Karhunen-Loēve theorem to approximate the pattern process $\{x(t)\}$ assured the minimal MISER achievable.

The eigenfunctions were determined by Youla's method [4] in the Laplace domain. Capon's method [5] was shown to be unsuitable for our purposes. Analytical techniques were used to get the time domain eigenfunctions, though numerical techniques could probably be used. Numerical inversion of the Laplace transform, which is required to obtain the time domain solution, is risky and unstable [16] . Numerical methods, to be reasonably good, have to be developed for the particular case to be studied.

The SGM process was considered as the signal process, mainly because results in a concise form could be expected. Admittedly, this hypothetical process does not exist in nature. However, as the purpose of studies of this nature are to analyze a problem without losing sight of the significance of the results, the signal used is not entirely without justification.

It is hoped that this dissertation will encourage others interested to do more work in this area. One could use a more realistic signal process available in nature, for example, or could proceed to analyze a second order transducer with the SGM process as the signal. The use of an analog computer with an analog-to-digital converter may also be explored.

BIBLIOGRAPHY

1. G. R. Cooper and C. D. McGillem, Methods of Signal and System Analysis. New York: Holt, Rinehart, and Winston, Inc. 1967. pp. 345-374; 338; 306-307; 310-311; 392-393.

2. A. Papoulis, Probability, Random Variables and Stochastic Processes. New York: McGraw-Hill Book Co., Inc. 1965, pp. 385-429; 368, 453-454; 300-302; 537.

3. W. B. Davenport Jr. and W. L. Root, An Introduction to the Theory of Random Signals and Noise. New York: McGraw-Hill Book Co., Inc. 1958. pp. 171-200; 59; 91; 145-167; 60-61; 91-99; 97; 371-376.

4. D. C. Youla, "The Solution of a Homogeneous Wiener-Hopf Integral Equation Occurring in the Expansion of Second Order Stationary Random Functions". IRE Transactions on Information Theory, Vol. IT-3, pp. 187-193, September 1957.

5. J. Capon, "Asymptotic Eigenfunctions and Eigenvalues of a Homogeneous Integral Equation", IRE Transactions on Information Theory, pp. 2-4, January 1962.

6. D. J. Sakrison, Communication Theory: Transmission of Waveforms and Digital Information, New York: John Wiley and Sons, Inc. 1968, pp. 9-13; 111-112; 119-120.

7. D. Middleton, An Introduction to Statistical Communication Theory. New York: McGraw-Hill Book Co., Inc. 1960. pp. 96-97; 724-727.

8. P. M. DeRusso, R. J. Joy and C. M. Close, State Variables for Engineers. New York: John Wiley and Sons, Inc. 1967, pp. 118.

9. R. Courant and D. Hilbert, Methods of Mathematical Physics vol. 1. New York: Interscience Publishers, Inc. 1963, pp. 112-136.

10. P. A. Wintz and A. J. Kurtenbach, "Analysis and Minimization of Message Error in PCM Telemetry Systems" TR-EE67-19. Purdue University, Lafayatte, Indiana. December 1967. pp. 114-117.

11. F. B. Hildebrand, Methods of Applied Mathematics. New Jersey: Prentice-Hall, Inc. 1965, pp. 81-88.

12. G. A. Korn and T. M. Korn, Mathematical Handbook for Scientist and Engineers. New York: McGraw-Hill Book Co., Inc. 1968. pp. 122-124.

13. R. Courant, Differential and Integral Calculus vol. 1. New York: Interscience Publishers, Inc. 1957, pp. 338-340.

14. R. Courant, Differential and Integral Calculus vol. 2. New York: Interscience Publishers, Inc. 1957, pp. 239-240; 310-312.

15. A. W. Langill Jr., Automatic Control Systems Engineering. New Jersey: Prentice-Hall, Inc. 1965, pp. 355.

16. R. E. Bellman, R. E. Kalaba and J. Lockett, Numerical Inversion of the Laplace Transform. New York: American Elsevier Publishing Co., Inc. 1966,

APPENDICES

APPENDIX A

On the Interchangability of the Expectation and Integration Operations

One often comes across the problem of finding the expected value of time integration. Though this can be done in most of the cases met in practice, it may be advisable to know the conditions which are prerequisites to interchange the expectation and time integral operations.

Let x(t) be a sample function from a random process and f(t) a non-random time function. Then,

$$E \int_{-\infty}^{\infty} x(t)\ f(t)\ dt = \int_{-\infty}^{\infty} E\left\{x(t)\right\}\ f(t)\ dt \qquad (A.1)$$

if

$$\int_{-\infty}^{\infty} E\left\{|x(t)|\right\}\ |f(t)|\ dt < \infty \qquad (A.2)$$

Sufficient conditions that this inequality be satisfied are that,

$$E\left\{|x(t)|\right\} \leq M < \infty\ , \qquad \forall\ t \qquad (A.3)$$

and that the system be stable,

$$\int_{\infty}^{\infty} |f(t)|\ dt < \infty \qquad (A.4)$$

Fortunately, these conditions are met in all meaningful communications applications [1], and one does not have to verify that these conditions are satisfied in each case.

APPENDIX B

On the Interchangability of the Order of Integration

The requirement to interchange the order of integration is common. It has been done several times in this paper. The following are the necessary and sufficient conditions, stated without proof, when it is possible to interchange the order of integration.

If any one of the following three conditions

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)| \, dx \, dy < \infty \tag{B.1}$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x,y)| \, dy < \infty \tag{B.2}$$

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y)| \, dx < \infty \tag{B.3}$$

is satisfied, then it is true that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \, dx \, dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y) \, dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y) \, dx \tag{B.4}$$

This result is known as Fubini's Theorem. Further reference to [6] and [14] may be helpful.

APPENDIX C

Analysis of a Short Circuited Transducer

During the study of the main problem in hand, it was noted that two possible solutions of the Homogeneous Integral Equation

$$\int_a^b R(t,u)\, \varnothing_i(u)\, du = \mu_i\, \varnothing_i(t) \qquad \text{(C-1)}$$

$$a \leq t \leq b; \quad i = i,2,3,\ldots\ldots,n.$$

were available. One of the solutions, used in this paper, was due to D. C. Youla [4] and the other by J. Capon [5]. The solution in the Laplace domain by Youla was preferred over the solution in the time domain by Capon. This appendix attempts to justify the choice.

Consider a short circuited transducer illustrated in Figure C.1. Let the signal be w(t) and the pattern x(t). The signal is an assumed SGM process. As a consequence of the short circuited filter, the pattern is an exact replica of the signal. Let the autocorrelation function of the signal be

$$R_{ww}(\tau) = e^{-|\tau|} \qquad \text{(C-2)}$$

Obviously,

$$R_{xx}(\tau) = R_{ww}(\tau) \qquad \text{(C-3)}$$

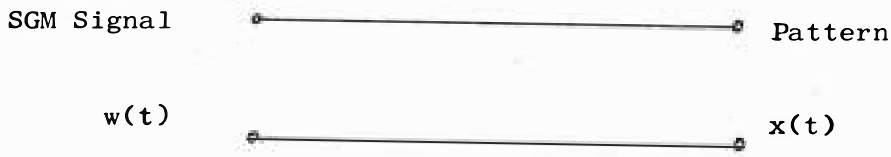


Figure C.1. A short circuited transducer

Table C.1

Eigenvalues of the integral equation and the approximations for

$$R(\tau) = e^{-|\tau|}$$

i \ Source	Integral Equation	Youla's Method	Capon's Method
1	0.7388	0.7392	0.1522
2	0.1380	0.1493	0.0558
3	0.0451	0.0521	0.0369
4	0.0213	0.0360	0.0206
5	0.0123	0.0201	0.0235
6	0.0079	0.0232	0.0139
7	0.0056	0.0137	0.0199
8	0.0041	0.0197	0.0118
9	0.0031	0.0117	0.0188
10	0.0025	0.0186	0.0163
Sum	0.9787	1.0835	0.3697

and $$R_{xw}(\tau) = R_{xx}(\tau) \qquad \text{(C-4)}$$

Substitution of these in

$$\sigma_i = \int_0^1 \int_0^1 \left[R_{xx}(\tau) - 2R_{xw}(\tau) \right] \phi_i(t)\, \phi_i(u)\, dt\, du$$

yields

$$\sigma_i = - \int_0^1 \int_0^1 e^{-|\tau|} \phi_i(t)\, \phi_i(u)\, dt\, du \qquad \text{(C-5)}$$

Using the technique advocated by Youla [4], it can be shown that

$$\phi_i(t) = \frac{1}{C_i} \left\{ \frac{1}{\beta_i^2} \sin \beta_i t + \cos \beta_i t \right\}, \qquad \text{(C-6)}$$

where C_i is the normalizing constant determined by [10]

$$C_i^{\,2} = \frac{1}{\beta_i^{\,2}} \sin^2 \beta_i + \frac{\beta_i^2 - 1}{4\,\beta_i^2} \sin 2\beta_i + \frac{1 + \beta_i^2}{2\,\beta_i^2}$$

(C-7)

The β_i are solutions of

$$\tan \beta_i/2 = 1/\beta_i\,, \qquad i \text{ is odd} \qquad \text{(C-8)}$$

and $$\cot \beta_i/2 = -1/\beta_i\,, \qquad i \text{ is even} \qquad \text{(C-9)}$$

Capon's method [5] leads to

$$\phi_i(t) = 2 \cos \pi i t , \qquad 0 < t < 1 \qquad \text{(C-10)}$$

The other set of equations is obtained by replacing the dummy variable t by u.

Table C.1 gives a comparison of the results obtained by three different methods ---- Analytic solution, Youla's method and Capon's method. It will be observed that Capon's method leads to results grossly in error for small i. As the first few eigenvalues are the major contributors, errors in these eigenvalues will make the MISER even more erroneous. The numerical approximation to both the techniques described above were obtained by rectangular approximations $\Delta u = \Delta t = 0.10$. A smaller width of a rectangle would, undoubtedly, improve results obtained by Youla's method.

APPENDIX D
COMPUTER PROGRAM TO CALCULATE EIGENVALUES

```
      COMMON A1,A2,B1,B2,ALPHA,CAY,D0,D1,D2,T,W2S,W1,X,TAN
      DIMENSION RLAMDA(50), SLAMDA(50), ERROR(50)
  100 READ (11,1) A1,B1,A2,B2,ALPHA,T,A,B
    1 FORMAT(8F9.3)
      ROFZER=(A*A*ALPHA*B)/(B2*B2-ALPHA*ALPHA*A2*A2)
      ROFZER=ROFZER*(((B1*B1-A1*A1*ALPHA*ALPHA)/ALPHA)-((A2*A2*B1*B1-
     *A1*A1*B2*B2)/(A2*B2)))
      DO=ALPHA*B2
      D1=B2+ALPHA*A2
      D2=A2
      CAY=2*ALPHA*A**2*B
      WRITE (12,2) T,ALPHA,A1,A2,B1,B2,D0,D1,D2,CAY,ROFZER,A,B
    2 FORMAT(1H1,8X,2HT=,F10.3/5X,6HALPHA=,F10.3/8X,3HA1=,F10.3/8X,
     *3HA2=,F10.3/8X,3HB1=,F10.3/8X,3HB2=,F10.3/8X,3HDO=,F10.3/8X,
     13HD1=,F10.3,/8X,3HD2=,F10.3/7X,4HCAY=F10.3/4X,7HROFZER=,F10.3/
     29X,2HA=F10.3/9X,2HB=F10.3//)
      WRITE(12,12)
   12 FORMAT(30X,39HS(OMEGA)=2.*ALPHA*B/(ALPHA**2+OMEGA**2)//
     130X,70H(ABS(H(J*OMEGA)))**2=A**2(A1**2*OMEGA**2+B1**2)/(A2**2+OMEG
     2A**2&B2**2)    //30X,18HCAY=2*ALPHA*A**2*B)
      X=1.0
      DO 4 I=1,50
    6 RLAMDA(I)=X
   60 GLAMDA=((B2*B2&A2*A2*ALPHA*ALPHA)-RLAMDA(I)*CAY*A1*A1)/(2.*A2*A2)
      FLAMDA=(ALPHA*ALPHA*B2*B2-RLAMDA(I)*CAY*B1*B1)/(A2*A2)*(-1.0)
C     WRITE (12,42) RLAMDA(I),I
   42 FORMAT(1H ,7HRLAMDA=,1PE14.7,3X,2HI=,I3)
      IF (GLAMDA**2&FLAMDA) 3,3,11
    3 IF (RLAMDA(I)-100000.) 5,5,10
    5 RLAMDA(I) = RLAMDA(I)&0.001*RLAMDA(I)
      GO TO 60
   10 STOP
   11 W1=SQRT(GLAMDA&SQRT(GLAMDA*GLAMDA&FLAMDA))
      W2S=GLAMDA-SQRT(GLAMDA*GLAMDA&FLAMDA)
C     WRITE (12,43) W2S,W1
   43 FORMAT(1H ,4HW2S=,1PE14.7)
C     WRITE (12,41)
   41 FORMAT(1H ,9HINTO EQT2)
      CALL EQT2(RLAMDA(I))
      I=I&1
```

```
      RLAMDA(I)=RLAMDA(I-1)
      CALL INCR(RLAMDA(I),W2S,W1)
C     WRITE (12,40)
   40 FORMAT(1H ,9HINTO EQT1)
      CALL EQT1(RLAMDA(I))
      X=X+.005
    4 CONTINUE
      WRITE (12,15)
   15 FORMAT(1H1,10X,23HLAMDA FOR ZERO CROSSING,5X,
     *24HSLAMDA FOR ZERO CROSSING,5X,21HERROR FOR LAMDA TERMS/)
      S=O.
      DO 18 J=1,50
      SLAMDA(J)=1./(RLAMDA(J)*ROFZER)
      S=S&SLAMDA(J)
      ERROR(J)=1.-S
   18 WRITE (12,20) J,RLAMDA(J),SLAMDA(J),ERROR(J)
   20 FORMAT(1H ,2X,12,12X,E14.7,14X,E14.7,14X,E14.7).
      GO TO 100
      END
```

```
      SUBROUTINE EQT1(XX)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
      X=XX
      CALL CALC1(W2S,W1,Y)
      INDEX = 0
   11 CALL INCR(X2,A,B)
      INDEX = INDEX&1
    4 CALL CALC1(A,B,Y2)
      IF (Y*Y2) 5,6,6
    5 IF (ABS(Y-Y2)-100.) 7,6,6
    6 Y=Y2
      X=X2
      GO TO  11
    7 CALL FALSE1(X,Y,X2,Y2,X3)
C     WRITE (12,20) X3,INDEX
   20 FORMAT(1H ,2X,3HX3=,1PE14.7,20X,6HINDEX=,I7)
      XX=X3
      RETURN
      END
```

```
      SUBROUTINE EQT2(XX)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
      X=XX
      CALL CALC2(W2S,W1,Y)
      INDEX = 0
   11 CALL INCR(X2,A,B)
      INDEX = INDEX&1
    4 CALL CALC2(A,B,Y2)
      IF (Y*Y2) 5,6,6
    5 IF (ABS(Y-Y2)-100.) 7,6,6
    6 Y=Y2
      X=X2
      GO TO  11
    7 CALL FALSE2(X,Y,X2,Y2,X3)
C     WRITE (12,20) X3,INDEX
   20 FORMAT(1H ,2X,3HX3=,1PE14.7,20X,6HINDEX=,I7)
      XX=X3
      RETURN
      END
```

```
      SUBROUTINE CALC1(W2,W,Y)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
      IF (W2) 1,1,2
    1 SQR  = SQRT(-W2)
      SIGN = -1.0
      TAN = SIN (SQR *T/2.0)/COS(SQR *T/2.0)
      GO TO 3
    2 SQR  = SQRT(W2)
      SIGN = 1.0
      TAN = TANH(SQR *T/2.0)
    3 SLEFT = ((DO&D2*W2)&SIGN*D1*SQR *TAN)*SQR
      SLEFT = SLEFT/((DO&D2*W2)*TAN&D1*SQR)
      SRIGHT=((DO&D2*W**2)&D1*W*TANH((W*T)/2.))*W
      SRIGHT=SRIGHT/((DO&D2*W**2)*TANH((W*T)/2.)&D1*W)
      Y=SLEFT-SRIGHT
      RETURN
      END
```

```
      SUBROUTINE CALC2(W2,W,Y)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
      IF (W2) 1,1,2
    1 SQR  = SQRT(-W2)
      SIGN = -1.0
      TAN = SIN (SQR *T/2.0)/COS(SQR *T/2.0)
      GO TO 3
    2 SQR  = SQRT(W2)
      SIGN = 1.0
      TAN = TANH(SQR *T/2.0)
    3 SLEFT = ((DO&D2*W2)*TAN&D1*SQR )*SIGN*SQR
      SLEFT = SLEFT/((DO&D2*W2)&SIGN*SQR *TAN*D1)
      SRIGHT=((DO&D2*W*W)*TANH((W*T)/2.)&D1*W)*W
      SRIGHT=SRIGHT/((DO&D2*W**2)&D1*W*TANH((W*T)/2.1))
      Y=SLEFT-SRIGHT
      RETURN
      END
```

```
      SUBROUTINE FALSE1(X1,Y1,X2,Y2,X3)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
   40 FORMAT(1H ,11HINTO FALSE1)
C     WRITE (12,40)
      X=X1
    2 X3=(X1*Y2-X2*Y1)/(Y2-Y1)
      GLAMDA=((B2*B2&A2*A2*ALPHA*ALPHA)-X3*CAY*A1*A1)/(2.*A2*A2)
      FLAMDA=(ALPHA*ALPHA*B2*B2-X3*CAY*B1*B1)/(A2*A2)*(-1.0)
      W11=SQRT(GLAMDA&SQRT(GLAMDA*GLAMDA&FLAMDA))
      W2S1=GLAMDA-SQRT(GLAMDA*GLAMDA&FLAMDA)
      CALL CALC1(W2S1,W11,Y3)
      DO 4 I=1,100
      IF (Y3*Y2) 10,20,20
   20 Y2=Y1
      X2=X1
   10 XNEW=(X2*X3-X3*Y2)/(Y3-Y2)
      X1=X2
      X2=X3
      X3=XNEW
      Y1=Y2
      Y2=Y3
      GLAMDA=((B2*B2&A2*A2*ALPHA*ALPHA)-X3*CAY*A1*A1)/(2.*A2*A2)
      FLAMDA=(ALPHA*ALPHA*B2*B2-X3*CAY*B1*B1)/(A2*A2)*(-1.0)
      W11=SQRT(GLAMDA&SQRT(GLAMDA*GLAMDA&FLAMDA))
      W2S1=GLAMDA-SQRT(GLAMDA*GLAMDA&FLAMDA)
      CALL CALC1(W2S1,W11,Y3)
    6 IF (ABS(X3-X2)/X3-.000001)11,4,4
    4 CONTINUE
      STOP
   11 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FALSE2(X1,Y1,X2,Y2,X3)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
C     WRITE (12,40)
   40 FORMAT(1H ,11HINTO FALSE2)
      X=X1
    2 X3=(X1*Y2*Y1)/(Y2-Y1)
      GLAMDA=((B2*B2&A2*A2*ALPHA*ALPHA)-X3*CAY*A1*A1)/(2.*A2*A2)
      FLAMDA=(ALPHA*ALPHA*B2*B2-X3*CAY*B1*B1)/(A2*A2)*(-1.0)
      W11=SQRT(GLAMDA&SQRT(GLAMDA*GLAMDA&FLAMDA))
      W2S1=GLAMDA-SQRT(GLAMDA*GLAMDA&FLAMDA)
      CALL CALC2(W2S1,W11,Y3)
      DO 4 I=1,100
      IF (Y3*Y2) 10,20,20
   20 Y2=Y1
      X2=X1
   10 XNEW=(X2*Y3-X3*Y2)/(Y3-Y2)
      X1=X2
      X2=X3
      X3=XNEW
      Y1=Y2
      Y2=Y3
      GLAMDA=((B2*B2&A2*A2*ALPHA*ALPHA)-X3*CAY*A1*A1)/(2.*A2*A2)
      FLAMDA=(ALPHA*ALPHA*B2*B2-X3*CAY*B1*B1)/(A2*A2)*(-1.0)
      W11=SQRT(GLAMDA&SQRT(GLAMDA*GLAMDA&FLAMDA)
      W2S1=GLAMDA-SQRT(GLAMDA*GLAMDA&FLAMDA)
      CALL CALC2(W2S1,W11,Y3)
    6 IF (ABS(X3-X2)/X3-.000001)11,4,4
    4 CONTINUE
      STOP
   11 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE INCR(X2,A,B)
      COMMON A1,A2,B1,B2,ALPHA,CAY,DO,D1,D2,T,W2S,W1,X,TAN
      IF (X-1.0E7) 1,1,10
   10 STOP
    1 X2 = X&0.005*X
    6 G=((B2*B2&A2*A2*ALPHA*ALPHA)-X2*CAY*A1*A1)/(2.*A2*A2)
      F=(ALPHA*ALPHA*B2*B2-X2*CAY*B1*B1)/(A2*A2)*(-1.0)
      B=SQRT(G&SQRT(G*G&F)
      A=G-SQRT(G*G&F))
      RETURN
      END
```

T=	1.000
ALPHA=	1.000
A1=	1.000
A2=	1.000
B1=	0.500
B2=	10.000
D0=	10.000
D1=	11.000
D2=	1.000
CAY=	800.000
ROFZER=	37.273
A=	20.000
B=	1.000

	LAMDA FOR ZERO CROSSING	SLAMDA FOR ZERO CROSSING	ERROR FOR LAMDA TERMS
1	0.1411393E 01	0.1900908E-01	0.9809909E 00
2	0.1668593E 01	0.1607898E-01	0.9649119E 00
3	0.1954645E 01	0.1372591E-01	0.9511861E 00
4	0.2261170E 01	0.1186522E-01	0.9393209E 00
5	0.2596444E 01	0.1033309E-01	0.9289877E 00
6	0.2952309E 01	0.9087559E-02	0.9199002E 00
7	0.3336850E 01	0.8040302E-02	0.9118600E 00
8	0.3742055E 01	0.7169664E-02	0.9046903E 00
9	0.4175894E 01	0.6424800E-02	0.8982655E 00
10	0.4630448E 01	0.5794100E-02	0.8924715E 00
11	0.5113602E 01	0.5246650E-02	0.8872249E 00
12	0.5617506E 01	0.4776012E-02	0.8824489E 00
13	0.6149988E 01	0.4362494E-02	0.8780864E 00
14	0.6703234E 01	0.4002437E-02	0.8740840E 00
15	0.7285048E 01	0.3682791E-02	0.8704013E 00
16	0.7887643E 01	0.3401434E-02	0.8669999E 00
17	0.8518782E 01	0.3149429E-02	0.8638505E 00
18	0.9170735E 01	0.2925534E-02	0.8609250E 00
19	0.9851212E 01	0.2723451E-02	0.8582016E 00
20	0.1055253E 02	0.2542451E-02	0.8556591E 00
21	0.1128236E 02	0.2377986E-02	0.8532812E 00
22	0.1203299E 02	0.2229643E-02	0.8510516E 00
23	0.1281217E 02	0.2094047E-02	0.8489575E 00
24	0.1361215E 02	0.1970980E-02	0.8469866E 00
25	0.1444066E 02	0.1857899E-02	0.8451287E 00

APPENDIX E

FLOW DIAGRAM TO CALCULATE MISER

The following pages indicate a method to compute the MISER for any case studied in this thesis. The flow diagram is adaptable to FORTRAN IV programming. To make use of the program in Appendix D, it must be modified to instruct the computer so that the output is obtained on punched cards.

It was stated in section 4-2d that $(\beta^2 + \tau^2)$ is the coefficient of s^2 and $\beta^2\tau^2$ is the constant term in $D(s^2) - \lambda_i N(s^2)$. In order to separate β and τ, the coefficients of s^2 and the constant term are denoted by D_2 and D_0, respectively. So that

$$D_2 = \beta^2 + \tau^2 \tag{E.1}$$

and

$$D_0 = \beta^2 \times \tau^2 \tag{E.2}$$

Knowing D_0, D_2; β and τ may be determined. It is stressed that the flow diagram is to be read with the main body of the dissertation.

Using the rectangular approximation method of evaluating an integral Eq. (4.4-1) may be approximated as

$$\epsilon^2 = 1 + \frac{1}{C_i^2} \sum_{i=1}^{10} \left[\sum_u \phi_i(u)\Delta u \quad \sum_t \left\{ R_{xx}(\Delta) - 2R_{xw}(\Delta) \right\} \phi_i(t)\Delta t \right] \tag{E.3}$$

In words, the two cases $t > u$ and $u > t$ are evaluated separately within the constraints. For an i, u is selected and t is allowed to take on various values. The value obtained after the iteration of t is multiplied by the $\phi_i(u) \cdot \Delta u$. This is repeated till u is iterated. The next value of i is chosen and the procedure repeated till all the values of i are exhausted.

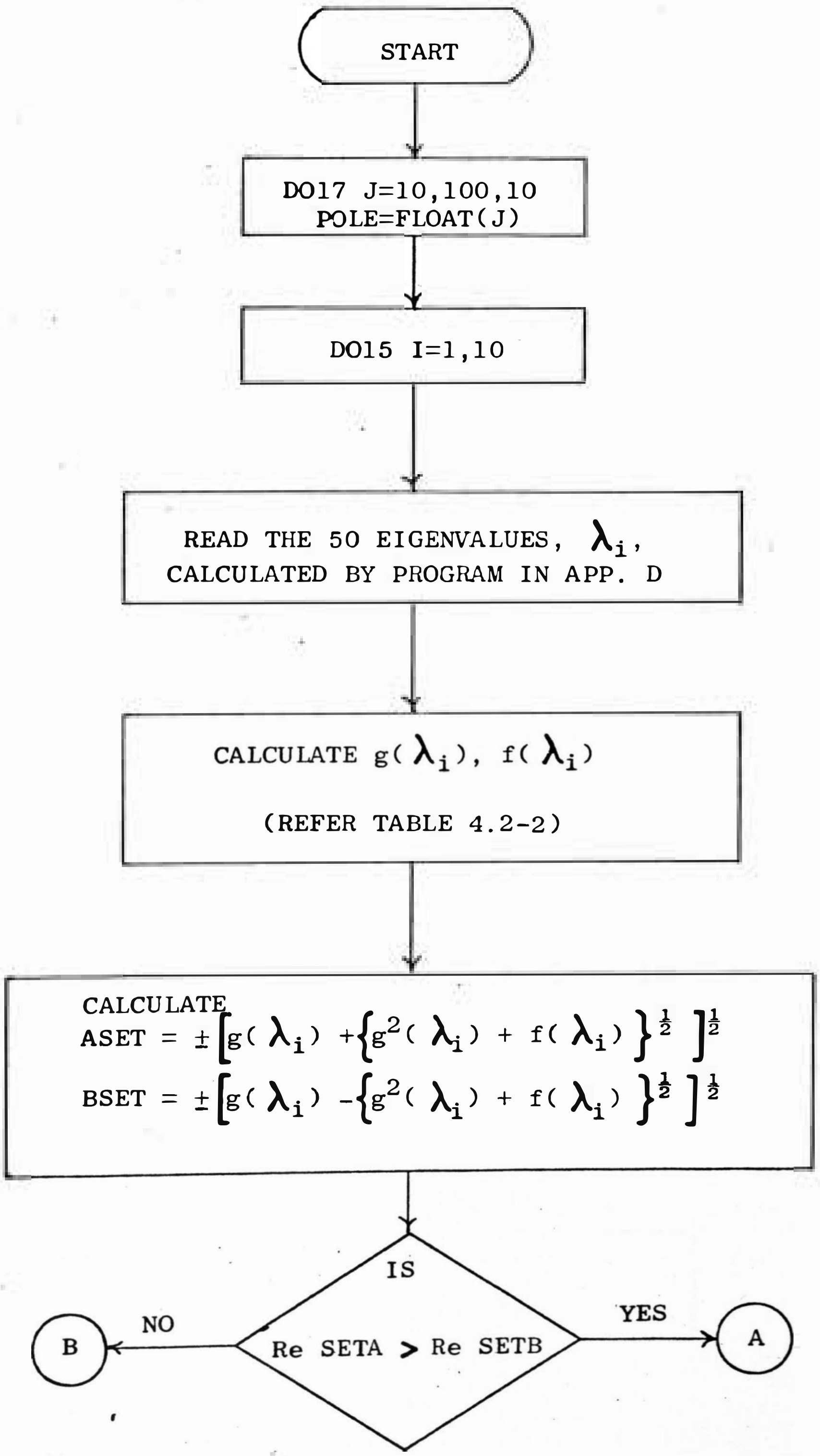
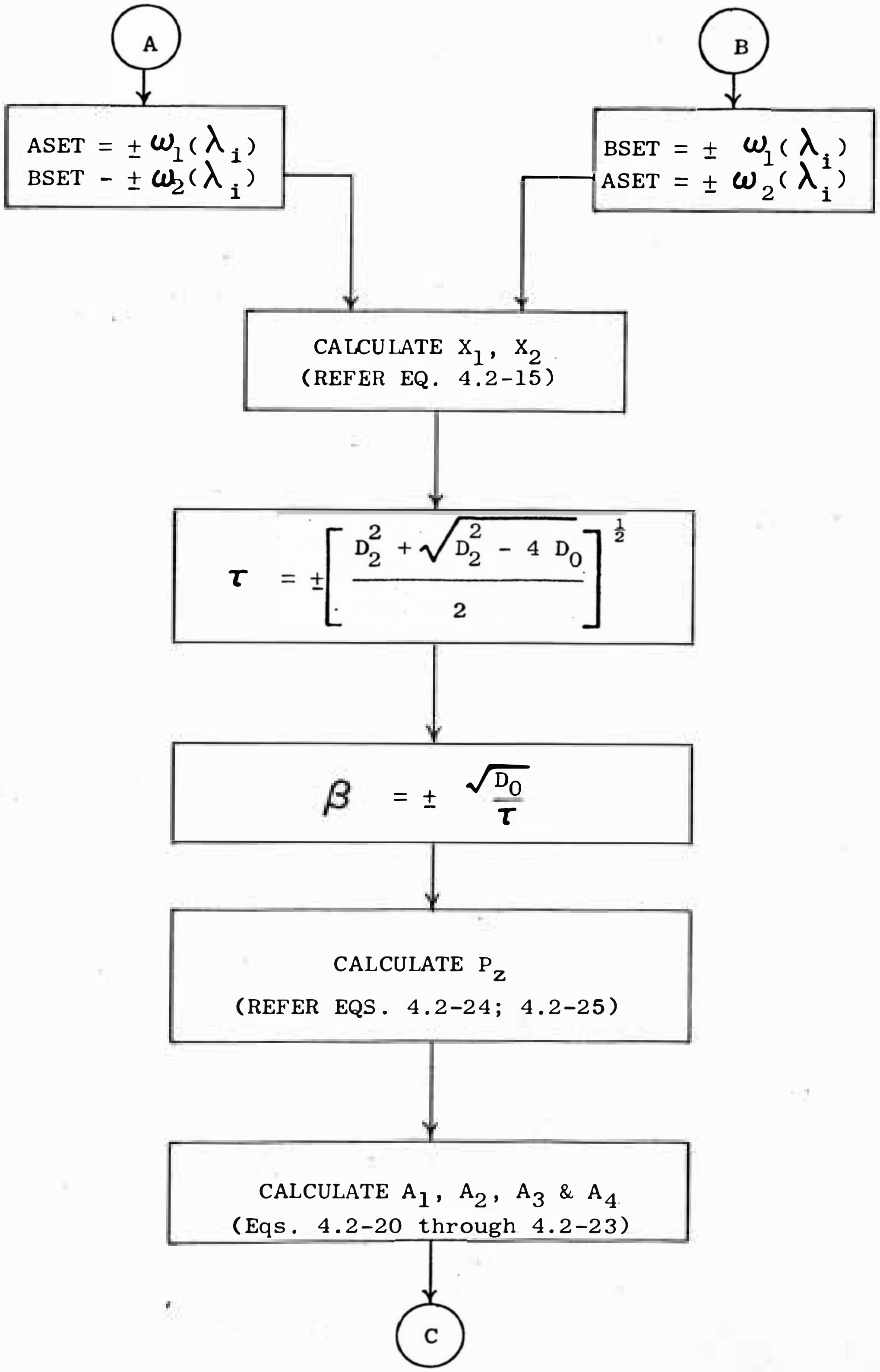
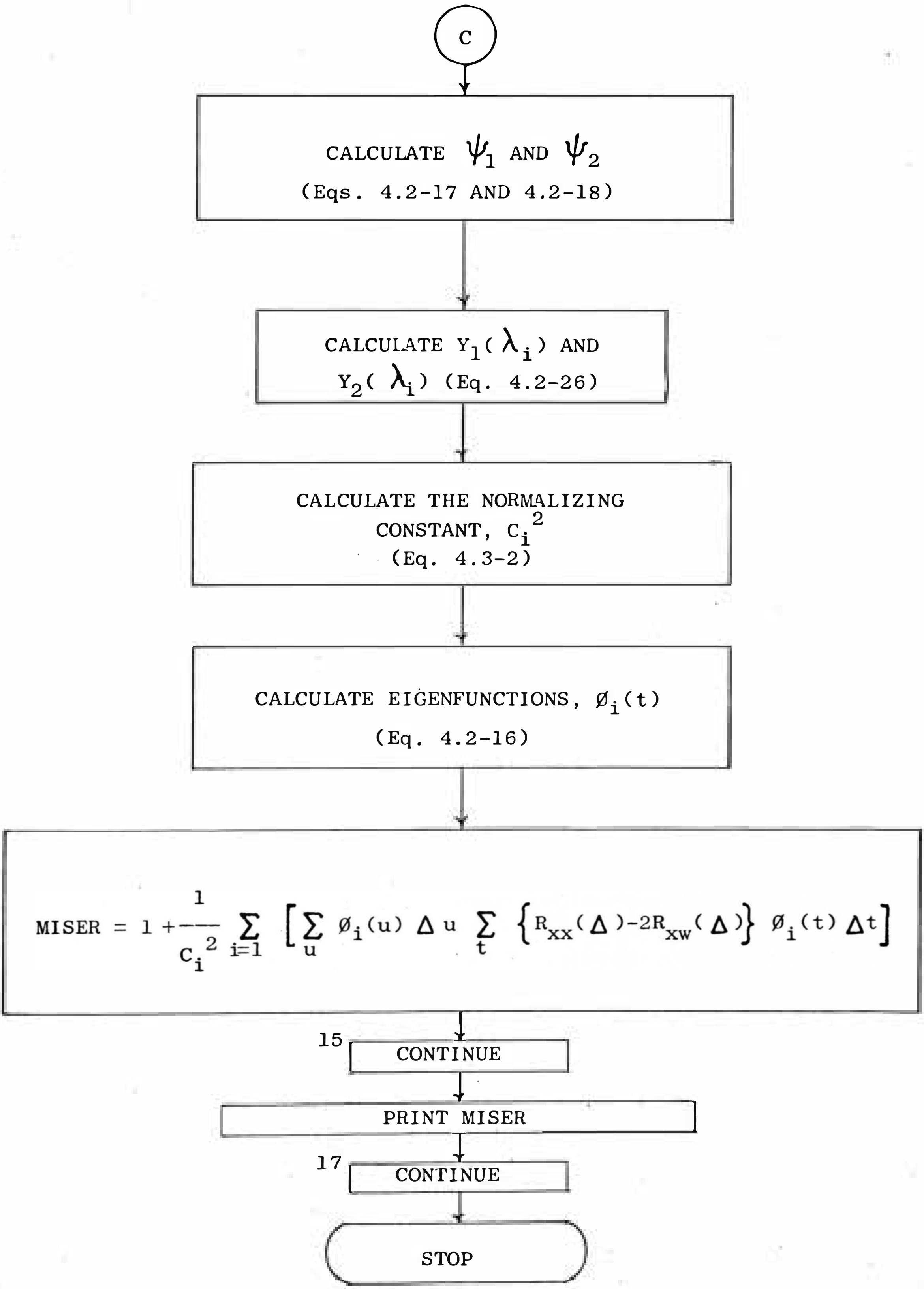


Figure E.1. Flow diagram to compute the value of MISER


A
B
ASET = $\pm\omega_1(\lambda_i)$
BSET - $\pm\omega_2(\lambda_i)$
BSET = $\pm\omega_1(\lambda_i)$
ASET = $\pm\omega_2(\lambda_i)$
CALCULATE X_1, X_2
(REFER EQ. 4.2-15)
$\tau = \pm\left[\frac{D_2^2 + \sqrt{D_2^2 - 4D_0}}{2}\right]^{\frac{1}{2}}$
$\beta = \pm\frac{\sqrt{D_0}}{\tau}$
CALCULATE P_z
(REFER EQS. 4.2-24; 4.2-25)
CALCULATE A_1, A_2, A_3 & A_4
(Eqs. 4.2-20 through 4.2-23)
C


C
CALCULATE ψ1 AND ψ2
(Eqs. 4.2-17 AND 4.2-18)
CALCULATE Y1(λi) AND
Y2(λi) (Eq. 4.2-26)
CALCULATE THE NORMALIZING
CONSTANT, Ci2
(Eq. 4.3-2)
CALCULATE EIGENFUNCTIONS, Øi(t)
(Eq. 4.2-16)
MISER = 1 + 1/Ci2 Σi=1 [Σu Øi(u) Δu Σt {Rxx(Δ)-2Rxw(Δ)} Øi(t) Δt]
15
CONTINUE
PRINT MISER
17
CONTINUE
STOP