*P-Finsler spaces with vanishing Douglas tensor

S. BÁCSÓ, I. PAPP

Abstract. The purpose of the present paper is to prove that a *P-Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater then three.

1. Introduction

Let $F^n(M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x,y)$ is the fundamental function defined on the manifold $T(M)\setminus 0$ of nonzero tangent vectors. Let us consider a geodesic curve $x^i = x^i(t)$,[1] $(t_0 \leq t \leq t_1)$. The system of differential equations for geodesic curves of F^n with respect to canonical parameter t is given by

$$\frac{d^2x^i}{dt^2} = -2G^i(x,y), \quad y^i = \frac{dx^i}{dt},$$

where

$$G^i = \frac{1}{4}g^{ir}\left(y^s\left(\frac{\partial L^2_{(r)}}{\partial x^s}\right) - \frac{\partial L^2}{\partial x^r}\right),$$

$$g_{ij} = \frac{1}{2}L^2_{(i)(j)}, \quad (i) = \frac{\partial}{\partial y^i}, \text{ and } \quad (g^{ij}) = (g_{ij})^{-1}.$$

The Berwald connection coefficients $G^i_j(x,y)$, $G^i_{jk}(x,y)$ can be derived from the function G^i, namely $G^i_j = G^i_{(j)}$ and $G^i_{jk} = G^i_{j(k)}$. The Berwald covariant derivative with respect to the Berwald connection can be written as

$$(1) \qquad T^i_{j;k} = \partial T^i_j/\partial x^k - T^i_{j(r)}G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

(Throughout the present paper we shall use the terminology and definitions described in Matsumoto's monograph [6].)

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[1] The Roman indices run over the range $1,\ldots,n$.

2. Douglas tensor, Randers metric, *P-space

Let us consider two Finsler space $F^n(M^n, L)$ and $\overline{F}^n(M^n, \overline{L})$ on a common underlying manifold M^n. A diffeomorphism $F^n \to \overline{F}^n$ is called geodesic if it maps an arbitrary geodesic of F^n to a geodesic of $\overline{F}^n$. In this case the change $L \to \overline{L}$ of the metric is called projective. It is well-known that the mapping $F^n \to \overline{F}^n$ is geodesic iff there exist a scalar field $p(x, y)$ satisfying the following equation

$$\overline{G}^i = G^i + p(x, y)y^i, \quad p \neq 0. \tag{2}$$

The projective factor $p(x, y)$ is a positive homogeneous function of degree one in y. From (2) we obtain the following equations

$$\overline{G}^i_j = G^i_j + p\delta^i_j + p_j y^i, \quad p_j = p_{(j)}, \tag{3}$$

$$\overline{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i, \quad p_{jk} = p_{j(k)}, \tag{4}$$

$$\overline{G}^i_{jkl} = G^i_{jkl} + p_{jk}\delta^i_l + p_{jl}\delta^i_k + p_{kl}\delta^i_j + p_{jkl}y^i, \quad p_{jkl} = p_{jk(l)}. \tag{5}$$

Substituting $p_{ij} = \left(\overline{G}_{ij} - G_{ij}\right)/(n+1)$ and $p_{ijk} = \left(\overline{G}_{ij(k)} - G_{ij(k)}\right)/(n+1)$ into (5) we obtain the so called Douglas tensor which is invariant under geodesic mappings, that is

$$D^i_{jkl} = G^i_{jkl} - \left(y^i G_{jk(l)} + \delta^i_j G_{kl} + \delta^i_k G_{jl} + \delta^i_l G_{jk}\right)/(n+1), \tag{6}$$

which is invariant under geodesic mappings, that is

$$D^i_{jkl} = \overline{D}^i_{jkl}. \tag{7}$$

We now consider some notions and theorems for special Finsler spaces.

Definition 1. ([1]) In an n-dimensional differentiable manifold M^n a Finsler metric $L(x, y) = \alpha(x, y) + \beta(x, y)$ is called Randers metric, where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric in M^n and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form in M^n. The Finsler space $F^n = (M^n, L) = \alpha + \beta$ with Randers metric is called Randers space.

Definition 2. ([1]) The Finsler metric $L = \alpha^2/\beta$ is called Kropina metric. The Finsler space $F^n = (M^n, L) = \alpha^2/\beta$ with Kropina metric is called Kropina space.

Definition 3. ([1], [6]) A Finsler space of dimension $n > 2$ is called C-reducible, if the tensor $C_{ijk} = \frac{1}{2} g_{ij(k)}$ can be written in the form

$$C_{ijk} = \frac{1}{n+1} \left(h_{ij} C_k + h_{ik} C_j + h_{jk} C_i \right), \tag{8}$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric tensor and $l_i = L_{(i)}$.

Theorem 1. ([7]) *A Finsler space F^n, $n \geq 3$, is C-reducible iff the metric is a Randers metric or a Kropina metric.*

Definition 4. ([4], [5]) A Finsler space F^n is called *P-Finsler space, if the tensor $P_{ijk} = \frac{1}{2} g_{ij;k}$ can be written in the form

$$P_{ijk} = \lambda(x,y) C_{ijk}. \tag{9}$$

Theorem 2. ([4]) *For $n > 3$ in a C-reducible *P-Finsler space $\lambda(x,y) = k(x) L(x,y)$ holds and $k(x)$ is only the function of position.*

3. *P-Randers space with vanishing Douglas tensor

Definition 5. ([3]) A Finsler space is said to be of Douglas type or Douglas space, iff the functions $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Theorem 3. ([3]) *A Finsler space is of Douglas type iff the Douglas tensor vanishes identically.*

Theorem 4. ([5]) *For $n > 3$, in a C-reducible *P-Finsler space $D^i_{jkl} = 0$ holds.*

If we consider a Randers change

$$\overline{L}(x,y) \to L(x,y) + \beta(x,y),$$

where $\beta(x,y)$ is a closed one-form, then this change $\overline{L} \to L$ is projective.

Definition 6. ([1]) A Finsler space is called Landsberg space if the condition $P_{ijk} = 0$ holds.

Theorem 5. ([2]) *If there exist a Randers change with respect to a projective scalar $p(x,y)$ between a Landsberg and a *P-Finsler space (fulfilling the condition $\overline{P}_{ijk} = p(x,y) \overline{C}_{ijk}$), then $p(x,y)$ can be given by the equation*

$$p(x,y) = e^{\varphi(x)} \overline{L}(x,y). \tag{10}$$

It is well-known that the Riemannian space is a special case of the Landsberg space. In a Riemannian space we have $D^i_{jkl} = 0$, and a *P-Randers space with a closed one-form $\beta(x,y)$ is a Finsler space with vanishing Douglas tensor

Theorem 6. ([3]) *A Randers space is a Douglas space iff $\beta(x,y)$ is a closed form. Then*

$$2G^i = \gamma^i_{jk}y^j y^k + \frac{r_{lm}y^l y^m}{\alpha+\beta}y^i, \tag{11}$$

where $\gamma^i_{jk}(x)$ is the Levi–Civita connection of a Riemannian space, r_{lm} is equal to $b_{i;j}$ hence r_{lm} depends only on position.

From the Theorem 6. and (10) follows that

$$\frac{r_{lm}y^l y^m}{\alpha+\beta} = e^{\varphi(x)}(\alpha+\beta)$$

that is

$$\frac{r_{lm}y^l y^m}{\overline{L}} = e^{\varphi(x)}\overline{L}.$$

From the last equation we obtain

$$r_{lm}y^l y^m = e^{\varphi(x)}\overline{L}^2.$$

Differentiating twice this equation by y^l and y^m we get

$$b_{i;j} = e^{\varphi(x)}\overline{g}_{ij}.$$

This means that the metrical tensor $\overline{g}_{ij}$ depends only on x, so we get the following

Theorem. *A *P-Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three.*

4. Further possibilities

From Theorem 1, Theorem 4 and our Theorem follows that only the *P-Kropina spaces can be *P-C reducible spaces with vanishing Douglas tensor which are different from Riemannian spaces. We would like to investigate this letter case in a forthcoming paper.

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Sándor Bácsó
Lajos Kossuth University
Institute of Mathematics and Informatics
4010 Debrecen P.O. Box 12
Hungary
E-mail: sbacso@math.klte.hu

Ildikó Papp
Lajos Kossuth University
Institute of Mathematics and Informatics
4010 Debrecen P.O. Box 12
Hungary
E-mail: ipapp@math.klte.hu