Jaeger's Strong 3-Flow Conjecture for Graphs in Low Genus Surfaces

by

Jamie de Jong

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Cun-Quan Zhang Professor, Dept. of Mathematics West Virginia University
Supervisor:	Bruce Richter Professor, Dept. of Combinatorics and Optimization University of Waterloo
Internal Members:	Chris Godsil Professor, Dept. of Combinatorics and Optimization University of Waterloo
	Penny Haxell Professor, Dept. of Combinatorics and Optimization University of Waterloo
Internal-External Member:	Therese Biedl Professor, David R. Cheriton School of Computer Science University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In 1972, Tutte posed the 3-Flow Conjecture: that all 4-edge-connected graphs have a nowhere zero 3-flow. This was extended by Jaeger et al.[1] to allow vertices to have a prescribed, possibly non-zero difference (modulo 3) between the inflow and outflow. He conjectured that all 5-edge-connected graphs with a valid prescription function have a nowhere zero 3-flow meeting that prescription. Kochol[2] showed that replacing 4-edge-connected with 5-edge-connected would suffice to prove the 3-Flow Conjecture and Lovász et al.[3] showed that both conjectures hold if the edge connectivity condition is relaxed to 6-edge-connected. Both problems are still open for 5-edge-connected graphs.

The 3-Flow Conjecture was known to hold for planar graphs, as it is the dual of Grötzsch's Colouring Theorem. Steinberg and Younger[4] provided the first direct proof using flows for planar graphs, as well as a proof for projective planar graphs. Richter et al.[5] provided the first direct proof using flows of Jaeger's Strong 3-Flow Conjecture for planar graphs. We extend their result to graphs embedded in the projective plane.

Lai[6] showed that Jaeger's Strong 3-Flow Conjecture cannot be extended to 4-edge-connected graphs by constructing an infinite family of 4-edge-connected graphs that do not have a nowhere zero 3-flow meeting their prescribed net flow. We prove that graphs with arbitrarily many non-crossing 4-edge-cuts sufficiently far apart have a nowhere zero 3-flow, regardless of their prescription function. This is a step toward answering the question of which 4-edge-connected graphs have this property.

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Chapter 1

Introduction

Tutte (cf. Bondy and Murty [3]) conjectured that every 4-edge-connected graph has a nowhere zero 3-flow. This is known as the 3-Flow Conjecture, and while progress has been made for many classes of graphs, it is still an open problem. For planar graphs the 3-Flow Conjecture is equivalent to Grötzsch's Theorem, and Steinberg and Younger [25] provided a direct proof using flows. Steinberg and Younger [25] also proved that the 3-Flow Conjecture holds for graphs embedded in the projective plane.

As an extension of $\mathbb{Z}_3$-flows, we can consider $\mathbb{Z}_3$-connectivity, where each vertex in the graph is assigned a prescription in $\mathbb{Z}_3$, which defines the net flow through the vertex. The prescriptions of the vertices in the graph must sum to zero in $\mathbb{Z}_3$. This led Jaeger et al. [13] to pose the following conjecture.

Conjecture 3.0.5 (Jaeger's Strong 3-Flow Conjecture)**.** *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

Lai and Li [19] proved that Jaeger's Strong 3-Flow Conjecture holds for planar graphs using the duality with graph colouring. Richter et al. [23] provided the first direct proof of this result using flows. We prove Jaeger's Strong 3-Flow Conjecture for two main classes of graphs. Our first result is Theorem 4.5.1.

Theorem 4.5.1. *Let G be a 5-edge-connected graph embedded in the projective plane. Then G is $\mathbb{Z}_3$-connected.*

This extends the result of Richter et al. [23] to apply to projective planar graphs. The proof of Theorem 4.5.1 appears in Chapter 4, following preliminary results regarding planar graphs.

Our second main result is to extend the result of Richter et al. [23] to allow non-crossing 4-edge-cuts provided they are sufficiently far apart; i.e. not incident with adjacent faces. This is motivated by an example due to Lai [18] of an infinite family of 4-edge-connected graphs that do not have a nowhere zero 3-flow for all valid prescription functions. This raises the question of which 4-edge-connected graphs do have such a flow. Our result is a step toward answering this question, as we provide a class of graphs allowing arbitrarily many 4-edge-cuts that do have a nowhere zero 3-flow for all valid prescription functions. The proof of this result appears in Chapter 5.

In Chapter 2 we summarise the relevant background material in graph theory required to understand the results in this thesis. In Chapter 3 we discuss the background and history of the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture.

Chapter 2

Background Material

2.1 Basic Graph Theory

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a vertex set $V(G)$ and an edge multiset $E(G)$ of pairs of vertices in $V(G)$. For any $x, y \in V(G)$, x and y are *adjacent* if $\{x, y\} \in E(G)$; x and y are the *endpoints* of $\{x, y\}$. We often use xy to specify an edge with endpoints x and y, or refer to the edge directly (for example, as e) if the endpoints are less relevant. An edge $e \in E(G)$ and a vertex $v \in V(G)$ are *incident* if v is an endpoint of e. Edges e and f are *adjacent* if they have an endpoint in common. Edges $e, f \in E(G)$ are *parallel* if they have the same endpoints. An edge $\{x, x\}$ is called a *loop*. A graph is *simple* if it has no loops or parallel edges. A *directed graph* D is an ordered pair $(V(D), E(D))$ consisting of a vertex set $V(D)$ and a directed edge multiset $E(D)$ of ordered pairs of vertices (x, y) in $V(D)$.

The *degree* of a vertex $v \in V(G)$, denoted $deg(v)$, is the number of times v appears as an endpoint of an edge $e \in E(G)$. Note that this means a loop contributes two to the degree of a vertex. The *neighbour set* of v, denoted $N(v)$, is given by

$$N(v) = \{u \in V(G) : \ u \text{ and } v \text{ are adjacent}\}.$$

Then $deg(v) \geq |N(v)|$. If G is a directed graph, then the *indegree* of a vertex $v \in V(G)$, denoted $indeg(v)$, is the number of directed edges in G of the form (u, v). The *outdegree* of a vertex $v \in V(G)$, denoted $outdeg(v)$, is the number of directed edges in G of the form (v, u). Note that loops contribute one to both the indegree and outdegree of a vertex.

Let G and H be graphs. Then H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore H is a *proper subgraph* of G, denoted $H \subset G$, if $H \subseteq G$, and either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. The subgraph $G[X]$ of G *induced* by the vertex set $X \subseteq V(G)$ is the subgraph such that

$$V(G[X]) = X,$$

$$E(G[X]) = \{\{x, y\} \in E(G) : x, y \in X\}.$$

Let $X, Y \subseteq V(G)$ where $X \cap Y = \emptyset$. We use the notation $[X : Y]$ to denote the set of edges in $E(G)$ with one endpoint in X and one endpoint in Y.

A *path* P of length k is a sequence of distinct vertices $\{v_0, v_1, ..., v_k\}$ such that for all $i \in \{0, 1, ..., k-1\}$, v_i and v_{i+1} are adjacent. A *cycle* of length k in G is a subgraph consisting of a sequence of distinct vertices $\{v_0, v_1, ..., v_{k-1}\}$ such that for all $i \in \{0, 1, ..., k-1\}$, v_i and v_{i+1} are adjacent (where we work modulo k). The *girth* of G is the length of the shortest cycle in G. Let C be a cycle consisting of vertices v_0, v_1, ..., v_{k-1}. Then an edge e is a *chord* of C if e is not an edge in the cycle, but both endpoints of e are in $\{v_0, v_1, ..., v_{k-1}\}$. A *circulant* graph is a graph consisting of a sequence of distinct vertices $\{v_0, v_1, ..., v_{k-1}\}$ and a distance set $J \subseteq \{0, ..., \lfloor \frac{k}{2} \rfloor\}$ such that for all $i \in \{0, 1, ..., k-1\}$ and $j \in J$, v_i and v_{i+j} are adjacent (where we work modulo k). Note that a cycle is a circulant graph with distance set $J = \{1\}$.

Let P_1 and P_2 be paths in G with endpoints x and y. Then P_1 and P_2 are *vertex-disjoint* if $V(P_1) \cap V(P_2) = \{x, y\}$. Similarly, P_1 and P_2 are *edge-disjoint* if $E(P_1) \cap E(P_2) = \emptyset$. A set $\mathcal{P}$ of paths in G is *edge-disjoint* (*vertex-disjoint*) if each pair of paths in $\mathcal{P}$ is edge-disjoint (vertex-disjoint).

A *walk* W of length k in G is a sequence $\{v_0 e_1 v_1 e_2 ... e_k v_k\}$ of vertices and edges such that for all $i \in \{1, 2, ..., k\}$, v_{i-1} and v_i are the endpoints of e_i. If all edges and vertices of W are distinct, then W is a path. A *circuit* is a walk in which the first and last vertices are the same.

The *complete graph* K_n is the simple graph consisting of n vertices, where every pair of vertices is adjacent. A *bipartite graph* G is a graph with two sets of vertices X and Y, where $X \cap Y = \emptyset$, $X \cup Y = V(G)$, and every edge in G has one endpoint in X and one endpoint in Y. Then G has the *partition* (X, Y). The *complete bipartite graph* $K_{x,y}$ is the simple bipartite graph with partition (X, Y) such that $|X| = x$, $|Y| = y$, and for all $u \in X$, $v \in Y$, u and v are adjacent.

2.2 Deletion, Contraction, and Subdivision

Let G be a graph. If we *delete an edge* $e \in E(G)$, we obtain the graph $G - e$ such that $V(G - e) = V(G)$, and $E(G - e) = E(G) - e$. If we *delete a vertex* $v \in V(G)$, we obtain the graph $G - v$ such that $V(G - v) = V(G) - v$, and

$$E(G - v) = E(G) - \{e \in E(G) : \ e \text{ is incident to } v\}.$$

Let X be a set of vertices and edges in G. Then we *delete* X to obtain the graph $G - X$, by deleting all vertices and edges in X from G.

Consider an edge $e = \{x, y\} \in E(G)$. We *contract* e to obtain the graph G/e such that

$$V(G/e) = (V(G) \cup \{z\}) - \{x, y\},$$

$$\begin{aligned} E(G/e) = &E(G - \{x, y\}) \cup \{\{z, u\} : \text{for each } \{v, u\} \in [\{x, y\} : V(G - \{x, y\})]\} \\ &\cup \{\{z, z\} : \text{for each } f \in E(G[\{x, y\}]), f \neq e\}. \end{aligned}$$

In other words, G/e is obtained by replacing x and y with a new vertex z, and replacing x and y with z as endpoints of edges in $E(G)$.

Let H be an induced subgraph of G. We *contract* H to obtain the graph G/H, where

$$V(G/H) = (V(G) \cup \{z\}) - V(H),$$

$$E(G/H) = E(G - H) \cup \{\{z, u\} : \text{for each } \{v, u\} \in [V(H) : V(G - H)]\}.$$

If H is connected, this operation can also be viewed as a sequence of contractions of edges in H, followed by the deletion of any subsequent loops at the vertex of contraction. We will also use the notation G/X where $X \subseteq V(G)$ to refer to $G/G[X]$ in cases where the vertex set is defined first.

Let $\{x, y\}$ be an edge of a graph G. We *subdivide* $\{x, y\}$ to obtain the graph G' such that

$$V(G') = V(G) \cup \{z\},$$

and

$$E(G') = (E(G) \cup \{\{x, z\}, \{y, z\}\}) - \{x, y\}.$$

In other words, a new vertex z and edges $\{x, z\}$ and $\{y, z\}$ replace the edge $\{x, y\}$. A graph H is a *subdivision* of G if there is a sequence G_0, G_1,..., G_k of graphs such that $G_0 = G$, $G_k = H$, and for $1 \leq i \leq k$, G_i is obtained from G_{i-1} by subdividing an edge of G_{i-1}.

Let u, v, and w be vertices in a graph G where both u and v are adjacent to w. The graph G' obtained from G by *lifting* the edges $\{u, w\}$ and $\{v, w\}$ is defined by

$$V(G') = V(G),$$

$$E(G') = (E(G) - \{\{u, w\}, \{v, w\}\}) \cup \{\{u, v\}\}.$$

2.3 Connectivity

A graph G is *connected* if for any pair of vertices $x, y \in V(G)$, there is a path in G containing x and y. Otherwise, G is *disconnected*. A *component* of G is a maximal connected subgraph of G.

Let G be a connected graph. A *cut* in G is a set $X \subseteq V(G)$ such that $G - X$ is disconnected. If a cut X consists of a single vertex x, then we refer to x as a *cut vertex*. The *connectivity* of G is the size of a minimum cut. The graph G is k*-connected* if its connectivity is at least k. A *block* of G is a maximal 2-connected subgraph of G.

An *edge-cut* in G, denoted $\delta_G(X)$ where $X \subset V(G)$, is $[X : V(G) - X]$. Thus $G - \delta_G(X)$ is disconnected. We also use the notation $\delta(H)$ where $H \subseteq G$ to refer to $\delta(V(H))$ in cases where the subgraph is defined first. If $\delta_G(X)$ consists of a single edge e, then e is a *bridge*. The *edge-connectivity* of G is the size of a minimum edge-cut. A graph is k*-edge-connected* if its edge-connectivity is at least k. In cases where the graph in question is clear we will drop the subscript and simply refer to an edge-cut as $\delta(X)$.

An edge-cut $\delta(A)$ is *non-peripheral* if $|A|, |G - A| \geq 2$. This and related concepts will be used frequently throughout this thesis, as we work with graphs containing small non-peripheral edge-cuts.

We consider the effect of deletion and contraction on the edge-connectivity of a graph. These effects will be particularly relevant in Chapters 4 and 5, as we delete and contract edges or subgraphs and check that the resulting graph maintains the edge-connectivity requirement of our result.

Lemma 2.3.1. *Let G be a k-edge-connected graph. Let $e \in E(G)$, $v \in V(G)$, and H be a proper, connected subgraph of G. Then*

1. *$G - e$ is $(k-1)$-edge-connected,*
2. *$G - v$ is $(k-d)$-edge-connected where $d = \lfloor \frac{deg(v)}{2} \rfloor$,*
3. *G/e is k-edge-connected,*
4. *G/H is k-edge-connected, and*
5. *$G - H$ is $(k-d)$-edge-connected where $d = \lfloor \frac{|\delta(H)|}{2} \rfloor$.*

Proof.

1. Suppose for a contradiction that $G - e$ contains an edge-cut $\delta_{G-e}(A)$ where we have $|\delta_{G-e}(A)| \leq k-2$. Let x and y be the endpoints of e. If $x, y \in A$, it follows that $\delta_G(A) = \delta_{G-e}(A)$, contradicting the k-edge-connectivity of G. The case $x, y \in G - A$ is equivalent. Suppose that $x \in A$ and $y \in G - A$. Then $\delta_G(A) = \delta_{G-e}(A) \cup \{e\}$, so $|\delta_G(A)| \leq k-1$, a contradiction. Hence $G - e$ is $(k-1)$-edge-connected.

2. Suppose for a contradiction that $G - v$ contains an edge-cut $\delta_{G-v}(A)$ where we have $|\delta_{G-v}(A)| \leq k-d-1$. Each neighbour of v is in either A or $G - (\{v\} \cup A)$. By the pigeonhole principle, either $|[\{v\} : A]|$ or $|[\{v\} : V(G - (\{v\} \cup A))]|$ is at most $d = \lfloor \frac{deg(v)}{2} \rfloor$. Suppose without loss of generality that $|[\{v\} : A]| \leq d$. Then
$$\delta_G(A) = \delta_{G-v}(A) \cup [\{v\} : A].$$
Therefore
$$|\delta_G(A)| = |\delta_{G-v}(A)| + |[\{v\} : A]| \leq k-1,$$
a contradiction.

3. Suppose for a contradiction that G/e contains an edge-cut $\delta_{G/e}(A)$ where we have $|\delta_{G/e}(A)| \leq k-1$. Let x and y be the endpoints of e and let z be the vertex of contraction in G/e. Without loss of generality, suppose that $z \in G/e - A$. Then in G, x and y are in $G - A$. Hence $\delta_G(A) = \delta_{G/e}(A)$, contradicting the k-edge-connectivity of G.

4. Suppose for a contradiction that G/H contains an edge-cut $\delta_{G/H}(A)$ where we have $|\delta_{G/H}(A)| \leq k-1$. Let z be the vertex of contraction in G/H. Without loss of generality, suppose that $z \in (G/H) - A$. Then in G, $V(H)$ is in $G - A$. Hence $\delta_G(A) = \delta_{G/H}(A)$, contradicting the k-edge-connectivity of G.

5. This follows from (4) and (2). □

In Chapters 4 and 5 we will consider the possibility of our graphs containing cut vertices. The following lemma is used to show that we may consider the blocks of the graph, as they maintain the edge-connectivity requirement.

Lemma 2.3.2. *Let G be a k-edge-connected graph. Suppose that G contains a cut vertex v. Let H and K be maximal subgraphs of G such that $H \cap K = (\{v\}, \emptyset)$ and $H \cup K = G$. Then H and K are k-edge-connected.*

Proof. Suppose for a contradiction that H is not k-edge-connected. Then there exists an edge-cut $\delta_H(A)$ where $|\delta_H(A)| \leq k-1$. Suppose without loss of generality that $v \in H - A$. Then $\delta_G(A) = \delta_H(A)$, contradicting the k-edge-connectivity of G. Hence H is k-edge-connected. The same is true of K. □

2.4 Surfaces

While a graph is defined by its vertices and edges, there are multiple ways to represent the same graph. A *drawing* of a graph G is a mapping from the vertices of the graph to distinct points in a surface S, and from edges to curves connecting their two endpoints, that do not intersect other vertices. A *crossing* in a drawing of a graph is an intersection between the curves representing two edges. An *embedding* is a drawing without crossings.

If ϕ is an embedding of G in a surface S, then the *faces* of ϕ are the components of $S \setminus \phi(G)$. Throughout this thesis the graph G will always come with an embedding in a surface, so we will refer to the faces of the embedding as the faces of G. We denote the set of faces of G as $F(G)$.

The *length* of a face f of a graph G is length of its boundary walk. We say that faces $f_1, f_2 \in F(G)$ are *adjacent* if their boundaries have an edge in common.

A graph is *planar* if it can be embedded on the plane. The infinite face is referred to as the *outer face* or the *unbounded face*. A vertex that is not on the boundary of the outer face is called an *internal vertex*. The dual of G, denoted G^*, is given by

$$V(G^*) = F(G),$$

$$E(G^*) = \{\{F_1, F_2\} : \text{ for each edge } e \in E(G) \text{ incident with } F_1 \text{ and } F_2\}.$$

Both the plane and graphs embedded on the plane have been studied extensively, yielding many results that can be used to classify planar graphs. Two of the most well known results are stated below.

Theorem 2.4.1 (Euler's Formula). *Let G be a connected graph embedded on the plane. Let $n = |V(G)|$, $m = |E(G)|$, and f be the number of faces in the embedding of G. Then $n - m + f = 2$.*

Theorem 2.4.2 (Kuratowski's Theorem). *A graph G is planar if and only if it has no subgraph that is a subdivision of $K_{3,3}$ or K_5.*

Due to the ease of characterising planar graphs, for many problems in graph theory, such as flows and colouring, planar graphs are the first consideration. However, graphs can be embedded on other surfaces, several of which we consider here, beginning with the sphere. Consider the following proposition.

Proposition 2.4.3. *A graph G can be embedded on the sphere if and only if G is planar.*

Proof. Let G be a graph embedded in the plane. Consider the sphere being bisected by the plane. Let x be one of the two points on the sphere that is furthest from the plane. For each point y in the plane, let ℓ be the line in $\mathbb{R}^3$ that contains both x and y. Let z be the point of intersection between the sphere and ℓ that is not x, and map y to z. It is clear that this mapping from the plane to the sphere without x is a continuous surjection. Suppose that two points y_1 and y_2 map to z. Then the unique line ℓ' in $\mathbb{R}^3$ containing both x and z, contains both y_1 and y_2. Since x is not on the plane, ℓ intersects the plane in at most one point. Since y_1 and y_2 are both on the plane, they must therefore be the same point. Hence the mapping is bijective. $\square$

Note that to obtain a planar embedding of G, given an embedding on the sphere, we chose a face f arbitrarily to be the outer face and reverse the given mapping. This means that given a planar embedding of G, and any face f in the embedding, there is a planar embedding of

G where f is the outer face.

We now consider how to construct other surfaces from the sphere. A *handle* can be added to the sphere by cutting two circular holes in the surface of the sphere, and joining them with a cylinder. A *crosscap* can be added to the sphere by cutting a circular hole, and sewing each pair of opposite points together. Any connected, closed surface can be obtained from the sphere by adding some combination of handles and crosscaps. Two of the most commonly considered surfaces aside from the plane and sphere are the *projective plane* and the *torus*, which are obtained from the sphere by adding a crosscap and a handle respectively. If a graph can be embedded on the projective plane or torus, the graph is *projective planar* or *toroidal* respectively.

A surface is *orientable* if it does not contain a crosscap. A curve on a surface is *contractible* if it can be contracted to a point. A simple closed curve C on a surface S is contractible if and only if $S - C$ has two components, one of which is a disc. Thus in the plane, all closed curves are contractible. In the projective plane a closed curve is non-contractible if and only if every neighbourhood of it contains a Möbius band. If a graph is embedded in the projective plane so that all its cycles are contractible, then the given embedding is planar.

In later results we will discuss cuts in the projective plane. Lemma 2.4.5 is a well known result that will be necessary for reducing these cuts. We require the preliminary result in Lemma 2.4.4. For more detail or for a generalisation to other surfaces see Appendix B of Diestel [4].

Lemma 2.4.4. *Let G be a graph embedded in the projective plane. Then G does not contain two disjoint non-contractible cycles.*

Proof. Suppose for a contradiction that G contains two disjoint non-contractible cycles C_1 and C_2. Note that any neighbourhood of C_1 contains a Möbius band. Cut the surface along C_1, deleting C_1. Identify the equivalent (opposite) points on the crosscap, and fill the hole with a disk. The resulting surface is planar. Since C_2 is disjoint from C_1, C_2 is a non-contractible cycle in the plane, a contradiction. Thus G does not contain two disjoint non-contractible cycles. □

Lemma 2.4.5. *Let G be a connected graph embedded in the projective plane, and let $\delta(A)$ be a minimal edge-cut in G. Then either G/A or $G/(G-A)$ is planar.*

Proof. Suppose that both $G[A]$ and $G - A$ contain a non-contractible cycle. Then G contains disjoint non-contractible cycles, contradicting Lemma 2.4.4. Thus at least one of $G[A]$ and $G - A$ does not contain a non-contractible cycle. Without loss of generality, suppose that $G[A]$ does not contain a non-contractible cycle. Then the given embedding of $G[A]$ is a planar embedding. Let F be the face of $G[A]$ that contains $G - A$. Then all edges of $\delta(A)$ are also in F. Delete $G - A$ and insert a single vertex incident with all edges in $\delta(A)$. The resulting graph is planar by construction, and is G/A. □

2.5 Vertex Colouring

Let S be a set of colours. A *vertex colouring* of a graph G is a function $c : V(G) \to S$ such that for all $e = \{u, v\} \in E(G)$, $c(u) \neq c(v)$. A *k-colouring* is a vertex colouring $c : V(G) \to \{1, ..., k\}$. A graph G is *k-colourable* if it has a k-colouring. The *chromatic number* of G is the minimum k for which G is k-colourable.

We briefly consider three of the most well known results in colouring, as they have dual results in the area of flows.

The Four Colour Theorem is believed to have been posed as a conjecture by Francis Guthrie in 1852, when he noticed that only four colours were needed to colour the different counties in England so that no two adjacent counties had the same colour. He suggested that this was true for all maps. This translates to a problem in graph theory by placing a vertex in each region of the map, and adding an edge between two regions if they share a border along a non-trivial segment.

Theorem 2.5.1 (Four Colour Theorem)**.** *Every loopless planar graph is* 4*-colourable.*

Kempe [14] provided a claimed proof of the Four Colour Theorem that was widely believed to be correct until Heawood [8] showed it to be incorrect. This proof attempt used the idea of Kempe chains: maximal connected subgraphs containing vertices of only two colours. While Kempe's proof of the Four Colour Theorem was incorrect, the same techniques were used by Heawood [8] to prove the simpler Five Colour Theorem. Appel and Haken [1] provided a computer assisted proof of the Four Colour Theorem, over 100 years after it was posed as a conjecture.

Theorem 2.5.2 (Five Colour Theorem)**.** *Every loopless planar graph is* 5*-colourable.*

Grötzsch's Theorem is a similar result for 3-colouring graphs, and is the dual of the Three Flow Conjecture. It was proven by Grötzsch [6]. A simpler proof was provided by Thomassen [26], using list colouring.

Theorem 2.5.3 (Grötzsch's Theorem)**.** *Every loopless planar graph not containing a cycle of length* 3 *is* 3*-colourable.*

Let Γ be an abelian group. Then G is Γ-*colourable* if for every function f assigning to each edge an ordered pair consisting of a direction, and a value $f(e) \in \Gamma$, there exists a function $c : V(G) \to \Gamma$ such that if D is the resulting directed graph, then for each $e = (u, v) \in E(D)$, $c(u) - c(v) \neq f(e)$. If $\Gamma = \mathbb{Z}_k$ and $f(e) = 0$ for all $e \in E(G)$, then Γ-colourability is the same as k-colourability. The *group chromatic number* of G is the smallest positive interger k for which G is Γ-colourable for every abelian group Γ of order at least k.

Let S be a set of colours. A graph G is k*-choosable* if for every set

$$\{C_v \subseteq S : v \in V(G), |C_v| = k\},$$

G has a vertex colouring c such that for all $v \in V(G)$, $c(v) \in C_v$.

2.6 Flows

A k*-flow* on a graph G is a function that assigns to each edge $e \in E(G)$ an ordered pair consisting of a direction, and a value $f(e) \in \{0, ..., k-1\}$, such that if D is the resulting directed graph, then, for each vertex $v \in V(G)$,

$$\sum_{e=(u,v)\in E(D)} f(e) - \sum_{e=(v,w)\in E(D)} f(e) = 0.$$

It is easy to see that every graph G has a k-flow for every value of k: set $f(e) = 0$ for all $e \in E(G)$. Therefore it is typical to use the following more restrictive concept. A *nowhere zero* k*-flow* on G is a k-flow on G such that no edge is assigned the value zero. To see that this is indeed more restrictive, consider the following theorem (see Diestel [4] for a proof).

Theorem 2.6.1. *A graph has a nowhere zero* 2*-flow if and only if all of its vertices have even degree.*

The *flow number* of G, denoted $\varphi(G)$, is the smallest positive integer k for which G has a nowhere zero k-flow. The interest in flows came out of the study of graph colourings, via Theorem 2.6.2 [28]. Here $\chi(G)$ is the chromatic number of G.

Theorem 2.6.2. *Let G be a planar graph, and let G^* be its dual. Then $\chi(G) = \varphi(G^*)$.*

Thus, the following theorem is equivalent to the Four Colour Theorem.

Theorem 2.6.3. *Let G be a bridgeless planar graph. Then G has a nowhere zero 4-flow.*

It was shown by König [16] that this is not true for all graphs, as the Petersen graph is a counterexample. Based on planar duality, Tutte proposed the following three conjectures as extensions of the Five Colour Theorem [28], the Four Colour Theorem [29], and Grötzsch's Theorem (cf. [3]) respectively.

Conjecture 2.6.4 (5-Flow Conjecture)**.** *Let G be a bridgeless graph. Then G has a nowhere zero 5-flow.*

Conjecture 2.6.5 (4-Flow Conjecture)**.** *Let G be a bridgeless graph not containing the Petersen graph as a minor. Then G has a nowhere zero 4-flow.*

Conjecture 2.6.6 (3-Flow Conjecture)**.** *Let G be a 4-edge connected graph. Then G has a nowhere zero 3-flow.*

The 3-Flow Conjecture and an extension of it, Jaeger's Strong 3-Flow Conjecture, form the basis of this thesis. These conjectures will be discussed in more depth in Chapter 3.

2.7 $\mathbb{Z}_\mathbf{k}$-Flows and Valid Orientations

In the previous section we considered flows that assign each edge a value between 0 and k. However, we can also consider flows that assign each edge a value from a group, such as $\mathbb{Z}_k$.

Let G be a graph. A $\mathbb{Z}_k$*-flow* on G is a function that assigns to each edge $e \in E(G)$ an ordered pair consisting of a direction and a value $f(e) \in \mathbb{Z}_k$, such that if D is the resulting directed graph, then for each vertex $v \in V(G)$,

$$\sum_{e=(u,v)\in E(D)} f(e) - \sum_{e=(v,w)\in E(D)} f(e) = 0 \pmod{k}.$$

In 1950, Tutte proved that the two concepts are equivalent [4].

Theorem 2.7.1. *A graph has a nowhere zero k-flow if and only if it has a nowhere zero $\mathbb{Z}_k$-flow.*

Thus for the remainder of this thesis we will use the term k-flow to refer to a $\mathbb{Z}_k$-flow. Note that as we are working with 3-flows, it is not necessary to actually assign values to the edges.

We define an *orientation* of a graph G to be the directed graph D obtained by adding a direction to each edge. With reference to finding a 3-flow on G, an orientation is *valid* if for each vertex $v \in V(G)$,

$$\sum_{e=(u,v)\in E(D)} 1 - \sum_{e=(v,w)\in E(D)} 1 = 0 \pmod 3.$$

Lemma 2.7.2. *A graph G has a valid orientation if and only if there exists a nowhere zero 3-flow on G.*

Proof. First, suppose that G has a valid orientation. Let $f(e) = 1$ for all edges $e \in E(G)$. This yields a nowhere zero 3-flow on G. Now suppose that G has a nowhere zero 3-flow. For each edge $e \in E(G)$ where $f(e) = -1$, reverse the direction of e. The resulting orientation is a valid orientation of G. □

A vertex d in a graph G is a *directed vertex* if all its incident edges are directed. We call this an *orientation* of d. We say that an orientation of G *extends* the orientation of d if the direction of the edges at d is maintained. In cases involving a directed vertex we take the term *valid orientation* to include that the orientation extends that of d.

Chapter 3

The 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture

In this chapter we discuss several major conjectures regarding k-flows that motivate our work, especially surrounding 3-flows. In particular, we state Jaeger's Strong 3-Flow Conjecture, which forms the basis of our work. In Sections 3.2, 3.3, 3.4, and 3.5 we discuss prior progress toward these conjectures and summarise the techniques used, many of which we adapt in the proofs throughout this thesis. Section 3.1 introduces the preliminary ideas that are required to understand these techniques. We restate the 3-Flow Conjecture here for completeness.

Conjecture 3.0.1 (3-Flow Conjecture)**.** *Let G be a 4-edge-connected graph. Then G has a nowhere zero 3-flow.*

Following Tutte's 3-Flow Conjecture, Jaeger [12] posed the following weaker conjecture.

Conjecture 3.0.2 (Weak 3-Flow Conjecture)**.** *There is a natural number h such that every h-edge-connected graph has a nowhere zero 3-flow.*

He also defined the concept of a modular orientation. A graph G has a *modulo k orientation* if its edges can be directed to yield a directed graph D so that for all $v \in V(G)$,

$$\sum_{e=(u,v)\in E(D)} 1 = \sum_{e=(v,w)\in E(D)} 1 \pmod{k}.$$

Jaeger [12] showed that orientations can be used to prove results about flows.

Theorem 3.0.3 (Jaeger [12]). *A graph G has a modulo $2k+1$ orientation if and only if it has a circular $\left(2+\frac{1}{k}\right)$-flow.*

(A *circular k-flow* where $k \in \mathbb{R}_{\geq 2}$ is a real-valued flow modulo k, where every edge has flow value between 1 and $k-1$. This concept is not used elsewhere in this thesis.) Note that a modulo 3 orientation is equivalent to a 3-flow, which is what Theorem 3.0.3 says when $k = 1$. However, this result enables other flows to be treated as orientations also. Based on this equivalence, Jaeger [12] posed the following generalisation of the 3-Flow Conjecture.

Conjecture 3.0.4 (Circular Flow Conjecture). *Every $4k$-edge-connected graph admits a modulo $2k+1$ orientation.*

When $k = 1$ the Circular Flow Conjecture implies the 3-Flow Conjecture.

We define a *prescription function* for a graph G to be $p : V(G) \to \Gamma$, where Γ is an abelian group. A prescription function is *valid* if

$$\sum_{v \in V(G)} p(v) = 0.$$

Throughout this thesis, we will assume that for all prescription functions, $\Gamma = \mathbb{Z}_3$, unless indicated otherwise.

A flow f on G *satisfies* the prescription function p if for all $v \in V(G)$,

$$F(v) := \sum_{e=(u,v) \in E(D)} f(e) - \sum_{e=(v,w) \in E(D)} f(e) = p(v).$$

A graph is Γ-*connected* if for every valid prescription function p on Γ, G has a flow satisfying p.

The idea of a prescription function led Jaeger et al. [13] to pose the following conjecture.

Conjecture 3.0.5 (Jaeger's Strong 3-Flow Conjecture). *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

We define a modulo 3 orientation to be a *valid orientation* (with respect to a prescription function p), if for each vertex $v \in V(G)$,

$$p(v) = \sum_{e=(u,v) \in E(D)} 1 - \sum_{e=(v,w) \in E(D)} 1 \pmod 3.$$

In results where we are considering a 3-flow rather than $\mathbb{Z}_3$-connectivity, $p(v)$ is implicitly zero for every $v \in V(G)$.

We *orient* a vertex v by adding a direction to each edge incident with v so that the prescription at v is satisfied. If G has directed vertices, then we take the term *valid orientation* to include that the orientation extends the existing partial orientation.

3.1 Preliminary Ideas

We now discuss some preliminary ideas that have been used to prove numerous results regarding the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture. They will also be used throughout this thesis.

Let G be a graph (that may contain a directed vertex) and let p_G be a valid prescription function for G. We define the prescription function $p_{G'}$ for a graph G' obtained from G as follows:

- If G' is obtained from G by deleting a directed edge $e = (u, v)$, then $p_{G'}(u) = p_G(u)+1$, $p_{G'}(v) = p_G(v) - 1$, and $p_{G'}(w) = p_G(w)$ for all $w \in V(G) - \{u, v\}$.
- If G' is obtained from G by orienting an edge $e = \{u, v\}$ from u to v, and deleting e, then $p_{G'}(u) = p_G(u) + 1$, $p_{G'}(v) = p_G(v) - 1$, and $p_{G'}(w) = p_G(w)$ for all $w \in V(G) - \{u, v\}$.
- If G' is obtained from G by adding a directed edge from u to v, then $p_{G'}(u) = p_G(u)-1$, $p_{G'}(v) = p_G(v) + 1$, and $p_{G'}(w) = p_G(w)$ for all $w \in V(G) - \{u, v\}$.
- If G' is obtained from G by contracting a set of vertices $X \subseteq V(G)$ to a single vertex v, then $p_{G'}(v) = \sum_{x \in X} p_G(x)$, and $p_{G'}(w) = p_G(w)$ for all $w \in V(G) - X$.
- If G' is obtained from G by lifting the pair of edges $e_1 = \{u, v\}$ and $e_2 = \{v, w\}$, then $p_{G'}(w) = p_G(w)$ for all $w \in V(G)$.

Throughout the remainder of this thesis we will automatically adjust the prescription function as defined here, without mention. Note that if the prescription function is zero, then contraction and lifting are the only operations listed that maintain this property.

Lemma 3.1.1. *Let G be a graph with prescription function p, and let $v \in V(G)$. If v has at least two undirected incident edges that are not loops, then v can be oriented to satisfy $p(v)$.*

Proof. Arbitrarily orient all but two non-loop edges incident with v in G. Let G' be the graph obtained from G by deleting all directed edges incident with v. Direct both remaining edges in or out to satisfy prescriptions of -1 and $+1$ respectively, and one in and one out to satisfy a prescription of 0. □

Lemma 3.1.2. *If G has an orientation f, then either f is valid, or there exist $u, v \in V(G)$ such that $F(u) \neq p(u)$ and $F(v) \neq p(v)$.*

Proof. We may assume that f is not valid. Suppose that $F(v) \neq p(v)$ but $F(u) = p(u)$ for all vertices $u \neq v$. We have

$$\begin{aligned} p(v) &= -\sum_{u \in V(G)-\{v\}} p(u) \\ &= -\sum_{u \in V(G)-\{v\}} F(u) \\ &= -\left(\sum_{\substack{e=\{a,b\}\in E(G) \\ a,b\neq v}} f(e) - \sum_{\substack{e=\{a,b\}\in E(G) \\ a,b\neq v}} f(e) + \sum_{e=\{v,b\}\in E(G)} f(e) - \sum_{e=\{a,v\}\in E(G)} f(e) \right) \\ &= F(v), \end{aligned}$$

a contradiction. Thus at least two vertices do not meet their prescription. □

Therefore, it suffices to show that the prescription of all but one vertex is met, as this implies a valid orientation.

3.2 Direct Results

We first note that if the 3-Flow Conjecture is true, then it is tight. Consider the following well known result (see Diestel [4]). Note that a graph is *cubic* if every vertex has degree 3.

Lemma 3.2.1. *A cubic graph has a nowhere-zero 3-flow if and only if it is bipartite.*

We note that, in particular, this means that K_4 does not have a nowhere-zero 3-flow. Thus is clear that not all 3-edge-connected graphs have a nowhere zero 3-flow.

Kochol [15] proved the following result.

Theorem 3.2.2. *The following statements are pairwise equivalent.*

1. *Every* 4*-edge-connected graph has a nowhere zero* 3*-flow.*
2. *Every* 5*-edge-connected graph has a nowhere zero* 3*-flow.*
3. *Every* 5*-edge-connected graph with no non-peripheral* 5*-edge cut, and a directed vertex* d *of degree* 5 *has a valid orientation.*
4. *Every* 4*-edge-connected graph with only vertices of degree* 4 *and* 5*, and a directed vertex* d *has a valid orientation.*

This means that in order to prove the 3-Flow Conjecture it suffices to prove that every 5-edge-connected graph has a nowhere zero 3-flow. The proof focusses on the cases (3) $\implies$ (4) and (2) $\implies$ (3), as (1) $\implies$ (2) is trivial and (4) $\implies$ (1) is a well known result.

For the former, we may take a minimal counterexample to (4). If a non-peripheral 4 or 5-edge-cut $\delta(X)$ exists, then at least one of the graphs obtained by contracting X and $G - X$ can be shown to be a smaller counterexample. If an unoriented degree 4 vertex v exists, two pairs of edges may be lifted from it, deleting the vertex while maintaining a 4-edge-connected graph. If d has degree 4, a parallel edge may simply be added to one of its existing edges. Hence (3) implies (4).

For the latter, we may split d into three vertices joined in a path so that all have degree 3. This can be done in such a way that a valid orientation of the path implies the directions assigned to d originally. Kochol [15] replaced two disjoint edges of a copy of K_4 with this graph to form a graph H, and then replaced all edges of a copy of K_4 with H to form a graph G' that is 5-edge-connected. He shows that the graph obtained by replacing an edge of a graph that does not have a nowhere zero 3-flow with a graph that does not have a nowhere zero 3-flow, also does not have a nowhere zero 3-flow. Since K_4 does not have a nowhere zero 3-flow, the result follows.

In addition, Kochol [15] states the following conjecture, a slight weakening of Conjecture 3.0.4.

Conjecture 3.2.3. *Every* $(4k+1)$*-edge-connected graph admits a modulo* $2k+1$ *orientation.*

For $k = 1$, Theorem 3.2.2 shows that this is equivalent to the Circular Flow Conjecture.

The Weak 3-Flow Conjecture remained open until Thomassen [27] proved that $h = 8$ sufficed.

Theorem 3.2.4. *Every* 8*-edge-connected graph is* $\mathbb{Z}_3$*-connected.*

In order to prove this, Thomassen [27] introduced a directed vertex d and applied induction to the number of edges in the graph. The purpose of introducing d is to allow one to reduce certain cuts in the graph, by contracting one side of the cut, applying the induction hypothesis, transferring this orientation to the original graph, contracting the other side of the cut, and applying the induction hypothesis once more. An example of this method can be seen in Section 3.3, and it is a technique that will be used throughout this thesis. If no directed vertex is permitted, the second contraction does not produce a graph that the induction hypothesis applies to.

Thomassen [27] also showed the following result toward the Circular Flow Conjecture.

Theorem 3.2.5. *Let* k *be an odd integer. Every* $(2k^2 + k)$*-edge-connected graph admits a modulo* k *orientation.*

Lovász et al. [21] extended this to the following result.

Theorem 3.2.6. *Every* $6k$*-edge-connected graph admits a modulo* $2k + 1$ *orientation.*

For $k = 1$ this is equivalent to the following result.

Theorem 3.2.7. *If* G *is a* 6*-edge-connected graph, then* G *is* $\mathbb{Z}_3$*-connected.*

Thus Lovász et al. [21] proved results analogous to the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture for 6-edge-connected graphs. Their proof uses similar techniques to that of Thomassen [27], introducing a directed vertex of small degree in order to reduce small edge-cuts.

Recently, Han et al. [7] proved that for every $k \geq 3$, there exists a $4k$-edge-connected graph without a modulo $2k+1$ orientation, and for every $k \geq 5$, there exists a $(4k+1)$-edge-connected graph without a modulo $2k+1$ orientation. Hence Jaeger's Circular Flow Conjecture is false for $k \geq 3$, and the simpler conjecture (Conjecture 3.2.3) of Kochol [15] is false for $k \geq 5$.

3.3 Flows in Graphs on a Surface

Since the 3-Flow Conjecture for planar graphs is equivalent to Grötzsch's Theorem, it was shown by Grötzsch [6] thirteen years before the conjecture was posed by Tutte (cf. [3]). The first direct proof of the 3-Flow Conjecture for planar graphs was by Steinberg and Younger [25], who also proved that the 3-Flow Conjecture is true for projective planar graphs. Their result is summarised in Theorem 3.3.1.

Theorem 3.3.1. *Let G be a 3-edge-connected graph that is*

1. *planar and has at most three 3-edge-cuts,*
2. *projective planar and has at most one 3-edge-cut, or*
3. *planar, has a distinguished oriented vertex d with degree 4 or 5 and at most one 3-edge-cut, and the minority edge at a degree 5 d does not lie in a 3-edge-cut.*

Then G has a valid orientation.

The *minority edge* at a degree 5 d is defined to be the edge that has the opposite direction from the remaining four. This is guaranteed to exist as the net flow at d is zero. The third condition here is only added to allow an inductive proof; the first two imply the 3-Flow Conjecture in the plane and projective plane respectively. We will discuss the outline of their proof technique here, as it has many similarities with the ideas used in this thesis.

Steinberg and Younger [25] use induction on the number of edges in G. They consider a series of configurations in G (subgraphs) that can be reduced to give a smaller graph that meets the conditions of the theorem. Configurations such as loops, parallel edges, and cut vertices are straightforward to reduce, as we will see in Chapter 4. The purpose behind introducing an oriented vertex d, is to allow a small edge-cut to be reduced.

Reducing Small Cuts. Let $\delta(A)$ be a minimal (with respect to the number of edges) non-peripheral edge-cut in G of size at most 5. Suppose that G is planar, $d \in A$ if applicable, and $G - A$ has at most one 3-edge-cut. Let G' be the graph obtained from G by contracting $G - A$ to a vertex v. It is clear that G' is planar. By Lemma 2.3.1, G' has at most three 3-edge-cuts, and if $d \in G'$, it has at most one 3-edge-cut. Hence G' has a valid orientation by the induction hypothesis.

Transfer this orientation to G. Now let G'' be the graph obtained from G by contracting A to a vertex d'. It is clear that G'' is planar. By definition, G'' has at most one 3-edge-cut aside from $\delta(\{d'\})$, and d' is an oriented vertex. If d' has degree 3, then ignore the orientation of d; the graph G'' has a valid orientation by the induction hypothesis, and if necessary we may reverse the direction of every edge to match the orientation of G'. If d' has degree 4, or does not have a minority edge in a 3-edge-cut, then G'' has a valid orientation by the induction hypothesis. Transfer this orientation to G to obtain a valid orientation of G.

Hence we may assume that d' has degree 5 and has a minority edge e in a 3-edge-cut. Then by the minimality of $\delta(A)$, e is incident with a degree 3 vertex t. Assume that $\delta(A)$ is chosen to minimise $|A|$. Let e_1, e_2, e_3, e_4, e_5 be the edges in $\delta(A)$ in order, where $e_1 = e$ is incident with t.

If d has degree 5, and three of e_1, e_2, e_3, e_4, e_5 are incident with d, then $\delta(A)$ is not minimal, a contradiction. We may lift a pair of consecutive edges in G' at v (the vertex of contraction) that does not include e_1 and does include the minority edge incident with d, if applicable. Let $\bar{G}'$ be the resulting graph. This graph can be seen in Figure 3.1. Then $\bar{G}'$ is 3-edge-connected by the minimality of A. Since $t \notin G'$, G' has at most two 3-edge-cuts (none if d exists), and thus $\bar{G}'$ has at most three (one if d exists). If d has degree 5 in $\bar{G}'$, then it is not incident with v via its minority edge, and thus does not have a minority edge in a 3-edge-cut. Hence $\bar{G}'$ has a valid orientation by the induction hypothesis. This transfers to G, to give an orientation of d' in G'' where $e_1 = e$ is not the minority edge. Hence G'' has a valid orientation by the induction hypothesis. Again, this leads to a valid orientation of G.

Now assume that G is projective planar. Let G' and G'' be the graphs obtained from G by contracting A and $G - A$ respectively. By Lemma 2.4.5, at least one of G' and G'' is a planar graph. Without loss of generality, assume that G'' is planar. We may reduce a cut in the same way as for a planar graph, beginning with an orientation of G', which may be

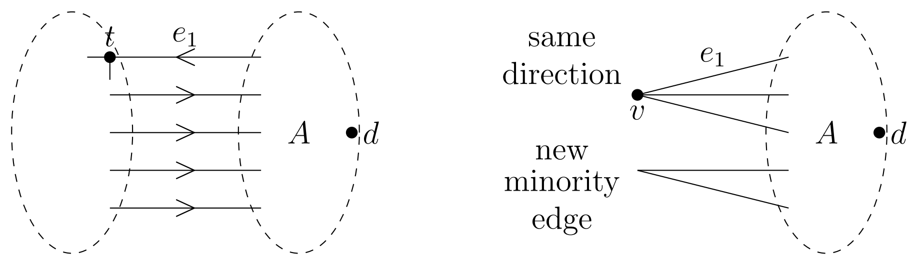


Figure 3.1: Lifting at v in G' so the minority edge does not appear in a 3-edge-cut in G''.

projective planar, followed by an orientation of G'' which has a directed vertex d', but is a planar graph. □

We will use similar techniques in Chapters 4 and 5 to reduce small edge-cuts.

After reducing all at most 5-edge-cuts in G, Steinberg and Younger [25] are able to reduce vertices of degree at least 6 by lifting a pair of edges without creating an edge-cut of size at most 3. They can do the same with a degree 4 vertex by lifting two pairs of edges.

The remaining two simple reductions they perform reduce i) a length 3 face containing a degree 3 vertex, and ii) a 6-edge-cut containing a zigzag (a path of length three where the edges are not all on the boundary of a common face, each adjacent pair of edges is consecutive at their common vertex, and each internal vertex has degree 5 and is not d). Finally, they are able to prove that the graph must contain a Grötzsch configuration [6]. This configuration can be seen in Figure 3.2, and consists of a vertex of degree 5 for which at most one incident face has length greater than 3. This is proven directly using Euler's Formula. It can also be shown using a discharging argument, as we will see in Section 4.5. Steinberg and Younger [25] provide a reduction for a Grötzsch configuration.

For planar graphs, Jaeger's Conjecture is equivalent to the following theorem.

Theorem 3.3.2. *Every planar graph with girth at least* 5 *has group chromatic number at most* 3.

Theorem 3.3.2 was proven by Lai and Li [19]. Their proof modified the list-colourability proof of Grötzsch's Theorem by Thomassen [26].

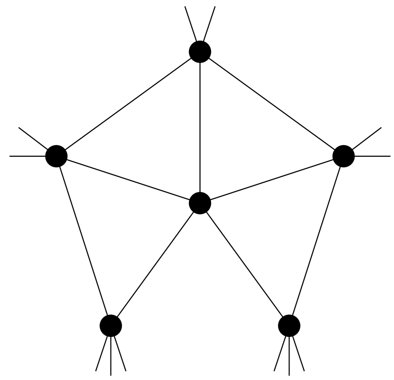


Figure 3.2: A Grötzsch Configuration.

Richter et al. [23] provided a proof of Jaeger's Conjecture for graphs in the plane using flows, by first proving Theorem 3.3.3.

Theorem 3.3.3. *Let G be a 3-edge-connected graph embedded in the plane with at most two specified vertices d and t such that*

- *if d exists, then it has degree 3, 4, or 5, has its incident edges oriented and labelled with elements in $\mathbb{Z}_3 \setminus \{0\}$, and is in the boundary of the unbounded face,*
- *if t exists, then it has degree 3 and is in the boundary of the unbounded face,*
- *there are at most two 3-cuts, which can only be $\delta(\{d\})$ and $\delta(\{t\})$,*
- *if d has degree 5, then t does not exist, and*
- *every vertex not in the boundary of the unbounded face has five edge-disjoint paths to the boundary of the unbounded face.*

If G has a valid prescription function, then G has a valid orientation.

The 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture for planar graphs are corollaries of Theorem 3.3.3. The techniques used by Richter et al. [23] are similar to those of Thomassen [27], Lovász et al. [21] and Steinberg and Younger [25]. In Chapter 4 we prove several generalisations of Theorem 3.3.3 in the plane and the projective plane. Their proofs form the basis of the techniques used throughout the results in this thesis.

3.4 4-Edge-Connected Graphs

While Tutte's 3-Flow Conjecture is for 4-edge-connected graphs, Jaeger et al. [13] conjectured that 5-edge-connected graphs (as opposed to 4-edge-connected) have a modulo 3 orientation that meets their prescription. Lai [18] provided a construction of an infinite family of 4-edge-connected simple planar graphs that demonstrates that Jaeger's Strong 3-Flow Conjecture is not true for 4-edge-connected graphs.

This family of graphs was constructed in order to disprove a conjecture of Barat and Thomassen [2]; that all 4-edge-connected, simple, planar graphs with $|E(G)| = 0 \pmod 3$ have a claw decomposition. A *claw decomposition* is a decomposition of a graph G into subgraphs that are isomorphic to $K_{1,3}$. The conjecture would have implied that any simple planar graph with $|E(G)| = 0 \pmod 3$ and prescription function $p : V(G) \to \mathbb{Z}_3$ where $p(v) = deg(v) \pmod 3$ for all $v \in V(G)$ has a valid orientation meeting p.

The construction of Lai [18] shows that this conjecture is false, and thus the edge-connectivity condition of Jaeger's Strong 3-Flow Conjecture cannot be relaxed to 4-edge-connected. This family of graphs consists of $3k$ incident copies of a graph H, where k is a positive integer. The construction can be seen in Figure 3.3.

We call the resulting graph G. Now $|V(H)| = 9$, so $|V(G)| = 8(3k) = 24k$. We define $p(G)$ where $p(v) = 1$ for all $v \in V(G)$ (the prescription function that is implied by the conjecture of Barat and Thomassen [2]). Then

$$\sum_{v \in V(G)} p(v) = 24k \equiv 0 \mod 3.$$

Hence p is a valid prescription function for G. The following argument that G has no valid orientation is due to Bruce Richter (private communication).

Lemma 3.4.1. *The graph G (as defined in Figure 3.3) with prescription function p does not have a valid orientation.*

Proof. Suppose for a contradiction that G has a valid orientation that meets the prescription function p. Each vertex has degree 4 and has prescription 1. Hence at each vertex v, either all edges are directed into v, or 3 edges are directed out from v and one is directed in. We define T to be the set of vertices v in G with all edges directed into v, and $t = |T|$. Then

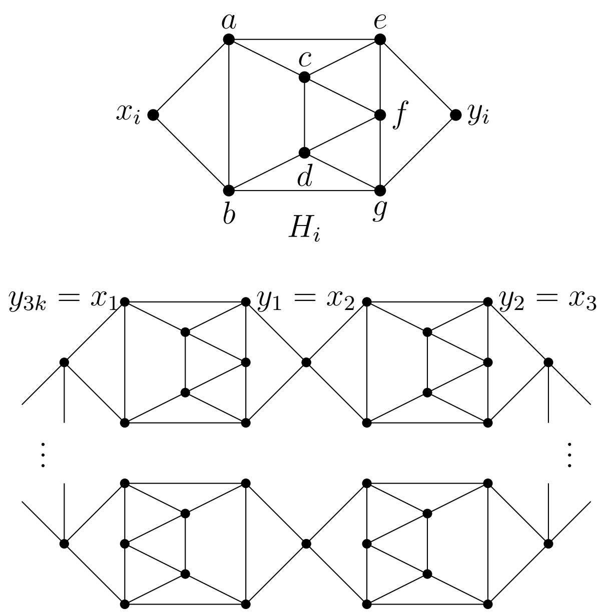


Figure 3.3: Lai's example.

G has $24k - t$ vertices v that have 3 edges directed out from v and one directed in. The number of edges in G is $2(24k) = 48k$. Therefore,

$$4t + (24k - t) = 48k.$$

We conclude that $t = 8k$.

No two vertices in T are adjacent, else an edge points in to both its endpoints. Let G_i be the subgraph $H_i \cup (H_{i+1} - y_{i+1})$ of G. Then $|V(G_i)| = 16$. We show that G_i contains at most 5 vertices in T. Consider $H_i - y_i$. If $x_i, a, b \notin T$, then what remains is a cycle of length 5, which may only have two vertices in T. If $a \in T$, then $x_i, b, c, e \notin T$, and what remains is a cycle of length 3, which may only have one vertex in T. Thus if $x_i \notin T$, at most two vertices of H_i are in T. We now suppose that $x_i \in T$. Then a and b are not. What remains is a cycle of length 5, which may only have two vertices in T. Since c, d, f lie in a cycle of length 3, at least one of these two vertices must be e or g. Hence $H_i - y_i$ has at most three vertices in T.

Now consider G_i. If $H_i - y_i$ has at most two vertices in T, then since $H_{i+1} - y_{i+1}$ has at most three vertices in T, it is clear that G_i has at most 5 vertices in T. If $H_i - y_i$ has three vertices in T, then at least one of e and g is in T, and so $y_i \notin T$. Therefore $H_{i+1} - y_{i+1}$ has at most two vertices in T, and so G_i has at most 5 vertices in T. Each vertex of G appears in two of the subgraphs G_i, $1 \leq i \leq 3k$. Thus

$$2t \leq 5(3k) = 15k.$$

We have $16k = 2t \leq 15k$, a contradiction. Hence G does not have an orientation that meets p. □

This raises the question of which 4-edge-connected graphs have a modulo 3 orientation for every valid prescription function. In Chapter 5 we make some progress toward answering this question by proving a generalisation of Theorem 3.3.3 that allows arbitrarily many internal degree 4 vertices with pairwise restrictions.

3.5 Other Results

Here we briefly discuss results regarding 3-flows and orientations in other subsets of graphs.

Independence Number

An *independent set* in a graph G is a set of vertices, no two of which are adjacent. The *independence number* of G, denoted $\alpha(G)$, is the size of the largest independent set in G. Li et al. [20] proved the following results regarding graphs with independence number at most 4.

Theorem 3.5.1. *Every 4-edge-connected graph G where $|V(G)| \geq 21$ and $\alpha(G) \leq 4$ has a nowhere zero 3-flow.*

Theorem 3.5.2. *Every 4-edge-connected graph G where $\alpha(G) \leq 3$ has a nowhere zero 3-flow.*

These extend earlier results of Luo et al. [22] and Yang et al. [30] characterising bridgeless graphs with independence number 2, and 3-edge-connected graphs with independence number 2.

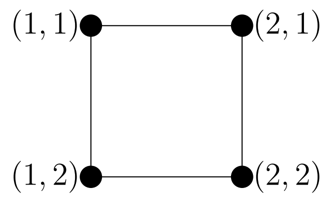


Figure 3.4: The graph $K_2 \times K_2$.

Locally Connected Graphs

A graph G is *locally k-edge-connected* if for all $v \in V(G)$, $G[N(v)]$ is k-edge-connected. Lai [17] proved the following result about locally 3-edge-connected graphs.

Theorem 3.5.3. *Let G be a 2-edge-connected, locally 3-edge-connected graph with valid prescription function p. Then G has a valid orientation.*

Lai [17] also constructed an infinite family of graphs without a nowhere zero 3-flow that are 2-edge-connected and locally 2-edge-connected, so this condition cannot be relaxed.

Products of Graphs

Let G and H be graphs. The *cartesian product* of G and H, denoted $G \times H$ is defined such that

$$V(G \times H) = V(G) \times V(H),$$
$$E(G \times H) = \{\{(g,h),(g',h')\} : h = h', \{g,g'\} \in E(G) \text{ or } g = g', \{h,h'\} \in E(H)\}.$$

Recently it has become common to use the notation $G \,\square\, H$ instead of $G \times H$ to denote the cartesian product. Figure 3.4 shows the graph $K_2 \times K_2$.

Imrich and Skrekovski [9] proved the following result regarding 3-flows on products of graphs.

Theorem 3.5.4. *Let G and H be graphs. Then $G \times H$ has a nowhere-zero 3-flow if both G and H are bipartite.*

This was generalised to the following results by Shu and Zhang [24] and Yao [31]. Note that a *tree* is a connected graph with no cycles.

Theorem 3.5.5. *Let G and H be two connected non-trivial graphs where G does not have more odd degree vertices than H. Then $G \times H$ has a nowhere-zero 3-flow unless H has a bridge and every block of G is a circuit of odd length.*

Theorem 3.5.6. *Let G and H be connected simple graphs. Then $G \times H$ has a nowhere zero 3-flow unless G and H are trees and* $\min\{|V(G)|, |V(H)|\} = 2$.

Triangularly Connected Graphs

Let G be a graph. A *triangle-path* in G is a sequence of distinct cycles of length 3, T_1, T_2,..., T_m such that

$$|E(T_i) \cap E(T_{i+1})| = 1 \text{ for all } 1 \leq i \leq m-1,$$
$$E(T_i) \cap E(T_j) = \emptyset \text{ if } j > i+1.$$

Then G is *triangularly connected* if for any pair of edges $e_1, e_2 \in E(G)$ that are not parallel, G has a triangle-path T_1, T_2,...,T_m such that $e_1 \in E(T_1)$ and $e_2 \in E(T_m)$. Fan et al. [5] proved that the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture hold for triangularly connected graphs.

Theorem 3.5.7. *Let G be a triangularly connected graph with $|V(G)| \geq 3$. If G contains at most three 3-cuts, then G is $\mathbb{Z}_3$-connected.*

Chapter 4

Projective Plane

In this chapter we prove Jaeger's Strong 3-Flow Conjecture for graphs embedded in the projective plane. The 3-Flow Conjecture for graphs embedded in the projective plane is an immediate corollary. This result is similar in form to that of Richter et al. [23] and appears in Section 4.4. In Sections 4.2 and 4.3 we prove two required preliminary results, the first extending the planar result of Richter et al. [23] to allow two unoriented vertices of degree 3, and the second extending the result under certain conditions to allow vertices of low degree on two specified faces of the embedding, instead of only the outer face. In Section 4.1 we first discuss some of the ideas that will be used throughout the proofs in this chapter.

4.1 Preliminaries

Specified Face(s)

First, we consider the idea of a specified face. Theorem 3.3.3 [23] allows vertices of degree 4 and potentially degree 3 on the boundary of the outer face. However, a graph embedded in the projective plane does not have a defined outer face, so the result cannot be directly extended to the projective plane. While the results in Sections 4.2 and 4.3 deal with planar graphs, and thus could refer to the outer face, to simplify the use of these results in proving Jaeger's Strong 3-Flow Conjecture for projective planar graphs, we define specified faces for all three results.

Let G be a graph with specified faces F_G and F^*_G. Note that these are not required to both exist. Let G' be a graph obtained from G by one or more operations. Unless otherwise stated, the specified faces $F_{G'}$ and $F^*_{G'}$ are defined as follows:

1. **Suppose that $\mathbf{G'}$ is obtained from G by deleting or contracting a connected subgraph of G that has no edge in common with the boundary of $\mathbf{F_G}$ or $\mathbf{F^*_G}$.** Then $F_{G'} = F_G$ and $F^*_{G'} = F^*_G$.

2. **Suppose that $\mathbf{G'}$ is obtained from G by contracting a connected subgraph of G that contains the boundaries of $\mathbf{F_G}$ and $\mathbf{F^*_G}$.** Then G' has no specified face. Both can be chosen arbitrarily; in general we will chose a specified face incident with the vertex of contraction.

3. **Suppose that $\mathbf{G'}$ is obtained from G by deleting an edge e in the boundary of $\mathbf{F_G}$.** Let F be the other face incident with e. Then $F_{G'}$ is the face formed by the union of the boundaries of F and F_G (without e), and $F^*_{G'} = F^*_G$. Note that if $F = F^*_G$, then G' has only one specified face. The second can be chosen arbitrarily if necessary.

4. **Suppose that $\mathbf{G'}$ is obtained from G by deleting a vertex v in the boundary of $\mathbf{F_G}$.** Let F_1, F_2,...,F_k be the other faces incident with v. Then $F_{G'}$ is the face formed by the union of the boundaries of F_1, F_2,...,F_k, and F_G (without the edges incident with v), and $F^*_{G'} = F^*_G$. Note that if $F^*_G \in \{F_1, F_2, ..., F_k\}$, then G' has only one specified face. The second can be chosen arbitrarily if necessary.

5. **Suppose that $\mathbf{G'}$ is obtained from G by contracting a connected subgraph H of G whose intersection with $\mathbf{F_G}$ is a path P of length at least one.** Then $F_{G'}$ is the face formed by the boundary of F_G without P, and $F^*_{G'} = F^*_G$. If the intersection of H with F_G consists of more than one path, this contraction can simply be completed in multiple steps.

6. **Suppose that $\mathbf{G'}$ is obtained from G by lifting a pair of adjacent edges $\mathbf{e_1}$, $\mathbf{e_2}$, where $\mathbf{e_1}$ is in the boundary of $\mathbf{F_G}$, $\mathbf{e_2}$ is not, and $\mathbf{e_1}$ and $\mathbf{e_2}$ are consecutive at their common vertex.** Let F_1 be the other face incident with e_1. Note that F_1 is incident with e_2. Let F_2 be the other face incident with e_2. Then $F_{G'}$ is the face formed by the union of the boundaries of F_G and F_2 (using the lifted edge instead of e_1 and e_2), and $F^*_{G'} = F^*_G$. Note that if $F_1 = F^*_G$, then $F^*_{G'}$ will use the lifted edge instead of e_1 and e_2, and if $F_2 = F^*_G$, then G' has only one specified face. The second can be chosen arbitrarily if necessary.

When performing these operations we will not explicitly state the new specified faces unless necessary; for example, if we must define a second specified face.

We define an edge-cut $\delta(A)$ in G to be *internal* if either A or $G - A$ does not intersect the boundary of the specified face(s) of G.

Edge-Disjoint Paths to the Boundary

As in Richter et al. [23], throughout this chapter we are working with graphs for which all vertices not on the boundary of the specified face(s) have at least 5 edge-disjoint paths to the boundary of the specified face(s). We consider here how reductions to the graph affect this condition.

Lemma 4.1.1. *Let G be a graph with specified face F_G such that all vertices not on the boundary of F_G have 5 edge-disjoint paths to the boundary of F_G. Let G' be a graph obtained from G by*

1. *contracting a subgraph X of G that does not intersect the boundary of F_G to a vertex x,*
2. *deleting a boundary edge e of F_G,*
3. *deleting a boundary vertex x of F_G,*
4. *lifting a pair of adjacent edges e_1, e_2, where e_1 is in the boundary of F_G, e_2 is not, and e_1 and e_2 are consecutive at their common vertex, or*
5. *contracting a subgraph X of G whose intersection with F_G is a path P.*

Then all vertices not on the boundary of $F_{G'}$ have 5 edge-disjoint paths to the boundary of $F_{G'}$.

Proof. Suppose for a contradiction that G' has a vertex v that is not on the boundary of $F_{G'}$, and does not have 5 edge-disjoint paths to the boundary of $F_{G'}$.

1. Suppose that $v \neq x$. Then in G, v has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. We note that $F_{G'} = F_G$. Suppose that P_i intersects X. Let P_i' and P_i'' be the minimal subpaths of P_i such that $v \in P_i'$; P_i'' intersects the boundary of F_G; and both P_i' and P_i'' have an endpoint in X. Then $P_i'xP_i''$ is a path in G''. Let

$P_i^* = P_i' x P_i''$. If P_i does not intersect X, then let $P_i^* = P_i$. Then $\{P_1^*, P_2^*, P_3^*, P_4^*, P_5^*\}$ is a set of 5 edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction.

Now suppose that $v = x$. Let x' be a vertex in X. Then in G, x' has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. We define P_i' to be the minimal subpath of P_i such that P_i' has one endpoint at a vertex in X and one endpoint on the boundary of F_G. Then $\{P_1', P_2', P_3', P_4', P_5'\}$ is a set of 5 edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction.

2. In G, v has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. These paths exist in G', and are edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction.

3. In G, v has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. Suppose that P_i contains x. Then x is an endpoint of P_i. Let x' be the vertex of P_i adjacent to x. By definition, x' is on the boundary of $F_{G'}$. Let P_i' be the subpath of P_i with endpoints v and x'. If P_i does not contain x, then let $P_i' = P_i$. Then $\{P_1', P_2', P_3', P_4', P_5'\}$ is a set of 5 edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction.

4. In G, v has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. Suppose that P_i contains e_2. Let x and y be the endpoints of e_2 where x is on the boundary of F_G. Let P_i' be the subpath of P_i with endpoints v and y. If P_i does not contain e_2, then let $P_i' = P_i$. Then $\{P_1', P_2', P_3', P_4', P_5'\}$ is a set of 5 edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction.

5. In G, v has 5 edge-disjoint paths P_1, P_2, P_3, P_4, and P_5 to the boundary of F_G. Suppose that P_i intersects X. Let P_i' be the minimal subpath of P_i with endpoints v and a vertex in X. If P_i does not intersect X, then let $P_i' = P_i$. Then $\{P_1', P_2', P_3', P_4', P_5'\}$ is a set of 5 edge-disjoint paths from v to the boundary of $F_{G'}$, a contradiction. □

We therefore only discuss the preservation of this property in cases where Lemma 4.1.1 does not apply. In Section 4.3 we consider graphs with two specified faces, having a vertex in common. Lemma 4.1.1 applies analogously to the union of these specified faces.

Minimal Cuts

Let G be a graph with a directed vertex d. An edge-cut $\delta(A)$ in G with $d \in A$ is *k-robust* if $|A| \geq 2$ and $|G - A| \geq k$. Note that when $k = 2$ this is the same as non-peripheral.

Throughout this chapter we will perform local reductions on graphs. Many of these reductions will involve considering 2-robust edge-cuts, either because G has a small edge-cut that must be reduced, or because we must verify that the graph resulting from a reduction does not have any small edge-cuts. In all cases, we first consider the smallest possible edge-cuts. Thus, if we consider a 2-robust k-edge-cut $\delta(A)$ in a graph G, we may assume that G has no 2-robust at most $(k-1)$-edge-cut. Thus it may be assumed that either $G[A]$ is connected, or it consists of two isolated vertices whose degrees sum to $|\delta(A)|$. The same is true of $G-A$. Given the size of the cuts we consider, generally both $G[A]$ and $G-A$ will be connected.

Non-Crossing 3-Edge-Cuts

Let $\delta(A)$ and $\delta(B)$ be distinct edge-cuts in G. We say that $\delta(A)$ and $\delta(B)$ *cross* if $A\cap B$, $A\setminus B$, $B\setminus A$, and $\overline{A\cup B}$ are all non-empty. Throughout this chapter we consider graphs that are allowed to have non-crossing 2-robust 3-edge-cuts under certain restrictions. We discuss here why we may assume that such cuts are non-crossing.

Lemma 4.1.2. *Let G be a 3-edge-connected graph and let $\delta(A)$ and $\delta(B)$ be 3-edge-cuts in G. Then $\delta(A)$ and $\delta(B)$ do not cross.*

Proof. Suppose that $\delta(A)$ and $\delta(B)$ cross. Then $A\cap B$, $A\setminus B$, $B\setminus A$, and $\overline{A\cup B}$ are all non-empty. Since G is 3-edge-connected, $\delta(A\cap B)$, $\delta(A\setminus B)$, $\delta(B\setminus A)$, and $\delta(\overline{A\cup B})$ all have size at least 3. Consider the graph G' obtained from G by contracting these four sets of vertices to vertices a, b, c, and d respectively (and deleting any resulting loops). Then $deg(a), deg(b), deg(c), deg(d)\geq 3$. Since the edges of G' are the edges of $\delta(A)\cup\delta(B)$, $|\delta(A)\cup\delta(B)|\geq 6$. Since $|\delta(A)\cup\delta(B)|\leq 6$, we conclude that $\delta(A)$ and $\delta(B)$ do not share any edges, and that $deg(a), deg(b), deg(c), deg(d)=3$.

Let u, v, w, and x be the number of edges with endpoints at a and c, b and c, a and d, and b and d, respectively. Then

$$v+w=|\delta(A)|=3,$$

$$u+x=|\delta(B)|=3,$$

$$u+w=|\delta(A\cap B)|=3,$$

$$v+x=|\delta(\overline{A\cup B})|=3,$$

$$u + v = |\delta(A - B)| = 3,$$
$$w + x = |\delta(B - A)| = 3.$$

Then we have

$$u = 3 - w = v = 3 - x = w = 3 - u = x.$$

The only solution to this system of equations is $u = v = w = x = \frac{3}{2}$, which is not an integer solution. $\square$

Face Boundaries in the Projective Plane

Let G be a graph embedded in the plane, and let F_G be a specified face of G. If G does not contain a cut vertex, then we may assume that F_G is bounded by a cycle. If G is embedded in the projective plane, this is not true. Suppose that v is a vertex that appears more than once in the boundary walk of F_G, and assume that v is not a cut vertex. Then there exists a non-contractible curve that passes through only F_G and v. Cut along this curve, and draw the graph on the plane. The result is a planar graph with one specified face containing two copies of v. See Figure 4.1 for an illustration. Contract the two copies of v to a single vertex. Then G is a planar graph with two specified faces, each containing v. In Section 4.3 we prove a result for such graphs analogous to that of Richter et al. [23].

Chords in the Projective Plane

Let G be a graph embedded in the plane and let F_G be a specified face of G. Suppose that F_G has a chord e with endpoints u and v. Then there exist subgraphs H and K of G

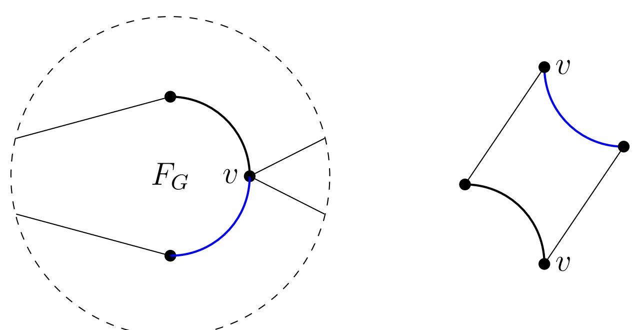


Figure 4.1: Redrawing G in the plane.

such that $H \cap K = \{\{u, v\}, \{e\}\}$ and $H \cup K = G$. This is a property that Richter et al. [23] exploited when proving Jaeger's Strong 3-Flow Conjecture for planar graphs, and a property that we use throughout this chapter.

Now suppose that G is a graph embedded in the projective plane with a cycle C bounding a closed disk. Let e be a chord of C. If $C+e$ is contained in an open disk, then e is *contractible*. Otherwise it is a *non-contractible* chord. This is relevant in the case where the specified face F_G is bounded by a cycle. If F_G has a contractible chord, we may use techniques analogous to those in the plane to reduce the graph. In the case of a non-contractible chord, the graph is not split into subgraphs as it is in the plane, and we require different techniques. We see here that the deletion of such a chord and its endpoints results in a planar graph.

Lemma 4.1.3. *Let G be a graph embedded in the projective plane with a specified face F_G bounded by a cycle. Suppose that e is a non-contractible chord of F_G with endpoints u and v. Let $G' = G - \{u, v\}$. Then G' is planarly embedded in the projective plane with one specified face; namely the one containing F_G.*

Proof. Let $u, v_1, v_2, ..., v_i, v, v_j, ..., v_k, u$ be the boundary of F_G. There exist two faces F_1 and F_2 of G that are incident with e. The boundary of $F_{G'}$ is the union of a set of faces including F_G, F_1, and F_2, without the edges incident with u and v. This boundary is $v_1, v_2, ..., v_i P_1 v_k, ..., v_j P_2 v_1$, where P_1 is the path in F_1 between v_i and V_k not containing e, and P_2 is the path in F_2 between v_j and v_1 not containing e.

Consider the projective plane as represented by a circle of radius 2, where opposite points are identified. Draw the given embedding of G such that the boundary of F_G lies on the unit circle (where the origin is in the specified face), and e is contained in the line $y = 0$. In G', the graph does not intersect the line $y = 0$. Cut the projective plane along the line $y = 0$ and identify the opposite points on the circle. The result is a planar embedding of G' where $F_{G'}$ is the outer face. □

Loops, Parallel Edges, and Cut Vertices

In the proofs throughout this chapter, we wish to show that our graph (a minimum counterexample) does not have loops, unoriented parallel edges, or cut vertices. As the proofs of these results are equivalent in all cases, we write a general proof here. It relies on the fact that the set of graphs is closed under loop deletion and edge contraction, which we will show in each case.

Lemma 4.1.4. *Let $\mathcal{G}$ be a set of connected graphs that is closed under loop deletion and edge contraction. If $G \in \mathcal{G}$ is a minimum counterexample (with respect to the number of edges in G) to the statement 'all graphs in $\mathcal{G}$ have a valid orientation for every valid prescription function', then G has:*

1. *no loop,*
2. *no unoriented parallel edges, and*
3. *no cut vertex.*

Proof.

1. Suppose that G contains a loop ℓ. Then $G - \ell \in \mathcal{G}$, so $G - \ell$ has a valid orientation by the minimality of G. Orient ℓ in either direction to yield a valid orientation of G, a contradiction. Thus G has no loops.

2. Suppose that G has unoriented parallel edges e and f. Then $G/\{e, f\} = (G/e) - f \in \mathcal{G}$ and thus has a valid orientation by the minimality of G. Let u and v be the endpoints of e and f. Orient e and f to satisfy $p(u)$. Since v cannot be the only vertex whose prescription is not met, this yields a valid orientation of G, a contradiction. Thus G has no unoriented parallel edges.

3. Suppose that G has a cut vertex v. Then there exist subgraphs H and K of G such that $G = H \cup K$ and $H \cap K = (\{v\}, \emptyset)$. Then $H = G/K$ and $K = G/H$ and so $H, K \in \mathcal{G}$. By the minimality of G, H has a valid orientation. Transfer this orientation to G and adjust the prescription of v in K accordingly. Then by the minimality of G, K has a valid orientation. This leads to a valid orientation of G, a contradiction. Hence G has no cut vertex. □

4.2 Increasing the Number of Degree 3 Vertices

Lemma 4.1.3 shows that we may reduce graphs in the projective plane that have a non-contractible chord to planar graphs. However, this process will, in general, introduce new degree 3 vertices. Therefore, in order to make use of this property, we first show that such graphs have a valid orientation. This extends the result in Theorem 3.3.3 [23] to allow a directed vertex d and two other vertices of degree 3. An analogous result with three degree 3 vertices and no directed vertex follows as an immediate corollary.

Definition 4.2.1. *A* DTS graph *is a graph G embedded in the plane, together with a valid $\mathbb{Z}_3$-prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is 3-edge-connected,*
2. *G has a specified face F_G, and at most three specified vertices d, t, and s,*
3. *if d exists, then it has degree 3, 4, or 5, is oriented, and is on the boundary of F_G,*
4. *if t or s exists, then it has degree 3 and is on the boundary of F_G,*
5. *d has degree at most $5 - a$ where a is the number of unoriented degree 3 vertices in G,*
6. *G has at most three 3-edge-cuts, which can only be $\delta(d)$, $\delta(t)$, and $\delta(s)$, and*
7. *every vertex not in the boundary of F_G has 5 edge-disjoint paths to the boundary of F_G.*

We define all 3-edge-connected graphs on at most two vertices to be DTS graphs, regardless of vertex degrees.

A 3DTS graph *is a graph G with the above definition, where (6) is replaced by*

6'. all vertices other than d, t, and s have degree at least 4, and if d, t, and s all exist, then every 3-edge-cut in G separates one of d, t, and s from the other two.

We note that (7) implies that the 3-edge-cuts allowed by (6') are not internal.

Our main result for this section is Theorem 4.2.2.

Theorem 4.2.2. *Every DTS graph has a valid orientation.*

Proof. Let G be a minimal counterexample with respect to the lexicographic ordering of the pairs $(|E(G)|, |E(G)| - deg(d))$. If $|E(G)| = 0$, then G consists of only an isolated vertex, and thus has a trivial valid orientation. If $|E(G)| - deg(d) = 0$ then G has an existing valid orientation. Thus we may assume G has at least one unoriented edge.

We will prove the following series of properties of G.

DTS1: The graph G does not contain a loop, unoriented parallel edges, or a cut vertex.

DTS2: The graph G does not contain

a) a 2-robust 4-edge-cut, $\delta_G(A)$, where $d \in A$ and $G - A$ contains at most one of s and t,

b) a 2-robust 5-edge-cut, $\delta_G(A)$, where $d \in A$ and $G - A$ contains neither s nor t, or

c) an internal 2-robust 6-edge-cut.

DTS3: If $e = uv$ is a chord of F_G incident with a vertex u of degree at most 4, then $deg(u) = 4$, e separates d from both s and t, and u is incident with e and one other edge in the side containing d, while on the side containing s and t, u is incident with e and two other edges.

DTS4: Vertices d, s, and t exist in G.

DTS5: Vertices s and t are not adjacent.

Let u and v be the boundary vertices adjacent to t, and let w be the remaining vertex adjacent to t. Let x and y be the boundary vertices adjacent to s, and let z be the remaining vertex adjacent to s.

DTS6: Vertices u, v, x, and y have degree 4.

DTS7: Edges uw, vw, xz, and yz exist, and w and z have degree 5.

DTS8: The vertices d, s, t, u, v, x, and y form the boundary of F_G, where either $v = x$ or $u = z$ (up to renaming).

The proofs of these properties form the bulk of the proof of Theorem 4.2.2.

DTS1. *The graph G does not contain a loop, unoriented parallel edges, or a cut vertex.*

Proof. By Lemma 4.1.4 it suffices to show that the set of DTS graphs is closed under loop deletion and edge contraction. We may assume that G has at least three vertices. Suppose that G contains a loop e incident with a vertex v. If $G - e$ contains a vertex of degree at most 3 that does not have degree at most 3 in G, this vertex is v. Then $|\delta_G(\{v\})| = 3$, a contradiction. If $v = d$, then since $|\delta(\{d\})| \geq 3$, $deg_{G-e}(d) \geq 3$. The edge connectivity requirements follow from Lemma 2.3.1.

Let $e = \{x, y\}$ be an edge in G and consider G/e. Let v be the vertex of contraction. If G/e contains a vertex of degree at most 3 that does not have degree at most 3 in G, this vertex is v. Then $|\delta_G(\{x, y\})| = 3$. If $|V(G)| = 3$, then G/e consists of two vertices joined by 3 parallel edges, and thus is a DTS graph. Otherwise, $|\delta_G(\{x, y\})|$ is 2-robust, a contradiction. The edge connectivity requirements follow from Lemma 2.3.1. □

Since G has no cut vertex, every face in G is bounded by a cycle. We now consider the presence of small 2-robust edge-cuts in G. Recall that by definition, G has no 2-robust at most 3-edge-cut. Property DTS2 categorises three cases that can always be reduced. This is not a comprehensive list, but other small cuts arise arise in special circumstances and will be reduced on a case by case basis.

DTS2. *The graph G does not contain*

a) a 2-robust 4-edge-cut, $\delta_G(A)$, where $d \in A$ and $G - A$ contains at most one of s and t,

b) a 2-robust 5-edge-cut, $\delta_G(A)$, where $d \in A$ and $G - A$ contains neither s nor t, or

c) an internal 2-robust 6-edge-cut.

Proof.

a) Suppose that G does contain a 2-robust 4-edge-cut, $\delta_G(A)$, with A chosen so that $d \in A$ and $G - A$ contains at most one of s and t. Let G' be the graph obtained from G by contracting $G - A$ to a single vertex. The resulting vertex v has degree 4. Since every vertex not incident with F_G has 5 edge-disjoint paths to the boundary of F_G, F_G is incident with edges in $\delta_G(A)$. Therefore, v is incident with $F_{G'}$. If G' contains a cut $\delta_{G'}(B)$ of size at most 3, then such a cut also exists in G by Lemma 2.3.1, a contradiction unless it is one of the specified vertices. Hence G' is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G.

   Let G'' be the graph obtained from G by contracting A to a single vertex d'. This vertex has degree 4 and is oriented. Since $d \in V(A)$, G'' has only one oriented vertex, which is d'. Since F_G is incident with edges in $\delta_G(A)$, d' is incident with $F_{G''}$. If G'' has a cut $\delta_{G''}(B)$ of size at most 3, then such a cut also exists in G by Lemma 2.3.1, a contradiction unless it is one of the specified vertices. Thus G'' is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G to obtain a valid orientation of G, a contradiction. Hence any 4-edge-cut in G separates d from both s and t.

b) This case works in the same way as a). In G'', there is a degree 5 oriented vertex, and no degree 3 vertices. Therefore, any 5-edge-cut in G separates d from an unoriented degree 3 vertex.

c) Contract $G - A$ to a vertex v, calling the resulting graph G'. As in a), it is clear that G' is a DTS graph. Therefore, by the minimality of G, G' has a valid orientation. Transfer this orientation to G.

   Contract A to a vertex d', delete a boundary edge e incident with d', and call the resulting graph G''. Since the boundary of F_G is contained in A, the endpoint of e in $G - A$ has degree at least 5 in G, and therefore has degree at least 4 in G''. If G'' contains an edge-cut $\delta_{G''}(B)$, then either $\delta_G(B)$ or $\delta_G(G - B)$ is a cut in G of size at most one greater. If G'' contains a 2-robust edge cut $\delta_{G''}(B)$ of size three or less, then G contains an analogous internal edge-cut of size four or less. Such a cut does not exist by definition. Thus G'' contains no such at most 3-edge-cut, and is a DTS graph. Therefore, by the minimality of G, G'' has a valid orientation. Transfer this orientation to G to obtain a valid orientation of G, a contradiction. Hence any 6-edge-cut in G contains edges in the boundary of F_G. □

In the absence of structures such as loops, parallel edges, cut vertices and small edge-cuts, we reduce at low degree vertices in G. It is useful to consider the possible adjacencies of such vertices. We say that a chord $e = \{u, v\}$ of F_G *separates* vertices x and y if the components of $F_G - \{u, v\}$ are P_1 and P_2, where for some $i \in \{1, 2\}$, $x \in P_i$ and $y \in P_{3-i}$.

DTS3. *If $e = uv$ is a chord of F_G incident with a vertex u of degree at most 4, then $deg(u) = 4$, e separates d from both s and t, and u is incident with e and one other edge in the side containing d, while on the side containing s and t, u is incident with e and two other edges.*

Proof. Suppose that a chord uv exists, where $deg_G(u) \in \{3, 4\}$. Let H and K be subgraphs of G such that $H \cap K = \{\{u, v\}, \{uv\}\}$, $H \cup K = G$, and d, if it exists, is in H.

Suppose that $\delta(H)$ is not 2-robust. Then $|V(K)| = 3$, and K contains d, else G has unoriented parallel edges, contradicting DTS1. By definition, either u or v is d. Since G

does not contain unoriented parallel edges $deg_H(d) = 2$, and $|\delta((H - \{u, v\}) \cup \{d\})| = 3$, a contradiction. Hence we may assume that $\delta(H)$ is 2-robust.

Suppose that $\delta(K)$ is not 2-robust. Then $|V(H)| = 3$. If u or v is d, the same argument applies. Thus we may assume that d is in $V(H) - \{u, v\}$. If there are parallel edges with endpoints d and u, then $\delta(\{d, u\})$ is an at most 5-edge-cut. Orient u and contract the parallel edges between d and u, calling the resulting graph G'. Note that the vertex of contraction has the same degree as d. Hence it is clear that G' is a DTS graph, and thus has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Suppose that there are not parallel edges with endpoints d and u. Then there are parallel edges with endpoints d and v. Since $|\delta(\{d, v\})| \geq 4$, $deg_K(v) \geq 3$. If $deg(u) = 3$, then $deg_K(v) \geq 4$, else $|\delta(H)| = 3$, a contradiction. Orient u and add a directed edge from u to v in K. Then K is a DTS graph, and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Thus $deg(u) = 4$. If $K - V(H)$ does not contain both s and t the same argument applies. Thus $s, t \in K - V(H)$, and the chord separates d from s and t. Since there are not parallel edges with endpoints u, and d, $deg_K(u) = 3$. We may now assume that $\delta(K)$ is 2-robust.

By definition, $\delta(H)$ and $\delta(K)$ have size at least 4. Hence $deg_G(v) \geq 6$, and so v is not d, s, or t. In addition, v has degree at least 3 in both H and K.

Suppose that $u \neq d$. Then in H, contract uv. The graph H/uv is a DTS graph, and so by the minimality of G, H/uv has a valid orientation. Transfer this orientation to G, and orient u. In K, if u has degree 2, add an edge e directed from u to v (in the boundary of F_K). Suppose that v has degree 3 in K and K does not contain both s and t. Since $deg_G(u) \in \{3, 4\}$ it is clear that u, v are incident with F_K, and thus $K + e$ is a DTS graph. By the minimality of G, $K + e$ has a valid orientation. This leads to a valid orientation of G, a contradiction. The remaining case is the one where v has degree 3 in K, and K contains both s and t. Then e separates d from s and t, as required.

Thus $u = d$. Then in both H and K if $deg(u) = 2$, add a directed edge e from u to v (in the boundary of F_H and F_K). If K contains both s and t, then $deg(u) = 3$, so in K, $deg(v) \geq 4$. It is clear that $H + e$ and $K + e$ are DTS graphs, so by the minimality of G, they have valid orientations. Transfer the orientations of $H + e$ and $K + e$ to G to obtain a valid orientation of G, a contradiction. Thus no such chord exists. $\square$

Since we reduce at low degree vertices in G, we are especially interested in the specified vertices d, s, and t. We must first establish the existence of these vertices. The following two claims are needed in order to prove DTS4.

Claim 4.2.3. *If d exists, then it is not adjacent to a vertex of degree at most 4 via parallel edges.*

Proof. Suppose that d is adjacent to t via parallel edges. Then $|\delta(\{d,t\})| = deg(d) - 1 \leq 3$ (since t exists, $deg(d) \leq 4$). Since $|\delta(\{d,t\})| \geq 3$, equality holds. Thus $deg(d) = 4$ and s does not exist. It follows that $\delta(\{d,t\})$ is a 2-robust 3-edge-cut, a contradiction. The same is true of d and s.

Suppose that d is adjacent to a vertex v of degree 4 via parallel edges. If they are adjacent via at least 3 parallel edges, then either $deg(d) = 4$ and $\delta(\{d,v\})$ is an at most 2-edge-cut, or $deg(d) = 5$ and $\delta(\{d,v\})$ is an at most 3-edge-cut, a contradiction. If s and t exist, then $deg(d) = 3$, and $\delta(\{d,v\})$ is a 2-robust at most 3-edge-cut, a contradiction. Otherwise, orient v and contract $\{d,v\}$, calling the resulting graph G'. Then G' has a directed vertex of degree at most $deg_G(d)$, and thus is a DTS graph. By the minimality of G, G' has a valid orientation. This leads to a valid orientation of G, a contradiction. $\square$

Claim 4.2.4. *If d exists, then it is not adjacent to a vertex of degree at most 3.*

Proof. Suppose that d is adjacent to t. Orient t and contract $\{d,t\}$ calling the resulting graph G'. Then G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

DTS4. *Vertices d, s, and t exist in G.*

Proof. Suppose that d does not exist. Let v be a vertex in the boundary of F_G of minimum degree. If $deg(v) \leq 5$, orient v, calling the resulting graph G'. Then G' is a DTS graph, and has a valid orientation by the minimality of G. This is a valid orientation of G, a contradiction. If $deg(v) \geq 6$, orient and delete a boundary edge incident with v, calling the resulting graph G'. If G' has a 2-robust at most 3-edge-cut, then G has a 2-robust at most 4-edge-cut, a contradiction. Thus G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence d exists.

Suppose that G has at most one unoriented vertex (t) of degree 3. If d has degree 4 or 5, let v be a vertex adjacent to d on the boundary of F_G that has degree at least 4 (such a vertex exists by Claim 4.2.3). Let G' be the graph obtained by deleting dv. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Thus d has degree 3. By Claim 4.2.4, d is not adjacent to t. If d has at most one adjacent vertex of degree 4, delete d, calling the resulting graph G'. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is a DTS graph and has a valid orientation by the minimality of G.

Otherwise, d has two adjacent vertices of degree 4. Delete d and orient one neighbour of d, calling the resulting graph G'. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is a DTS graph and has a valid orientation by the minimality of G. All cases lead to a valid orientation of G, a contradiction. Hence s and t exist. □

Since s and t exist, $deg(d) = 3$.

In the following reductions, the graphs produced may be 3DTS graphs. We show that such graphs have a valid orientation.

Claim 4.2.5. *If G' is a 3DTS graph with $|E(G')| < |E(G)|$, then G' has a valid orientation.*

Proof. Let G' be a minimal counterexample with respect to $|E(G')|$. Suppose that G' has no 2-robust 3-edge-cut. Then G' is a DTS graph and has a valid orientation by the minimality of G, a contradiction. Thus we may assume that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$, where A is chosen so that $d \in A$. We may assume that $t \notin A$.

Let G_1 be the graph obtained from G' by contracting $G - A$. Then G_1 is a 3DTS graph and has a valid orientation by the minimality of G'. Transfer this orientation to G'. Let G_2 be the graph obtained from G' by contracting A to a directed vertex d'. Then G_2 is a 3DTS graph and has a valid orientation by the minimality of G'. This leads to a valid orientation of G', a contradiction. □

DTS5. *Vertices s and t are not adjacent.*

Proof. Suppose for a contradiction that s and t are adjacent. Let u and v be the boundary neighbours of s and t respectively. We prove the following claims:

a. Vertices s and t have a common internal neighbour w of degree 5, $deg(u) = 4$, and $deg(v) = 4$.

b. Vertices u and w are adjacent.

Claim DTS5a. *Vertices s and t have a common internal neighbour w of degree 5, $deg(u) = 4$, and $deg(v) = 4$.*

Proof. Assume that either s and t do not have a common internal neighbour of degree 5, or at least one of the vertices u and v has degree at least 5. Let G' be the graph obtained from G by orienting and deleting s and t. Then G' has at most two vertices of degree 3: two of u, v, and a possible common neighbour of s and t.

We now check that G' has no 2-robust 2- or 3-edge-cuts. The cases are indicated in italics. We follow a similar process in most future cases. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where u and v are in A.* By definition this cut does not exist in G, and so we may assume that $\delta_G(G' - A)$ is an internal cut. We make this assumption without mention in future cases. Then $\delta_G(A \cup \{s, t\})$ is a 2-robust at most 4-edge-cut that does not separate d from s or t, a contradiction. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $u \in A$ and $v \notin A$.* Then $\delta_G(A \cup \{s\})$ is a 2-robust at most 4-edge-cut that separates s from t, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where u and v are in A. Then $\delta_G(A \cup \{s, t\})$ is a 2-robust at most 5-edge-cut that does not separate d from s or t, a contradiction. *Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $u \in A$ and $v \notin A$.* Then all such cuts separate u and v, so by Claim 4.2.5, if s and t do not have a common neighbour of degree 5, then G' has a valid orientation. Hence we may assume that s and t have a common internal neighbour w of degree 5. Note that in G', w has degree 3. Suppose that $d \in A$. By Claim 4.2.5, $w \in A$, and u is the other vertex of degree 3 in G'. Then $\delta_G(G' - A)$ is a 2-robust 4-edge-cut in G that does not separate d from s or t, a contradiction. Figure 4.2 shows an analysis of these cuts.

We conclude that G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

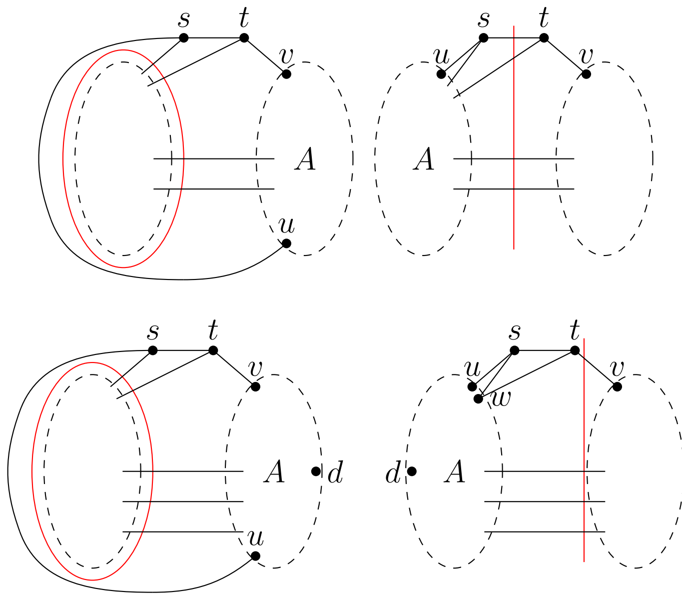


Figure 4.2: DTS5: Analysis of cuts (1).

Therefore s and t have a common internal neighbour w of degree 5, $deg(u) = 4$, and $deg(v) = 4$. Let the edges incident with u be e_1, e_2, e_3, e_s in order, where e_1 is on the boundary of F_G and $e_s = su$. By DTS3, e_3 is not a chord. The same is true at v.

Claim DTS5b. *Vertices u and w are adjacent.*

Proof. Suppose that u and w are not adjacent. Lift the pair of edges e_1, e_2, and orient and delete u, s, and t, calling the resulting graph G'. This reduction can be seen in Figure 4.3. Then G' has at most two unoriented vertices of degree 3 (v and w).

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where v and the lifted edge are in A. Then $\delta_G(A \cup \{s, t, u\})$ is an internal 2-robust at most 5-edge-cut, a contradiction. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where $v \in A$ and the lifted edge is in $G' - A$.* Then $w \in A$, else $\delta_G(A)$ is a 2-robust 3-edge-cut. Similarly, the endpoint of e_3 is in A, else $\delta_G(A \cup \{s, t\})$ is a 2-robust 3-edge-cut. Now $\delta_G(A \cup \{u, s, t\})$ is a 4-edge-cut. By DTS2, $d \in G' - A$. Contract A in G to form a graph $\bar{G}$. Then $\bar{G}$ is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G, contract $G - A$, and delete tv and tw to form a graph $\bar{G}'$ with an oriented degree 4 vertex. If $\bar{G}'$ contains

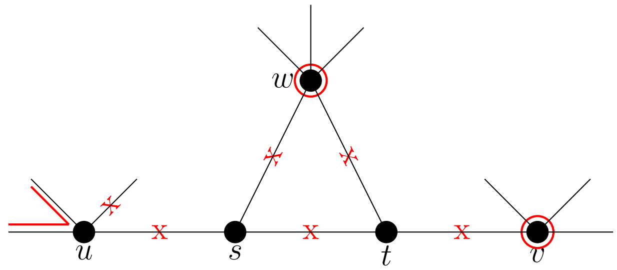


Figure 4.3: DTS5: Reduction when u and w are not adjacent.

an at most 3-edge-cut, then G contains an at most 5-edge-cut that does not separate d, s, and t, a contradiction. Hence $\bar{G}'$ is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Therefore G' is 3-edge-connected.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where v and the lifted edge are in A. Then $\delta_G(A \cup \{s,t,u\})$ is an internal 2-robust at most 6-edge-cut, a contradiction. *Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $v \in A$, and the lifted edge is in $G' - A$.* If all such cuts have the property that $d \in G' - A$, then G' is a 3DTS graph and has a valid orientation by Claim 4.2.5. This leads to a valid orientation of G, a contradiction. Hence we may assume that $d \in A$. The endpoint of e_3 is in $G' - A$, else $\delta_G(A \cup \{u,s,t\})$ is a 2-robust 5-edge-cut that does not separate d from s or t, a contradiction. Also, $w \in G' - A$, else $\delta_G(A \cup \{s,t\})$ is a 2-robust 4-edge-cut that does not separate d from s or t, a contradiction. Then all such 3-edge-cuts in G' separate w from v and d, and so G' has a valid orientation by Claim 4.2.5. Therefore G' has no such 2-robust at most 3-edge-cuts. Figure 4.4 shows an analysis of these cuts.

Thus far we have omitted consideration of the case where the lifted edge is an edge in the cut $\delta(A)$. *Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is an edge of $\delta(A)$.* If the endpoint of e_1 is in $G' - A$, then $\delta_G(G' - A)$ is a 2-robust at most 3-edge-cut, a contradiction. If the endpoint of e_1 is in A, then the argument is analogous to (but simpler than) that for the case where v and the lifted edge are on the same side of the cut. In future reductions we omit discussion of the cases where the lifted edge is an edge of the cut.

Hence G' is a DTS graph and has a valid orientation by the minimality of G. This leads to

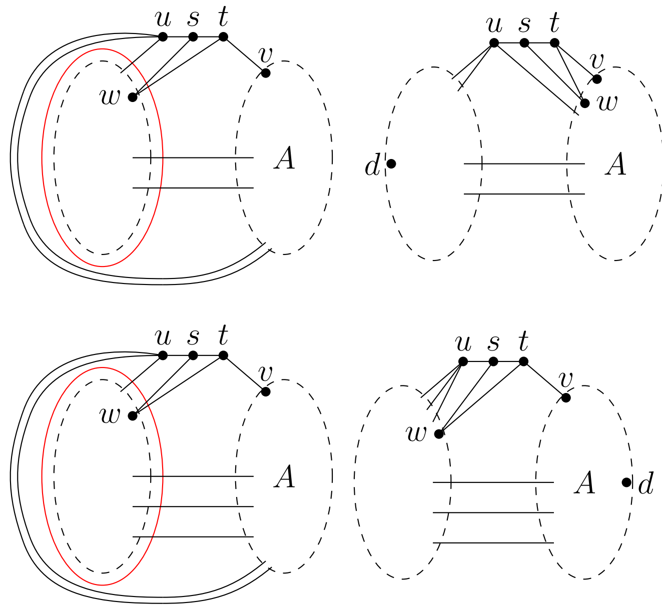


Figure 4.4: DTS5: Analysis of cuts (2).

a valid orientation of G, a contradiction. $\square$

By symmetry, v and w are adjacent. This provides sufficient structure to complete the proof. Suppose that e_2 and the analogous edge f_2 incident with v are both chords. By DTS3, e_2 and f_2 separate d from s and t. Thus $e_2 = f_2$. Then G has a 1-edge-cut: the remaining edge incident with w, a contradiction. Hence e_2 and f_2 are not both chords.

By DTS2, u and v are not both adjacent to d. Without loss of generality, assume that u is not adjacent to d. Suppose that e_2 is a chord. Then by DTS3, e_2 separates d from s and t. Since e_2 is not incident with v, it also separates d from v, so v is not adjacent to d. Therefore at least one of u and v is not adjacent to d and is not incident with a chord. Without loss of generality, we assume this vertex is u. Let z be the other endpoint of e_1.

Orient and delete e_1 and e_2 to satisfy $p(u)$ (which lifts ut and uw at u), and contract $\{u, s, t, v, w\}$ to a single vertex c of degree 3, calling the resulting graph G'. Then G' has at most two unoriented degree 3 vertices (c and z). This reduction is shown in Figure 4.5.

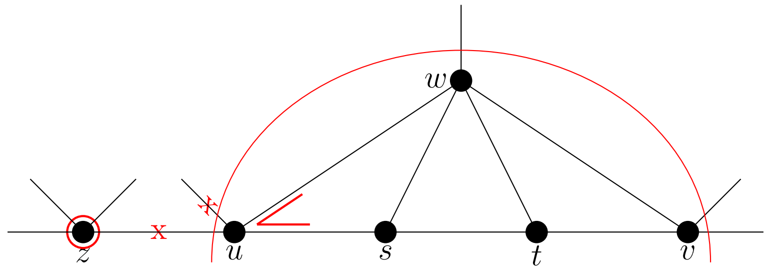


Figure 4.5: DTS5: Reduction when u and w are adjacent.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where z and c are in A. Then G contains a 2-robust internal at most 4-edge-cut, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $z \in G' - A$ and $c \in A$. Since $\delta_G(A)$ and $\delta_G(A \cup \{u\})$ are 4-edge-cuts, $d \notin A$. If v is adjacent to d or incident with a chord, then G has an internal at most 3-edge-cut $\delta_G(A - c)$, a contradiction. Hence v is not adjacent to d or incident with a chord. We consider applying the same reduction at v. If an analogous 4-edge-cut $\delta_G(B)$ exists using edges incident with v, then these cuts cross.

We have $\{s, t, u, v, w\} \subseteq A \cap B$ in G (where we replace c with $\{s, t, u, v, w\}$ in G), and $d \in G - (A \cup B)$. By construction, $A - B$ and $B - A$ each contain at least two vertices: the neighbours of u and v. Thus DTS2 implies that $|\delta_G(A - B)|, |\delta_G(B - A)| \geq 6$. This is not possible given that $|\delta_G(A)| = |\delta_G(B)| = 4$. Thus we may apply the same reduction at v without producing a 2-edge-cut. Hence we may assume that no such cut exists, and G' is 3-edge-connected.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $z \in G' - A$ and $c \in A$. Then G' is a 3DTS graph and has a valid orientation by Claim 4.2.5. This leads to a valid orientation of G, a contradiction. Hence no such cut exists.

We conclude that G' is a DTS graph, and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence s and t are not adjacent. $\square$

Let u and v be the boundary vertices adjacent to t, and let w be the remaining vertex adjacent to t. Let x and y be the boundary vertices adjacent to s, and let z be the remaining vertex adjacent to s. Note that w and z are internal and have degree at least 5, and that none of u, v, w, x, y, and z are in the set $\{d, s, t\}$.

DTS6. *Vertices u, v, x, and y have degree* 4.

Proof. Suppose without loss of generality that v has degree at least 5. Let G' be the graph obtained from G by orienting and deleting t. Then s and u are the only possible unoriented degree 3 vertices in G'. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$.* Then $\delta_G(A)$ or $\delta_G(A \cup \{t\})$ is a 2-robust at most 3-edge-cut, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$. If u and v are both in A, then G contains an internal 4-edge-cut, a contradiction. Let $u \in A$ and $v \in G' - A$. If all such 3-edge-cuts separate d from u, then by Claim 4.2.5 G' has a valid orientation. This leads to a valid orientation of G, a contradiction. Hence we may assume that $d \in A$. If all such cuts separate s from u, then similarly, G' has a valid orientation. This leads to a valid orientation of G, a contradiction. Hence we may assume that $s \in A$. Then $\delta_G(A)$ or $\delta_G(A \cup \{t\})$ is a 4-edge-cut that does not separate s from d, a contradiction. Hence no such cut exists. Thus G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence v has degree 4. The same argument applies for u, x, and y. □

DTS7. *The edges uw, vw, xz, and yz exist, and w and z have degree* 5.

Proof. Suppose without loss of generality that either u and w are not adjacent or that $deg(w) \geq 6$. Let e_1, e_2, e_3, and e_t be the edges incident with u in order, where e_1 is on the boundary of F_G and $e_t = ut$. By DTS3, e_3 is not a chord. Let G' be the graph obtained from G by lifting the pair of edges e_1, e_2, and orienting and deleting u and t. Then v and s are the only possible unoriented degree 3 vertices in G'.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where v and the lifted edge are in A. Then $\delta_G(A)$ is a 2-robust internal at most 5-edge-cut, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is in $G' - A$. Since G does not contain a 2-robust at most 3-edge-cut, w and the endpoint of e_3 are in A. Since $\delta_G(A \cup \{u, t\})$ is a 2-robust 4-edge-cut, $d \in G' - A$, and $s \in A$. Figure 4.6 shows this graph. In G, contract A to form a graph $\hat{G}$. Then $\hat{G}$ is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation

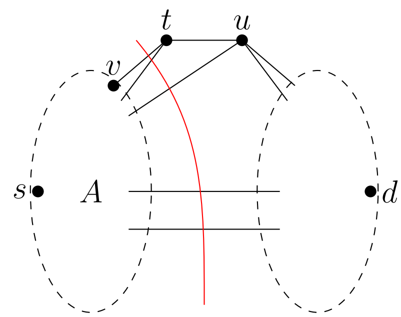


Figure 4.6: DTS7: Reduce the 5-edge-cut $\delta(A)$.

to G. Contract $G - A$ and delete tv and tw to form a graph $\hat{G}'$. Then $\hat{G}'$ has an oriented degree 3 vertex d' and two vertices of degree 3 (s and v). If $\hat{G}'$ contains a 2-robust at most 2-edge-cut, then G contains a 2-robust at most 3-edge-cut, a contradiction. If $\hat{G}'$ contains a 2-robust 3-edge-cut, it necessarily separates d' from s or v (else G has a 2-robust at most 5-edge-cut with d, s, and t on the same side), so $\hat{G}'$ has a valid orientation by Claim 4.2.5. Thus $\hat{G}'$ is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence G' is 3-edge-connected.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is in $G' - A$. If all such 3-edge-cuts separate d from v, then by Claim 4.2.5 G' has a valid orientation. This leads to a valid orientation of G, a contradiction. Hence we may assume that $d \in A$. If $s \in A$, then either $\delta_G(A \cup \{t\})$ or $\delta_G(A \cup \{u, t\})$ is a 2-robust at most 5-edge-cut that does not separate d from s or t, contradicting DTS2. Hence $s \in G' - A$. Therefore $\delta_{G'}(A)$ separates s from d and v. By Claim 4.2.5, G' has a valid orientation. This leads to a valid orientation of G, a contradiction. Thus G' has no 2-robust at most 3-edge-cut.

Therefore G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. We conclude that u and w are adjacent and $deg(w) = 5$. The other cases are equivalent. □

DTS8. *The vertices d, s, t, u, v, x, and y form the boundary of F_G, where either $v = x$ or $u = z$ (up to renaming).*

Proof. Suppose without loss of generality that u is not adjacent to s or to d. Suppose that u is incident with a chord. If u and v are both incident with chords, then by DTS3 these chords cross, a contradiction. Thus we may assume that v is not incident with a chord. If

v is adjacent to d, then u is not incident with a chord, by DTS3, a contradiction. If v is adjacent to s, then $v = x$. If u and y are both incident with chords, then by DTS3 these chords cross, a contradiction. Thus we may assume that y is not incident with a chord. If y is adjacent to d, then u is not incident with a chord, by DTS3. Then y is neither incident with a chord nor adjacent to d. Hence there exists a vertex in the set $\{u, v, x, y\}$ that is adjacent to exactly one vertex in $\{s, t, d\}$ and is not incident with a chord. Without loss of generality, assume this vertex is u. Let e_1, e_2, e_3, and e_t be the edges incident with u in order, where e_1 is on the boundary of F_G and $e_t = ut$. Let q be the other endpoint of e_1.

Let G' be the graph obtained from G by orienting and deleting e_1 and e_2 to satisfy $p(u)$, and contracting $\{u, t, v, w\}$ to a single vertex c of degree 4. Then G' has at most two vertices of degree 3 (s and q). *Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where c and q are in A.* Then $\delta_G(G' - A)$ is a 2-robust internal at most 4-edge-cut in G, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut where $c \in A$ and $q \in G' - A$. Then $s \in A$ and $d \in G' - A$, else G has a 2-robust at most 4-edge-cut that does not separate d from s and t, a contradiction. Figure 4.7 shows this graph. In G, contract $(A - \{c\}) \cup \{v, w\}$ to a vertex, calling the resulting graph $\bar{G}$. Then $\bar{G}$ is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G, contract $(G' - A) \cup \{u, t\}$ to a vertex, and delete tv and tw, calling the resulting graph $\bar{G}'$. Then $\bar{G}'$ has a directed degree 3 vertex d' and two vertices of degree 3 (s and v). If $\bar{G}'$ contains a 2-robust at most 2-edge-cut, then G contains a 2-robust at most 4-edge-cut that does not separate d from s and t, a contradiction. If $\bar{G}'$ contains a 2-robust 3-edge-cut, it necessarily separates d' from s or v, so $\bar{G}'$ has a valid orientation by Claim 4.2.5. Thus $\bar{G}'$ is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence G' is 3-edge-connected.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $c \in A$ and $q \in G' - A$. If all such cuts separate d from q, then G' has a valid orientation by Claim 4.2.5. This leads to a valid orientation of G, a contradiction. Thus we may assume that $d \in G' - A$. If all such cuts separate s from q, then G' has a valid orientation by Claim 4.2.5. This leads to a valid orientation of G, a contradiction. Thus we may assume that $s \in G' - A$. In G, $\delta_G(G' - A)$ is not a 4-edge-cut, else it separates d and s from t, a contradiction. Hence it is a 5-edge-cut. Figure 4.8 shows this graph.

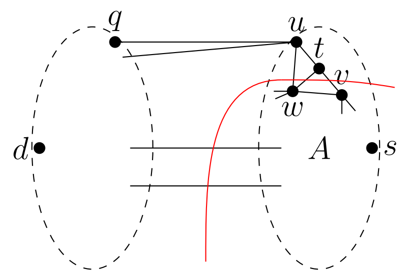


Figure 4.7: DTS8: Reduce the 5-edge-cut $\delta(A - \{u,t\}))$.

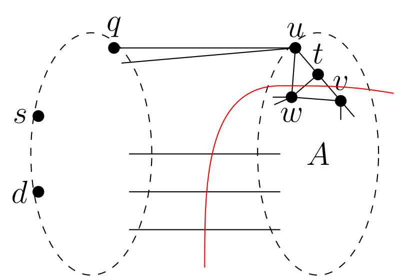


Figure 4.8: DTS8: Reduce the 6-edge-cut $\delta(A - \{u,t\})$.

In G, contract $(A - \{r\}) \cup \{v, w\}$ to a vertex, calling the resulting graph $\bar{G}$. Then $\bar{G}$ is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G, contract $(G' - A) \cup \{u, t\}$ to a vertex, and delete tv and tw, calling the resulting graph $\bar{G}'$. Then $\bar{G}'$ has a directed degree 4 vertex d' and one vertex of degree 3 (v). If $\bar{G}'$ contains a 2-robust at most 2-edge-cut, then G contains a 2-robust at most 4-edge-cut that does not separate d from s and t, a contradiction. If $\bar{G}'$ contains a 2-robust 3-edge-cut, it necessarily separates d' from v (else G contains an internal 2-robust at most 4-edge-cut), so $\bar{G}'$ has a valid orientation by Claim 4.2.5. Thus $\bar{G}'$ is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence G' has no 2-robust at most 3-edge-cuts.

Therefore G' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. The result follows. □

This concludes the proof of the properties of DTS graphs. Without loss of generality, the boundary of F_G consists of the vertices $u, t, v = x, s, y, d$ in order. This graph is shown in Figure 4.9. Let $A = \{u, t, v, s, y, d, w, z\}$. Then $\delta(A)$ is an internal 7-edge-cut. If $G - A$ contains only one vertex, then the graph contains unoriented parallel edges, a contradiction. Hence $\delta(A)$ is 2-robust. Contract $G - A$ to a vertex, calling the resulting graph G'. Then G' is a DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G. Contract A to a single vertex d' and delete the two edges incident with w, calling the resulting graph G''. Then d' is a directed vertex of degree 5, and G'' contains no degree 3 vertices, since in G the neighbours of w are distinct internal vertices of degree at least 5. If G'' contains a 2-robust at most 3-edge-cut, then G contains a 2-robust internal at most 5-edge-cut, a contradiction. Hence G'' is a DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Therefore no minimum counterexample exists, and Theorem 4.2.2 follows. □

We conclude this section with some simple consequences of Theorem 4.2.2. In proving Jaeger's Strong 3-Flow Conjecture for projective planar graphs, our reductions will result in DTS graphs, and also graphs with three vertices of degree 3 and no directed vertex. Theorem 4.2.2 implies that such graphs also have a valid orientation.

Definition 4.2.6. *An* RST graph *is a graph G embedded in the plane, together with a valid $\mathbb{Z}_3$-prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is* 3*-edge-connected,*

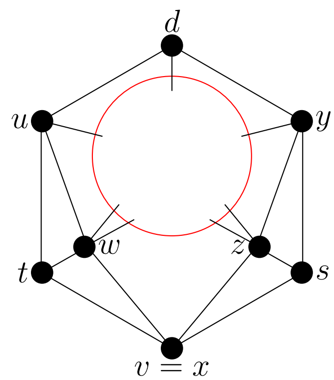


Figure 4.9: The boundary consists of only the vertices $u, t, v = x, s, y, d$.

2. G has a specified face F_G, and at most three specified vertices r, s, and t,

3. if r, s, and t exist, then they have degree 3 and are in the boundary of F_G,

4. G has at most three 3-edge-cuts, which can only be $\delta(\{r\})$, $\delta(\{s\})$, and $\delta(\{t\})$, and

5. every vertex not in the boundary of F_G has 5 edge-disjoint paths to the boundary of F_G.

A 3RST graph *is a graph G with the above definition, where (4) is replaced by*

4'. all vertices aside from r, s, and t have degree at least 4, and if r, s, and t exist, then every 3-edge-cut in G separates one of r, s, and t from the other two.

Corollary 4.2.7. *Every RST graph has a valid orientation.*

Proof. Orient r to satisfy $p(r)$. The resulting graph is a DTS graph and has a valid orientation by Theorem 4.2.2. □

The result in Claim 4.2.5 will also be required in later proofs, along with an analogous result for 3RST graphs.

Lemma 4.2.8. *All 3DTS graphs have a valid orientation.*

Proof. Let G be a minimal counterexample with respect to $|E(G)|$. Suppose that G has no 2-robust 3-edge-cut. Then G is a DTS graph and has a valid orientation by Theorem 4.2.2, a contradiction. Thus we may assume that G has a 2-robust 3-edge-cut $\delta(A)$, where A is chosen so that $d \in A$. We may assume that $t \notin A$.

Let G_1 be the graph obtained from G by contracting $G - A$. Then G_1 is a 3DTS graph and has a valid orientation by the minimality of G. Transfer this orientation to G. Let G_2 be the graph obtained from G by contracting A to a directed vertex d'. Then G_2 is a 3DTS graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Corollary 4.2.9. *All 3RST graphs have a valid orientation.*

Proof. Orient r to satisfy $p(r)$. The resulting graph is a 3DTS graph and has a valid orientation by Lemma 4.2.8. □

4.3 Two Faces

As discussed in Section 4.1, if the boundary of the specified face of a graph embedded in the projective plane is not bounded by a cycle, then the graph can be embedded in the plane with two specified faces that have a common vertex. A proof of this appears in Section 4.4. Theorem 4.3.2 is the variant of Theorem 3.3.3 that we require for this situation.

Definition 4.3.1. *An* FT graph *is a graph G embedded in the plane, together with a valid prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is 3-edge-connected,*
2. *G has two specified faces F_G and F_G^*, and at most one specified vertex d or t,*
3. *there is at least one vertex in common between F_G and F_G^*,*
4. *if d exists, then it has degree 3, 4, or 5, is oriented, and is in the boundary of both F_G and F_G^*,*
5. *if t exists, then it has degree 3 and is in the boundary of at least one of F_G and F_G^*,*
6. *G has at most one 3-edge-cut, which can only be $\delta(\{d\})$ or $\delta(\{t\})$, and*

7. *every vertex not in the boundary of F_G or F_G^* has 5 edge-disjoint paths to the union of the boundaries of F_G and F_G^*.*

We define all 3-edge-connected graphs on at most two vertices to be FT graphs, regardless of vertex degrees.

We note that a DTS graph where at most one of d, t, and s exists is an FT graph.

Theorem 4.3.2. *Every FT graph has a valid orientation.*

Proof. Let G be a minimal counterexample with respect to the number of edges, followed by the number of unoriented edges. If $|E(G)| = 0$, then G consists of only an isolated vertex, and thus has a trivial valid orientation. If $|E(G)| - deg(d) = 0$ then G has an existing valid orientation. Thus we may assume G has at least one unoriented edge.

We will prove the following series of properties of G.

FT1: The graph G does not contain a loop, unoriented parallel edges, or a cut vertex.

We define a *Type 1 cut* to be an edge-cut $\delta(A)$ that does not intersect the boundary of F_G or F_G^*. Since F_G and F_G^* have a common vertex, it follows that they are either both contained in A or both contained in $G - A$. Hence this is an internal cut. We define a *Type 2 cut* to be an edge-cut $\delta(A)$ that intersects the boundary of exactly one of F_G and F_G^*. Finally, we define a *Type 3 cut* to be an edge-cut $\delta(A)$ that intersects the boundary of both F_G and F_G^*.

FT2: The graph G does not contain

a) a 2-robust at most 5-edge-cut of Type 1 or 3,

b) a 2-robust 4-edge-cut,

c) a 2-robust 5-edge-cut of Type 2 where t is on the side containing the boundary of both F_G and F_G^*, or

d) a 2-robust 6-edge-cut of Type 1.

FT3: The graph G does not have a chord of the cycle bounding F_G, that is incident with a vertex of degree 3 or 4.

FT4: The vertex t exists.

FT5: There is no edge in common between F_G and F_G^*.

The proofs of these properties form the bulk of the proof of Theorem 4.3.2. We then consider a vertex in common between F_G and F_G^* of least degree, to complete the proof.

FT1. *The graph G does not contain a loop, unoriented parallel edges, or a cut vertex.*

Proof. This proof is identical to that of DTS1. □

We consider edge-cuts in G. We define a *Type 1 cut* to be an edge-cut $\delta(A)$ that does not intersect the boundary of F_G or F_G^*. Since F_G and F_G^* have a common vertex, it follows that they are either both contained in A or both contained in $G - A$. Hence this is an internal cut. We define a *Type 2 cut* to be an edge-cut $\delta(A)$ that intersects the boundary of exactly one of F_G and F_G^*. Finally, we define a *Type 3 cut* to be an edge-cut $\delta(A)$ that intersects the boundary of both F_G and F_G^*.

FT2. *The graph G does not contain*

a) *a 2-robust at most 5-edge-cut of Type 1 or 3,*

b) *a 2-robust 4-edge-cut,*

c) *a 2-robust 5-edge-cut of Type 2 where t is on the side containing the boundary of both F_G and F_G^*, or*

d) *a 2-robust 6-edge-cut of Type 1.*

Proof.

a) Suppose that G does contain a 2-robust at most 5-edge-cut $\delta_G(A)$ of Type 1 or 3. Assume that $d, t \notin G - A$, and if the cut is of Type 1, the boundaries of F_G and F_G^* are in A. Let G' be the graph obtained from G by contracting $G - A$ to a single vertex. The resulting vertex v has degree 4 or 5. If $\delta_G(A)$ is of Type 1, then v has degree 5. If v has degree 4, then the cut is of Type 3, and v is on the boundary of both specified faces in G'. If G' contains a 2-robust cut $\delta_{G'}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Hence G' is an FT graph and has a valid orientation by the minimality of G. Transfer this orientation to G.

Let G'' be the graph obtained from G by contracting A to a single vertex v. This vertex has degree 4 or 5 and is oriented. If $\delta_G(A)$ is of Type 3, then v is on the boundary of both specified faces in G''. If $\delta_G(A)$ is of Type 1, then we can choose both specified faces to be incident with v. If G'' has a 2-robust cut $\delta_{G''}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Thus G'' is an FT graph and has a valid orientation by the minimality of G. Transfer this orientation to G to obtain a valid orientation of G, a contradiction.

b) Suppose that G does contain a 2-robust 4-edge-cut $\delta_G(A)$. By a) $\delta_G(A)$ is of Type 2. Let A be the side containing (part of) the boundaries of both F_G and F^*_G. Then $d \in A$ if it exists. Let G' be the graph obtained from G by contracting $G - A$ to a single vertex. The resulting vertex v has degree 4 and is on the boundary of a specified face. If G' contains a 2-robust cut $\delta_{G'}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Hence G' is an FT graph and has a valid orientation by the minimality of G. Transfer this orientation to G.

Let G'' be the graph obtained from G by contracting A to a single vertex v. This vertex has degree 4 and is oriented. There is only one specified face, which may contain t. If G'' has a 2-robust cut $\delta_{G''}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Thus G'' is a DTS graph and has a valid orientation by Theorem 4.2.2. Transfer this orientation to G to obtain a valid orientation of G, a contradiction.

c) This case works in the same way as b). In G'' there is a degree 5 oriented vertex and no degree 3 vertex. Thus G'' is a DTS graph and has a valid orientation by Theorem 4.2.2. Transfer this orientation to G to obtain a valid orientation of G, a contradiction.

d) This case works in the same way as a). In G'', there is a degree 6 oriented vertex, and no degree 3 or 4 vertices. Hence we may delete one boundary edge incident with v to obtain a graph G''' with a degree 5 oriented vertex and no degree 3 vertex. If G''' contains a 2-robust cut of size at most 3, then G has a corresponding cut of size 4, contradicting a). Thus G''' is an FT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence no such cut exists. □

We now consider the local properties of the graphs at vertices of low degree.

FT3. *The graph G does not have a chord of the cycle bounding F_G that is incident with a vertex u of degree 3 or 4.*

Proof. Suppose that such a chord uv exists, where $deg_G(u) \in \{3, 4\}$. Let H and K be subgraphs of G such that $H \cap K = \{\{u, v\}, \{uv\}\}$, $H \cup K = G$, and F_G^* is in H. Note that this implies $d \in H$ if it exists.

Suppose that $\delta(H)$ is not 2-robust. Then K contains d, else G has unoriented parallel edges, and thus a valid orientation by FT1. By definition, either u or v is d. Suppose that $v = d$. Since G has no unoriented degree 3 vertex, $|\delta(H - \{u\})| \leq 3$, a contradiction. Now consider the case where $u = d$. Since G has no unoriented parallel edges, $\delta(H)$ is a 3-edge-cut and G contains a degree 3 vertex, a contradiction. Hence we may assume that $\delta(H)$ is 2-robust.

Suppose that $\delta(K)$ is not 2-robust. Then $|V(H)| = 3$. If u or v is d, the above argument applies. Thus we may assume that d is in $V(H) - \{u, v\}$. If there are parallel edges with endpoints d and u, then $\delta(\{d, u\})$ is an at most 5-edge-cut. Orient u and contract the parallel edges between d and u, calling the resulting graph G'. Note that the vertex of contraction has the same degree as d. Thus it is clear that G' is an FT graph, and thus has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Suppose that there are not parallel edges with endpoints d and u. Then there are parallel edges with endpoints d and v. Since $|\delta(\{d, v\})| \geq 5$, $deg_K(v) \geq 4$. Orient u. Then K is an FT graph, and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. We may now assume that $\delta(K)$ is 2-robust.

By definition, $\delta(H)$ and $\delta(K)$ have size at least 4. Hence $deg_G(v) \geq 6$, and so v is not d or t. In addition, v has degree at least 3 in both H and K.

Suppose that $u \neq d$. Then in H, contract uv. The graph H/uv is an FT graph, and so by the minimality of G, H/uv has a valid orientation. Transfer this orientation to G, and orient u. In K, if u has degree 2, add an edge e directed from u to v. Then u is a directed vertex of degree 3. Since F_G^* is in H, we can choose the second specified face to be incident with u. Hence $K + e$ is an FT graph. By the minimality of G, K' has a valid orientation. This leads to a valid orientation of G, a contradiction.

Thus $u = d$. Then in both H and K add a directed edge from u to v. It is clear that $H + e$ and $K + e$ are FT graphs, so by the minimality of G, they have valid orientations. Transfer the orientations of $H + e$ and $K + e$ to G to obtain a valid orientation of G, a contradiction. Thus no such chord exists. □

Similarly, no such chord of F_G^* exists.

Claim 4.3.3. *The graph G contains d or t.*

Proof. Suppose for a contradiction that neither d nor t exists in G. Let v be a vertex in both F_G and F_G^*. If $deg(v) \leq 5$, orient v, calling the resulting graph G'. Then it is clear that G' is an FT graph, and has a valid orientation by the minimality of G. This is a valid orientation of G, a contradiction.

We may assume $deg(v) \geq 6$. Orient and delete a boundary edge incident with v, calling the resulting graph G'. Then G' has at most one vertex of degree 3. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 4-edge-cut, a contradiction. Hence G' is an FT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

FT4. *The vertex t exists.*

Proof. Suppose that d exists. Then t does not. By definition, d is on the boundaries of both F_G and F_G^*. Suppose that d has degree 3. Let G' be the graph obtained from G by deleting d. Then G' has at most three vertices of degree 3 and no oriented vertex. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 4-edge-cut, a contradiction. Hence G' is an RST graph, and has a valid orientation by Theorem 4.2.7. This leads to a valid orientation of G, a contradiction.

We may assume that d has degree 4 or 5. Suppose that an edge e incident with d is in the boundary of F_G and F_G^*. Let G' be the graph obtained from G by deleting e. Then G' has at most one vertex of degree 3 and an oriented vertex of degree 3 or 4. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 4-edge-cut, a contradiction. Hence G' is a DTS graph, and has a valid orientation by Theorem 4.2.2. This leads to a valid orientation of G, a contradiction.

We now assume that F_G and F_G^* do not have an edge in common incident with d. Let e_1, e_2, e_3, e_4, and (possibly) e_5 be the edges incident with d in cyclic order, where e_1 is on the boundary of F_G and e_2 is on the boundary of F_G^*. Let G' be the graph obtained from G

by deleting e_1 and e_2. Then G' contains at most two vertices of degree 3 and an oriented vertex of degree 2 or 3. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding either 2-robust at most 4-edge-cut, or 5-edge-cut of Type 3, a contradiction. Let G'' be the graph obtained from G' by adding a directed edge from d to the other endpoint of e_3 if $deg_{G'}(d) = 2$. Then G' is a DTS graph, and has a valid orientation by Theorem 4.2.2. This leads to a valid orientation of G, a contradiction. □

Claim 4.3.4. *The vertex t is not in the boundary of both F_G and F^*_G.*

Proof. Suppose that t is in the boundary of both F_G and F^*_G. Let G' be the graph obtained from G by orienting the edges incident with t to satisfy $p(t)$. Then G' is an FT graph and has a valid orientation by the minimality of G. This is a valid orientation of G, a contradiction. □

FT5. *There is no edge in common between F_G and F^*_G.*

Proof. Suppose for a contradiction that G has an edge e in the boundary of F_G and F^*_G. Then e is not incident with t by Claim 4.3.4. Let G' be the graph obtained from G by deleting e. Then G' has at most three vertices of degree 3 and no oriented vertex. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 4-edge-cut, a contradiction. Hence G' is an RST graph, and has a valid orientation by Theorem 4.2.7. This leads to a valid orientation of G, a contradiction. □

By definition, there exists a vertex in the boundaries of both F_G and F^*_G. Among all such vertices, let v have the least degree, say k. Let e_1, e_2, ..., e_k be the edges incident with v in cyclic order, where e_1 and e_k are on the boundary of F_G. Let i be such that e_i, e_{i+1} are on the boundary of F^*_G. Note that $i \neq 1$ and $i + 1 \neq k$ by FT5.

Claim 4.3.5. *We have $k \geq 5$.*

Proof. The alternative is that $k = 4$. Since G does not contain unoriented parallel edges, at most one of the edges e_1, e_2, e_3, e_4 is incident with t. Hence without loss of generality, we may assume that e_1 and e_2 are not incident with t. Let G' be the graph obtained from G by lifting e_3 and e_4, and orienting and deleting e_1 and e_2. Then G' contains at most three vertices of degree 3. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 4-edge-cut, or a 2-robust at most 5-edge-cut of Type 3, a contradiction. Then G' is an RST graph, and has a valid orientation by Theorem 4.2.7. This leads to a valid orientation of G, a contradiction. □

Claim 4.3.6. *We have $i \neq 2$.*

Proof. Suppose that $i = 2$. Let G' be the graph obtained from G by lifting e_1 and e_2. Then G' contains at most two vertices of degree 3: v and t. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding 2-robust at most 5-edge-cut of Type 3, a contradiction. Then G' is an RST graph, and has a valid orientation by Theorem 4.2.7. This leads to a valid orientation of G, a contradiction. Hence $i > 2$. □

Similarly we may assume that $i + 1 < k - 1$. It follows that $k \geq 6$.

Let G' be the graph obtained from G by lifting e_1 and e_2. Then $deg_{G'}(v) \geq 4$, so G' has at most one degree 3 vertex: t. If G' has a 2-robust at most 2-edge-cut, then G has a corresponding 2-robust at most 4-edge-cut, a contradiction. If G' is an FT graph and has a valid orientation by the minimality of G, this leads to a valid orientation of G, a contradiction. Hence we may assume that G' is not an FT graph. Then G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$. Now $\delta_G(A)$ is a 2-robust 5-edge-cut using e_1 and e_2. By FT2, $\delta_G(A)$ is of Type 2 and separates t from v.

Let G'' be the graph obtained from G by lifting e_i and e_{i-1}. Similarly, G'' is an FT graph and has a valid orientation by the minimality of G, unless G'' has a 2-robust 5-edge-cut $\delta_{G''}(B)$ of Type 2 using e_i and e_{i-1} that separates t from v.

But $\delta_G(A)$ intersects only the boundary of F_G, so t is on the boundary of F_G and not F^*_G. Similarly $\delta_G(B)$ intersects only the boundary of F^*_G, so t is on the boundary of F^*_G and not F_G, a contradiction.

We conclude that no such counterexample exists. Therefore all FT graphs have a valid orientation. □

4.4 Projective Plane

Sections 4.2 and 4.3 provide the necessary planar results to show Jaeger's Strong 3-Flow Conjecture for graphs embedded in the projective plane. We prove this result here.

Definition 4.4.1. *A* PT graph *is a graph G embedded in the projective plane, together with a valid prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is 3-edge-connected,*

2. *G has a specified face F_G, and at most one specified vertex t,*

3. *if t exists, then it has degree 3 and is in the boundary of F_G,*

4. *G has at most one 3-edge-cut, which can only by $\delta(\{t\})$, and*

5. *every vertex not in the boundary of F_G has 5 edge-disjoint paths to the boundary of F_G.*

We define all 3-edge-connected graphs on at most two vertices to be PT graphs, regardless of vertex degrees.

A 3PT graph *is a graph G with the above definition, where (4) is replaced by*

4'. *all vertices aside from t have degree at least 4, and if t exists, then every 3-edge-cut $\delta(A)$ in G where A is contained in an open disk has $t \in A$.*

Theorem 4.4.2. *All PT graphs have a valid orientation.*

Proof. Let G be a minimal counterexample with respect to the number of edges. If $|E(G)| = 0$, then G consists of only an isolated vertex, and thus has a trivial valid orientation. Thus we may assume G has at least one edge.

We will prove the following series of properties of G.

PT1: The graph G does not contain a loop, unoriented parallel edges, or a cut vertex.

PT2: The face F_G is bounded by a cycle.

PT3: There is no contractible chord of F_G incident with a degree 3 or 4 vertex.

We define a *Type 1 cut* to be an edge-cut $\delta(A)$ that does not intersect the boundary of F_G. We define a *Type 2 cut* to be an edge-cut $\delta(A)$ that has exactly two edges in the boundary of F_G. We define all remaining edge-cuts (with at least 4 edges in the boundary of F_G) to be of *Type 3*.

PT4: 1. The graph G does not contain a 2-robust at most 4-edge-cut.

2. If G contains a 2-robust at most 5-edge-cut $\delta(A)$, then it is either
   (a) of Type 1, where A is contained in an open disk and the boundary of F_G is in A, or
   (b) of Type 2, where A is contained in an open disk, $t \in A$, and t does not have an incident edge in $\delta(A)$.
3. If G has a 2-robust Type 1 6-edge-cut $\delta(A)$ where A is contained in an open disk, then the boundary of F_G is in A.

PT5: The vertex t exists.

Let u and v be the boundary vertices adjacent to t, and let w be the remaining vertex adjacent to t.

PT6: Vertices u and v have degree 4.

PT7: Vertex w has degree 5.

PT8: The edge tw is a chord.

PT9: Both u and v are adjacent to w.

The proofs of these properties form the bulk of the proof of Theorem 4.4.2.

PT1. *The graph G does not contain a loop, parallel edges, or a cut vertex.*

Proof. This proof is identical to that of DTS1. $\square$

PT2. *The face F_G is bounded by a cycle.*

Proof. Suppose that F_G is not bounded by a cycle. Then there exists a vertex v that appears twice on the boundary walk of F_G. By PT1, v is not a cut vertex. As shown in Section 4.1, G is a planar graph with two specified faces, each containing v, and G has at most one vertex of degree 3. Hence G is an FT graph and has a valid orientation by Theorem 4.3.2, a contradiction. $\square$

We classify the 2-robust cuts in G into three types. Let $\delta(A)$ be an edge-cut in G. First, we note that since F_G is bounded by a cycle, any edge-cut contains an even number of edges from the boundary of F_G. We say that $\delta(A)$ is:

1. Type 1, if it does not contain an edge in the boundary of F_G,
2. Type 2, if it has precisely two edges in the boundary of F_G, and
3. Type 3 otherwise.

Claim 4.4.3. *The graph G has no 2-robust Type 1 cut $\delta(A)$ of size at most 6 where A is contained in an open disk and the boundary of F_G is in $G - A$.*

Proof. Suppose such a cut $\delta(A)$ exists in G. Then by definition, $\delta_G(A)$ is at least a 5-edge-cut. Let G' be the graph obtained from G by contracting A to a vertex. It is clear that G' is a PT graph, and thus has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting $G - A$ to a vertex. Then G'' is a graph where all vertices have degree at least 5, with a directed vertex d' of degree 5 or 6. Now $G[A]$ is planarly embedded by construction, where every vertex adjacent to d' is on the outer face. Therefore, G'' can be embedded on the plane by inserting d' into the outer face of $G[A]$. Choose the specified face to be incident with d'.

If d' has degree 5, then G'' is a DTS graph and has a valid orientation by Theorem 4.2.2. If d' has degree 6, delete one boundary edge incident with d' to form a graph $\bar{G}$. If $\bar{G}$ has a 2-robust at most 3-edge-cut, then G has a 2-robust at most 4-edge-cut of Type 1, a contradiction. Thus $\bar{G}$ is a DTS graph and has a valid orientation by Theorem 4.2.2. This yields a valid orientation of G, a contradiction. Hence no such cut exists. □

Claim 4.4.4. *The graph G has no 2-robust Type 2 cut of size 4.*

Proof. Suppose such a cut $\delta(A)$ exists in G. Either A or $G - A$ is contained in an open disk. Without loss of generality, suppose that A is contained in an open disk. Let G' be the graph obtained from G by contracting A to a vertex. It is clear that G' is a PT graph, and thus has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting $G - A$ to a vertex v. Now $G[A]$ is planarly embedded by construction, where every vertex adjacent to v is on the outer face. Therefore, G'' can be embedded on the plane by inserting v into the outer face of $G[A]$. Choose the specified face to be incident with v and all vertices in A incident with F_G.

Then G'' is a planar graph with a specified face containing a directed vertex of degree 4. Hence G'' is a DTS graph and has a valid orientation by Theorem 4.2.2. This yields a valid orientation of G, a contradiction. Hence no such cut exists. □

Claim 4.4.5. *The graph G has no 2-robust Type 2 cut of size 5 where A is contained in an open disk and $t \in G - A$.*

Proof. Suppose such a cut exists in G. Let G' be the graph obtained from G by contracting A to a vertex. It is clear that G' is a PT graph, and thus has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting $G - A$ to a vertex. Then G'' can be embedded as a planar graph with a specified face containing a directed vertex of degree 5 but no degree 3 vertex. Hence G'' is a DTS graph and has a valid orientation by Theorem 4.2.2. This yields a valid orientation of G, a contradiction. Hence no such cut exists. □

In order to complete our analysis of Type 2 cuts, we must consider contractible chords.

PT3. *There is no contractible chord of F_G incident with a degree 3 or 4 vertex.*

Proof. Suppose that such a chord $e = uv$ exists where the degree of u is 3 or 4. Let H and K be subgraphs of G such that $H \cap K = \{\{u, v\}, \{e\}\}$, $H \cup K = G$, and K is contained in an open disk.

Both $\delta(H)$ and $\delta(K)$ are 2-robust, else G has unoriented parallel edges. Both cuts are of Type 2. Therefore, $|\delta(H)|, |\delta(K)| \geq 5$, and so $deg(v) \geq 8$.

Contract K to a single vertex. It is clear that G/K is a PT graph and has a valid orientation by the minimality of G. Transfer this orientation to G, and orient u. Add a directed edge e from u to v in K (in the boundary of F_K). Then $K + e$ can be embedded as a planar graph with a single specified face, a directed vertex of degree 3 or 4, and one other possible degree 3 vertex t. Hence $K + e$ is a DTS graph and has a valid orientation by Theorem 4.2.2. This leads to a valid orientation of G, a contradiction. Hence any chord in G incident with a vertex of degree 3 or 4 is non-contractible. □

Claim 4.4.6. *The graph G has no 2-robust Type 2 cut of size 5 where A is contained in an open disk, $t \in A$, and t is incident with a single edge in $\delta(A)$ which must be a boundary edge of F_G.*

Proof. Suppose such a cut exists in G. Let G' be the graph obtained from G by contracting A to a vertex. Since G' is a PT graph, it has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting $G - A$ to a vertex. Then G'' can be embedded as a planar graph with a specified face containing a directed vertex of degree 5 adjacent to t. Since A is contained in an open disk, t is not incident with a chord by PT3. Let $\hat{G}$ be the graph obtained from G'' by orienting and deleting t. Then $\hat{G}$ has a directed vertex of degree 4 and at most one vertex of degree 3. If $\hat{G}$ contains a 2-robust at most 3-edge-cut, then G contains an at most 4-edge-cut of Type 1 or 2, a contradiction. Hence $\hat{G}$ is a DTS graph and has a valid orientation by Theorem 4.2.2. This yields a valid orientation of G, a contradiction. Hence no such cut exists. □

Claim 4.4.7. *The graph G has no 2-robust Type 3 cut of size at most 5.*

Proof. Suppose that such a cut exists. Without loss of generality, suppose that $t \in A$. Let G' be the graph obtained from G by contracting A to a single vertex. Since G' is a PT graph, it has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting $G - A$ to a single directed vertex d of degree 4 or 5. Then d appears twice in the boundary walk of $F_{G''}$. As in the proof of PT2, G'' is a planar graph with two specified faces that both contain the vertex d. Since $t \in A$, G'' has no degree 3 vertex. Therefore G'' is an FT graph and has a valid orientation by Claim 4.3.2. This leads to a valid orientation of G, a contradiction. Hence no such cut exists. □

We summarise the reducible/irreducible cuts in G.

PT4.

1. *The graph G does not contain a 2-robust at most 4-edge-cut.*
2. *If G contains a 2-robust at most 5-edge-cut $\delta(A)$, then it is either*
   (a) *of Type 1, where A is contained in an open disk and the boundary of F_G is in A, or*
   (b) *of Type 2, where A is contained in an open disk, $t \in A$, and t does not have an incident edge in $\delta(A)$.*
3. *If G has a 2-robust Type 1 6-edge-cut $\delta(A)$ where A is contained in an open disk, then the boundary of F_G is in A.*

Proof.

1. By definition, G has no 2-robust either at most 3-edge-cut, or Type 1 4-edge-cut. Claims 4.4.4 and 4.4.7 show that G has no 2-robust 4-edge-cut of Type 2 or 3.
2. This is implied by Claims 4.4.3, 4.4.5, 4.4.6, and 4.4.7.

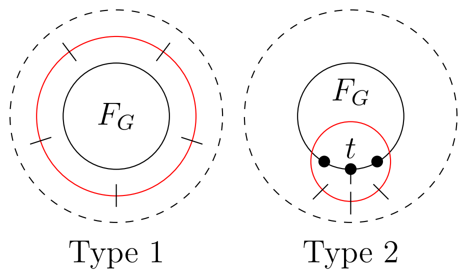


Figure 4.10: The 2-robust at most 5-edge-cuts that can exist in G.

3. This is given by Claim 4.4.3. □

Figure 4.10 shows the 2-robust at most 5-edge-cuts that exist in G. The dashed line is used to represent the crosscap. Again, we wish to reduce at low degree vertices in G. We establish the existence of t and consider its adjacent vertices.

PT5. *The vertex t exists.*

Proof. Suppose that t does not exist. We prove the following claims:

a. Every vertex on the boundary of F_G has degree 4.

b. All vertices in G are on the boundary of F_G.

Let $v_1, v_2, ..., v_k$ be the vertices on the boundary of F_G in order.

c. The graph G is a circulant graph, where k is odd, and for all $1 \leq i \leq k$, v_i is adjacent to v_{i-1}, v_{i+1}, $v_{i+\frac{k-1}{2}}$ and $v_{i-\frac{k-1}{2}}$, where all operations are performed modulo k.

These properties provide the necessary structure to obtain a contradiction.

Claim PT5a. *Every vertex on the boundary of F_G has degree 4.*

Proof. Consider the case where the boundary of F_G contains a vertex v of degree at least 5. Orient and delete a boundary edge incident with v, and call the resulting graph G'. Then G' has at most one degree 3 vertex. Suppose that G' contains a 2-robust at most 3-edge-cut. Then G contains a 2-robust at most 4-edge-cut, a contradiction. Thus G' is a PT graph. By the minimality of G, G' has a valid orientation. This yields a valid orientation of G, a contradiction. □

Claim PT5b. *All vertices in G are on the boundary of F_G.*

Proof. Suppose there exists a vertex v on the boundary of F_G that has an adjacent vertex u not on the boundary of F_G. Then $deg_G(u) \geq 5$. Let e_1, e_2, e_3, e_4 be the edges incident with v in order, where e_1 and e_4 are on the boundary of F_G, and e_2 is incident with u. Lift the pair of edges e_3, e_4, orient the remaining two edges incident with v to satisfy $p(v)$, and delete v, calling the resulting graph G'. Then G' has at most one degree three vertex (the other endpoint of e_1). If G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, then G contains a 2-robust at most 4-edge-cut, or a 2-robust 5-edge-cut that uses a boundary edge of F_G and thus is of Type 2 or Type 3, a contradiction (since t does not exist). Thus G' is a PT graph, and so by the minimality of G, G' has a valid orientation. This yields a valid orientation of G, a contradiction. □

Thus all vertices lie on the boundary of F_G and have degree 4. Let $v_1, v_2, ..., v_k$ be the vertices on the boundary of F_G in order.

Claim PT5c. *The graph G is a circulant graph, where k is odd, and for all $1 \leq i \leq k$, v_i is adjacent to v_{i-1}, v_{i+1}, $v_{i+\frac{k-1}{2}}$ and $v_{i-\frac{k-1}{2}}$, where all operations are performed modulo k.*

Proof. Consider a vertex v_j. It is clear from the construction that v_j is adjacent to v_{j-1} and v_{j+1}. Let v_a and v_b be the remaining two vertices adjacent to v_j. Suppose that v_a and v_b are not adjacent. Then $\delta_G(\{v_{a+1}, v_{a+2}, ..., v_{b-1}\})$ is an at most 4-edge-cut of Type 2. If it is not a 2-robust 4-edge-cut, then G contains parallel edges, a contradiction. Hence G contains a 2-robust 4-edge-cut of Type 2, a contradiction. Thus we may assume that v_a and v_b are adjacent.

Without loss of generality, assume that $j - b > j - a$. Let

$$S = \{v_a, v_{a+1}, v_{a+2}, ..., v_{j-1}\},$$

$$T = \{v_{j+1}, v_{j+2} ..., v_{b-1}, v_b\}.$$

If there exists an edge not on the boundary of F_G with both endpoints in S, then either G contains parallel edges, or G contains a contractible chord incident with a degree 4 vertex, a contradiction. The same is true of edges with both endpoints in T. Hence every edge not in the boundary of F_G and not incident with v_j has one endpoint in S and the other endpoint in T. Since every vertex has degree 4, $|S| = |T|$. If k is even, then the non-contractible chords incident with a vertex are parallel edges. Hence k is odd and G is a circulant graph where for all $1 \leq i \leq k$, v_i is adjacent to v_{i-1}, v_{i+1}, $v_{i+\frac{k-1}{2}}$ and $v_{i-\frac{k-1}{2}}$, where all operations are performed modulo k. □

We now show that such graphs have a valid orientation. Suppose that there exists a vertex $v \in V(G)$ where $p(v) = 0$. Lift two pairs of edges at v, calling the resulting graph G'. If G' contains a 2-robust at most 3-edge-cut, then G contains a corresponding at most 5-edge-cut containing a boundary edge of F_G, and is therefore of Type 2 or Type 3, a contradiction. Hence G' is a PT graph, and thus has a valid orientation by the minimality of G. This yields a valid orientation of G, a contradiction. Hence we may assume that all vertices have non-zero prescription. Since k is odd, there are two vertices with the same prescription that are adjacent via a boundary edge of F_G. Without loss of generality, label the graph so that these vertices are v_1 and v_2, and assume that their prescriptions are -1.

Orient all edges incident with v_1 out from v_1. Orient $v_2 v_{\frac{k+3}{2}}$ into v_2 to meet $p(v_2)$. Orient the remaining two edges incident with the following vertices in order:

$$v_{\frac{k+3}{2}}, v_{\frac{k+1}{2}}, \left(v_k, v_{\frac{k-1}{2}}\right), \left(v_{k-1}, v_{\frac{k-3}{2}}\right), \ldots, \left(v_{\frac{k+7}{2}}, v_3\right).$$

There is only one unoriented edge at v_2 ($v_2 v_{\frac{k+5}{2}}$), which by construction must have the opposite direction to $v_2 v_3$. Since $v_{\frac{k+5}{2}}$ cannot be the only vertex whose prescription is not met, this is a valid orientation for G, a contradiction. □

Claim 4.4.8. *Let G' be a 3PT graph where $|E(G')| < |E(G)|$. Then G' has a valid orientation.*

Proof. Let G' be a minimal counterexample. If G' is a PT graph, then G' has a valid orientation by the minimality of G. Thus we may assume that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $t \in A$. By definition, A is contained in an open disk.

Let G_1 be the graph obtained from G' by contracting A to a vertex. Then G_1 is a 3PT graph and has a valid orientation by the minimality of G'. Transfer this orientation to G'. Let G_2 be the graph obtained from G' by contracting $G' - A$ to a vertex. Then G_2 is a 3DTS graph and has a valid orientation by Lemma 4.2.8. This leads to a valid orientation of G', a contradiction. □

Let u, v, and w be the vertices adjacent to t, where tu and tv are on the boundary of F_G. Since G has no parallel edges, these three vertices are distinct. Note that t is not incident with a contractible chord, so either w is not in the boundary of F_G, or tw is a non-contractible chord.

Claim 4.4.9. *At least two of u, v, and w have degree 4.*

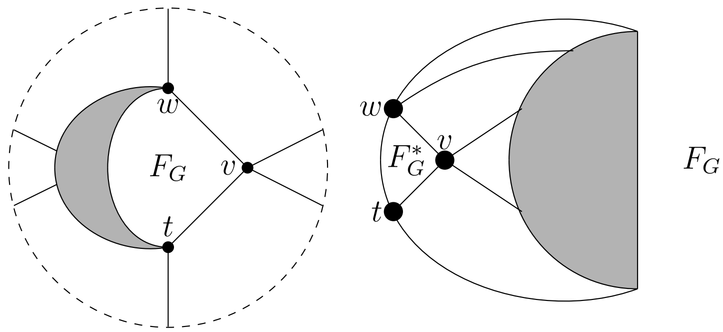


Figure 4.11: PT6: Planar drawing of G.

Proof. Suppose not. Let G' be the graph obtained from G by orienting and deleting t. Then G' contains at most one vertex of degree 3. Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$. Then G contains a 2-robust at most 4-edge-cut, a contradiction. Hence G' is a PT graph. Thus by the minimality of G, G' has a valid orientation. This yields a valid orientation of G, a contradiction. □

PT6. *Vertices u and v have degree* 4.

Proof. Without loss of generality, suppose that u has degree at least 5. We may assume, by Claim 4.4.9, that v and w have degree 4. Then tw is a non-contractible chord of F_G. Let g_1, g_2, g_t, g_3 be the edges incident with w in order, where g_1 and g_3 are on the boundary of F_G, and $g_t = wt$.

Suppose that $g_3 = wv$. Then G can be redrawn with tw inside F_G to yield an FT graph with t and w on both specified faces. Figure 4.11 shows this drawing. By Theorem 4.3.2, G has a valid orientation a contradiction.

We now assume that $g_3 \neq wv$. Lift the pair g_1, g_2, and orient the remaining edges incident with w to satisfy $p(w)$. Orient the remaining edges incident with t to satisfy $p(t)$. Delete w and t, and call the resulting graph G'. Then G' can be embedded as a planar graph with a single specified face. There are two possible degree three vertices in G': v, and the vertex incident with g_3 (which by assumption are distinct). Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$. Then G contains a 2-robust at most 4-edge-cut, a contradiction. Figure 4.12 shows an analysis of the possible cuts.

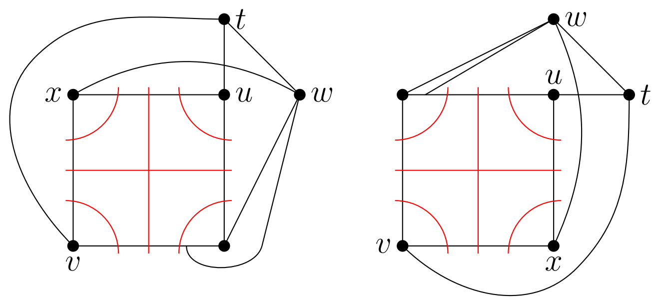


Figure 4.12: PT6: Analysis of cuts.

If G contains a 2-robust at most 3-edge-cut, then by Lemma 4.2.8, G' has a valid orientation, leading to a valid orientation of G, a contradiction. Thus G' is a DTS graph. By Theorem 4.2.2, G' has a valid orientation. This extends to a valid orientation of G, a contradiction. $\square$

Consider vertex w.

Claim 4.4.10. *Vertex w does not have degree 4.*

Proof. Suppose that $deg(w) = 4$. Note that tw is a non-contractible chord of F_G. Let g_1, g_2, g_t, g_3 be the edges incident with w in order, where g_1 and g_3 are on the boundary of F_G, and $g_t = wt$. We may choose the labelling of u and v so that the uw-path in the boundary of F_G contains g_1 but not t.

Suppose that $g_3 = wv$. Then G can be redrawn with tw inside F_G to yield an FT graph with t and w on both specified faces. By Theorem 4.3.2, G has a valid orientation, a contradiction.

We may now assume that $g_3 \neq wv$. Lift the pair g_1, g_2, and orient the remaining edges incident with w to satisfy $p(w)$. Orient the remaining edges incident with t to satisfy $p(t)$. Delete w and t, and call the resulting graph G'. Then G' is a planar graph. There are three possible degree three vertices in G': u, v, and the vertex incident with g_3 (which by assumption are distinct). If G' contains a 2-robust at most 2-edge-cut, then G contains a 2-robust at most 4-edge-cut as in PT6, a contradiction.

If all 2-robust 3-edge-cuts in G' separate two degree 3 vertices, then by Theorem 4.2.5, G' has a valid orientation that leads to a valid orientation of G, a contradiction. Suppose that

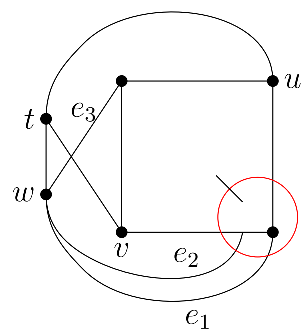


Figure 4.13: Claim 4.4.10: 2-robust Type 2 5-edge-cut.

G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ that does not separate the degree 3 vertices. We assume that $\delta_G(A)$ is not an at most 4-edge-cut. Then $\delta_G(A)$ is a 5-edge-cut using edges g_1 and g_2. This is a Type 2 cut, for which t is in the side not contained in an open disk, a contradiction. Figure 4.13 shows this cut. Hence G' is an RST graph. By Theorem 4.2.7, G' has a valid orientation. This extends to a valid orientation of G, a contradiction. $\square$

PT7. *Vertex w has degree 5.*

Proof. Suppose that $deg(w) \geq 6$. Let e_1, e_2, e_3, e_t be the edges incident with u in order, where e_1 is on the boundary of F_G, and $e_t = ut$. We prove the following claims

a. Edge e_3 is incident with two vertices of degree 4.

b. Edge tw is not a chord.

These provide the necessary structure to complete the proof.

Claim PT7a. *Edge e_3 is incident with two vertices of degree 4.*

Proof. Suppose that e_3 is not incident with two vertices of degree 4. Lift the pair e_1, e_2, and orient the remaining edges incident with u. Orient the remaining edges incident with t, and delete u and t, calling the resulting graph G'. Then G' contains at most one vertex of degree 3, which is v.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$. Then G contains a 2-robust at most 4-edge-cut, a contradiction. *Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$.*

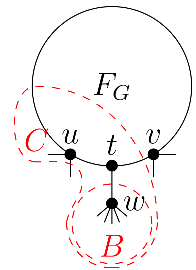


Figure 4.14: PT7: 2-robust Type 1 4-edge-cut.

Then G contains a corresponding 2-robust cut $\delta(C)$ of size at most 5. There are two options. First, $\delta_G(C)$ is of Type 2 and has t on the side that is contained in an open disk. We assume that this side is C. By Claim 4.4.8, $v \notin C$. Hence $tv \in \delta_G(C)$. We have $u, w \in C$, else a smaller cut exists. We conclude that $e_3 \in \delta_G(C)$. Let B be the maximal connected subgraph of C containing w but not t. Then $\delta_G(B)$ is an at most 4-edge-cut of Type 1, a contradiction. This cut can be seen in Figure 4.14. Second, $\delta_G(C)$ is of Type 1 in G. Then $\delta_G(C)$ has v on the side contained in a disk in G', and thus G' has a valid orientation by Claim 4.4.8. This leads to a valid orientation of G, a contradiction.

Hence G' is a PT graph, and thus has a valid orientation by the minimality of G. This extends to a valid orientation of G, a contradiction. □

Since e_3 is incident with two vertices of degree 4, e_3 is a non-contractible chord. Similarly, v must have an analogous incident chord. Neither chord is incident with w, since w has degree at least 6. Let x and y be the vertices adjacent to u and v respectively via a chord.

Claim PT7b. *Edge tw is not a chord.*

Proof. Suppose that tw is a chord. Let B be the set of vertices in the interior of the closed disc bounded by wt, tu, ux, and the wx-subpath of $F_G - t$. If B is empty, consider the same set with respect to v and y. Now $\delta_G(B)$ contains at most 2 edges incident with x. Hence it contains at least two edges incident with w, since $\delta(t)$ is the only 3-edge-cut in G. Since G has no parallel edges, $\delta_G(B)$ is 2-robust. This cut can be seen in Figure 4.15. Let C be the minimal 2-robust edge-cut where $C \subseteq B$ and $\delta_G(C)$ contains at most three edges not incident with w.

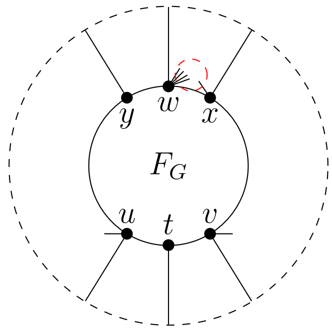


Figure 4.15: PT7: Small Type 1 or 2 cut.

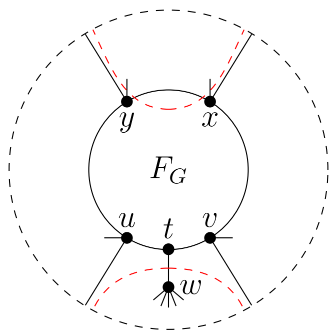


Figure 4.16: PT7: Small Type 2 cut.

Contract C to a vertex, calling the resulting graph G'. Then G' is a PT graph and has a valid orientation by the minimality of G. Transfer this orientation to G and contract $G - C$, calling the resulting graph G'. Delete edges incident with w to make the vertex of contraction a degree 4 vertex. Since G has no parallel edges, at most one degree 3 vertex results from this process. If G'' has a 2-robust at most 3-edge-cut, then C was not minimal, a contradiction. $\square$

Since tw is not a chord, G contains a non-peripheral cut of size at most 5, $\delta_G(A)$, using at most two edges incident with each x and y, and the edge tw. This cut can be seen in Figure 4.16. Then $t \in G - A$, and the graph obtained from contracting $G - A$ to a single vertex is planar. By Claim 4.4.5, G has a valid orientation. $\square$

Therefore, u and v have degree 4, and w has degree 5.

PT8. *The edge tw is a chord.*

Proof. Suppose that tw is not a chord. Let e_1, e_2, e_3, e_t be the edges incident with u in order, where $e_t = ut$, and e_1 is on the boundary of F_G. We first prove the following property.

Claim PT8a. *Either e_3 is incident with two degree four vertices or e_3 is incident with w.*

Proof. Suppose that e_3 is not incident with two degree four vertices, and is not incident with w. Lift the pair e_1, e_2, orient the remaining edges incident with u to satisfy $p(u)$, and orient the remaining edges incident with t to satisfy $p(t)$. Delete u and t, calling the resulting graph G'. Then in G', the only possible degree three vertex is v. The analysis that G' has no small cuts is equivalent to that in PT7. Hence G' is a PT graph, and thus has a valid orientation by the minimality of G. This extends to a valid orientation of G, a contradiction. $\square$

Therefore either e_3 is incident with two degree four vertices or e_3 is incident with w. The same must be true of the corresponding edge incident with v. There are three cases.

1. First, suppose that both u and v are adjacent to w. *Suppose that e_2 is not incident with two degree four vertices.* Then orient and delete e_1 and e_2 to satisfy $p(u)$, and contract the set of vertices $\{u, t, v, w\}$ to a single degree four vertex, calling the resulting graph G'. Then G' has at most one degree three vertex, which is incident with e_1 in G. If G' contains a 2-robust edge-cut of size at most 3, then G contains a 2-robust edge-cut of size at most 5 that contains a boundary edge incident with t, a contradiction. Hence G' is a PT graph. By the minimality of G, G' has a valid orientation. Transfer this orientation to G. Orient the remaining two edges incident with v to satisfy $p(v)$, and the remaining two edges incident with t to satisfy $p(t)$. Since $p(u)$ is satisfied by the orientations of e_1 and e_2, the direction of e_3 is determined to be the opposite (relative to u) of the direction of e_t. Since w cannot be the only vertex whose prescription is not met, this is a valid orientation of G, a contradiction.

   We now assume that e_2 is incident with two degree four vertices. The same must be true of the corresponding edge incident with v. Let these vertices be x and y respectively. Suppose that x and y are not adjacent. Let C_1 and C_2 be the components of $G - \{x, y, w\}$, where the labelling is chosen so that $t \in C_2$. Consider the cut $\delta_G(C_1)$. If C_1 contains a single vertex, then G has parallel edges, a contradiction. Hence $\delta_G(C_1)$ is 2-robust. Note that $\delta_G(C_1)$ is a Type 2 cut and has size at most 6. This cut is shown in Figure 4.17.

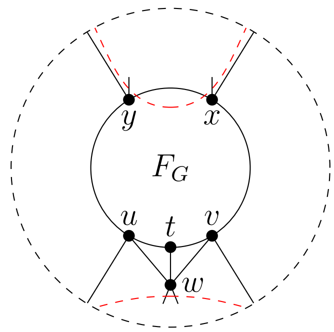


Figure 4.17: PT8: Small Type 2 cut (1).

Contract C_1 to a vertex, calling the resulting graph G'. It is clear that G' is a PT graph. Thus G' has a valid orientation by the minimality of G. Transfer this orientation to G and contract $G - C_1$ calling the resulting graph G''. Note that G'' can be embedded in the plane with a directed vertex d' of degree at most 6. Delete the edges in the cut that are incident with x, calling the resulting graph $\bar{G}$. Then $\bar{G}$ is planar, has at most one vertex of degree 3, and has a directed vertex of degree at most 4. If $\bar{G}$ has a 2-robust at most 3-edge-cut, then G has a 2-robust at most 5-edge-cut, which is of Type 1, or Type 2 with t in the side not in an open disk, a contradiction. Hence $\bar{G}$ is a DTS graph and has a valid orientation by Theorem 4.2.2. This leads to a valid orientation of G, a contradiction.

Hence we may assume that x and y are adjacent. Note that $G - \{t, u, v, w, x, y\}$ has two components: A with neighbours in G among x, y, and w, and B, with neighbours in G among u, v, x, and y. If $\delta_G(A)$ is a 2-robust cut, then it has size at most 4 and is of Type 1, a contradiction. If A contains a single vertex, then G has parallel edges incident with w, a contradiction. Hence x and y are adjacent to w. This graph is shown in Figure 4.18.

Let $A = \{t, u, v, w, x, y\}$. If $\delta_G(A)$ is a 2-robust cut, then it has size 4 and is of Type 3, a contradiction. If $|V(G - A)| = 1$, then this vertex is repeated in the boundary walk of F_G, so the boundary of F_G is not a cycle, a contradiction. Hence $|V(G - A)| = 0$. Lift the pair of edges tv, vw and orient v, x, y, u, t in order (each having at least two unoriented edges). This determines the direction of vw, and since w cannot be the only vertex to not meet its prescription, the result is a valid orientation of G, a contradiction.

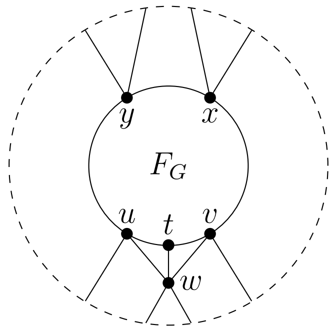


Figure 4.18: PT8: x and y adjacent to w

2. Now suppose that e_3 and the corresponding edge incident with v each have two endpoints of degree 4, x and y respectively. Then x and y are on the boundary of F_G. Then G contains a non-peripheral edge-cut $\delta_G(A)$ of size at most 5 (containing tw, one or two edges incident with x, and one or two edges incident with y). If the labelling is chosen so that $t \in G - A$, then the graph obtained by contracting $G - A$ to a single vertex is planar, so by Claim 4.4.5, G has a valid orientation.

3. Finally, suppose without loss of generality that e_3 has two endpoints of degree 4, and v is adjacent to w. Let x be the other endpoint of e_3. Let f_1 and f_2 be the edges incident with v that are not vt or vw, where f_1 is on the boundary of F_G. *Assume that f_2 does not have two incident vertices of degree* 4. Orient and delete f_1 and f_2 to satisfy $p(v)$, and contract the set of vertices $\{v, t, w\}$ to a single vertex of degree 4, calling the resulting graph G'. If G' contains a 2-robust at most 3-edge-cut, then G contains a 2-robust at most 5-edge-cut containing a boundary edge incident with t, a contradiction. Then G' is a PT graph, so by the minimality of G, G' has a valid orientation. Transfer this orientation to G. Orient the remaining two edges incident with t to satisfy $p(t)$. Since $p(v)$ is satisfied by the orientations of f_1 and f_2, the direction of vw is determined to be the opposite (relative to v) of the direction of vt. Since w cannot be the only vertex whose prescription is not met, this is a valid orientation of G, a contradiction.

   We now assume that f_2 has two incident vertices of degree 4. Then f_2 is a chord. Let y be the other endpoint of f_2. Consider the cut $\delta_G(A)$ where $x, y, t, v \in A$, $w \in G - A$, and $G - A$ is connected and maximised. If $G - A$ contains only one vertex, then G has parallel edges, a contradiction. Hence $\delta_G(A)$ is 2-robust. Note that $\delta_G(A)$ is a

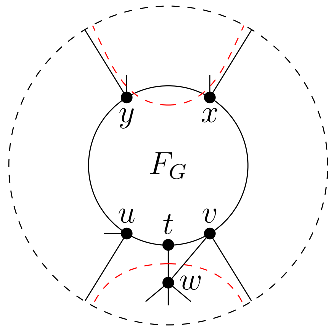


Figure 4.19: PT8: Small Type 2 cut (2).

Type 2 cut and has size at most 6. This cut is shown in Figure 4.19.

Contract $G - A$ to a vertex, calling the resulting graph G'. It is clear that G' is a PT graph. Thus G' has a valid orientation by the minimality of G. Transfer this orientation to G and contract A calling the resulting graph G''. Note that G'' is planar and has a directed vertex d' of degree at most 6. Delete two consecutive edges incident with d' where one is a boundary edge, calling the resulting graph $\bar{G}$. Then $\bar{G}$ is planar and has at most one vertex of degree 3. If $\bar{G}$ has a 2-robust at most 3-edge-cut, then G has a 2-robust at most 5-edge-cut, either of Type 1, or Type 2 where t is on the side not contained in an open disk, a contradiction. Hence $\bar{G}$ is a DTS graph and has a valid orientation by Theorem 4.2.2. This leads to a valid orientation of G, a contradiction. □

We are left with the case where tw is a chord. Let the edges incident with u be e_1, e_2, e_3, e_t in order, where $e_t = ut$, and e_1 is a boundary edge of F_G. Suppose that e_3 is not incident with two degree four vertices, and is not incident with w. Lift the pair e_1, e_2, orient the remaining edges incident with u to satisfy $p(u)$, and orient the remaining edges incident with t to satisfy $p(t)$. Delete u and t, calling the resulting graph G'. Then in G', the only degree three vertex is v. The argument that G' contains no small cuts is equivalent to previous arguments. Hence G' is a PT graph. By the minimality of G, G' has a valid orientation. This extends to a valid orientation of G, a contradiction.

Therefore e_3 is incident with either two degree four vertices or with w. The same must be true of the corresponding edge incident with v.

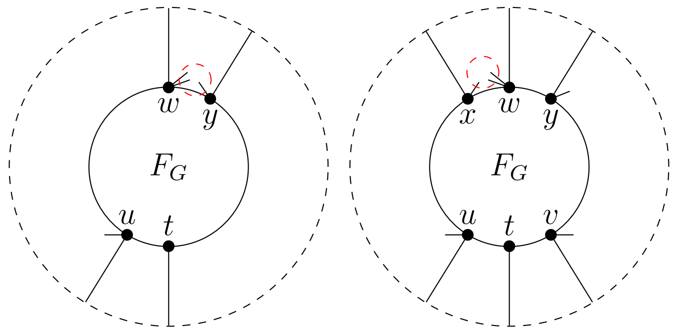


Figure 4.20: Edge tw is a chord: Small Type 2 cuts.

PT9. *Both u and v are adjacent to w.*

Proof. Consider the case where e_3 is incident with two degree four vertices. Then e_3 is a chord. Let y be the other endpoint of e_3. Then y is adjacent to w on the boundary of F_G, else G has a 2-robust at most 5-edge-cut of Type 2, where t is not in the side contained in a disk, a contradiction. This cut is shown in Figure 4.20. If the corresponding edge incident with v is a chord, the same is true. Then G has a 2-robust at most 3-edge-cut, a contradiction. This cut can also be seen in Figure 4.20. Thus we may assume that v and w are adjacent.

Let f_1, f_2, f_w, f_t be the edges incident with v in order, where $f_t = vt$, $f_w = vw$, and f_1 is a boundary edge of F_G. Now f_2 is not incident with two vertices of degree 4, else G has 2-robust at most 4-edge-cut, or parallel edges. Also, f_2 cannot be adjacent to w via parallel edges. Orient and delete f_1 and f_2 to satisfy $p(v)$, and contract the set of vertices $\{t, w, v\}$ to a single vertex of degree 4, calling the resulting graph G'. Now G' has only one possible degree three vertex, in G it is incident with f_1. If G' contains a 2-robust edge-cut of size at most 3, then G contains a 2-robust edge-cut of size at most 5 containing a boundary edge incident with t, a contradiction. Thus G' is a PT graph, and has a valid orientation by the minimality of G. Transfer this orientation to G. Orient the remaining edges incident with t. Since f_1 and f_2 satisfy $p(v)$, f_w is known to have the opposite direction (relative to v) from f_t. Since w cannot be the only vertex whose prescription is not met, this is a valid orientation of G. □

The remaining case is that both u and v are adjacent to w. Let $P_1 = tu_1u_2...u_iw$ be the path on the boundary of F_G from t to w that includes u, and let $P_2 = tv_1v_2...v_jw$ be the path on the boundary of F_G from t to w that includes v. Let S_1 and S_2 be the subsets of vertices in P_1 and P_2 respectively that have an adjacent vertex of degree at least 5 that is not w.

If $S_1 \neq \emptyset$, then let k be such that $u_k \in S_1$ and $\min\{k, i-k+1.5\}$ is minimised. If $S_2 \neq \emptyset$, then let ℓ be such that $v_\ell \in S_2$ and $\min\{\ell, j-\ell+1.5\}$ is minimised. Without loss of generality, suppose that $\min\{k, i-k+1.5\} \leq \min\{\ell, j-\ell+1.5\}$. Let e_1, e_2, e_3, e_4 be the edges incident with u_k in order, where if $k > \frac{i+1}{2}$, $e_1 = u_{k-1}u_k$ and $e_4 = u_k u_{k+1}$ (with the convention that $t = u_0$ and $w = u_{i+1}$ if necessary), and otherwise, $e_4 = u_{k-1}u_k$ and $e_1 = u_k u_{k+1}$. At least one of e_1 and e_2 is incident with a degree 5 vertex, that is not w, by definition.

If, for example, $k = 3$, then $u_1 = u$ is adjacent to w and v_j, v_j is adjacent to u_2, u_2 is adjacent to v_{j-1} and v_{j-1} is adjacent to u_3. Likewise $v_1 = v$ is adjacent to w and u_i, u_i to v_2, v_2 to u_{i-1}, and u_{i-1} to v_3. More generally, the set X defined next, consists of those vertices whose adjacencies are determined in this fashion.

If $k > \frac{i+1}{2}$, let $X = \{u_m : m \geq k \text{ or } m \leq i-k+1\} \cup \{v_m : m \geq j-i+k \text{ or } m \leq i-k+1\}$. Otherwise, let $X = \{u_m : m \leq k \text{ or } m > i-k+1\} \cup \{v_m : m \leq k \text{ or } m > j-k+1\}$. Orient and delete e_1 and e_2 to satisfy $p(u_k)$, and contract the set of vertices $X \cup \{t, w\}$ to a single vertex of degree 4. Call the resulting graph G'. By definition, G' has at most one degree three vertex. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding 2-robust at most 5-edge-cut that does not separate t from w, a contradiction. Hence G' is a PT graph. By the minimality of G, G' has a valid orientation. Transfer this orientation to G.

Suppose that $k > \frac{i+1}{2}$. Orient and delete the two edges incident with the following vertices in order:

$$v_{j-i+k}, u_{i-k+1}, v_{j-i+k+1}, u_{i-k}, ..., v_j, u_1, t, w, v_1, u_i, v_2, u_{i-1}, ..., v_{i-k}, u_{k+1}.$$

There is only one unoriented edge at u_k ($u_k v_{i-k+1}$), which by construction must have the opposite direction (relative to u_k) from $u_k u_{k+1}$. Since v_{i-k+1} cannot be the only vertex whose prescription is not met, this is a valid orientation for G.

Suppose that $k \leq \frac{i+1}{2}$. Orient and delete the two edges incident with the following vertices in order:

$$v_k, u_{i-k+2}, v_{k-1}, u_{i-k+3}, ..., u_i, v_1, t, w, w_1, v_j, u_2, v_{j-1}, ..., v_{j-k+3}, u_{k-1}.$$

There is only one unoriented edge at u_k ($u_k v_{j-k+2}$), which by construction must have the opposite direction (relative to u_k) from $u_k u_{k+1}$. Since v_{j-k+2} cannot be the only vertex whose prescription is not met, this is a valid orientation for G.

Now suppose that $S_1, S_2 = \emptyset$. Then every vertex in G is on the boundary of F_G, and aside from t and w, all vertices have degree 4. Note that P_1 and P_2 have the same length, so $j = i$, and the edges in the graph that are not on the boundary of F_G are

$$\{tw, u_1w, v_1w, u_k v_{i-k+1} \text{ where } 1 \leq k \leq i, u_\ell v_{i-\ell} \text{ where } 2 \leq \ell \leq i\}$$

(See PT5).

Suppose $p(t) \neq 0$. Lift tu and tw, and orient and delete tv to satisfy $p(t)$, calling the resulting graph G'. Then v is the only possible degree three vertex in G', and it is clear that G' is PT graph, and thus has a valid orientation by the minimality of G. This yields a valid orientation of G. Hence we may assume that $p(t) = 0$.

Suppose $p(u_1) = 0$. Lift two pairs of edges at u_1. The resulting graph is a PT graph, and thus has a valid orientation by the minimality of G. This yields a valid orientation of G. Hence we may assume that $p(u_1) \neq 0$.

Without loss of generality, suppose $p(u_1) = 1$. Orient all three edges incident with t into t. Orient u_1w from u_1 to w. Orient the remaining three edges incident with w to satisfy $p(w)$. Orient the remaining two edges incident with the following vertices in order:

$$v_1, u_i, v_2, u_{i-1}, \ldots, v_{i-1}, u_2.$$

There is only one unoriented edge at u_1 ($u_1 v_i$), which by construction must have the opposite direction to $u_1 u_2$. Since v_i cannot be the only vertex whose prescription is not met, this is a valid orientation for G. $\square$

4.5 Discussion

In this section we relate the theorems of this chapter to the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture, and consider possible extensions of these results. The following result (Jaeger's Strong 3-Flow Conjecture for projective planar graphs) is a direct corollary of Theorem 4.4.2.

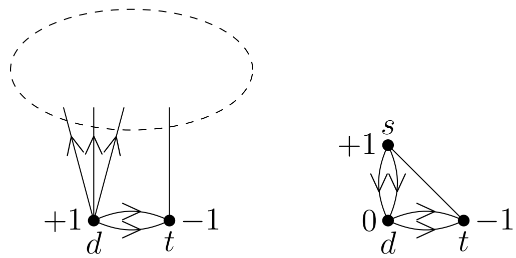


Figure 4.21: A graph with a directed vertex of degree 5 (4), one (two) degree 3 vertex (vertices), and no valid orientation.

Theorem 4.5.1. *Let G be a 5-edge-connected graph embedded in the projective plane. Then G is $\mathbb{Z}_3$-connected.*

The case where the prescription function p is such that $p(v) = 0$ for all $v \in V(G)$, is the 3-Flow Conjecture for projective planar graphs, shown by Steinberg and Younger [25]. We note that the 3-Flow Conjecture and Jaeger's Strong 3-Flow Conjecture for planar graphs follow from Theorem 4.2.2.

Planar Graphs

Theorem 3.3.3 allows a directed vertex of degree 5, or a directed vertex of degree 4 and a degree 3 vertex, but not both a directed vertex of degree 5 and a degree 3 vertex. The reason for this is that any graph with a degree 5 directed vertex d adjacent via parallel edges to a vertex t of degree 3 does not have a valid orientation for some prescription functions. For example, if the edges incident with d and t are directed into t, and $p(t) = -1$, then G has no valid orientation. Such a graph can be seen in Figure 4.21.

In Theorem 4.2.2 we allow a directed vertex of degree 3 with two other degree 3 vertices. Again, a directed vertex of degree 4 with two degree 3 vertices is not possible. If d is a degree 4 directed vertex adjacent via parallel edges to a vertex t of degree 3, then there may not be an orientation of t that extends the existing orientation of d and meets $p(t)$. Now $\delta(\{d, t\})$ is a 3-edge-cut, but since the graph has a second degree 3 vertex, such a 3-edge-cut need not be 2-robust. This graph can also be seen in Figure 4.21.

When increasing the number of unoriented degree 3 vertices to three, we know of no graph or family of graphs that would rule out a directed vertex of degree 3. While the example

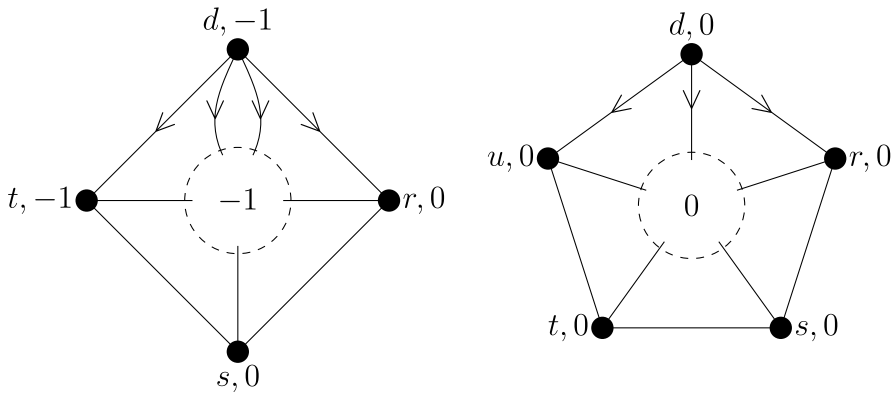


Figure 4.22: A graph with a directed vertex of degree 4 (3), three (four) degree 3 vertices, and no valid orientation.

in Figure 4.21 does not extend to this case, we note that there is an infinite family of graphs with three degree 3 vertices and an oriented degree 4 vertex that do not have a valid prescription. Let G be a graph where the boundary of the outer face consists of a directed degree 4 vertex d, and three degree 3 vertices r, s, and t. Let $p(d) = p(t) = -1$, $p(r) = p(s) = 0$, and assume that all edges incident with d are directed out from d. Let $A = G - \{d, r, s, t\}$. Then $\delta(A)$ is an internal 5-edge-cut. We assume that $p(A) = -1$. Then $p(G) = 0$, so p is a valid prescription function. Since rd is directed into r, all edges incident with r must be directed into r. Since rs is directed out of s, all edges incident with s must be directed out of s. Then st and dt are directed into t. No direction of the remaining edge incident with t meets $p(t)$, so G does not have a valid orientation that meets p. This family of graphs can be seen in Figure 4.22.

If we allow a directed vertex d of degree 3 and four vertices of degree 3, then we obtain a similar family of graphs. Let the boundary of the outer face of G consist of d and the four vertices of degree 3, producing an internal 5-edge-cut. Assume that all vertices have prescription zero. Each of the five vertices on the boundary of the outer face has either all edges pointing into the vertex, or all edges pointing out. It is clear that with an odd length boundary, this is not possible. Hence G does not have a 3-flow (as opposed to simply a modulo 3 orientation meeting p). This family of graphs can also be seen in Figure 4.22.

Similarly, we consider extending Theorem 4.2.7 to allow further degree 3 vertices. It is clear that a family of graphs with five vertices of degree 3 vertices similar to that with four and a directed degree 3 vertex can be constructed, that do not have nowhere zero 3-flows. We conjecture that such graphs with four degree 3 vertices or three degree 3 vertices and a directed degree 3 vertex have a valid orientation to meet a given valid prescription function. If true, then this would be the best possible such result.

Conjecture 4.5.2. *Let G be a graph embedded in the plane, together with a valid $\mathbb{Z}_3$-prescription function $p : V(G) \to \{-1, 0, 1\}$ such that:*

1. *G is 3-edge-connected,*
2. *G has a specified face F_G, and at most four specified vertices d, r, s, and t,*
3. *if d exists, then it has degree 3, is in the boundary of F_G, and may be oriented,*
4. *if r, s, or t exists, then it has degree 3 and is in the boundary of F_G,*
5. *G has at most four 3-edge-cuts, which can only be $\delta(\{d\})$, $\delta(\{r\})$, $\delta(\{s\})$, and $\delta(\{t\})$, and*
6. *every vertex not in the boundary of F_G has 5 edge-disjoint paths to the boundary of F_G.*

Then G has a valid orientation.

In Chapter 5 we consider a relaxation of the requirement that every vertex not in the boundary of F_G has 5 edge-disjoint paths to the boundary of F_G.

Theorem 4.3.2 requires d, if it exists, to be in the boundaries of both F_G and F_G^*. In the following conjecture we remove this hypothesis, and the requirement that the two specified faces have a vertex in common.

Conjecture 4.5.3. *Let G be a graph embedded in the plane, together with a valid $\mathbb{Z}_3$-prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is 3-edge-connected,*
2. *G has two specified faces F_G and F_G^*, and at most one specified vertex d or t,*
3. *if d exists, then it has degree 3, 4, or 5, is oriented, and is in the boundary of F_G or F_G^*,*

4. *if t exists, then it has degree 3 and is in the boundary of F_G or F_G^*,*

5. *G has at most one 3-edge-cut, which can only be $\delta(\{d\})$ or $\delta(\{t\})$, and*

6. *every vertex not in the boundary of F_G or F_G^* has 5 edge-disjoint paths to the union of the boundaries of F_G and F_G^*.*

Then G has a valid orientation.

This statement does not hold if both d and t are allowed. Consider a graph G with two specified faces F_G and F_G^* such that every vertex is on the boundary of F_G or F_G^*. Let t be a degree 3 vertex on the boundary of F_G^*, and w a degree 5 vertex adjacent to t on the boundary of F_G. Let G have a cycle $P = wv_0v_1...v_nw$ where $n = 6k$ for some $k \in \mathbb{Z}^+$, v_i has degree 4 for all $1 \leq i \leq n$, v_1, v_2, ..., v_n alternate between the boundaries of F_G and F_G^*, and $V(G) = \{t, w, v_1, v_2, ..., v_n\}$. Let $d = v_{\frac{n}{2}}$. We set $w = v_{-1} = v_{n+1}$ and $t = v_{-2} = v_{n+2}$. See Figure 4.23.

Suppose that $p(d) = -1$ and all edges incident with d are directed out from d. Let $p(v_{\frac{n}{2}-1}) = p(v_{\frac{n}{2}-2}) = p(v_{\frac{n}{2}+1}) = p(v_{\frac{n}{2}+2}) = +1$. Let $p(v_j) = +1$ for $j < \frac{n}{2} - 2$ and $p(v_j) = -1$ for $j > \frac{n}{2} + 2$. Let $p(t) = p(w) = 0$. Then it is clear that p is a valid prescription function for G.

Lemma 4.5.4. *The graph G does not have a valid orientation that meets p.*

Proof. We first show that for all $0 \leq j \leq k$, $v_{3(k-j)}v_{3(k-j)-1}$ and $v_{3(k-j)}v_{3(k-j)-2}$ point out of $v_{3(k-j)}$. We proceed by induction on j. The base case is determined by the directions of the edges incident with d. Suppose that for some j where $0 \leq j \leq k-1$, $v_{3(k-j)}v_{3(k-j)-1}$ and $v_{3(k-j)}v_{3(k-j)-2}$ point out of $v_{3(k-j)}$. We consider $v_{3(k-j-1)}$. Since $v_{3(k-j)}v_{3(k-j)-1}$ and $v_{3(k-j)}v_{3(k-j)-2}$ point out from $v_{3(k-j)}$, the remaining edges at each of $v_{3(k-j)-1}$ and $v_{3(k-j)-2}$ point all in or all out. Since these vertices are adjacent, $v_{3(k-j)-3}v_{3(k-j)-1}$ and $v_{3(k-j)-3}v_{3(k-j)-2}$ point one in and one out of $v_{3(k-j)-3}$. Hence the remaining two edges at $v_{3(k-j)-3}$: $v_{3(k-j)-4}v_{3(k-j)-4}$ and $v_{3(k-j)-3}v_{3(k-j)-5}$, must satisfy the prescription of $v_{3(k-j)-3}$ and thus point out of $v_{3(k-j)-3} = v_{3(k-j-1)}$ as required.

Similarly for all $0 \leq j \leq k$, $v_{3(k+j)}v_{3(k+j)+1}$ and $v_{3(k+j)}v_{3(k+j)+2}$ point into $v_{3(k+j)}$. Then $v_{6k}t$ points out of t and v_0t points into t. Hence no direction of tw meets $p(t)$. Thus G has no valid orientation that meets p. $\square$

Such a result may be possible if d is restricted to degree 3.

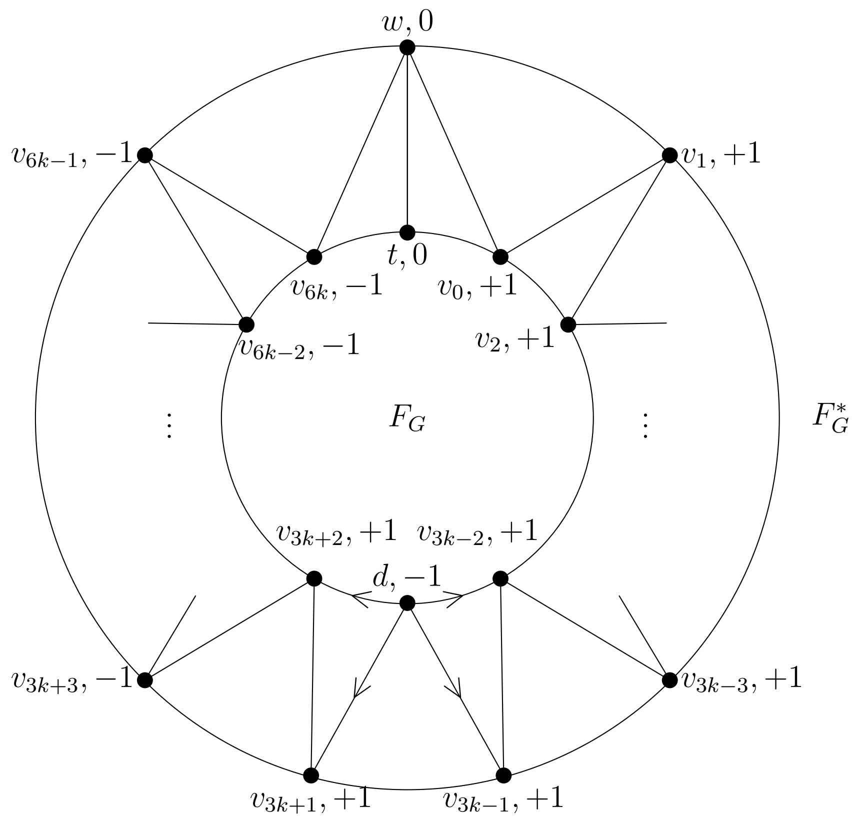


Figure 4.23: A graph with two special faces, a directed vertex of degree 4, and no valid orientation.

Projective Planar Graphs

Unlike in the plane, in the projective plane (Theorem 4.4.2) we do not allow a directed vertex. In the plane this is necessary, in order to reduce small edge-cuts. In the projective plane we utilise the property that one of the two graphs resulting from contracting the sides of an edge-cut is a planar graph, and thus are able to apply planar results.

In fact, adding a directed vertex d of degree 4 to the result is not possible. Consider a graph G with specified face F_G such that every vertex is on the boundary of F_G. Let t be a degree 3 vertex on the boundary of F_G, and w a degree 5 vertex on the boundary of F_G, adjacent to t via a non-contractible chord. Let $k \in \mathbb{Z}^{\geq 0}$. Let the two paths between t and

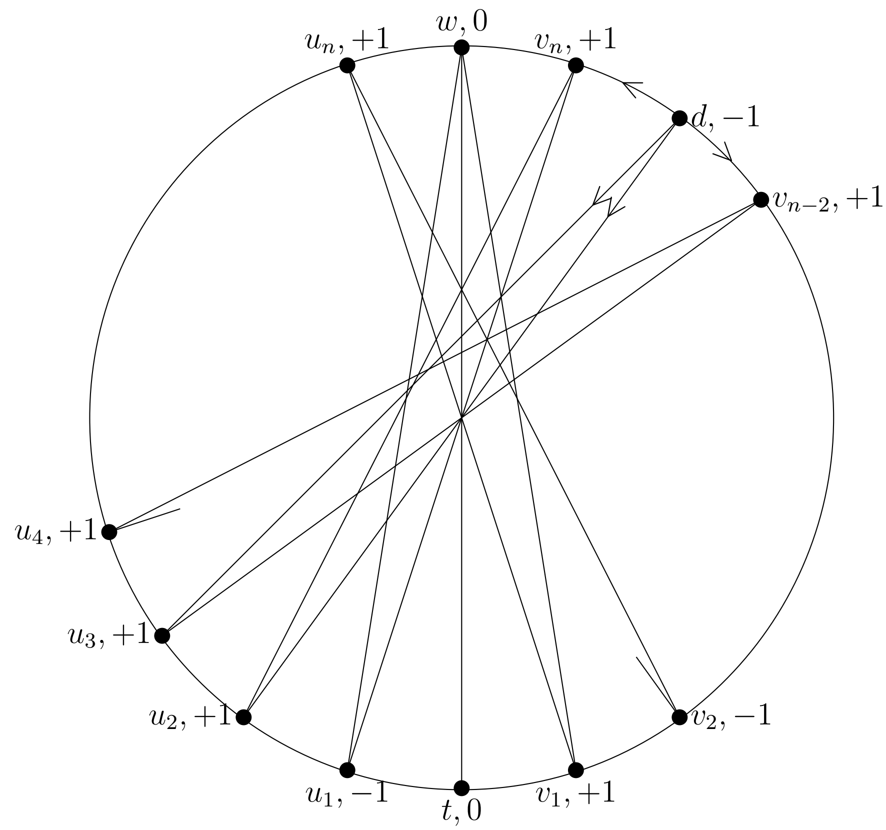


Figure 4.24: A projective planar graph drawn in the plane with a directed vertex of degree 4 and no valid orientation.

w on the boundary of F_G be $P_1 = tu_1u_2...u_nw$ and $P_2 = tv_1v_2...v_nw$ where $n = 3k+5$ and all vertices in G aside from t and w have degree 4. We assume that for all $1 \leq i \leq n$, u_i is adjacent to u_{i-1}, u_{i+1}, v_{n-i+1}, and v_{n-i+2}, where we set $u_0 = v_0 = t$ and $u_{n+1} = v_{n+1} = w$. Finally, w and v_1 are adjacent. We let d be v_{n-1}. See Figure 4.24.

Suppose that $p(d) = -1$ and all edges incident with d are directed out from d. Let $p(v_n) = p(v_{n-2}) = p(u_2) = p(u_3) = +1$. Let $p(u_1) = -1$. Let $p(t) = p(w) = 0$. Finally, let all other vertices have prescription $+1$. We have

$$p(G) = 4(+1) + 2(-1) + 2(0) + (2n+2-8)(+1) = 2n-4 = 6k+6 \equiv 0 \mod 3.$$

Then it is clear that p is a valid prescription function for G.

Lemma 4.5.5. *The graph G does not have a valid orientation that meets p.*

Proof. The edges incident with d meet the prescription of v_n, v_{n-2}, u_2, and u_3, hence each of these vertices has all remaining edges directed in or all remaining edges directed out. Since v_n and u_2 are adjacent, one has all edges directed in and the other has all edges directed out. Consider u_1. The edges u_1u_2 and u_1v_n are directed one in and one out of u_1. Thus the remaining edges incident with u_1: u_1t and u_1w, are directed into u_1 to satisfy $p(u_1)$.

We show that for all $1 \leq j \leq \frac{n-2}{3}$, $u_{3j+1}u_{3j+2}$ and $u_{3j+1}u_{n-3j}$ are directed out of u_{3j+1}, and that for all $0 \leq i \leq \frac{n-2}{3}$, $v_{n-(3i+1)}v_{n-(3i+1)-1}$ and $v_{n-(3i+1)}u_{3i+3}$ are directed out of $v_{n-(3i+1)}$. We consider the sequence $d, u_4, v_{n-4}, u_7, v_{n-7}, ..., u_{n-1}, v_1$. The property is known to hold for d. Let x be the first vertex in this sequence for which this property does not hold.

Suppose that $x = u_{3j+1}$ for some $1 \leq j \leq \frac{n-2}{3}$. Then by definition, the property holds for $v_{n-(3j-2)}$. Hence $v_{n-(3j-2)}v_{n-(3j-2)-1}$ and $v_{n-(3j-2)}u_{3j}$ are directed out of $v_{n-(3j-2)}$. This satisfies the prescription of $v_{n-(3j-2)-1}$ and u_{3j}, which are adjacent, and so the remaining three edges at one of these vertices are directed in, and the remaining three edges at the other are directed out. Both are adjacent to u_{3j+1}, so the edges $u_{3j+1}v_{n-(3j-2)-1}$ and $u_{3j+1}u_{3j}$ are directed one in and one out of u_{3j+1}. Hence the remaining two edges incident with u_{3j+1} are directed out of u_{3j+1}, as required.

Now suppose that $x = v_{n-(3j+1)}$ for some $1 \leq j \leq \frac{n-2}{3}$. Then by definition, the property holds for u_{3j+1}. Hence $u_{3j+1}u_{3j+2}$ and $u_{3j+1}v_{n-3j}$ are directed out of u_{3j+1}. This satisfies the prescription of u_{3j+2} and v_{n-3j}, which are adjacent, and so the remaining three edges at one of these vertices are directed in, and the remaining three edges at the other are directed out. Both are adjacent to $v_{n-(3j+1)}$, so the edges $v_{n-(3j+1)}u_{3j+2}$ and $v_{n-(3j+1)}v_{n-3j}$ are directed one in and one out of $v_{n-(3j+1)}$. Hence the remaining two edges incident with $v_{n-(3j+1)}$ are directed out of $v_{n-(3j+1)}$, as required.

Therefore, u_1t is directed out of t, and v_1t is directed into t. Hence no direction of tw meets $p(t)$. Thus G has no valid orientation that meets p. □

As in the case of a planar graph with two specified faces, such a result may be possible if the degree of the directed vertex d is restricted to 3.

Toroidal Graphs

Jaeger's Strong 3-Flow Conjecture and the 3-Flow Conjecture remain open for surfaces with genus larger than that of the projective plane. The natural question to ask is whether this can be extended to the torus.

In the projective plane we were able to make use of two properties of graphs embedded on the surface. The first is that of the graphs G' and G'' obtained from a graph G with edge-cut $\delta(A)$ by contracting A and $G - A$ respectively, at least one is planar. This not true of edge-cuts in graphs embedded on the torus. Therefore, in order to reduce small edge-cuts as we have throughout the results in this chapter, we require the toroidal graph to have a directed vertex in addition to a degree 3 vertex.

The second property is that of Lemma 4.1.3; that the graph G' obtained from G by deleting a non-contractible chord (and its endpoints) is a planar graph with one specified face. We were able to reduce graphs with non-contractible chords to planar graphs meeting the conditions of Theorem 4.2.7. In the torus, such a reduction produces a planar graph with two specified faces. Since this may produce up to two additional degree 3 vertices, we would require a result for planar graphs, allowing two specified faces with three degree 3 vertices and a directed vertex. Lemma 4.23 shows that this is false.

Thus the techniques used to prove Theorem 4.4.2 are not extendable to toroidal graphs.

We further note that the techniques used by Steinberg and Younger [25] to prove the 3-Flow Conjecture for the projective plane do not extend to the torus. As discussed in Section 3.3, Steinberg and Younger [25] make use of the existance of a Grötzsch configuration in the plane and projective plane.

Such techniques do not extend to Jaeger's Strong 3-Flow Conjecture, as they make use of the assumption that all vertices have degree at most 5; with every vertex having prescription zero it is possible to make several lifts at a vertex. When a degree 6 vertex does not have prescription zero, such a lift cannot be made without creating a degree 4 vertex with non-zero prescription, which is not allowed. Thus we may only guarantee the existence of a similar configuration where vertices may have degree 6 and non-zero prescription.

In the torus, the discharging argument used to prove the existence of the Grötzsch configuration does not apply. To see this, consider a 5-regular simple graph G embedded in the plane, projective plane, or torus.

Lemma 4.5.6. *Either G contains a Grötzsch configuration or G is toroidal and every vertex is incident with three faces of length three and two of length four.*

Proof. Assign

$$ch_i(v) = deg(v) - 6 \text{ for all } v \in V(G),$$
$$ch_i(f) = 2(|f| - 3) \text{ for all } f \in F(G).$$

Let f be a face of G. If f has length at least four, send charge $\frac{1}{2}$ from f to each incident vertex. We call the final charges ch_f.

Now

$$\begin{aligned}\sum_{v\in V(G)} ch_f(v) + \sum_{f\in F(G)} ch_f(f) &= \sum_{v\in V(G)} ch_i(v) + \sum_{f\in F(G)} ch_i(f)\\ &= \sum_{v\in V(G)} (deg(v) - 6) + 2\sum_{f\in F(G)} (|f| - 3)\\ &= 6|E(G)| - 6|V(G)| - 6|F(G)|.\end{aligned}$$

Suppose that G does not have a Grötzsch configuration. We consider the sum of the final charges. Let f be a face of G. If $|f| = 3$ then $ch_f(f) = ch_i(f) = 0$. Suppose $|f| > 3$. Then f sends charge $\frac{1}{2}$ to each of its $|f|$ incident vertices. We have

$$ch_f(f) = ch_i(f) - \frac{|f|}{2} = \frac{3|f|}{2} - 6 \geq 0.$$

Now let $v \in V(G)$. Then $ch_i(v) = deg(v) - 6 = -1$. Since G does not have a Grötzsch configuration, v has at least two incident faces that have length greater than three. Hence

$$ch_f(v) \geq ch_i(v) + 1 = 0.$$

We conclude that

$$6|E(G)| - 6|V(G)| - 6|F(G)| = \sum_{v\in V(G)} ch_f(v) + \sum_{f\in F(G)} ch_f(f) \geq 0.$$

Since we required G to be planar, projective planar, or toroidal, this is possible only if G is a toroidal graph. Hence

$$\sum_{v \in V(G)} ch_f(v) + \sum_{f \in F(G)} ch_f(f) = 6|E(G)| - 6|V(G)| - 6|F(G)| = 0.$$

Since every term on the left hand side is non-negative, it is necessary that $ch_f(v) = 0$ for all $v \in V(G)$ and $ch_f(f) = 0$ for all $f \in F(G)$.

Consider a face $f \in F(G)$. If $|f| \geq 5$, then

$$ch_f(f) = ch_i(f) - \frac{|f|}{2} = \frac{3|f|}{2} - 6 > 0.$$

Hence we may assume all faces in G have length 3 or 4. Let $v \in V(G)$ and suppose that v is not incident with three faces of length 4 and two of length 4. Since a Grötzsch configuration does not exist, v is incident with at least three faces of length 4. Then

$$ch_f(v) \geq ch_i(v) + \frac{3}{2} = \frac{1}{2} > 0,$$

a contradiction. Thus all faces in G have length 3 or 4 and every vertex is incident with three faces of length 3 and two of length 4. □

While Steinberg and Younger [25] reduce the Grötzsch configuration, it is not clear that a similar reduction exists when a vertex is incident with three faces of length 3 and two of length 4. The inclusion of a vertex of degree 3 or a directed vertex naturally complicates this argument further.

Chapter 5

Degree 4 Vertices

In Section 3.4 we discussed the family of graphs constructed by Lai [18] showing that the edge-connectivity condition in Jaeger's Strong 3-Flow Conjecture cannot be relaxed to include all 4-edge-connected graphs. This raised the question of which 4-edge-connected graphs do have a nowhere-zero 3-flow for all valid prescription functions. In this chapter we modify Theorem 3.3.3 to allow graphs with arbitrarily many degree 4 vertices, if they do not appear on adjacent faces. Recall that two faces are adjacent if they have a common boundary edge. In this way we relax the requirement that all internal vertices must have 5 edge-disjoint paths to the boundary of the outer face. We define an *internal* edge-cut to be one that does not intersect the boundary of the outer face.

Definition 5.0.1. *An* FDT Graph *is a graph G embedded in the plane, together with a valid prescription function $p : V(G) \to \{-1, 0, 1\}$, such that:*

1. *G is 3-edge-connected,*
2. *G has at most two specified vertices d and t,*
3. *if d exists, then it has degree 3, 4, or 5, is oriented, and is in the boundary of the outer face,*
4. *if t exists, then it has degree 3 and is in the boundary of the outer face,*
5. *if d and t both exist, then d has degree at most 4,*
6. *G has at most two 3-edge-cuts, which can only be $\delta(\{d\})$, and $\delta(\{t\})$,*

7. *internal degree 4 vertices in G are not on the boundary of adjacent faces,*

8. *if $\delta(A)$ is a 2-robust internal cut in G, then $|\delta(A)| \geq 5$.*

A 3FDT graph *is a graph G with the above definition, where (6) is replaced by*

6'. *all vertices aside from d and t have degree at least 4, and if d and t both exist, then every 3-edge-cut in G separates d from t.*

The main result for this chapter is Theorem 5.0.2. The proof appears in Section 5.2. In Section 5.1 we discuss the ideas that will be used in the proof that differ from those used in Chapter 4.

Theorem 5.0.2. *All FDT graphs have a valid orientation.*

5.1 Preliminaries

Outer Face

We first note that in this chapter, since we only consider graphs embedded in the plane, we may simply choose the outer face to be the specified face. The discussion in Section 4.1 of how the specified face of a new graph is found after a reduction applies to yield the outer face in all cases.

Not on the Boundary of Adjacent Faces

In this chapter we drop the restriction that internal vertices have at least 5 edge-disjoint paths to the boundary of the outer face. This is replaced by the following conditions:

7. Internal degree 4 vertices in G are not on the boundary of adjacent faces.

8. If $\delta(A)$ is a 2-robust internal cut in G, then $|\delta(A)| \geq 5$.

It is clear that (8) is simply a modification to allow degree 4 vertices, but no other internal 4-edge-cuts. It ensures that any set X of internal vertices where $|X| \geq 2$ has at least 5 edge-disjoint paths to the boundary of the outer face.

Now (7) is the condition that restricts the degree 4 vertices that may exist. We first consider here what this restriction says about the proximity of internal degree 4 vertices.

Lemma 5.1.1. *Let v be an internal vertex of degree k in an FDT graph G. Let $f_G(v)$ denote the number of internal degree 4 vertices adjacent to v. If v has degree 4, then $f(v) = 0$. Otherwise $f(v) \leq \left\lfloor \frac{deg(v)}{3} \right\rfloor$.*

Proof. Adjacent vertices have a face in common, and thus it is clear that G does not have adjacent internal degree 4 vertices. It follows that if $deg(v) = 4$, then $f(v) = 0$. Suppose that $deg(v) = k \geq 5$, and let v_1, v_2, ..., v_k be the vertices adjacent to v in cyclic order. Let F_i be the face of G that has v_i, v, and v_{i+1} on its boundary (where we let $v_{k+1} = v_1$). Then for all $1 \leq i \leq k$, v_i is on the boundaries of F_i and F_{i-1}. Hence v_i is on the same face as v_{i-1} and v_{i+1} and on an adjacent face to v_{i-2} and v_{i+2}. The result follows. □

This implies that internal degree 4 vertices are not adjacent, and may only be distance two apart if the vertex on the path between them has degree at least 6. The key fact that will be used throughout this chapter is that a degree 5 vertex may only have one adjacent internal degree 4 vertex.

The reductions in the proof of Theorem 5.0.2 are more complex than those in Chapter 4 and involve deleting more vertices. It therefore becomes necessary to consider whether a deleted vertex may be adjacent to multiple degree 4 vertices. Since many of the internal vertices we are concerned with are degree at most 5, by Lemma 5.1.1 the only way this can occur is if some are on the boundary of the outer face. Suppose that G has an internal vertex v of degree d_1 adjacent to non-adjacent boundary vertices x and y of degree d_2 and d_3 respectively. Define A_v and B_v to be the vertex sets of the components of $G - \{v, x, y\}$. Then

$$|\delta(A_v)| + |\delta(B_v)| \leq (d_1 - 2) + (d_2 - 1) + (d_3 - 1) = d_1 + d_2 + d_3 - 4.$$

Once we consider the reduction of small cuts, such configurations with small d_1, d_2, and d_3 will not exist.

We now consider how reductions to the graph affect the condition on the proximity of internal degree 4 vertices.

Lemma 5.1.2. *Let G be a graph where internal degree 4 vertices are not on adjacent faces. Let G' be a graph obtained from G by*

1. *deleting a boundary edge e of the outer face,*
2. *deleting a boundary vertex x of the outer face,*

3. *lifting a pair of adjacent edges e_1, e_2, where e_1 is in the boundary of the outer face, or*
4. *contracting a subgraph X where $|\delta(X)| \geq 5$ to a vertex x.*

Then internal degree 4 vertices in G' are not on adjacent faces.

Proof. Suppose for a contradiction that G' has internal degree 4 vertices u and v that are on adjacent faces F_1 and F_2 respectively. Note that it is possible that $F_1 = F_2$. Evidently neither F_1 nor F_2 is the outer face of G'. Let F_G be the outer face of G and $F_{G'}$ be the outer face of G'.

1. In G, u and v are internal degree 4 vertices. Let F' be the face incident with e in G that is not F_G. We have

$$F(G') = (F(G) - \{F_G, F'\}) \cup \{F_{G'}\}.$$

   Thus F_1 and F_2 are faces in G that are not F_G, and are adjacent. Hence u and v are internal degree 4 vertices of G on adjacent faces, a contradiction.

2. This case follows by sequentially deleting boundary edges incident with v until v is an isolated vertex. It is clear that the deletion of an isolated vertex in F_G does not affect the existence or proximity of internal degree 4 vertices.

3. As in (1), it is clear that F_1 and F_2 are faces in G that are not F_G, and u and v are internal degree 4 vertices in G. Thus u and v are internal degree 4 vertices of G on adjacent faces, a contradiction.

4. Let $\mathcal{F}$ be the set of internal faces of $G[X]$. Let $F \in F(G) - \mathcal{F}$. If the boundary of F does not intersect $G[X]$, let $F' = F$. Otherwise, let F' be the face whose boundary is obtained from the face boundary of F by contracting all paths in $G[X]$. It is clear that this operation preserves adjacency of faces. Then

$$F(G') = \{F' : F \in F(G) - \mathcal{F}\}.$$

   Let F_1' and F_2' be the adjacent faces containing u and v. Since $|\delta(X)| \geq 5$, $x \notin \{u, v\}$. Thus u and v are internal vertices of degree 4 in G, on the boundaries of F_1 and F_2, which are adjacent, a contradiction. $\square$

We therefore only discuss the preservation of this property in cases where Lemma 5.1.2 does not apply.

5.2 Allowing Degree 4 Vertices

Here we prove Theorem 5.0.2.

Proof. Let G be a minimal counterexample with respect to the lexicographic ordering of the pair $(|E(G)|, |E(G)| - deg(d))$. We will prove the following series of properties of G.

FDT1: The graph G has no cut vertex.

FDT2: The graph G has no loops or unoriented parallel edges.

FDT3: The graph G has no 2-robust 4-edge-cut.

FDT4: Every 2-robust 5-edge-cut in G separates d from t.

FDT5: The graph G has no chord of the outer face incident with a vertex of degree 3 or 4.

FDT6: Vertices d and t exist.

FDT7: Vertices d and t are not adjacent.

FDT8: Every 4-robust 6-edge-cut in G separates d from t.

Let u and v be the boundary vertices adjacent to t, and let w be the remaining vertex adjacent to t.

FDT9: Vertices u and v have degree 4.

FDT10: Vertex w has degree 5.

Let e_1, e_2, e_3, e_t be the edges incident with u in order, where $e_t = ut$ and e_1 is on the boundary of the outer face. Let f_1, f_2, f_3, f_t be the edges incident with v in order, where $f_t = vt$ and f_1 is on the boundary of the outer face.

FDT11: One of the following holds:

1. the edge e_3 is incident with an internal degree 4 vertex, and v and w are adjacent, or
2. the edge f_3 is incident with an internal degree 4 vertex, and u and w are adjacent.

Without loss of generality, we assume that the edge e_3 is incident with an internal degree 4 vertex, which we call x, and v and w are adjacent.

FDT12: Vertices x and w are adjacent.

FDT13: Vertices u and d are adjacent.

The proofs of these properties form the bulk of the proof of Theorem 4.2.2.

Unlike in Chapter 4, we may not apply Lemma 4.1.4, as loop deletion may affect the property that internal degree 4 vertices are not on the boundary of adjacent faces.

FDT1. *The graph G has no cut vertex.*

Proof. Suppose that G contains a cut vertex v. Let H and K be subgraphs of G such that $H \cap K = \{\{v\}, \emptyset\}$ and $H \cup K = G$. Since G is 3-edge-connected, $deg_H(v), deg_K(v) \geq 3$. Suppose that $deg_H(v) \geq 5$. Then H is an FDT graph, and has a valid orientation by the minimality of G. Suppose that $deg_H(v) = 4$. Then $\delta_G(K)$ is a 4-edge-cut. If $\delta_G(K)$ is not an internal cut, then H is an FDT graph, and has a valid orientation by the minimality of G. If $\delta_G(K)$ is an internal cut, then $|V(H)| = 2$. If $d \in V(H)$, then H has a valid orientation. Otherwise, orient v to give a valid orientation of H.

Suppose that $deg_H(v) = 3$. Since G has no 3-edge-cuts except for $\delta(d)$ and $\delta(t)$, $V(H) = \{v, w\}$ where $w \in \{d, t\}$. If $w = d$, then H is an oriented graph. Suppose that $w = t$. Orient t. Thus H has a valid orientation. Transfer this orientation to G and adjust the prescription of v in K. An equivalent argument shows that K has a valid orientation. This yields a valid orientation of G, a contradiction. Hence G has no cut vertices. □

FDT2. *The graph G has no loops or unoriented parallel edges.*

Proof. First, suppose that G contains a loop e incident with a vertex v. Since v is not a cut vertex, the deletion of e does not affect the property that internal degree 4 vertices are not on adjacent faces. Orient (if necessary) and delete e, calling the resulting graph G'. Suppose that G' contains an edge-cut $\delta_{G'}(A)$. Then $\delta_G(A)$ is an edge-cut of the same size. Hence G' is 3-edge-connected. It also follows that $\delta(d)$ and $\delta(t)$ are the only possible 3-edge-cuts in G'. If $v = d$, then since $\delta_G(d) \geq 3$, $deg_{G'}(d) \geq 3$. If $v = t$, then $\delta_G(v) = 1$, a contradiction. Otherwise, since $\delta_G(v) \geq 4$, $deg_{G'}(v) \geq 4$. We conclude that G' is an FDT graph. Since G is a minimal counterexample, G' has a valid orientation, which extends to a valid orientation of G, a contradiction. Hence G does not contain any loops.

Now suppose that G contains unoriented parallel edges. Suppose that vertices $u, v \in V(G)$ are incident with a 2-cycle consisting of unoriented edges. Assume that the resulting cycle of length 2 does not separate vertices of G. Let G' be the graph obtained from G by contracting $G[\{u, v\}]$, and let w be the resulting vertex. Since the edges are unoriented, neither u nor v is d. Suppose that $\deg_{G'}(w) \leq 3$. Then $\delta_G(\{u, v\}) \leq 3$, a contradiction unless $|V(G)| = 3$ (in this case, orient the degree 3 vertex, followed by u; v cannot be the only vertex whose prescription is not met). Thus the only possible vertices of degree three in G' are d and t. If $deg_{G'}(w) = 4$, then w is on the boundary of the outer face, else $\delta_G(\{u, v\})$ is an internal 4-edge-cut, a contradiction. Suppose that G' contains an edge-cut $\delta_{G'}(A)$. Then either $\delta_G(A)$ or $\delta_G(G' - A)$ is an edge-cut of the same size. Hence G' is 3-edge-connected and has no internal 4-edge-cuts. If there exist internal degree 4 vertices on adjacent face boundaries in G, then since no edge was deleted, this is also true in G, a contradiction. Therefore, G' is an FDT graph. Since G is a minimal counterexample, G' has a valid orientation. Transfer this orientation to G, and orient u. Since v cannot be the only vertex whose orientation is not met, this yields a valid orientation of G, a contradiction.

Now assume that the cycle formed by u, v, and the parallel edges separates vertices of G. Let A be the set of vertices in a connected component of $G - \{u, v\}$ that does not contain a vertex in the boundary of F_G. Let e_1 and e_2 be edges with endpoints u and v, such that the cycle $C = (\{u, v\}, \{e_1, e_2\})$ separates A from $G - (A \cup C)$. Define G_A to be the subgraph of G consisting of C and its interior. Define G_{G-A} to be the subgraph of G consisting of C and its exterior. Let G' be the graph obtained from G by contracting G_{G-A}, and let w be the resulting vertex. As above, G' is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G. Let G'' be the graph obtained from G by contracting G_A, and let w' be the resulting vertex (note that all directed edges have been contracted, so d is the only possible directed vertex). Similarly, G'' is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G. Orient u. Since v cannot be the only vertex whose orientation is not met, this yields a valid orientation of G, a contradiction. □

As in Chapter 4 we now consider some of the reducible small edge-cuts in G.

FDT3. *The graph G has no 2-robust 4-edge-cut.*

Proof. Suppose that G does contain a 2-robust 4-edge-cut, $\delta(A)$. Then $\delta(A)$ is not an internal cut by definition. Assume that $d \notin G - A$. Let G' be the graph obtained from G

by contracting $G - A$ to a single vertex. The resulting vertex v has degree 4 and is on the boundary of the outer face. If G' contains a 2-robust edge-cut $\delta_{G'}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Hence G' is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G.

Let G'' be the graph obtained from G by contracting A to a single vertex v. This vertex has degree 4, is oriented, and is in the boundary of the outer face. Since $d \notin V(G' - A)$, G'' has only one oriented vertex, namely v. If G'' has a 2-robust edge-cut $\delta_{G'}(B)$ of size at most 3, then such a cut also exists in G, a contradiction unless it is one of the specified vertices. Thus G'' is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G to obtain a valid orientation of G, a contradiction. Hence G has no 2-robust 4-edge-cut. $\square$

Claim 5.2.1. *Let $\delta(A)$ be a 2-robust internal cut in G, where A contains the boundary of the outer face. Then G/A is an FDT graph.*

Proof. All cuts in G/A are cuts in G, hence G/A is 3-edge-connected and has no internal 2-robust 4-edge-cuts. The contraction does not introduce any new vertices of small degree. Suppose that two internal degree 4-vertices in G/A are on adjacent faces. Then since no edge was deleted, this is also the case in G, a contradiction. Hence G/A is an FDT graph. $\square$

FDT4. *Every 2-robust 5-edge-cut in G separates d from t.*

Proof. Suppose that G does contain a 2-robust 5-edge-cut, $\delta(A)$, that does not separate d from t. This works in the same way as the proof of FDT3. In G'' there is an oriented vertex of degree 5 on the boundary of the outer face, and no degree 3 vertex. Hence G'' has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

FDT5. *The graph G has no chord of the outer face incident with a vertex of degree 3 or 4.*

Proof. Suppose that such a chord uv exists, where $deg_G(u) \in \{3, 4\}$. Let H and K be subgraphs of G such that $H \cap K = \{\{u, v\}, \{uv\}\}$, $H \cup K = G$, and d, if it exists, is in H.

Suppose that $\delta(H)$ is not 2-robust. Then K contains d, else G has unoriented parallel edges, and thus a valid orientation by FDT2. By definition, either u or v is d. Let w be the third vertex in K. Since G has no unoriented parallel edges, $|\delta(\{d, w\})| \leq 4$, a contradiction. Hence we may assume that $\delta(H)$ is 2-robust.

Suppose that $\delta(K)$ is not 2-robust. Then $|V(H)| = 3$. If u or v is d, the above argument applies. Thus we may assume that d is in $V(H) - \{u, v\}$. If there are parallel edges with endpoints d and u, then $\delta(\{d, u\})$ is an at most 5-edge-cut. Orient u and contract the parallel edges between d and u, calling the resulting graph G'. Note that the vertex of contraction has the same degree as d. Thus it is clear that G' is an FDT graph, and thus has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Suppose that there are not parallel edges with endpoints d and u. Then there are parallel edges with endpoints d and v. Since $|\delta(\{d, v\})| \geq 4$, $deg_K(v) \geq 3$. Orient u and add a directed edge from u to v in K. Then K is an FDT graph, and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. We may now assume that $\delta(K)$ is 2-robust.

By FDT3, $\delta(H)$ and $\delta(K)$ have size at least 5. Hence $deg_G(v) \geq 8$, and so v is not d or t. In addition, v has degree at least 4 in both H and K.

Suppose that $u \neq d$. Then in H, contract uv. Now H/uv is a DTS graph, and so by the minimality of G, H/uv has a valid orientation. Transfer this orientation to G, and orient u. Add an edge e directed from u to v (in the boundary of the outer face). Since $deg_G(u) \in \{3, 4\}$ it is clear that u, v are incident with the outer face, and thus $K + e$ is an FDT graph. By the minimality of G, $K + e$ has a valid orientation. This leads to a valid orientation of G, a contradiction.

Thus $u = d$. Then add a directed edge e from u to v, to obtain graphs $H + e$ and $K + e$ respectively. It is clear that $H + e$ and $K + e$ are FDT graphs, so by the minimality of G, they have valid orientations. Transfer the orientations of $H + e$ and $K + e$ to G to obtain a valid orientation of G, a contradiction. Thus no such chord exists. □

Again, we wish to reduce at low degree vertices in G. We begin by showing that d and t exist.

FDT6. *Vertices d and t exist.*

Proof. Suppose that d does not exist. Let v be a vertex in the boundary of the outer face of minimum degree. If $deg(v) \leq 5$, orient v, calling the resulting graph G'. Then G' is an FDT graph, and has a valid orientation by the minimality of G. This is a valid orientation of G, a contradiction. If $deg(v) \geq 6$, orient and delete a boundary edge incident with v, calling the resulting graph G'. If G' has a 2-robust at most 3-edge-cut, then G has a 2-robust at most 4-edge-cut, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Thus we may assume d exists.

Suppose that t does not exist. If d has degree 4 or 5, let v be a vertex adjacent to d on the boundary of the outer face. Let G' be the graph obtained by deleting dv. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Therefore, d has degree 3. Suppose that d has an adjacent vertex of degree 5. Delete d. The resulting graph has at most two degree 3 vertices. If there are two, orient one, calling the resulting graph G'. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Hence we may assume that d has three adjacent vertices of degree 4. Let these neighbours be u, v, and w, where u and v are on the boundary of the outer face. Suppose that w is not adjacent to u or v. If w has an adjacent vertex x of degree 4, then x is in the boundary of the outer face by definition. Then $|\delta(A_w)| + |\delta(B_w)| \leq 7$. Either $\delta(A)$ or $\delta(B)$ is a 2-robust at most 3-edge-cut, a contradiction. Hence the non-d neighbours of w have degree at least 5. Delete d and orient and delete w, calling the resulting graph G'. Then u and v are the only possible degree 3 vertices in G'. Orient one, calling the resulting graph G''.

Suppose that G'' has a 2-robust at most 3-edge-cut $\delta_{G''}(A)$ where $u \in A$ and $v \notin A$. Then either $\delta_G(A)$ or $\delta_G(G - A)$ is a 2-robust at most 5-edge-cut, a contradiction. Suppose that G'' has a 2-robust at most 3-edge-cut $\delta_{G''}(A)$ where $u, v \in A$. Then $\delta_G(A \cup \{d\})$ or $\delta_G(A \cup \{d, w\})$ is an internal 2-robust at most 5-edge-cut, a contradiction. An analysis of these cuts can be seen in Figure 5.1. Thus G'' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Therefore, w is adjacent to u or v. Without loss of generality suppose that w is adjacent to u. Delete d, orient u and w, and contract uw to a single vertex x, calling the resulting

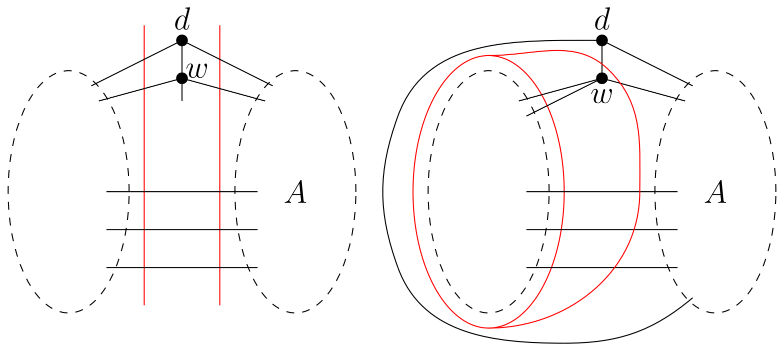


Figure 5.1: FDT6: Analysis of cuts.

graph G'. Then x is a directed vertex of degree 4 on the boundary of the outer face, and v is the only possible vertex of degree 3. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. Thus G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

FDT7. *Vertices d and t are not adjacent.*

Proof. Suppose that d is adjacent to t. Then d has degree 3 or 4. Orient t and contract dt calling the resulting graph G'. The vertex of contraction has degree 4 or 5, and G' has no vertex of degree 3. Then G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Claim 5.2.2. *If G' is a 3FDT graph with $|E(G')| < |E(G)|$, then G' has a valid orientation.*

Proof. Let $\hat{G}$ be a minimal counterexample, and let $\delta(A)$ be a 2-robust 3-edge-cut in $\hat{G}$ where $t \in A$ and $d \in G - A$. Let $\hat{G}'$ be the graph obtained from $\hat{G}$ by contracting A to a vertex. Then $\hat{G}'$ is a 3FDT graph and has a valid orientation by the minimality of $\hat{G}$. Transfer this orientation to $\hat{G}$. Let $\hat{G}''$ be the graph obtained from $\hat{G}$ by contracting $\hat{G} - A$ to a vertex. Then $\hat{G}''$ is a 3FDT graph and has a valid orientation by the minimality of $\hat{G}$. This leads to a valid orientation of $\hat{G}$, a contradiction. Otherwise, $\hat{G}$ is an FDT graph and has a valid orientation by the minimality of G, a contradiction. □

Due to the presence of internal degree 4 vertices in G, the reductions used in this proof involve deleting more vertices than those in Chapter 4. As a result, it is necessary to consider 6-edge-cuts in G. We show here that 4-robust 6-edge-cuts that do not separate d from t can be reduced. Since this proof is by minimal counterexample, it is useful to consider the non-4-robust 6-edge-cuts that may exist.

Suppose $A \subseteq V(G)$ is such that $\delta(G-A)$ is a 2-robust 6-edge-cut in G that is not 4-robust, and assume that $t \notin A$ (note that $d \notin A$ by definition). Let $|A| = n \leq 3$. Since every vertex in A has degree at least 4,

$$\sum_{v \in A} deg(v) \geq 4n.$$

Since G has no unoriented parallel edges,

$$\sum_{v \in A} deg(v) \leq 2\binom{n}{2} + 6.$$

When $n = 2$, this implies that

$$8 \leq \sum_{v \in A} deg(v) \leq 8.$$

When $n = 3$, this implies that

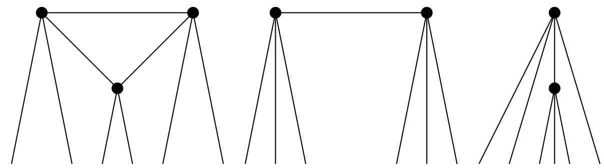
$$12 \leq \sum_{v \in A} deg(v) \leq 12.$$

Hence every vertex in A has degree exactly 4, and the vertices of A are pairwise adjacent. It follows that $\delta(A)$ is not an internal cut.

Suppose that $n = 3$. Then at least one vertex in A is an internal degree 4 vertex. By definition, A contains at most one internal degree 4 vertex. Thus the remaining two vertices in A are on the boundary of the outer face, and all three vertices have two incident edges in $\delta(A)$. We call this a 6-edge-cut of Type 1.

Suppose that $n = 2$. Then both vertices in A have three incident edges in $\delta(A)$. If both are on the boundary of the outer face, we call this a 6-edge-cut of Type 2. If one is an internal degree 4 vertex, we call this a 6-edge-cut of Type 3. Figure 5.2 shows the three types of non-4-robust 6-edge-cuts. We note that if a graph has a Type 1 cut, then it also has a Type 2 cut, but the distinction is important when analysing a given cut.

FDT8. *Every* 4*-robust* 6*-edge-cut in* G *separates* d *from* t*.*



(a) Type 1 (b) Type 2 (c) Type 3

Figure 5.2: The non-4-robust 6-edge-cuts that can exist in G.

Proof. Suppose for a contradiction that G has a 4-robust 6-edge-cut $\delta(A)$ such that $d, t \in A$. Choose the cut to minimise $|V(G-A)|$. Let G' be the graph obtained from G by contracting $G - A$ to a vertex. Then G' is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G and contract A to a vertex d', calling the resulting graph G''. Then $deg(d') = 6$ and G'' has no vertex of degree 3. Let the vertices adjacent to d' be u, v, w, x, y, and z in order, where u and z are boundary vertices.

We prove the following claims:

a. We have $deg(u) = deg(z) = 4$.

b. We have $u \neq v$, $y \neq z$, $u \neq z$.

c. We have $deg(v) = deg(y) = 4$.

d. Vertices u and v are adjacent.

e. We have $v \neq w$.

Let m and n be the remaining vertices adjacent to u, where m is on the boundary of the outer face. Let p and q be the remaining vertices adjacent to v.

f. We have $deg(m) = 4$, $n = p$, and $deg(n = p) = 5$.

g. Vertices m and n are adjacent.

These provide sufficient structure to complete the argument.

Claim FDT8a. *We have $deg(u) = deg(z) = 4$.*

Proof. Suppose that $deg(u) \geq 5$. Then delete $d'u$, calling the resulting graph G_1. It is clear that G_1 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence $deg(u) = 4$. Similarly, $deg(z) = 4$. □

Claim FDT8b. *We have $u \neq v$, $y \neq z$, $u \neq z$.*

Proof. Suppose that $u = v$. Then $\delta(A \cup \{u\})$ is a 6-edge-cut in G that does not separate d from t. By assumption, $\delta(A \cup \{u\})$ is not 4-robust, and so $|V(G - A)| = 4$. Then $\delta(A \cup \{u\})$ is a 6-edge-cut of Type 1. There is a vertex in $G - (A \cup \{u\})$ that is adjacent to u via parallel edges, a contradiction. Thus we may assume that $u \neq v$. Similarly $y \neq z$.

Suppose that $u = z$. Then $\delta(A \cup \{u\})$ is a 6-edge-cut in G. The argument that this cut is 4-robust is analogous to the argument above. Hence $u \neq z$. □

Delete $d'u$ and $d'v$, calling the resulting graph G_2

Claim FDT8c. *We have $deg(v) = deg(y) = 4$.*

Proof. Suppose that $deg_{G''}(v) \geq 5$. It is clear that G_2 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence $deg(v) = 4$. Similarly, $deg(y) = 4$. □

We may now assume that G_2 has two vertices of degree 3: u and v, and a directed vertex of degree 4: d'. Note that v is an internal degree 4 vertex in G. Suppose that v has an adjacent vertex c of degree 4 that is not u. Then by definition, c is on the boundary of the outer face. We have $|\delta_{G''}(A_v)| + |\delta_{G''}(B_v)| \leq 10$, so we may assume that $|\delta_{G''}(A_v)| \leq 5$. Thus $|\delta_G(A_v)| \leq 5$ also (where d' is replaced by A if necessary). Now $\delta_G(A_v)$ does not separate d from t, so it must not be 2-robust. Hence $|A_v| = 1$. Thus $w, x, y, z \in B_v$ and $u \in A_v$, a contradiction since u has degree 4 and only three adjacent vertices, leading to unoriented parallel edges. Thus we may assume that v has no degree 4 neighbour in G except for possibly u.

Claim FDT8d. *Vertices u and v are adjacent.*

Proof. Suppose that u and v are not adjacent. Let G_3 be the graph obtained from G_2 by deleting the edges incident with v. Then G_3 has one possible degree 3 vertex: u.

Suppose that G_3 contains a 2-robust at most 3-edge-cut $\delta_{G_3}(B)$ where $d', u \in B$. Note that when referring to G, we will use B as shorthand for the set $(B - d') \cup A$ for simplicity. Then

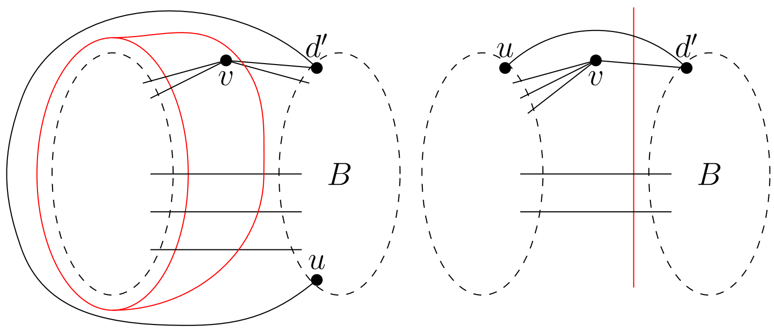


Figure 5.3: FDT8: Analysis of cuts (1).

G contains an internal 2-robust at most 5-edge-cut $\delta_G(B \cup \{v\})$ or $\delta_G(B)$, a contradiction. *Suppose that G_3 contains a 2-robust at most 2-edge-cut $\delta_{G_3}(B)$ where $d' \in B$ and $u \notin B$.* All neighbours of v are in $G' - B$, else $\delta_G(B \cup \{v\})$ is a 2-robust at most 5-edge-cut that does not separate d from t. Then $\delta_G(B)$ is a 2-robust at most 4-edge-cut in G, a contradiction. *By Claim 5.2.2, G_3 does not contain a 2-robust 3-edge-cut separating d' from u.* An analysis of these cuts is shown in Figure 5.3. Hence G_3 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Claim FDT8e. *We have $v \neq w$.*

Proof. Suppose that $v = w$. Then $\delta(A \cup \{u, v\})$ is an at most 6-edge-cut in G. By assumption, $\delta(A \cup \{u, v\})$ is not 4-robust, so $|V(G - (A \cup \{u, v\}))| \in \{2, 3\}$. Since v is an internal degree 4 vertex, $G - (A \cup \{u, v\})$ does not contain an internal degree 4 vertex. Thus $\delta(A \cup \{u, v\})$ is a 6-edge-cut of Type 2, and G has unoriented parallel edges incident with u, a contradiction. Hence $v \neq w$. □

Let m and n be the remaining vertices adjacent to u, where m is on the boundary of the outer face. Let p and q be the remaining vertices adjacent to v. Note that $deg(n), deg(p), deg(q) \geq 5$.

Claim FDT8f. *We have $deg(m) = 4$, $n = p$, and $deg(n = p) = 5$.*

Proof. If $deg(m) \geq 5$, $n \neq p$ or $deg(n = p) \geq 6$, then the graph G_4 obtained from G_2 by orienting and deleting u and v has at most one vertex of degree 3. *Suppose that G_4 has a 2-robust at most 3-edge-cut $\delta_{G_4}(B)$ where $m, d' \in B$.* We may assume that $\delta_G(B \cup \{u, v\})$ is not a 5-edge-cut. Thus $\delta_G(B \cup \{u\})$ is an internal 2-robust at most 6-edge-cut that does not separate d from t. Recall that if $\delta_G(B \cup \{u\})$ is a 6-edge-cut it is necessarily 4-robust, contradicting the minimality of A. Hence no such cut exists.

Suppose that G_4 has a 2-robust at most 3-edge-cut $\delta_{G_4}(B)$ where $d' \in B$ and $m \notin B$. Then n and p are in $G' - B$, else $\delta_G(B \cup \{u, v\})$ is a 2-robust at most 5-edge-cut that does not separate d from t. We have $\delta_G(B)$, a 4-robust at most 6-edge-cut, that does not separate d from t, contradicting the minimality of $G - A$. Hence no such cut exists. Then G_4 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

We now have $deg(m) = 4$, $n = p$, and $deg(n = p) = 5$. Then G_4 as defined above would have two vertices of degree 3. Hence we consider different reductions. Let e_1, e_2, e_3, e_u be the edges incident with m in order, where e_1 is on the boundary of the outer face and $e_u = \{m, u\}$.

Claim FDT8g. *Vertices m and n are adjacent.*

Proof. Suppose that m and n are not adjacent. Let G_5 be the graph obtained from G_2 by lifting e_1 and e_2, and orienting and deleting m, u, and v. Then n is the only possible degree 3 vertex in G_5.

Suppose that G_5 has a 2-robust at most 2-edge-cut $\delta_{G_5}(B)$ where d' and the lifted edge are in B. Then $\delta_G(B \cup \{u, v, m\})$ is an internal 2-robust at most 6-edge-cut in G, a contradiction. *Suppose that G_5 has a 2-robust at most 2-edge-cut $\delta_{G_5}(B)$ where $d' \in B$ and the lifted edge is not in B.* Then p and q are in B, else $\delta_G(B)$ is a 4-robust at most 5-edge-cut that does not separate d from t, contradicting the minimality of $\delta(A)$. We have $\delta_G(B \cup \{u, v\})$, a 2-robust at most 4-edge-cut, a contradiction.

Suppose that G_5 contains a 2-robust 3-edge-cut $\delta_{G_5}(B)$ where $d' \in B$. By Claim 5.2.2, we may assume that $p \in B$. If the lifted edge is also in B, then $\delta_G(B \cup \{u, v, m\})$ is an internal 2-robust at most 4-edge-cut, a contradiction. Hence the lifted edge is in $G_5 - B$. Then $\delta_G(B \cup \{u, v\})$ is a 2-robust at most 5-edge-cut that does not separate d from t, a contradiction. An analysis of these cuts is shown in Figure 5.4. Therefore, G_5 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

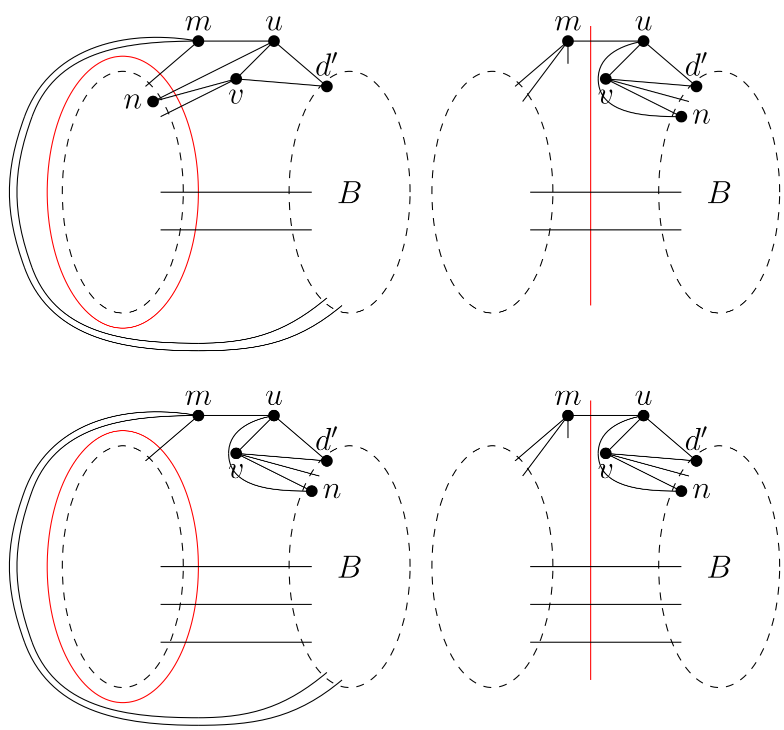


Figure 5.4: FDT8: Analysis of cuts (2).

Therefore m and n are adjacent. This series of claims provides sufficient structure to complete the proof.

Note that the only internal degree 4 neighbour of n is b. Suppose that n is adjacent to a degree 4 vertex c that is not m or v. Then c is on the boundary of the outer face and is not adjacent to m. We have

$$|\delta(A_n)| + |\delta(B_n)| = 9,$$

so either $\delta(A_n)$ or $\delta(B_n)$ is an at most 4-edge-cut in G. It is easily verified that this at most 4-edge-cut is 2-robust, contradicting FDT3. Thus we may assume that m and v are the only neighbours of n that have degree at most 4.

Let G_6 be the graph obtained from G_2 by lifting e_1 and e_2, and orienting and deleting m, u, v, and n. Then q is the only possible degree 3 vertex in G_6 (if n and q are adjacent).

Suppose that G_6 has a 2-robust at most 2-edge-cut $\delta_{G_6}(B)$ where d' and the lifted edge are in B. Then G has an internal 2-robust at most 5-edge-cut $\delta_G(B \cup \{u, v, m, n\})$, a contradiction. *Suppose that G_6 has a 2-robust at most 2-edge-cut $\delta_{G_6}(B)$ where $d' \in B$ and the lifted edge is not in B.* Then q and both neighbours of n are in B, else $\delta_G(B)$ is a 4-robust at most 6-edge-cut that does not separate d from t. Then G contains a 2-robust at most 4-edge-cut $\delta_G(B \cup \{u, v, m, n\})$, a contradiction.

Suppose that G_6 has a 2-robust 3-edge-cut $\delta_{G_6}(B)$ where $d' \in B$. Then by Claim 5.2.2, n and q are adjacent and $q \in B$. Suppose the lifted edge is in B. Then $\delta_G(B \cup \{u, v, m, n\})$ is an internal 2-robust at most 4-edge-cut in G, a contradiction. Thus the lifted edge is not in B. The remaining neighbour of n is in $G' - B$, else $\delta_G(B \cup \{u, v, m, n\})$ is a 2-robust at most 5-edge-cut that does not separate d from t. Then $\delta_G(B \cup \{u, v, n\})$ is a 3-robust 6-edge-cut. If it is not 4-robust, then it is of Type 1, and n has two adjacent internal degree 4 vertices, a contradiction. Therefore, G_6 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence no such 6-edge-cut exists. □

We now consider the vertices adjacent to t. Let u, v, and w be adjacent to t such that u and v are on the boundary of the outer face. Many of the arguments used in the proof of FDT8 will arise again in FDT11, where we consider a configuration with u and v adjacent to w, and replace G_2 in the proof of FDT8 with the the graph obtained by deleting the remaining edges incident with either u or v, and contracting $\{t, u, v, w\}$. We now consider properties of u, v, and w, in order to establish such a structure.

Claim 5.2.3. *At least two of u, v, and w have degree 4.*

Proof. Suppose that two of the vertices u, v, and w have degree at least 5. Let G' be the graph obtained from G by orienting and deleting t. Then G' has at most one vertex of degree 3. Suppose that G' contains a 2-robust at most 3-edge-cut. Then G contains a 2-robust at most 4-edge-cut, a contradiction. Hence G' is an FDT graph, and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Claim 5.2.4. *If a degree 4 vertex v in G is incident with parallel edges, then $v = d$.*

Proof. Suppose that such a vertex v exists where $v \neq d$. Then there exist oriented parallel edges with endpoints v and d. Orient v and contract d and v to a single vertex d', calling the resulting graph G'. Then d' is an oriented vertex on the boundary of the outer face of G', with the same degree as d. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

FDT9. *Vertices u and v have degree 4.*

Proof. Suppose that v has degree 5. Then u and w have degree 4. Let e_1, e_2, e_3, e_t be the edges incident with u in order, where e_1 is on the boundary of the outer face. We first prove the following claim:

Claim FDT9a. *Vertices u and w are adjacent.*

Proof. Suppose that u and w are not adjacent. Let G' be the graph obtained from lifting e_1 and e_2, and orienting and deleting u and t. The remaining endpoint of e_3 is not a degree 4 vertex, as it shares a face with w. Hence w is the only possible degree 3 vertex in G'.

Suppose that G' has a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where v and the lifted edge are in A. Then $\delta_G(G - A)$ is an internal 2-robust at most 5-edge-cut, a contradiction. *Suppose that G' has a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A.* Then w and the endpoint of e_3 are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Hence $\delta_G(G - A)$ is a 2-robust 4-edge-cut, a contradiction. *Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A.* Then $w \in A$, else $\delta_G(A)$ is a 2-robust 4-edge-cut. By Claim 5.2.2, we may assume that $d \in A$. Then $\delta_G(A \cup \{t\})$ is

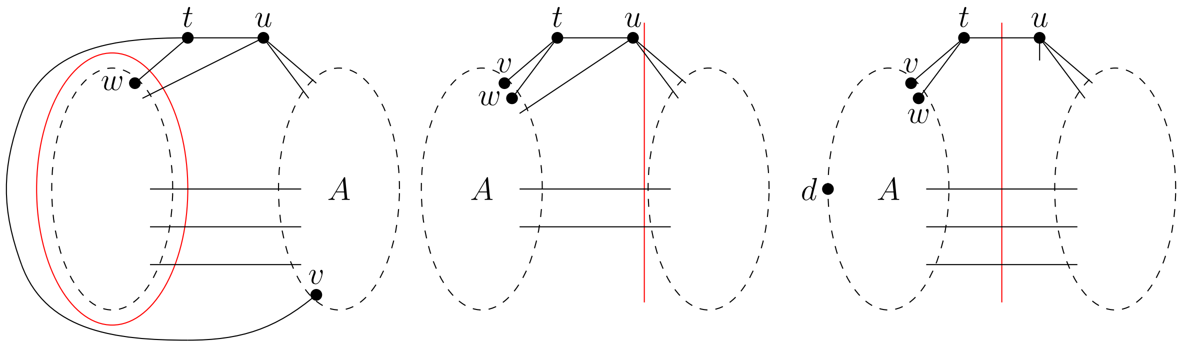


Figure 5.5: FDT9: Analysis of cuts (1).

a 2-robust at most 5-edge-cut in G that does not separate d from t, a contradiction. An analysis of these cuts can be seen in Figure 5.5.

We conclude that G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

Suppose that w is adjacent to a vertex x of degree 4 that is not u. Then x is on the boundary of the outer face by definition. We have $|\delta(A_w)| + |\delta(B_w)| = 7$. Then $\delta(A_w)$ or $\delta(B_w)$ is a 3-edge-cut that is not $\delta(t)$ or $\delta(d)$, a contradiction. Hence we may assume that w has no adjacent vertex of degree at most 4 except u and t. An analogous argument shows that w is not adjacent to d. Let G' be the graph obtained from G by lifting e_1 and e_2, and orienting and deleting u, t, and w. Then v is the only possible vertex of degree 3 in G' (if it is adjacent to w).

Suppose that G' has a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, where v and the lifted edge are in A. Then G has an internal 2-robust at most 5-edge-cut, a contradiction. *Suppose that G' has a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where $v \in A$ and the lifted edge is in $G' - A$.* Then G contains a 2-robust at most 4-edge-cut $\delta_G(A)$ or $\delta_G(G' - A)$, a contradiction.

Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $v \in A$ and the lifted edge is in $G' - A$. If G' has no degree 3 vertex, then it has a valid orientation by Claim 5.2.2. This leads to a valid orientation of G, a contradiction. Hence we may assume that v is a degree 3 vertex. Then $d \in A$ by Claim 5.2.2. It follows that $\delta_G(A \cup \{t, u, w\})$ is a 2-robust at

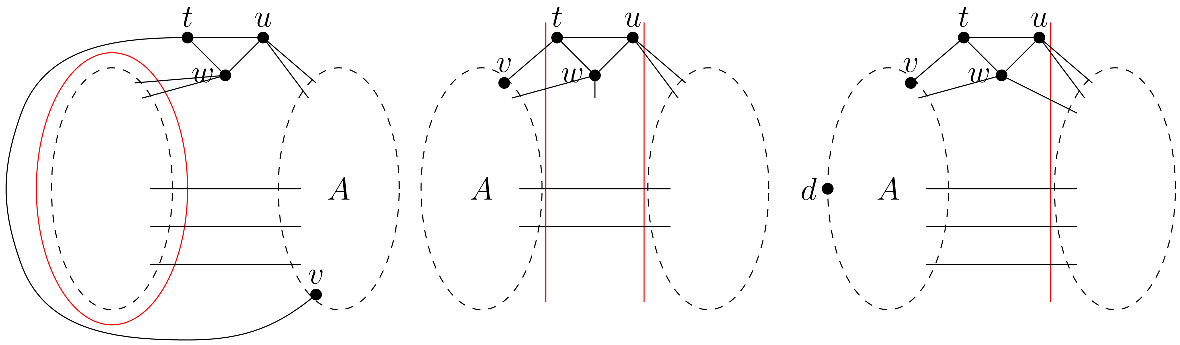


Figure 5.6: FDT9: Analysis of cuts (2).

most 6-edge-cut in G that does not separate d from t. If it is a 6-edge-cut, then since w is an internal degree 4 vertex, $G - A - \{t, u, w\}$ does not have an internal degree 4 vertex, so the 6-edge-cut is of Type 2. Then G has unoriented parallel edges incident with u, a contradiction. Hence G' has no such cut. An analysis of these cuts can be seen in Figure 5.6.

Therefore, G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, the final contradiction. □

Claim 5.2.5. *Vertex w has degree at most 5.*

Proof. Suppose that w has degree at least 6. Let e_1, e_2, e_3, e_t be the edges incident with u in order, where e_1 is on the boundary of the outer face. If e_3 has two incident degree 4 vertices, then FDT5 implies that the corresponding edge incident with v does not, else G has two internal degree 4 vertices on adjacent faces, a contradiction. Hence without loss of generality, we may assume that e_3 has only one endpoint of degree 4. Let G' be the graph obtained from G by lifting e_1 and e_2, and orienting and deleting u and t. Since $deg(w) \geq 6$, v is the only degree 3 vertex in G'.

Suppose that G' has a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where v and the lifted edge are in A. Then $\delta_G(G - A)$ is an internal 2-robust at most 5-edge-cut, a contradiction. *Suppose that G' has a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A.* Then w and the endpoint of e_3 are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Hence $\delta_G(G - A)$ is a 2-robust 4-edge-cut a contradiction.

Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A. Then $w \in A$, else $\delta_G(A)$ is a 2-robust 4-edge-cut. By Claim 5.2.2, we may assume that $d \in A$. Then $\delta_G(A \cup \{t\})$ is a 2-robust at most 5-edge-cut that does not separate d from t, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Claim 5.2.6. *We have $p(t) = 0$, $p(u) = \pm 1$, and $p(v) = \pm 1$.*

Proof. Suppose that $p(t) \neq 0$. Let G' be the graph obtained from G by lifting tu and tw and orienting and deleting t. Then v is the only degree 3 vertex in G'. If G' has a 2-robust at most 3-edge-cut, then G has a corresponding 2-robust at most 4-edge-cut, a contradiction. Thus G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence $p(t) = 0$.

Suppose that $p(u) = 0$. Lift the pairs of edges e_1, e_2 and e_3, e_t, calling the resulting graph G'. Then t is the only vertex of degree 3 in G'. If G' has a 2-robust at most 2-edge-cut, then G has a corresponding 2-robust at most 4-edge-cut, a contradiction. Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $t \in A$. By Claim 5.2.2, $d \in A$. Then $\delta_G(A)$ is a 2-robust at most 5-edge-cut that does not separate d from t, a contradiction. Therefore G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence $p(u) \neq 0$. Similarly $p(v) \neq 0$. □

Claim 5.2.7. *If deg$(w) = 4$, then w is adjacent to at least one of u and v.*

Proof. Suppose that w has degree 4 and assume that w is not adjacent to either u or v. Suppose that w has a degree 4 neighbour x. Then x is in the boundary of the outer face by definition. We have $|\delta(A_w)| + |\delta(B_w)| \leq 7$, so G has an at most 3-edge-cut that is not $\delta(d)$ or $\delta(t)$, a contradiction. Hence we may assume that w has no neighbours of degree 4. An analogous argument shows that w is not adjacent to d.

First, suppose that u and v have the same prescription. Let f_1, f_2, f_3, f_t be the edges incident with v in order, where f_1 is on the boundary of the outer face. Lift e_1 and e_2, and f_1 and f_2, and orient and delete u, v, and t, calling the resulting graph G'. Then w is the only possible vertex of degree 3 in G'.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where both lifted edges are in A. Then G contains an internal 2-robust at most 5-edge-cut $\delta_G(G' - A)$, a contradiction.

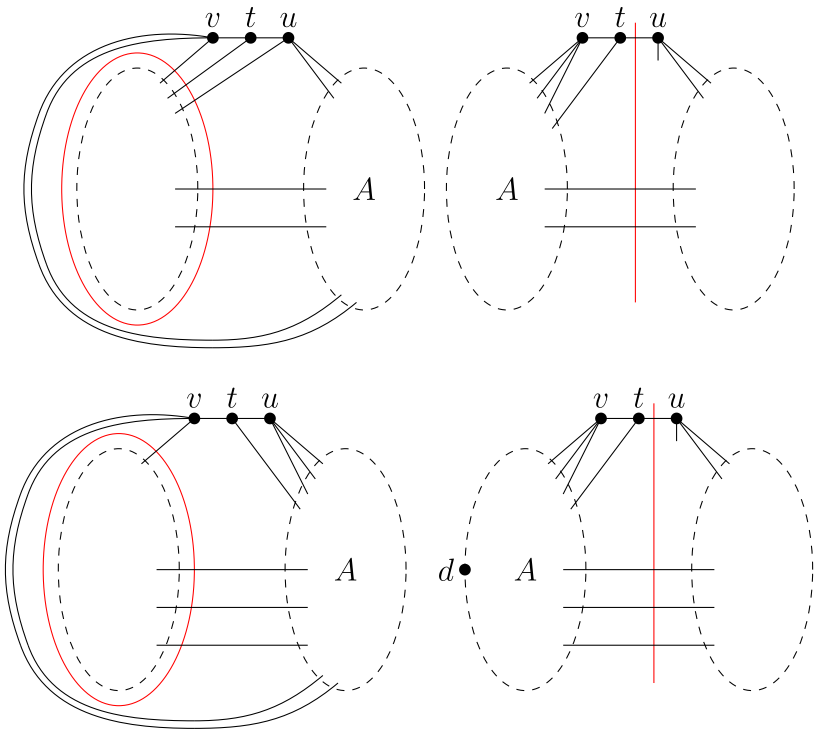


Figure 5.7: Claim 5.2.7: Analysis of cuts (1).

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where the lifted edge from u is in A and the remaining lifted edge is in $G' - A$. Then $w \in A$, else $\delta_G(A \cup \{u\})$ is a 2-robust at most 4-edge-cut. We have $\delta_G(A \cup \{u, t\})$ a 2-robust at most 4-edge-cut, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where both lifted edges are in A. Then $d \in A$ by definition. By Claim 5.2.2, $w \in A$. Then G contains an internal 2-robust at most 4-edge-cut, a contradiction. *Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where the lifted edge from u is in A and the remaining lifted edge is in $G' - A$.* Without loss of generality let $w \in A$. Then $d \in A$ by Claim 5.2.2. But then G contains a 2-robust at most 5-edge-cut $\delta_G(A \cup \{u, t\})$ that does not separate d from t, a contradiction. Hence no such cut exists. An analysis of these cuts can be seen in Figure 5.7.

Then G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

We now assume that u and v do not have the same prescription. Then there exists $x \in \{u, v\}$ such that $p(x) \neq -p(w)$. Without loss of generality, assume that $x = v$.

Let g_1, g_2, g_3, g_t be the edges incident with w where g_1 is on the same face as e_3. Lift e_1 and e_2, and g_1 and g_2, and orient and delete u, w, and t, calling the resulting graph G'. When orienting w, if w has the same prescription as u, the directions of edges are clear. If w has prescription zero, then the remaining edges are directed to allow the $p(t)$ to be satisfied. Then v is the only possible vertex of degree 3 in G'.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where v and the lifted edge from u are in A. If e_3 does not have an endpoint in $G' - A$, then G contains an internal 2-robust at most 5-edge-cut $\delta_G(A \cup \{u, t\})$ or $\delta_G(A \cup \{u, t, w\})$, a contradiction. Thus e_3 has an endpoint in $G' - A$. All remaining vertices adjacent to w are in $G' - A$, else $\delta_G(G' - A)$ is an internal 2-robust at most 6-edge-cut. Then $\delta_G(A \cup \{t, u\})$ is an internal 2-robust 5-edge-cut, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge from u is not in A. The neighbours of w via g_2 and g_3 are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut in G. Then $\delta_G(A \cup \{t, w\})$ is a 2-robust at most 4-edge-cut in G, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge from u is not in A. Then $d \in A$ by Claim 5.2.2. The endpoints of g_1 and g_2 are in $G' - A$, else $\delta_G(A \cup \{t, w\})$ is a 2-robust at most 5-edge-cut that does not separate d from t. The endpoint of g_3 is in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. We have $\delta_G(A \cup \{t, w\})$ a 3-robust 6-edge-cut in G that does not separate d from t. This cut is of Type 1, which implies that u is adjacent to w, a contradiction.

Therefore no such cut exists. An analysis of these cuts can be seen in Figure 5.8. Thus G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. We conclude that w is adjacent to u or v. $\square$

Claim 5.2.8. *If $deg(w) = 4$, then w is adjacent to both u and v.*

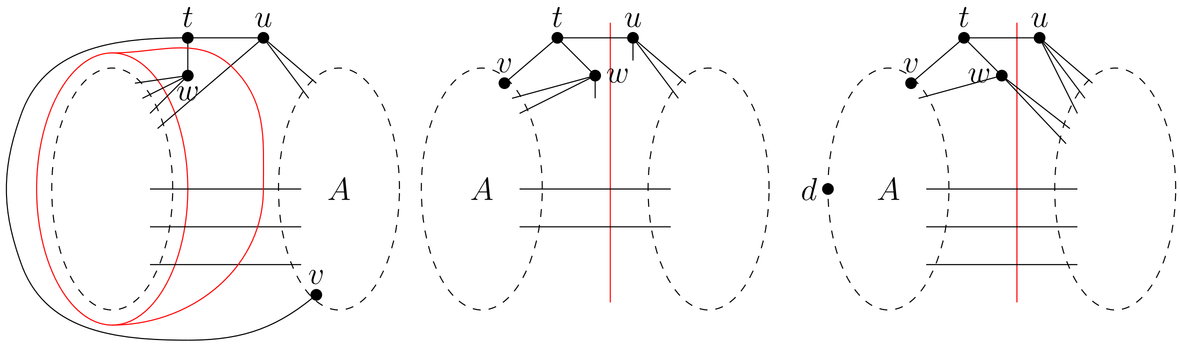


Figure 5.8: Claim 5.2.7: Analysis of cuts (2).

Proof. Suppose that w is adjacent to u but not v. Suppose that w has a degree 4 neighbour x aside from u. Then x is in the boundary of the outer face by definition. We have $|\delta(A_w)| + |\delta(B_w)| \leq 7$, so G has an at most 3-edge-cut that is not $\delta(d)$ or $\delta(t)$, a contradiction. Hence we may assume that w has no neighbours of degree 4 aside from u. The argument that w is not adjacent to d is analogous.

Let G' be the graph obtained from G by lifting e_1 and e_2 and orienting and deleting u, t, and w. Then v is the only possible vertex of degree 3 in G'. *Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ that does not separate the lifted edge from v.* Then $\delta_G(A)$ is an internal 2-robust at most 5-edge-cut, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A. The neighbours of w are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut, a contradiction. We have $\delta_G(G' - A)$ a 2-robust 4-edge-cut, a contradiction.

Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A. Then $d \in A$ by Claim 5.2.2. At least one neighbour of w is in A, else $\delta_G(A)$ is a 2-robust 4-edge-cut. Then $\delta_G(A \cup \{t, w, u\})$ is a 2-robust at most 6-edge-cut. If it is a 6-edge-cut, then since w is an internal degree 4 vertex, the cut is of Type 2, and G contains unoriented parallel edges, a contradiction. Therefore G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

Claim FDT10. *Vertex w has degree 5.*

Proof. The only alternative is that $deg(w) = 4$. By Claim 5.2.8, w is adjacent to both u and v. As in previous arguments, we may assume that the remaining neighbour y of w has degree at least 5. Then y is not adjacent to both u and v, else $\delta(\{t, u, v, w, y\})$ is a 2-robust 4-edge-cut in G, a contradiction.

We prove the following claims:

a. Exactly one of the following holds:

   1. u is adjacent to y and v is adjacent to d, or
   2. v is adjacent to y and u is adjacent to d.

Without loss of generality, we assume that u is adjacent to y and v is adjacent to d.

b. We have $deg(d) = 4$.

Let x be the remaining endpoint (not u) of e_1.

c. Vertices x and y are adjacent, and $deg(x) = 4$.

Claim FDT10a. *Exactly one of the following holds:*

1. *u is adjacent to y and v is adjacent to d, or*
2. *v is adjacent to y and u is adjacent to d.*

Proof. Suppose that v is adjacent to neither y nor d. Let G' be the graph obtained from G by lifting e_1 and e_2, and orienting and deleting u, t, w, and v. Then the boundary neighbour z of v is the only possible vertex of degree 3 in G.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where z and the lifted edge are in A. Then G contains an internal 2-robust at most 5-edge-cut $\delta_G(G' - A)$. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $z \in A$ and the lifted edge is not in A.* Then G contains a 2-robust at most 4-edge-cut $\delta_G(A)$ or $\delta_G(G' - A)$, a contradiction.

Suppose that G' has a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $z \in A$ and the lifted edge is not in A. By Claim 5.2.2, $d \in A$. Now $\delta_G(A)$ cannot be a 4-edge-cut, so the endpoint of f_2 is in A. Then $\delta_G(G' - A)$ is a 2-robust at most 6-edge-cut. Since w is an internal degree 4 vertex, if $\delta_G(G' - A)$ is not 4-robust, then it is of Type 2, and G has unoriented parallel edges incident with u, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

We now assume that v is adjacent to at least one of y and d. The same is true of u. If u and v are both adjacent to d, then $\delta_G(\{t, u, v, w, d\})$ is an internal 2-robust at most 4-edge-cut, a contradiction. We conclude that exactly one of u and v is adjacent to y and exactly one of u and v is adjacent to d. ☐

We now assume without loss of generality that u is adjacent to y and v is adjacent to d.

Claim FDT10b. *We have $deg(d) = 4$.*

Proof. If $deg(d) = 3$, then orient v, t, w, u, contract $\{d, u, t, v, w\}$ and delete e_1 and e_2. The resulting graph is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence $deg(d) = 4$. ☐

Suppose that y has an adjacent vertex z of degree 4 that is not w, u, or adjacent to u. Then z is on the boundary by definition. We have $|\delta(A_y)| + |\delta(B_y)| = 9$. Then either $\delta(A_y)$ or $\delta(B_y)$ is an at most 4-edge-cut, which by FDT3 is not 2-robust. It is easy to verify that G has unoriented parallel edges, a contradiction. Let x be the remaining endpoint of e_1.

Claim FDT10c. *Vertices x and y are adjacent, and $deg(x) = 4$.*

Proof. Suppose that x and y are not adjacent or that $deg(x) \geq 5$. Note that the directions of two edges at w (wu and wy) may be determined (the pre-orientation of d may force vw and tw to be in opposite directions relative to w). Let h_1, h_2, h_3, yw, yu be the edges incident with y in order. If the edge vd achieves $p(v)$, lift h_2 and h_3, and orient and delete y, u, t, v, and w, calling the resulting graph G'. Then x is the only possible degree 3 vertex in G aside from d.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $d \in A$ and $x \in A$. Then y has a neighbour in A, else $\delta_G(A \cup \{t, u, v, w\})$ is an internal 2-robust at most 6-edge-cut that does not separate d from t. Similarly, $\delta_G(A \cup \{t, u, v, w, y\})$ is an internal 2-robust at most 6-edge-cut, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $d \in A$ and $x \notin A$. The endpoints of h_1, h_2, and h_3 are in $G' - A$, else $\delta_G(G' - A)$ is a 2-robust at most 5-edge-cut that does not separate d from t. Then $\delta_G(G' - A)$ is a 2-robust at most 4-edge-cut, a contradiction. *If G' contains a 3-robust 3-edge-cut separating d and x,* then G' has a valid orientation by Claim 5.2.2. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Hence the edge vd does not achieve $p(v)$. Lift vw, vt, contract $\{t, u, w, y\}$ to a single vertex of degree 4, and orient and delete v, calling the resulting graph G'. Then G' has no vertex of degree 3 aside from d. If G' has a 2-robust at most 2-edge-cut, then G has a corresponding at most 4-edge-cut, a contradiction. By Claim 5.2.2, G' does not have a 2-robust 3-edge-cut. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Therefore x and y are adjacent and $deg(x) = 4$. Let j_1 and j_2 be the remaining edges incident with x. If the edge vd meets $p(v)$, lift the pair of edges j_1 and j_2, and orient and delete x, u, t, v, w, and y, calling the resulting graph G'. Otherwise, lift j_1 and j_2, orient tv to satisfy $p(v)$, and orient and delete t, x, u, w, v, and y, calling the resulting graph G'. Then G' has a degree 3 directed vertex and only one other possible degree 3 vertex: a common neighbour of v and y.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $d \in A$ and the lifted edge is in A. Then G contains an internal 2-robust at most 6-edge-cut $\delta_G(G' - A)$, a contradiction. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $d \in A$ and the lifted edge is not in A.* The neighbour of f_2 and the neighbour of h_3 are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Thus $\delta_G(G' - A)$ is a 2-robust at most 5-edge-cut in G that does not separate d from t, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$ where $d \in A$ and the lifted edge is not in A. Then by Claim 5.2.2, G' has a degree 3 vertex, which must be in A. Then $\delta_G(G' - A)$ is a 2-robust at most 6-edge-cut in G that does not separate d from t. Suppose that $\delta_G(G' - A)$ is a 6-edge-cut that is not 4-robust. Then since w is an internal degree 4 vertex, $G' - A$ does not contain an internal degree 4 vertex, and thus is of Type 2. It follows that G has unoriented parallel edges incident with x, a contradiction. Hence no such cut exists, and G' is an FDT graph. Therefore G' has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

FDT11. *One of the following holds:*

1. *the edge e_3 is incident with an internal degree 4 vertex, and v and w are adjacent, or*
2. *the edge f_3 is incident with an internal degree 4 vertex, and u and w are adjacent.*

Proof. Due to the complexity of this proof, we provide a detailed summary in sections at the appropriate points. The overall idea of this proof is to:

1. establish that at least one of e_3 and f_3 is incident with w, and that the other is incident with either w or an internal degree 4 vertex,
2. assume (for a contradiction) that both e_3 and f_3 are incident with w,
3. check that there exists $z \in \{u, v\}$ with the property that no 6-edge-cut using certain edges incident with z separates d from t, and
4. apply reductions similar to those in FDT8 to obtain a valid orientation.

We first prove the following claims:

a. The edges e_3 and f_3 are each incident with an internal degree 4 vertex, or with w.

b. At least one of the edges e_3 and f_3 is incident with w.

Claim FDT11a. *The edges e_3 and f_3 are each incident with an internal degree 4 vertex, or with w.*

Proof. Suppose for a contradiction that e_3 is not incident with w or with an internal degree 4 vertex. Let G' be the graph obtained from G by lifting e_1 and e_2 and orienting and deleting u and t. Then v is the only possible degree 3 vertex in G.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is in A. Then G has an internal 2-robust at most 5-edge-cut $\delta_G(G' - A)$, a contradiction. *Suppose that G contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A.* Then $w \in A$, else $\delta_G(A)$ is a 2-robust 4-edge-cut. We have the endpoint of e_3 in A, else $\delta_G(A \cup \{t\})$ is a 2-robust 4-edge-cut. Then $\delta_G(A)$ and $\delta_G(A \cup \{t\})$ are both 2-robust 5-edge-cuts. One of them does not separate d from t, a contradiction. Therefore G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. The same is true of f_3. $\square$

Claim FDT11b. *At least one of the edges e_3 and f_3 is incident with w.*

Proof. If both e_3 and f_3 are incident with internal degree 4 vertices, these vertices are on adjacent faces, a contradiction. $\square$

Thus we may assume without loss of generality that v and w are adjacent. It remains to be shown that e_3 is incident with an internal degree 4 vertex. Suppose for a contradiction that u and w are adjacent. We prove the following claims:

c. Neither u nor v is adjacent to d.

d. The edges e_2 and f_2 each have an endpoint that is an internal degree 4 vertex.

Claim FDT11c. *Neither u nor v is adjacent to d.*

Proof. Suppose that both u and v are adjacent to d. Then $\delta(\{t, u, v, w, d\})$ is an internal 6-edge-cut in G. It is 2-robust, else w has unoriented parallel edges. The argument that this cut is 4-robust is analogous to previous such arguments.

Now assume that u is adjacent to d and v is not. If d has degree 3, orient u and t, delete tv and tw, and contract $\{t, u, d\}$ to a single vertex, calling the resulting graph G'. Then G' has a directed vertex of degree 3 and at most one degree 3 vertex, v. If G' contains a 2-robust at most 3-edge-cut, then G contains a 2-robust at most 4-edge-cut, a contradiction.

Thus we may assume that $deg(d) = 4$. If f_2 has an internal endpoint of degree at least 5, let G' be the graph obtained from G by orienting and deleting f_1 and f_2 to satisfy $p(v)$, and contracting $\{t, u, v, w\}$ to a vertex z of degree 4. Then G' has at most one vertex of degree 3. Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, where $z \in A$. If $d \in A$, then $\delta_G(A)$ is a 2-robust at most 5-edge-cut, not separating d from t, a contradiction. Hence $d \notin A$. Then $\delta_G((G' - A) \cup \{u, t\})$ is a 4-robust at most 6-edge-cut, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G by orienting the two remaining edges incident with u, and t; then vw's orientation is determined by vt, and w cannot be the only vertex whose prescription is not met, a contradiction.

Hence f_2 has an internal endpoint of degree 4. Thus w is not adjacent to an internal degree 4 vertex. Let G' be the graph obtained from G by lifting f_1 and f_2, and orienting v, t, u, and w, deleting v, t, and w, and contracting d and u to a single vertex. Then G' has no degree 3 vertex.

If G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where d and the lifted edge are in A, then G contains an internal 2-robust at most 5-edge-cut $\delta_G(A \cup \{t, u, v, w\})$, a contradiction. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $d \in A$ and the lifted edge is not in A.* Then w has a neighbour in A and a neighbour in $G' - A$, else $\delta_G(A)$ or $\delta_G(G' - A)$ is a 2-robust 4-edge-cut. Now $\delta_G(G' - A)$ is a 2-robust 5-edge-cut that does not separate d from t, a contradiction. By Claim 5.2.2, G' does not have a 2-robust 3-edge-cut. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. $\square$

Claim FDT11d. *The edges e_2 and f_2 each have an endpoint that is an internal degree 4 vertex.*

Proof. Suppose that e_2 has an endpoint of degree at least 5. Let G' be the graph obtained from G by orienting and deleting e_1 and e_2 to satisfy $p(u)$, and contracting $\{t, u, v, w\}$ to a vertex z of degree 4. Then G' has at most one vertex of degree 3. Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, where $z \in A$. If $d \in A$, then $\delta_G(A)$ is a 2-robust at most 5-edge-cut, not separating d from t, a contradiction. Hence $d \notin A$. Then $\delta_G((G' - A) \cup \{u, t\})$ is a 4-robust at most 6-edge-cut, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G by orienting the two remaining edges incident with v, and t; then uw's orientation is determined by ut, and w cannot be the only vertex whose prescription is not met, a contradiction. We conclude that e_2 has an endpoint that is an internal degree 4 vertex. The same is true of f_2. $\square$

Let a and b be the endpoints of e_1 and e_2 respectively that are not u. Let c and d be the corresponding endpoints of f_1 and f_2. Note that neither b nor d is adjacent to w, else they are on adjacent faces. It also follows that w is not adjacent to an internal degree 4 vertex.

We wish to apply a similar reduction to that of Claim FDT11d, however, this results in two vertices of degree 3. This situation is very similar to the proof of FDT8, however, the 5 and

6-edge-cuts that arise in G may separate d from t. We now establish that this cannot occur on both sides of t. Let e_2' be the edge incident with b on the boundary of a face including both e_1 and e_2. Let f_2' be the corresponding edge incident with the endpoint of f_2.

Specifically, we show that G has 2-robust at most 6-edge-cuts $\delta(A)$ and $\delta(B)$ where $\{t,u,v,w\} \subseteq A \cap B$, $d \in G-(A \cup B)$, and

- $A, B \neq \{t,u,v,w\}$,
- if $\delta(A)$ (or $\delta(B)$) is a 5-edge-cut, then e_1 and either e_2 or e_2' (or f_1 and either f_2 or f_2') is in the cut, and
- if $\delta(A)$ (or $\delta(B)$) is a 6-edge-cut, then e_1 and e_2 (or f_1 and f_2) are in the cut.

Let $C = A \cap B - \{t,u,v,w\}$. Suppose that both e_1 and f_1 are in $\delta(A)$. If $e_2 \in \delta(A)$, then $\delta(C)$ is an internal 2-robust at most 6-edge-cut in G, a contradiction. Thus $|\delta(A)| = 5$ and $e_2 \notin \delta(A)$. Then $\delta(C)$ is an internal 2-robust at most 7-edge-cut in G. Since w is not adjacent to an internal degree 4 vertex, the graph obtained from G by contracting $G-C$ and deleting the two edges incident with w is an FDT graph. Thus $\delta(C)$ is reducible in the same way as a smaller internal edge-cut, a contradiction. We conclude that f_1 is not in $\delta(A)$. Similarly, e_1 is not in $\delta(B)$. It follows that $a \in B-A$ and the analogous neighbour of v is in $A-B$.

Suppose that $|\mathbf{B}-\mathbf{A}| = \mathbf{1}$. Then $B - A = \{a\}$. Suppose that $e_2 \in \delta(A)$. Then $b \in G-(A \cup B)$. Hence all other neighbours of a are in $G-(A \cup B)$. It follows that $\delta(C)$ is a 2-robust at most 4-edge-cut in G, a contradiction. We may now assume that $\delta(A)$ is a 5-edge-cut and $e_2' \in \delta(A)$. If the other endpoint of e_2' is in $G-(A \cup B)$, then apply the same argument. Hence e_2' is incident with a. Suppose that the remaining edges incident with a are in $\delta(B)$. If $f_2 \in \delta(B)$, then $\delta(C)$ is a 2-robust internal at most 6-edge-cut, a contradiction. Otherwise, $\delta(C)$ is a 2-robust internal at most 7-edge-cut. Again, it can be reduced following the same argument as for smaller internal edge-cuts. Thus a has an adjacent vertex in $A \cap B$. Since 3 edges incident with A are in $\delta(A)$, we conclude that $\delta(G-(A \cup B))$ is a 3-edge-cut, and therefore $G-(A \cup B) = \{d\}$.

Let G' be the graph obtained from G by orienting and deleting the e_1 and e_2 to satisfy $p(u)$, orienting the remaining two edges incident with a and b, contracting $\{d,a,b\}$ to a vertex d', and contracting $\{t,u,v,w\}$ to a vertex z. Then G has an oriented vertex of degree 5, and no vertex of degree 3. It is clear that G' contains no 2-robust at most

3-edge-cut, and thus G' has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. We conclude that $|B-A| \geq 2$. Similarly $|A-B| \geq 2$.

Suppose that $|\mathbf{A} - \mathbf{B}| \geq \mathbf{4}$. Then $\delta(A - B)$ is an at least 7-edge-cut, as it does not separate d from t. Similarly, $|\delta(B - A)| \geq 6$. Then

$$|\delta(A)| + |\delta(B)| \geq |\delta(A - B)| + |\delta(B - A)| \geq 13,$$

a contradiction.

We conclude that $\mathbf{4} > |\mathbf{A} - \mathbf{B}|, |\mathbf{B} - \mathbf{A}| \geq \mathbf{2}$. Then $\delta(A - B)$ and $\delta(B - A)$ are at least 6-edge-cuts, as they do not separate d from t. Then

$$|\delta(A)| + |\delta(B)| \geq |\delta(A - B)| + |\delta(B - A)| \geq 12,$$

so $\delta(A - B)$ and $\delta(B - A)$ are exactly 6-edge-cuts, and $[A \cap B : \overline{A \cup B}]$ is empty.

If $\delta(A - B)$ is of Type 1 or 2, then G has unoriented parallel edges incident with u. Thus $\delta(A - B)$ is of Type 3. The same is true of $\delta(B - A)$. If $[A - B : B - A]$ is non-empty, then the internal degree 4 vertices in $A - B$ and $B - A$ are on adjacent faces, a contradiction. Hence $[A - B : B - A]$ is empty. Neither internal degree 4 vertex is adjacent to w, else they are on adjacent faces. Thus $\delta((A \cap B) - \{t, u, v, w\})$ is an internal at most 6-edge-cut, the final contradiction.

We may therefore assume without loss of generality that there is no 2-robust at most 6-edge-cut $\delta(C)$ in G where

- $C \neq \{t, u, v, w\}$,
- C separates d from $\{t, u, v, w\}$,
- if $|\delta(C)| = 5$, then $e_1 \in \delta(C)$ and e_2 or e_2' is in $\delta(C)$, and
- if $|\delta(C)| = 6$, then $e_1, e_2 \in \delta(C)$.

Having established that such 5 and 6-edge-cuts do not exist, we return to our main argument. Let G_2 be the graph obtained from G by orienting and deleting ua and ub to satisfy the prescription of u, and contracting $\{t, u, v, w\}$. Then both a and b have degree 3 in G_2. Let

z be the vertex of contraction.

If b has a degree 4 neighbour x that is not a or u, then x is in the boundary of the outer face, by definition. We have $|A_b| + |B_b| = 8$. Suppose that $a \in A_b$. Then $\delta(B_b)$ is 2-robust and must therefore be of size at least 5. Then A_b contains a single degree 3 vertex, which must be d. But $a \in A_b$, a contradiction. Therefore, u and a are the only possible degree 4 neighbours of b.

We now prove the following claims:

e. Vertices a and b are adjacent.

Let m and n be the remaining vertices adjacent to a, where m is on the boundary of the outer face. Let p and q be the remaining vertices adjacent to b.

f. We have $m \neq d$, $deg(m) = 4$, $n = p$, and $deg(n = p) = 5$.

g. Vertices m and n are adjacent.

These will provide us with sufficient structure to complete the proof.

Claim FDT11e. *Vertices a and b are adjacent.*

Proof. Suppose that a and b are not adjacent. Let G_3 be the graph obtained from G_2 by deleting the edges incident with b. Then G_3 has one possible degree 3 vertex: a.

Suppose that G_3 contains a 2-robust at most 3-edge-cut $\delta_{G_3}(A)$, where $a, z \in A$. Then $\delta_G(A)$ or $\delta_G(A \cup \{b\})$ is a 2-robust internal at most 5-edge-cut, a contradiction. *Suppose that G_3 contains a 2-robust at most 2-edge-cut $\delta_{G_3}(A)$ where $a \in A$ and $z \notin A$.* At least two neighbours of b are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Then $\delta_G(\cup\{b\})$ is a 2-robust at most 5-edge-cut that uses e_1 and e_2, a contradiction.

Suppose that G_3 contains a 2-robust 3-edge-cut $\delta_{G_3}(A)$ where $a \in A$ and $z \notin A$. By Claim 5.2.2, $d \in A$. At least one neighbour of b is in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. At least two neighbours of b are in A, else $\delta_G(A)$ is a 2-robust at most 5-edge-cut using e_1 and e_2'. Then $\delta_G(A \cup \{b\})$ is a 2-robust at most 6-edge-cut using e_1 and e_2, a contradiction. Hence G_3 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Let m and n be the remaining vertices adjacent to a, where m is on the boundary of the outer face. Let p and q be the remaining vertices adjacent to b. Note that $deg(n), deg(p), deg(q) \geq 5$.

Claim FDT11f. *We have $m \neq d$, $deg(m) = 4$, $n = p$, and $deg(n = p) = 5$.*

Proof. If any of the following holds: $deg(m) \geq 5$, $m = d$ and $deg(d) = 4$, $n \neq p$, or $deg(n = p) \geq 6$, then the graph G_4 obtained from G_2 by orienting and deleting a and b has at most one vertex of degree 3.

Suppose that G_4 has a 2-robust at most 3-edge-cut $\delta_{G_4}(A)$ where $m, z \in A$. Then $\delta_G(A \cup \{b\})$ is an internal 2-robust at most 6-edge-cut that does not separate d from t. The argument that such a 6-edge-cut is 4-robust is similar to previous such arguments.

Suppose that G_4 has a 2-robust at most 3-edge-cut $\delta_{G_4}(A)$ where $z \in A$ and $m \notin A$. Then p and q are in A, else $\delta_G(A)$ is a 2-robust at most 6-edge-cut using e_1 and e_2. Similarly $n \in A$, else $\delta_G(A \cup \{b\})$ is a 2-robust at most 5-edge-cut using e_1 and e_2'. Then $\delta_G(A \cup \{a, b\})$ is a 2-robust 4-edge-cut, a contradiction. Hence no such cut exists. Then G_4 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction.

Now suppose that $m = d$ and $deg(d) = 3$. Let G' be the graph obtained from G_2 by orienting the remaining edges incident with a and b and contracting $\{d, a, b\}$ to a vertex d'. Then G' has an oriented vertex of degree 5 and no degree 3 vertex. If G' has a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, then $\delta_G(A)$ is a 2-robust 5-edge-cut in G using e_1 and e_2, a contradiction. □

Claim FDT11g. *Vertices m and n are adjacent.*

Proof. Suppose that m and n are not adjacent. Let j_1 and j_2 be the remaining edges incident with m that do not share a face with m. Let G_5 be the graph obtained from G_2 by lifting j_1 and j_2, and orienting and deleting m, a, and b. Then n is the only possible degree 3 vertex in G_5.

Suppose that G_5 has a 2-robust at most 2-edge-cut $\delta_{G_5}(A)$ where z and the lifted edge are in A. Then G has a 2-robust at most 6-edge-cut $\delta_G(A \cup \{a, b, m\})$. The analysis that such a 6-edge-cut must be 4-robust is analogous to previous such arguments. *Suppose* *that G_5 has a 2-robust at most 2-edge-cut $\delta_{G_5}(A)$ where $z \in A$ and the lifted edge is not in A.* Then $n = p$ and q are in A, else $\delta_G(A)$ is a 4-robust at most 6-edge-cut using e_1 and e_2, a contradiction. Then $\delta_G(A \cup \{a, b\})$ is a 2-robust at most 4-edge-cut, a contradiction.

Suppose that G_5 has a 2-robust 3-edge-cut $\delta_{G_5}(A)$ where z and the lifted edge are in A. Then by Claim 5.2.2, $n \in A$. Then G has an internal 2-robust at most 4-edge-cut $\delta_G(A \cup \{a, b, m\})$, a contradiction. *Suppose that G_5 has a 2-robust 3-edge-cut $\delta_{G_5}(A)$ where $z \in A$ and the lifted edge is not in A.* If $n = p \in A$, then by Claim 5.2.2, $d \in A$. Then $\delta_G(A \cup \{a, b\})$ is a 2-robust at most 5-edge-cut that does not separate d from t, a contradiction. We may assume that $n = p \notin A$. Then by Claim 5.2.2, $d \notin A$. Then $\delta_G(A)$ is a 2-robust at most 6-edge-cut using e_1 and e_2, a contradiction. Therefore, G_5 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

We now have sufficient information about the structure of the graph to complete the proof. Note that the only internal degree 4 neighbour of n is b. Suppose that n has another boundary neighbour x of degree 4. Then $|A_n| + |B_n| = 9$. We may assume that $|\delta(A_n)| \leq 4$. Hence A_n contains only a single vertex, which must be m. Then $x = d$ is adjacent to m via parallel edges, else G contains unoriented parallel edges. This is not possible. Hence we may assume that all other neighbours of n have degree at least 5.

Let G_6 be the graph obtained from G_2 by lifting j_1 and j_2, and orienting and deleting m, a, b, and n. Then q is the only possible degree 3 vertex in G_6.

Suppose that G_6 has a 2-robust at most 3-edge-cut $\delta_{G_6}(A)$ where z and the lifted edge are in A. Then G has an internal 2-robust at most 6-edge-cut, a contradiction. The argument that such a 6-edge-cut is 4-robust is analogous to previous such arguments.

Suppose that G_6 has a 2-robust at most 2-edge-cut $\delta_{G_6}(A)$ where $z \in A$ and the lifted edge is not in A. Then $q \in A$ and both neighbours of n are in A, else $\delta_G(A)$ is a 4-robust at most 6-edge-cut that uses e_1 and e_2. Hence $\delta_G(A \cup \{a, b, m, n\})$ is a 2-robust 4-edge-cut in G, a contradiction.

Suppose that G_6 has a 2-robust 3-edge-cut $\delta_{G_6}(A)$ where $z \in A$ and the lifted edge is not in A. By Claim 5.2.2, q has degree 3 in G_6. Suppose that $q \in A$. Then by Claim 5.2.2, $d \in A$. Then $\delta_G(A \cup \{a, b, m, n\})$ is a 2-robust at most 6-edge-cut in G that does not separate d from t. Since b is an internal degree 4 vertex, $G - A - \{a, b, m, n\}$ does not have an internal degree 4 vertex, and is therefore of Type 2. Thus G has unoriented parallel edges incident with m, a contradiction. Hence $q \notin A$. By Claim 5.2.2, $d \notin A$. Then $\delta_G(A)$ is a 2-robust 5-edge-cut in G using e_1 and e_2, a contradiction.

Therefore, G_6 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Suppose that x is adjacent to a degree 4 vertex z aside from u. Then z is in the boundary of the outer face by definition. We have $|\delta(A_x)| + |\delta(B_x)| = 8$, so at least one of $\delta(A_x)$ and $\delta(B_x)$ is an at most 4-edge-cut; suppose this cut is $\delta(A_x)$. By FDT3, $\delta(A_w)$ is not 2-robust. Let y be the vertex in A_w. Then y and u are adjacent via parallel edges, a contradiction. Thus x has no adjacent degree 4 vertex aside from u. An analogous argument shows that x is not adjacent to d.

FDT12. *Vertices x and w are adjacent.*

Proof. Suppose that x and w are not adjacent. Let G' be the graph obtained from G by lifting e_1 and e_2, and orienting and deleting u, t, and x. Then v is the only possible degree 3 vertex in G'.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$ where v and the lifted edge are in A. Then $\delta_G(A)$ or $\delta_G(A \cup \{u, t\})$ is an internal 2-robust at most 6-edge-cut, a contradiction. *Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$ where $v \in A$ and the lifted edge is not in A.* Then w and at least one vertex adjacent to x are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. At least one more neighbour of x is in A, else $\delta_G(A \cup \{t\})$ is a 2-robust at most 4-edge-cut. The remaining neighbour of x is in $G' - A$, else $\delta_G(A \cup \{t, u, x\})$ is a 2-robust at most 4-edge-cut. If $d \in A$, then $\delta_G(A \cup \{t, u, x\})$ is a 2-robust at most 5-edge-cut that does not separate d from t. Hence $d \notin A$. Then $\delta_G(A)$ is a 2-robust 6-edge-cut that does not separate d from t. Since x is an internal degree 4 vertex, A does not contain an internal degree 4 vertex, and thus has Type 2. Then G has unoriented parallel edges incident with t (we have $v = w$), a contradiction.

Suppose that G contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $v \in A$ and the lifted edge is not in A. Then by Claim 5.2.2, $d \in A$. We have $w \in A$, else $\delta_G(A)$ is a 2-robust 4-edge-cut. Similarly, x has three neighbours in A, else $\delta_G(A \cup \{t\})$ is a 4-robust at most 6-edge-cut that does not separate d from t. Then $\delta_G(G' - A)$ is a 2-robust 5-edge-cut that does not

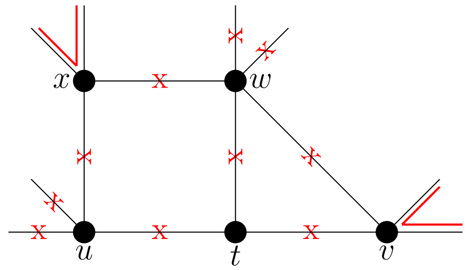


Figure 5.9: FTD13: Reduction when u and d are not adjacent.

separate d from t, a contradiction. Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

Note that x is the only internal degree 4 vertex adjacent to w.

FTD13. *Vertices u and d are adjacent.*

Proof. Suppose that u and d are not adjacent. Let G' be the graph obtained from G by lifting f_1 and f_2, lifting j_1 and j_2, and orienting and deleting v, t, x, w, and u. Then G' has only one possible degree 3 vertex: the endpoint z of e_1. This reduction can be seen in Figure 5.9.

Suppose that G' contains a 2-robust at most 3-edge-cut $\delta_{G'}(A)$, where z and the lifted edge (from f_1 and f_2) are both in A. The endpoints of j_1 and j_2 are in $G' - A$, else $\delta_G(G' - A)$ is an internal 2-robust at most 5-edge-cut. Similarly, w has a neighbour in $G' - A$, else $\delta_G(G' - A)$ is an internal 2-robust at most 6-edge-cut. Also, the endpoint of e_2 is in $G' - A$, else either $\delta_G(G' - A)$ or $\delta_G(A \cup \{t, u, v\})$ is an internal 2-robust at most 6-edge-cut.

Suppose that the remaining neighbour of w is in $G' - A$. Then $\delta_G((G' - A) \cup \{w, x\})$ is a 2-robust internal 7-edge-cut containing parallel edges incident with u. Contract $(G' - A) \cup \{w, x\}$ to a vertex in G calling the resulting graph G_1. Then G_1 is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G. Contract $A \cup \{t, u, v\}$ to a vertex in G, and deleting e_2, e_3, and tw, calling the resulting graph G_2. Then G_2 is has a directed vertex of degree 4 and one degree 3 vertex: x. If G_2 has a 2-robust at most 3-edge-cut, then G has an internal 2-robust at most 6-edge-cut, a contradiction. Thus G_2 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Thus the remaining neighbour of w is in A.

Then $\delta_G(G' - A)$ is a 2-robust 7-edge-cut in G. Contract $G' - A$ to a single vertex calling the resulting graph G_1. Then G_1 is an FDT graph and has a valid orientation by the minimality of G. Transfer this orientation to G, contract $A \cup \{t, u, v, w, x\}$ to a vertex, and delete the two edges incident with w, calling the resulting graph G_2. Then G_2 has no vertex of degree 3, and a degree 5 oriented vertex. If G_2 has a 2-robust at most 3-edge-cut, then G has an internal 2-robust at most 5-edge-cut, a contradiction. Hence G_2 is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence no such cut exists.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where $z \in A$ and the lifted edge (from f_1 and f_2) is not in A. Then the endpoints of e_2, j_1, and j_2 are in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Similarly, w has a neighbour in A, else $\delta_G(A \cup \{u, x\})$ is a 2-robust at most 4-edge-cut. The remaining neighbour of w is in $G' - A$, else $\delta_G(G' - A)$ is a 2-robust at most 4-edge-cut. Now $\delta_G(A \cup \{u, x\})$ and $\delta_G(A \cup \{u, x, t, w\})$ are 2-robust 5-edge-cuts. At most one separates d from t, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $z \in A$ and the lifted edge (from f_1 and f_2) is not in A. Then by Claim 5.2.2, $d \in A$. Both neighbours of w are in $G' - A$, else $\delta_G(G' - A)$ is a 2-robust at most 6-edge-cut that does not separate d from t. The argument that such a 6-edge-cut is 4-robust is analogous to previous such arguments. The endpoints of j_1 and j_2 are in $G' - A$, else $\delta_G(A \cup \{u, x, t\})$ is a 4-robust at most 6-edge-cut that does not separate d from t. The same is true of e_2 and $\delta_G(A \cup \{u, t\})$. Then $\delta_G(A)$ is a 2-robust 4-edge-cut, a contradiction. An analysis of these cuts is shown in Figure 5.10.

Hence G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. □

We now have enough of the structure of G to complete the proof. Suppose that d has degree 3. Then $\delta(\{d, u, t\})$ is a 4-robust 6-edge-cut, a contradiction. Hence we may assume that d has degree 4.

Let G' be the graph obtained from G by lifting f_1 and f_2, orienting v, t, and u, deleting vw, tw, and ux, contracting $\{d, u, t, v\}$, and orienting and deleting w and x. Then a possible common neighbour of x and w is the only degree 3 vertex in G'. This reduction can be

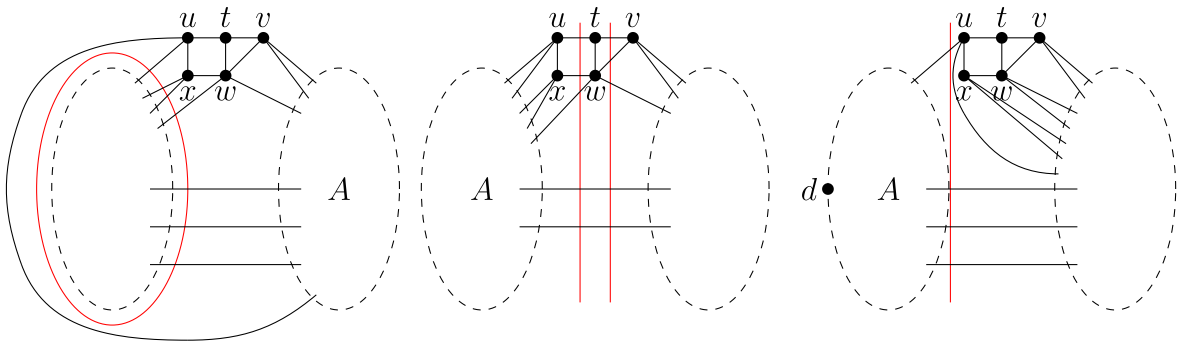


Figure 5.10: FTD13: Analysis of cuts.

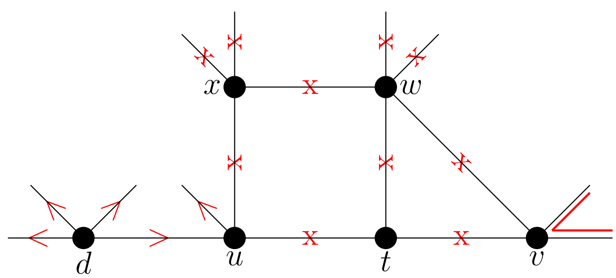


Figure 5.11: Reduction when u and d are adjacent.

seen in Figure 5.11.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where d and the lifted edge are both in A. Then G contains an internal 2-robust at most 6-edge-cut $\delta_G(G' - A)$, a contradiction.

Suppose that G' contains a 2-robust at most 2-edge-cut $\delta_{G'}(A)$, where $d \in A$ and the lifted edge is not in A. Then the vertices adjacent to w, and the endpoint of j_2 are in $G' - A$, else $\delta_G(G' - A)$ is a 2-robust at most 6-edge-cut that does not separate d from t. The argument that such a 6-edge-cut is 4-robust is analogous to previous such arguments. The endpoint of j_1 is in A, else $\delta_G(A)$ is a 2-robust at most 4-edge-cut. Then $\delta_G(A \cup \{u, x, t\})$ is a 4-robust 6-edge-cut that does not separate d from t, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where d and the lifted edge are both in A. Then by Claim 5.2.2 the common neighbour of x and w exists and is in A. Then G has an internal 2-robust at most 4-edge-cut $\delta_G(G' - A)$, a contradiction.

Suppose that G' contains a 2-robust 3-edge-cut $\delta_{G'}(A)$, where $d \in A$ and the lifted edge is not in A. Then by Claim 5.2.2 the common neighbour of x and w exists and is in A. Then $\delta_G(G' - A)$ is a 2-robust at most 6-edge-cut. Again, such a 6-edge-cut is necessarily 4-robust. Hence no such cut exists. Therefore G' is an FDT graph and has a valid orientation by the minimality of G. This leads to a valid orientation of G, a contradiction. Hence no such graph exists. □

5.3 Discussion

In this section we relate Theorem 5.0.2 to Jaeger's Strong 3-Flow Conjecture and the proof of Lai [18] that not all 4-edge-connected graphs have a nowhere zero 3-flow for all prescription functions. We also consider possible extensions of the result in Theorem 5.0.2.

Theorem 5.0.2 extends Theorem 3.3.3 to allow internal degree 4 vertices if they do not appear on adjacent faces. Corollary 5.3.1 is an immediate consequence.

Corollary 5.3.1. *Let G be a 4-edge-connected graph embedded in the plane where the internal 4-edge-cuts are non-crossing, and no two adjacent faces are incident with edges of distinct internal 4-edge-cuts. Suppose that G may have a pre-oriented vertex d of degree 4 or 5 on the boundary of the outer face. Then G is $\mathbb{Z}_3$-connected.*

Proof. Let G be a minimal counterexample with respect to the number of 2-robust 4-edge-cuts. If no 4-edge-cut in G is 2-robust, then G is an FDT graph and has a valid orientation by Theorem 5.0.2. Thus we may assume that G has a 2-robust 4-edge-cut $\delta(A)$. Assume that d, if it exists, is in $G - A$. If d doesn't exist, and $\delta(A)$ is an internal edge-cut, then choose A so that A is internal.

Let G' be the graph obtained from G by contracting A to a vertex v. Then v has degree 4. If v is on the boundary of the outer face, then it is clear that in G' the internal 4-edge-cuts are non-crossing, and no two adjacent faces are incident with edges of distinct internal 4-edge-cuts. If v is an internal vertex, it is incident in G' with each face that is incident with an edge of $\delta(A)$ in G. Therefore, v is not on a face adjacent to a face containing an edge of any other internal 4-edge-cut in G. By the minimality of G, G' has a valid orientation.

Transfer this orientation to G and let G'' be the graph obtained from G by contracting $G - A$ to a vertex d'. Then G'' has one pre-oriented vertex, d', which is on the boundary of the outer face and has degree 4. As for G', it is clear that G'' has all internal 4-edge-cuts non-crossing, and no two adjacent faces incident with edges of distinct internal 4-edge-cuts. Hence by the minimality of G, G'' has a valid orientation. This leads to a valid orientation of G, a contradiction. □

While Lai [18] showed that Jaeger's Strong 3-Flow Conjecture cannot be extended to allow all 4-edge-connected graphs, Corollary 5.3.1 shows that there are graphs with arbitrarily many 4-edge-cuts that have valid orientations. This is a step toward answering the question we posed in Section 3.4; which 4-edge-connected graphs have a modulo 3 orientation for any valid prescription function?

We briefly discuss two possible extensions to this result. The first is that, instead of excluding internal degree 4 vertices from being on the boundary of adjacent faces, we instead exclude them from being on the boundary of the same face. In this case, internal degree 4 vertices would still not be allowed to be adjacent, but an internal degree 5 vertex may have up to two internal degree 4 neighbours. The limitation that an internal degree 5 vertex could only have one internal degree 4 neighbour was one we made significant use of in the proof of Theorem 5.0.2.

It may also be possible to extend this result to simply restrict the distance between internal degree 4 vertices. The difficulty here is that contraction of internal edge-cuts (or parallel edges) does not preserve the distance between internal degree 4 vertices, unless the requirement is only that they are at least distance two apart. This restriction would allow internal vertices of degree at least 5 to have all their neighbours be internal degree 4 vertices, and it is not clear how the reductions in the proof of Theorem 5.0.2 could be modified to allow this. If we only permit internal degree 4 vertices at least distance 3 apart, then new techniques will be required to account for the fact that we cannot easily reduce small internal edge-cuts.

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