ABELIAN, AMENABLE OPERATOR ALGEBRAS ARE SIMILAR TO C^*-ALGEBRAS

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Abstract. Suppose that H is a complex Hilbert space and that $\mathcal{B}(H)$ denotes the bounded linear operators on H. We show that every abelian, amenable operator algebra is similar to a C^*-algebra. We do this by showing that if $\mathcal{A} \subseteq \mathcal{B}(H)$ is an abelian algebra with the property that given any bounded representation $\varrho : \mathcal{A} \to \mathcal{B}(H_\varrho)$ of $\mathcal{A}$ on a Hilbert space H_ϱ, every invariant subspace of $\varrho(\mathcal{A})$ is topologically complemented by another invariant subspace of $\varrho(\mathcal{A})$, then $\mathcal{A}$ is similar to an abelian C^*-algebra.

1. Introduction.

1.1. Let $\mathcal{A}$ be a Banach algebra and X be a Banach space which is also a bimodule over $\mathcal{A}$. We say that X is a **Banach bimodule** over $\mathcal{A}$ if the module operations are continuous; that is, if there exists $\kappa > 0$ so that $\|ax\| \leqslant \kappa\|a\|\,\|x\|$, and $\|xb\| \leqslant \kappa\|x\|\,\|b\|$ for all $a, b \in \mathcal{A}$ and $x \in X$.

Given a Banach bimodule X over $\mathcal{A}$, we introduce an action of $\mathcal{A}$ upon the dual space X^* of X under which X^* becomes a **dual Banach $\mathcal{A}$-bimodule**. This is the so-called **dual action**:

$$(ax^*)(x) = x^*(xa) \qquad \text{and} \qquad (x^*a)(x) = x^*(ax)$$

for all $a \in \mathcal{A}$, $x \in X$, $x^* \in X^*$.

A (continuous) **derivation** from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is a continuous linear map $\delta : \mathcal{A} \to X$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. For any fixed $z \in X$, the map $\delta_z : \mathcal{A} \to X$ defined by $\delta_z(a) = az - za$ is a derivation with $\|\delta_z\| \leqslant 2\|z\|$. Derivations of this type are said to be **inner**, and the algebra $\mathcal{A}$ is said to be **amenable** if every continuous derivation of $\mathcal{A}$ into a dual Banach bimodule X is inner.

The notion of amenability of Banach algebras was introduced by B. Johnson in his 1972 monograph [18]. He showed that a locally compact topological group G is amenable as a group - that is, G admits a left translation-invariant mean - if and only if the corresponding group algebra $(L^1(G), \|\cdot\|_1)$ is amenable as a Banach algebra. It is a standard and relatively straightforward exercise to show that if $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range, and if $\mathcal{A}$ is amenable, then $\mathcal{B}$ is amenable also.

For C^*-algebras acting on a Hilbert space, the notion of amenability coincides with that of *nuclearity*. A C^*-algebra $\mathcal{B}$ is said to be **nuclear** if there exists a directed set Λ and two families $\varphi_\lambda : \mathcal{B} \to \mathbb{M}_{k(\lambda)}(\mathbb{C})$ and $\psi_\lambda : \mathbb{M}_{k(\lambda)}(\mathbb{C}) \to \mathcal{B}$, $\lambda \in \Lambda$ of completely positive contractions, where $k(\lambda) \in \mathbb{N}$ for all $\lambda \in \Lambda$, so that

$$\lim_\lambda \|\psi_\lambda \circ \varphi_\lambda(b) - b\| = 0 \text{ for all } b \in \mathcal{B}.$$

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It was shown by A. Connes [6] and by E. Effros and E. Lance [9] that every amenable C^*-algebra is nuclear, while the converse - namely that every nuclear C^*-algebra is amenable - was established by U. Haagerup [16].

Let H be a complex Hilbert space and denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators acting on H. It follows from our observation above that if $\mathcal{D}$ is a nuclear C^*-algebra and if $\varrho : \mathcal{D} \to \mathcal{B}(H)$ is a continuous representation of $\mathcal{D}$, then $\overline{\varrho(\mathcal{D})}$ is an amenable algebra of operators in $\mathcal{B}(H)$. It is also known that any abelian C^*-algebra is nuclear (cf. [2], Proposition 2.4.2), as is the algebra $\mathcal{K}(H)$ of compact operators on H (cf. [2], Proposition 2.4.1). In 1955, R.V. Kadison raised the following question, now known as **Kadison's Similarity Problem** [19]: Let $\mathcal{D}$ be a C^*-algebra, and suppose that $\varrho : \mathcal{D} \to \mathcal{B}(H_\varrho)$ is a continuous representation of $\mathcal{D}$ on some Hilbert space H_ϱ. For $S \in \mathcal{B}(H)$ invertible, denote by $\mathrm{Ad}_S : \mathcal{B}(H) \to \mathcal{B}(H)$ the map $\mathrm{Ad}_S(X) = S^{-1}XS$. Does there exist an invertible operator $S \in \mathcal{B}(H_\varrho)$ so that $\tau := \mathrm{Ad}_S \circ \varrho$ is a *-homomorphism of $\mathcal{D}$?

While the problem in this generality remains unsolved, it has been shown by E. Christensen [5] to admit a positive answer whenever $\mathcal{D}$ is irreducible (i.e. $\mathcal{D}$ admits no invariant subspaces) and when $\mathcal{D}$ is nuclear. In particular, therefore, it holds when $\mathcal{A}$ is abelian. Haagerup [15] showed that if $\mathcal{D}$ admits a **cyclic** vector, (i.e. there exists $x \in H$ so that $H = \overline{\mathcal{D}x}$, then again, every continuous representation of $\mathcal{D}$ is similar to a *-representation.

It follows from Christensen's work that if a closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is a homomorphic image of an abelian C^*-algebra, then $\mathcal{A}$ is necessarily amenable (and abelian), and that $\mathcal{A}$ is similar to a C^*-algebra.

The converse problem is the following:

> **Question A.** Is every amenable algebra of Hilbert space operators a continuous, homomorphic image of (and hence similar to) a nuclear C^*-algebra?

This problem has circulated since the 1980s. It has been ascribed to Pisier, to Curtis and Loy, to Šeĭnberg, and to Helemskii, amongst others. For certain special classes of algebras, the question has been answered affirmatively.

Observe that if an amenable algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is similar to a C^*-algebra, then it must necessarily be semisimple. In that regard, it is interesting to note that C.J. Read [28] has constructed an example of an abelian, radical, amenable Banach algebra. As a consequence of Corollary 3.3 below, the only continuous representation of Read's algebra on a Hilbert space is the trivial representation. Thus ours is very much a result about amenable, abelian operator algebras, as opposed to amenable, abelian Banach algebras.

The first positive result with respect to Question A is due to M.V. Šeĭnberg [32]:

1.2. Theorem. [M.V. Šeĭnberg] *If Ω is a compact Hausdorff space and $\mathcal{A} \subseteq \mathcal{C}(\Omega)$ is an amenable, uniform algebra that separates points, then $\mathcal{A} = \mathcal{C}(\Omega)$.*

For $T \in \mathcal{B}(H)$, we denote by $\mathcal{A}_T$ the norm-closed unital subalgebra of $\mathcal{B}(H)$ generated by T.

1.3. Theorem. [G. Willis] [34] *Let $K \in \mathcal{K}(H)$. If $\mathcal{A}_K$ is amenable, then K is similar to a diagonal operator.*

The norm-closed algebra generated by a compact diagonal operator is self-adjoint. As such, an immediate corollary to this Theorem is that if $K \in \mathcal{K}(H)$ and $\mathcal{A}_K$ is amenable, then $\mathcal{A}_K$ is similar to a C^*-algebra.

P.C. Curtis and R.J. Loy [7] have proven that if $\mathcal{A} \subseteq \mathcal{B}(H)$ is amenable and generated by its normal elements, then $\mathcal{A} = \mathcal{A}^*$ is a C^*-algebra.

In [10, 11], D. Farenick, B.E. Forrest and the first author showed that if $T \in \mathcal{B}(H)$ generates an amenable algebra $\mathcal{A}_T$, and if H admits an orthonormal basis $\{e_n\}_{n=1}^\infty$ under which the matrix $[T] := [t_{ij}] = [\langle Te_j, e_i\rangle]$ is upper triangular, then again, T is similar to a normal operator N with **Lavrientieff** spectrum. That is, the spectrum $\sigma(T)$ of T does not have interior, and it does not disconnect the complex plane. As was shown by Lavrentieff [21], this is precisely the property of the spectrum needed to ensure that the algebra of polynomials on $\sigma(T)$ is dense in the space of continuous functions on $\sigma(T)$ with respect to the uniform norm, which implies that the algebra $\mathcal{A}_N$ generated by N is a C^*-algebra, and hence that $\mathcal{A}_T$ is similar to $C^*(N)$.

More recently, Y. Choi [3] has shown (amongst other things) that if $\mathcal{A}$ is a closed, commutative amenable subalgebra of a finite von Neumann algebra $\mathcal{M}$, then $\mathcal{A}$ must be similar to a C^*-algebra. We also mention in passing that N.C. Phillips [24] has constructed certain "L^2 UHF algebras of tensor product type" for which he has shown that amenability implies similarity to a C^*-algebra. His examples, while quite specific, are of a rather different nature from previous examples.

In a recent paper of Y. Choi, I. Farah, and N. Ozawa [4], Question A above has finally been resolved (in the negative). There, the authors construct an ingenious example of a nonseparable and nonabelian amenable subalgebra of $\ell_\infty(\mathbb{N}, \mathbb{M}_2(\mathbb{C}))$ which is not isomorphic to a nuclear C^*-algebra. As they point out, their counterexample is "inevitably nonseparable", and as we shall see, "inevitably nonabelian". The existence or nonexistence of a separable, amenable operator algebra which is not similar to a C^*-algebra remains an open problem.

1.4. The current work is motivated by this problem in the case where the algebra in question is *abelian*. Our main result is Theorem 2.10, which states that

> every *abelian*, amenable operator algebra is similar to a (necessarily abelian, hence nuclear) C^*-algebra.

This result stands in stark contrast to the counterexample of Choi, Farah and Ozawa mentioned above. One should also note that the counterexample is, in a sense, not too far from being commutative. In fact, an examination of the example shows that the quotient of the algebra in it with respect to compact operators is commutative. Our approach, however, takes us away from the notion of amenability proper, and is heavily influenced by the remarkable thesis of J.A. Gifford [13] and his subsequent paper [14].

A particularly useful device in studying an operator algebra $\mathcal{A}$ (i.e. a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H) is to examine its lattice of closed invariant subspaces, $\operatorname{Lat}\mathcal{A}$. It is elementary to see that the lattice $\operatorname{Lat}\mathcal{D}$ of a C^*-algebra $\mathcal{D} \subseteq \mathcal{B}(H)$ has the property that if $M \in \operatorname{Lat}\mathcal{D}$, then $M^\perp \in \operatorname{Lat}\mathcal{D}$; in other words, every element of $\operatorname{Lat}\mathcal{D}$ is orthogonally complemented. We shall write $H = M \oplus M^\perp$ to denote the *orthogonal* direct sum of the subspace M and of $M^\perp$. Given two closed subspaces V and W of H, we shall reserve the notation $H = V \dot{+} W$ to mean that V and W are **topological complements** in H; that is, $H = V + W$, while $V \cap W = \{0\}$.

Suppose now that $\mathcal{D}$ is a nuclear C^*-algebra, that $\varrho : \mathcal{D} \to \mathcal{B}(H_\varrho)$ is a continuous representation of $\mathcal{B}$ and that $\mathcal{A} := \overline{\varrho(\mathcal{D})}$. By Christensen's Theorem [5], there exists an invertible operator $S \in \mathcal{B}(H_\varrho)$ so that $\tau := \mathrm{Ad}_S \circ \varrho$ is a *-homomorphism. From this it follows that the range of ϱ is closed and that $\mathcal{B} := \tau(\mathcal{D}) = S^{-1}\mathcal{A}S$ is a C^*-algebra. A quick calculation shows that $\operatorname{Lat}\mathcal{A} = S^{-1}\operatorname{Lat}\mathcal{B}$.

As such, given $M \in \operatorname{Lat}\mathcal{A}$, we have that $SM \in \operatorname{Lat}\mathcal{B}$, and thus $(SM)^\perp \in \operatorname{Lat}\mathcal{B}$. But then $H = S^{-1}H = S^{-1}((SM) \oplus (SM)^\perp) = M \dot{+} S^{-1}(SM)^\perp$ shows that M is topologically complemented in $\operatorname{Lat}\mathcal{A}$ by the element $S^{-1}(SM)^\perp$ of $\operatorname{Lat}\mathcal{A}$.

We say that an operator algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ has the **reduction property** if every element of its invariant subspace lattice $\operatorname{Lat}\mathcal{A}$ is topologically complemented in $\operatorname{Lat}\mathcal{A}$. The above argument shows that if $\mathcal{A}$ is the homomorphic image of a nuclear C^*-algebra, or more generally if $\mathcal{A}$ is similar to a C^*-algebra, then $\mathcal{A}$ has the reduction property.

That the lattice of invariant subspaces of an operator algebra being complemented reveals a great deal of structure about the algebra and its generators has been the theme of more than one paper. For example, C.K. Fong [12] closely examined the relationship between the reduction property of an operator algebra $\mathcal{A}$ and the boundedness of certain graph transformations for $\mathcal{A}$. Later, S. Rosenoer [29, 30] showed amongst other things that if $T \in \mathcal{B}(H)$ is an operator for which $\mathcal{A}_T$ has the reduction property, and if T commutes with an injective compact operator with dense range, then T is similar to a normal operator. Furthermore, he showed that every unital, strongly closed operator algebra $\mathcal{A}$ with the reduction property and with the property that the ranges of the compact operators in $\mathcal{A}$ span the underlying Hilbert space is reflexive: that is, $\mathcal{A}$ coincides with the algebra $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ of all operators on H which leave invariant each element of $\operatorname{Lat}\mathcal{A}$. (Both Fong's and Rosenoer's results are actually stated for operators on a Banach space - we shall not require those results here.)

In his thesis [13] (alternatively, see [14]), J.A. Gifford defined a stronger version of the reduction property which he refers to as the *total reduction property*:

1.5. Definition. *Let $\mathcal{A}$ be a Banach algebra of operators acting on a Hilbert space H. We say that $\mathcal{A}$ has the* ***total reduction property (TRP)*** *if, for every continuous representation $\varrho : \mathcal{A} \to \mathcal{B}(H_\varrho)$ of $\mathcal{A}$ as bounded linear operators on a Hilbert space H_ϱ, we have that the operator algebra $\overline{\varrho(\mathcal{A})}$ has the reduction property as a subalgebra of $\mathcal{B}(H_\varrho)$.*

Following [10], we shall say that an operator T has the **total reduction property** if $\mathcal{A}_T$ does.

Insofar as we are concerned, a particularly attractive relationship exists between the total reduction property and amenability. The following theorem, which is a special case of Helemskii's splitting theorem (see [17]; see also [8] and [31, Theorem 2.3.13]), was first proved by Šeinberg in [32] and later reproved by Gifford in his thesis [13].

1.6. Theorem. [Šeinberg] [32] *If $\mathcal{A} \subseteq \mathcal{B}(H)$ is an amenable Banach algebra of operators on a Hilbert space H, then $\mathcal{A}$ has the total reduction property.*

Armed with this notion, Gifford obtained a far-reaching and beautiful generalization of Willis's result.

1.7. Theorem. [J.A. Gifford] [14] *If $\mathcal{A} \subseteq \mathcal{K}(H)$ is a subalgebra of compact operators, then $\mathcal{A}$ has the total reduction property if and only if $\mathcal{A}$ is similar to a C^*-algebra. As a consequence, every amenable subalgebra of $\mathcal{K}(H)$ is similar to a C^*-algebra.*

In fact, Gifford proved this result under a slightly weaker hypothesis for $\mathcal{A}$, namely that $\mathcal{A}$ has the *complete reduction property*, which is the statement that the algebra $\mathcal{A}^{(\infty)} := \{A \oplus A \oplus \cdots : A \in \mathcal{A}\} \subseteq \mathcal{B}(H^{(\infty)})$ has the reduction property.

Suppose that an abelian algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is similar to a C^*-algebra $\mathcal{D}$, say $\mathcal{A} = S^{-1}\mathcal{D}S$ for some invertible operator $S \in \mathcal{B}(H)$. Let $\varrho : \mathcal{A} \to \mathcal{B}(H_\varrho)$ be a (continuous) representation of $\mathcal{A}$. Then $\tau : \mathcal{D} \to \mathcal{B}(H_\varrho)$ defined by $\tau(D) = \varrho(S^{-1}DS)$ defines a continuous representation of $\mathcal{D}$. The argument of Section 1.4 above shows that the lattice $\operatorname{Lat}\overline{\tau(D)} = \operatorname{Lat}\overline{\varrho(A)}$ is topologically complemented, and thus $\mathcal{A}$ has the TRP.

Our main result, Theorem 2.10 establishes the converse: if $\mathcal{A} \subseteq \mathcal{B}(H)$ is an abelian Banach algebra which has the TRP, then $\mathcal{A}$ is similar to a C^*-algebra. In particular, this confirms a conjecture of Gifford [14] in the abelian setting.

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2. The main result.

2.1. Our ultimate goal is to show that if an abelian operator algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ has the total reduction property, and if $\Sigma_{\mathcal{A}}$ denotes the maximal ideal space of $\mathcal{A}$, then the Gelfand Transform $\Gamma : \mathcal{A} \to \mathcal{C}(\Sigma_{\mathcal{A}})$ is a topological isomorphism. This approach is motivated by the following.

In his thesis, J.A. Gifford provides the following analogue of Šeĭnberg's Theorem 1.2 for total reduction algebras (part (a) below). As he mentions there, his proof owes much to the original. (Note that part (b) below also follows part (a), combined with the previously mentioned result of Christensen.)

2.2. Theorem. [J.A. Gifford] [13] *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an abelian, total reduction algebra.*

- (a) *If $\mathcal{A}$ is contained in an abelian C^*-algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is self-adjoint.*
- (b) *If $\mathcal{A}$ is isomorphic to a closed subalgebra of an abelian C^*-algebra, then $\mathcal{A}$ is similar to a C^*-algebra.*

The next result, again due to Gifford, shows that operator algebras $\mathcal{A}$ with the total reduction property have a very rigid invariant subspace lattice *under any continuous representation.* Following the terminology in [14], we refer to idempotents in $\mathcal{B}(H)$ as *projections*, and we refer to self-adjoint projections as *orthogonal projections.*

2.3. Theorem. [J.A. Gifford] Lemma 1.7 [14] *Let $\mathcal{A}$ be an operator algebra with the total reduction property. Then there exists an increasing function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\theta : \mathcal{A} \to \mathcal{B}(H_\theta)$ is a continuous representation of $\mathcal{A}$ and if $M \subseteq H_\theta$ is an invariant subspace for $\theta(\mathcal{A})$, then there exists a projection $E \in (\theta(\mathcal{A}))' = \{T \in \mathcal{B}(H_\theta) : \theta(A)T = T\theta(A)$ for all $A \in \mathcal{A}\}$ such that $\operatorname{ran} E = M$ and $\|E\| \leqslant \kappa(\|\theta\|)$.*

Note: For the sake of convenience below, we may and do assume that $\kappa(t) > 1$ for all $t \geqslant 0$.

Upon fixing a representation $\theta : \mathcal{A} \to \mathcal{B}(H_\theta)$, the corresponding real number $\kappa(\|\theta\|)$ is referred to as the **projection constant** for the representation θ (or the projection constant for $\overline{\theta(\mathcal{A})}$). Our strategy is to show that the projection constant imposes a fixed bound on the norm of T in terms of the norm of T^2 for all $T \in \mathcal{A}$, which we then show to be precisely the result required to prove that the spectral radius on $\mathcal{A}$ is a norm on $\mathcal{A}$ which is equivalent to the operator norm.

2.4. The following proposition is motivated by results of Arveson [1]. Recall that if $\mathcal{S} \subseteq \mathcal{B}(H)$ is a non-empty set, then $\mathcal{S}^{(2)} = \{S \oplus S : S \in \mathcal{S}\} \subseteq \mathcal{B}(H^{(2)}) = \mathcal{B}(H \oplus H)$. By a **linear manifold** in a Hilbert space H, we mean a vector subspace L of H which need not be closed in the norm topology on H.

2.5. Proposition. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an algebra with the total reduction property. Let $\kappa(\cdot)$ denote the projection function for $\mathcal{A}$, and let $\kappa := \kappa(1)$. If $\mathcal{N} \in \operatorname{Lat} \mathcal{A}^{(2)}$, then there exist $Y \in \operatorname{Lat} \mathcal{A}$, an $\mathcal{A}$-invariant linear manifold $L \subseteq H$, and a closed linear map $R : L \to H$ satisfying $RTz = TRz$ for all $T \in \mathcal{A}$ and $z \in L$ such that*

$$\mathcal{N} = (0 \oplus Y) \dot{+} \{(z, Rz) : z \in L\}.$$

Moreover, the projection P_Y of $\mathcal{N}$ onto $0 \oplus Y$ along $\{(z, Rz) : z \in L\}$ has norm at most κ.

Proof. Consider $\theta : \mathcal{A} \to \mathcal{B}(\mathcal{N})$ defined by $\theta(T) = (T \oplus T)|_{\mathcal{N}}$. Then θ is a representation of $\mathcal{A}$ satisfying $\|\theta\| \leqslant 1$. Let $Y = \{y \in \mathcal{H} : (0, y) \in \mathcal{N}\}$, so that $0 \oplus Y = \mathcal{N} \cap (0 \oplus H)$. Since $0 \oplus Y \in \operatorname{Lat} \theta(\mathcal{A})$, we have that $Y \in \operatorname{Lat} \mathcal{A}$. It follows from Theorem 2.3 that there is a projection $P_Y = P_Y^2 \in (\theta(\mathcal{A}))'$ such that $P_Y \mathcal{N} = (0 \oplus Y)$ and $\|P_Y\| \leqslant \kappa$.

Let $\mathcal{N}_0 = \ker P_Y$, and observe that $\mathcal{N}_0 \in \operatorname{Lat} \theta(\mathcal{A})$. Furthermore, $\mathcal{N} = \operatorname{ran} P_Y \dot{+} \ker P_Y = (0 \oplus Y) \dot{+} \mathcal{N}_0$. Define

$$L = \{x \in H : (x, y) \in \mathcal{N}_0 \text{ for some } y \in H\}.$$

We claim that for each $x \in L$, there is a unique $y \in H$ such that $(x, y) \in \mathcal{N}_0$. Indeed, if $y_1, y_2 \in H$ are such that

$$(x, y_1) \quad \text{and} \quad (x, y_2) \in \mathcal{N}_0,$$

then

$$(x, y_1) - (x, y_2) = (0, y_1 - y_2) \in \mathcal{N}_0.$$

However, from the definition of Y, we also have that $(0, y_1 - y_2) \in (0 \oplus Y)$. Since $(0 \oplus Y) \cap \mathcal{N}_0 = \{0\}$, we find that $y_1 = y_2$.

It follows that we can define a map $R : L \to H$ by letting Rx be equal to the unique $y \in H$ for which $(x, y) \in \mathcal{N}_0$. It is routine to verify that R is a linear map. By the definition of L, we get

$$\mathcal{N}_0 = \{(x, Rx) : x \in L\},$$

and since $\mathcal{N}_0$ is closed as a subspace of $\mathcal{N}$, R is closed as a linear map. Finally, if $x \in L$ and $T \in \mathcal{A}$, it follows from the fact that $\mathcal{N}_0$ is $\mathcal{A}$-invariant that

$$(Tx, TRx) \in \mathcal{N}_0.$$

Since $Tx \in L$ and RTx is the unique element of H so that $(Tx, RTx) \in \mathcal{N}_0$, we may conclude that $TRx = RTx$. □

The next result provides the key estimate we shall require to prove our main theorem.

2.6. Theorem. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an abelian operator algebra with the total reduction property. Then there exists $\mu > 0$ so that for all $S \in \mathcal{A}$,*

$$\|S\|^2 \leqslant \mu \, \|S^2\|.$$

Proof. As before, we denote Gifford's projection function by $\kappa(\cdot)$, and we let $\kappa := \kappa(1)$. Let $S \in \mathcal{A}$ be such that $\|S^2\| \leqslant 1$. We will prove that $\|S\| \leqslant 8\kappa^4$.

Define

$$\mathcal{M} = \{(h, Sh) \, : \, h \in H\}.$$

Since S is continuous, $\mathcal{M}$ is a closed subspace of $H^{(2)}$, being the graph of S. Since $\mathcal{A}$ is abelian, $\mathcal{M} \in \operatorname{Lat} \mathcal{A}^{(2)}$. By the total reduction property, there exists a projection $P \in (\mathcal{A}^{(2)})'$ so that $PH^{(2)} = \mathcal{M}$ and $\|P\| \leqslant \kappa$. Let $\mathcal{N} := \ker P \in \operatorname{Lat} \mathcal{A}^{(2)}$. Then $H^{(2)} = \mathcal{M} \dotplus \mathcal{N}$.

By Proposition 2.5, $\mathcal{N}$ decomposes into a topological direct sum of $\mathcal{A}^{(2)}$-invariant subspaces as

$$\mathcal{N} = (0 \oplus Y) \dotplus \{(z, Rz) \, : \, z \in L\},$$

where Y, L and R are as described in that Proposition. Moreover, the projection P_Y of $\mathcal{N}$ onto $0 \oplus Y$ along $\{(z, Rz) : z \in L\}$ corresponding to this decomposition is of norm at most κ. Thus $H^{(2)}$ decomposes into a topological direct sum of $\mathcal{A}^{(2)}$-invariant subspaces as

$$H^{(2)} = \mathcal{M} \dotplus (0 \oplus Y) \dotplus \{(z, Rz) \, : \, z \in L\}.$$

Take $x \in H$ such that $\|x\| = 1$ and write

$$(x, 0) = (h, Sh) + (0, y) + (z, Rz)$$

according to the decomposition of $H^{(2)}$ above. Since $\|P\| \leqslant \kappa$, we have

$$\|h\| \leqslant \kappa$$

and

$$\|Sh\| \leqslant \kappa.$$

Also, $\|I - P\| \leqslant \kappa + 1 \leqslant 2\kappa$ and $\|P_Y\| \leqslant \kappa$, hence

$$\|y\| \leqslant 2\kappa^2.$$

Again, as $\|I - P_Y\| \leqslant \kappa + 1 \leqslant 2\kappa$, we get

$$\|z\| \leqslant 4\kappa^2.$$

Next, write

$$(y, 0) = (y, Sy) + (0, -Sy).$$

By the same reasoning as above, we get $\|Sy\| \leqslant \|(y, Sy)\| \leqslant \kappa\|y\| \leqslant 2\kappa^3$. Hence

$$\|SRz\| = \|-S^2h - Sy\| \leqslant \|S^2\|\|h\| + \|Sy\| \leqslant \kappa + 2\kappa^3 \leqslant 3\kappa^3.$$

Now, for any $w \in \operatorname{dom}(R)$ we can write

$$(0, (S - R)w) = (w, Sw) + (-w, -Rw),$$

hence getting $\|w\| \leqslant \kappa\|(S - R)w\|$. Recall now that $z \in L = \operatorname{dom}(R)$, and that L is S-invariant. So, $Sz \in \operatorname{dom}(R)$, and we can use the above inequality with $w = Sz$. Using also the fact that R commutes with S on L, we obtain

$$\|Sz\| \leqslant \kappa\|(S - R)Sz\| \leqslant \kappa(\|S^2\|\|z\| + \|SRz\|) \leqslant \kappa(4\kappa^2 + 3\kappa^3) \leqslant 7\kappa^4.$$

It follows that

$$\|Sx\| = \|Sh + Sz\| \leqslant \|Sh\| + \|Sz\| \leqslant \kappa + 7\kappa^4 \leqslant 8\kappa^4.$$

This proves the theorem. □

2.7. Remarks.

(a) A careful examination of the proof of Theorem 2.6 shows that the only place where we used the fact that the algebra $\mathcal{A}$ is abelian was to conclude that the space $\mathcal{M} := \{(h, Sh) : h \in H\}$ is invariant for $\mathcal{A}$. For this, however, it is sufficient that S lie in the centre $\mathcal{Z}(\mathcal{A}) := \{Z \in \mathcal{A} : ZA = AZ \text{ for all } A \in \mathcal{A}\}$ of $\mathcal{A}$. Thus, even if $\mathcal{A}$ is not abelian, so long as it has the total reduction property, the proof of Theorem 2.6 asserts the existence of a universal constant $\mu > 0$ so that if $S \in \mathcal{Z}(\mathcal{A})$, then $\|S\|^2 \leqslant \mu \|S^2\|$.

Now suppose that $\mathcal{A}$ is a non-abelian, amenable operator algebra and that $0 \neq T$ lies both in $\mathcal{Z}(\mathcal{A})$ and in the Jacobson radical of $\mathcal{A}$. By virtue of the fact that T is quasinilpotent, given $\varepsilon > 0$, there exists some $n \geqslant 1$ so that $\|T^{2^{n+1}}\| < \varepsilon \|T^{2n}\|^2$. But then with $S = T^{2^n} \in \mathcal{Z}(\mathcal{A})$, we see that $\|S^2\| < \varepsilon \|S\|^2$. Since $\varepsilon > 0$ is arbitrary, this leads to a contradiction.

The conclusion is that if $\mathcal{A}$ is an amenable operator algebra, then the intersection of the centre of $\mathcal{A}$ with the radical of $\mathcal{A}$ is $\{0\}$. In the case where $\mathcal{A}$ is abelian, this is the statement that $\mathcal{A}$ is semisimple. But as we shall now see, in the abelian case, much more is true.

(b) The only place where the total reduction property was used in Theorem 2.6 is the application of Proposition 2.5. This proposition can, in fact, be established under the weaker assumption of the complete reduction property. Indeed, the total reduction property was used in Proposition 2.5 to get a uniform bound on the norm of projections corresponding to invariant subspaces of the algebra $\mathcal{A}^{(2)}$. The existence of such a bound follows from the complete reduction property, see [13, Lemma 1.5]. The constant κ in Proposition 2.5 can in this case be replaced with any number $\alpha > 1$ such that, given an invariant subspace $\mathcal{M} \in \operatorname{Lat} \mathcal{A}^{(2)}$, there is a projection $E \in (\mathcal{A}^{(2)})'$ with range $\mathcal{M}$ and $\|E\| \leqslant \alpha$.

We remark, however, that the main result where Theorem 2.6 will be used, Theorem 2.10, will require the full power of the total reduction property.

Recall now the following standard fact about commutative Banach algebras (see, e.g., [20, Theorem VIII.3.7]).

2.8. Proposition. *Let $(\mathcal{A}, \|\cdot\|)$ be an abelian Banach algebra. A necessary and sufficient condition for the norm $\|\cdot\|$ of $\mathcal{A}$ to be equivalent to the spectral radius function* $\operatorname{spr}(\cdot)$ *is the existence of a constant μ such that*

$$\|x\|^2 \leqslant \mu \, \|x^2\| \quad \text{for all } x \in \mathcal{A}.$$

2.9. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an abelian algebra with the total reduction property. Recall that $\Gamma : \mathcal{A} \to \mathcal{C}(\Sigma_{\mathcal{A}})$ denotes the Gelfand Transform of $\mathcal{A}$ into the space of continuous functions on the maximal ideal space $\Sigma_{\mathcal{A}}$ of $\mathcal{A}$ and that $\operatorname{spr}(x) = \|\Gamma(x)\|$ for all $x \in \mathcal{A}$.

We are now in a position to prove our Main Theorem.

2.10. Theorem. *Let H be a complex Hilbert space and $\mathcal{A}$ be a closed, abelian subalgebra of $\mathcal{B}(H)$. The following conditions are equivalent:*

(a) *$\mathcal{A}$ is amenable;*
(b) *$\mathcal{A}$ has the total reduction property;*
(c) *$\mathcal{A}$ is similar to a C^*-algebra.*

Proof. (a) implies (b): This is Theorem 1.6 above, due to Gifford.

(b) implies (c): By Theorem 2.6, there exists $\mu > 0$ so that $\|x\|^2 \leqslant \mu \|x^2\|$ for all $x \in \mathcal{A}$. By Proposition 2.8, the spectral radius is a norm on $\mathcal{A}$ which is equivalent to the operator norm on $\mathcal{A}$.

As mentioned above, it follows that the Gelfand Transform $\Gamma : \mathcal{A} \to \mathcal{C}(\Sigma_{\mathcal{A}})$ is not only injective, but the range of Γ is closed. That is, $\mathcal{A}$ is topologically isomorphic to the closed subalgebra $\Gamma(\mathcal{A})$ of $\mathcal{C}(\Sigma_{\mathcal{A}})$. Since $\mathcal{A}$ has the total reduction property, so does $\Gamma(\mathcal{A})$, and we can now apply Theorem 2.2 to conclude that $\mathcal{A}$ is similar to a C^*-algebra.

(c) implies (a): Since $\mathcal{A}$ is abelian, if $\mathcal{A}$ is similar to a C^*-algebra $\mathcal{B}$, then $\mathcal{B}$ must be abelian as well. Thus $\mathcal{B}$ is amenable by, for instance, [31, Example 2.3.4], and so $\mathcal{A}$ is amenable, being similar to, and hence a homomorphic image of, an amenable algebra.

□

2.11. Corollary. *Let H be a complex Hilbert space, and let $T \in \mathcal{B}(H)$. The following conditions are equivalent:*

(a) *$\mathcal{A}_T$ is amenable.*
(b) *$\mathcal{A}_T$ has the total reduction property.*
(c) *T is similar to a normal operator and the spectrum of T is a Lavrentieff set.*

Proof. (a) implies (b): As before, this is Theorem 1.6.

(b) implies (c): Since $\mathcal{A}_T$ is clearly abelian, Theorem 2.10 implies that $\mathcal{A}_T$ is similar to a C^*-algebra $\mathcal{B}$, say
$$\mathcal{A}_T = S^{-1} \mathcal{B} S.$$
But then $\mathcal{B} = S \mathcal{A}_T S^{-1} = \mathcal{A}_{STS^{-1}}$. Since $\mathcal{B}$ is selfadjoint and abelian, $N := STS^{-1}$ is normal. That the spectrum of T is a Lavrentieff set is Proposition 3.6 of [22].

(c) implies (a): Suppose that $T = S^{-1} N S$, where $S \in \mathcal{B}(H)$ is invertible and N is normal. Since $\sigma(T) = \sigma(N)$ is a Lavrentieff set, $\mathcal{A}_N = C^*(N)$([10], Theorem 2.7). But then $\mathcal{A}_T = S^{-1} \mathcal{A}_N S = S^{-1} C^*(N) S$ is similar to an abelian C^*-algebra, so that $\mathcal{A}_T$ is amenable.

□

3. Consequences of the Main Theorem

3.1. The article [10] contained a number of results about singly generated, amenable operator algebras which relied upon the equivalence of conditions (a) and (c) of Corollary 2.11 above. Unfortunately, although that paper claimed a proof of this equivalence, an error was later discovered (see [11]), and as a consequence, the results of Section 5 of [10] had to be withdrawn as well. Now that the validity of Corollary 2.11 has been established, we are able to retrieve some of those results, and to extend them beyond the singly generated case. This having been said, the proofs here are often very similar to the original proofs.

The following result provides a partial answer to a question of G. Pisier [26], p. 13.

3.2. Corollary. *Let $\mathcal{A}$ be a unital, abelian, amenable algebra. If $\varphi : \mathcal{A} \to \mathcal{B}(H)$ is a bounded, unital homomorphism, then there exists a contractive homomorphism $\rho : \mathcal{A} \to \mathcal{B}(H)$ and an invertible operator $S \in \mathcal{B}(H)$ such that $\varphi(x) = \mathrm{Ad}_S \circ \rho(x) = S^{-1} \rho(x) S$ for all $x \in \mathcal{A}$.*

Proof. Let $\mathcal{B} = \overline{\varphi(\mathcal{A})}$. Then $\mathcal{B}$ is an abelian, amenable subalgebra of $\mathcal{B}(H)$, and so by Theorem 2.10, $\mathcal{B}$ is similar to an abelian C^*-algebra $\mathcal{C}$, say $\mathcal{B} = S^{-1} \mathcal{C} S$ for some invertible operator $S \in \mathcal{B}(H)$.

Consider $\rho : \mathcal{A} \to \mathcal{C}$ defined by $\rho(x) = S\varphi(x)S^{-1}$. Then ρ is clearly a bounded homomorphism, and for each $x \in \mathcal{A}$, $\rho(x) \in \mathcal{C}$ implies that $\|\rho(x)\| = \mathrm{spr}(\rho(x)) \leqslant \mathrm{spr}(x) \leqslant \|x\|$. $\square$

3.3. Corollary. *Let $\mathcal{A}$ be an abelian, amenable Banach algebra, and suppose that $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a continuous representation of $\mathcal{A}$. Then $\rho(q) = 0$ for all $q \in \mathrm{Rad}\,(\mathcal{A})$.*

Proof. If $\mathcal{B} = \overline{\rho(\mathcal{A})}$, then $\mathcal{B}$ is an abelian, amenable operator algebra, and by Theorem 2.10, $\mathcal{B}$ is semisimple. Since $\sigma(\rho(q)) \subseteq \sigma(q) = \{0\}$ for each $q \in \mathrm{Rad}\,(\mathcal{A})$, it follows that $\rho(q) = 0$. $\square$

We conclude by listing (without proof) a couple of relatively straightforward consequences of Theorem 2.10.

3.4. Corollary. *Suppose that $\mathcal{A} \subseteq \mathcal{B}(H)$ is a unital, abelian and amenable subalgebra. Then*

(a) *$\mathcal{A} + \mathcal{K}(H)$ is norm-closed and amenable.*
(b) *Every continuous representation $\varrho : \mathcal{A} \to \mathcal{B}(H_\varrho)$ is completely bounded.*
(c) *The "similarity degree" (or "length") of $\mathcal{A}$ (as developed in the rich and deep theory of G. Pisier in [25], [26], [27]) is at most 2 (and is equal to 1 if and only if $\mathcal{A}$ is finite-dimensional).*

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