LINEAR PRESERVERS OF POLYNOMIAL NUMERICAL HULLS OF MATRICES

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Abstract. Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices, $1 \leq k \leq n-1$ be an integer, and $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ be a linear operator. In this paper, it is shown that φ preserves the polynomial numerical hull of order k if and only if there exists a unitary matrix $U \in \mathbb{M}_n$ such that either $\varphi(A) = U^*AU$ for all $A \in \mathbb{M}_n$, or $\varphi(A) = U^*A^tU$ for all $A \in \mathbb{M}_n$.

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1. Introduction and preliminaries

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices, and $A \in \mathbb{M}_n$. The *numerical range*, or the *field of values*, of A is defined as $W(A) = \{x^*Ax \ : \ x \in \mathbb{C}^n, \ x^*x = 1\}$. This is useful in studying and understanding both complex matrices and Hilbert space operators, and has many applications in numerical analysis, differential equations, systems theory, etc; (e.g. see [4, 8] and their references). It has been shown (see [1, Lemma 6.22.1]) that

$$W(A) = \{\lambda \in \mathbb{C} \ : \ |\lambda - \mu| \leq \|A - \mu I\|, \quad \forall \mu \in \mathbb{C}\}, \tag{1}$$

where $\|.\|$ is the usual operator norm on $\mathbb{M}_n$ (i.e., the norm on $\mathbb{M}_n$ obtained through its action on $\mathbb{C}^n$, where $\mathbb{C}^n$ carries the usual Euclidean norm), and I is the $n \times n$ identity matrix. Now, let k be a positive integer and denote by $\mathbb{P}_k$ the set of all scalar polynomials of degree k or less. Using the formulation of $W(A)$ given in (1), the concept of *numerical range* of A has been generalized to that of *the polynomial numerical hull of order k* of A, which is defined and denoted (e.g., see [12]) by

$$V^k(A) = \{\lambda \in \mathbb{C} : \ |p(\lambda)| \leq \|p(A)\| \text{ for all } p \in \mathbb{P}_k\}.$$

This is a set designed to give more information than the spectrum and numerical range alone can provide about the behaviour of the matrix A under the action of polynomials and other functions

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of A, and has many applications in the study of convergence of iterative methods in solving linear systems. For more information, we refer the reader to [6] and [7] and their references (see also [12]).

In the following proposition, we state some properties of polynomial numerical hulls of matrices which will be useful in our discussion. For proofs of these and a number of other properties, we suggest [3] and [6].

Proposition 1.1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) $V^k(A)$ *is a compact set in* $\mathbb{C}$*;*

(ii) $\sigma(A) = V^m(A) \subseteq \cdots \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq \cdots \subseteq V^1(A) = W(A)$, *where* $m \geq n$*;*

(iii) $V^k(\alpha A + \beta I) = \alpha V^k(A) + \beta$, *where* $\alpha, \beta \in \mathbb{C}$*;*

(iv) $V^k(U^*AU) = V^k(A)$, *where* $U \in \mathbb{M}_n$ *is unitary;*

(v) $V^k(A^T) = V^k(A)$ *and* $V^k(\overline{A}) = \overline{V^k(A)} := \{\overline{\lambda} \ : \ \lambda \in V^k(A)\}$. *Consequently,* $V^k(A^*) = \overline{V^k(A)}$*;*

(vi) $V^k(A) = \{\lambda \in \mathbb{C} : (\lambda, \lambda^2, ..., \lambda^k) \in conv(W(A, A^2, ..., A^k))\}$, *where* $conv(\cdot)$ *denotes the convex hull, and* $W(A_1, A_2, ..., A_k) := \{(x^*A_1x, x^*A_2x, ..., x^*A_kx) : x \in \mathbb{C}^n, \ x^*x = 1\}$ *is the joint numerical range of* $(A_1, A_2, ..., A_k) \in \mathbb{M}_n^k$*;*

(vii) If A is Hermitian, then $V^k(A) = \begin{cases} conv(\sigma(A)) & for\ k = 1, \\ \sigma(A) & for\ k \geq 2. \end{cases}$

Regarding the polynomial numerical hulls of Jordan blocks, we have the following result which can be found in [5, Section 2].

Proposition 1.2. *Let J be an $n \times n$ Jordan block with eigenvalue 0. If $k = 1, 2, \ldots, n-1$, then $V^k(J)$ is a circular disk about the origin of radius $0 < r_k < 1$, and for $k \geq n$, $V^k(J) = \{0\}$.*

An active and popular research area in matrix and operator theory is the study of linear preserver problems. Typically, one attempts to classify those linear maps $\varphi : \mathbb{M}_n \to \mathbb{M}_n$ which preserve some property of matrices (such as the rank, or the spectrum, etc.). Examples of such problems can be found in [10]. The purpose of this paper is to characterize linear operators preserving the polynomial numerical hull of order k of matrices. Let $k \geq n$, and $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ be a linear operator satisfying $V^k(\varphi(A)) = V^k(A)$ for all $A \in \mathbb{M}_n$. In this case, for every $A \in \mathbb{M}_n$, $V^k(A)$ reduces to $\sigma(A)$, and then, by [11, Theorem 3], there exists a nonsingular matrix $S \in \mathbb{M}_n$ such that either $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n$, or $\varphi(A) = S^{-1}A^tS$ for all $A \in \mathbb{M}_n$. This reduces our problem to the case where $1 \leq k \leq n-1$, and we characterize linear preservers of $V^k(\cdot)$ on $\mathbb{M}_n$ in Section 2 of this paper (see Theorem 2.5 below).

2. Main results

Let $k \in \mathbb{N}$. A linear operator $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is called a **linear preserver of the polynomial numerical hull of order** k if $V^k(\varphi(A)) = V^k(A)$ for all $A \in \mathbb{M}_n$. Our first goal is to show that φ is a unital bijective map. For this, we require the following lemma.

Lemma 2.1. *Let $A \in \mathbb{M}_n$, $\alpha \in \mathbb{C}$, and $k \in \mathbb{N}$. Then*

$$V^k(A+X) = V^k(X) + \alpha \quad \textit{for all } X \in \mathbb{M}_n \Longleftrightarrow A = \alpha I.$$

Proof. In view of Proposition 1.1(iii), without loss of generality, we assume that $\alpha = 0$.

Now, suppose that $V^k(A+X) = V^k(X)$ for all $X \in \mathbb{M}_n$. We will show that $A = 0$. By setting $X = 0$, we see that $\sigma(A) \subseteq V^k(A) = \{0\}$. Moreover, $\sigma(A + A^*) \subseteq V^k(A + A^*) = V^k(A^*) = \{0\}$ (by Proposition 1.1 above). Since $A + A^*$ is hermitian, we conclude that $A = -A^*$, so that A is normal. But A normal and $\sigma(A) = \{0\}$ implies that $A = 0$, as required.

The converse is a trivial consequence of Proposition 1.1(iii). This completes the proof. □

Lemma 2.2. *Let $k \in \mathbb{N}$, and $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ be a linear operator preserving the polynomial numerical hull of order k. Then φ is bijective and $\varphi(I) = I$.*

Proof. To prove that φ is bijective, we suppose, for $A \in \mathbb{M}_n$, that $\varphi(A) = 0$, and we will show that $A = 0$. In view of Lemma 2.1, it is enough to show that for every $X \in \mathbb{M}_n$, $V^k(A+X) = V^k(X)$.

Let $X \in \mathbb{M}_n$ be given. Then, by our assumption, we have that

$$V^k(A+X) = V^k(\varphi(A+X)) = V^k(\varphi(A) + \varphi(X)) = V^k(\varphi(X)) = V^k(X).$$

To prove the second assertion, we show that $\varphi(I) - I = 0$. Again, using Lemma 2.1, it suffices to show that given $X \in \mathbb{M}_n$ arbitrary, we have $V^k((\varphi(I) - I) + X) = V^k(X)$. Since φ is surjective, there exists $C \in \mathbb{M}_n$ such that $\varphi(C) = X$. Now, by Proposition 1.1(iii), we have

$$V^k((\varphi(I) - I) + X) = V^k(\varphi(I) + X) - 1 = V^k(\varphi(I + C)) - 1 = V^k(C) = V^k(\varphi(C)) = V^k(X).$$

This completes the proof. □

We shall require the following notation. We denote by $\mathcal{U}_n := \{U \in \mathbb{M}_n \ : \ U^*U = I\}$ the group of all $n \times n$ unitary matrices, by $GL_n(\mathbb{C}) := \{A \in \mathbb{M}_n \ : \ det(A) \neq 0\}$ the general linear group of $n \times n$ nonsingular complex matrices, and by $\mathbb{C}^*$ the set of all non-zero complex numbers. Given $\gamma \in \mathbb{C}^*$, we denote by $\widehat{\gamma}$ the vector $(1, \gamma, \gamma^2, \ldots, \gamma^{n-1}) \in (\mathbb{C}^*)^n$. We also write $\mathfrak{S}_n$ to denote the symmetric group of all permutations of $\{1, 2, \ldots, n\}$. Given a vector $d = (d_1, d_2, \ldots, d_n) \in \mathbb{C}^n$, we write $D_d = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, and given $\sigma \in \mathfrak{S}_n$, we shall denote by D_d^σ the diagonal matrix $D_d^\sigma = \mathrm{diag}(d_{\sigma(1)}, d_{\sigma(2)}, \ldots, d_{\sigma(n)})$. Finally, let us also introduce the sets:

$$\mathcal{G}_k := \{X \in GL_n(\mathbb{C}) \ : \ V^k(X^{-1}AX) = V^k(A) \ for \ all \ A \in \mathbb{M}_n\}, \tag{2}$$

where $k \in \mathbb{N}$, and

$$\mathbb{C}^*\mathcal{U}_n := \{\alpha U \ : \ \alpha \in \mathbb{C}^*, \ U \in \mathcal{U}_n\}. \tag{3}$$

It is easy to see that $\mathcal{G}_k$ and $\mathbb{C}^*\mathcal{U}_n$ are subgroups of $GL_n(\mathbb{C})$, and by Proposition 1.1(iv), $\mathbb{C}^*\mathcal{U}_n \subseteq \mathcal{G}_k$. In fact, we have the following result.

Lemma 2.3. *Let n, k be two positive integers such that $n \geq 2$ and $k \leq n-1$. Moreover, let $\mathcal{G}_k$ and $\mathbb{C}^*\mathcal{U}_n$ be the groups defined as in (2) and (3), respectively. Then $\mathcal{G}_k = \mathbb{C}^*\mathcal{U}_n$.*

Proof. It is enough to show that $\mathcal{G}_k \subseteq \mathbb{C}^*\mathcal{U}_n$. Suppose that $S \in \mathcal{G}_k$, and that $S \notin \mathbb{C}^*\mathcal{U}_n$. Let $S = U|S|$ be the polar decomposition of S. Note that $U \in \mathcal{U}_n$ and $|S| = (S^*S)^{\frac{1}{2}}$. Since $\mathcal{U}_n \subseteq \mathcal{G}_k$, $|S| = U^*S$, and $\mathcal{G}_k$ is a group, $|S| \in \mathcal{G}_k$, and without loss of generality, we may assume that $|S| = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, where $0 < s_1 \leq s_2 \leq \cdots \leq s_n$. Moreover, since $S \notin \mathbb{C}^*\mathcal{U}_n$, it follows that $|S| \neq \alpha I$ for all $\alpha \in \mathbb{C}^*$, and thus $s_n > s_1$. Furthermore, since $s_1^{-1}I \in \mathcal{G}_k$ and the latter is a group, it follows that $D := s_1^{-1}|S| \in \mathcal{G}_k$. Let $r_i = s_1^{-1}s_i$ for $i = 2, 3, \ldots, n$, $r = (1, r_2, r_3, \ldots, r_n)$, so that $D_r = \mathrm{diag}(1, r_2, r_3, \ldots, r_n)$. Consider the permutation $\sigma \in \mathfrak{S}_n$ defined by:

$$\sigma(i) = \begin{cases} 1 & \text{if } i = 1 \\ i+1 & \text{if } 2 \leq i < n \\ 2 & \text{if } i = n \end{cases} \cdot$$

Then $D_r^{\sigma^j}$ is unitarily equivalent to D. Since $D \in \mathcal{G}_k$ and $\mathcal{G}_k$ is a group containing all unitary matrices, it follows that $D_r^{\sigma^j} \in \mathcal{G}_k$ for all $j \geq 1$, and also

$$M := D \cdot D_r^{\sigma} \cdot D_r^{\sigma^2} \cdots D_r^{\sigma^{n-1}} \in \mathcal{G}_k.$$

Observe that

$$M = \mathrm{diag}(1, \alpha, \alpha, \ldots, \alpha) \in \mathbb{M}_n,$$

where $\alpha = r_2 r_3 \cdots r_n > 1$. Again, the fact that $\mathcal{G}_k$ is a group containing $\mathbb{C}^*I$ implies that

$$P := \alpha M^{-1} = \mathrm{diag}(\alpha, 1, 1, \ldots, 1) \in \mathcal{G}_k.$$

By setting $P_j := \mathrm{diag}(1, 1, \ldots, 1, \alpha, 1, 1, \ldots, 1) \in \mathcal{G}_k$, where $j = 2, 3, \ldots, n-1$, and the unique α appears in the j^{th} coordinate, we see that P_j is unitarily equivalent to P, and so $P_j \in \mathcal{G}_k$. Also, observe that again, as $\mathcal{G}_k$ is a group, we have

$$D_{\widehat{\alpha}} = P_2 P_3^2 P_4^3 \cdots P_n^{n-1} \in \mathcal{G}_k.$$

This implies that

$$V^k(D_{\widehat{\alpha}}^{-1} J D_{\widehat{\alpha}}) = V^k(J), \tag{4}$$

where J is the $n \times n$ Jordan block with eigenvalue 0. By Proposition 1.2 and the fact that $k < n$, there exists $\rho > 0$ such that $V^k(J) = \{z \in \mathbb{C} \ : \ |z| \leq \rho\}$. Observe that $D_{\widehat{\alpha}}^{-1} J D_{\widehat{\alpha}} = \alpha J$, and so, by Proposition 1.1(iii),

$$V^k(\alpha J) = \alpha V^k(J) = \{z \in \mathbb{C} \ : \ |z| \leq \alpha\rho\},$$

which contradicts (4) because $\alpha > 1$. Thus $\mathcal{G}_k \subseteq \mathbb{C}^* \mathcal{U}_n$, and so the proof is complete. □

To reach our goal, we also need the following lemma.

Lemma 2.4. *Let $2 \leq k \in \mathbb{N}$, and $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ be a linear operator preserving the polynomial numerical hull of order k. Then $tr(\varphi(H)) = tr(H)$ for all Hermitian matrices $H \in \mathbb{M}_n$.*

Proof. Consider the following two steps:

Step 1: Let P and Q be two nonzero rank-one orthogonal projections in $\mathbb{M}_n$ such that $PQ = QP = 0$. Then $tr(\varphi(P)) = tr(\varphi(Q))$.

To see this, observe that by our assumptions on P and Q, we have $\sigma(P) = \sigma(Q) = \{0, 1\}$, and clearly there exists a unitary matrix $U \in \mathbb{M}_n$ such that $U^*PU = Q$. Since $\mathcal{U}_n$ is a connected set in $\mathbb{M}_n$, there exists a continuous path $\{U_t \ : 0 \leq t \leq 1\}$ of unitary matrices in $\mathbb{M}_n$ such that $U_0 = I$ and $U_1 = U$. Since $k \geq 2$, Proposition 1.1((ii) *and* (vii)) and our assumption on φ show that for every $0 \leq t \leq 1$, $\sigma(\varphi(P_t)) \subseteq V^k(\varphi(P_t)) = V^k(P_t) = \sigma(P_t) = \{0, 1\}$, where $P_t = U_t^* P U_t$. By the continuity of φ, we find that $\sigma(\varphi(P_t)) = \sigma(\varphi(P_0))$, counting multiplicities, for all $0 \leq t \leq 1$. Consequently, $\sigma(\varphi(P)) = \sigma(\varphi(Q))$, counting multiplicities, and hence, $tr(\varphi(P)) = tr(\varphi(Q))$.

Step 2: Let P be a nonzero rank-one orthogonal projection in $\mathbb{M}_n$. Then $tr(\varphi(P)) = 1$.

To prove the assertion in *Step 2*, let $\{x_1, x_2, \ldots, x_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that $P = x_1^* x_1$. By setting $P_1 = P = x_1^* x_1$, $P_2 = x_2^* x_2, \ldots, P_n = x_n^* x_n$, we see that $P_1, P_2, \ldots, P_n$ are nonzero rank-one orthogonal projections such that $P_i P_j = P_j P_i = 0$ for every $i \neq j$, and $P_1 + P_2 + \cdots + P_n = I$.

Now by Lemma 2.2 and *Step 1*, we have

$$n = tr(I) = tr(\varphi(I)) = tr(\varphi(\sum_{i=1}^{n} P_i)) = \sum_{i=1}^{n} tr(\varphi(P_i)) = n\ tr(\varphi(P)).$$

This shows that $tr(\varphi(P)) = 1$.

Next, let $H \in \mathbb{M}_n$ be a Hermitian matrix. Then there exist real numbers $d_1, d_2, \ldots, d_n$ and nonzero rank-one orthogonal projections $P_1, P_2, \ldots, P_n$ such that $H = \sum_{i=1}^{n} d_i P_i$. By *Step 2*, we have

$$tr(\varphi(H)) = \sum_{i=1}^{n} d_i\ tr(\varphi(P_i)) = \sum_{i=1}^{n} d_i = tr(H).$$

This completes the proof. □

We are now ready to characterize the linear preservers of polynomial numerical hulls of matrices.

Theorem 2.5. *Let n, k be two positive integers, $n \geq 2$ and $k \leq n-1$. Moreover, let $\varphi : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ be a linear operator. Then $V^k(\varphi(A)) = V^k(A)$ for all $A \in \mathbb{M}_n$ if and only if there exists a unitary matrix $U \in \mathbb{M}_n$ such that either $\varphi(A) = U^*AU$ for all $A \in \mathbb{M}_n$, or $\varphi(A) = U^*A^tU$ for all $A \in \mathbb{M}_n$.*

Proof. The assertion for the cases where $k = 1$ or $n = 2$ follow from [9, Theorem 3] and the fact that $V^1(\cdot)$ coincides with the numerical range. As such, we may assume that $n \geq 3$ and $2 \leq k \leq n-1$.

Let $H \in \mathbb{M}_n$ be an arbitrary Hermitian matrix. By Proposition 1.1((*ii*) *and* (*vii*)) and our hypotheses, we have

$$\sigma(\varphi(H)) \subseteq V^k(\varphi(H)) = V^k(H) = \sigma(H).$$

Using Lemma 2.4, we may argue in the same manner as in the proof of Lemma 3 of [2, p. 2677] to deduce that $\sigma(\varphi(H)) = \sigma(H)$ for any arbitrary Hermitian matrix $H \in \mathbb{M}_n$. So, by [11, Theorem 3], there exists a nonsingular matrix $S \in GL_n(\mathbb{C})$ such that either $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n$, or $\varphi(A) = S^{-1}A^tS$ for all $A \in \mathbb{M}_n$.

Suppose, as a first case, that $\varphi(A) = S^{-1}AS$ for all $A \in \mathbb{M}_n$. Since $V^k(\varphi(A)) = V^k(A)$ for all $A \in \mathbb{M}_n$, $S \in \mathcal{G}_k$, where $\mathcal{G}_k$ is the group defined in (2) above. Since $k < n$, Lemma 2.3 implies that $S \in \mathbb{C}^*\mathcal{U}_n$, where $\mathbb{C}^*\mathcal{U}_n$ is the group as in (3), and so, there exist $\alpha \in \mathbb{C}^*$ and a unitary matrix $U \in \mathcal{U}_n$ such that $S = \alpha U$. Therefore, for every $A \in \mathbb{M}_n$, $\varphi(A) = S^{-1}AS = U^*AU$, and so the result holds.

The result in the second case, i.e., $\varphi(A) = S^{-1}A^tS$ for all $A \in \mathbb{M}_n$, follows from Proposition 1.1(v) and an argument similar to that used in the proof of the first case above.

Finally, the converse of the assertion follows easily from Proposition 1.1((*iv*) *and* (*v*)), completing the proof. □

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