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LIMIT CYCLES OF PIECEWISE SMOOTH DIFFERENTIAL EQUATIONS ON TWO DIMENSIONAL TORUS

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Abstract. In this paper we study the limit cycles of some classes of piecewise smooth vector fields defined in the two dimensional torus. The piecewise smooth vector fields that we consider are composed by linear, Ricatti and perturbations of these two classes. For these kind of piecewise smooth vector fields we study their global dynamics, their upper bounds for the maximum number of limit cycles that they can exhibit, and the existence of non-trivial recurrences and of a continuum of periodic orbits. We also present a family of piecewise smooth vector fields that possesses a finite number of fold points, and that for any positive integer k there are values of the parameters of this family for which the piecewise smooth vector field exhibit k limit cycles,

1. Introduction

The theory of piecewise smooth vector fields (PSVF) has been studied intensively in these last years, mainly due to its strong relation with branches of applied sciences. These PSVF are in the boundary between mathematics, physics and engineering, for more details see for instance the two recent surveys [7] and [13], and the two books [4] and [12] on this subject, where also models of PSVF from control theory are considered. Roughly speaking the PSVF are formed by several smooth differential systems defined in different regions of the global domain of definition of the PSVF. The common frontier between the regions that separate the different smooth vector fields is called switching manifold (or discontinuity manifold).

Let $\mathbb{T}$ be the two dimensional torus. We decomposed $\mathbb{T}$ as the union of $\mathbb{T}^+$ with $\mathbb{T}^-$, where $\mathbb{T}^+$ denotes the closed upper half part of the torus $\mathbb{T}$ (homeomorphic to a closed annulus), and $\mathbb{T}^-$ the closed bottom half part of this torus (also homeomorphic to a closed annulus). We denote by $\Sigma = \mathbb{T}^+ \cap \mathbb{T}^-$ a smooth curve, formed by two circles, which separates $\mathbb{T}$ into two connected components, each one homeomorphic to an open annulus. Let X^+ and X^- be smooth vector fields on $\mathbb{T}^+$ and $\mathbb{T}^-$, respectively. A precise definition of $\mathbb{T}$, $\mathbb{T}^+$, $\mathbb{T}^-$ and Σ is given at the beginning of Section 2.

In this paper we consider piecewise smooth differential equations of the form

$$(1) \qquad \dot{\mathbf{x}} = \begin{cases} X^+(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \mathbb{T}^+, \\ X^-(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \mathbb{T}^-. \end{cases}$$

The dynamics over Σ is defined following the Filippov's convention (see [5]). For simplicity a differential system (1) will be denoted by (X^+, X^-), and referred as vector field (1).

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The study of piecewise smooth dynamical systems defined on torus is not new, but as far as we know it has been restricted to the case of discrete dynamical systems. There are a large number of results for piecewise maps [1], [2], [3] and [14], but there is a lack of theoretical results for the case of piecewise dynamical systems where the flow is the solution of a piecewise differential system.

The research of the number and stability of limit cycles for some classes of vector fields is one of the most relevant problems of the qualitative theory of the dynamical systems. This kind of studies started with Poincaré in [11] and [10]. The main objective of this paper is to start this research first for the PSVF (1) when the smooth vector fields on $\mathbb{T}^+$ and $\mathbb{T}^-$ are either linear, or Ricatti, or some families of perturbations of them coming from the applications (see (3)), and after for a family of PSVF presenting a finite number of fold points (see (6)).

This paper is organized as follows. In Section 2 we formalize some basic concepts on the PSVF, as the first return map in this scenario and present some techniques that we shall use in the proof of the main results. In Section 3 the main results are presented, in Section 4 we prove these results, and in Section 5 we end this paper presenting some numerical examples of PSVF with the maximum number of limit cycles that they can exhibit.

2. Basic Theory

2.1. **Filippov's convention.** In this work the two dimensional torus $\mathbb{T}$ that we consider is defined by the following equivalence relation in the square $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$:

$$(x, y) \sim (z, w) \Leftrightarrow (x - z, y - w) \in \mathbb{Z} \times \mathbb{Z}. \tag{2}$$

Consider $\Sigma_1 = \{(x, y) \in Q : y = 0\}$ and $\Sigma_2 = \{(x, y) \in Q : y = 1/2\}$. We denote by $h_1(x, y) = y$ and $h_2(x, y) = y - 1/2$, in this way we can write $\Sigma_1 = h_1^{-1}(0)$ and $\Sigma_2 = h_2^{-1}(0)$. Clearly the switching manifold $\Sigma = \Sigma_1 \cup \Sigma_2$ is the common boundary between the two regions $\mathbb{T}^- = \{(x, y) \in \mathbb{T}; 0 \le y \le 1/2\}$ and $\mathbb{T}^+ = \{(x, y) \in \mathbb{T}; 1/2 \le y \le 1\}$.

Designate by $\mathfrak{X}^r$ the space of C^r-vector fields on $\mathbb{T}$ endowed with the C^r-topology with $r = \infty$ or $r \ge 1$ large enough for our purposes. Call Ω^r the space of PSVF $X : \mathbb{T} \to \mathbb{T}$ such that

$$X(x, y) = \begin{cases} X^+(x, y) & \text{for} \quad (x, y) \in \mathbb{T}^+, \\ X^-(x, y) & \text{for} \quad (x, y) \in \mathbb{T}^-, \end{cases}$$

where $X^+ = (X_1^+, X_2^+)$ and $X^- = (X_1^-, X_2^-)$ are in $\mathfrak{X}^r$. Let $h \in \{h_1, h_2\}$. We denote by $X^\pm h(p) = \langle X^\pm(p), \nabla h(p)\rangle$ and $(X^\pm)^n h(p) = \langle X^\pm(p), \nabla (X^\pm)^{n-1} h(p)\rangle$ the Lie's derivatives, where $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. We may consider $\Omega^r = \mathfrak{X}^r \times \mathfrak{X}^r$ endowed with the product topology and denote any element in Ω^r by $X = (X^+, X^-)$, which we will accept to be multivalued in points of Σ. In this context the basic results on the PSVF were stated by Filippov in [5]. Related theories can be found in [4, 9, 13] and references therein.

On Σ we generically distinguish three regions: the *crossing region* $\Sigma^c = \{p \in \Sigma : X_2^+(p) X_2^-(p) > 0\}$, the *stable sliding region* $\Sigma^s = \{p \in \Sigma : X_2^+(p) < 0, X_2^-(p) > 0\}$, and the *unstable sliding region* $\Sigma^u = \{p \in \Sigma : X_2^+(p) > 0, X_2^-(p) < 0\}$.

Following the Filippov's convention if $q \in \Sigma^s$ the sliding vector field associated to $X \in \Omega^r$ is the vector field $\widehat{X}^s$ tangent to Σ^s, expressed in coordinates as

$$\widehat{X}^s(q) = \frac{1}{(X_2^- - X_2^+)(q)}((X_1^+ - X_1^-)(q), 0),$$

which, after a time rescaling, is topologically equivalent to the *normalized sliding vector field*

$$X^s(q) = (X_1^+ - X_1^-)(q).$$

A point $q \in \Sigma$ such that $X^s(q) = 0$ is called a *pseudo equilibrium* of X, and a point $p \in \Sigma$ such that $X^+h(p)X^-h(p) = 0$ is called a *tangential singularity* of X (i.e. the trajectory through p is tangent to Σ). We say that a point $q \in \Sigma$ is a *regular point* if $q \in \Sigma^c$ or $q \in \Sigma^s$ and $X^s(q) \neq 0$.

A tangential singularity $q \in \Sigma$ of X^+ is a *fold point* of X^+ if $X^+h(q) = 0$ but $(X^+)^2h(q) \neq 0$, visible tangency if $(X^+)^2h(q) > 0$ and invisible tangency if $(X^+)^2h(q) < 0$.

The *flow* ϕ_X of $X \in \Omega^r$ is obtained by the concatenation of flows of X^+, X^- and X^s, denoted by ϕ_{X^+}, ϕ_{X^-} and ϕ_{X^s}, respectively.

Let be $X = (X^+, X^-) \in \Omega^r$, we say that $p \in \Sigma$ is a *fold-regular point* of X if p is a fold point of X^+ and $X^-(p)$ is transversal to Σ at p.

2.2. **Extended Chebyshev systems.** Let I be a proper real interval. A ordered set of functions $\mathcal{F} = \{g_j : I \to \mathbb{R} \text{ for } j = 0, 1, \dots, k\}$ is an *extended Chebyshev* system on I if and only if every nontrivial linear combination of functions of $\mathcal{F}$ has at most k zeros taking into account their multiplicities. $\mathcal{F}$ is an *extended complete Chebyshev* system on I if and only if for any s, $0 \leq s \leq k$, we get that $(g_0, g_1, \dots, g_s)$ is an extended system. For details and proofs see [6].

It is necessary and sufficient for proving that $\mathcal{F}$ is an extended Chebyshev system on I that $W(g_0, g_1, \dots, g_s)(t) \neq 0$ on I for $0 \leq s \leq k$, where $W_s(t) = W(g_0, g_1, \dots, g_s)(t)$ is the Wronskian of the functions $(g_0, g_1, \dots, g_s)$ with respect to t.

In [8] the authors proved that for a family of $n+1$ linearly independent analytical functions where at least one of that possess constant sign in its domain, there exists a linear combination of these functions having at least n simple zeros. Precisely, they proved the following result:

Theorem B Let $\mathcal{F} = \{g_0, g_1, \dots, g_n\}$ be an ordered set of real C^∞ functions on (a, b) for which there exists $\xi \in (a, b)$ with $W(g_0, g_1, \dots, g_{n-1})(\xi) = W_{n-1}(\xi) \neq 0$. Then the following statements hold.

(a) If $W_n(\xi) \neq 0$ then for each configuration of $m \leq n$ zeros, taking into account their multiplicity, there exists a linear combination of the functions of $\mathcal{F}$ having this configuration of zeros.

(b) If $W_n(\xi) = 0$ and $W_n'(\xi) \neq 0$ then for each configuration of $m \leq n+1$ zeros, taking into account their multiplicity, there exists a linear combination of the functions of $\mathcal{F}$ having this configuration of zeros.

3. Main results for PSVF in the two dimensional torus

One of the main objectives of this paper is to study the linear and Ricatti vector fields in $\mathbb{T}$, that we denote by

$$\begin{aligned} X_L^\omega(x, y) &= (a^\omega y + b^\omega, c^\omega y + d^\omega), \\ X_R^\omega(x, y) &= (1, e^\omega + f^\omega y + g^\omega y^2), \end{aligned}$$

respectively, where $a^\omega, b^\omega, c^\omega, d^\omega, e^\omega, f^\omega, g^\omega \in \mathbb{R}$ and either $\omega = +$ or $\omega = -$, if the vector field is defined either in $\mathbb{T}^+$ or in $\mathbb{T}^-$. The special case of X_L^ω where $a^\omega = c^\omega = 0$ in X_L^ω will be denoted by X_C^ω (constant vector field).

In the following we shall perturb these PSVF considering the functions defined in $\mathbb{T}$:

$$F_1(x,y) = (-x+x^2,0), \quad F_2(x,y) = (\eta_1 y + \eta_2 y^2, 0), \quad F_3(x,y) = (\cos(2\pi x), 0), \tag{3}$$

where $\eta_1, \eta_2 \in \mathbb{R}$ and are small. We denote by X_{LL} the PSVF composed by two linear vector fields in each half torus, by X_{LR} the PSVF composed by a linear vector field in $\mathbb{T}^-$ and Ricatti vector field on $\mathbb{T}^+$, and by X_{RR} the PSVF composed by the Ricatti vector fields in each half torus. Considering the PSVFs X_{LL}, X_{LR} and X_{RR} we perform the following perturbations:

$$\begin{aligned}
&X_{LL2+} = X_{LL} + (F_2, \overrightarrow{0}), && X_{RR1+} = X_{RR} + \varepsilon(F_1, \overrightarrow{0}), && X_{RR2+} = X_{RR} + (F_2, \overrightarrow{0}),\\
&X_{RR3+} = X_{RR} + \varepsilon(F_3, \overrightarrow{0}), && X_{LR1-} = X_{LR} + \varepsilon(\overrightarrow{0}, F_1), && X_{LR2+} = X_{LR} + (F_2, \overrightarrow{0}),\\
&X_{LR2-} = X_{LR} + (\overrightarrow{0}, F_2), && X_{LR3-} = X_{LR} + \varepsilon(\overrightarrow{0}, F_3), &&
\end{aligned} \tag{4}$$

where $\overrightarrow{0}$ denotes the null vector field $(0,0)$ in $\mathbb{T}$.

Remark 1. *We only consider perturbations of X_{LL} and X_{RR} in $\mathbb{T}^+$, because due to the symmetry of the problem we should obtain the same results if we consider perturbations in $\mathbb{T}^-$.*

For each of the families presented in (4) we consider the following subfamilies

$$\begin{aligned}
\Omega^1_{L^\omega} &= \{X_{L^\omega}; c^\omega > 0, d^\omega > -\tfrac{c^\omega}{2}\},\\
\Omega^2_{L^\omega} &= \{X_{L^\omega}; c^\omega < 0, -1 < \tfrac{c^\omega}{(c^\omega + 2d^\omega)} < 0\},\\
\Omega^3_{L^\omega} &= \{X_{L^\omega}; c^\omega < 0, -1 < \tfrac{c^\omega}{d^\omega} < 0\},\\
\Omega^4_{L^\omega} &= \{X_{L^\omega}; c^\omega > 0, d^\omega > 0\},\\
\Omega^5_{L^\omega} &= \{X_{L^\omega}; a^\omega > 0, b^\omega > 0\},\\
\Omega^6_{L^\omega} &= \{X_{L^\omega}; a^\omega < 0, -1 < \tfrac{a^\omega}{(2b^\omega)} < 0\},\\
\Omega^1_{R^\omega} &= \left\{X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left(\tfrac{f^\omega}{2}\right)^2, \tan^{-1}(\theta^\omega_1) > \tan^{-1}(\theta^\omega_2)\right\},\\
\Omega^2_{R^\omega} &= \left\{X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left(\tfrac{f^\omega}{2}\right)^2, \tan^{-1}(\theta^\omega_2) > \sec^{-1}(\theta^\omega_3)\right\},\\
\Omega^3_{R^\omega} &= \left\{X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left(\tfrac{f^\omega}{2}\right)^2, \tan^{-1}(\theta^\omega_1) > \sec^{-1}(\theta^\omega_3)\right\},
\end{aligned} \tag{5}$$

where $\theta^\omega_1 = \dfrac{f^\omega + 2g^\omega}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}}, \theta^\omega_2 = \dfrac{f^\omega + g^\omega}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}}$ and $\theta^\omega_3 = \dfrac{2\sqrt{e^\omega g^\omega}}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}}$. Prior to present the theorem we define the following real numbers

$$\begin{aligned}
\Delta_{LL} &= \frac{1}{2(c^-c^+)^2}\Big(c^-\left(c^+(a^-c^+ + a^+c^-) + 2c^- \log\left(\tfrac{c^+}{c^+ + 2d^+} + 1\right)(b^+c^+ - a^+d^+)\right)\\
&\quad + 2(c^+)^2 \log\left(\tfrac{c^-}{2d^-} + 1\right)(b^-c^- - a^-d^-)\Big),
\end{aligned}$$

$$
\begin{aligned}
\Delta_{LR} &= \frac{1}{2(c^-)^2}\Big(a^-c^- + (4(c^-)^2(-\tan^{-1}(\theta_2^+) + \tan^{-1}(\theta_1^+))/\sqrt{-(f^+)^2 + 4e^+g^+}\\
&+2b^-c^- \log(1 + c^-/(2d^-)) - 2a^-d^- \log(1 + c^-/(2d^-))),\\
\Delta_{RR} &= \frac{\sec^{-1}\left(\theta_3^-\right) - \tan^{-1}\left(\theta_2^-\right)}{\sqrt{4e^-g^- - (f^-)^2}} + \frac{\tan^{-1}\left(\theta_2^+\right) - \tan^{-1}\left(\theta_1^+\right)}{\sqrt{4e^+g^+ - (f^+)^2}},\\
\Delta_{LL2+} &= \frac{1}{8(c^+)^3}\Big(8\log\left(\tfrac{c^+}{c^++2d^+} + 1\right)\left(d^+(d^+\eta_2 - c^+(a^+ + \eta_1)) + b^+(c^+)^2\right) +\\
&c^+(4c^+(a^+ + c^+ + \eta_1) + \eta_2(3c^+ - 4d^+))\Big),\\
\Delta_{RR2+} &= \frac{1}{2(g^+)^2\sqrt{4e^-g^- - (f^-)^2}\sqrt{4e^+g^+ - (f^+)^2}}\Big(\sqrt{4e^-g^- - (f^-)^2}\\
&(\tan^{-1}(\theta_2^+)(4e^+\eta_2 g^+ - 2(f^+)^2\eta_2 + 2f^+\eta_1 g^+ - 4(g^+)^2)+\\
&2\tan^{-1}\left(\theta_1^+\right)\left(-g^+(2e^+\eta_2 + f^+\eta_1) + (f^+)^2\eta_2 + 2(g^+)^2\right) +\\
&\sqrt{4e^+g^+ - (f^+)^2}(\log\left(\tfrac{g^+(e^++f^++g^+)}{4e^+g^+-(f^+)^2}\right)(\eta_1 g^+ - f^+\eta_2)+\\
&\log\left(1 + (\theta_2^+)^2\right)\left(f^+\eta_2 - \eta_1 g^+\right) - f^+\eta_2\log(4) + \eta_1 g^+\log(4) + \eta_2 g^+))+\\
&4(g^+)^2\sqrt{4e^+g^+ - (f^+)^2}\Big(\tan^{-1}\left(\theta_2^-\right) - \sec^{-1}\left(\theta_3^-\right)\Big)\Big),\\
\Delta_{LR2-} &= \frac{1}{2(c^+)^2(g^-)^2\sqrt{4e^-g^- - (f^-)^2}}\Big(\sqrt{4e^-g^- - (f^-)^2}\Big(2(g^-)^2\log\left(\tfrac{c^+}{c^++2d^+} + 1\right)\\
&(b^+c^+ - a^+d^+) + c^+\Big(a^+(g^-)^2 + 2c^+\eta_1 g^-\log\left(\tfrac{\sqrt{4e^-g^--(f^-)^2}}{\sqrt{e^-}\sqrt{g^-}}\right) -\\
&2c^+\eta_1 g^-\log\left(\tfrac{1}{\sqrt{1+(\theta_2^-)^2}}\right) - c^+\sqrt{e^-}\eta_2\sqrt{g^-}\sqrt{\tfrac{(f^-)^2}{e^-g^-}} -\\
&2c^+f^-\eta_2\log\left(\tfrac{\sqrt{4e^-g^--(f^-)^2}}{2\sqrt{e^-g^-}}\right) + 2c^+f^-\eta_2\log\left(\tfrac{1}{\sqrt{1+(\theta_2^+)^2}}\right) +\\
&c^+f^-\eta_2 - c^+\eta_1 g^-\log(4) + c^+\eta_2 g^-\Big)\Big) + \Big(2(c^+)^2\Big(-g^-(2e^-\eta_2 + f^-\eta_1)+\\
&(f^-)^2\eta_2 + 2(g^-)^2\Big)\Big)\Big(\tan^{-1}\left(\theta_2^-\right) - \sec^{-1}\left(\theta_3^-\right)\Big)\Big),\\
\Delta_{LR2+} &= \frac{1}{8(c^+)^3\left((f^-)^2 - 4e^-g^-\right)}\Big(\left((f^-)^2 - 4e^-g^-\right)\left(8\log\left(\tfrac{c^+}{c^++2d^+} + 1\right)\right.\\
&\left.\Big(c^+(b^+c^+ - d^+(a^+ + \eta_1)) + (d^+)^2\eta_2\Big) + c^+(4c^+a^+ + \eta_1) + \eta_2(3c^+ - 4d^+)\right)\Big)+\\
&16(c^+)^3\sqrt{4e^-g^- - (f^-)^2}\Big(\sec^{-1}\left(\theta_3^-\right) - \tan^{-1}\left(\theta_2^-\right)\Big)\Big).
\end{aligned}
$$

In Theorem 2 we prove that these subfamilies correspond the piecewise smooth vector fields where the first return map $P : \Sigma_1 \to \Sigma_1$ is defined.

Theorem 2. *Consider the PSVFs defined in* (5).

(*a*) *If* $\Delta_{LL} \in \mathbb{Q}$ *then* X_{LL} *has a continuum of periodic orbits, and if* $\Delta_{LL} \notin \mathbb{Q}$ *then all trajectories of* X_{LL} *are dense.*

(*b*) *If* $\Delta_{LR} \in \mathbb{Q}$ *then* X_{LR} *has a continuum of periodic orbits, and if* $\Delta_{LR} \notin \mathbb{Q}$ *then all trajectories of* X_{LR} *are dense.*

(c) *If* $\Delta_{RR} \in \mathbb{Q}$ *then* X_{RR} *has a continuum of periodic orbits, and if* $\Delta_{RR} \notin \mathbb{Q}$ *then all trajectories of* X_{RR} *are dense.*

Considering the perturbations F_i^ω *we have*

(d) *If* $\Delta_{LL2+} \in \mathbb{Q}$ *then* X_{LL2+} *has a continuum of periodic orbits, and if* $\Delta_{LL2+} \notin \mathbb{Q}$ *then all trajectories of* X_{LL2+} *are dense.*

(e) *If* $\varepsilon > 0$ *then the maximum number of limit cycles for* X_{RR1+} *is two, and this upper bound is reached.*

(f) *If* $\Delta_{RR2+} \in \mathbb{Q}$ *then* X_{RR2+} *has a continuum of periodic orbits, and if* $\Delta_{RR2+} \notin \mathbb{Q}$ *then all trajectories of* X_{RR2+} *are dense.*

(g) *The maximum number of limit cycles of* X_{RR3+} *is two, and this upper bound is reached.*

(h) *The maximum number of limit cycles of* X_{LR1-} *is two, and this upper bound is reached if* $X_{LR1-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$, $\varepsilon < 0$, $a^+c^+ > 0$ *and* $b^+c^+ > a^+d^+$.

(i) *If* $\Delta_{LR2-} \in \mathbb{Q}$ *then* X_{LR2-} *has a continuum of periodic orbits, and if* $\Delta_{LR2-} \notin \mathbb{Q}$ *then all trajectories of* X_{LR2-} *are dense.*

(j) *The maximum number of limit cycles of* X_{LR3-} *is two, and this upper bound is reached.*

(l) *If* $\Delta_{LR2+} \in \mathbb{Q}$ *then* X_{LR2+} *has a continuum of periodic orbits, and if* $\Delta_{LR2+} \notin \mathbb{Q}$ *then all trajectories of* X_{LR2+} *are dense.*

In what follows we consider a PSVF $X_{Ck} = (X_C, X_k)$ in $\mathbb{T}$ having a finite number of fold-regular points in Σ, where

$$X_k(x, y) = (\alpha, \beta\cos(2k\pi x)), \tag{6}$$

is defined in $\mathbb{T}^-$ and $X_C(x, y) = (b^+, d^+)$ is defined in $\mathbb{T}^+$, with $b^+, d^+ \in \mathbb{R}$, k is a positive integer and $\alpha, \beta \in \mathbb{R}$. For this PSVF there exists a choice of the parameters of X_{Ck} such that X_{Ck} exhibits a finite number of limit cycles depending on k. More precisely we have the following result.

Theorem 3. *The PSVF* X_{Ck} *has at most* k *limit cycles, and this upper bound is reached for every* $k \geq 1$.

Remark 4. *Note that the vector field in the family* X_{Ck} *can have no limit cycles. In such case there are sliding regions over the switching manifold and* X_{Ck} *may present a chaotic behavior, see for instance* [15].

4. Proof of main results

4.1. **Preliminary results.** Before to prove the main results of this paper we need some auxiliary results. The next lemma provides the expression of the first return map for the PSVFs X_{LL}, X_{RR}, X_{LR} and their perturbations.

Lemma 5. *Consider the PSVFs defined in* (5) *and the functions defined in* (3).

(a) *If* $X_{LL} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \left[\Omega^1_{L^-} \cup \Omega^2_{L^-}\right]$ *then the first return map* $P_{LL} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{LL}(x_0) = x_0 + \Delta_{LL}$.

(b) *If* $X_{LR} \in \left[\Omega^1_{L^-} \cup \Omega^2_{L^+}\right] \cap \Omega^1_{R^+}$ *then the first return map* $P_{LR} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{LR}(x_0) = x_0 + \Delta_{LR}$.

(c) *If* $X_{RR} \in \left[\Omega^1_{R^+} \cap \Omega^2_{R^-}\right]$ *then the first return map* $P_{RR} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{RR}(x_0) = x_0 + \Delta_{RR}$.

(d) *If* $X_{LL2+} \in \left[\Omega^5_{L^-} \cup \Omega^6_{L^-}\right] \cap \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right]$ *then the first return map* $P_{LL2+} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{LL2+}(x_0) = x_0 + \Delta_{LL2+}$

(e) *If* $X_{RR1+} \in \left[\Omega^2_{R^-} \cap \Omega^3_{R^+}\right]$ *and* ε *is a small positive number then the first return map* $P_{RR1+} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by*

$$\begin{aligned} P_{RR1+}(x_0) = \quad & \tfrac{1}{2\sqrt{-(\varepsilon-4)\varepsilon}}\Bigg(\sqrt{(4-\varepsilon)\varepsilon} + (4-\varepsilon)\tan\Bigg(\tfrac{\sqrt{-(\varepsilon-4)\varepsilon}\left(\tan^{-1}\left(\theta_1^+\right)-\sec^{-1}\left(\theta_3^+\right)\right)}{\sqrt{4e^+g^+-(f^+)^2}} - \\ & \tan^{-1}\Bigg(\tfrac{1}{\sqrt{4-\varepsilon}\left(((f^-)^2-4e^-g^-\right)}\sqrt{\varepsilon}\Big((2x_0-1)\left(\left(-(f^-)^2+4e^-g^-\right)\right) + \\ & 4\sqrt{4e^-g^- - (f^-)^2}\tan^{-1}\left(\theta_2^-\right) - 4\sqrt{4e^-g^- - (f^-)^2}\sec^{-1}(\theta_3^-)\Big)\Bigg)\Bigg)\Bigg). \end{aligned}$$

(f) *If* $X_{RR2+} \in \left[\Omega^1_{R^-} \cap \Omega^2_{R^+}\right]$ *then the first return map* $P_{RR2+} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{RR2+}(x_0) = x_0 + \Delta_{RR2+}$

(g) *If* $X_{RR3+} \in \left[\Omega^2_{R^-} \cap \Omega^1_{R^+}\right]$ *and* ε *is a small positive number then the first return map* $P_{RR3+} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by*

$$\begin{aligned} P_{RR3+}(x_0) = \quad & -\tfrac{1}{\pi}\tan^{-1}\left(\tfrac{\sqrt{\varepsilon+1}}{\sqrt{1-\varepsilon}}\left(\tan\left(\tfrac{2\pi\sqrt{1-\varepsilon^2}(\tan^{-1}(\theta_2^+)-\tan^{-1}(\theta_1^+))}{\sqrt{4e^+g^+-(f^+)^2}}\right)\right)\right) - \\ & \tan^{-1}\Bigg(\tfrac{(\varepsilon-1)}{\sqrt{1-\varepsilon^2}}\tan\Bigg(\tfrac{\pi}{(f^-)^2-4e^-g^-}\Big(-(f^-)^2 + 2\sqrt{4e^-g^- - (f^-)^2} \\ & \tan^{-1}(\theta_2^+) - 2\sqrt{4e^-g^- - (f^-)^2}\sec^{-1}(\theta_3^-) + 4e^-g^-x_0 - (f^-)^2x_0\Big)\Bigg)\Bigg) \end{aligned}$$

(h) *If* $X_{LR1-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$ *and* ε *is a small negative number then the first return map* $P_{LR1-} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by*

$$\begin{aligned} P_{LR1-}(x_0) \quad & = \tfrac{1}{2}\Bigg(\tfrac{2\log\left(\frac{c^+}{c^++2d^+}+1\right)(b^+c^+-a^+d^+)+c^+(a^++c^+)}{(c^+)^2} + \\ & \tfrac{\tanh\left(\frac{\sqrt{4-\varepsilon}\sqrt{-\varepsilon}\left(\tan^{-1}\left(\theta_2^+\right)-\sec^{-1}\left(\theta_3^-\right)\right)}{\sqrt{4e^-g^--(f^-)^2}}+\tanh^{-1}\left((2x_0-1)\sqrt{\frac{\varepsilon}{\varepsilon-4}}\right)\right)}{\sqrt{\frac{\varepsilon}{\varepsilon-4}}}\Bigg). \end{aligned}$$

(i) *If* $X_{LR2-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$ *then the first return map* $P_{LR2-} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{LR2-}(x_0) = x_0 + \Delta_{LR2-}$.

(j) *If* $X_{LR3-} \in \left[\Omega^1_{L+} \cup \Omega^2_{L+}\right] \cap \Omega^2_{R-}$ *and* ε *is a small positive number then the first return map* $P_{LR3-} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by*

$$
\begin{aligned}
P_{LR3-}(x_0) \quad &= \tfrac{1}{2\pi(c^+)^2}\Big(\pi\left(2\log\left(\tfrac{c^+}{c^++2d^+}+1\right)(b^+c^+-a^+d^+)+a^+c^+\right)+\\
&2(c^+)^2\tan^{-1}\left(\tfrac{\sqrt{\varepsilon+1}}{\sqrt{1-\varepsilon}}\tan\left(\tfrac{2\pi\sqrt{1-\varepsilon^2}\left(\tan^{-1}\left(\theta_2^-\right)-\sec^{-1}\left(\theta_3^-\right)\right)}{\sqrt{4e^-g^--(f^-)^2}}-\right.\right.\\
&\left.\left.\tan^{-1}\left(\tfrac{(\varepsilon-1)\tan(\pi x_0)}{\sqrt{1-\varepsilon^2}}\right)\right)\right)\Big).
\end{aligned}
$$

(l) *If* $X_{LR2+} \in \left[\Omega^1_{L+} \cup \Omega^2_{L+}\right] \cap \Omega^2_{R-}$ *then the first return map* $P_{LR2+} : \Sigma_1 \to \Sigma_1$ *is well defined and is given by* $P_{LR2+}(x_0) = x_0 + \Delta_{LR2+}$.

Proof. The flow $\phi_X(t)$ where X is one of the vector fields $X_{L\omega}, X_{R\omega}, X_{L2\omega}, X_{R1\omega}, X_{R2\omega}, X_{R3\omega}$ passing through the point $p = (x_0, y_0)$ when $t = 0$ is given by

$$
\begin{aligned}
\phi_{X_{L\omega}}(t) \quad &= \left(\frac{1}{(c^\omega)^2}\Big(-a^\omega c^\omega d^\omega t + a^\omega d^\omega e^{c^\omega t} + a^\omega c^\omega y_0 e^{c^\omega t} - a^\omega c^\omega y_0 - a^\omega d^\omega +\right.\\
&\left. b^\omega(c^\omega)^2 t + (c^\omega)^2 x_0\Big), \tfrac{d^\omega e^{c^\omega t}+c^\omega y_0 e^{c^\omega t}-d^\omega}{c^\omega}\right),\\
\phi_{X_R}(t) \quad &= \Big(t + x_0, \tfrac{1}{2g^\omega}\Big(\sqrt{4e^\omega g^\omega-(f^\omega)^2}\tan\Big(\tfrac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}+\\
&\tan^{-1}\Big(\tfrac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\Big)\Big)-f^\omega\Big)\Big),\\
\phi_{X_{L2}}(t) \quad &= \Big(\tfrac{e^{c^\omega t}(c^\omega y_0+d^\omega)-d^\omega}{c^\omega}, \tfrac{1}{2(c^\omega)^3}\Big(-2a^\omega(c^\omega)^2 d^\omega t - 2a^\omega(c^\omega)^2 y_0 +\\
&2e^{c^\omega t}(c^\omega y_0 + d^\omega)(c^\omega(a^\omega+\eta_1) - 2d^\omega\eta_2) - 2a^\omega c^\omega d^\omega + 2b^\omega(c^\omega)^3 t +\\
&2(c^\omega)^3 x_0 - 2(c^\omega)^2 d^\omega\eta_1 t - (c^\omega)^2\eta_2 y_0^2 - 2(c^\omega)^2\eta_1 y_0 + 2c^\omega(d^\omega)^2\eta_2 t -\\
&2c^\omega d^\omega\eta_1 + \eta_2 e^{2c^\omega t}(c^\omega y_0 + (d^\omega)^2 + 2c^\omega d^\omega\eta_2 y_0 + 3(d^\omega)^2\eta_2\Big)\Big),\\
\phi_{X_{R1}}(t) \quad &= \Big(\tfrac{1}{2\varepsilon}\Big(\sqrt{-(\varepsilon-4)\varepsilon}\tan\Big(\tfrac{1}{2}t\sqrt{-(\varepsilon-4)\varepsilon}+\tan^{-1}\Big(\tfrac{(2x_0-1)\sqrt{\varepsilon}}{\sqrt{4-\varepsilon}}\Big)\Big)+\varepsilon\Big),\\
&\tfrac{1}{2g^\omega}\Big(\sqrt{4e^\omega g^\omega-(f^\omega)^2}\tan\Big(\tfrac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}\\
&+\tan^{-1}\Big(\tfrac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\Big)\Big)-f^\omega\Big)\Big), \phi_{X_{R2}}(t)\\
&= \Big(\tfrac{1}{2(g^\omega)^2}\Big(-(\eta_1 g^\omega - f^\omega\eta_2)\Big(2\log\Big(\cos\Big(\tfrac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}+\\
&\tan^{-1}\Big(\tfrac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\Big)\Big)\Big)+\log\Big(1-\tfrac{(f^\omega+2g^\omega y_0)^2}{(f^\omega)^2-4e^\omega g^\omega}\Big)\Big)+\\
&\eta_2\sqrt{4e^\omega g^\omega-(f^\omega)^2}\tan\Big(\tfrac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}+\tan^{-1}\Big(\tfrac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\Big)\Big)-\\
&g^\omega(2\eta_2(e^\omega t + y_0) + f^\omega\eta_1 t) + f^\omega\eta_2(f^\omega t - 1) + 2(g^\omega)^2(t + x_0)\Big),\\
&\tfrac{1}{2g^\omega}\Big(\sqrt{4e^\omega g^\omega-(f^\omega)^2}\tan\Big(\tfrac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}+\\
&\tan^{-1}\Big(\tfrac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\Big)\Big)-f^\omega\Big)\Big),
\end{aligned}
$$

$$\begin{aligned}\phi_{X_{R3}}(t) &= \left(\frac{1}{\pi}\tan^{-1}\left(\frac{(\varepsilon+1)\tanh\left(\pi t\sqrt{\varepsilon^2-1}+\tanh^{-1}\left(\frac{(\varepsilon-1)\tan(\pi x_0)}{\sqrt{\varepsilon^2-1}}\right)\right)}{\sqrt{\varepsilon^2-1}}\right),\right.\\ &\frac{1}{2g^\omega}\left(\sqrt{4e^\omega g^\omega-(f^\omega)^2}\tan\left(\frac{1}{2}t\sqrt{4e^\omega g^\omega-(f^\omega)^2}+\right.\right.\\ &\left.\left.\left.\tan^{-1}\left(\frac{f^\omega+2g^\omega y_0}{\sqrt{4e^\omega g^\omega-(f^\omega)^2}}\right)\right)-f^\omega\right)\right),\end{aligned}$$

respectively.

In the following we detail the proof for X_{LL}. In this case considering the flow $\phi_{X_{L-}}(t) = (x_1(t), y_1(t))$ starting at the point $p = (x_0, 0) \in \Sigma_1$, the smallest positive time $t_1(p)$ such that $\phi_{X_{L-}}(t_1(p)) \in \Sigma_2$ is

$$t_1(p) = \frac{\log\left(\frac{c^-}{2d^-}+1\right)}{c^-}.$$

In this way we obtain the half first return map $P_L^- : \Sigma_1 \to \Sigma_2$ given by $P_1^-(x_0, 0) = \phi_{X_{L-}}(t_1(p))$ $= (x_1, 1/2)$.

Considering now the flow $\phi_{X_{L+}}(t) = (x_2(t), y_2(t))$ and the initial condition $p_1 = (x_1, 1/2)$ the smallest positive time such that $\phi_{X_{L+}}(t(p_1)) \in \Sigma_2$ is

$$t_2(p_1) = \frac{\log\left(\frac{c^+}{c^++2d^+}+1\right)}{c^+},$$

that provides the upper half first return map $P_1^+ : \Sigma_2 \to \Sigma_1$ given by $P_L^+(x_1, 1/2) = \phi_{X_{L+}}(t_2(p_1)$ $= (x_2, 0)$. Note that a sufficient condition in order that $t_1(p)$ and $t_2(p_1)$ are the smallest positive times is that $X_{LL} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \left[\Omega^1_{L^-} \cup \Omega^2_{L^-}\right]$.

Finally the first return map $P_{LL} : \Sigma_1 \to \Sigma_1$ is given by the composition $P_{LL}(x_0) = (P_L^+ \circ P_L^-)(x_0) = x_0 + \Delta_{LL}$.

Working in a similar way as in the computation of the first return map P_{LL}, we have obtained for the other first return maps their domains of definition and their expressions. □

4.2. **Proof of Theorem 2.** Now we are able to perform the proof of Theorem 2. Let P_X be the first return map for each PSVF X consider in this paper. Now we define the displacement map

$$d_X(x_0) = P_X(x_0) - x_0.$$

The limit cycles of X are given by simple zeros of d_X. Lemma 5 provides the first return map for PSVF. Thus the proof of statements $(a), (b), (c), (d), (f), (i)$ and (l) follows directly because in each one of these cases the first return map is given by $P_X(x_0) = x_0 + \Delta_X$ where Δ_X is a real number given in function of coefficients of X. Therefore the iterates of P_X are $P_X^k(x_0) = x_0 + k\Delta_X$, or equivalently the k-iterate of the displacement map is $d_X^k(x_0) = k\Delta_X$, where k is an integer number. Considering the equivalence relation (2) that defines the two dimensional torus, we have that $d_X^k(x_0)$ return to x_0 if and only if there exists an integer k_0 such that $k_0\Delta_X \in \mathbb{Z}$, or equivalently Δ_X is a rational number. Otherwise if Δ_X is not a rational number then the trajectory passing through X_0 never closes. In other words, P_X^k is a rotation on the circle with irrational rotation number, so we conclude that all trajectories are dense in torus and the proof follows for these cases.

Jaume, por favor, veja se o argumento usado acima esta bom.

In the following we detail the proofs of statements (e) and (h).

The first return map for X_{RR1+} is given in statement (e) of Lemma 5, so the displacement map in this case is

$$d_{XRR1+}(x_0) = \frac{1}{2}\left(\frac{\sqrt{4-\varepsilon}\tan\left(\xi_1-\tan^{-1}\left(\xi_2+\frac{(1-2x_0)\sqrt{\varepsilon}}{\sqrt{4-\varepsilon}}\right)\right)}{\sqrt{\varepsilon}} - 2x_0 + 1\right),$$

where $\xi_1 = \frac{\sqrt{(4-\varepsilon)\varepsilon}\left(\tan^{-1}(\theta_1^+)-\sec^{-1}(\theta_3^+)\right)}{\sqrt{4e^+g^+-(f^+)^2}}$ and $\xi_2 = \frac{4\sqrt{\varepsilon}\left(\sec^{-1}(\theta_3^-)-\tan^{-1}(\theta_2^+)\right)}{\sqrt{4-\varepsilon}\sqrt{4e^-g^--(f^-)^2}}$ if $\varepsilon > 0$. As $X_{RR1+} \in \left[\Omega^2_{R-} \cap \Omega^3_{R+}\right]$ then $\xi_1 > 0$ and $\xi_2 > 0$. Solving directly the equation $d_{XRR1+}(x_0) = 0$ we obtain the values

$$x_0^{\pm} = \frac{\pm\csc(\xi_1)\sqrt{(4-\varepsilon)\varepsilon\sin^2(\xi_1)\left(4\xi_2\cot(\xi_1)+\xi_2^2-4\right)} + \xi_2\sqrt{(4-\varepsilon)\varepsilon} + 2\varepsilon}{4\varepsilon},$$

recall that $\varepsilon > 0$.

In fact, the radical $R_{RR1+} = (4-\varepsilon)\varepsilon\sin^2(\xi_1)\left(4\xi_2\cot(\xi_1)+\xi_2^2-4\right)$ is given in function of ε and can be written as

$$R_{RR1+}(\varepsilon) = \frac{16\varepsilon^2\left(\left(\sec^{-1}(\theta_3^+)-\tan^{-1}(\theta_1^+)\right)^2\left(\frac{4\sqrt{4e^+g^++(f^+)^2}\left(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\right)}{\sqrt{4e^-g^--(f^-)^2}\left(\tan^{-1}(\theta_1^+)-\sec^{-1}(\theta_3^+)\right)}-4\right)\right)}{(f^+)^2-4e^+g^+} + \mathcal{O}(\varepsilon^{5/2}).$$

Since $X_{RR1+} \in \left[\Omega^2_{R-} \cap \Omega^3_{R+}\right]$ we have that R_{RR1+} is positive. Therefore if $\varepsilon > 0$ there exists two simple zeros of d_{XRR1+}, or equivalently two limit cycles of X_{RR1+}. In Example 6 we show a PSVF X_{RR1+} presenting exactly two limit cycles.

For the case (h) the displacement map of X_{LR1-} is

$$d_{XLR1-}(x_0) = \frac{1}{2}\left(\xi_3 - 2x_0 + \frac{\tanh\left(\xi_4+\tanh^{-1}\left((2x_0-1)\sqrt{\frac{\varepsilon}{\varepsilon-4}}\right)\right)}{\sqrt{\frac{\varepsilon}{\varepsilon-4}}}\right),$$

where $\xi_3 = \frac{2\log\left(\frac{c^+}{c^++2d^+}+1\right)(b^+c^+-a^+d^+)+c^+(a^++c^+)}{(c^+)^2}$ and $\xi_4 = \frac{\sqrt{4-\varepsilon}\sqrt{-\varepsilon}\left(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\right)}{\sqrt{4e^-g^--(f^-)^2}}$ for $X_{LR1-} \in \left[\Omega^1_{L+}\cup\Omega^2_{L+}\right]\cap\Omega^2_{R-}$ and $\varepsilon < 0$. Note that $\xi_3 > 0$. Solving the equation $d_{XLR1-}(x_0) = 0$ we obtain for x_0 the values

$$x_0^{\pm} = \pm\frac{\sqrt{\varepsilon\left(\frac{4(\xi_3-1)\varepsilon\coth(\xi_4)}{\sqrt{\frac{\varepsilon}{\varepsilon-4}}} + ((\xi_3-2)\xi_3+5)\varepsilon - 16\right)} + \xi_3\varepsilon + \varepsilon}{4\varepsilon}.$$

These values of x_0 are real numbers because $\varepsilon < 0$, $a^+c^+ > 0$ and $b^+c^+ > a^+d^+$. Moreover, the radical $R_{LR1-} = \varepsilon(\frac{4(\xi_3-1)\varepsilon\coth(\xi_4)}{\sqrt{\frac{\varepsilon}{\varepsilon-4}}} + ((\xi_3-2)\xi_3+5)\varepsilon-16)$ in terms of ε can be written as

$$\sqrt{-\varepsilon}\left(\sqrt{\frac{4\sqrt{4e^-g^--(f^-)^2}\left(2\log\left(\frac{c^+}{c^++2d^+}+1\right)(b^+c^+-a^+d^+)+a^+c^+\right)}{(c^+)^2\left(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\right)}} + 16\right) + \mathcal{O}(\varepsilon^{3/2}).$$

So assuming that $X_{LR1-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$, $\varepsilon < 0$, $a^+c^+ > 0$ and $b^+c^+ > a^+d^+$, then $R_{LR1-} > 0$. A vector field X_{LR1-} with exactly two limit cycle is presented in Example 7.

For the case (g), similarly to the previous cases, the displacement map for X_{RR3+} is

$$d_{XRR3+}(x_0) = -\frac{1}{\pi}\left(\tan^{-1}\left(\frac{\sqrt{\varepsilon+1}\tan\left(\xi_5 + \tan^{-1}\left(\frac{(\varepsilon-1)\tan(\xi_6+\pi x_0)}{\sqrt{1-\varepsilon^2}}\right)\right)}{\sqrt{1-\varepsilon}}\right)\right) - x_0,$$

where $\xi_5 = \frac{2\pi\sqrt{1-\varepsilon^2}\left(\tan^{-1}(\theta_2^+)-\tan^{-1}(\theta_1^+)\right)}{\sqrt{4e^+g^+-(f^+)^2}}$ and $\xi_6 = \frac{2\pi\left(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\right)}{\sqrt{4e^-g^--(f^-)^2}}$. As $X_{RR3+} \in \left[\Omega^2_{R^-} \cap \Omega^1_{R^+}\right]$ then $\xi_5 < 0$ and $\xi_6 > 0$. The solutions of equation $d_{XRR3+}(x_0) = 0$ are

$$\begin{aligned} x_0^1 &= \frac{\cos^{-1}\left(\frac{\sqrt{1-\varepsilon^2}\cot(\xi_5)\sin(\xi_6)-\cos(\xi_6)}{\varepsilon}\right) - \xi_6}{2\pi} + k_1,\\ x_0^2 &= \frac{-\cos^{-1}\left(\frac{\sqrt{1-\varepsilon^2}\cot(\xi_5)\sin(\xi_6)-\cos(\xi_6)}{\varepsilon}\right) - \xi_6}{2\pi} + k_2, \end{aligned}$$

where k_1, k_2 are integer numbers. In the torus we obtain only two distinct points and the integers k_1 and k_2 are the smallest such that $x_0^1, x_0^2 \in [0, 1]$. In Example 8 we perform a PSVF with two limit cycles.

Remain to prove the statement (j) when $X_{LR3-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$. In this case the displacement map is

$$d_{XLR3-}(x_0) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{\sqrt{\varepsilon+1}\tan\left(\xi_7 - \tan^{-1}\left(\frac{(\varepsilon-1)\tan(\pi x_0)}{\sqrt{1-\varepsilon^2}}\right)\right)}{\sqrt{1-\varepsilon}}\right)\right) + \xi_8 - x_0,$$

where $\xi_7 = \frac{2\pi\sqrt{1-\varepsilon^2}\left(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\right)}{\sqrt{4e^-g^--(f^-)^2}}$ and $\xi_8 = \frac{2\log\left(\frac{c^+}{c^++2d^+}+1\right)(b^+c^+-a^+d^+)+a^+c^+}{2(c^+)^2}$. As $X_{LR3-} \in \left[\Omega^1_{L^+} \cup \Omega^2_{L^+}\right] \cap \Omega^2_{R^-}$ then $\xi_7 > 0$.

Considering the change of coordinates $z = \tan(\pi x_0)$ the map d_{XLR3-} can be written

$$d_{XLR3-}(z) = \frac{C_1 z(\varepsilon+1)}{C_1\varepsilon + C_1 - z\sqrt{1-\varepsilon^2}} + \frac{\varepsilon+1}{C_1\sqrt{1-\varepsilon^2} + z(\varepsilon-1)} - \frac{C_2 z}{C_2+z} + \frac{1}{C_2+z},$$

where we denote by $C_1 = \cot(\xi_7)$ and $C_2 = \cot(\xi_8\pi)$. Observe that $d_{XLR3-}(z)$ is given as a linear combination of the functions $g_0(z) = 1/(C_2+z)$, $g_1(z) = z/(C_2+z)$ and $g_2(z) = 1/(C_1\sqrt{1-\varepsilon^2} + z(\varepsilon-1))$. In fact, it is sufficient to prove that the function $g_3(z) = z/(C_1\varepsilon + C_1 - z\sqrt{1-\varepsilon^2})$ is given as a linear combination of the functions g_0, g_1 and g_2. By a direct computation we obtain that the Wronskian $W_3(g_0, g_1, g_2, g_3)(z)$ is zero. Therefore the set of functions $\{g_0(z), g_1(z), g_2(z), g_3(z)\}$ is linearly dependent.

Besides than if we consider the ordered set of functions $\mathcal{F} = \{g_0(z), g_1(z), g_2(z)\}$, the Wronskians $W_1(z)$ and $W_2(z)$ are

$$\begin{aligned} W_1(z) &= \frac{1}{(C_2+z)^2},\\ W_2(z) &= \frac{2(\varepsilon-1)\left(C_2(\varepsilon-1) - C_1\sqrt{1-\varepsilon^2}\right)}{(C_2+z)^3\left(C_1\sqrt{1-\varepsilon^2} + z(\varepsilon-1)\right)^3}. \end{aligned}$$

So $W_1(z) \neq 0$ and $W_2(z) \neq 0$, because $\cot(\xi_8\pi) \neq -\cot(\xi_7)\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$, i.e.

$$\cot\left(\frac{\pi\Big(2\log\Big(\frac{c^+}{c^++2d^+}+1\Big)(b^+c^+-a^+d^+)+a^+c^+\Big)}{2(c^+)^2}\right) \neq$$
$$\cot\left(\frac{2\pi\sqrt{1-\varepsilon^2}\big(\tan^{-1}(\theta_2^-)-\sec^{-1}(\theta_3^-)\big)}{\sqrt{4e^-g^--(f^-)^2}}\right)\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right).$$

Therefore, by Theorem B we obtain that the upper bound of zeros of any linear combination of functions in $\mathcal{F}$ is two and besides that there exists a linear combination of $\mathcal{F}$ presenting exactly two zeros.

In this way, as the displacement map d_{XLR3-} is given as a *specific* linear combination of functions of $\mathcal{F}$ we guarantee that the upper bound of zeros of d_{XLR3-} is two (as a function of $z = \tan(\pi x_0)$). But if z_1, z_2 are the zeros of d_{XLR3-} then there exists $x_0^1 + k_1$ and $x_0^2 + k_2$ real numbers where k_1, k_2 are integer numbers such that $\tan(\pi(x_0^1+k_1)) = z_1$ and $\tan(\pi(x_0^2+k_2)) = z_2$. We choose the integers k_1 and k_2 such that $x_0^1, x_0^2 \in [0,1]$. Despite the displacement map d_{XLR3-} is a *specific* linear combination of g_0, g_1 and g_2 in Example 9 we are able to present values of the parameters such that X_{LR3-} presents exactly two limit cycles. $\square$

4.3. **Proof of Theorem 3.** Recall that $X_{Ck}(x,y) = (X_C(x,y), X_k(x,y)$ is defined on the torus, with $X_C(x,y) = X_C^+(x,y) = (b^+, d^+)$ defined on Σ^+ and $X_k(x,y) = (\alpha, \beta\cos(2k\pi x)$ is defined on Σ^-.

It is straightforward to obtain the expressions of the flows $\phi_{X_k}(t)$ and $\phi_{X_C}(t)$ of X_k and X_C^+ respectively, passing through the point $p = (x_0, y_0)$ when $t = 0$, namely

$$\begin{aligned} \phi_{X_k}(t) &= \left(\alpha t + x_0, y_0 - \frac{\beta\sin(2\pi k x_0)}{2k\pi\alpha}\frac{\beta\sin(2\pi\alpha k t + 2\pi k x_0)}{2k\pi\alpha}\right), \\ \phi_{X_C^+}(t) &= (b^+ t + x_0, d^+ t + y_0)\,. \end{aligned}$$

The fly maps $P_{X_k} : \{(x,y) \in Q;\ y = 0\} \to \{(x,y) \in Q;\ y = 1/2\}$ of ϕ_{X_k} and $P_{X_C} : \{(x,y) \in Q;\ y = 1/2\} \to \{(x,y) \in Q;\ y = 1\}$ of $\phi_{X_C^+}$ are given by

$$P_{X_k}(x_0, 0) = \left(\frac{\arcsin\left(\dfrac{k\pi\alpha + \beta\sin(2\pi k x_0)}{\beta}\right)}{2k\pi}, \frac{1}{2}\right),$$
$$P_{X_C^+}(x_0, 1/2) = \left(\frac{b^+ + 2d^+ x_0}{2b}, 1\right),$$

where $m \in \{0, \ldots, k\}$ is such that $x_0 \in [m/k, (m+1)/k]$.

Thus the Poincaré map $P_{X_{Ck}} : \{(x,y) \in Q;\ y = 0\} \to \{(x,y) \in Q;\ y = 1\}$ is given by $P_{X_{Ck}}(x_0, 0) = (P_{X_C} \circ P_{X_k})(x_0, 0) = (P_1(x_0), 1)$, with

$$P_1(x_0) = \frac{1}{2k\pi}\arcsin\Big(\sin(2k\pi x_0) + k\alpha\pi/\beta\Big) + \frac{b^+}{2d^+} + \frac{m}{k}.$$

So to find limit cycles we have to find the simple zeros of the displacement map

$$d_{X_{Lk}}(x_0) = P_1(x_0) - x_0, \tag{7}$$

for $x_0 \in [0,1]$. Now we show that for every $m = 0, \dots, k-1$, there are at most one solution for (7) with $x_0 \in (m/k, (m+1)/k)$, thus there is at most k limit cycles for X_{Lk}.

Now we study the solutions of

$$d^+ \arcsin\left(\frac{k\pi\alpha + \beta\sin(2\pi k x_0)}{\beta}\right) = 2x_0 d^+ k\pi - b^+ k\pi - 2md^+\pi \tag{8}$$

for $x \in [0,1]$, where $m \in \mathbb{Z}$ and $x \in (m/k, (m+1)/k)$.

Fix $m = 0$ and $k = 1$ without loss of generality (we can always restrict ourselves to $x \in [0, 1/k]$).

Before we conclude the analysis we discuss the tangency points of X_{Ck}. For $k = 1$, the tangency points are $(1/4, 0)$, $(3/4, 0)$, $(1/4, 1/2)$ and $(3/4, 1/2)$. According to the signs of b^+, d^+, α, and β, the segments between these points vary between (stable or unstable) sliding segments and crossing segments.

Suppose $b^+\alpha < 0$ and $d^+\beta < 0$. Then the segments $[1/4, 3/4] \times \{0\}$ and $[1/4, 3/4] \times \{1/2\}$ are crossing regions. Solutions passing outside these segments cannot be limit cycles.

Note that the function $g(x_0) = d^+ \arcsin\left(\frac{k\pi\alpha + \beta\sin(2\pi k x_0)}{\beta}\right)$ has two critical points, $1/4$ and $3/4$, so it is monotone on $(1/4, 3/4)$. Therefore the straight line $2x_0 d^+ k\pi - b^+ k\pi - 2md^+\pi$ meets the graph of g at most in one point. Thus there is at most 1 limit cycles for X_{Ck} with $k = 1$. It is easy to see that there are at most k limit cycles for X_{Ck}. In Example 10 we provided values of the coefficients α, β, b^+ and d^+ for which the PSVF X_{Ck} presents one limit cycle for $k = 1$.

5. Final remarks and some examples

In the present section we exhibit explicit values for the parameters of the PSVFs X_{RR1+}, X_{RR3+}, X_{LR1-} and X_{LR3-} for which they realize their upper bound on the maximum number of limit cycles.

Example 6. *If* $e^+ = 0.466532$, $f^+ = 0.1$, $g^+ = 0.541227$, $e^- = -0.35481$, $f^- = 0.4$, $g^- = -0.817339$ *and* $\varepsilon = 0.02$*, then the displacement map associated to* $X_{RR1+}(x, y)$ *given in* (4) *is*

$$d_{XRR1+}(x_0) = 7.05337 \tan\left(0.217 - \tan^{-1}(0.290888 - 0.141776\, x_0)\right) - x_0 + 0.5.$$

Solving the equation $d_{XRR1+}(x_0) = 0$ *we obtain the points* $x_0^1 = 0.571897$ *and* $x_0^2 = 1.97984 = 1 + 0.97984$ *which represents the points in the torus:* $y_0^1 = 0.571897$ *and* $y_0^2 = 0.97984$*. In other words, this means that the trajectory passing through* y_0^2 *rotates one time before return to* y_0^2*.*

Example 7. *If* $a^+ = -4.47442$,$b^+ = 0$, $c^+ = 1$, $d^+ = 1$, $e^- = 0.0355785$, $f^- = \sqrt{3}$, $g^- = 28.1069$ *and* $\varepsilon = -0.08$*, then the displacement map associated to* $X_{LR11}(x, y)$ *given in* (4) *is*

$$d_{XLR1-}(x_0) = -x_0 + 3.57071 \tanh\left(\tanh^{-1}(0.140028(2x_0 - 1)) + 0.28\right) - 0.45,$$

and its solutions of $d_{XLR1-}(x_0) = 0$ *in the torus are* $x_0^1 = 0.286257$ *and* $x_0^2 = 0.763743$*.*

Example 8. *If* $e^+ = 0.584555$, $f^+ = 0.130158$, $g^+ = 0.434921$, $e^- = 0.670355$, $f^- = \sqrt{3}$, $g^- = 1.49175$ *and* $\varepsilon = 0.06$*, then the displacement map associated to* $X_{RR3+}(x, y)$ *given in* (4) *is*

$$d_{XRR3+}(x_0) = \frac{\tan^{-1}\left(1.06191 \tan\left(\tan^{-1}(0.941697 \tan(\pi x_0 + 1.4)) + 1.7\right)\right)}{\pi} - x_0,$$

and the solutions of $d_{XRR3+}(x_0) = 0$ *in the torus are* $x_0^1 = 0.15119$ *and* $x_0^2 = 0.403176$*.*

Example 9. *If* $a^+ = 1.21371$, $b^+ = 1$, $c^+ = 1$, $d^+ = 0$, $e^- = 0.250971$, $f^- = \sqrt{3}$, $g^- = 3.98453$ *and* $\varepsilon = 0.04$, *then the displacement map associated to* $X_{LR3-}(x, y)$ *given in* (4) *is*

$$d_{XLR3-}(x_0) = -x_0 + 0.31831 \tan^{-1}\left(1.04083 \tan\left(\tan^{-1}(0.960769 \tan(\pi x_0)) + 2.2\right)\right) + 1.3,$$

and the solutions of $d_{XLR3-}(x_0) = 0$ *in the torus are* $x_0^1 = 0.397519$ *and* $x_0^2 = 0.902481$, *see Figure 1.*

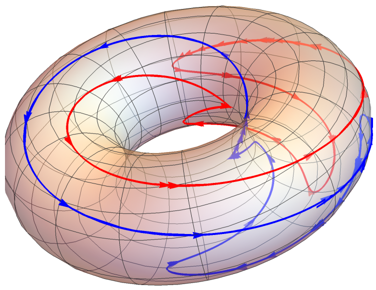

FIGURE 1. The two limit cycles presented in Example 9.

Example 10. *Finally we provide an example with exactly* k *limit cycles for* X_{Ck} *(see Theorem 3) for* $k = 1$. *Given the vector field* $X_k(x, y) = (\alpha, \beta \cos(2k\pi x)$ *with* $\alpha, \beta > 0$ *and* $k > 0$ *an integer, we will construct a vector field* $X_C(x, y) = (b^+, d^+)$ *with a limit cycle. Note that for every* $m = 0, \ldots, k-1$ *we have*

$$P_{X_k}\left(\frac{1}{8k} + \frac{m}{k}, \frac{1}{2}\right) = \left(\frac{1}{2k\pi} \arcsin\left(\frac{k\pi\alpha}{\beta} + \frac{\sqrt{2}}{2}\right) + \frac{m}{k}, \frac{1}{2}\right).$$

So the first restriction is $-1 < \dfrac{k\pi\alpha}{\beta} + \dfrac{\sqrt{2}}{2} < 1$. *Now we fix* $m = 0$ *and prove that we have at least limit cycles for* $x \in [0, 1/k]$.

Let $\Delta_{Ck} = \dfrac{1}{8k} - \dfrac{1}{2k\pi} \arcsin\left(\dfrac{2k\pi\alpha + \beta\sqrt{2}}{2\beta}\right)$ *and consider* $X_C(x, y) = (\Delta_{Ck}, 1/2)$, *i.e.* $b^+ = \Delta_{Ck}$ *and* $d^+ = 1/2$. *By construction,* $P_{X_{Ck}}(1/8k, 0) = (1/8k, 1)$, *so we have a fixed point of the Poincaré map of* X_{Ck}, *and consequently a limit cycle. The derivative of the Poincaré map on this fixed point is*

$$P_1'(1/8k) = \left(\sqrt{\frac{\beta^2 - 2\sqrt{2}\pi\alpha\beta k - 2\pi^2\alpha^2 k^2}{\beta^2}}\right)^{-1},$$

that is not zero under generic conditions. So this is an isolated fixed point, providing a limit cycle, see Figure 2. Thus we have exactly $k = 1$ *limit cycles.*

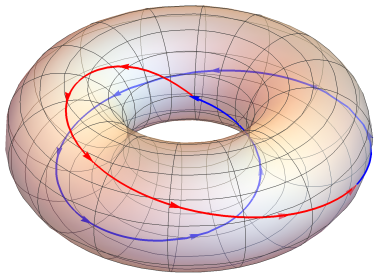

FIGURE 2. The only limit cycle for X_{Ck} with $k = 1$ in Example 10.

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