

GLOBAL DYNAMICS OF THE REAL SECANT METHOD

ANTONIO GARIJO AND XAVIER JARQUE

ABSTRACT. We investigate the root finding algorithm given by the secant method applied to a real polynomial p as a discrete dynamical system defined on \mathbb{R}^2 . We study the main dynamical properties associated to the basins of attraction of the roots of p and show the existence of stable dynamics not related to them. We extend the secant map to the punctured torus \mathbb{T}_∞^2 and the real projective plane $\mathbb{R}\mathbb{P}^2$ which allow us to better understand the dynamics of the secant method near ∞ .

Keywords: Root finding algorithms, iteration, secant method.

1. INTRODUCTION

Root finding algorithms are commonly used as an efficient way to find numerical solutions of non linear equations which cannot be solve explicitly. Consequently its range of applicability is wide over all areas like engineering, economics, sociology or biology.

The common idea behind any such algorithm is to consider an initial *guess or seed* of the unknown solution and construct a sequence converging to the actual solution. There are three immediate questions. If the equation has more than one solution, as it happens in most cases, how to find different seeds converging to all of them? Is it possible to have open regions where seeds do not converge to any solution? What can be said about the speed (number of steps) to have a reasonable approximation of the solution?

One way to find out wide and deep answers to these questions is to study the root finding algorithms as discrete dynamical systems. Set $X = \{\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}\mathbb{P}^n, \mathbb{C}\mathbb{P}^n, \mathbb{T}^n\}$. Roughly speaking a discrete dynamical system over X , known as *phase space*, is a map $f : X \rightarrow X$ and the orbits induced by this map starting at $x_0 \in X$, that is, $\{x_n := f^n(x_0)\}_{n \in \mathbb{N}}$ where

$$f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}.$$

The main goal when studding dynamical systems is to describe the global picture of the asymptotic behaviour of those orbits when x_0 runs over all X . In particular the study of *fixed points*, i.e. x_0 in X such that $f(x_0) = x_0$, or *periodic points of (minimal) period q* or *q -periodic points*, i.e. x_0 in X such that there exist $q \geq 2$ satisfying $f^q(x_0) = x_0$ and $f^\ell(x_0) \neq x_0$ for all $\ell < q$. Those points, which can be either attracting or repelling (or none) depending if all nearby seeds correspond to orbits converging or diverging (or both) to them, play a key role on the global dynamics. In fact, given an attracting fixed ($q = 1$) or q -periodic point (orbit) $x_0 \in X$, its *basin of attraction* is denoted by $A(x_0)$ and given by

$$A(x_0) = \{x \in X \mid f^{nq}(x) \rightarrow x_0, \text{ as } n \rightarrow \infty\},$$

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or, in other words, $A(x_0)$ could be defined as the maximal open set where orbits converge to x_0 . The connected component $A^*(x_0) \subset A(x_0)$ which contains x_0 is called the *immediate basin of attraction of x_0* .

How this connect with root finding algorithms is almost direct. Let p be a polynomial and suppose we want to solve $p(x) = 0$ where x belongs to X . A root finding algorithm is a discrete dynamical system $f_p : X \rightarrow X$ so that its orbits $\{x_n := f_p^n(x_0)\}_{n \in \mathbb{N}}$, converge to the roots of p for *most* initial conditions.

The most well-known and universal root finding algorithm is Newton's method. Assume, to simplify the presentation, that p is a real polynomial with $\deg(p) \geq 2$. The Newton's method, is defined as

$$N_p : \mathbb{C} \mapsto \mathbb{C}, \quad N_p(z) = z - \frac{p(z)}{p'(z)}.$$

Observe that zeroes of p coincide with the fixed points of N_p . Moreover easy computations show that if $\zeta \in \mathbb{C}$ is a simple root of p then $N_p'(\zeta) = 0$. This implies that every root ζ belongs to $A(\zeta)$ the immediate basin of attraction of ζ , for which the sequences $\{z_n := N_p^n(z_0)\}_{n \in \mathbb{N}}$ tend to ζ ; so Newton's method is a root finding algorithm. Certainly the dynamical system is not well defined at the critical points of p . To go over this problem we extend the phase space from \mathbb{C} to $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. One can show by the use of the charts defined on $\hat{\mathbb{C}}$ that the Newton's map is well defined at the whole Riemann sphere, and ∞ is a repelling fixed point of N .

Since Newton's method is one of the most powerful tools for solving polynomial or transcendental equations, the literature about it, from a dynamical system point of view, is extremely large and it was the starting point of *holomorphic dynamics* (see [Cay79a, Cay79b, Cay80]). This approach has had important numerical implications. For instance we refer to M. Shishikura [Shi09] who proved a remarkable theorem implying the simple connectivity of the immediate basins of attraction (see also [Prz89]), and we refer to J. Hubbard, D. Schleicher and S. Sutherland [HSS01], who provided a universal set (only depending on the degree of the polynomial) of initial conditions to find out all roots of a polynomial. As a counterpart it is known that for certain polynomials of degree larger than two there are open sets of initial conditions for which N_p do not converge to any root of p (see [McM87]) and so Newton's method might have problems as a root finding algorithm. Finally we refer to [BFJK14] and [BFJK15] for Newton's method applied to transcendental maps.

Alternative to Newton's method, another well known root finding algorithm is the *secant method*. Surprisingly, there are few references about the secant method as a dynamical system. This is in fact the main topic of this paper. More precisely, if again we denote by p a real polynomial, with $\deg(p) \geq 2$, the secant method is the root finding algorithm associated to the function

$$S_p : \mathbb{C}^2 \mapsto \mathbb{C}^2, \quad S_p : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} w \\ w - p(w) \frac{w-z}{p(w)-p(z)} \end{pmatrix}.$$

One major difference with respect to Newton's method is that Newton's map depends on one variable (real or complex) while secant method depends on two variables (real or complex). Thus, the natural phase space for the secant method is \mathbb{R}^2 or \mathbb{C}^2 .

Of course there are several papers in the literature studding discrete dynamical systems on \mathbb{C}^2 (or $\mathbb{C}\mathbb{P}^2$). For example there are many papers on polynomial automorphisms of \mathbb{C}^2 (see for instance [BS91a, BS91b, BS92, Duj04, DL15]) or, more focussed, on the complex version of the (polynomial) Hénon map (see for instance [HOV94, HOV95, HPV00, BS06, FsS92, ABFsP17]).

There is also a wide literature on birational maps (see [CMn11, BD05, Bed03, CZ14]) but essentially all of these works refer to dynamical systems generated by diffeomorphisms. In contrast the secant map defined in the \mathbb{R}^2 or in \mathbb{C}^2 is not an injective map at all, introducing new and rich potential dynamics.

A first step towards the understanding of the complexity of this dynamical system is to restrict the attention to the *real version* of the secant method. More precisely, we will assume through the whole paper that p is a monic, real polynomial of degree $k \geq 2$ having $n \leq k$ simple real roots denoted by $\alpha_1 < \alpha_2 < \dots < \alpha_n$. Then we consider the secant method acting on \mathbb{R}^2 which is the root finding algorithm generated by the real secant map

$$(1) \quad S := S_p : \mathbb{R}^2 \mapsto \mathbb{R}^2, \quad S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ y - p(y) \frac{y-x}{p(y)-p(x)} \end{pmatrix},$$

where the orbit of $(x_0, y_0) \in \mathbb{R}^2$ is given by the points $\{(x_n, y_n) = S^n(x_0, y_0) \in \mathbb{R}^2\}_{n \in \mathbb{N}}$. Observe that the real versions of Newton and secant methods are based on a similar idea. In the Newton case, for a given seed $x_0 \in \mathbb{R}$, the next value $x_1 = N_p(x_0)$ is the intersection between the x -axis and the tangent line through the point $(x_0, p(x_0))$ with the while in the secant case for a given seed $(x_0, y_0) \in \mathbb{R}^2$, the next value is $(y_0, x_1) = S(x_0, y_0)$ where x_1 is the intersection between the x -axis and the secant line through the points $(x_0, p(x_0))$ and $(y_0, p(y_0))$. It is also worth to be noticed that S has no definition at points where the denominator of the second component is zero. We will run over with this difficulty later.

The natural framework for studying S as a plane dynamical system is the iteration of rational-like maps on \mathbb{R}^2 (see [BGM99, BGM03, BGM05], and reference therein, for a more complete discussion). We introduce here the notation we need to state our main results. Consider the iteration of a map defined in the plane given by

$$(2) \quad T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} F(x, y) \\ N(x, y)/D(x, y) \end{pmatrix},$$

where F , N and D are assumed to be differentiable functions. Of course (1) is a particular case. Let

$$\delta_T = \{(x, y) \in \mathbb{R}^2 \mid D(x, y) = 0\},$$

be the subset of \mathbb{R}^2 where T is not, a priori, well-defined. Therefore the map $T : E \rightarrow E$ defines a smooth dynamical system on E , where

$$E := E_T = \mathbb{R}^2 \setminus \bigcup_{n \geq 0} T^{-n}(\delta_T).$$

From definition E corresponds to points whose infinite T -orbit is well defined, or equivalently, its complement in \mathbb{R}^2 are all seeds for which the orbit gets eventually truncated. Roughly speaking T sends points of δ_T to infinity since the denominator D is zero, except at those points of δ_T where also the numerator N is zero and hence the value of T is uncertain.

We say that a point Q in δ_T is a *focal point* if the second component of T evaluated at Q takes the form $0/0$ (i.e. $N(Q) = D(Q) = 0$), and there exists a smooth simple arc $\gamma(\tau)$, $\tau \in [0, \epsilon)$, with $\gamma(0) = Q$, such that $\lim_{\tau \rightarrow 0^+} T(\gamma(\tau))$ exists and it is finite. We denote by \mathcal{Q} the set of all focal points of the dynamical system given by (2). Focal points are point of discontinuity of the map T (see Figure 1). By definition the set of focal points \mathcal{Q} is a subset of the set of no definition of the secant map δ_T .

We are ready to state our main results. The first one is about the shape and distribution of the basins of attraction of the fixed points of S , in particular we show that any focal point belongs to the boundary of the basin of attraction of all the roots of the polynomial p . We recall that p is a monic, real polynomial of degree $k \geq 2$ having $n \leq k$ simple real roots denoted by $\alpha_1 < \alpha_2 < \dots < \alpha_n$.

Theorem A. *The secant map S defined in (1) induces a smooth dynamical system on E . Moreover the following statements hold.*

- (a) *The only fixed points of S are the points $(\alpha_\ell, \alpha_\ell)$, $\ell = 1, \dots, n$, and they are all attracting.*
 - (b) *Each basin of attraction $A(\alpha_\ell) := A(\alpha_\ell, \alpha_\ell)$, $\ell = 1, \dots, n$, is unbounded.*
 - (c) *If $n = 1$ or $n = 2$, then $A^*(\alpha_1)$ (and $A^*(\alpha_2)$, if $n = 2$) are unbounded. If $n \geq 3$ then $A^*(\alpha_1)$ and $A^*(\alpha_n)$ are unbounded while $A^*(\alpha_\ell)$ with $2 \leq \ell \leq n - 1$ are bounded.*
 - (d) *The set of focal points of the secant map is $\mathcal{Q} = \{(\alpha_i, \alpha_j), i, j = 1 \dots n, i \neq j\}$.*
- Moreover

$$Q_{i,j} \in \bigcap_{1 \leq \ell \leq n} \partial A(\alpha_\ell) \quad \text{and} \quad Q_{j-1,j} \in \partial A^*(\alpha_j)$$

A natural question when studying root finding algorithms is to have a control of the possible *stable* dynamics different from the attracting basins of the roots of the polynomial, studied above (see again [McM87]). The existence of those open sets will bound the effectiveness of the method as a root finding algorithm.

Theorem B. *The following statements hold.*

- (a) *The secant map S defined in (1) has no periodic orbits of minimal period either two or three.*
- (b) *There exists a polynomial p such that the secant map applied to p exhibits an attracting periodic orbit of minimal period four. In particular the dynamical plane has open regions of initial conditions for which S does not converge to any root of the polynomial p .*

We turn our attention to the behavior of the secant map near infinity. Using the expression of the secant map it is possible to continuously extend the secant map on the limit points of vertical lines (parallel to $x = 0$), horizontal lines (parallel to $y = 0$), and lines passing through the origin.

In Section 3 we study the extension of the secant map, denoted by \hat{S} , to the space \mathbb{T}_∞^2 . Using this approach we are able to define \hat{S} at all points in \mathbb{R}^2 except at the set \mathcal{Q} of focal points. As we will see points in $\delta_S \setminus \mathcal{Q}$ firstly go to infinity in a particular direction and secondly come back to the finite plane in a continuous way. We can also prove the existence of a periodic orbit of minimal period three using this way to go and come back to infinity. This special phenomenon occurs at the set of points $\delta_{S_2} = \{(\beta, \beta) \in \mathbb{R}^2, p'(\beta) = 0\}$. Numerical experiments illustrate an stable asymptotic dynamics near the points in δ_{S_2} , following this periodic orbit of minimal period three. It turns out, however, that the extension \hat{S} is non smooth at those *critical* points and hence we cannot apply the differential matrix to study their stability. However, some numerical experiments seem to indicate that there are nearby points whose orbit converge to this three periodic orbit. In the next theorem we collect the main result on the dynamics of \hat{S} defined on \mathbb{T}_∞^2 .

Theorem C. *The map $\hat{S} : \mathbb{T}_\infty^2 \setminus \mathcal{Q} \mapsto \mathbb{T}_\infty^2 \setminus \mathcal{Q}$ defines a continuous extension of S . The map \hat{S} is smooth except at the points in $\delta_{S_2} = \{(\beta, \beta) \in \mathbb{R}^2 \text{ with } p'(\beta) = 0\}$. Moreover at these points \hat{S} exhibits a periodic orbit of minimal period three.*

In Section 4 we extend S to the real projective plane \mathbb{RP}^2 . Using homogeneous coordinates, we built a new extension of S in such a way the line at infinity $\ell_\infty \subset \mathbb{RP}^2$ is invariant. The behaviour of the restricted dynamical system at the line of infinity allow us to conclude important dynamical properties of S as a plane dynamical system. Precisely we prove the following result.

Theorem D. *Let $k = \deg(p)$. Let $\varphi(x) = \frac{x^{k-1}-1}{x^k-1}$, where $x \in \mathbb{R}$. The secant map admits an extension \tilde{S} over \mathbb{RP}^2 . In particular $\tilde{S} : \ell_\infty \rightarrow \ell_\infty$ defines the smooth dynamical system (in homogeneous coordinates)*

$$(3) \quad \tilde{S}|_{\ell_\infty} = \begin{cases} \tilde{S}(x) := \tilde{S}[1 : x : 0] = [1 : \varphi(x) : 0], \\ \tilde{S}(\infty) := \tilde{S}[0 : 1 : 0] = [1 : 0 : 0], \end{cases}$$

having a unique attracting fixed point $\eta_k > 0$. Moreover the following statements hold.

- (a) *If k is even, then \tilde{S} has a repelling fixed point $\tau_k < 0$. Moreover, $\tilde{S}^n(x) \rightarrow \eta_k$ as $n \rightarrow \infty$, for all $x \in \mathbb{R} \cup \{\infty\}$ such that $x \neq \tau_k$.*
- (b) *If k is odd, then $\tilde{S}^n(x) \rightarrow \eta_k$ as $n \rightarrow \infty$, for all $x \in \mathbb{R} \cup \{\infty\}$.*

The acknowledge of the behaviour of S near infinity given by this result allow us to conclude some relevant information of the dynamics of S on the plane (see Section 5).

We organize the paper as follows. In Section 2 we prove Theorems A and B. In Section 3 we present the first extension of S over the torus \mathbb{T}_∞^2 and prove Theorem C. In section 4 we introduce the homogeneous coordinates for the projective plane, extend S to it and prove Theorem D. In Section 5 we conclude and present some final remarks.

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2. THE SECANT MAP ON THE REAL PLANE: PROOF OF THEOREMS A AND B

In this section, as well as in the next sections, we will describe the phase portrait of the dynamical system generated by the iterates of the secant map S applied to a real polynomial p of degree $k \geq 2$ having $1 \leq n \leq k$ simple real roots denoted by $\alpha_1 < \alpha_2 < \dots < \alpha_n$. Its primer explicit formula is given in (1). It is clear that, due to the rational character of S there is an implicit uncertainty on how to define the image at points where the denominator of the second component is zero. In order to partially run over this problem we set the following sets

$$(4) \quad \begin{aligned} \delta_{S_1} &= \{(x, y) \in \mathbb{R}^2 \mid p(x) = p(y) \text{ with } x \neq y\}, \quad \delta_{S_2} = \{(x, x) \in \mathbb{R}^2 \mid p'(x) = 0\}, \\ \delta_S &= \delta_{S_1} \cup \delta_{S_2} \quad \text{and} \quad E = \mathbb{R}^2 \setminus \bigcup_{n \geq 1} S^{-n}(\delta_S) \end{aligned}$$

The following lemma shows that S extends easily to the points $(x_0, x_0) \in \mathbb{R}^2$, since they are removable singularities of S , unless $p'(x_0) = 0$.

Lemma 2.1. *The secant map defined in (1) admits the following well defined expression on $\mathbb{R}^2 \setminus \delta_S$*

$$(5) \quad S(x, y) = \begin{cases} \left(y, \frac{yq(x,y)-p(y)}{q(x,y)} \right) & \text{if } y \neq x, \\ \left(x, \frac{xp'(x)-p(x)}{p'(x)} \right) & \text{if } y = x, \end{cases}$$

where $q(x, y)$ is a symmetric polynomial such that $p(x) - p(y) = (x - y)q(x, y)$. Moreover, $q(x, x) = p'(x)$ and for any (x_0, y_0) such that $x_0 \neq y_0$ we have that

$$(6) \quad \frac{\partial q}{\partial x}(x_0, y_0) = \frac{p'(x_0) - q(x_0, y_0)}{x_0 - y_0} \quad \text{and} \quad \frac{\partial q}{\partial y}(x_0, y_0) = \frac{q(x_0, y_0) - p'(y_0)}{x_0 - y_0}.$$

In particular (5) defines a smooth dynamical system on E .

Remark 1. *Observe that for a given $x, y \in \mathbb{R}$ with $x \neq y$ the value of $q(x, y)$ is the slope of the secant line through the points $(x, p(x))$ and $(y, p(y))$. Moreover $S_p(x, x) = (x, N_p(x))$ where N_p is the Newton method applied to p .*

Proof. Fix a natural $j \geq 1$. Simple computations show that

$$x^j - y^j = (x - y)(x^{j-1} + x^{j-2}y + \dots + xy^{j-2} + y^{j-1}) := (x - y)q_j(x, y)$$

Thus if we write

$$(7) \quad \begin{aligned} q_j(x, y) &:= x^{j-1} + x^{j-2}y + \dots + xy^{j-2} + y^{j-1} \\ q(x, y) &:= \sum_{j=1}^k a_j q_j(x, y) \end{aligned}$$

we easily conclude

$$(8) \quad p(x) - p(y) = \sum_{j=1}^k a_j (x^j - y^j) = (x - y) \sum_{j=1}^k a_j q_j(x, y) = (x - y) q(x, y).$$

In other words the factor $(x - y)$ divides $p(x) - p(y)$ and the resultant polynomial is symmetric. Moreover, since $q_j(x, x) = x^{j-1} + x^{j-2}x + \dots + xx^{j-2} + x^{j-1} = jx^{j-1}$, we get

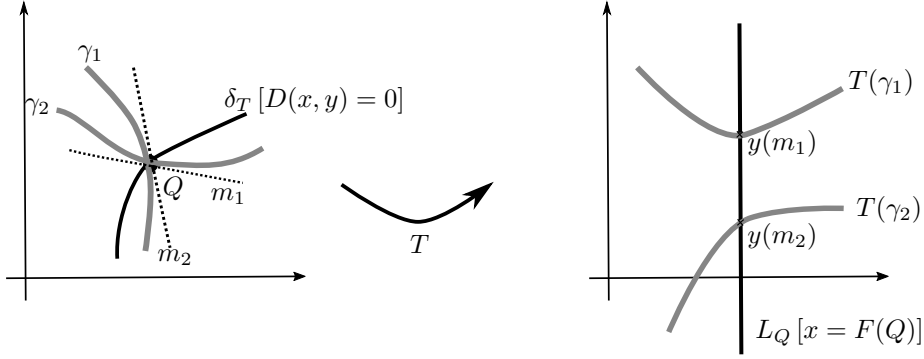
$$q(x, x) = \sum_{i=1}^k a_i q_i(x, x) = \sum_{i=1}^k a_i i x^{i-1} = p'(x).$$

According to this new notation it is immediate to see that (1) writes as (5), while deriving at both sides of the expression (8) we obtain (6). \square

According to (2), and the definition of focal point, by taking different paths, $\gamma(\tau)$ landing at Q , the value of $\lim_{\tau \rightarrow 0^+} T(\gamma(\tau))$ might take different finite values on the, so called, **focal line** $L_Q = \{(x, y) \in \mathbb{R}^2 \mid x = F(Q)\}$. In other words, when we land at a focal point Q with a curve $\gamma(\tau)$ with tangent slope m at Q then the image by T tends to a concrete value $(F(Q), y(m))$, where

$$(9) \quad y(m) = \lim_{\tau \rightarrow 0} \frac{N(\gamma(\tau))}{D(\gamma(\tau))}.$$

In particular, the operator T is not continuous at the focal points. See Figure 1. A natural question to ask is the relation between the slope m and the value of the limit (9). Next result gives an answer for the generic case.

FIGURE 1. Discontinuity of T at a focal point Q .

Theorem 2.2 ([BGM99]). *Let T be the rational map described in (2). Let Q be one of its focal points and assume $N_x(Q)D_y(Q) - N_y(Q)D_x(Q) \neq 0$. Then there is a one-to-one correspondence between the non δ_T -tangent slopes $m \in \mathbb{R}$ of arcs $\gamma(\tau)$ landing at Q , and the points $(F(Q), y) \in L_Q$. The correspondence writes as (see Figure 1)*

$$(10) \quad \begin{aligned} m &\mapsto (F(Q), y(m)) \quad \text{with} \quad y(m) = \frac{N_x(Q) + mN_y(Q)}{D_x(Q) + mD_y(Q)} \\ (F(Q), y) &\mapsto m(y) \quad \text{with} \quad m(y) = \frac{D_x(Q)y - N_x(Q)}{N_y(Q) - D_y(Q)y} \end{aligned}$$

The focal points belong to $\mathbb{R}^2 \setminus E$ but they, and their focal lines, play a key role on the understanding of the global dynamics of the dynamical system generated by T , and so S

Lemma 2.3. *The set \mathcal{Q} of all the focal points of S contains $n(n-1)$ points located at the points $Q_{i,j} = (\alpha_i, \alpha_j)$, $i \neq j$. Moreover, for a given focal point $Q_{i,j}$, the focal line is given by $L_{Q_{i,j}} = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\}$, and hence, for a given i , the $n-1$ focal points $Q_{i,j}$, $j = 1, \dots, n$, $j \neq i$ share the same focal line.*

Proof. As stated if $Q = (x_0, y_0) \in \mathbb{R}^2$ is a focal point of S then the evaluation of its second component at the point Q takes the form $0/0$. According to (5) there are no focal points at the line $x = y$, otherwise we would have $p(Q) = p'(Q) = 0$, a contradiction with the assumption that p has no multiple roots.

Therefore, again from (5), focal points should be solutions of the system of equations

$$\begin{cases} yq(x, y) - p(y) &= 0, \\ q(x, y) &= 0, \end{cases}$$

with $x \neq y$. If $q(x, y) = 0$, we conclude that $p(y) = 0$, and we recall that $p(x) - p(y) = (x - y)q(x, y) = 0$, then a focal point should satisfy $p(x) = p(y) = 0$ and $x \neq y$. Therefore we conclude that S has $n(n-1)$ focal points and they are located at $Q_{i,j} = (\alpha_i, \alpha_j)$, $i, j = 1, \dots, n$, $i \neq j$. It is easy to check that

$$N_x(Q_{i,j})D_y(Q_{i,j}) - N_y(Q_{i,j})D_x(Q_{i,j}) = \frac{p'(\alpha_i)p'(\alpha_j)}{\alpha_i - \alpha_j} \neq 0,$$

since, by hypothesis, the zeros of p are simples. From definition of focal line, for a given $Q_{i,j}$, $i, j = 1, \dots, n$, $i \neq j$, the focal line is given by

$$L_{Q_{i,j}} = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\},$$

and the lemma follows. In Figure 2 we sketch the situation for $n = 3$. □

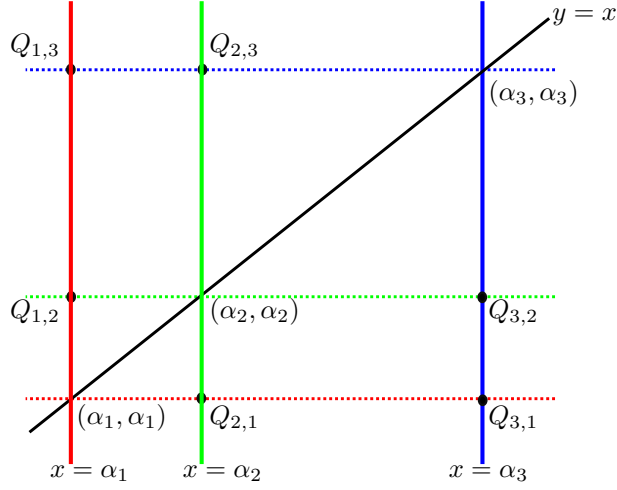


FIGURE 2. Sketch of the dynamical plane of S_p where p is a polynomial with three simple real roots $\alpha_1 < \alpha_2 < \alpha_3$. The focal focal points $Q_{2,1}$ and $Q_{3,1}$ share the focal line $x = \alpha_1$. The focal points $Q_{1,2}$ and $Q_{3,2}$ share the focal line $x = \alpha_2$, and finally, the focal points $Q_{1,3}$ and $Q_{2,3}$ share the focal line $x = \alpha_3$. Red points converge to (α_1, α_1) , green points converge to (α_2, α_2) and blue points converge to (α_3, α_3) .

The following consequence of Hôpital's rule will be needed in the proof of Theorem A.

Lemma 2.4. *Let $f, g : (-\varepsilon, \varepsilon) \rightarrow (-\varepsilon, \varepsilon)$ be two smooth functions such that for some $a \in \mathbb{R}$ and $b \neq 0$ we have*

$$\lim_{\tau \rightarrow 0^+} f(\tau) = \lim_{\tau \rightarrow 0^+} g(\tau) = 0, \quad \lim_{\tau \rightarrow 0^+} f'(\tau) = a \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} g'(\tau) = b.$$

Then

$$\lim_{\tau \rightarrow 0^+} \left(\frac{f(\tau)}{g(\tau)} \right)' = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

Proof. Since

$$\lim_{\tau \rightarrow 0^+} \left(\frac{f(\tau)}{g(\tau)} \right)' = \lim_{\tau \rightarrow 0^+} \frac{f'(\tau)g(\tau) - g'(\tau)f(\tau)}{g^2(\tau)} = \frac{0}{0},$$

we apply Hôpital's rule to obtain

$$\lim_{\tau \rightarrow 0^+} \left(\frac{f(\tau)}{g(\tau)} \right)' = \lim_{\tau \rightarrow 0^+} \left(\frac{f''(\tau)}{2g'(\tau)} - \frac{g''(\tau)f(\tau)}{2g'(\tau)g(\tau)} \right) = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

□

2.1. Proof of Theorem A. First we prove that the map $S : E \rightarrow E$ is of class \mathcal{C}^1 . We denote the two components of $S = (S^1, S^2)$. From the expression of S (5) it is clear that we only need to check the smoothness of the second component of S on the straight line $y = x$ when $p'(x) \neq 0$. Fix a point (x_0, x_0) satisfying this condition. We claim that

$$\frac{\partial S^2}{\partial x}(x_0, x_0) = \frac{\partial S^2}{\partial y}(x_0, x_0) = \frac{1}{2} \cdot \frac{p(x_0)p''(x_0)}{p'(x_0)^2}.$$

From definition we have that

$$\begin{aligned} \frac{\partial S^2}{\partial x}(x_0, x_0) &= \lim_{h \rightarrow 0} \frac{1}{h} [S^2(x_0 + h, x_0) - S^2(x_0, x_0)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[x_0 - p(x_0) \frac{h}{p(x_0+h) - p(x_0)} - \left(x_0 - \frac{p(x_0)}{p'(x_0)} \right) \right] \\ &= \lim_{h \rightarrow 0} -\frac{p(x_0)}{h} \left[\frac{h}{p(x_0+h) - p(x_0)} - \frac{1}{p'(x_0)} \right] \\ &= \lim_{h \rightarrow 0} -\frac{p(x_0)}{h} \left[\frac{1}{p'(x_0) + \frac{p''(x_0)}{2}h + \mathcal{O}(h^2)} - \frac{1}{p'(x_0)} \right], \end{aligned}$$

where the last equality follows from the fact that $p(x_0+h) = p(x_0) + p'(x_0)h + \frac{p''(x_0)}{2}h^2 + \mathcal{O}(h^3)$. Moreover, taking series, we also have

$$\frac{1}{p'(x_0) + \frac{p''(x_0)}{2}h + \mathcal{O}(h^2)} - \frac{1}{p'(x_0)} = -\frac{p'(x_0)^2}{2p''(x_0)}h + \mathcal{O}(h^2).$$

All together imply

$$\frac{\partial S^2}{\partial x}(x_0, x_0) = \frac{1}{2} \cdot \frac{p(x_0)p''(x_0)}{p'(x_0)^2}.$$

Using the fact that $S^2(x, y) = S^2(y, x)$ we conclude that $\frac{\partial S^2}{\partial x}(x_0, x_0) = \frac{\partial S^2}{\partial y}(x_0, x_0)$ using the definition of these partial derivatives. Now we show that the partial derivative $\frac{\partial S^2}{\partial x}$ is a continuous map in a neighborhood of (x_0, x_0) (by symmetry $\frac{\partial S^2}{\partial y}$ will be also a continuous map). Let $(x_1, y_1) \in E$ with $x_1 \neq y_1$. We have

$$\frac{\partial S^2}{\partial x}(x_1, y_1) = \frac{p(y_1)}{q^2(x_1, y_1)} \frac{\partial q}{\partial x}(x_1, y_1).$$

From (7) we have

$$\begin{aligned} \lim_{(x_1, y_1) \rightarrow (x_0, x_0)} \frac{\partial q_j}{\partial x}(x_1, y_1) &= x_0^{j-2} (1 + \dots + j - 1) = \frac{j(j-1)}{2} x_0^{j-2}, \text{ and} \\ \lim_{(x_1, y_1) \rightarrow (x_0, x_0)} \frac{\partial q}{\partial x}(x_1, y_1) &= \frac{p''(x_0)}{2}. \end{aligned}$$

Finally, we get

$$\lim_{(x_1, y_1) \rightarrow (x_0, x_0)} \frac{\partial S^2}{\partial x}(x_1, y_1) = \frac{1}{2} \frac{p(x_0)p''(x_0)}{p'(x_0)^2} = S^2(x_0, x_0).$$

Since the maps S^1 and S^2 are continuous maps in a neighbourhood of (x_0, x_0) we have that S is a differentiable map at (x_0, x_0) with

$$DS(x_0, x_0) = \begin{pmatrix} 0 & 1 \\ \frac{p(x_0)p''(x_0)}{2p'(x_0)^2} & \frac{p(x_0)p''(x_0)}{2p'(x_0)^2} \end{pmatrix}.$$

Now we prove (a). From (5) we see that if (x_0, y_0) is a fixed point of S then $x_0 = y_0$. Moreover it should satisfy

$$(11) \quad p'(x) \neq 0 \quad \text{and} \quad p'(x)x = xp'(x) - p(x),$$

so x should be a zero of p . Fix $(\alpha_\ell, \alpha_\ell)$, $1 \leq \ell \leq n$. Since p has all its roots simple, its differential matrix writes as

$$DS(\alpha_\ell, \alpha_\ell) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1 \leq \ell \leq n$$

So the two eigenvalues of $DS(\alpha_\ell, \alpha_\ell)$ are equal to 0, proving thus that the fixed point $(\alpha_\ell, \alpha_\ell)$ is attracting. This proves (a).

Fix $\ell = 1, \dots, n$. It is easy to see from (5) that, on the one hand, if $j \neq \ell$ and $x \neq \alpha_j$ then $S(x, \alpha_\ell) = (\alpha_\ell, \alpha_\ell)$; and, on the other hand, if $y \in \mathbb{R}$, then $S(\alpha_\ell, y) = (y, \alpha_\ell)$. This proves (b) since the attracting basin of each fixed point $(\alpha_\ell, \alpha_\ell)$ of S contains unbounded segments of the straight lines $r_H := \{(x, y) \in \mathbb{R}^2 \mid y = \alpha_\ell\}$ and $r_V := \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_\ell\}$.

The above arguments also prove (c) if $n = 1$ or $n = 2$. Assume $n \geq 3$. We observe that if $\ell = 2, \dots, n-1$ the fixed point $(\alpha_\ell, \alpha_\ell)$ belongs to the rectangle R_ℓ with

$$(12) \quad \partial R_\ell = [(\alpha_{\ell-1}, \alpha_{\ell-1}), (\alpha_{\ell+1}, \alpha_{\ell-1})] \cup [(\alpha_{\ell+1}, \alpha_{\ell+1}), (\alpha_{\ell-1}, \alpha_{\ell+1})] \cup [(\alpha_{\ell-1}, \alpha_{\ell-1}), (\alpha_{\ell-1}, \alpha_{\ell+1})] \cup [(\alpha_{\ell+1}, \alpha_{\ell-1}), (\alpha_{\ell+1}, \alpha_{\ell+1})]$$

where $[a, b]$, $a, b \in \mathbb{R}^2$ denotes the straight segment from a to b . See Figure 2. Statement (c) now follows from the fact that $\partial R_\ell \cap A^*(\alpha_\ell) = \emptyset$, according to the arguments of the paragraph above.

We now will prove statement (d). Fix $Q := Q_{i,j} = (\alpha_i, \alpha_j) \in \mathbb{R}^2$, a focal point. First we claim that here is a one-to-one correspondence between the non δ_S -tangent slopes of the paths landing at Q and the points of the prefocal line $L_Q = \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_j\}$. Let

$$(13) \quad F(x, y) = y, \quad N(x, y) = yq(x, y) - p(y) \quad \text{and} \quad D(x, y) = q(x, y).$$

From Lemma 2.1 and easy computations we have

$$(14) \quad N_x(Q) = \frac{\alpha_j p'(\alpha_i)}{\alpha_i - \alpha_j} \quad N_y(Q) = \frac{-\alpha_i p'(\alpha_j)}{\alpha_i - \alpha_j} \quad D_x(Q) = \frac{p'(\alpha_i)}{\alpha_i - \alpha_j} \quad \text{and} \quad D_y(Q) = -\frac{p'(\alpha_j)}{\alpha_i - \alpha_j},$$

which, in particular, implies

$$N_x(Q)D_y(Q) - N_y(Q)D_x(Q) = \frac{p'(\alpha_i)p'(\alpha_j)}{\alpha_i - \alpha_j} \neq 0.$$

Now Theorem 2.2 tells us that under these hypothesis, there is a one-to-one correspondence between the slope, m , of any non δ_{S_p} -tangent path $\gamma(\tau)$ landing at Q , and the points of L_Q . Moreover, from (10) and (14) it is easy to see that the bijection is given by

$$(15) \quad y(m) = \frac{\alpha_j p'(\alpha_i) - \alpha_i p'(\alpha_j)m}{p'(\alpha_i) - p'(\alpha_j)m} \quad \text{or} \quad m(y) = \frac{p'(\alpha_i)(\alpha_j - y)}{p'(\alpha_j)(\alpha_i - y)}.$$

In particular, $m = 0$ corresponds to $y = \alpha_j$ and $m = \infty$ corresponds to $y = \alpha_i$. Observe that we know that the horizontal line $r_H := \{(x, y) \in \mathbb{R}^2 \mid y = \alpha_j\}$, outside the focal points, is mapped to the fixed point (α_j, α_j) , and the prefocal line $L_{Q_{j,i}} := \{(x, y) \in \mathbb{R}^2 \mid x = \alpha_i\}$, outside the focal points, maps in two iterates to the fixed point (α_i, α_i) . This implies that $Q \in \partial A(\alpha_j) \cap \partial A(\alpha_i)$. To finish the proof we need to show that $Q \in \partial A(\alpha_\ell)$ for all $\ell \neq j, i$. Clearly the non δ_S -tangent slopes given by

$$(16) \quad m_k = \frac{p'(\alpha_i)(\alpha_j - \alpha_k)}{p'(\alpha_j)(\alpha_i - \alpha_k)} \quad \text{for } 1 \leq k \leq n \text{ with } k \neq i, j,$$

correspond, through the one-to-one correspondence, to the focal points $Q_{j,\ell} = (\alpha_j, \alpha_\ell) \in L_Q$ with $\ell \neq j, i$. So, every path, $\gamma(\tau), \tau \in (0, \varepsilon)$, landing at Q with finite slope $m \in \mathbb{R}$ with $m \neq m_k, k = 1, \dots, n, k \neq i, j$, satisfies that $S(\gamma(\tau))$ lands at a non focal point of L_Q and so, it eventually belongs to the basin of attraction of the fixed point (α_j, α_j) . Therefore the only paths landing at Q with finite slope which could eventually belong to $A(\alpha_\ell)$ for $\ell \neq i, j$ are the ones defined in (16). We claim that for every m_k we have paths landing at Q with slope m_k which eventually belong to each $A(\alpha_\ell)$ for $\ell \neq i, j$. To see the claim let us fix m_ℓ one of the *singular* slopes defined in (16), and let us consider the following κ -family of paths parametrized by $\tau \in (-\varepsilon, \varepsilon)$

$$\gamma_\kappa(\tau) =: (x(\tau), y(\tau)) = (\alpha_i, \alpha_j) + (1, m_\ell)\tau + \frac{1}{2}(1, \kappa)\tau^2 = \left(\alpha_i + \tau + \frac{1}{2}\tau^2, \alpha_j + m_\ell\tau + \frac{1}{2}\kappa\tau^2 \right).$$

Certainly $\gamma_\kappa(0) = Q$, $\gamma'_\kappa(0) = (1, m_\ell)$ and $\gamma''_\kappa(0) = (1, \kappa)$. Our aim is to prove that there is a one-to-one correspondence between κ and $(S \circ \gamma)'(0)$, or equivalently, we want to show that choosing different values of κ we are landing at the corresponding focal point (according to m_0) with all possible slopes. Observe that

$$(S \circ \gamma)'(0) = \left(m_\ell, \left(\frac{f(\tau)}{g(\tau)} \right)' \Big|_{\tau=0} \right),$$

where $f(\tau) := N(x(\tau), y(\tau))$ and $g(\tau) := D(x(\tau), y(\tau))$, and N and D are defined in (13). Since Q is a focal point we have

$$\lim_{\tau \rightarrow 0^+} \frac{f(\tau)}{g(\tau)} = \frac{0}{0}.$$

Some computations show that

$$\lim_{\tau \rightarrow 0^+} f'(\tau) = \alpha_j p'(\alpha_i) - \alpha_i p'(\alpha_j) m_0 := a \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} g'(\tau) = p'(\alpha_i) - p'(\alpha_j) m_0 := b.$$

We claim that $b \neq 0$. Indeed, otherwise, $y(m_\ell) = \infty$ (see (10)) while singular slopes correspond to focal points of the form (α_j, α_ℓ) , $\ell \neq j$. Hence, we are on the hypothesis of Lemma 2.4 to get

$$(17) \quad \lim_{\tau \rightarrow 0^+} \left(\frac{f}{g} \right)'(\tau) = \frac{1}{2b} \left(f''(0) - g''(0) \frac{a}{b} \right).$$

Finally some computations show that

$$f''(\tau) = \begin{bmatrix} N_{xx}(x(\tau), y(\tau)) & x'(\tau) + N_{xy}(x(\tau), y(\tau)) y'(\tau) \\ N_{yx}(x(\tau), y(\tau)) & x'(\tau) + N_{yy}(x(\tau), y(\tau)) y'(\tau) \end{bmatrix} \begin{bmatrix} x'(\tau) + N_x(x(\tau), y(\tau)) x''(\tau) \\ y'(\tau) + N_y(x(\tau), y(\tau)) y''(\tau) \end{bmatrix},$$

and

$$g''(\tau) = \begin{bmatrix} D_{xx}(x(\tau), y(\tau)) & x'(\tau) + D_{xy}(x(\tau), y(\tau)) y'(\tau) \\ D_{yx}(x(\tau), y(\tau)) & x'(\tau) + D_{yy}(x(\tau), y(\tau)) y'(\tau) \end{bmatrix} \begin{bmatrix} x'(\tau) + D_x(x(\tau), y(\tau)) x''(\tau) \\ y'(\tau) + D_y(x(\tau), y(\tau)) y''(\tau) \end{bmatrix}.$$

Thus,

$$(18) \quad \begin{aligned} f''(0) &= N_{xx}(Q) + 2N_{xy}(Q)m_\ell + N_{yy}(Q)m_\ell^2 + N_x(Q) + N_y(Q)\kappa \\ g''(0) &= D_{xx}(Q) + 2D_{xy}(Q)m_\ell + D_{yy}(Q)m_\ell^2 + D_x(Q) + D_y(Q)\kappa \end{aligned}$$

Substituting on the right hand side expression of (17) we see that the κ -coefficient is given by

$$p'(\alpha_i)(\alpha_j - \alpha_i) \neq 0,$$

as desired. Hence statement (d) follows.

In Figure 3 we illustrate Theorem A for a concrete polynomial of degree three with three roots. In Figure 3(b) we can see that in a neighbourhood of the focal point $Q_{3,2}$ all directions except two correspond to green (basin of attraction of the fixed point (α_2, α_2)). One direction correspond to blue (basin of attraction of (α_3, α_3)) and there is one *singular* direction with red (basin of attraction of the fixed point (α_1, α_1)). In fact the picture shows that with $m = \infty$ there is also red. But in any event Figure 3(b) illustrate that, for this example,

$$Q_{3,2} \in \partial A(\alpha_1) \cap \partial A(\alpha_2) \cap \partial A(\alpha_3)$$

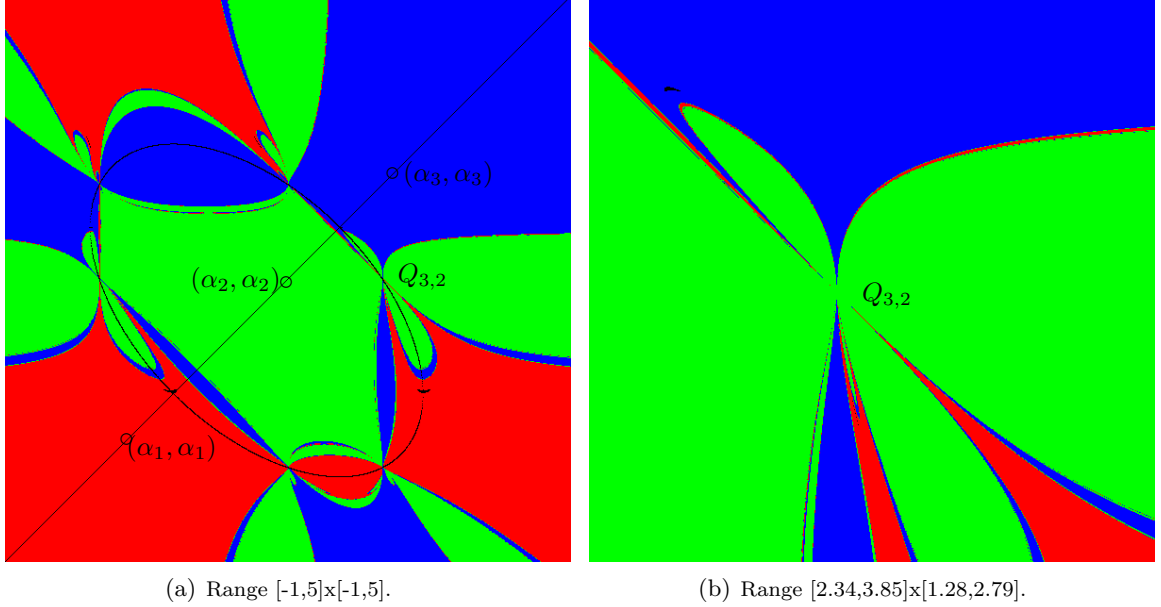


FIGURE 3. Dynamical plane of S applied to the polynomial $p(x) = x(x-2)(x-3)$. We show in red the basin of attraction of $(0,0)$, in green the basin of attraction of $(2,2)$ and in blue the basin of attraction of $(3,3)$. We show the set δ_S where the map S is not well defined. We also plot the line $y = x$ where belong the three attracting points of S (a). Zoom in the dynamical plane around the focal point $Q_{2,3}$ (b).

2.2. Proof of Theorem B. We show statement (a) by contradiction. We first assume the existence of a periodic orbit of minimal period 2 in E , that is $S(a,b) = (c,d)$ and $S(c,d) = (a,b)$ for some $a,b,c,d \in \mathbb{R}$ such that (a,b) and (c,d) are in E . From (5) we conclude that $c = b$ and $d = a$. So, $S(a,b) = (b,a)$ and $S(b,a) = (a,b)$ with $a \neq b$ (otherwise we would have a fixed point). From (5), if $S(a,b) = (b,a)$ we conclude that

$$a = b - p(b) \frac{b-a}{p(b) - p(a)}.$$

Notice that $p(a) \neq p(b)$ since $(a,b) \in E$. The above equation writes as

$$0 = (b-a) \left[1 - \frac{p(b)}{p(b) - p(a)} \right] = p(a) \frac{(b-a)}{p(b) - p(a)}.$$

Since $a \neq b$ the above equation concludes $p(a) = 0$. Interchanging the role of a and b we also conclude $p(b) = 0$. All together imply $(a,b) \notin E$.

We secondly assume the existence of a periodic orbit of minimal period 3 in E , that is $S(a, b) = (c, d)$, $S(c, d) = (e, f)$ and $S(e, f) = (a, b)$ for some $a, b, c, d, e, f \in \mathbb{R}$ such that (a, b) , (c, d) and (e, f) are in E . Arguing in a similar way as before we have that $c = b$, $e = d$ and $f = a$, so we have that $S(a, b) = (b, d)$, $S(b, d) = (d, a)$ and $S(d, a) = (a, b)$. Since the minimal period of the orbit is three we conclude that the three real numbers a, b and d are different. Without loss of generality we might assume $a < b < d$ (otherwise we rename the letters). Since the secant line through $(a, p(a))$ and $(d, p(d))$ should cut the line $y = 0$ at the point $x = b$ (observe that $S(d, a) = (a, b)$) we know that $p(a)p(d) < 0$. Assume $p(a) > 0$ and $p(d) < 0$ (the other case is similar). This force $p(b) > 0$, since the secant line passing through $(a, p(a))$ and $(b, p(b))$ should intersect the line $y = 0$ at $x = d$ (observe that $S(a, b) = (b, d)$). Accordingly the secant line through $(b, p(b))$ and $(d, p(d))$ will intersect the line $y = 0$ at a point $\eta \in (b, d)$, a contradiction with $S(b, d) = (d, a)$ and $a < b$. This finish the proof of statement (a).

Now we deal with statement (b) by showing of the existence of (attracting) periodic S -orbits of minimal period 4. We denote by a, b, c and d four real numbers such that $a < b < c < d$.

Arguing in a similar way as we did above, after relabelling the real numbers involved in the construction of the periodic orbit of period 4 the configuration should be (see Figure 4),

$$(19) \quad \begin{aligned} & a < b < c < d \quad \text{and} \\ & S(a, b) = (b, d) \quad S(b, d) = (d, c) \quad S(d, c) = (c, a) \quad S(c, a) = (a, b). \end{aligned}$$

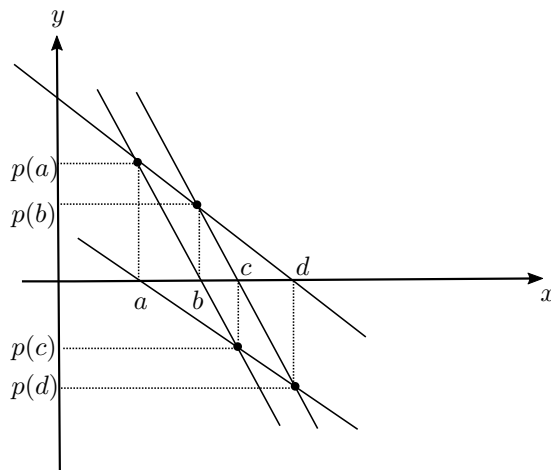


FIGURE 4. Configuration of the period 4 cycle.

To simplify the construction we assume that the secant lines passing through $(a, p(a))$ and $(b, p(b))$, and through $(c, p(c))$ and $(d, p(d))$ have slope equal to -1 . Under this assumption and the fact that we $S(a, b) = (b, d)$ and $S(d, c) = (c, a)$ we get

$$(20) \quad p(a) = d - a > 0, \quad p(b) = d - b > 0, \quad p(c) = a - c < 0 \quad \text{and} \quad p(d) = a - d < 0.$$

In order to satisfy all equalities in (19) we need to still force two conditions: that the secant line passing through $(b, p(b))$ and $(d, p(d))$ crosses the line $y = 0$ at the point $x = c$, and that the line passing through $(a, p(a))$ and $(c, p(c))$ crosses the line $y = 0$ at the point $x = b$. Easy

computations show that these two conditions write as

$$(21) \quad c = d - \frac{(a-d)(d-b)}{a+b-2d} \quad \text{and} \quad b = a - \frac{(d-a)(a-c)}{c+d-2a}.$$

Choosing the following concrete values

$$\bar{a} = 1 < \bar{b} = 2 < \bar{c} = \frac{1}{2} (3 + \sqrt{5}) \approx 2.618 < \bar{d} = \frac{1}{2} (5 + \sqrt{5}) \approx 3.618$$

the polynomial p satisfying

$$(22) \quad p(\bar{a}) = \bar{d} - \bar{a} > 0, \quad p(\bar{b}) = \bar{d} - \bar{b} > 0, \quad p(\bar{c}) = \bar{a} - \bar{c} < 0 \quad \text{and} \quad p(\bar{d}) = \bar{a} - \bar{d} < 0$$

is such that S has a periodic orbit of period four given by the points

$$\{(\bar{a}, \bar{b}), (\bar{b}, \bar{d}), (\bar{d}, \bar{c}), (\bar{c}, \bar{a})\}.$$

The unique interpolating polynomial of degree three through these points writes as

$$p_{\bar{a}, \bar{b}, \bar{c}, \bar{d}}(x) = 2.61803 - (x-1) - 2.61803(x-1)(x-2) + 2(x-1)(x-2)(x-2.61803)$$

Observe that the arguments used above implicitly provide a huge family of polynomials for which the secant method has a four periodic orbit. Our aim is to find one having an attracting four periodic orbit. The strategy will be to keep the parameters $\bar{a} < \bar{b} < \bar{c} < \bar{d}$, as well as their images in (20), but modify the values of the derivative (so, the new polynomial will have a higher degree).

Due to the fact that our dynamical system in E is smooth (see Theorem A) the local behaviour (attracting, repelling or other) of the cycle is governed by the eigenvalues of the matrix Λ where

$$(23) \quad \Lambda = DS(\bar{a}, \bar{b}) \, DS(\bar{b}, \bar{d}) \, DS(\bar{d}, \bar{c}) \, DS(\bar{c}, \bar{a}),$$

with $DS(x_0, y_0)$ denoting the differential matrix of the map S at the point (x_0, y_0) . In particular if the eigenvalues of Λ have both absolute value less than one then the cycle is attracting.

Taking partial derivatives from (5) when $x \neq y$ (which is our case here) we have

$$(24) \quad DS(x_0, y_0) = \begin{pmatrix} 0 & 1 \\ A(x_0, y_0) & B(x_0, y_0) \end{pmatrix}$$

where

$$A(x_0, y_0) = \frac{p(y_0) [p'(x_0) - q(x_0, y_0)]}{q^2(x_0, y_0) [x_0 - y_0]} \quad \text{and} \quad B(x_0, y_0) = \frac{p(x_0) [q(x_0, y_0) - p'(y_0)]}{q^2(x_0, y_0) [x_0 - y_0]}.$$

According with our parameters $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ given by (22), their images given by (20) and Remark 1 we know that $q(\bar{a}, \bar{b}) = q(\bar{d}, \bar{c}) = -1$ and $q(\bar{c}, \bar{a}) = q(\bar{b}, \bar{d}) = -1/2 (3 - \sqrt{5})$. Substituting we have

$$\begin{aligned} A(\bar{a}, \bar{b}) &= -\frac{1}{2} (1 + \sqrt{5}) (1 + p'(\bar{a})) & B(\bar{a}, \bar{b}) &= \frac{1}{2} (3 + \sqrt{5}) (1 + p'(\bar{b})) \\ A(\bar{b}, \bar{d}) &= \frac{1}{2} (-2 + \sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{b})) & B(\bar{d}, \bar{d}) &= \frac{1}{4} (7 - 3\sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{d})) \\ A(\bar{d}, \bar{c}) &= -\frac{1}{2} (1 + \sqrt{5}) (1 + p'(\bar{d})) & B(\bar{d}, \bar{c}) &= \frac{1}{2} (3 + \sqrt{5}) (1 + p'(\bar{c})) \\ A(\bar{c}, \bar{a}) &= \frac{1}{2} (-2 + \sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{c})) & B(\bar{c}, \bar{a}) &= \frac{1}{4} (7 - 3\sqrt{5}) (3 + \sqrt{5} + 2p'(\bar{a})). \end{aligned}$$

All we need to do is to choose suitable values of $p'(\bar{a})$, $p'(\bar{b})$, $p'(\bar{c})$ and $p'(\bar{d})$ so that the eigenvalues of Λ are of modulus less than one.

Fixing a polynomial p such that $p'(\bar{a}) = p'(\bar{b}) = p'(\bar{c}) = p'(\bar{d}) = -1$ we get

$$DS(\bar{a}, \bar{b}) = DS(\bar{d}, \bar{c}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad DS(\bar{b}, \bar{d}) = DS(\bar{c}, \bar{a}) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}(3 - \sqrt{5}) & \sqrt{5} - 2 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \frac{1}{4}(3 - \sqrt{5})^2 & \frac{1}{2}(5\sqrt{5} - 11) \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of Λ are 0 and $\frac{1}{4}(3 - \sqrt{5})^2 \approx 0,14589803 < 1$. Therefore using the Hermite interpolation we obtain the existence of a polynomial of degree seven for which the secant map exhibits an attracting periodic orbit of period four $(\bar{a}, \bar{b}) \mapsto (\bar{b}, \bar{d}) \mapsto (\bar{d}, \bar{c}) \mapsto (\bar{c}, \bar{a}) \mapsto (\bar{a}, \bar{b})$. In Figure 5 we show the dynamical plane of the secant map applied to this interpolating polynomial.

Remark 2. *The strategy for proving Theorem B (b) could be modify to find out polynomials p satisfying the statement with lower degree. For instance one can modify some of the parameters used in the proof of Theorem B (b) to produce a degree six polynomial with an attracting period four cycle. But we do not know if six is the lower degree for a polynomial with an attracting periodic orbit of period four. In Figure 5 we show the dynamical plane of S applied to a polynomial of degree 7 (the one constructed in the proof of Theorem B). In the dynamical plane we observe the basin of attraction of each one of the seven real roots of the polynomial and the attracting cycle of period four.*

3. THE SECANT MAP ON A TORUS: PROOF OF THEOREM C

In the previous section the secant map is defined on $\mathbb{R}^2 \setminus \delta_S$, where δ_S is the set of no definition of S . More precisely δ_S contains all the points (x, y) with $x \neq y$ such that $p(x) = p(y)$ and all the points (x, x) with $p'(x) = 0$ (see Equation 4). There is no hope to extend continuously the dynamical system generated by S on the whole plane since focal points are intrinsically points of discontinuity of S . We recall that the set \mathcal{Q} containing all the focal points is a subset of δ_S . However a continuous extension of S is possible outside \mathcal{Q} , as long as *infinity* is added to the domain of definition of an extension of S . A first step in this direction is to enlarge \mathbb{R}^2 to get a torus \mathbb{T}_∞^2 minus one point by adding *certain directions* to ∞ with identifications. Moreover, this extension is smooth on $\mathbb{T}_\infty^2 \setminus \mathcal{Q}$ except at the points in $\delta_{S_2} = \{(x, x) \in \mathbb{R}^2 \mid p'(x) = 0\}$.

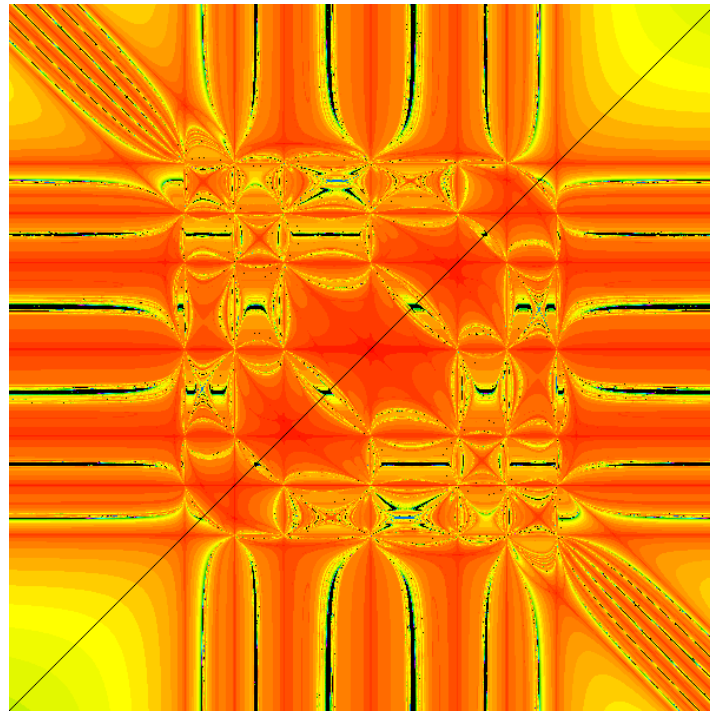
Before entering in the definition of the space \mathbb{T}_∞^2 and the extension of the secant map we can compute the dynamics *near* infinity. Concretely, we can evaluate the limit of the secant map at some specific directions.

Lemma 3.1. *Let p be a polynomial of degree $k \geq 2$, and let r be a real number. Then,*

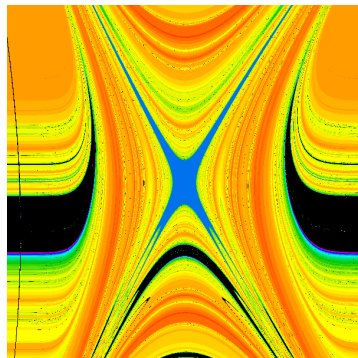
- (a) $\lim_{y \rightarrow \pm\infty} S(r, y) = (\pm\infty, r)$,
- (b) $\lim_{x \rightarrow \pm\infty} S(x, r) = (r, r)$.

Proof. From the definition of the secant map (1) we have that

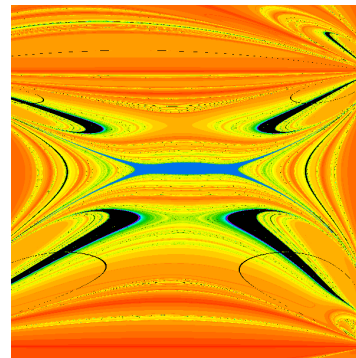
$$S(r, y) = \left(y, y - p(y) \frac{y - r}{p(y) - p(r)} \right) = \left(y, \frac{p(y)r - yp(r)}{p(y) - p(r)} \right),$$



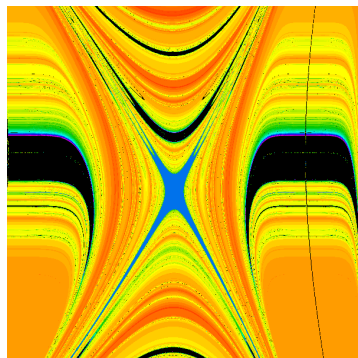
(a)



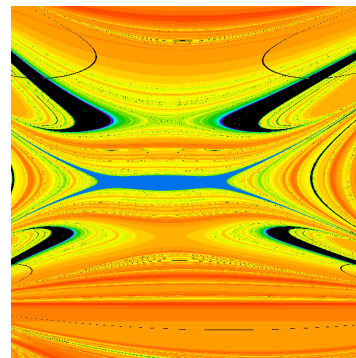
(b)



(c)



(d)



(e)

FIGURE 5. (a) Dynamical plane of the Secant method applied to a polynomial of degree 7 with seven real roots (red) and an attracting 4-cycle (blue). See the proof of Theorem B for the construction of this polynomial. In the picture we show the line $y = x$ and a small circle around each one of the seven fixed points of S . (b) Zoom near $(1, 2)$. (c) Zoom near $(2, 5/2 + \sqrt{5}/2)$. (d) Zoom near $(5/2 + \sqrt{5}/2, 3/2 + \sqrt{5}/2)$. (e) Zoom near $(3/2 + \sqrt{5}/2, 1)$.

and thus $S(r, y) \rightarrow (\pm\infty, r)$ as the variable y tends to $\pm\infty$, since the polynomial p has degree $k \geq 2$. In a similar way computing $S(x, r)$ we obtain

$$S(x, r) = \left(r, r - p(r) \frac{x - r}{p(x) - p(r)} \right) = \left(r, \frac{p(x)r - p(r)x}{p(x) - p(r)} \right),$$

concluding that $S(x, r) \rightarrow (r, r)$ as the variable x tends to $\pm\infty$. □

The idea of the extension of S is the following. Firstly we can extend the space \mathbb{R}^2 adding an external boundary of the form $(x, \pm\infty)$ and $(\pm\infty, y)$ and later on we identify the symbols $+\infty$ and $-\infty$ obtaining an space homeomorphic to the torus minus one point. Secondly we continuously extend the map S using Lemma 3.1, that says that the image under S of the vertical line $x = r$ near infinity is the horizontal line $y = r$ near infinity, and the image under S of the horizontal line $y = r$ near infinity is close to the point (r, r) . Thus we have a way to go to infinity, using vertical lines, and a way to come back from infinity using horizontal lines. Now we formalize this extension.

We first recall the construction of the two-dimensional torus \mathbb{T}^2 using the identification of the unit square $Q = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1\}$, then a natural topological model for \mathbb{T}^2 is given by $\mathbb{T}^2 := \overline{Q} / \sim$ with the identifications $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. Notice that $\mathbb{T}_0^2 := Q / \sim$ is the torus minus one point $(0, 0) \sim (1, 0) \sim (0, 1) \sim (1, 1)$. See Figure 6.

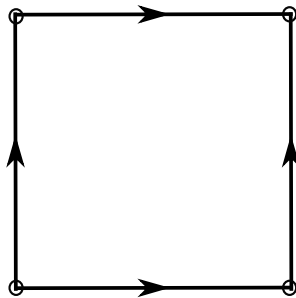


FIGURE 6. The topological model of the Torus \mathbb{T}^2 and the torus minus one point (erasing the four extreme boundary points).

Inspired on this model, our goal is to enlarge \mathbb{R}^2 adding certain points at infinity. More precisely, we take

$$Q_\infty = \{(x, y) \in \mathbb{R}^2\} \cup \{(x, \pm\infty), x \in \mathbb{R}\} \cup \{(\pm\infty, y), y \in \mathbb{R}\}$$

with the following identifications $(x, +\infty) \sim (x, -\infty)$ and $(+\infty, y) \sim (-\infty, y)$. Observe that the resultant object that we obtain after identifying the points at infinity is $\mathbb{T}_\infty^2 := Q_\infty / \sim$ which correspond precisely to the torus minus one point

$$(-\infty, -\infty) \sim (-\infty, \infty) \sim (\infty, -\infty) \sim (\infty, \infty).$$

To simplify notation we write

$$\mathbb{T}_\infty^2 = \{(x, y) \in \mathbb{R}^2\} \cup \{(x, \infty), x \in \mathbb{R}\} \cup \{(\infty, y), y \in \mathbb{R}\},$$

and the following three charts define an atlas on the surface

$$(25) \quad \varphi_1(x, y) := \text{Id}(x, y) = (x, y) \text{ if } (x, y) \in \mathbb{R}^2$$

$$\varphi_2(x, y) = \begin{cases} (x, 0) & \text{if } y = \infty \\ \left(x, \frac{1}{y}\right) & \text{if } y \neq \{0, \infty\} \end{cases} \quad \text{and} \quad \varphi_3(x, y) = \begin{cases} (0, y) & \text{if } x = \infty \\ \left(\frac{1}{x}, y\right) & \text{if } x \neq \{0, \infty\}. \end{cases}$$

Given a point $p \in \mathbb{T}_\infty^2$ we denote by \mathcal{U}_p a small enough open neighborhood of p on the surface and U_p the open image by the corresponding chart on \mathbb{R}^2 .

Using this atlas we are able to define an extension of S on $\mathbb{T}_\infty^2 \setminus \mathcal{Q}$. For all $(x, y) \in \mathbb{R}^2$, set

$$(26) \quad G(x, y) := \begin{cases} \left(y, \frac{yq(x,y)-p(y)}{q(x,y)}\right) & \text{if } y \neq x, \\ \left(x, \frac{xp'(x)-p(x)}{p'(x)}\right) & \text{if } y = x, \end{cases}$$

Observe this is in fact the precise expression (5) of the map S , well defined in $\mathbb{R}^2 \setminus \delta_S$. Then

$$(27) \quad \hat{S}(x, y) := \begin{cases} G(x, y) & \text{if } (x, y) \in U_p \text{ where } p \in \mathbb{R}^2 \setminus \delta_S \\ (\varphi_2^{-1} \circ G)(x, y) & \text{if } (x, y) \in U_p \text{ where } p \in \delta_S \setminus \mathcal{Q} \\ (\varphi_3^{-1} \circ G \circ \varphi_2)(x, y) & \text{if } (x, y) \in U_p \text{ where } p = (x_0, \infty) \\ (G \circ \varphi_3)(x, y) & \text{if } (x, y) \in U_p \text{ where } p = (\infty, y_0) \end{cases}$$

3.1. Proof of Theorem C. Observe that the focal points are points of discontinuity of the map \hat{S} since they are already points of discontinuity of the map S . On the other hand since the map \hat{S} coincides with G at the set $\mathbb{R}^2 \setminus \delta_S$ we conclude that \hat{S} is differentiable on $\mathbb{R}^2 \setminus \delta_S$ (see Theorem A). So, we only need to study the differentiability of \hat{S} at the points in $\delta_S \setminus (\mathcal{Q} \cup \delta_{S_2})$, at the points (x, ∞) , $x \in \mathbb{R}$, and at the points (∞, y) , $y \in \mathbb{R}$. Thus, we split the proof into three cases.

Case (i). Let $p = (x_0, y_0)$ be a point in $\delta_S \setminus (\mathcal{Q} \cup \delta_{S_2}) \subset \mathbb{T}_\infty^2$ and let U_p be a neighbourhood of (x_0, y_0) , or in other words a point (x_0, y_0) such that $p(x_0) = p(y_0)$ with $x_0 \neq y_0$. Easy computations from (25), (26) and (27) show that

$$\hat{S}(x, y) = \left(y, \frac{p(x) - p(y)}{yp(x) - xp(y)}\right), \quad (x, y) \in U_p.$$

Since \hat{S} has a rational expression and the denominator does not vanished by taking a neighborhood of p small enough, we conclude that \hat{S} is a smooth map with $\hat{S}(x_0, y_0) = (y_0, 0)$.

Case (ii). Let $p = (x_0, \infty)$ be a point in \mathbb{T}_∞^2 . Using the expressions of the charts (25) and the map \hat{S} (27) we obtain

$$\hat{S}(x, y) = \left(y, \frac{q(y)x - p(x)y^{k-1}}{q(y) - p(x)y^k}\right),$$

where

$$p(z) = a_0 + a_1z + \cdots + a_{k-1}z^{k-1} + z^k, z \in \{x, y\}, \text{ and}$$

$$q(y) = 1 + a_{k-1}y + \cdots + a_1y^{k-1} + a_0y^k.$$

We notice that $\hat{S}(x_0, 0) = (0, x_0)$ and that \hat{S} is a smooth map whenever it is defined (the denominator is not zero near $(x_0, 0)$).

Case (iii). Let $p = (\infty, y_0) \in \mathbb{T}_\infty^2$. Using the corresponding charts and the definition of the secant map we have that \hat{S} writes as,

$$(28) \quad \hat{S}(x, y) = \frac{p(y)x^{k-1} - yq(x)}{p(y)x^k - q(x)}.$$

As before, considering \hat{S} near the point $(0, y_0)$, we have $\hat{S}(0, y_0) = (y_0, y_0)$ and \hat{S} is also smooth in a sufficiently small neighborhood of p .

This finish the proof of the smoothness of the extension of the real secant map S on the torus.

The existence of a periodic orbit of minimal period three on \mathbb{T}_∞^2 (compare with Theorem B) is a direct application of the definition of \hat{S} . Indeed, if x_0 is such that $p'(x_0) = 0$ then $(x_0, x_0) \in \delta_S \setminus \mathcal{Q}$ and its \hat{S} -orbit is given by,

$$\hat{S}(x_0, x_0) = (x_0, \infty), \quad \hat{S}(x_0, \infty) = (\infty, x_0) \quad \text{and} \quad \hat{S}(\infty, x_0) = (x_0, x_0).$$

This finish the proof of Theorem C.

Remark 3. *As a consequence of Theorem C every point (x_0, x_0) with $p'(x_0) = 0$ generates a periodic orbit $(x_0, x_0) \mapsto (x_0, \infty) \mapsto (\infty, x_0) \mapsto (x_0, x_0)$, and the natural question is to investigate the local behavior of this cycle. One can see that \hat{S} is not differentiable at (x_0, x_0) and so the local behaviour cannot be explained by the eigenvalues of the differential matrix. We have explored numerically this phenomenon and we conjecture that there exists a simply connected region $U_{(x_0, x_0)}$ and for every point w in $U_{(x_0, x_0)}$ the sequence $\{S^{3n}(w)\}_{n \geq 0}$ converges (very slowly) toward (x_0, x_0) as n tends to infinity. Moreover, the point (x_0, x_0) belongs to the boundary of $U_{(x_0, x_0)}$. If this conjecture is true we obviously have countably many images and preimages of $U_{(x_0, x_0)}$. We have observed this phenomenon around every critical point (x_0, x_0) such that $p'(x_0) = 0$, see for example the small black sets (that is points not converging to the roots of the polynomial) in Figures 3 and 5. We also have constructed a detailed numerical analysis for the polynomial $p(x) = \frac{1}{3}x^3 - 4x + 3$, focusing in a small neighborhood of the critical point $(2, 2)$, since $p'(2) = 0$. In Figure 7 we show in black a numerical approximation of the region $U_{(2,2)}$.*

4. THE SECANT MAP ON THE LINE AT INFINITY: PROOF OF THEOREM D

In order to give a more detailed global behaviour of the dynamical system defined by the secant map S we consider now another extension of S on the projective plane $\mathbb{RP}^2 = \mathbb{R}^2 \sqcup \mathbb{RP}^1$, where \sqcup denotes the disjoint union.

Recall that \mathbb{RP}^2 is a compact, non orientable space. Indeed it can be defined as the space of lines through the origin in \mathbb{R}^3 . Using this approach we can visualize \mathbb{RP}^2 as the union of the plane \mathbb{R}^2 and the compact line \mathbb{RP}^1 . Consider the plane $z = 1$. We identify every line through the origin with the intersecting point between this line and the plane $z = 1$. Observe that all lines contained in the plane $z = 0$ are excluded. See Figure 8(a). So, we might parametrized these points by $[x : y : 1]$.

The set of lines excluded in this construction forms a line called *the line at infinity* of the real projective plane and it is denoted by $\ell_\infty := \mathbb{RP}^1$ (see Figure 8(b)). To induce coordinates on ℓ_∞ we use the slope of the lines $m := y/x$ when $x \neq 0$ and the remaining point directly by $[0 : 1 : 0]$. In Figure 8(b) we show four examples of points of the type $[1 : m : 0]$ (black) and the point $[0 : 1 : 0]$ (grey). We notice that when $m \rightarrow \infty$ we obtain precisely the point

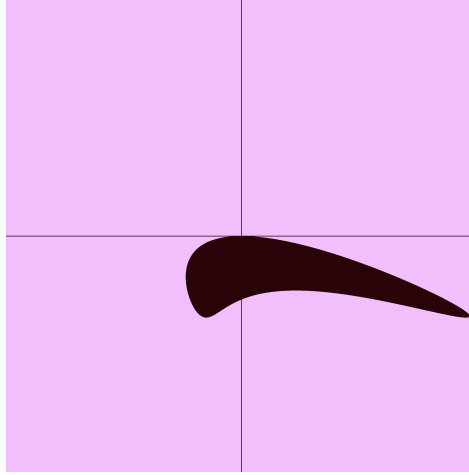


FIGURE 7. Dynamical plane of the secant map applied to the polynomial $p(x) = \frac{1}{3}x^3 - 4x + 3$ near the critical point $(2,2)$. Range of the picture $[1.92, 2.08] \times [1.92, 2.08]$. Points in color black denote points in the set $U_{(2,2)}$ converging to $(2,2)$ under S^3 , while points in color pink denote points attracted by the root $x = 3$ of p . We also show the lines $x = 2$ and $y = 2$, thus the point $(2,2)$ is located at the center of the picture.

$[0 : 1 : 0]$. Finally \mathbb{RP}^2 admits the following coordinates,

$$(29) \quad \mathbb{RP}^2 = \begin{cases} [x : y : 1] & \text{with } (x, y) \in \mathbb{R}^2 \\ [1 : m : 0] & \text{with } m \in \mathbb{R} \\ [0 : 1 : 0] \end{cases}$$

We notice that points in the finite plane are characterized by the third coordinate equal to 1, while points at the line at infinity have the third coordinate equal to 0. Notice also that antipodal points are identified.

4.1. Proof of Theorem D. We firstly deduce the expression of the extended secant map defined on ℓ_∞ , using homogeneous coordinates, and secondly we deal with the dynamical system defined on it.

In order to define $\tilde{S}[1 : m : 0]$, $m \neq 0$ we consider a general sequence of points (u_n, v_n) over the line $v_n = mu_n$ with $\lim_{n \rightarrow \infty} u_n = \infty$. Then if we write $(r_n, s_n) := S(u_n, v_n)$ we will define

$$\tilde{S}[1 : m : 0] = [1 : \lim_{n \rightarrow \infty} \frac{s_n}{r_n} : 0].$$

From the definition of the secant map (1) we have

$$(r_n, s_n) = \left(v_n, v_n - p(v_n) \frac{u_n - v_n}{p(u_n) - p(v_n)} \right).$$

Clearly $r_n = v_n = mu_n$ and using the highest term of the monic polynomial p , i.e. the one leading near infinity, we have that $p(u_n) \approx u_n^k$ and $p(v_n) = p(mu_n) \approx (mu_n)^k$. So

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(mu_n - (mu_n)^k \frac{u_n - mu_n}{u_n^k - (mu_n)^k} \right) = \lim_{n \rightarrow \infty} mu_n \varphi(m),$$

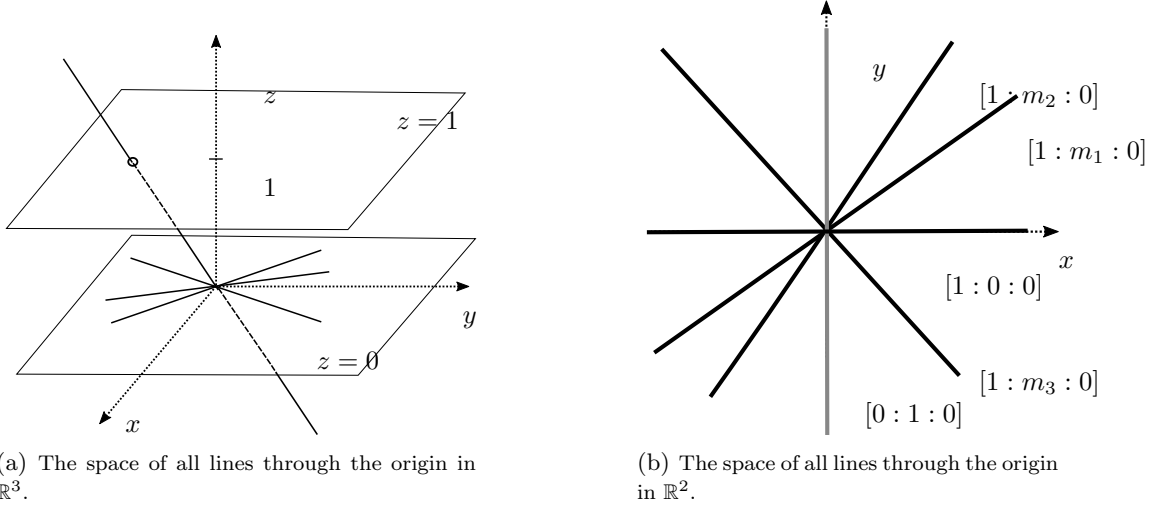


FIGURE 8. A model for the real projective plane $\mathbb{R}\mathbb{P}^2 = \mathbb{R}^2 \sqcup \ell_\infty$ (a). The line of infinity ℓ_∞ (b).

as desired. We notice that

$$\lim_{m \rightarrow 1} \varphi(m) = \lim_{m \rightarrow 1} \frac{m^{k-1} - 1}{m^k - 1} = \lim_{m \rightarrow 1} \frac{(k-1)m^{k-2}}{km^{k-1}} = \frac{k-1}{k},$$

and so the map φ has a removable singularity at $m = 1$. Similarly when $m = 1$ we can compute directly $S(u_n, u_n)$ using the expression of $q(x, x) = p'(x)$ from Lemma 2.1 obtaining the same result. To define the map \tilde{S} at the points $[1 : 0 : 0]$ ($m = 0$) and $[0 : 1 : 0]$ ($m = \infty$) we simply extend \tilde{S} continuously on ℓ_∞ . In particular $\tilde{S}[0 : 1 : 0] = [1 : 0 : 0]$. In fact if we consider a vertical line $x = 0$ and we take $(0, v_n)$ with $\lim_{n \rightarrow \infty} v_n = \infty$ we obtain

$$(r_n, s_n) = \left(v_n, v_n - p(v_n) \frac{v_n}{p(v_n) - p(0)} \right) = \left(v_n, -\frac{v_n p(0)}{p(v_n) - p(0)} \right)$$

and so,

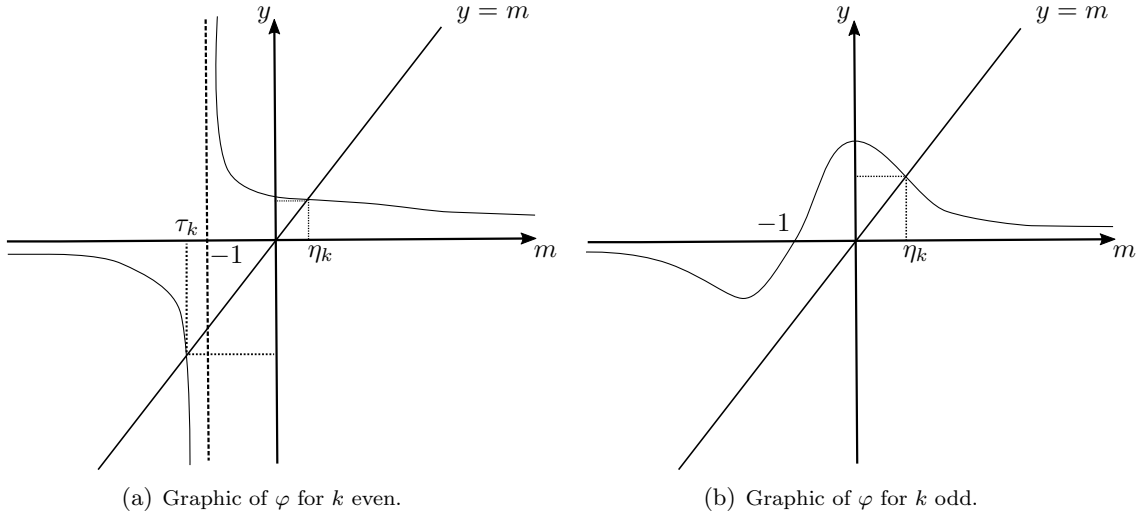
$$\lim_{n \rightarrow \infty} \frac{s_n}{r_n} = 0$$

Thus the image of a vertical line $x = 0$ is mapped to the horizontal line $y = 0$ close to infinity showing the compatibility with the definition of the extended map at infinity.

Remark 4. We notice that the map \tilde{S} can be considered a “continuous” extension of the real secant map S . The only discontinuity happens to be at $[0 : 1 : 0]$, since at this point we have $\tilde{S}[0 : 1 : 0] = [1 : 0 : 0] \in \ell_\infty$ while $S(x, \alpha) \equiv (\alpha, \alpha)$ for all $x \in \mathbb{R}$, and for all α such that $p(\alpha) = 0$. We also remark that the definition of \hat{S} and \tilde{S} coincide where defined except precisely at this point.

Now we turn to investigate the dynamical system governed by the map

$$\varphi(m) = \frac{m^{k-1} - 1}{m^k - 1}.$$

FIGURE 9. Dynamics of the secant method on the line at infinity ℓ_∞ .

We split the proof depending on the oddity of k . We proof statement (a) which correspond to the case where k is even and left the proof of statement (b) to the reader since follows similarly. See Figure 9.

We assume that k is an even number. Clearly the map $\varphi(m)$ has a unique vertical asymptote at $m = -1$, and a horizontal asymptote since $\varphi(m) \rightarrow 0^\pm$ as $m \rightarrow \pm\infty$. Easy computations show

$$\varphi'(m) = \frac{-1}{(m^k - 1)^2} m^{k-2} \psi(m),$$

where $\psi(m) = (m^k - [k(m-1) + 1])$. Since k is even, we have $\text{sign}(\varphi'(m)) = -\text{sign}(\psi)$. But observe that $\psi(m)$ measures the difference from the graph of the function $y = m^k$ and the graph of the function $y = k(m-1) + 1$ which is precisely the tangent line to the graph of $y = m^k$ at the point $m = 1$. Hence $\psi(m) \geq 0$ for all $m \in \mathbb{R}$ and $\psi(m) = 0$ if and only if $m = 1$. In particular this implies that φ is strictly decreasing in its domain of definition.

All together prove that φ has a unique fixed point $\tau_k \in (-\infty, -1)$ and, since $\varphi(1/2) > 1/2$ and $\varphi(1) = (k-1)/k < 1$, φ has a unique fixed point $\eta_k \in (1/2, 1)$. See Figure 9 (a).

We claim that the positive fixed point η_k is locally attracting, that is, $|\varphi'(\eta_k)| < 1$. To see this we notice that $\varphi'(m)$ after simplification (removing the indetermination at $m = 1$) also writes as

$$\varphi'(m) = - \frac{m^{k-2} (m^{k-2} + 2m^{k-3} + \dots + (k-2)m + k-1)}{(m^{k-1} + m^{k-2} + \dots + m + 1)^2}.$$

Doing some computations we get

$$\varphi'(m) = - \frac{\sum_{j=4}^{k+1} (j-3)m^{2k-j} + (k-1)m^{k-2}}{\zeta(m) + \sum_{j=4}^{k+1} (j-1)m^{2k-j} + (k-1)m^{k-2}},$$

where $\zeta(m) > 0$ for all $m > 0$ (in fact is a polynomial with positive coefficients). Accordingly $|\varphi'(m)| < 1$ for all $m > 0$ since $j-3 < j-1$ and so the positive fixed point η_k of φ is locally attracting.

To finish the proof we will see that the fixed point $\tau := \tau_k < -1$ is repelling. Observe that

$$\varphi'(m) = \frac{1}{(m^k - 1)^2} \left[(k-1)m^{k-2}(m^k - 1) - km^{k-1}(m^{k-1} - 1) \right]$$

and since τ is a fixed point of φ we conclude that $\tau^{k-1} - 1 = \tau(\tau^k - 1)$. So some computations give

$$\varphi'(\tau) = -\frac{\tau^{k-2}}{\tau^k - 1} [k(\tau^2 - 1) + 1].$$

Since $\tau < -1$ and k is even it is clear that $\varphi'(\tau) < 0$. We claim that $\varphi'(\tau) < -1$ and so τ is a repelling fixed point of φ . Indeed

$$-\varphi'(\tau) > k + \frac{1}{\tau^2} - \frac{k}{\tau^2} > 1.$$

In Table 1 we compute numerically the fixed points of φ for several values of k .

k	attracting fixed point η_k of φ	repelling fixed point τ_k of φ
2	$\approx 0,61803399$	$\approx -1,61803399$
3	$\approx 0,75487766$	—
4	$\approx 0,81917251$	$\approx -1,38027757$
5	$\approx 0,85667488$	—
6	$\approx 0,88127146$	$\approx -1,28519903$
7	$\approx 0,89865371$	—
8	$\approx 0,91159235$	$\approx -1,23205463$
9	$\approx 0,92159932$	—
...
20	$\approx 0,9650705$	$\approx -1,1186991$
...
50	$\approx 0,9860941$	$\approx -1,05933705$

TABLE 1. Fixed points of the map \tilde{S} on ℓ_∞ .

Finally we prove that $\tilde{S}^n(x) \rightarrow \eta_k$ as $n \rightarrow \infty$, for all $x \in \mathbb{R} \cup \{\infty\}$ such that $x \neq \tau_k$. By the graphical analysis of φ , see Figure 9 (a), we conclude that for $x > -1$ we have that $\tilde{S}^n(x) \rightarrow \eta_k$ as $n \rightarrow \infty$, in fact any point $x_0 \in (-1, \eta_k)$ “jumps” to the interval $(\eta_k, +\infty)$ and viceversa forming two monotone sequence of iterates $\{\tilde{S}^{2m}(x_0)\}_{m \geq 0}$ and $\{\tilde{S}^{2m+1}(x_0)\}_{m \geq 0}$ both tending to x_0 , the first one strictly increasing and the second one strictly decreasing. The point corresponding to $m = \infty$ is mapped by \tilde{S} to the corresponding point with $m = 0$ and hence tending to η_k , the same happens with $m = -1$ since $\varphi(-1) = \infty$. Now let w_1 the unique premiere of -1 under \tilde{S} , then all $x \in (-\infty, w_1]$ are mapped into the interval $[-1, 0)$ and then attracted by η_k . Arguing similarly we denote by w_2 the unique preimage of w_1 in the interval $(\tau_k, -1)$, then all point in the interval $[w_2, -1)$ “jumps” to the interval $(-\infty, w_1]$ and then converge to η_k . Taking inductively preimages of w 's we conclude that all points in $\mathbb{R} \cup \{\infty\}$ converge toward η_k under \tilde{S} except the repelling fixed point τ_k .

5. CONCLUSIONS

In this paper we give a global description of the possible dynamical behaviours that might occur in the dynamical plane of the secant method S applied to real polynomials with simple roots. To do so we have extended in a natural way the secant map to *infinity* and we have used these further information to have a better understanding of the global dynamics of S as a plane map.

On the one hand we have shown from the extension of S to a map \hat{S} on a torus allows us to prove the existence of *virtual* period three cycles, that is, regions of the dynamical plane where orbits follows a periodic three cycle formed by one point in \mathbb{R} and two points at infinity. So, the *wrong* seeds not converging to the roots of the polynomial not only belong to attracting basins of higher period cycles not related with the zeros of the polynomial (Theorem B) but they also belong to regions where points *use* infinity (Theorem C).

On the other hand we also consider the extension of S to the circle at infinity $\ell_\infty \subset \mathbb{RP}^2$ and show how its dynamical behaviour *only* depends of the parity of k , the degree of the polynomial p . The basins of attraction of the roots of a polynomial p extend to infinity in the direction of the horizontal lines $y = \alpha$ where α is a root of p . This horizontal lines correspond to the point $[1 : 0 : 0]$, or $m = 0$ in the homogeneous coordinates, at the line of infinity. When k is odd there exist two preimage of $m = 0$ under the dynamics induced by the secant map at ℓ_∞ which are $m = \infty$ and $m = -1$. So, in the dynamical plane of the secant map S_p where p is a polynomial of degree odd we can see preimages of the basins of attraction at the direction of the lines $y = -x$ and the vertical line $y = 0$. Moreover, when k is even the unique preimage of $m = 0$ is $m = \infty$. In Figure 10(a) we have drawn the dynamical plane of the polynomial $p(x) = x(x-1)(x-2)(x-3)$ while in Figure 10(b) we have drawn the dynamical plane of the polynomial $p(x) = x(x-1)(x-2)$. In particular we deduce that if p is a degree k polynomial having at least one root, initial conditions on the line $y = \tau_k x$ (for x large enough in absolute value) S -converge with high speed to the greatest and smallest root of p .

During all the present work we have assumed that the roots of the polynomial p are simple. Many of the results can be done also for the case of multiple roots. However at a multiple root of the polynomial different phenomena occurs at the same point. Let β be a multiple root of p then, on the one hand, the point (β, β) is not a fixed point of S_p , in fact it is worst since the map S_p is not defined in \mathbb{R}^2 at this point! and, on the other hand, now the point (β, β) exhibits a period three cycle since $p'(\beta) = 0$.

REFERENCES

- [ABFsP17] Leandro Arosio, Anna Miriam Benini, John Erik Fornæss, and Han Peters. Dynamics of transcendental Hénon maps. preprint, 2017.
- [BD05] Eric Bedford and Jeffrey Diller. Real and complex dynamics of a family of birational maps of the plane: the golden mean subshift. *Amer. J. Math.*, 127(3):595–646, 2005.
- [Bed03] Eric Bedford. On the dynamics of birational mappings of the plane. *J. Korean Math. Soc.*, 40(3):373–390, 2003.
- [BFJK14] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska. On the connectivity of the Julia sets of meromorphic functions. *Invent. Math.*, 198(3):591–636, 2014.
- [BFJK15] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska. Connectivity of julia sets of newton maps: A unified approach. *to appear Revista Iberoamericana*, 01 2015.
- [BGM99] Gian-Italo Bischi, Laura Gardini, and Christian Mira. Plane maps with denominator. I. Some generic properties. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 9(1):119–153, 1999.

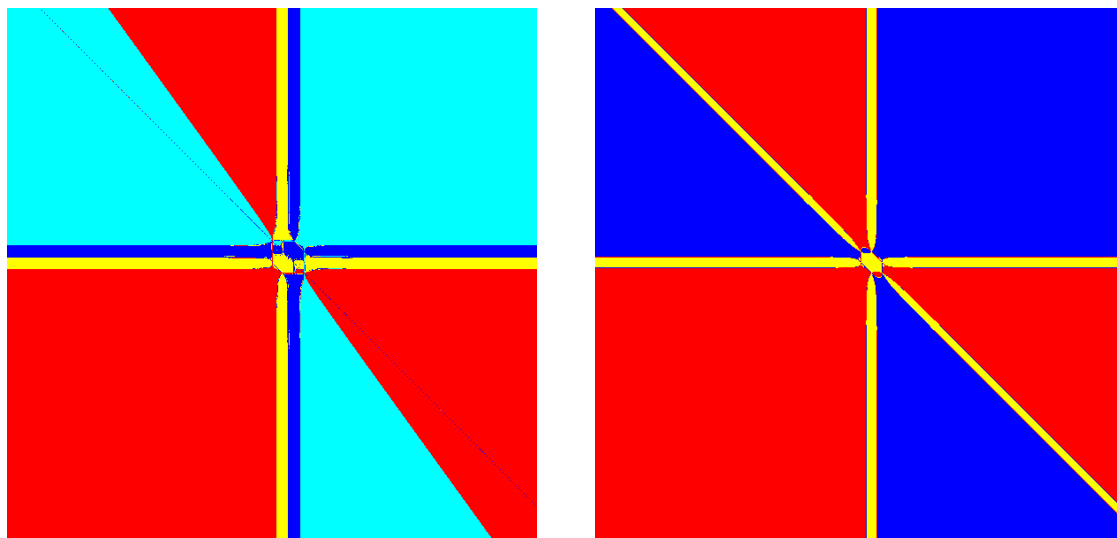
(a) Dynamical plane of S_p with $\deg(p)$ odd.(b) Dynamical plane of S_p with $\deg(p)$ even.

FIGURE 10. We have proven that the line $y = \alpha_j$ ($m = 0$) belong to the basin of attraction of the fixed point (α_j, α_j) , $j = 1, \dots, n$. This explains the horizontal colors associated to the different basins of attraction. Moreover since $x = \alpha_j$ ($m = \infty$) is a preimage of $y = \alpha_j$ we also have those colors at the y -axis. For polynomials of odd degree (figure (b)) since $m = 0$ has $m = -1$ as a (further) finite φ -preimage, we see near infinity, copies of the basins of attraction drawn with colors. The range of the pictures is $[-25, 25] \times [-25, 25]$.

- [BGM03] Gian-Italo Bischi, Laura Gardini, and Christian Mira. Plane maps with denominator. II. Noninvertible maps with simple focal points. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(8):2253–2277, 2003.
- [BGM05] Gian-Italo Bischi, Laura Gardini, and Christian Mira. Plane maps with denominator. III. Nonsimple focal points and related bifurcations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15(2):451–496, 2005.
- [BS91a] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 : currents, equilibrium measure and hyperbolicity. *Invent. Math.*, 103(1):69–99, 1991.
- [BS91b] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 . II. Stable manifolds and recurrence. *J. Amer. Math. Soc.*, 4(4):657–679, 1991.
- [BS92] Eric Bedford and John Smillie. Polynomial diffeomorphisms of \mathbf{C}^2 . III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.*, 294(3):395–420, 1992.
- [BS06] Eric Bedford and John Smillie. The Hénon family: the complex horseshoe locus and real parameter space. In *Complex dynamics*, volume 396 of *Contemp. Math.*, pages 21–36. Amer. Math. Soc., Providence, RI, 2006.
- [Cay79a] Arthur Cayley. Applications of the Newton-Fourier Method to an imaginary root of an equation. *Quat. J. of Pure and App. Math.*, 16:179–185, 1879.
- [Cay79b] Arthur Cayley. The Newton-Fourier Method imaginary problem. *Am. J. of Math.*, 2:97, 1879.
- [Cay80] Arthur Cayley. On the Newton-Fourier imaginary problem. *Proc. Camb. Phil. Soc.*, 3:231–232, 1880.
- [CMn11] Anna Cima and Francesc Mañosas. Real dynamics of integrable birational maps. *Qual. Theory Dyn. Syst.*, 10(2):247–275, 2011.
- [CZ14] Anna Cima and Sundus Zafar. Integrability and algebraic entropy of k -periodic non-autonomous Lyness recurrences. *J. Math. Anal. Appl.*, 413(1):20–34, 2014.

- [DL15] Romain Dujardin and Mikhail Lyubich. Stability and bifurcations for dissipative polynomial automorphisms of \mathbb{C}^2 . *Invent. Math.*, 200(2):439–511, 2015.
- [Duj04] Romain Dujardin. Hénon-like mappings in \mathbb{C}^2 . *Amer. J. Math.*, 126(2):439–472, 2004.
- [FsS92] John Erik Fornæss and Nessim Sibony. Complex Hénon mappings in \mathbb{C}^2 and Fatou-Bieberbach domains. *Duke Math. J.*, 65(2):345–380, 1992.
- [HOV94] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain. I. The global topology of dynamical space. *Inst. Hautes Études Sci. Publ. Math.*, (79):5–46, 1994.
- [HOV95] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. In *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 89–132. Kluwer Acad. Publ., Dordrecht, 1995.
- [HPV00] John Hubbard, Peter Papadopol, and Vladimir Veselov. A compactification of Hénon mappings in \mathbb{C}^2 as dynamical systems. *Acta Math.*, 184(2):203–270, 2000.
- [HSS01] John Hubbard, Dierk Schleicher, and Scott Sutherland. How to find all roots of complex polynomials by Newton’s method. *Invent. Math.*, 146(1):1–33, 2001.
- [McM87] Curt McMullen. Families of rational maps and iterative root-finding algorithms. *Ann. of Math. (2)*, 125(3):467–493, 1987.
- [Prz89] Feliks Przytycki. Remarks on the simple connectedness of basins of sinks for iterations of rational maps. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 229–235. PWN, Warsaw, 1989.
- [Shi09] Mitsuhiro Shishikura. The connectivity of the Julia set and fixed points. In *Complex dynamics*, pages 257–276. A K Peters, Wellesley, MA, 2009.

DEPARTAMENT D’ENGINYERIA INFORMÀTICA I MATEMÀTIQUES, UNIVERSITAT ROVIRA I VIRGILI, 43007 TARRAGONA, CATALONIA.

E-mail address: antonio.garijo@urv.cat

DEPARTAMENT DE MATEMÀTIQUES I INFORMÀTICA AT UNIVERSITAT DE BARELONA AND BARCELONA GRADUATE SCHOOL OF MATHEMATICS, 08007 BARCELONA, CATALONIA.

E-mail address: xavier.jarque@ub.edu