

Difference equations everywhere: some motivating examples

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Abstract This work collects several situations where discrete dynamical systems or difference equations appear. Most of them are different from the examples used in textbooks and from the usual mathematical models appearing in Biology or Economy. The examples are presented in detail, including some appropriate references. Although most of them are known, the fact of collecting all together aims to be a source of motivation for studying DDS and difference equations and to facilitate teaching these subjects.

1 Introduction

The main goal of the theory of discrete dynamical systems (DDS) is to study the limit behavior of the sequence $\{x_n\}_n$, defined iteratively as $x_{n+1} = F(x_n)$, in terms of the initial condition x_0 , where F is an invertible map defined on a given space. When F is not invertible sometimes it is said that it defines a semi-DDS. In particular, many difference equations and recurrences can be interpreted as semi-DDS or DDS.

They also appear frequently in problems of other branches of Mathematics. Without aiming to be exhaustive, some examples are: the iterative methods proposed to approximate the solutions of linear or non-linear systems, the Bernoulli iterative method to find the dominant root of a polynomial, the numerical schemes like the Euler, Taylor or Runge-Kutta methods designed to approximate the solutions of ordinary differential equations, the schemes of differences used to approximate the solutions of partial differential equations, the study of discrete Markov chains, the complex dynamics, ...

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Another source of examples comes from the mathematical models of the real world, see [34, 37]. For instance, two of the most famous models are the Verhulst and the Beverton-Holt models:

$$x_{n+1} = rx_n \left(1 - \frac{x_n}{K}\right) \quad \text{and} \quad x_{n+1} = \frac{rx_n}{1 + \frac{x_n}{K}},$$

where r, K and x_0 are all positive. They are the discrete analogues of the well-known continuous logistic growth model. Many other models, much more complicated and realistic, including several populations or some delays, have been considered in Mathematical Ecology, see for instance [26, Ch. 6] or [65, Ch. 2]. Also the so called Leslie models, although linear, give a wide range of applications for age structured populations, see [46, Ch. 21&22]. Moving to another discipline, discrete models are also used for instance in both theoretical and empirical economics, see [21, 72, 77].

This work collects some less known situations where DDS or difference equations appear. Most of them are different from the examples appearing in textbooks on these subjects and from the ones listed above. We will consider some recreational problems, the Titius-Bode law, the $3x + 1$ problem, the Landen maps and the study of some generalized means, several questions on probability and random walks like the gambler's ruin problem, the comparison of several methods to compute square roots, the study of several algorithms to approach π , the computation of the greatest common divisor as the ω -limit of a DDS, some questions of algebraic geometry about the existence of points with rational coordinates on planar elliptic curves, the study of the speed of the divide-and-conquer algorithm QUICKSORT, the deduction of some lower bounds for the so called Hilbert numbers (that count the number of limit cycles of polynomial planar differential equations in terms of their degrees), and the so called Coxeter recurrences.

The results are presented in detail, including some appropriate references. Although most of them are known, the fact of collecting all together aims to be a source of motivation to study DDS and difference equations and to facilitate teaching these subjects.

2 Some warming up examples

2.1 Harmonic series and the Fibonacci numbers

It is well-known that the so called harmonic numbers $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ tend to infinity and so the harmonic series diverges. There are many demonstrations of this fact; in the papers [43, 44, 45] the authors collect more than forty proofs.

The oldest one (around 1350) is attributed to the French philosopher Nicole Oresme (1323-1382) and is the following:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \frac{1}{9} + \dots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty.$$

In a more formal way, $h_{2^k} \geq 1 + \frac{k}{2}$, and so $\lim_{k \rightarrow \infty} h_{2^k} = \infty$. Here we include a more sophisticated proof ([43]) based on the knowledge of the Fibonacci numbers, F_n . These famous numbers were introduced in 1202 by the Italian mathematician Leonardo of Pisa, known as Fibonacci, to model a rabbit population. They also appear in many biological settings and are related with plant patterns, see for instance [68] and their references. These numbers are the solutions of the linear difference equation

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, \quad F_1 = 1.$$

The first ones are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... and it is well-known that

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{2\phi - 1},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean, which satisfies $\phi^2 - \phi - 1 = 0$. We refer to [26, 30] for the methods used to obtain the explicit solution of the difference equations solved in this paper. Hence,

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \phi, \quad \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_{n+1}} = \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} \frac{F_n}{F_{n+1}} = \left(\frac{1}{\phi}\right)^2 \neq 0.$$

Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{5}\right) + \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{13}\right) \\ &+ \left(\frac{1}{14} + \dots + \frac{1}{21}\right) + \left(\frac{1}{22} + \dots + \frac{1}{34}\right) + \left(\frac{1}{35} + \dots + \frac{1}{55}\right) + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{3} + \frac{2}{5} + \frac{3}{8} + \frac{5}{13} + \frac{8}{21} + \frac{13}{34} + \frac{21}{55} + \dots = 1 + \sum_{n=2}^{\infty} \frac{F_{n-1}}{F_{n+1}}, \end{aligned}$$

is divergent because the general term of the latter series does not tend to zero.

2.2 A couple of puzzles

Édouard Lucas presented in 1883 one of the most famous and funny puzzles, the known as *tower of Hanoi puzzle*, see for instance [53]. It consists of three rods and n disks of different sizes, which can slide onto any rod. The puzzle starts with the disks as in Figure 1 (there $n = 8$) and the objective is to translate this tower to the right rod, moving each time only one disk and following these two rules:

- Each move consists of taking the upper disk from one of the stacks and placing it on top of a different stack.
- Any disk must be placed either on top of bigger disks or on the basis of a rod.



Fig. 1 The tower of Hanoi puzzle with 8 pieces and the Chinese rings with 6 pieces.

Call T_n the (minimum) number of moves to solve the puzzle. It is clear that before solving it we must pass by the position where the bigger $n - 1$ disks are placed in the middle rod (this needs T_{n-1} moves). Then we only need one move to exchange the biggest disk from the left to the right, plus again T_{n-1} more moves to change the middle stack (that has $n - 1$ disks) to the right stack. In short,

$$T_n = 2T_{n-1} + 1, \quad T_1 = 1.$$

Hence $T_n = 2^n - 1$ increases exponentially. In [26, Ch. 9] it can be seen how the similar difference equation

$$T_n = 2T_{n-1} + g_n, \quad T_1 = t_1,$$

where $\{g_n\}_n$ is a given sequence, like for instance $g_n = c \log(n)$, can be used to study the computational complexity of implementing the described algorithm in a computer.

Another similar puzzle is the so called *Chinese rings* (also known as *Bague-naudier*, which means "time-waster" in French). It seems that it goes back to the Chinese dynasty Sung (960-1279), see Figure 1. It can be seen that if S_n denotes the minimum number of moves to take all the rings off (we do not give here neither its description nor the study of S_n), see for instance [5, Ch. 10] or [54] for the details, it holds that

$$S_n = S_{n-1} + 2S_{n-2} + 1, \quad S_1 = 1, \quad S_2 = 2.$$

Its solution, again with exponential growth, is

$$S_n = \frac{2^{n+1}}{3} + \frac{(-1)^{n+1} - 3}{6}.$$

2.3 Tossing n coins

Take n coins, not necessarily fair, and assume that for each of them the probability of getting head is p and the one of getting tail is $q = 1 - p$.

We toss all of them together and want to know the probability P_n of getting an even number of heads, where 0 counts as an even number.

It is well-known that the number of heads can be modeled by a random variable with Binomial distribution $X_n = B(n, p)$. Moreover this X_n can be seen as the sum of n independent random variables $Y_j \sim Y$, each one with Bernoulli distribution with parameter p , that is defined as $P(Y = 1) = p$, $P(Y = 0) = q$. Thus $X_n = \sum_{k=1}^n Y_k$.

Hence the probability of getting exactly k heads is

$$P(X_n = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

and in consequence,

$$P_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} (1-p)^{n-2k}.$$

An easy alternative way of getting a compact expression for P_n comes from a difference equation. Notice that

$$\begin{aligned} P_{n+1} &= P(X_{n+1} \text{ is even}) \\ &= P(X_{n+1} \text{ is even and } Y_{n+1} \text{ is 0}) + P(X_{n+1} \text{ is even and } Y_{n+1} \text{ is 1}) \\ &= P(X_n \text{ is even})P(Y_{n+1} = 0) + P(X_n \text{ is odd})P(Y_{n+1} = 1) \\ &= P_n(1-p) + (1-P_n)p = (1-2p)P_n + p, \end{aligned}$$

where we have used twice that $P(A \cap B) = P(A/B)P(B)$. Hence

$$P_{n+1} = (1-2p)P_n + p, \quad P_0 = 1.$$

Solving it we obtain the nice compact expression $P_n = \frac{1}{2}(1 + (1-2p)^n)$. When p is 0; $\frac{1}{2}$; or 1 we get the very intuitive results: 1, 1, 1, ...; $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots$; or 1, 0, 1, 0, 1, ...

2.4 The Titius-Bode law

This law is attributed to the German astronomers J. D. Titius (1729-1796) and J. E. Bode (1747-1826). If we define d_n as

$$d_{n+1} = 2d_n - 0.4, \quad d_0 = 0.7,$$

it holds that

$$d_n = 0.3 \times 2^n + 0.4.$$

In particular, denoting $d_{-\infty} := \lim_{n \rightarrow -\infty} d_n = 0.4$, the sequence $d_{-\infty}, d_0, d_1, d_2, \dots$ is

$$0.4, 0.7, 1.0, 1.6, 2.8, 5.2, 10.0, 19.6, 38.8, \dots$$

They observed that the sequence of the major axis of the elliptic trajectories of the known planets at that time (Mercury, Venus, Earth, Mars, Jupiter and Saturn), that is

$$0.39, 0.72, 1.00, 1.52, 5.20, 9.55,$$

in astronomical unities (AU), remarkably coincides with the first terms of the sequence $\{d_n\}_n$ defined above, except for one gap, the one corresponding to 2.8. When in 1781, William Herschel discovered Uranus, and the corresponding distance was found to be 19.22 AU (also near the mathematical prediction), people started to believe that this law holds in the whole Solar system. For this reason, in 1800 many astronomers began the research of the “lost” planet. In 1801, G. Piazzi found a minor-planet, named Ceres (today classified as a dwarf planet) which precisely was at a distance 2.76 AU, confirming once more the Titius-Bode law. In fact, already in 1807 four more minor-planets were found at a similar distance. Nowadays, thousands of bodies are localized in that region forming the so called *asteroid belt*.

When Neptune was discovered in 1846, thanks to the mathematical predictions of Urbain J. J. Le Verrier, the Titus-Bode law was broken because its distance to the Sun was 30.11 AU. The former planet Pluto, today a dwarf planet, is at distance 39.54 AU, again in good agreement with the mathematical law.

Although Titius-Bode law was only empirical, without physical or mathematical explanations, some people have tried to find some reasons for the good predictions for the actual positions of the planets in the Solar system. For instance, in the nice paper [59] the authors propose a four body problem (center of masses of the Galaxy, Sun and two consecutive planets, the farthest planet with mass much smaller than the closest one). In that situation they prove that if the closest planet follows a circular orbit of radius R , one of the “more stable” solutions of this four body problem (a possible trajectory for the other planet) is at distance $3^{2/3}R \simeq 2.08R$, quite similar to the $\simeq 2$ given for n big by the law. That paper also proposes an explanation of the fail of the mathematical law for Neptune. Normalizing the masses such that the mass of the Earth is 1, the masses of the planets Jupiter, Saturn, Uranus, Neptune and Pluto

are 317.8, 95.2, 14.5, 17.1, 0.0025. The couple Uranus, Neptune is the only one for which the masses are not decreasing with their distance to the Sun.

2.5 Factorial and subfactorial functions

A very classical problem of recreational mathematics (see for instance [5, App. G]), that goes back to the French mathematician Pierre de Montmort (1678-1719) is the next one: consider n persons, each one wearing a different hat. Which is the probability that if the hats and the persons are coupled randomly, no person wear its own hat?

To solve it, people introduced the so called subfactorial function $!n$. In fact, both the factorial and the subfactorial functions can be defined like solutions of similar difference equations:

$$\begin{aligned} x_{n+1} &= (n+1)x_n, & x_0 &= 1 & \Rightarrow & n! := x_n, \\ y_{n+1} &= (n+1)y_n + (-1)^{n+1}, & y_0 &= 1 & \Rightarrow & !n := y_n. \end{aligned}$$

By using induction it is not difficult to prove that

$$!n = n! \left(1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

Let us relate this function with our problem. Given n persons and n hats it is clear that the total number of different possibilities of people wearing a hat is $n!$. On the other hand let us prove that the numbers of possibilities z_n of people wearing a hat that is not its own hat is $!n$.

First we will prove that z_n satisfies the following non-autonomous second order difference equation:

$$z_{n+1} = n(z_n + z_{n-1}), \quad z_0 = 1, \quad z_1 = 0. \quad (1)$$

To prove it, fix one of the $n+1$ persons, say person **1**, and assume that it wears a hat that is not yours, say hat **k**, $k \neq 1$. This hat can be selected of n different ways. Then consider person **k**. There are two possibilities:

- The hat of person **k** is hat **1**. Then there are z_{n-1} possibilities of wearing the $n-1$ remaining hats without coincidences.
- The hat of person **k** can be any hat but hat **1**. Then, there are n remaining people and n remaining hats, and each person only has a forbidden hat. There are z_n ways of wearing them.

Then equation (1) holds. Finally, let us prove by induction that $y_n = z_n$. It is clear that $y_0 = z_0 = 1$. Assume that $y_j = z_j$ for $j = 0, 1, \dots, n-1$ and let us prove it for $j = n$. Notice first that $y_{n-2} = \frac{y_{n-1} + (-1)^n}{n-1}$. Thus

$$\begin{aligned} z_n &= (n-1)(z_{n-1} + z_{n-2}) = (n-1)(y_{n-1} + y_{n-2}) \\ &= (n-1)\left(y_{n-1} + \frac{y_{n-1} + (-1)^n}{n-1}\right) = ny_{n-1} + (-1)^n = y_n. \end{aligned}$$

Hence the probability of having no coincidences is

$$\frac{z_n}{n!} = \frac{!n}{n!} = 1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!},$$

which clearly tends to $\frac{1}{e}$ when n goes to infinity. Similarly, the probability that at least one person wears its own hat tends to $1 - \frac{1}{e} \simeq 0.63$. In fact, the value 0.63 is already attained for $n = 6$ and does not essentially vary increasing n .

2.6 Computation of definite integrals

When studying the propagation of the error, several textbooks illustrate a surprising fact that happens computing recursively some definite integrals. By using the same algorithm, either forward or backwards, one is well-conditioned while the other one is not, see for instance [22, Ch. 1]. Consider

$$I_n = \frac{1}{e} \int_0^1 x^n e^x dx.$$

By using integration by parts we get

$$I_n = 1 - nI_{n-1}, \quad I_0 = 1 - \frac{1}{e}.$$

The above difference equation is useful for obtaining exact expressions of I_n . For applying numerically it, take as initial condition an approximation of I_0 , say 0.6321, that is $I_0 = 0.6321 \pm \varepsilon$, with $0 < \varepsilon < 3 \times 10^{-5}$. Then, from the algorithm we get I_n with the quite big absolute error, $n!\varepsilon$. On the other hand, take it backwards and with initial condition for $n = 100$, 0,

$$I_{n-1} = \frac{1 - I_n}{n}, \quad I_{100} = 0.$$

This has sense because I_n tends to zero when n goes to infinity, and $I_{100} = \delta$, with $0 < \delta < 1/101$, because

$$0 < I_n = \int_0^1 x^n e^{x-1} dx \leq \int_0^1 x^n dx = \frac{1}{n+1}.$$

Then all I_k , for k from 90 to 0, can be obtained with the reasonable small absolute error $\frac{\delta}{100 \times 99 \times 98 \times \dots \times (k+1)}$.

2.7 The $3x + 1$ problem

Consider the simple difference equation, defined on the positive integers,

$$x_{n+1} = g(x_n) = \begin{cases} \frac{3x_n+1}{2}, & \text{when } x_n \text{ is odd,} \\ \frac{x_n}{2}, & \text{when } x_n \text{ is even,} \end{cases} \quad x_0 \in \mathbb{N}.$$

The sequence $\{x_n\}_n$ is called the orbit of x_0 . For instance, the orbit corresponding to $x_0 = 11$ is 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, ... Until now, it has been checked that for any x_0 smaller than 87×2^{60} , after finitely many steps the orbit arrives to the 2-periodic behavior 2, 1, 2, 1, ...

The so called $3x + 1$ problem asks whether the above situation happens or not for any positive integer initial condition. See [50, 51] for more information about it. According to J. Lagarias “this is an extraordinarily difficult problem, completely out of reach of present day mathematics.” This problem is one of the simpler to state open problems in mathematics and it is frequently used in the talks addressed to young people for motivating them on science. It seems that it was studied for first time by the German mathematician Lothar Collatz, around 1930.

It is also known with many other different names: $3x + 1$ conjecture, Collatz conjecture, Ulam conjecture, Kakutani’s problem, Syracuse problem, ...

The answer is “no” for negative integer initial conditions. For instance, starting with $x_0 \in \{0, -1, -5, -17\}$ four other final periodic behaviors appear. For instance: $-5, -7, -10, -5, -7, \dots$. For the moment no other final behaviors have been found.

The map g can also be extended to the reals or to the complex, as

$$g(z) = \frac{z}{2} \cos^2\left(\frac{\pi}{2}z\right) + \frac{3z+1}{2} \sin^2\left(\frac{\pi}{2}z\right) = \frac{1+4z-(1+2z)\cos(\pi z)}{4},$$

giving rise to a complicated dynamical system, see [16, 55].

2.8 Proofs without words

Famous sequences of numbers like the Fibonacci ones, F_n , or the triangular numbers $T_n = 1 + 2 + \dots + n$, satisfy some difference equations. For instance

$$F_{n+1} = \sqrt{4F_n F_{n-1} + F_{n-2}^2}, \quad T_{2n} = 3T_n + T_{n-1}.$$

Although it is not difficult to prove them analytically, we show in next figure their *proofs without words* borrowed from the nice books [66, p. 104] and [67, p. 107].

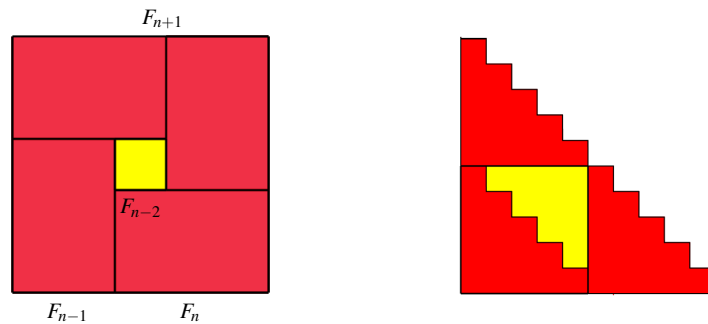


Fig. 2 Two proofs without words from [66, 67].

3 Newton and Chebyshev methods

To find a simple solution s of a non-linear equation $f(x) = 0$, with f smooth, people use the iterative methods $x_{n+1} = g(x_n)$, where

$$g(x) = g_N(x) = x - \frac{f(x)}{f'(x)} \quad \text{or} \quad g(x) = g_C(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2} \frac{f^2(x)f''(x)}{(f'(x))^3},$$

and x_0 is an approximation of s . They correspond to the Newton and Chebyshev methods. Recall that it is said that an iterative method has order p towards s if $\lim_{n \rightarrow \infty} x_n = s$ and

$$\lim_{n \rightarrow \infty} \frac{s - x_{n+1}}{(s - x_n)^p} = \alpha \neq 0.$$

It is well-known that Newton method is quadratic ($p = 2$) and Chebyshev method is cubic ($p = 3$) toward simple solutions. In fact, for some specific functions f , any of them can be even of higher order.

Sometimes you can read that Chebyshev method is faster than Newton method. This assertion could led to some misunderstandings. Let us clarify why. Consider the sequence given by Newton method $\{x_n\}_n$ and define a new sequence $z_n = x_{2n}$, $n \in \mathbb{N}$. It is clear that $z_{n+1} = g_N(g_N(z_n))$, $z_0 = x_0$; and we will call this “new” method bi-Newton. Thus Newton method has $p = 2$ and bi-Newton method has $p = 4$, but clearly the one with higher order is not better than the other one. Both are equal.

The above situation leads to the definition of *efficiency* of a method, associated to a solution s of a given equation $f(x) = 0$. This efficiency E , should take into account not only the order of convergence of the method towards s but also the *time* t needed for computing x_{n+1} from x_n . Hence $E = E(p, t)$. The reasoning of the above paragraph implies that this function must satisfy $E(p^2, 2t) = E(p, t)$, since applying $g \circ g$ one uses twice the time of applying g . In fact, in general

$$E(p^k, kt) = E(p, t), \quad k \in \mathbb{N}, p \in \mathbb{R}^+. \quad (2)$$

First, we will prove that the “simplest” smooth solution of the above functional equation is $E(p, t) = p^{1/t}$, which precisely is the definition of efficiency of a method towards s , see also [73].

To prove this assertion, assume first that (2) holds for all $k \in \mathbb{R}^+$. Taking derivatives with respect to k it holds that

$$\frac{\partial}{\partial p} E(p^k, kt) p^k \ln(p) + \frac{\partial}{\partial t} E(p^k, kt) t = 0.$$

Replacing $k = 1$ in the above equation we get that E must be a solution of the linear partial differential equation,

$$\frac{\partial}{\partial p} E(p, t) p \ln(p) + \frac{\partial}{\partial t} E(p, t) t = 0. \quad (3)$$

To find all its solutions we will us the method of characteristics. That is, first we have to find two functionally independent first integrals of the system of ordinary differential equations

$$p' = p \ln(p), \quad t' = t, \quad E' = 0.$$

From the first two equations we get the new ordinary differential equation $dp/dt = p \ln(p)/t$, that has the first integral $H(p, t, E) = \ln(p)/t$. From the third one we get the fist integral $H_2(p, t, E) = E$. Hence the general implicit solution of (3) is $\Phi(\ln(p)/t, E) = 0$ and its general explicit solution is $E = \phi(\ln(p)/t)$, for any smooth functions Φ and ϕ . Hence the “simplest” solution comes taking $\phi = \exp$ and then $E = \exp(\ln(p)/t) = p^{1/t}$. Another natural and equivalent definition would be possible: simply take $\phi = \text{Id}$, and then the efficiency would be $\ln(p)/t$.

Then the most efficient method among a list of methods, in the sense that it is the one that requires less computational time to get s with a desired accuracy, is the method with biggest efficiency.

As an illustration, let us compare several methods, with increasing orders, to compute \sqrt{a} . Applying Newton and Chebyshev methods to $f(x) = x^2 - a = 0$ gives

$$g_N(x) = \frac{1}{2} \left(x + \frac{a}{x} \right) \quad \text{and} \quad g_C(x) = \frac{x}{8} \left(3 + 6 \frac{a}{x^2} - \left(\frac{a}{x^2} \right)^2 \right).$$

In fact, for $x \approx \sqrt{a}$ and $(|(a-x^2)/x^2| < 1)$, it holds that

$$\begin{aligned} \sqrt{a} &= \sqrt{x^2 + (a-x^2)} = x \left(1 + \frac{a-x^2}{x^2} \right)^{\frac{1}{2}} = x \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{a-x^2}{x^2} \right)^k \\ &= x \sum_{k=0}^m \binom{\frac{1}{2}}{k} \left(\frac{a-x^2}{x^2} \right)^k + O \left(\left(\frac{a-x^2}{x^2} \right)^{m+1} \right) = g_m(x) + O \left((\sqrt{a}-x)^{m+1} \right), \end{aligned}$$

where

$$g_m(x) = x \sum_{k=0}^m \binom{\frac{1}{2}}{k} \left(\frac{a-x^2}{x^2} \right)^k.$$

Hence, for each $0 < m \in \mathbb{N}$, the iterative method $x_{n+1} = g_m(x_n)$ has order $p = m + 1$ towards \sqrt{a} , because

$$\lim_{n \rightarrow \infty} \frac{\sqrt{a} - x_{n+1}}{(\sqrt{a} - x_n)^{m+1}} = 2 \binom{\frac{1}{2}}{m+1} \left(\frac{2}{\sqrt{a}} \right)^m \neq 0.$$

In particular, $g_1 = g_N, g_2 = g_C$ and, for instance,

$$\begin{aligned} g_3(x) &= x \left(1 + \frac{1}{2} \frac{a-x^2}{x^2} - \frac{1}{8} \left(\frac{a-x^2}{x^2} \right)^2 + \frac{1}{16} \left(\frac{a-x^2}{x^2} \right)^3 \right) \\ &= \frac{x}{16} \left(5 + 15 \frac{a}{x^2} - 5 \left(\frac{a}{x^2} \right)^2 + \left(\frac{a}{x^2} \right)^3 \right). \end{aligned}$$

To know the efficiency of these three methods we must know the respective times $t_m, m = 1, 2, 3$ needed to compute an iteration. They only perform additions, subtractions, multiplications and divisions. Since the most consuming time ones are the last two, to simplify the problem we only will take into account the number of multiplications and divisions at each step. We will call τ the time used for each of these operations. For the Newton method, at each step we only need two divisions: a/x and $1/2$, so $t_1 = 2\tau$.

For the Chebyshev method and the one associated to g_3 we use the following procedures based on the Horner's evaluation of polynomials. First we need two divisions for computing $W := a/x^2$, and then

$$g_2(x) = \frac{x}{8} \left(3 + W(6 - W) \right) \quad \text{and} \quad g_3(x) = \frac{x}{16} \left(5 + W(15 + W(-5 + W)) \right).$$

Hence $t_2 = (2+3)\tau = 5\tau$ and $t_3 = (2+4)\tau = 6\tau$. Therefore the efficiencies of g_1, g_2 and g_3 are

$$E_1 = 2^{\frac{1}{2\tau}} = \left(2^{\frac{1}{2}}\right)^{\frac{1}{\tau}}, \quad E_2 = 3^{\frac{1}{5\tau}} = \left(3^{\frac{1}{5}}\right)^{\frac{1}{\tau}}, \quad E_3 = 4^{\frac{1}{6\tau}} = \left(4^{\frac{1}{6}}\right)^{\frac{1}{\tau}},$$

and it holds that $E_1 > E_3 > E_2$. In general it can be seen that $E_1 > E_m, m \geq 2$ and in consequence although the methods that we have introduced have increasing orders the most efficient one is in fact the one with lower order, the celebrated Newton method.

It is also worth to comment that the bi-Newton method can also be written using the value W introduced above as

$$g_N(g_N(x)) = \frac{x^4 + 6ax^2 + a^2}{4x(x^2 + a)} = \frac{x}{4} \left(\frac{1 + W(6 + W)}{1 + W} \right), \quad (4)$$

but it is more efficient to use it applying twice the Newton expression.

We end this section with some historical comments about the Newton method for computing \sqrt{a} ,

$$x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}, \quad x_0 \approx \sqrt{a}.$$

and also about bi-Newton method. Most probably, this Newton method is the first recurrence developed by humanity. The babylonians, around (2000-1700 BC), already proposed it as a method for computing (with enough precision for their interests) the square root of a number, see [15, Ch. 7]. This method was also described by Heron of Alexandria in the first century AC. They only perform the first steps of the method and they deduce it with a beautiful geometrical reasoning that I can not resist to include here: Let x_0 be a good approximation of \sqrt{a} . Then construct the rectangle with one side x_0 and the other one such that its area is a . If it is a square we are done and x_0 is the searched square root. Otherwise it is a rectangle with the sides of lengths x_0 and a/x_0 . One length is bigger than the square root and the other one is smaller. Therefore it is natural to take as a better approximation for \sqrt{a} their average, that is $x_1 = (x_0 + a/x_0)/2$, and the (Newton) method appears naturally.

The bi-Newton method, presented in a similar manner that in (4), was found in an ancient Indian mathematical manuscript called the Bakhshali manuscript (of around 200-400) and so, nowadays is known as Bakhshali method.

4 Approximating π .

This section includes three different algorithms to approximate π , all them based on recurrent procedures.

4.1 Archimedes approximations

The method devised by Archimedes to approximate π consists in computing the perimeters of the regular polygons with 6×2^n sides circumscribing and inscribed in a circle of diameter one, denoted as q_n and p_n , respectively. Then, $p_n < \pi < q_n$ and $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} p_n = \pi$.

We will first deduce a simple recurrence for p_n . For the inscribed hexagon ($n = 0$) it is clear that $p_0 = 3$. Let us call x the length of a side of a given regular polygon, and let us compute the length y of the side of the regular polygon with the double number of sides, see Figure 3.

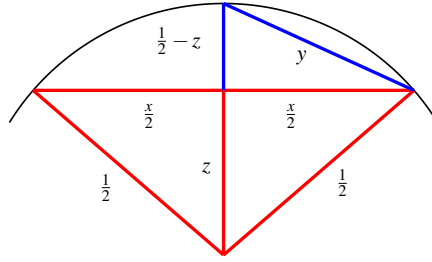


Fig. 3 Relation between the lengths of the sides of polygons with m and $2m$ sides.

By using twice Pythagoras Theorem we get that

$$\frac{x^2}{4} + z^2 = \frac{1}{4}, \quad \frac{x^2}{4} + \left(\frac{1}{2} - z\right)^2 = y^2.$$

Consequently, from the right-hand expression $\frac{x^2}{4} + \frac{1}{4} + z^2 - z = y^2$ and, substituting in the left-hand one, $\frac{1}{2} - z = y^2$. Since $z = \sqrt{\frac{1-x^2}{4}}$, we conclude that $y = \sqrt{\frac{1}{2}(1 - \sqrt{1-x^2})}$.

Hence, if ℓ_n is the length of a side of a regular polygon with 6×2^n it holds that $\ell_0 = \frac{1}{2}$, $p_0 = 3$,

$$\ell_{n+1} = \sqrt{\frac{1}{2}(1 - \sqrt{1 - \ell_n^2})}, \quad \text{and} \quad p_{n+1} = 6 \cdot 2^{n+1} \ell_{n+1}.$$

Notice, that by definition $\lim_{n \rightarrow \infty} p_n = \pi$. This algorithm produces round-off errors due to cancelations when computing $\sqrt{1 - \sqrt{1 - u}}$, for $u = \ell_n^2$ smaller and smaller. It is convenient to transform it using that

$$(1 - \sqrt{1 - u}) \frac{1 + \sqrt{1 - u}}{1 + \sqrt{1 - u}} = \frac{u}{1 + \sqrt{1 - u}}.$$

and the final algorithm is

$$\begin{aligned} \ell_{n+1} &= \frac{\ell_n}{\sqrt{2(1 + \sqrt{1 - \ell_n^2})}}, & \ell_0 &= \frac{1}{2}, \\ p_{n+1} &= 6 \cdot 2^{n+1} \ell_{n+1}, & p_0 &= 3. \end{aligned}$$

We get that $p_1 = \underline{3.1}0\dots$, $p_2 = \underline{3.13}0\dots$, $p_3 = \underline{3.139}\dots$, $p_4 = \underline{3.1410}\dots$, $p_{15} = \underline{3.1415926534}\dots$, where we underline the correct digits.

There is a different expression of Archimedes approach, see [32, Chp. 1] for the details, that computes both q_n and p_n simultaneously. It holds that

$$q_{n+1} = \frac{2q_n p_n}{q_n + p_n}, \quad p_{n+1} = \sqrt{q_{n+1} p_n} = p_n \sqrt{\frac{2q_n}{q_n + p_n}}, \quad q_0 = 2\sqrt{3}, \quad p_0 = 3.$$

Moreover $q_{n+1} - p_{n+1} < (q_n - p_n)/3$. For instance, taking the polygon with 96 sides ($n = 4$) we get $p_4 = 3.1410\dots = p_4 < \pi < q_4 = 3.1427\dots$ and we recover the classical Archimedes bounds

$$3 + \frac{10}{71} < p_4 < \pi < q_4 < 3 + \frac{10}{70}.$$

4.2 A new simple algorithm

The starting point of this algorithm is the simple equality

$$0 < \int_0^1 \frac{(3x^2 - 1)^2}{1 + x^2} dx = 4(\pi - 3),$$

Another similar and very nice equality, given by Dalzell ([27]) in 1944, is

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

Both together prove easily that $3 < \pi < 22/7$. Notice that 3 and $\frac{22}{7}$ are precisely the first two convergents of the development of π in continuous fractions, see (7). In [3, 8, 28, 60, 61, 69] there appear many other similar relations and some of them have already been used to get algorithms for approaching π , faster than the one we deduce here.

Let us start the deduction of the algorithm. It holds that

$$\frac{(3x^2-1)^2}{1+x^2} = 9x^2 - 15 + \frac{16}{1+x^2},$$

or, equivalently,

$$\frac{16 - (3x^2-1)^2}{1+x^2} = 15 - 9x^2.$$

So, for $0 \leq x \leq 1$,

$$\frac{4}{1+x^2} = \frac{15-9x^2}{4 - \frac{(3x^2-1)^2}{4}} = \frac{\frac{15-9x^2}{4}}{1 - \left(\frac{3x^2-1}{4}\right)^2} = \sum_{k=0}^{\infty} \frac{15-9x^2}{4} \left(\frac{3x^2-1}{4}\right)^{2k},$$

because $|(3x^2-1)/4| \leq 1/2 < 1$, and we use that for $|u| < 1$, $1/(1-u) = \sum_{k \geq 0} u^k$. Since for $|u| \leq 1/2$ the convergence is uniform,

$$\begin{aligned} \pi &= \int_0^1 \frac{4}{1+x^2} dx = \int_0^1 \sum_{k=0}^{\infty} \frac{15-9x^2}{4} \left(\frac{3x^2-1}{4}\right)^{2k} dx \\ &= \sum_{k=0}^{\infty} \int_0^1 \frac{(15-9x^2)(3x^2-1)^{2k}}{4^{2k+1}} dx. \end{aligned}$$

Hence, if we define

$$J_n := \int_0^1 \frac{(15-9x^2)(3x^2-1)^n}{4^{n+1}} dx, \quad \pi = \sum_{k=0}^{\infty} J_{2k}.$$

We will prove below that

$$J_n = I_n - I_{n+1}, \quad \text{where} \quad I_n := 3 \int_0^1 \left(\frac{3x^2-1}{4}\right)^n dx. \quad (5)$$

Therefore,

$$\pi = \sum_{k=0}^{\infty} J_{2k} = \sum_{k=0}^{\infty} (I_{2k} - I_{2k+1}) = \sum_{n=0}^{\infty} (-1)^n I_n,$$

Moreover,

$$I_n = \frac{3}{2^n(2n+1)} - \frac{n}{2(2n+1)}I_{n-1}, \quad I_0 = 3 \quad \text{and} \quad \pi = \sum_{n=0}^{\infty} (-1)^n I_n. \quad (6)$$

Hence,

$$\pi = 3 - 0 + \frac{3}{20} - \frac{3}{140} + \frac{9}{560} - \frac{3}{616} + \frac{159}{64064} - \dots$$

By taking as approximations for π the n th partial sums of this alternating sum we get alternatively upper and lower bounds for π . For instance, if $s_m := \sum_{n=0}^m (-1)^n I_n$, $s_0 = s_1 = 3$, $s_2 = \frac{63}{20} = 3.15$, $s_3 = \frac{219}{70} = 3.12\dots$, $s_4 = \frac{1761}{560} = 3.144\dots$. Similarly, $s_{10} = 3.1416\dots$, $s_{20} = 3.14159266\dots$, $s_{30} = 3.14159265359\dots$.

Finally, let us prove (5) and (6). The first formula follows because

$$\begin{aligned} J_n &= \int_0^1 \frac{(15-9x^2)(3x^2-1)^n}{4^{n+1}} dx = \int_0^1 \frac{3(4-(3x^2-1))(3x^2-1)^n}{4^{n+1}} dx \\ &= \int_0^1 \left(\frac{3(3x^2-1)^n}{4^n} - \frac{3(3x^2-1)^{n+1}}{4^{n+1}} \right) dx = I_n - I_{n+1}. \end{aligned}$$

To prove the second one, notice first that integrating by parts,

$$I_n = 3 \int_0^1 \left(\frac{3x^2-1}{4} \right)^n dx = 3x \left(\frac{3x^2-1}{4} \right)^n \Big|_0^1 - \frac{18n}{4} \int_0^1 x^2 \left(\frac{3x^2-1}{4} \right)^{n-1} dx.$$

Equivalently,

$$I_n = \frac{3}{2^n} - 6nK_n, \quad \text{with} \quad K_n = \frac{3}{4} \int_0^1 x^2 \left(\frac{3x^2-1}{4} \right)^{n-1} dx.$$

On the other hand

$$I_n = 3 \int_0^1 \left(\frac{3x^2-1}{4} \right)^n dx = 3 \int_0^1 \frac{3x^2-1}{4} \left(\frac{3x^2-1}{4} \right)^{n-1} dx = 3K_n - \frac{1}{3}I_{n-1}.$$

Joining both expressions, canceling K_n , we get $(2n+1)I_n = \frac{3}{2^n} - \frac{n}{2}I_{n-1}$, which clearly implies (6).

4.3 Brent-Salamin algorithm

In 1973, independently, Salamin ([74]) and Brent ([12]) found a quadratic method to approximate π . It is based on some classical equalities involving the so called

arithmetical-geometrical mean, that already appears in the works of Gauss and Legendre, complemented with efficient algorithms for computation of multiplications and square roots. The proofs of these equalities are based on the theory of elliptic integrals, see [9, 70]. In fact, it is not difficult to prove that if we consider $a_0 > 0$ and $b_0 > 0$ and we construct the sequences $a_{k+1} = (a_k + b_k)/2$, $b_{k+1} = \sqrt{a_k b_k}$, then $\lim_{k \rightarrow \infty} a_k$ and $\lim_{k \rightarrow \infty} b_k$ exist and coincide. This value is the arithmetic-geometric mean of a_0 and b_0 and it is denoted by $\text{AGM}(a_0, b_0)$. The difficult and remarkable equality is

$$\pi = \frac{4 \text{AGM}^2(1, 1/\sqrt{2})}{1 - \sum_{k=1}^{\infty} 2^{k+1}(a_k^2 - b_k^2)}, \quad \text{where}$$

$$a_{k+1} = \frac{a_k + b_k}{2}, \quad b_{k+1} = \sqrt{a_k b_k}, \quad a_0 = 1, \quad a_1 = \frac{1}{\sqrt{2}}.$$

Thus, taking $a_0 = 1$ and $b_0 = \frac{1}{\sqrt{2}}$, The Brent-Salamin algorithm computes

$$z_n = \frac{(a_n + b_n)^2}{1 - \sum_{k=1}^n 2^{k+1}(a_k^2 - b_k^2)},$$

where a_k and b_k are given above and proves that it is an algorithm that converges quadratically to π . Notice that to get it, the series is truncated and moreover it is used that for n big enough $\text{AGM}(1, 1/\sqrt{2}) \approx a_{n+1} = (a_n + b_n)/2$.

Hence $z_1 = 3.140\dots$, $z_2 = 3.14159264\dots$, $|z_3 - \pi| < 2 \times 10^{-19}$, $|z_4 - \pi| < 6 \times 10^{-41}$, $|z_5 - \pi| < 3 \times 10^{-84}$. Nowadays there are other similar, and even faster, algorithms, see for instance [2, 10, 11, 29, 38].

5 The GCD as a dynamical system

In his beautiful paper *Cooking the Classics* ([76]), Ian Stewart presents a way of computing the greatest common divisor (GCD) of two integer numbers by using a continuous DDS that we reproduce here. As he already commented, this way of obtaining the GCD is a dynamical reformulation of the classical way already used by the ancient Greeks called *anthyphaeresis*. This method consists of removing (the biggest possible) squares of a rectangle until arriving to a new rectangle for which this size of squares cannot be further removed, and then continue with the same procedure with the new rectangle and smaller squares and so on, until arriving to the empty set.

Dynamically, consider the non-invertible map $F : \mathbb{N}^2 \rightarrow \mathbb{N}^2$,

$$F(x, y) = (\max(x, y) - \min(x, y), \min(x, y)),$$

which defines a semi-DDS. As usual $F^0 = \text{Id}$ and $F^n = F \circ F^{n-1}$. The following holds:

Theorem 1. ([76]) *For each $(a, b) \in \mathbb{N}^2$, $ab \neq 0$, there exists $m = m(a, b)$ such that $F^m(a, b) = (\text{gcd}(a, b), 0)$.*

Proof. We start proving that if we define the two functions $V, W : \Omega \rightarrow \mathbb{N}$ as $V(x, y) = x + y$ and $W(x, y) = \text{gcd}(x, y)$, where $\Omega = \mathbb{N}^2 \setminus \{(x, y) \in \mathbb{N}^2 : xy = 0\}$, they are a *strict Lyapunov function* and a *first integral*, respectively, for the semi-DDS generated by F , on Ω . In fact, for $(x, y) \in \Omega$,

$$V(F(x, y)) = \max(x, y) < x + y = V(x, y).$$

Observe also that $V(F(x, y)) = V(x, y)$ when $xy = 0$.

To prove that $W(F(x, y)) = W(x, y)$ notice that if z divides both, x and y , it also divides $\max(x, y)$ and $\min(x, y)$ and so does with the two components of F . Conversely, if z divides $\max(x, y) - \min(x, y)$ and $\min(x, y)$ it also divides their sum, $\max(x, y)$, and so it divides x and y .

Notice that $F(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$. Therefore, since V is strictly decreasing and non-negative, given an initial pair $(x, y) \in \Omega$ it must exist a $k \in \mathbb{N}$ such that $V(F^k(x, y)) = (z, 0)$ or $(0, z)$. Since $F(0, z) = (z, 0)$ the result follows and m is either k or $k + 1$. \square

This result gives the following recurrence:

$$\begin{aligned} a_{n+1} &= \max(a_n, b_n) - \min(a_n, b_n), \\ b_{n+1} &= \min(a_n, b_n), \quad (a_0, b_0) \in \mathbb{N}^2, \quad a_0 b_0 \neq 0, \end{aligned}$$

for which there exists $m = m(a_0, b_0) \in \mathbb{N}$ such that $a_n = \text{gcd}(a_0, b_0)$ and $b_n = 0$ for all $n \geq m$.

It can be seen that this procedure is also equivalent to the Euclides algorithm, with the advantage that there is no need of introducing divisions. Moreover, it allows to extend the definition of GCD to positive rational numbers. Set $\mathbb{Q}^+ = \mathbb{Q} \cap \{x \in \mathbb{R} : x \geq 0\}$, then:

Corollary 1. *For each $a = p/q$, $b = r/s$ with $(a, b) \in (\mathbb{Q}^+)^2$ and $\text{gcd}(p, q) = \text{gcd}(r, s) = 1$ there exists $m = m(a, b)$ and a rational number c such that $F^m(a, b) = (c, 0)$. This c will be called the greatest common divisor of a and b , $\text{gcd}(a, b)$. Moreover*

$$c = \text{gcd}(a, b) = \frac{\text{gcd}(p, r)}{\text{lcm}(q, s)} \quad \text{and} \quad \text{lcm}(a, b) := \frac{ab}{\text{gcd}(a, b)}.$$

Proof. By linearity, $F(a, b) = \frac{1}{qs} F(qsa, qsb) = \frac{1}{qs} F(ps, qr)$. Hence, by Theorem 1 there exists $m \in \mathbb{N}$ such that

$$F^m(a, b) = \frac{1}{qs} F^m(ps, qr) = \frac{1}{qs} (\gcd(ps, qr), 0)$$

$$\stackrel{(*)}{=} \left(\frac{\gcd(p, r) \gcd(q, s)}{\gcd(q, s) \operatorname{lcm}(q, s)}, 0 \right) = \left(\frac{\gcd(p, r)}{\operatorname{lcm}(q, s)}, 0 \right),$$

where in $(*)$ we have used that $qs = \gcd(q, s) \operatorname{lcm}(q, s)$ and that $\gcd(ps, qr) = \gcd(p, r) \gcd(q, s)$. These points are already fixed points for F . \square

For instance $F^{11}(22/91, 55/63) = (11/819, 0)$. Therefore $\gcd(22/91, 55/63) = 11/819$ and $\operatorname{lcm}(22/91, 55/63) = 110/7$. Notice that all quotients

$$\frac{22/91}{11/819} = 18, \quad \frac{55/63}{11/819} = 65, \quad \frac{110/7}{22/91} = 65, \quad \frac{110/7}{55/63} = 18,$$

are natural numbers.

Finally, it can be extended to initial conditions in $\mathbb{R}^+ \times \mathbb{R}^+$ and it provides the continued fraction expansions of a number.

Let us consider an example. Recall that the elements of the continued fraction of π are $[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots]$ and that their associated convergents are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \dots \quad (7)$$

Now if we compute $F^n(\pi, 1)$, for $n \in \{3, 3+7=10, 10+15=25, 25+1=26, 26+292=318, 318+1=319, \dots\}$ we recover them:

$$a_3 = \pi - 3, \quad a_{10} = 22 - 7\pi, \quad a_{25} = 106\pi - 333,$$

$$a_{26} = 355 - 113\pi, \quad a_{318} = -103993 + 33102\pi, \quad a_{319} = 104348 - 33215\pi.$$

6 Landen maps

Loosely speaking, when given a family of definite integrals depending on parameters there is a map among these parameters in such a way that the integrals remain invariant, then this map is called a *Landen map*. In other words, the integrals are first integrals of the semi-DDS generated by this Landen map.

The paradigmatic example goes back to the works of Gauss and Landen (1719-1790, a British amateur mathematician) on elliptic integrals. For $a > 0$, $b > 0$, consider the elliptic integral

$$I(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}. \quad (8)$$

They proved that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\left(\frac{a+b}{2}\right)^2 \cos^2 \theta + ab \sin^2 \theta}}, \quad (9)$$

or, in other words, that if

$$F(a, b) = \left(\frac{a+b}{2}, \sqrt{ab} \right),$$

then $I(a, b) = I(F(a, b))$ and so I is a first integral or an invariant of the semi-DDS generated by F , see [9, 23, 36, 52]. In Section 6.1 we will prove an extension of this result due to Carlson ([14]).

As we have already explained in Section 4.3, for $a > 0, b > 0$ it holds that

$$\lim_{n \rightarrow \infty} F^n(a, b) = (\text{AGM}(a, b), \text{AGM}(a, b))$$

where $\text{AGM}(a, b)$ is the arithmetic-geometric mean of a and b . Let us prove that

$$I(a, b) = \frac{1}{\text{AGM}(a, b)} \frac{\pi}{2}. \quad (10)$$

This holds because $I(a, b)$ is invariant by F and so, for all $n \in \mathbb{N}$,

$$\begin{aligned} I(a, b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{(F_1^n(a, b))^2 \cos^2 \theta + (F_2^n(a, b))^2 \sin^2 \theta}} \\ &= \lim_{n \rightarrow \infty} \int_0^{\pi/2} \frac{d\theta}{\sqrt{(F_1^n(a, b))^2 \cos^2 \theta + (F_2^n(a, b))^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{\text{AGM}^2(a, b) \cos^2 \theta + \text{AGM}^2(a, b) \sin^2 \theta}} \\ &= \frac{1}{\text{AGM}(a, b)} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{1}{\text{AGM}(a, b)} \frac{\pi}{2}. \end{aligned}$$

Hence, we have the following algorithm for computing elliptic integrals

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad a_0 = a > 0, \quad b_0 = b > 0,$$

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \lim_{n \rightarrow \infty} \frac{\pi}{2a_n}.$$

Let us prove that it converges quadratically. Without loss of generality, we can consider $a > b > 0$. Then, for all $n \geq 1$, $b < b_n < a_n < a$ and $a_{n+1}^2 - b_{n+1}^2 = \frac{1}{4}(a_n - b_n)^2$.

Hence

$$0 < a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{4(a_{n+1} + b_{n+1})} < \frac{1}{8b}(a_n - b_n)^2.$$

Similar procedures can be used to compute also quadratically other elementary functions like $\log(x)$, e^x , \dots , see [9, 12].

Other interesting examples are given by some rational improper integrals. The simplest one is

$$I(a, b, c) = \int_0^\infty \frac{bx^2 + c}{x^4 + ax^2 + 1} dx, \quad a > -2.$$

If we consider

$$F(a, b, c) = \left(2, \frac{b+c}{\sqrt{a+2}}, \frac{b+c}{\sqrt{a+2}}\right),$$

it is not difficult to prove that $I(a, b, c) = I(F(a, b, c))$. Hence

$$\begin{aligned} I(a, b, c) &= I(F(a, b, c)) = \frac{b+c}{\sqrt{a+2}} \int_0^\infty \frac{x^2 + 1}{x^4 + 2x^2 + 1} dx \\ &= \frac{b+c}{\sqrt{a+2}} \int_0^\infty \frac{dx}{x^2 + 1} = \frac{(b+c)\pi}{2\sqrt{a+2}}. \end{aligned}$$

Notice that in this case there is no need to reach the limit to compute the integral because all points reach fixed points in two steps. Other rational examples, much more involved, are studied in [7, 17, 62, 63].

6.1 Computation of other means

In [14], Carlson extends (9) to other types of means. Given a and b positive, consider for each couple (i, j) with $i, j \in \{1, 2, 3, 4\}$, the next sequences:

$$\begin{aligned} a_0 &:= a, & b_0 &:= b, \\ a_{n+1} &:= f_i(a_n, b_n), & b_{n+1} &:= f_j(a_n, b_n), \quad n \geq 0, \quad \text{where} \\ f_1(a, b) &= \frac{a+b}{2}, & f_2(a, b) &= \sqrt{ab}, \\ f_3(a, b) &= \sqrt{a \frac{a+b}{2}}, & f_4(a, b) &= \sqrt{\frac{a+b}{2} b}. \end{aligned}$$

It is not difficult to see that given i and j , the sequences $\{a_n\}_n$ and $\{b_n\}_n$ converge to a common limit that we will denote as $\ell_{i,j}(a_0, b_0)$. Clearly these functions are some kind of means.

In next result it will appear the Beta function, that is

$$B(m, n) = \int_0^1 s^{n-1}(1-s)^{m-1} ds = \int_0^\infty t^{m-1}(1+t)^{-(m+n)} dt,$$

for each m and n positive. It satisfies $B(m, n) = B(n, m)$ and $B(1/2, 1/2) = \pi$, see for instance [1].

Theorem 2. ([14]) Consider the function

$$R(r; s, s'; a^2, b^2) := \frac{1}{B(r, r')} \int_0^\infty t^{r'-1}(t+a^2)^{-s}(t+b^2)^{-s'} dt, \quad (11)$$

with $r' = s + s' - r$. By taking the parameters (r, s, s') according to next table:

(i, j)	$F_{i,j}(a, b)$	(r, s, s')
$(1, 2)$	$\left(\frac{a+b}{2}, \sqrt{ab}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
$(1, 3)$	$\left(\frac{a+b}{2}, \sqrt{a\frac{a+b}{2}}\right)$	$\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}\right)$
$(1, 4)$	$\left(\frac{a+b}{2}, \sqrt{\frac{a+b}{2}b}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 1\right)$
$(2, 3)$	$\left(\sqrt{ab}, \sqrt{a\frac{a+b}{2}}\right)$	$\left(1, \frac{3}{4}, \frac{1}{2}\right)$
$(2, 4)$	$\left(\sqrt{ab}, \sqrt{\frac{a+b}{2}b}\right)$	$\left(1, \frac{1}{2}, 1\right)$
$(3, 4)$	$\left(\sqrt{a\frac{a+b}{2}}, \sqrt{\frac{a+b}{2}b}\right)$	$(1, 1, 1)$

it holds that $R(r; s, s'; a^2, b^2)$ is a first integral of the DDS associated to $F_{i,j}(a, b) := (f_i(a, b), f_j(a, b))$, $i < j$. That is, for the (r, s, s') corresponding to the $i < j$ considered,

$$R(r; s, s'; a^2, b^2) = R(r; s, s'; f_i^2(a, b), f_j^2(a, b)).$$

Moreover,

$$\ell_{i,j}(a, b) = \left(R(r; s, s'; a^2, b^2)\right)^{-\frac{1}{2r}}. \quad (12)$$

Proof. Given a and b , take $t = s(s + f_2^2)/(s + f_1^2)$, where $f_k = f_k(a, b)$. Then,

$$\frac{dt}{ds} = \frac{(s + f_3^2)(s + f_4^2)}{(s + f_1^2)^2}, \quad t + a^2 = \frac{(s + f_3^2)^2}{s + f_1^2}, \quad t + b^2 = \frac{(s + f_4^2)^2}{s + f_1^2}.$$

By using the above three equalities, the function (11) can be written as

$$\begin{aligned}
R(r; s, s'; a^2, b^2) &= \frac{1}{B(r, r')} \int_0^\infty s'^{r'-1} (s + f_1^2)^{r-1} (s + f_2^2)^{r'-1} (s + f_3^2)^{1-2s} (s + f_4^2)^{1-2s'} ds. \quad (13)
\end{aligned}$$

To prove that (11) is a first integral of $F_{i,j}$, both expressions (11) and (13) must coincide. For instance, consider $i = 1$ and $j = 2$, (case corresponding to the AGM). Then the parts involving f_3 and f_4 must disappear from (13). Thus $1 - 2s = 0$, $1 - 2s' = 0$, $r - 1 = -s$ and $r' - 1 = -s'$. Hence $s = s' = r = r' = 1/2$, which are the values appearing in the table of the statement. The other cases can be proved similarly.

To get $\ell_{i,j}(a, b)$, write $H(a^2, b^2) = R(r; s, s'; a^2, b^2)$ for the values of r, s and s' given in the table. Since $H(a^2, b^2) = H((f_i^n(a, b))^2, (f_j^n(a, b))^2)$ for all $n \in \mathbb{N}$ we get

$$\begin{aligned}
H(a^2, b^2) &= H(\ell_{i,j}^2(a, b), \ell_{i,j}^2(a, b)) = \frac{1}{B(r, r')} \int_0^\infty t^{r'-1} (t + \ell_{i,j}^2(a, b))^{-s-s'} dt \\
&= \frac{1}{B(r, r')} \ell_{i,j}^{2r'-2s-2s'}(a, b) \int_0^\infty u^{r'-1} (u+1)^{-s-s'} du \\
&= \frac{B(r', r)}{B(r, r')} \ell_{i,j}^{-2r}(a, b) = \ell_{i,j}^{-2r}(a, b).
\end{aligned}$$

Hence (12) holds. \square

Notice that for $(i, j) = (1, 2)$,

$$R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; a^2, b^2\right) = \frac{1}{B(1/2, 1/2)} \int_0^\infty \frac{dt}{\sqrt{t(t+a^2)(t+b^2)}} = \frac{2}{\pi} I(a, b).$$

Thus, modulus a multiplicative constant, the given first integral coincides with the first integral given by Gauss and Landen.

We end this section with the computation of the harmonic-geometric mean of $a > 0$ and $b > 0$, $\text{HGM}(a, b)$. It is natural to introduce it as the common limit of the two components of $(a_{n+1}, b_{n+1}) = G(a_n, b_n)$, where

$$G(x, y) = \left(\frac{2ab}{a+b}, \sqrt{ab} \right).$$

It is easy to see that if I is a first integral of the SDD given by f , that is, if $I \circ F = I$, then given any bijection φ , it holds that $J = I \circ \varphi$ is a first integral of the DDS generated by $g = \varphi^{-1} \circ f \circ \varphi$. Effectively,

$$J \circ g = I \circ \varphi \circ \varphi^{-1} \circ f \circ \varphi = I \circ f \circ \varphi = I \circ \varphi = J.$$

By taking $f = F$, with F given in (9), I as in (8) and $\varphi(a, b) = (1/a, 1/b)$, we get that

$$J(a, b) := \int_0^{\pi/2} \frac{abd\theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} = abI(b, a) = abI(a, b), \quad (14)$$

is a first integral of G . Arguing as in the proof of (10) it holds that $J(a, b) = \text{HGM}(a, b) \frac{a}{2}$. By using this last equality, (10) and (14) we get that

$$\text{AGM}(a, b) \cdot \text{HGM}(a, b) = ab.$$

7 Gambler's ruin

Assume that a gambler starts playing a game having a capital $N \in \mathbb{N}$. He wins a match with probability p , and then its capital increases by 1, and losses a match with probability $q = 1 - p$, and in this situation his capital passes to be $N - 1$. The game consists of successive matches and ends when either he arrives to a priori fixed capital, $A \in \mathbb{N}$, $A > N$, or when he losses all its capital arriving to 0, see for instance [33, Ch. XIV]. We want to know the probability R_N that the player get ruined starting with the capital N . It seems that this problem goes back to a letter from Blaise Pascal to Pierre Fermat in 1656.

Clearly the above game is equivalent to another one: two players, one against the other following similar rules and one with initial capital $N \geq 0$ and the other one with initial capital $A - N \geq 0$. This equivalent game stops when one of the two losses all its money.

The first game can be modeled by using the random variable $M_m = N + S_m$, where S_m is a simple random walk. More concretely, $S_m = \sum_{j=1}^m X_j$, where all X_j are independent, identically distributed random variables, with $X_j \sim X$ and X a Bernoulli type random variable such that $P(X = 1) = p$, $P(X = -1) = q$.

Call \mathcal{B}_n the event "get ruined if we start the game with a capital $N = n$ ". This happens if $M_m = 0$ for some $m \in \mathbb{N}$ and $M_i < A$ for all $i < m$. It holds that

$$\begin{aligned} R_n &= P(\mathcal{B}_n) = P(\mathcal{B}_n \cap \{X_1 = 1\}) + P(\mathcal{B}_n \cap \{X_1 = -1\}) \\ &= P(\mathcal{B}_n / X_1 = 1)P(X_1 = 1) + P(\mathcal{B}_n / X_1 = -1)P(X_1 = -1) \\ &= P(\mathcal{B}_n / X_1 = 1)p + P(\mathcal{B}_n / X_1 = -1)q \\ &\stackrel{(*)}{=} P(\mathcal{B}_{n+1})p + P(\mathcal{B}_{n-1})q = R_{n+1}p + R_{n-1}q, \end{aligned}$$

where the equality $(*)$ is intuitively clear and can be proved without major difficulties. The case $p = 0$ is trivial and can be treated apart. For $p \neq 0$, we get that

$$R_{n+1} = \frac{1}{p}R_n - \frac{q}{p}R_{n-1}, \quad R_0 = 1, \quad R_A = 0.$$

Notice that the boundary conditions come naturally from the rules of the game. The general solution of this difference equation is $\alpha + \beta \left(\frac{q}{p}\right)^n$ when $p \neq q$ and $\alpha + \beta n$ when $p = q = 1/2$, for some real numbers α and β . Imposing the boundary conditions we get that for $p \neq q$, and $n = N$,

$$R_N = \frac{(q/p)^N - (q/p)^A}{1 - (q/p)^A}$$

and for $p = q = 1/2$, $R_N = 1 - \frac{N}{A}$.

The above result also proves that the probability that the game does not finish is zero. In fact, the game ends if either the player gets ruined, or he arrives to the capital N . The probability of the first event is R_N , while the other one S_N , corresponds to another player that starts with capital $A - N$, instead of N , and a new p equal to $q = 1 - p$. Then

$$S_N = R_N|_{N \rightarrow A-N, p \rightarrow q} = \frac{(p/q)^{A-N} - (p/q)^A}{1 - (p/q)^A} = \frac{1 - (q/p)^N}{1 - (q/p)^A},$$

which clearly gives $R_N + S_N = 1$. Obviously, both maps $N \rightarrow A - N$ and $p \rightarrow 1 - p$, are involutions. Similarly, it can be seen that if we take the random variable \mathcal{D}_N , “duration of the game,” and we call E_N its expectation, $E_N = E(\mathcal{D}_N)$, that can be proved to be finite, then for $p \neq 0$,

$$E_{n+1} = \frac{1}{p}E_n - \frac{q}{p}E_{n-1} - \frac{1}{p}, \quad E_0 = 0, \quad E_A = 0$$

and its explicit solution can be found similarly.

8 Rational points on algebraic curves

Consider the DDS generated by the rational map

$$F(x, y) = \left(\frac{x(9x^3 - 8y^2)}{4y^2}, \frac{27x^6 - 36x^3y^2 + 8y^4}{8y^3} \right).$$

Some computations prove that $V(x, y) = y^2 - x^3$ is a first integral of F , that is, for all points where F is defined, $V(F(x, y)) = V(x, y)$.

Hence the map F provides a way of finding rational points on algebraic curves of the form $y^2 - x^3 = k$, where $k = y_0^2 - x_0^3$, for some $(x_0, y_0) \in \mathbb{Q}^2$. For instance, taking $(x_0, y_0) = (2, 2)$ we get that $k = -4$ and since

$$F(2, 2) = (5, 11), \quad F^2(2, 2) = \left(\frac{785}{484}, -\frac{5497}{10648} \right),$$

$$F^3(2, 2) = \left(\frac{3227836439105}{58500129424}, -\frac{5799120182710629023}{14149309303524032} \right),$$

we have obtained four rational solutions of equation $y^2 - x^3 = -4$. It can be seen that this procedure never ends and gives always different points, providing in consequence infinitely many rational solutions of this equation, see [64, 75]. In fact, it is known that if $x_0 y_0 \neq 0$, $y_0^2 - x_0^3 \notin \{1, -432\}$ and $y_0^2 - x_0^3$ is free from sixth power prime factors, then

$$(x_{n+1}, y_{n+1}) = F(x_n, y_n), \quad (x_0, y_0) \in \mathbb{Z}^2,$$

gives rise to infinitely many rational solutions of $y^2 - x^3 = y_0^2 - x_0^3$. Notice that starting at $(x_0, y_0) = (2m^2, 3m^3)$ we get that $F(2m^2, 3m^3) = (0, -m^3)$ which is a fixed point for F . For this initial condition $y_0^2 - x_0^3 = m^6$. Similarly, $F(12m^2, 36m^3) = (12m^2, 36m^3)$ and for $(x_0, y_0) = (12m^2, 36m^3)$, $y_0^2 - x_0^3 = -432m^6$.

On the other hand, not for all $k \in \mathbb{Q}$ it holds that $y^2 - x^3 = k$ has rational solutions. For instance, although it is not easy, it can be seen there are no rational solutions for $k = \pm 6$, see again [75].

Let us explain how we have found this map F . In fact, this result goes back to the studies of 1621 of the French mathematician Claude Gaspard Bachet de Méziriac (1581-1638) about the rational solutions of the now called Bachet equation $y^2 - x^3 = k$. Nowadays, this equation is also known as Mordell curve, in honor of the American-born British mathematician Louis J. Mordell (1888-1972) who proved that for each $0 \neq k \in \mathbb{Z}$ it contains finitely many points with integer coordinates. This finiteness result was also proved in 1908 by the Norwegian mathematician Axel Thue. Bachet already showed that if (x, y) is a solution of this equation, then

$$G_k(x, y) = \left(\frac{x^4 - 8kx}{4y^2}, \frac{-x^6 - 20kx^3 + 8k^2}{8y^3} \right)$$

is also a solution. Although it is not clear how he found this result, today the more common explanation is a geometric one related with the group structure operation on the elliptic curves. After some computations it can be seen that if we consider the curve $\mathcal{C} := \{(x, y) \in \mathbb{R}^2 : y^2 - x^3 - k = 0\}$ and its tangent line \mathcal{L} at $P = (x, y) \in \mathcal{C}$ it holds that

$$\mathcal{C} \cap \mathcal{L} = \{G_k(x, y)\} \cup \{(x, y)\}.$$

Then, $G_k|_{k=y^2-x^3} = F$.

Another simpler and famous integrable rational map is the Lyness map

$$L(x, y) = \left(y, \frac{a+y}{x} \right).$$

It has the elliptic first integral

$$H(x, y) = \frac{(x+1)(y+1)(x+y+a)}{xy},$$

because $H(L(x,y)) = H(x,y)$. This map L generates the DDS associated to the second order difference equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}}, \quad x_0, \quad x_1 \in \mathbb{R}.$$

Assume for instance that $a \in \mathbb{Q}$. Then it is known that for any $a \notin \{0, 1\}$ the above recurrence generates (real) periodic sequences with infinitely many different prime periods, see [6, 79]. On the other hand it only generates periodic sequences of rational numbers with prime periods 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 or 12 and all these periods are possible for some x_0, y_0 and a . Moreover, if we restrict our attention to positive rational values of a and positive rational initial conditions the only possible periods are 1, 5 and 9. Taking $a = n^2 - n$ and $x_0 = x_1 = n \in \mathbb{N}$ we obtain trivially 1-periodic integer sequences. The existence of positive rational periodic points of period 5 is well-known and simple: they only exist when $a = 1$ and in this case all rational initial conditions give rise to them because the recurrence is globally 5-periodic. To prove the existence of the 9 periodic sequences was the goal of [35], disproving a Conjecture of Zeeman about this question. See also that paper and its references for more details about the subject.

These strong differences between real and rational dynamics for integrable birational maps are explained by Mazur's Torsion Theorem (see for example [75]) that provides a complete and short list of possible torsion subgroups for rational elliptic curves.

9 A divide-and-conquer algorithm: QUICKSORT

The so called QUICKSORT algorithm was proposed by the British computer scientist C. A. R. Hoare, in 1959 and it is, in average, one of the best ones for sorting n objects when n is big. Let us describe it and prove that although sometimes it needs $O\left(\frac{n^2}{2}\right)$ comparisons, in average it only needs $O(2n \ln(n))$. This section is based on [40, Ch. 13.8], see also [26, Ch. 9].

To order n different objects the algorithm follows next three steps:

1. We randomly choose one of the objects. Call r its ordered position in the list, $1 \leq r \leq n$. A priori we do not know this value r .
2. We compare each of the remainder $n - 1$ objects with this one. Those smaller than r are placed in its left-hand side (there are $r - 1$) and the biggest one in the right-hand side (there are $n - r$). We get two piles with $r - 1$ and $n - r$ objects, which in general are not ordered.
3. We take each of the piles (with $r - 1$ and $n - r$ objects) and we proceed in the same way, starting from step 1, until one of these values is 1. In this case, the corresponding branch of the algorithm stops.

Let us compute the average time needed for the algorithm to sort all the n objects. We normalize to 1 the time needed to compare two objects and put the biggest (or the smallest) in the right (or the left) pile. We also assign time 1 to the time needed to choose randomly one of the objects from a list.

To formalize the algorithm we introduce two random variables. The first one, X_n , measures “the time needed to sort n objects following the proposed algorithm”. Notice that its value depends on the values of each of the random selections done of the values r appearing in each of the steps of the algorithm. The second one, Y_n gives “the place of the chosen object in the ordered list formed by the n objects”. The random variable X_n takes values in \mathbb{R} while the second one takes values in $\{1, 2, \dots, n\}$, and each one of them with probability $\frac{1}{n}$.

Set T_n for the expectation of X_n , that is $T_n = E(X_n)$. It holds that

$$\begin{aligned} T_n = E(X_n) &= \sum_{k=1}^n E(X_n / Y_n = k) P(Y_n = k) = \frac{1}{n} \sum_{k=1}^n E(X_n / Y_n = k) \\ &= \frac{1}{n} \sum_{k=1}^n \left(n + E(X_{k-1}) + E(X_{n-k}) \right) = n + \frac{1}{n} \sum_{k=1}^n \left(T_{k-1} + T_{n-k} \right) = n + \frac{2}{n} \sum_{k=0}^{n-1} T_k. \end{aligned}$$

The key point in the above chain of equalities is step **2** of the algorithm that says that, once we know that we have chosen the k -th object of the list, the expected time needed to order all the list is the sum of the expected times needed to order the piles with $k-1$ and $n-k$ object, which are T_{k-1} and T_{n-k} , respectively, plus n , that precisely corresponds to the $n-1$ comparisons needed to do the two piles in step **1**, plus 1, that corresponds to the first selection. Hence, $E(X_n / Y_n = k) = n + T_{k-1} + T_{n-k}$.

The obtained relation and the one corresponding to $n-1$ write as

$$nT_n = n^2 + 2 \sum_{k=0}^{n-1} T_k \quad \text{and} \quad (n-1)T_{n-1} = (n-1)^2 + 2 \sum_{k=0}^{n-2} T_k.$$

Subtracting them we get that $nT_n - (n-1)T_{n-1} = 2n-1 + 2T_{n-1}$, or equivalently,

$$T_n = \frac{n+1}{n} T_{n-1} + \frac{2n-1}{n}, \quad \text{with} \quad T_0 = 0, T_1 = 1.$$

To compute T_n we introduce the change of variables $S_k = \frac{T_k}{k+1}$. Then the above difference equation writes as

$$S_n = \frac{T_n}{n+1} = \frac{T_{n-1}}{n} + \frac{2n-1}{n(n+1)} =: S_{n-1} + c_n, \quad S_0 = \frac{T_0}{1} = 0.$$

It is easy to solve because

$$\begin{aligned} S_n &= S_{n-1} + c_n = (S_{n-2} + c_{n-1}) + c_n = ((S_{n-3} + c_{n-2}) + c_{n-1}) + c_n \\ &= \dots = S_0 + \sum_{k=1}^n c_k = \sum_{k=1}^n c_k. \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \sum_{k=1}^n c_k = \sum_{k=1}^n \frac{2k-1}{k(k+1)} = \sum_{k=1}^n \left(\frac{3}{k+1} - \frac{1}{k} \right) = 3(h_{n+1} - 1) - h_n \\ &= 3 \left(\frac{1}{n+1} + h_n - 1 \right) - h_n = 2h_n + \frac{3}{n+1} - 3, \end{aligned}$$

where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Finally, recall the result of Euler (1731),

$$h_n = \ln(n) + \gamma + R(n), \quad \text{with} \quad \lim_{n \rightarrow \infty} R(n) = 0,$$

where $\gamma \simeq 0.577218$ is the Euler constant, see for instance [40, 49]. Hence,

$$\begin{aligned} T_n &= (n+1)S_n = 2(n+1)h_n - 3n = 2(n+1)(\gamma + \ln(n) + R(n)) - 3n \\ &= 2(n+1)\ln(n) + (2\gamma - 3)n + 2(n+1)R(n) + 2\gamma \approx 2n\ln(n), \end{aligned}$$

as we wanted to prove.

It is clear that if we have very bad luck choosing the r -th objects during the algorithm, it takes longer to order the n objects. The worst situation occurs when, at each step, the chosen element is always the first of the list. Then we would need to perform

$$n + (n-1) + (n-2) + \dots + 2 = \frac{(n+2)(n-1)}{2} \approx \frac{n^2}{2}$$

comparisons, that is $X_n \approx \frac{n^2}{2}$. Taking as a unity of time 10^{-6} seconds and $n = 10^6$,

$$\frac{n^2}{2} \times 10^{-6} \text{ seconds} \simeq 5.8 \text{ days} \quad \text{and} \quad 2n\ln(n) \times 10^{-6} \text{ seconds} \simeq 0.46 \text{ minutes},$$

and so, the expected time is much lesser than in the worst possible situation.

To decrease the possibilities that this worst situation happens there is a usual modification of the algorithm, that almost does not increase the number of needed comparisons, that consists in replacing step **1** by the similar one:

1'. We choose randomly 3 objects among all the n objects of the list. The one that is in the middle of these three is the one that gives the value of r .

In the ‘‘Appendix II: Solving recurrences’’ of [31] there is a systematic treatment of divide-and-conquer algorithms. The corresponding running-time functional equations are of the form

$$T(n) = aT(n/b) + f(n),$$

where a and b are constant parameters and f is a given function. For some values of b they give rise to difference equations.

10 Lower bounds for Hilbert numbers

Consider real planar polynomial systems of ordinary differential equations (ODE), $x' = P(x, y)$, $y' = Q(x, y)$ with P and Q polynomials of degree at most n . The Hilbert number $\mathcal{H}(n)$ is the maximum number of *limit cycles* that these families of ODE can have, or infinity if there is no upper for the number of limit cycles. Recall that limit cycles are the periodic orbits that are isolated in the set all of the periodic orbits of the ODE. It is well-known that $\mathcal{H}(0) = \mathcal{H}(1) = 0$, but even for $n = 2$ it is not known whether $\mathcal{H}(2)$ is finite. It is known that $\mathcal{H}(2) \geq 4$.

The knowledge of $\mathcal{H}(n)$ is one of the most elusive problems of the famous Hilbert's list and constitutes the second part of Hilbert's sixteenth problem.

Based on the ideas of [18] and by using very simple background on ODEs and difference equations we will provide quadratic lower bounds for $\mathcal{H}(n)$.

Proposition 1. *There exists a sequence of values n_k tending to infinity and a constant $K > 0$ such that $\mathcal{H}(n_k) > Kn_k^2$.*

Proof. The construction of the ODE that gives this lower bound is a recurrent process. Let $X_0 = (P_0, Q_0)$ be a given polynomial vector field, of degree n_0 with $c_0 > 0$ limit cycles. Since this number of limit cycles is finite, there exists a compact set containing all of them. Therefore, doing a translation if necessary, we can assume that all of them are in the first quadrant. By simplicity we continue calling X_0 this new translated vector field. From it, we construct a new vector field, by using the (non-bijective) transformation

$$x = u^2, \quad y = v^2.$$

The differential equation associated to X_0 is $\dot{x} = P_0(x, y)$, $\dot{y} = Q_0(x, y)$ and it writes in these new variables as

$$\dot{u} = \frac{P_0(u^2, v^2)}{2u}, \quad \dot{v} = \frac{Q_0(u^2, v^2)}{2v}.$$

By introducing a new time s , defined as $dt/ds = 2uv$, we get that this ODE is transformed into

$$u' = vP_0(u^2, v^2), \quad v' = uQ_0(u^2, v^2).$$

Since each point in the first quadrant (x, y) has four preimages $(\pm\sqrt{x}, \pm\sqrt{y})$, the new ODE has at each quadrant a diffeomorphic copy of the positive quadrant of the

vector field X_0 . Hence this new vector field, that we call X_1 , has degree $n_1 = 2n_0 + 1$ and at least $c_1 = 4c_0$ limit cycles. By repeating this process, starting now with X_1 and so on, we get a sequence of vector fields X_k , with respective degrees n_k , having at least c_k limit cycles, where

$$n_{k+1} = 2n_k + 1, \quad c_{k+1} = 4c_k,$$

see Figure 4. Solving the above linear difference equations we get that

$$n_k = 2^k(n_0 + 1) - 1, \quad c_k = 4^k c_0. \quad (15)$$

Hence, since

$$2^k = \frac{n_k + 1}{n_0 + 1} \quad \text{and} \quad 4^k = \frac{c_k}{c_0},$$

we obtain that

$$\frac{c_k}{c_0} = \left(\frac{n_k + 1}{n_0 + 1} \right)^2 > \frac{1}{(n_0 + 1)^2} n_k^2.$$

Consequently,

$$\mathcal{H}(n) > \frac{c_0}{(n_0 + 1)^2} n^2 \quad \text{and} \quad n = 2^k(n_0 + 1) - 1, \quad k \in \mathbb{N},$$

as we wanted to prove. \square

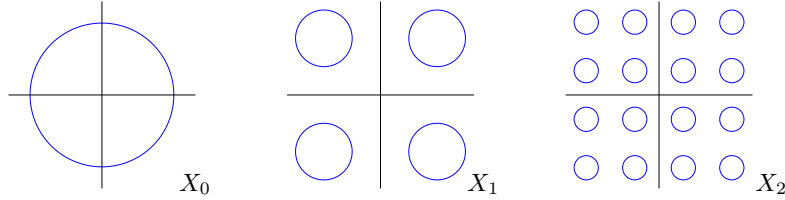


Fig. 4 Two steps of the construction of the vector fields X_k , starting from a vector field X_0 with a unique limit cycle.

Notice that the proof of Proposition 1 shows that the constant K depends on the seed of the procedure. For instance, take as X_0 the trivial example appearing in most textbooks of ODE, that in polar coordinates writes as

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

Clearly it has only the limit cycle $r = 1$. Thus $c_0 = 1$ and $n_0 = 3$. Hence, $K = 1/16$ and $n_k = 2^{k+2} - 1$. On the other hand if we consider as X_0 to be any quadratic system that reveals that $\mathcal{H}(2) \geq 4$, then $c_0 = 4$ and $n_0 = 2$. By using this seed, $K = 4/9$ and $n_k = 2^k 3 - 1$. By using the best known lower bounds for $\mathcal{H}(n)$ for n small, given in Table 1, see [39, 41, 42, 56, 58, 71, 78], we get that the best K obtained is $6/5$.

n	3	4	5	6	7	8	9	10
lower bounds for $\mathcal{H}(n)$	13	28	37	53	74	96	120	142
K	13/16	28/25	37/36	53/49	37/32	32/27	6/5	142/121

Table 1 Lower bounds for $\mathcal{H}(n)$ and corresponding values of K .

In fact, it is known that $\mathcal{H}(n) \geq K n^2 \log(n)$, and for the moment it is the best result on lower bounds for $\mathcal{H}(n)$, see again [18]. Our proof is totally inspired in that paper. In their study, the authors add at each step some limit cycles that appear by perturbing the centers created by the method on the axes $uv = 0$. They obtain (see also [56, 57]) that instead of (15), it holds that

$$n_{k+1} = 2n_k + 1, \quad c_{k+1} = 4c_k + d_k.$$

for a given sequence $\{d_k\}_k$. Studying it they got these better lower bounds.

11 Coxeter difference equations

Globally periodic recurrences have recently attracted the interest of many researchers, see for instance [4, 13, 19, 25, 47]. Here we recall one of the less known families of rational examples, the one introduced by Coxeter in 1971, see [24].

For each natural number $n \geq 2$, Coxeter proved that the recurrences

$$x_{n+m} = f_m(x_n, x_{n+1}, \dots, x_{n+m-1}) := 1 - \frac{x_{n+m-1}}{1 - \frac{x_{n+m-2}}{1 - \frac{x_{n+m-3}}{1 - \dots \frac{x_{n+1}}{1 - x_n}}}},$$

are globally $(m + 3)$ -periodic, that is for any admissible set of initial conditions, $x_{n+m+3} = x_n$, for all $n \geq 0$. For instance, for $m = 2, 3$, the recurrences are

$$x_{n+2} = 1 - \frac{x_{n+1}}{1-x_n}, \quad \text{and} \quad x_{n+3} = 1 - \frac{x_{n+2}}{1 - \frac{x_{n+1}}{1-x_n}} = \frac{1-x_n-x_{n+1}-x_{n+2}+x_n x_{n+2}}{1-x_n-x_{n+1}},$$

respectively. It is easy to see that for $m = 2$, in the new variables $u_n = x_n - 1$, it corresponds to the well-known 5-periodic Lyness difference equation

$$u_{n+2} = \frac{1+u_{n+1}}{u_n},$$

which has already appeared in Section 8. As usual, the study of the above recurrences can be reduced to the study of the discrete dynamical system given by the map

$$F_m(x_1, x_2, \dots, x_m) = (x_2, x_3, \dots, x_m, f_m(x_1, x_2, \dots, x_m)).$$

In his paper Coxeter gives a proof that these recurrences are globally $(m+3)$ -periodic, based on the properties of some cross-ratios. In [20] the authors gave a new algebraic proof of this result showing that $F_m^{m+3} = \text{Id}$. They also prove the surprising fact that “all” the possible geometrical behaviors that linear real globally periodic recurrences can have are present in the Coxeter map. We state their result in next results, where, as usual, $[s]$ denotes the integer part of s .

Lemma 1. *There are $2 \lfloor \frac{m+2}{2} \rfloor$ different types of globally $(m+3)$ -periodic real linear recurrences of order m when m is odd and $\lfloor \frac{m+2}{2} \rfloor$ types when m is even. Moreover there are only $\lfloor \frac{m+2}{2} \rfloor$ of them without the eigenvalue 1.*

Proof. Let $L : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the globally periodic linear map

$$L(x_1, \dots, x_m) = (x_2, \dots, x_m, a_1 x_1 + a_2 x_2 + \dots + a_m x_m)$$

associated to the periodic recurrence. It is known that the characteristic polynomial of L has not multiple roots, see [25, 48]. On the other hand it is a real polynomial of degree m and all its roots also must be $(m+3)$ -roots of the unity. So it divides $\lambda^{m+3} - 1$. Thus the proof follows by counting the different number of possibilities for removing a degree 3 real factor from $\lambda^{m+3} - 1$. \square

Theorem 3. *([20]) The Coxeter difference equations, given by the maps F_m , are globally $(m+3)$ -periodic and they have exactly $\lfloor \frac{m+2}{2} \rfloor$ fixed points, all of them with positive coordinates. At each of these fixed points F_m is locally conjugated to a linear $(m+3)$ -periodic recurrence which has no line of fixed points. Moreover, all these $m+3$ linear maps are not conjugated among them. As a consequence, all linear globally $(m+3)$ -periodic recurrences having no line of fixed points are present in the Coxeter difference equation.*

As an example, consider the Coxeter map for $m = 5$,

$$F_5(x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5, f_5(x_1, x_2, x_3, x_4, x_5))$$

with

$$f_5(x_1, x_2, x_3, x_4, x_5) = \frac{1 - \sum_{i=1}^5 x_i + x_1(x_3 + x_4 + x_5) + x_2(x_4 + x_5) + x_3 x_5(1 - x_1)}{1 - x_1 - x_2 - x_3 - x_4 + x_1 x_3 + x_1 x_4 + x_2 x_4},$$

which is globally 8-periodic. Following Lemma 1 we know that there are only three real 8-periodic linear recurrences of order 5 not having the eigenvalue 1. Since the polynomial $\lambda^8 - 1$ decomposes as

$$\lambda^8 - 1 = (\lambda - 1)(\lambda + 1)(\lambda^2 + 1)(\lambda^2 - \sqrt{2}\lambda + 1)(\lambda^2 + \sqrt{2}\lambda + 1),$$

they are the ones having associated characteristic polynomials:

$$\begin{aligned} p_1(\lambda) &= (\lambda + 1)(\lambda^2 + \sqrt{2}\lambda + 1)(\lambda^2 - \sqrt{2}\lambda + 1), \\ p_2(\lambda) &= (\lambda + 1)(\lambda^2 + 1)(\lambda^2 + \sqrt{2}\lambda + 1), \quad \text{and} \\ p_3(\lambda) &= (\lambda + 1)(\lambda^2 + 1)(\lambda^2 - \sqrt{2}\lambda + 1). \end{aligned}$$

On the other hand the map F has three fixed points $\mathbf{x}_i := (x_i, x_i, x_i, x_i, x_i)$, $i = 1, 2, 3$, with $x_1 = \frac{1}{2}$, $x_2 = 1 - \frac{\sqrt{2}}{2}$ and $x_3 = 1 + \frac{\sqrt{2}}{2}$. It is easy to prove that for each i , the characteristic polynomial of the differential matrix $d(F_5)$ at the point \mathbf{x}_i is exactly the polynomial $p_i(\lambda)$, as it is predicted by Theorem 3.

Acknowledgements The author is partially supported by Spanish Ministry of Economy and Competitiveness through grants MINECO MTM2013-40998-P and MTM2016-77278-P FEDER and by Generalitat de Catalunya through the SGR program.

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