Localizing Limit Cycles: from Numeric to Analytical Results

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Abstract: This note presents the results of [**4**]. It deals with the problem of location and existence of limit cycles for real planar polynomial differential systems. We provide a method to construct Poincaré–Bendixson regions by using transversal curves, that enables us to prove the existence of a limit cycle that has been numerically detected. We apply our results to several known systems, like the Brusselator one or some Liénard systems, to prove the existence of the limit cycles and to locate them very precisely in the phase space. Our method, combined with some other classical tools can be applied to obtain sharp bounds for the bifurcation values of a saddle-node bifurcation of limit cycles, as we do for the Rychkov system.

1 Introduction

We consider real planar polynomial differential systems of the form

$$(1) \qquad \dot{x} = dx/dt = P(x,y), \quad \dot{y} = dy/dt = Q(x,y),$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials. We denote by $X = (P,Q)$ the vector field associated to (1) and $z = (x,y)$. So, (1) can be written as $\dot{z} = X(z)$.

When dealing with system (1) one of the main problems is to determine the number and location of its limit cycles. Recall that a limit cycle is an isolated periodic orbit of the system. For a given vector field, when it is not very near of a bifurcation, the limit cycles can usually be detected by numerical methods. A bifurcation is a qualitative change in the behaviour of a vector field as a parameter of the system is varied. This phenomenon can involve a change in the stability of a limit cycle or the creation or destruction of one or more limit cycles. If a periodic orbit is stable (unstable), then forward (backward) numerical integration of a trajectory with an initial condition in its basin of attraction will converge to the periodic orbit as $t \to \infty$ ($t \to -\infty$). Once for a given vector field a limit cycle is numerically detected there is no general method to rigourously prove its existence. In this talk we present a procedure to do it. The method is based on a corollary of the Poincaré–Bendixson theorem, the so called *Poincaré-Bendixson annular Criterion* see for instance [**3, 8**] and also Theorem 1. It is very useful to prove the

existence of a limit cycle and to give a region where it is located. However, it is hardly found in applications due to the difficulty of constructing the boundaries of a suitable annular region. Our aim is to give a constructive procedure for finding transversal curves which define these annular Poincaré–Bendixson regions and, as a consequence, to prove the existence of limit cycles that have been numerically detected.

As usual, we will say that a smooth curve $C = \{z(s) = (x(s), y(s)) \,:\, s \in \mathcal{I} \subset \mathbb{R}\}$, is *transversal* with respect to the flow given by (1) if the scalar product

$$X(z(s)) \cdot (z'(s))^{\perp} = P\left(z(s)\right) y'(s) - Q\left(z(s)\right) x'(s)$$

does not change sign and vanishes only on finitely many contact points when $s \in \mathcal{I}$. A *contact point* with the flow given by (1) is a point $z(s)$ such that the tangent vector to C at this point, $z'(s)$ is parallel to $X(z(s))$. When the above scalar product does not vanish we will say that the curve is *strictly transversal.*

A transversal section of system (1) is an arc of a curve without contact points. Given a limit cycle Γ there always exist a transversal section Σ which can be parameterized by $r \in (-\rho, \rho)$ with $\rho > 0$ and $r = 0$ corresponding to a common point between Γ and Σ. Given $r \in (-\rho, \rho)$, we consider the flow of system (1) with initial point the one corresponding to r and we follow this flow for positive values of t. It can be shown, see for instance [**8**], that for ρ small enough, the flow cuts Σ again at some point corresponding to the parameter $\mathcal{P}(r)$. The map $r \longrightarrow \mathcal{P}(r)$ is called the Poincaré map associated to the limit cycle Γ of system (1). It is clear that $\mathcal{P}(0) = 0$. If $\mathcal{P}'(0) \neq 1$, the limit cycle Γ is said to be *hyperbolic.* A classical result, see for instance [**8**], states that if $\Gamma = \{\gamma(t) : t \in [0, T)\}$, where $\gamma(t)$ is the parametrization of the limit cycle in the time variable t of system (1) and $T > 0$ is the period of Γ, that is, the lowest positive value for which $\gamma(0) = \gamma(T)$, and $\gamma(0) = \Gamma \cap \Sigma$, then

$$\mathcal{P}'(0) = \exp\left\{\int_0^T \operatorname{div} X\left(\gamma(t)\right) dt\right\},$$

where

$$\operatorname{div} X(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)$$

is the *divergence* of X. Hence

$$k := \int_0^T \operatorname{div} X\left(\gamma(t)\right) dt \neq 0$$

is the condition for a limit cycle Γ to be hyperbolic. It is clear that if $k > 0$ (resp., $k < 0$), then Γ is an unstable (resp., stable) limit cycle.

The Poincaré–Bendixson theorem, which can be found for instance in [**3**, Sec. 1.7] or in [**8**, Sec. 3.7], has as a corollary the following result:

Theorem 1 [Poincaré-Bendixson annular Criterion] *Suppose that R is a finite region of the plane $\mathbb{R}^2$ lying between two $\mathcal{C}^1$ simple disjoint closed curves C_1 and C_2. If*

    (i) *the curves C_1 and C_2 are transversal for system (1) and the flow crosses them towards the interior of R, and*

    (ii) *R contains no critical points.*

Then, system (1) has an odd number of limit cycles (counted with multiplicity) lying inside R.

In such a case, we say that R is a *Poincaré–Bendixson annular region* for system (1).

2 Main results

In the present work we give an answer to the following question: if one numerically knows the existence of a hyperbolic limit cycle, can one analytically prove the existence of such limit cycle?

The following theorem is a key result. It gives the theoretical basis of the method described used in [**4**] to answer positively the above question.

Theorem 2 *Let* $\Gamma = \{(\gamma(t) \, : \, t \in [0, T]\}$ *be a T-periodic hyperbolic limit cycle of* (1), *parameterized by the time t. Define*

$$\tilde{z}_\varepsilon(t) = \gamma(t) + \varepsilon \tilde{u}(t)(\gamma'(t))^\perp,$$

where

$$\tilde{u}(t) = \frac{1}{||\gamma'(t)||^2} \exp\left\{ \int_0^t \operatorname{div} X\left(\gamma(s)\right) ds - \kappa\, t \right\} > 0$$

and $\kappa = \frac{k}{T} = \frac{1}{T} \int_0^T \operatorname{div} X(\gamma(t))\, dt$. *Then, the curve* $\{\tilde{z}_\varepsilon(t) \, : \, t \in [0, T]\}$ *is T-periodic and, for* $|\varepsilon| > 0$ *small enough, it is strictly transversal to the flow associated to system* (1).

Notice that as a consequence of the above result, the curve $\tilde{z}_\varepsilon(t)$ is a transversal oval close to the limit cycle Γ for $|\varepsilon| > 0$ small enough, which is inside or outside it depending on the sign of ε. The effective method for obtaining explicit Poincaré–Bendixson annular regions consists on following steps:

- Step 1: Find numerically the limit cycle.
- Step 2: Fix ε and use step 1 and Theorem 2 to find a numerical transversal curve.
- Step 3: Check numerically if the proposed curve is transversal. If yes, continue; if not, choose a smaller $|\varepsilon|$, with the same sign, and return to step 2.
- Step 4: Fix $m \in \mathbb{N}$ and approach, by interpolation, the curve given in step 2 by a couple of trigonometric polynomials of degree m.
- Step 5: Convert the above trigonometric polynomials to trigonometric polynomials with rational coefficients, close enough to the original ones.
- Step 6: Check analytically, with algebraic tools, if the curve given in step 5 is transversal. If yes, one of the boundaries of a Poincaré–Bendixson annular region is found and we have to start again the algorithm, with ε of different sign, to find the other boundary. If not, we have to choose a bigger m and return to step 4.

As an illustration of the effectiveness of our approach in [**4**] we apply it to locate the limit cycles in two celebrated planar differential systems, the van der Pol oscillator and the Brusselator system. We also we give there an explanation for the different level of difficulty for studying both limit cycles. It is hidden in the sizes of the respective Fourier coefficients of the two limit cycles, see [**4**, Thm 6]. That theorem also shows that our approach for detecting strictly transversal closed curves always works in finitely many steps.

Finally, to show the applicability of the method to detect bifurcation values, we use it to find a sharp interval for the bifurcation value for a saddle-node bifurcation of limit cycles for the Rychkov system. Recall that a saddle-node bifurcation of limit cycles occurs when a stable limit cycle and an unstable limit cycle coalesce and become a double semi-stable limit cycle.

In 1975 Rychkov [9] proved that the system

$$\dot{x} = y - \left(x^5 - \mu x^3 + \delta x\right), \quad \dot{y} = -x,$$

with $\delta, \mu \in \mathbb{R}$, has at most 2 limit cycles. Moreover, it is known that it has 2 limit cycles if and only if $\delta > 0$ and $0 < \delta < \Delta(\mu)$, for some unknown function Δ. For the value $\delta = \Delta(\mu)$ the system has a double limit cycle and, varying δ, it presents a saddle-node bifurcation of limit cycles. This system is also studied by Alsholm [1] and Odani [7]. In particular Odani proved that $\Delta(\mu) > \mu^2/5$. Here we will fix our attention on $\delta^* := \Delta(1)$. Notice that Odani's result implies that $\delta^* > 1/5 = 0.2$. We prove:

Theorem 3 *Let $\delta = \delta^*$ be the value for which the Rychkov system*

$$\dot{x} = y - \left(x^5 - x^3 + \delta x\right), \quad \dot{y} = -x$$

has a semi-stable limit cycle. Then $0.224 < \delta^* < 0.225$.

The lower bound for δ^* can be proved by using the tools introduced in this work. The upper bound is proved by constructing a polynomial function in (x, y) of very high degree such that its total derivative with respect to the vector field does not change sign. This method is proposed and already developed for general classical Liénard systems by Cherkas [2] and also by Giacomini-Neukirch [5, 6].

Acknowledgements

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