

**ON THE INTEGRABILITY OF HAMILTONIAN SYSTEMS  
WITH  $d$  DEGREES OF FREEDOM AND HOMOGENOUS  
POLYNOMIAL POTENTIAL OF DEGREE  $n$**

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ABSTRACT. We consider Hamiltonian systems with  $d$  degrees of freedom and a Hamiltonian of the form

$$H = \frac{1}{2} \sum_{i=1}^d p_i^2 + V(q_1, \dots, q_d),$$

where  $V$  is a homogenous polynomial of degree  $n \geq 3$ . We prove that such Hamiltonian systems have a Darboux first integral if and only if they have a polynomial first integral.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Here  $\mathbb{C}^{2d}$  is a symplectic linear space with canonical variables  $(q, p) = (q_1, \dots, q_d, p_1, \dots, p_d)$ , where  $q_i$  are the *positions* and  $p_i$  are the *momenta*. We study the Hamiltonian systems with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^d p_i^2 + V(q_1, \dots, q_d),$$

where  $V(q) = V(q_1, \dots, q_d)$  is a homogeneous polynomial of degree  $n$ , i.e. we study the Hamiltonian systems

$$(1) \quad \dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, \dots, d.$$

We write the vector field associated to system (1) as

$$X_H = \sum_{i=1}^d p_i \frac{\partial}{\partial q_i} - \sum_{i=1}^d \frac{\partial V}{\partial q_i} \frac{\partial}{\partial p_i}.$$

During the last thirty years the integrability of these Hamiltonian systems in the particular case of 2 degrees of freedom have been studied when  $V(q)$  is a polynomial of degree at most 4, see for instance [1, 2, 9, 10, 11, 12, 18, 19, 20, 21]).

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The functions  $A = A(p, q)$  and  $B = B(p, q)$  are *in involution* if  $\{A, B\} = 0$ , where

$$\{A, B\} = \sum_{i=1}^d \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right),$$

and they are *independent* if their gradients are linearly independent at all points  $\mathbb{C}^{2d}$  except perhaps in a zero Lebesgue measure set. A non-constant function  $F = F(q, p)$  is a *first integral* for the Hamiltonian system (1) if  $\{H, F\} = 0$ . The Hamiltonian system (1) is *completely integrable* if it has  $d$  functionally independent first integrals which are in involution. Note that one of these integrals can be the Hamiltonian.

We shall use the Darboux theory of integrability in dimension  $2d$  for studying the existence of first integrals of the Hamiltonian system (1). For the polynomial differential systems the Darboux theory of integrability is one of the best theories for studying the existence of first integrals. It uses the existence of invariant algebraic hypersurfaces, see for instance [4, 5, 16].

A polynomial  $f \in \mathbb{C}[p, q] \setminus \mathbb{C}$  is a *Darboux polynomial* of system (1) if

$$\sum_{i=1}^d \left( p_i \frac{\partial f}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial f}{\partial p_i} \right) = Kf$$

for some polynomial  $K$  called the *cofactor* of  $f$ , clearly  $K$  has degree at most  $n - 2$ . It is obvious that a Darboux polynomial  $f$  of system (1) with cofactor  $K$  can be written as  $X_H f = Kf$ .

Let  $f$  be a Darboux polynomial. Then the algebraic hypersurface  $f = 0$  is invariant by the flow of system (1), and if the cofactor of  $f$  is zero then it is a *polynomial first integral*.

A function of the form  $F = \exp(g_0/g_1) \notin \mathbb{C}$  with  $g_0, g_1 \in \mathbb{C}[p, q]$  coprime is an *exponential factor* of system (1) if it satisfies

$$\sum_{i=1}^d \left( p_i \frac{\partial F}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = LF,$$

for some polynomial  $L = L(p, q)$  with degree at most  $n - 2$  called the *cofactor* of  $F$ .

A first integral of system (1) of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

where  $f_1, \dots, f_p$  are Darboux polynomials and  $F_1, \dots, F_q$  are exponential factors and  $\lambda_j, \mu_k \in \mathbb{C}$  for  $j = 1, \dots, p$  and  $k = 1, \dots, q$ , is called a *Darboux first integral*.

The main result is the following.

**Theorem 1.** *Hamiltonian systems (1) with  $d$  degrees of freedom and a homogeneous polynomial potential of degree  $n \geq 3$  have a Darboux first integral if and only if they have a polynomial first integral.*

In view of Theorem 1, in order to show for systems (1) with  $n \geq 3$  the existence of an additional Darboux first integral it is enough to look for an additional polynomial first integral.

In section 2 we prove Theorem 1. Since the proof for arbitrary  $d \geq 2$  is exactly the same than the proof for  $d = 2$ , Theorem 1 is proved only for  $d = 2$ , in order to work with shorter expressions. The next result follows immediately from Theorem 1.

**Corollary 2.** *Hamiltonian systems (1) with 2 degrees of freedom and a homogeneous polynomial potential of degree  $n \geq 3$  are completely integrable with a Darboux first integral if and only if they are completely integrable with a polynomial first integral.*

The particular cases of Corollary 2 for  $n = 3, 4$  were proved in [14, 15] where the authors are also able to compute the cases which are completely integrable with polynomial first integrals and to provide the explicit expression of those polynomial first integrals.

Other results on the Darboux integrability of the Hamiltonian systems (1) can be found in [13].

## 2. PROOF OF THEOREM 1 FOR $d = 2$

**2.1. Quasi-homogeneous polynomial differential systems.** In what follows we present some results on the quasi-homogeneous polynomial differential systems that we shall use.

We study the polynomial integrability of the polynomial differential systems of the form

$$(2) \quad \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_4) \in \mathbb{C}^4,$$

with  $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \dots, P_4(\mathbf{x}))$  and  $P_i \in \mathbb{C}[x_1, \dots, x_4]$  for  $i = 1, \dots, 4$ . Here the independent variable  $t$  can be real or complex. As usual, we denote by  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  the sets of positive integers, real and complex numbers, respectively; and by  $\mathbb{C}[x_1, \dots, x_4]$  the polynomial ring over  $\mathbb{C}$  in the variables  $x_1, \dots, x_4$ .

The polynomial differential system (2) is *quasi-homogeneous* if for  $i = 1, \dots, 4$

$$P_i(\alpha^{s_1}x_1, \dots, \alpha^{s_4}x_4) = \alpha^{s_i-1+m}P_i(x_1, \dots, x_4),$$

for some  $\mathbf{s} = (s_1, \dots, s_4) \in \mathbb{N}^4$ ,  $m \in \mathbb{N}$  and arbitrary  $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R}, a > 0\}$ . Here  $\mathbf{s} = (s_1, \dots, s_4)$  is the *weight exponent* of system (2), and  $m$  is the *weight degree* with respect to the weight exponent  $\mathbf{s}$ . If  $\mathbf{s} = (1, \dots, 1)$  system (2) is a *homogeneous polynomial differential system of degree  $m$* .

Yoshida in [23, 24, 25], see also Furta [7], Goriely [8], Tsygvintsev [22] and Llibre and Zhang [16], provide some of the best results on the integrable quasi-homogeneous polynomial differential systems.

A function  $F(x_1, \dots, x_4)$  satisfying

$$F(\alpha^{s_1}x_1, \dots, \alpha^{s_4}x_4) = \alpha^m F(x_1, \dots, x_4),$$

for all  $\alpha \in \mathbb{R}^+$  is called *quasi-homogeneous of weight degree  $m$  with respect to the weight exponent  $\mathbf{s}$* .

**Lemma 3.** *Let  $V = V(q_1, q_2)$  be a homogeneous polynomial of degree  $n$  even. Then the Hamiltonian system (1) with  $d = 2$  is a quasi-homogeneous polynomial differential system (2) with  $\mathbf{x} = (q_1, q_2, p_1, p_2)$  and weight degree  $m = n/2$  with respect to the weight exponent  $\mathbf{s} = (s_1, s_2, s_3, s_4) = (1, 1, n/2, n/2)$ .*

*Proof.* It follows easily by direct computations.  $\square$

**Lemma 4.** *Let  $V = V(q_1, q_2)$  be a homogeneous polynomial of degree  $n$  odd. Then the Hamiltonian system (1) with  $d = 2$  is a quasi-homogeneous polynomial differential system (2) with  $\mathbf{x} = (q_1, q_2, p_1, p_2)$  and weight degree  $m = n - 1$  with respect to the weight exponent  $\mathbf{s} = (s_1, s_2, s_3, s_4) = (2, 2, n, n)$ .*

*Proof.* It follows easily by direct computations.  $\square$

**2.2. Darboux polynomials with non-zero cofactor.** The following result was proved in [3].

**Lemma 5.** *Let  $f$  be a polynomial and  $f = \prod_{j=1}^s f_j^{\alpha_j}$  its decomposition into irreducible factors in  $\mathbb{C}[q_1, q_2, p_1, p_2]$ . Then all the  $f_j$  are Darboux polynomials if and only if  $f$  is a Darboux polynomial. Furthermore, if  $K$  and  $K_j$  are the cofactors of  $f$  and  $f_j$ , then  $K = \sum_{j=1}^s \alpha_j K_j$ .*

By Lemma 5 we only need to consider irreducible Darboux polynomials.

Now we recall some properties of our Hamiltonian system (1) with homogeneous potential  $V$  of degree  $n$ .

**Proposition 6.** *Assume system (2) is a quasi-homogeneous polynomial differential system of weight exponent  $m$ . Let  $F$  be a Darboux polynomial of system (2) in the variables  $x_1, \dots, x_n$  with cofactor  $K$ . Let  $F = F_0 + F_1 + \dots + F_l$  and  $K = K_0 + K_1 + \dots + K_k$  be the decompositions of  $F$  and  $K$  into quasi-homogeneous polynomials of weight degrees  $i$  for  $i = 0, \dots, l$  and of weight degree  $j$  for  $j = 0, \dots, k$ , with respect to the weight exponent  $\mathbf{s}$ . Then  $F$  is a Darboux polynomial of the quasi-homogeneous polynomial differential system (2) if and only if each quasi-homogeneous part  $F_i$  is a Darboux polynomial with cofactor  $K_{m-1}$  of weight degree  $m - 1$  of system (2) and  $K_j = 0$  for  $j \neq m - 1$ .*

The proof of Proposition 6 it is easy and it is given in [15].

**Theorem 7.** *Consider system (1) with homogeneous potential of degree  $n$  even. Let  $f$  be an irreducible Darboux polynomial with cofactor  $K$ . Then  $K = 0$ .*

To prove Theorem 7 we will introduce some preliminary results. We define

$$Y = q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + \frac{n}{2} p_1 \frac{\partial}{\partial p_1} + \frac{n}{2} p_2 \frac{\partial}{\partial p_2}.$$

**Lemma 8.** *We have  $L_Y X_H = \frac{n-2}{2} X_H$ .*

*Proof.* Let  $f$  be a polynomial, then we have

$$\begin{aligned} L_Y(X_H(f)) &= [Y, X_H](f) = Y(X_H(f)) - X_H(Y(f)) \\ &= Y(p_1 f_{q_1} + p_2 f_{q_2} - V_{q_1} f_{p_1} - V_{q_2} f_{p_2}) \\ &\quad - X_H\left(q_1 f_{q_1} + q_2 f_{q_2} + \frac{n}{2} p_1 f_{p_1} + \frac{n}{2} p_2 f_{p_2}\right) \\ &= q_1(p_1 f_{q_1 q_1} + p_2 f_{q_1 q_2} - V_{q_1 q_1} f_{p_1} - V_{q_1} f_{q_1 p_1} - V_{q_1 q_2} f_{p_2} - V_{q_2} f_{q_1 p_2}) \\ &\quad + q_2(p_1 f_{q_1 q_2} + p_2 f_{q_2 q_2} - V_{q_1 q_2} f_{p_1} - V_{q_1} f_{q_2 p_1} - V_{q_2 q_2} f_{p_2} - V_{q_2} f_{q_2 p_2}) \\ &\quad + \frac{n}{2} p_1(f_{q_1} + p_1 f_{q_1 p_1} + p_2 f_{q_2 p_1} - V_{q_1} f_{p_1 p_1} - V_{q_2} f_{p_1 p_2}) \\ &\quad + \frac{n}{2} p_2(p_1 f_{q_1 p_2} + f_{q_2} + p_2 f_{q_2 p_2} - V_{q_1} f_{p_1 p_2} - V_{q_2} f_{p_2 p_2}) \\ &\quad - p_1\left(f_{q_1} + q_1 f_{q_1 q_1} + q_2 f_{q_1 q_2} + \frac{n}{2} p_1 f_{q_1 p_1} + \frac{n}{2} p_2 f_{q_1 p_2}\right) \\ &\quad - p_2\left(q_1 f_{q_1 q_2} + f_{q_2} + q_2 f_{q_2 q_2} + \frac{n}{2} p_1 f_{q_2 p_1} + \frac{n}{2} p_2 f_{q_2 p_2}\right) \\ &\quad + V_{q_1}\left(q_1 f_{q_1 p_1} + q_2 f_{q_2 p_1} + \frac{n}{2} f_{p_1} + \frac{n}{2} p_1 f_{p_1 p_1} + \frac{n}{2} p_2 f_{p_1 p_2}\right) \\ &\quad + V_{q_2}\left(q_1 f_{q_1 p_2} + q_2 f_{q_2 p_2} + \frac{n}{2} p_1 f_{p_1 p_2} + \frac{n}{2} f_{p_2} + \frac{n}{2} p_2 f_{p_2 p_2}\right), \end{aligned}$$

where we denote by  $f_x$  the derivative of  $f$  with respect to the variable  $x$ . Note that since  $V$  is a homogeneous polynomial of degree  $n$  we have

$$(3) \quad q_1 V_{q_1 q_1} + q_2 V_{q_1 q_2} = (n-1)V_{q_1} \quad \text{and} \quad q_1 V_{q_1 q_2} + q_2 V_{q_2 q_2} = (n-1)V_{q_2}$$

and so we have

$$L_Y(X_H(f)) = \frac{n-2}{2} \left( p_1 f_{q_1} + p_2 f_{q_2} - V_{q_1} f_{p_1} - V_{q_2} f_{p_2} \right) = \frac{n-2}{2} X_H(f),$$

which concludes the proof of the lemma.  $\square$

**Lemma 9.** *Let  $f = f(q_1, q_2, p_1, p_2)$  be a weight homogeneous polynomial with weight degree  $r$  with respect to the weight exponents  $(1, 1, n/2, n/2)$ . Then  $L_Y f = Y(f) = r f$ .*

*Proof.* We recall that if  $f = f(q_1, q_2, p_1, p_2)$  is a weight homogeneous polynomial with weight degree  $r$  with respect to the weight exponents  $(1, 1, n/2, n/2)$  then by definition

$$f(tq_1, tq_2, t^{n/2}p_1, t^{n/2}p_2) = t^r f(q_1, q_2, p_1, p_2).$$

Taking the derivative in  $t$  we get

$$(4) \quad \begin{aligned} & q_1 f_{q_1}(tq_1, tq_2, t^{n/2}p_1, t^{n/2}p_2) + q_2 f_{q_2}(tq_1, tq_2, t^{n/2}p_1, t^{n/2}p_2) \\ & + \frac{n}{2} t^{n/2-1} p_1 f_{p_1}(tq_1, tq_2, t^{n/2}p_1, t^{n/2}p_2) + \frac{n}{2} t^{n/2-1} p_2 f_{p_2}(tq_1, tq_2, t^{n/2}p_1, t^{n/2}p_2) \\ & = r t^{r-1} f(q_1, q_2, p_1, p_2). \end{aligned}$$

Evaluating (4) at  $t = 1$  we get

$$q_1 f_{q_1} + q_2 f_{q_2} + \frac{n}{2} p_1 f_{p_1} + \frac{n}{2} p_2 f_{p_2} = r f,$$

and so

$$L_Y f = Y(f) = q_1 \frac{\partial f}{\partial q_1} + q_2 \frac{\partial f}{\partial q_2} + \frac{n}{2} p_1 \frac{\partial f}{\partial p_1} + \frac{n}{2} p_2 \frac{\partial f}{\partial p_2} = r f.$$

This concludes the proof of the lemma.  $\square$

*Proof of Theorem 7.* Let  $f$  be a Darboux polynomial of system (1) with homogeneous potential of degree  $n$  even and with cofactor  $K$ . In view of Proposition 6 and Lemma 3 we have that  $K$  has weight degree  $m - 1 = n/2 - 1$ , and  $f$  can be considered of weight-degree  $r$  both with respect to the weight-exponents  $(1, 1, n/2, n/2)$ . In view of Lemma 8, and the definition of Darboux polynomial, we have

$$\frac{n-2}{2} K f = \frac{n-2}{2} X_H(f) = L_Y(X_H(f)) = L_Y(K f) = L_Y(K) f + K L_Y(f).$$

Moreover in view of Lemma 9 we get

$$L_Y(K) f = \frac{n-2}{2} K f \quad \text{and} \quad L_Y(f) K = r K f,$$

and so

$$\frac{n-2}{2} K f = \left( \frac{n-2}{2} + r \right) K f.$$

Since  $f \neq 0$  (otherwise we do not have a Darboux polynomial) we get that  $r \neq 0$ , so we must have  $K = 0$ . This concludes the proof of the theorem.  $\square$

**Theorem 10.** *Consider system (1) with homogeneous potential of degree  $n$  odd. Let  $f$  be an irreducible Darboux polynomial with cofactor  $K$ . Then  $K = 0$ .*

To prove Theorem 10 we will introduce some preliminary results. We define

$$Z = 2q_1 \frac{\partial}{\partial q_1} + 2q_2 \frac{\partial}{\partial q_2} + np_1 \frac{\partial}{\partial p_1} + np_2 \frac{\partial}{\partial p_2}.$$

**Lemma 11.** *We have  $L_Z X_H = (n-2) X_H$ .*

*Proof.* Let  $f$  be a polynomial and consider

$$\begin{aligned}
 L_Z(X_H(f)) &= [Z, X_H](f) = Z(X_H(f)) - X_H(Z(f)) \\
 &= Z(p_1 f_{q_1} + p_2 f_{q_2} - V_{q_1} f_{p_1} - V_{q_2} f_{p_2}) \\
 &\quad - X_H\left(2q_1 f_{q_1} + 2q_2 f_{q_2} + np_1 f_{p_1} + np_2 f_{p_2}\right) \\
 &= 2q_1(p_1 f_{q_1 q_1} + p_2 f_{q_1 q_2} - V_{q_1 q_1} f_{p_1} - V_{q_1} f_{q_1 p_1} - V_{q_1 q_2} f_{p_2} - V_{q_2} f_{q_1 p_2}) \\
 &\quad + 2q_2(p_1 f_{q_1 q_2} + p_2 f_{q_2 q_2} - V_{q_1 q_2} f_{p_1} - V_{q_1} f_{q_2 p_1} - V_{q_2 q_2} f_{p_2} - V_{q_2} f_{q_2 p_2}) \\
 &\quad + np_1(f_{q_1} + p_1 f_{q_1 p_1} + p_2 f_{q_2 p_1} - V_{q_1} f_{p_1 p_1} - V_{q_2} f_{p_1 p_2}) \\
 &\quad + np_2(p_1 f_{q_1 p_2} + f_{q_2} + p_2 f_{q_2 p_2} - V_{q_1} f_{p_1 p_2} - V_{q_2} f_{p_2 p_2}) \\
 &\quad - p_1\left(2f_{q_1} + 2q_1 f_{q_1 q_1} + 2q_2 f_{q_1 q_2} + np_1 f_{q_1 p_1} + np_2 f_{q_1 p_2}\right) \\
 &\quad - p_2\left(2q_1 f_{q_1 q_2} + 2f_{q_2} + 2q_2 f_{q_2 q_2} + np_1 f_{q_2 p_1} + np_2 f_{q_2 p_2}\right) \\
 &\quad + V_{q_1}\left(2q_1 f_{q_1 p_1} + 2q_2 f_{q_2 p_1} + n f_{p_1} + np_1 f_{p_1 p_1} + np_2 f_{p_1 p_2}\right) \\
 &\quad + V_{q_2}\left(2q_1 f_{q_1 p_2} + 2q_2 f_{q_2 p_2} + np_1 f_{p_1 p_2} + n f_{p_2} + np_2 f_{p_2 p_2}\right).
 \end{aligned}$$

Note that since  $V$  is a homogeneous polynomial of degree  $n$  we have that (3) holds and so

$$L_Z(X_H(f)) = (n-2)\left(p_1 f_{q_1} + p_2 f_{q_2} - V_{q_1} f_{p_1} - V_{q_2} f_{p_2}\right) = (n-2)X_H(f),$$

which concludes the proof of the lemma.  $\square$

Proceeding as in the proof of Lemma 9 taking  $Z$  instead of  $Y$  we obtain the following result.

**Lemma 12.** *Let  $f = f(q_1, q_2, p_1, p_2)$  be a weight homogeneous polynomial with weight degree  $r$  with respect to the weight exponents  $(2, 2, n, n)$ . Then  $L_Z f = r f$ .*

*Proof of Theorem 10.* Let  $f$  be a Darboux polynomial of system (1) with homogeneous potential of degree  $n$  odd and with cofactor  $K$ . In view of Proposition 6 and Lemma 4 we have that  $K$  has weight degree  $m-1 = n-2$  and  $f$  can be considered of weight-degree  $r$  both with respect to the weight-exponents  $(2, 2, n, n)$ . In view of Lemma 11, and the definition of a Darboux polynomial, we have

$$(n-2)Kf = (n-2)X_H(f) = L_Z(X_H(f)) = L_Z(K)f + KL_Z(f).$$

Moreover in view of Lemma 12 we get

$$L_Z(K)f = (n-2)Kf \quad \text{and} \quad L_Z(f)K = rKf,$$

and so

$$(n-2)Kf = (n-2+r)Kf.$$

As in the end of the proof of Theorem 7 we get  $K = 0$ .  $\square$

**2.3. Proof of Theorem 1.** To prove Theorem 1 we state some preliminary basic results. The first one is related with the exponential factors. Its proof and geometrical meaning is given in [3, 17].

**Proposition 13.** *The following statements hold.*

- (a) *If  $E = \exp(g_0/g)$  is an exponential factor for the polynomial system (1) and  $g$  is not a constant polynomial, then  $g = 0$  is an invariant algebraic hypersurface.*
- (b) *Eventually  $e^{g_0}$  can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.*

**Theorem 14.** *Suppose that a polynomial differential system defined in  $\mathbb{C}^4$  admits  $p$  invariant algebraic hypersurfaces  $f_i = 0$  with cofactors  $K_i$ , for  $i = 1, \dots, p$  and  $q$  exponential factors  $E_j = \exp(g_j/h_j)$  with cofactors  $L_j$ , for  $j = 1, \dots, q$ . Then there exists  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$$

*if and only if the function of Darboux type*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} E_1^{\mu_1} \dots E_q^{\mu_q}$$

*is a first integral of the polynomial differential system.*

Theorem 14 is proved in [6].

The following result is well-known.

**Lemma 15.** *Assume that  $\exp(g_1/h_1), \dots, \exp(g_r/h_r)$  are exponential factors of some polynomial differential system*

$$(5) \quad x' = P(x, y, p_x, p_y), \quad y' = Q(x, y, p_x, p_y), \quad p'_x = R(x, y, p_x, p_y), \quad p'_y = U(x, y, p_x, p_y)$$

*with  $P, Q, R, U \in \mathbb{C}[x, y, p_x, p_y]$  with cofactors  $L_j$  for  $j = 1, \dots, r$ . Then*

$$\exp(G) = \exp(g_1/h_1 + \dots + g_r/h_r)$$

*is also an exponential factor of system (5) with cofactor  $L = \sum_{j=1}^r L_j$ .*

Now we proceed with the proof of Theorem 1.

*Proof of Theorem 1.* It follows from Theorems 7, 10, 14 and Proposition 13 that in order to have a first integral of Darboux type which is not a polynomial we must have  $q$  exponential factors of the form  $E_j = \exp(g_j)$  with cofactors  $L_j$  such that  $\sum_{j=1}^q \mu_j L_j = 0$ . Let  $G = \sum_{j=1}^q \mu_j g_j$ , then  $E = \exp(G)$  is an exponential factor of the Hamiltonian system (1) with cofactor  $L = \sum_{j=1}^q \mu_j L_j = 0$  (see Lemma 15). So  $\exp(G)$  is a first integral, and consequently  $G$  is a polynomial first integral of the Hamiltonian system (1). This completes the proof.  $\square$



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