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Classification of linear skew-products of the complex plane and an affine route of fractalization

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Abstract

Linear skew products of the complex plane,

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta+\omega, \\ z & \mapsto & a(\theta)z, \end{array}\right\}$$

where $\theta \in \mathbb{T}$, $z \in \mathbb{C}$, $\frac{\omega}{2\pi}$ is irrational, and $\theta \mapsto a(\theta) \in \mathbb{C} \setminus \{0\}$ is a smooth map, appear naturally when linearizing dynamics around an invariant curve of a quasi-periodically forced complex map. In this paper we study linear and topological equivalence classes of such maps through conjugacies which preserve the skewed structure, relating them to the Lyapunov exponent and the winding number of $\theta \mapsto a(\theta)$. We analyze the transition between these classes by considering one parameter families of linear skew products. Finally, we show that, under suitable conditions, an affine variation of the maps above has a non-reducible invariant curve that undergoes a fractalization process when the parameter goes to a critical value. This phenomenon of fractalization of invariant curves is known to happen in nonlinear skew products, but it is remarkable that it also occurs in simple systems as the ones we present.

Keywords: Reducibility, winding number, Lyapunov exponent, complex fibered maps, topological classification.

1 Introduction

In this work we are concerned with the dynamics of linear skew products of the complex plane. These are maps $F_{\mu,\omega} : \mathbb{T} \times \mathbb{C} \to \mathbb{T} \times \mathbb{C}$ of the form

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta+\omega, \\ z & \mapsto & a(\theta)z, \end{array}\right\}$$

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where ω is Diophantine (see Section 2) and $\theta \in \mathbb{T} \mapsto a(\theta) \in \mathbb{C} \setminus \{0\}$ is a smooth map. They appear in a natural way when linearizing the dynamics around an invariant curve of a quasi-periodically forced complex map. As in many differentiable systems, the linear part of the dynamics determines, under certain conditions, the stability and the local behaviour of the system around the invariant curve ([Pon07]).

Our first goal is to classify linear skew products. We consider linear and topological conjugacies which preserve the skew product structure (see Definition 3.1) and we show the different equivalence classes. Two relevant indicators in this classification are the Lyapunov exponent (see Section 2) and the winding number of the skew product, defined as the winding number of the curve $\theta \mapsto a(\theta)$ w.r.t. the origin, and denoted by $\mathrm{wind}(a, 0)$. The simplest dynamics is the one given by the class of *reducible* skew products, which are those that can be reduced to a linear system with constant coefficients, under a linear change of variables (see Section 2). In this context, we show (Corollary 3.8) that reducibility can be characterized by the winding number being equal to zero. More generally, we show that any linear system can be written, via a linear change, as

$$\left.\begin{array}{rcl}\theta & \mapsto & \theta + \omega,\\ z & \mapsto & be^{im\omega}e^{in\theta}z,\end{array}\right\}$$

where $n = \mathrm{wind}(a, 0)$, m is an arbitrary integer (that depends on the choice of change of variables), $b \in \mathbb{C}$ and $\log|b|$ is the Lyapunov exponent (see Proposition 3.7). On the other hand, topological conjugacy classes are also determined by the winding number but only by the sign of the Lyapunov exponent, see Theorem 3.12 and 3.13.

In order to analyze the transitions between different conjugacy classes, we shall also consider one-parameter families of linear skew products of the form

$$\left.\begin{array}{rcl}\theta & \mapsto & \theta + \omega,\\ z & \mapsto & a(\theta, \mu)z,\end{array}\right\} \tag{1}$$

being μ a real parameter. Since we want the winding number to change when moving the parameter, we must allow a to have zeros, which means that in these cases the skew product will not be invertible. By means of suitable normal forms, we study how the Lyapunov exponent depends on the parameters of the system. In particular, we show that the dependence of the Lyapunov exponent w.r.t. parameters is only continuous (and never differentiable) when the winding number changes (see Section 4 for details).

The dynamics of general (nonlinear) skew products is a well known topic in dynamical systems that has been considered by several authors (see, for instance, [Sta97, Jor01, Gle02]), and very often specifically to study the existence of invariant curves, the fractalization phenomenon and the existence of Strange Non-chaotic Attractors (SNAs) [PMR98, PNR01, Jäg03, HS06, Jäg07, JNOT07, JT08, Bje09]. In linear systems, the invariant curve is given by $z = 0$ and, hence, it always exists. For this reason, let us consider a small modification of a linear skew product, given by the so called affine skew products,

$$\left.\begin{array}{rcl}\theta & \mapsto & \theta + \omega,\\ z & \mapsto & a(\theta, \mu)z + c(\theta, \mu).\end{array}\right\} \tag{2}$$

If (2) has an invariant curve, then it can be reduced to the form (1) by translating the curve to the origin and, hence, the classification we have obtained for linear systems can be extended to affine systems (see Section 5). An interesting situation happens when (2), for some value of the parameter μ, has no invariant curve. A natural question is then the following: if the parameter

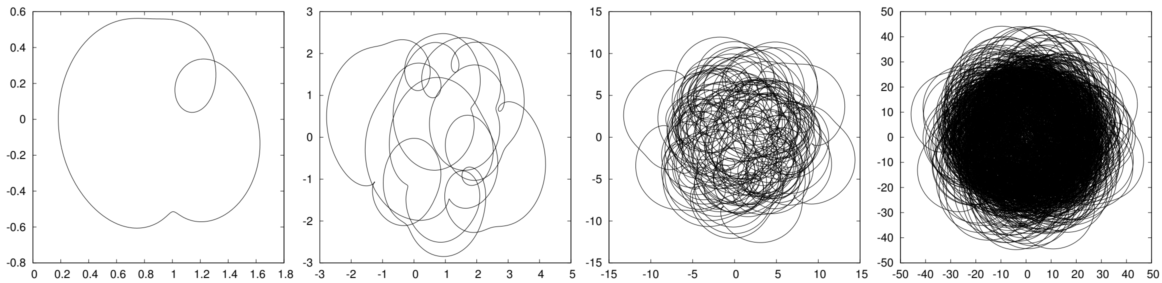

Figure 1: Invariant curve of (3) for $c = 1$. Plots for $\mu = 0.5$, $\mu = 0.9$, $\mu = 0.99$ and $\mu = 0.999$.

moves from a value for which there is an invariant curve to another value for which there is no invariant curve, how does this curve dissapear? We will see that, in this case, the curve may exhibit a fractalization process (see Section 5 for a precise definition) as the parameter varies. Even further, we show that this process appears in a family of of affine skew products as simple as

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta + \omega, \\ z & \mapsto & \mu e^{i\theta} z + c, \end{array}\right\} \tag{3}$$

where ω is the golden mean. The parameter μ is used to control the Lyapunov exponent and the value $c \neq 0$ controls the existence of a nontrivial invariant curve: as we will see, this system has an attracting invariant curve z_μ for $|\mu| < 1$ that we display in Figure 1 for several values of μ, and no invariant curves when $\mu = 1$. As indicators of the behaviour of the invariant curve when μ approaches 1 from below, we numerically compute its norm, the norm of its derivative, its length and its winding number w.r.t. the origin. All of them go to infinity and it is remarkable that their respective asymptotic behaviour seems very well fitted by quite simple functions, as shown in Figure 2. Note that the linearization of the dynamics at the invariant curve (which in this case is just the linear part of the system) is non-reducible.

We also study this fractalization phenomenon in a rigorous way, giving a proof of the asymptotic behaviour displayed in Figure 2. As a side result, to illustrate the "wild" behaviour of this curve, we prove that

$$\bigcup_{\mu_0 < \mu < 1} \{z_\mu(\psi) \mid \psi \in \mathbb{T}\} = \mathbb{C}$$

for any $\mu_0 \in [0, 1)$. In this direction note that if instead of the value $c = 1$ in (3) we use $c = \sqrt{1-\mu}$, then the invariant curve is scaled so that it stays bounded and then the union of curves for $\mu_0 < \mu < 1$ fills up a bounded domain for any value $\mu_0 < 1$. It is also remarkable that the curve disappears when μ reaches the value 1. Of course, the same phenomenon happens when μ approaches 1 from above, but then the invariant curve is repelling.

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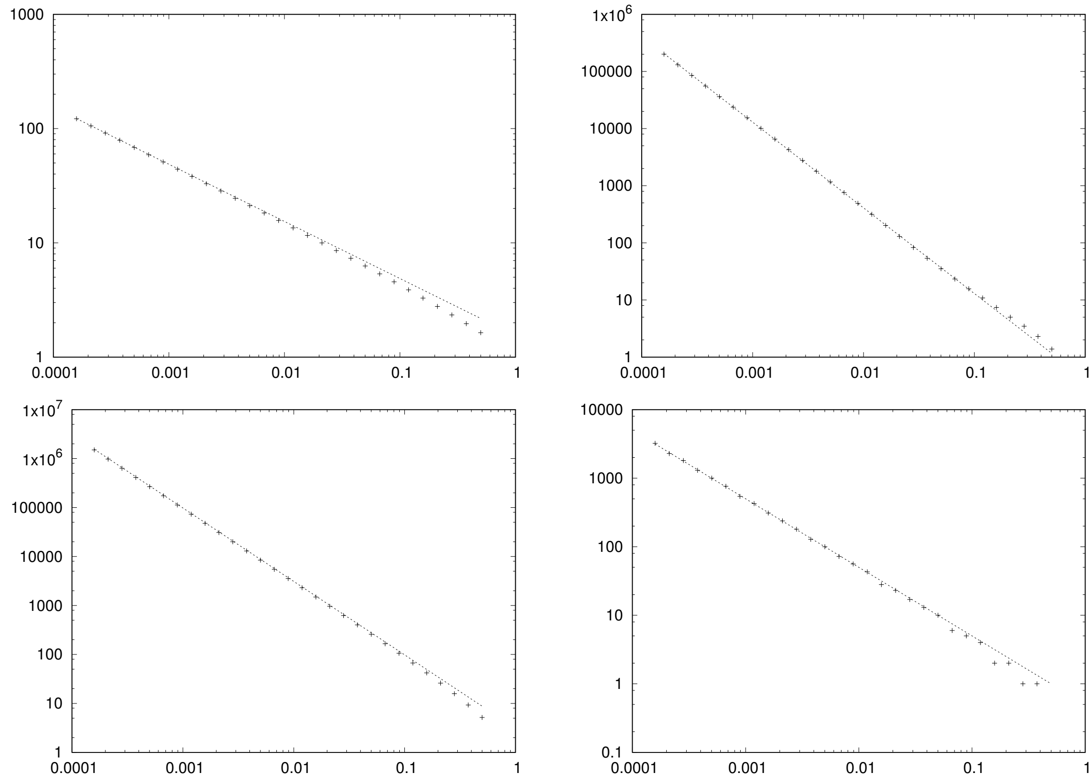

Figure 2: Asymptotic growth of the invariant curve of (3) w.r.t. μ when $\mu \nearrow 1$, for $c = 1$. The horizontal axis shows $1-\mu$ and the symbols "+" denote the computed values. The dotted line is the fitting function. **Top:** On the left, fitting $\|z_\mu\|_\infty$ by $1.54(1-\mu)^{-1/2}$. On the right, fitting of $\|z'_\mu\|_\infty$ by $0.41(1-\mu)^{-3/2}$. **Bottom:** On the left, fitting of the length of z_μ by $3.1(1-\mu)^{-3/2}$. On the right, fitting of $\mathrm{wind}(z_\mu, 0)$ by $0.5(1-\mu)^{-1}$.

2 Preliminaries

This section is a compendium of results and definitions that will be used during the work. This section is added in order to facilitate the reading of the paper.

The first notions we want to introduce concern the arithmetic properties of the frequency ω (see e.g. [Lan95, Khi97]). The set of Diophantine numbers is defined as follows.

Definition 2.1 (Diophantine numbers)**.** A number ω is called Diophantine of type (γ, τ) for $\gamma > 0$ and $\tau \geq 1$ if

$$\left|\omega - \frac{p}{q}\right| > \frac{\gamma}{|q|^\tau} \tag{4}$$

for all $\frac{p}{q} \in \mathbb{Q}$. We denote by $\mathcal{D}_{\gamma,\tau}$ the set of numbers that satisfy (4) for fixed $\gamma > 0$ and $\tau \geq 1$.

We define, as well, the set of numbers of constant type.

Definition 2.2 (Number of constant type)**.** A number ω is said to be of constant type if the coefficients of its continued fraction are bounded.

The numbers of constant type are Diophantine of type $(\gamma, 1)$. The Diophantine condition shall be used widely during Section 3. We shall require ω to be of constant type in Theorem 5.5 and related results.

An important dynamical observable is the Lyapunov exponent (see e.g. [KH95]).

Definition 2.3. Fixed $\theta \in \mathbb{T}$, we define the Lyapunov exponent at θ of the skew-product as

$$\lambda(\theta) = \limsup_{n\to\infty} \frac{1}{n} \log\left|\prod_{j=0}^{n-1} a(\theta + j\omega)\right|. \tag{5}$$

We define, as well, the quantity

$$\Lambda = \frac{1}{2\pi} \int_{\mathbb{T}} \log |a(\theta)| d\theta. \tag{6}$$

If Λ happens to be finite, then the Birkhoff Ergodic Theorem tells us that, for Lebesgue-a.e. $\theta \in \mathbb{T}$, the $\limsup$ (5) is in fact a limit and $\lambda(\theta) = \Lambda$. If $a(\theta)$ never vanishes, $\limsup$ is again a $\lim$ and coincides with Λ for all $\theta \in \mathbb{T}$. In this last case (5) converges uniformly. This follows from the fact that irrational rotations on $\mathbb{T}$ are uniquely ergodic.

The Argument principle is a standard result in complex analysis, see, for instance, [Ahl66]. This result will be used in Sections 4 and 5.

Theorem 2.4 (Argument principle)**.** *Let U be a bounded domain of $\mathbb{C}$ with piecewise regular positively oriented boundary. Let V be an open set $\bar{U} \subset V$, and f a meromorphic function on V. Let n be the number of zeros of f in V (counted with multiplicity). Suppose that there are no zeros nor poles of f on ∂U. Then*

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f'(z)}{f(z)} dz = n.$$

Another result we want to mention is the generalization to complex-valued functions of the Malgrange collocation Theorem given in [Nir71]. This will be used to construct a normal form at some specific critical values.

Theorem 2.5 (Malgrange-Nirenberg)**.** *Let $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ an open set containing the origin. Consider $f : (t, x) \in \mathcal{U} \mapsto f(t, x) \in \mathbb{C}$ of $\mathcal{C}^\infty$ class. Let $p > 0$ be the first integer such that*

$$\frac{\partial^p}{\partial t^p} f(0, 0) \neq 0.$$

Then, in an neighbourhood of the origin, one has the factorization

$$f(t, x) = Q(t, x)P(t, x),$$

where

$$P(t, x) = t^p + \sum_{j=1}^{p} \lambda_j(x) t^{p-j},$$

and Q and λ_j are $\mathcal{C}^\infty$ complex-valued functions with $Q(0, 0) \neq 0$. If f is real, Q and P can be chosen to be real.

When classifying linear skew products, we will use the following concept.

Definition 2.6. A homeomorphism $g : \mathbb{C} \to \mathbb{C}$ is isotopic to the identity if there exists a continuous map

$$\begin{array}{rccl} G: & [0,1]\times\mathbb{C} & \longrightarrow & \mathbb{C} \\ & (t,z) & \longmapsto & G(t,z) = G_t(z) \end{array}$$

such that

1. G_t is a homeomorphism of $\mathbb{C}$ for all $t \in [0,1]$,
2. $G_0 = g$ and G_1 is the identity map.

3 Linear invertible skew-products

In this section we focus on linear skew-products on the complex plane,

$$\begin{array}{rccc} F_a: & \mathbb{T}\times\mathbb{C} & \longrightarrow & \mathbb{T}\times\mathbb{C} \\ & (\theta,z) & \longmapsto & (\theta+\omega, a(\theta)z), \end{array} \tag{7}$$

where $a(\theta) \in \mathbb{C}\setminus\{0\}$ for all $\theta \in \mathbb{T}$. This means that (7) is an invertible map. Moreover, we assume that the map $\theta \mapsto a(\theta)$ is of class C^r $(r \geq 1)$ and that $\omega \in \mathcal{D}_{\gamma,\tau}$.

We are interested in classifying these linear skew-products and, to this end, we consider two different types of conjugacies.

Definition 3.1 (Topological and linear conjugacy as skew products)**.** Two linear skew-products F_a and F_b are *topologically conjugate as skew products* if there exists a change of coordinates of the form

$$\mathcal{H}(\theta,z) = (\theta+\nu, H(\theta,z))$$

where $\nu \in \mathbb{T}$ and, for each θ, $H(\theta,\cdot)$ is a homeomorphism of the plane verifying $H(\theta,0)=0$ and such that

$$\mathcal{H}^{-1}\circ F_a \circ \mathcal{H} = F_b.$$

When $H(\theta,z)$ can be chosen to be linear w.r.t. z, i.e. $H(\theta,z)=c(\theta)z$, with $c(\theta)$ continuous and different from zero for all θ, then F_a and F_b are said to be *linearly conjugate as skew products up to an angle translation.* If $\nu=0$ we simply say that F_a and F_b are linearly conjugate as skew products. Note that this is equivalent to

$$\frac{a(\theta)}{b(\theta)} = \frac{c(\theta+\omega)}{c(\theta)}, \qquad \text{for all } \theta\in\mathbb{T}.$$

In what follows, we will refer to “conjugacies as skew products” simply as “conjugacies”.

Definition 3.2. A topological conjugacy as above is isotopic to the identity if $H(\theta,\cdot)$ is isotopic to the identity for each $\theta \in \mathbb{T}$ (see Definition 2.6).

Note that linear conjugacies, in the sense described above, are always isotopic to the identity. An important category is the class of reducible skew products.

Definition 3.3 (Reducibility)**.** A linear skew product

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta+\omega, \\ z & \mapsto & a(\theta)z, \end{array}\right\}$$

is said to be *reducible* iff there exists a linear change of variables, $(\theta, z) = (\theta, e(\theta)u)$ such that the transformed system becomes uncoupled. That is, the transformed system takes the form

$$\left.\begin{array}{rcl}\theta & \mapsto & \theta+\omega, \\ u & \mapsto & bu,\end{array}\right\}$$

where $b = e(\theta+\omega)^{-1}a(\theta)e(\theta)$ does not depend on θ.

3.1 Linear conjugacy classes

Next, we give necessary and sufficient conditions for two systems to be linearly conjugate. Let $\mathrm{wind}(a(\theta), 0)$ denote the winding number of the closed curve $a(\theta)$ with respect to the point $z = 0$.

Proposition 3.4 (Linear conjugacy classes)**.** *Let $\omega \in \mathcal{D}_{\gamma,\tau}$. Then there exists $r = r(\tau) > 0$ such that if $a(\theta)$ and $b(\theta)$ are of class C^r then F_a and F_b are linearly conjugate if and only if the following two conditions are satisfied:*

(a) $\mathrm{wind}(a(\theta), 0) = \mathrm{wind}(b(\theta), 0)$.

(b) There exists $m \in \mathbb{Z}$ and a branch of the logarithm such that

$$\int_{\mathbb{T}} \log\left(e^{-im\omega}\frac{a(\theta)}{b(\theta)}\right) d\theta = 0.$$

Moreover, if such m exists, it is unique and the linear change of coordinates $H(\theta, z) = c(\theta)z$ satisfies that $\mathrm{wind}(c(\theta), 0) = m$.

It is easy to check that (a) and (b) are independent conditions.

Proof. ($\Longleftarrow$) Because of (a), the curve $\frac{a(\theta)}{b(\theta)}$ has winding number 0 and so does $e^{-im\omega}\frac{a(\theta)}{b(\theta)}$. Hence the curve defined by $l(\theta) = \log\left(e^{-im\omega}\frac{a(\theta)}{b(\theta)}\right)$ is also a closed C^r curve. Let us consider its Fourier series

$$l(\theta) = \log e^{-im\omega}\frac{a(\theta)}{b(\theta)} = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta},$$

where $\alpha_0 = 0$ because of (b). As $l(\theta) \in C^r$ we know that $|\alpha_k| = O(1/|k|^r)$. We now define for any $k \neq 0$

$$\widetilde{c_k} = \frac{\alpha_k}{\exp(ik\omega) - 1},$$

and set for example $\widetilde{c_0} = 0$. Note that the Diophantine condition on ω implies that $|\widetilde{c_k}| = \mathcal{O}(|k|^{\tau-r})$. So, if $r > \tau + 1$, the series $\sum_k \widetilde{c_k}e^{ik\theta}$ is absolutely and uniformly convergent and defines a $2\pi-$periodic function L,

$$L(\theta) = \sum_{k\in\mathbb{Z}} \widetilde{c_k}e^{ik\theta}.$$

Moreover, $L(\theta) \in \mathcal{C}^s$, with $s \geq s(\tau)$, where $s(\tau)$ is the integer part of $-1 + r - \tau$ if $r - \tau \notin \mathbb{N}$ or $s(\tau) = -2 + r - \tau$ if $-1 + r - \tau \in \mathbb{N}$. It is easy to check by comparing the coefficients in the Fourier series that, by construction,

$$l(\theta) = \log e^{-im\omega}\frac{a(\theta)}{b(\theta)} = L(\theta+\omega) - L(\theta).$$

Let $\widetilde{c}(\theta) = \exp(L(\theta))$ which has winding number zero with respect to $z = 0$ because $L(\theta)$ is a closed curve. It follows that

$$\exp\left(L(\theta+\omega) - L(\theta)\right) = \frac{\widetilde{c}(\theta+\omega)}{\widetilde{c}(\theta)},$$

and therefore

$$e^{-im\omega}\frac{a(\theta)}{b(\theta)} = \frac{\widetilde{c}(\theta+\omega)}{\widetilde{c}(\theta)}.$$

Finally, if we define $c(\theta) = e^{im\theta}\widetilde{c}(\theta)$, we have

$$\frac{a(\theta)}{b(\theta)} = \frac{c(\theta+\omega)}{c(\theta)}.$$

Hence, $c(\theta)$ provides the linear change and $\mathrm{wind}(c(\theta), 0) = m$.

($\Longrightarrow$) Suppose F_a and F_b are linearly conjugate by a change $c(\theta)$ and let m be the winding number of $c(\theta)$ with respect to 0. Then

$$\frac{a(\theta)}{b(\theta)} = \frac{c(\theta+\omega)}{c(\theta)}.$$

which already implies (a). Define $\tilde{c}(\theta) = e^{-im\theta}c(\theta)$ which makes $\mathrm{wind}(\tilde{c}(\theta), 0) = 0$ and also

$$e^{-im\omega}\frac{a(\theta)}{b(\theta)} = \frac{\widetilde{c}(\theta+\omega)}{\widetilde{c}(\theta)}.$$

Since $\tilde{c}$ has zero winding number, there exists a branch of the logarithm $\log(\tilde{c}(\theta))$. Then define

$$l(\theta) = \log(\tilde{c}(\theta+\omega)) - \log(\tilde{c}(\theta))$$

which is a branch of the logarithm of $\frac{\widetilde{c}(\theta+\omega)}{\widetilde{c}(\theta)}$, and hence it is a branch of $\log e^{-im\omega}\frac{a(\theta)}{b(\theta)}$. Moreover,

$$\int_{\mathbb{T}} l(\theta)d\theta = \int_{\mathbb{T}} \log(\tilde{c}(\theta+\omega)) - \log(\tilde{c}(\theta))d\theta = 0,$$

and (b) follows. $\square$

Remark 3.5. *The value of $r(\tau)$ given in the proof is not optimal, but in any case $r(\tau) > 0$ (see [Rüs76]).*

Remark 3.6. *If $a(\theta)$ and $b(\theta)$ are C^∞, then $c(\theta)$ is also C^∞.*

From now on, we will denote by $r(\tau) \in \mathbb{N}$ a value of r for which the previous proposition holds. We shall use the previous proposition to find canonical forms for these linear skew products. We need to differentiate cases depending on the winding number of the curve $\theta \mapsto a(\theta)$.

Proposition 3.7 (Linear normal form)**.** *Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ is $C^{r(\tau)}$ and $\mathrm{wind}(a(\theta), 0) = n$. Then, for any $m \in \mathbb{Z}$, there exists a linear change, of winding number $-m$, which conjugates F_a to*

$$F_{b(m,\theta)}(\theta, z) = (\theta+\omega, be^{im\omega}e^{in\theta}z),$$

where $b = |b|e^{i\rho} \in \mathbb{C}$ satisfies

$$|b| = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \log|a(\theta)|\, d\theta\right),$$

and

$$\rho = \frac{1}{2\pi}\mathrm{Im}\int_{\mathbb{T}} \log(a(\theta)e^{-in\theta})\, d\theta, \tag{8}$$

for any determination of the logarithm. Moreover, two such systems $(\theta+\omega, b_1e^{in\theta}z)$ and $(\theta+\omega, b_2e^{in\theta}z)$, with $b_1, b_2 \in \mathbb{C}$ are linearly conjugate if and only if $b_1 = b_2e^{im\omega}$ for some $m \in \mathbb{Z}$.

Note that $\log|b|$ is the Lyapunov exponent of the skew product.

Proof. Consider $m \in \mathbb{Z}$ fixed and choose a branch $l(\theta)$ of $\log(a(\theta)e^{-in\theta})$ which exists since this expression has winding number 0. We want to find $b = |b|e^{i\rho}$ such that condition (b) in Proposition 3.4 is satisfied (observe that condition (a) is already fulfilled). That is, we want that

$$\int_{\mathbb{T}} \log e^{im\omega}e^{i(-m\omega-\rho)}\frac{a(\theta)}{|b|e^{in\theta}}d\theta = 0,$$

for some branch of the logarithm. Given that $l(\theta) - \log|b| - i\rho$ is such a branch, we obtain

$$\int_{\mathbb{T}} l(\theta)d\theta = 2\pi\log|b| + 2\pi i\rho.$$

Separating real and imaginary part,

$$\int_{\mathbb{T}} \log|a(\theta)| = 2\pi\log|b| \quad \text{and} \quad \mathrm{Im}\int_{\mathbb{T}} l(\theta)d\theta = 2\pi\rho,$$

and the expressions follow. To finish the proof, observe that the systems are conjugate if and only if there exists $p \in \mathbb{Z}$ such that

$$\int_{\mathbb{T}} \log e^{-ip\omega}\frac{b_1}{b_2}d\theta = 0,$$

for some branch of the logarithm. But this is equivalent to requiring $e^{-ip\omega}\frac{b_1}{b_2} = 1$ for some $p \in \mathbb{Z}$. □

From the proposition above we obtain a trivial corollary about the reducibility of these systems, in the case of winding number 0.

Corollary 3.8 (Zero index and reducibility)**.** *Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ is $C^{r(\tau)}$. If $\mathrm{wind}(a(\theta), 0) = 0$, then the system is reducible. Moreover, the system is reducible to a system of the form $(\theta+\omega, bz)$ with $b \in \mathbb{R}$, if and only if there exists $m \in \mathbb{Z}$ and a branch of the argument such that*

$$\int_{\mathbb{T}} \arg(a(\theta))d\theta - m\omega = 0.$$

In such case, the change has winding number equal to $-m$.

In the nonzero winding number case, it turns out that we can always reduce to the case of $b \in \mathbb{R}$ by changing the phase. More precisely we have the following statement.

Proposition 3.9 (Nonzero index). *Assume $\omega \in \mathcal{D}_{\gamma,\tau}$, $a(\theta)$ is $C^r(\tau)$ and $\mathrm{wind}(a(\theta),0) = n \neq 0$. Then, there exists a unique $b \in \mathbb{R}$ such that F_a is linearly conjugate (up to angle translation) to*

$$F_b(\theta, z) = (\theta + \omega, be^{in\theta}z).$$

Similarly as before, the precise value of b is

$$b = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \log|a(\theta)|\, d\theta\right).$$

Proof. We apply Proposition 3.7 to conjugate F_a to a system $(\theta + \omega, |b|e^{i\rho}e^{in\theta}z)$ by considering $m = 0$. We now change the angle by $H(\theta, z) = (\theta + \rho/n, z)$. If we denote $\psi = \theta + \rho/n$, the transformed system becomes $(\psi + \omega, |b|e^{in\psi}z)$. □

3.2 Topological conjugacy classes

We now proceed to classify linear invertible skew-products of class $C^{r(\tau)}$ from a topological point of view. We recall from Definition 3.1 that if two linear skew products F_a and F_b are topologically conjugate then there exist a constant $\nu \in \mathbb{T}$ and a continuous map $H : \mathbb{T} \times \mathbb{C} \to \mathbb{C}$ with $H(\theta, 0) = 0$, such that

$$H(\theta + \omega, a(\theta)z) = b(\theta + \nu)H(\theta, z), \quad \forall \theta \in \mathbb{T},\ \forall z \in \mathbb{C}.$$

In the case that $H(\theta, \cdot)$ is isotopic to the identity we say that F_a and F_b are topologically conjugate via a homeomorphism isotopic to the identity.

Lemma 3.10 (Winding numbers). *Consider $f \in C^0(\mathbb{T} \times \mathbb{T}, \mathbb{C} \setminus \{0\})$ such that, $\forall \rho \in \mathbb{T}$, $\mathrm{wind}_\theta(f(\theta, \rho), 0) = p$ and, $\forall \theta \in \mathbb{T}$, $\mathrm{wind}_\rho(f(\theta, \rho), 0) = q$, where wind_θ and wind_ρ denote the winding numbers with respect to θ and ρ respectively. Then, $\mathrm{wind}_\theta(f(\theta, \theta), 0) = p + q$. In particular*

$$\mathrm{wind}(x(\theta)y(\theta), 0) = \mathrm{wind}(x(\theta), 0) + \mathrm{wind}(y(\theta), 0)$$

for all $x, y : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ continuous.

Proof. First of all notice that, since $f(\theta, \rho) \neq 0$ for each $(\theta, \rho) \in \mathbb{T} \times \mathbb{T}$, we can assume that $|f(\theta, \rho)| = 1$ $\forall (\theta, \rho) \in \mathbb{T} \times \mathbb{T}$. As the winding number of f w.r.t. a closed path on the torus remains constant under continuous deformations of this path, we can write the path (θ, θ) $(0 \le \theta \le 2\pi)$ as the composition of the paths $(\theta, 0)$ and $(0, \theta)$ and the result follows. □

Proposition 3.11 (Winding number and topological conjugacy). *If two linear skew products F_a and F_b are topologically conjugate via a homeomorphism isotopic to the identity then $\mathrm{wind}(a(\theta), 0) = \mathrm{wind}(b(\theta), 0)$.*

Proof. Since F_a and F_b are topologically conjugate we have the relation

$$H(\theta + \omega, a(\theta)z) = b(\theta + \nu)H(\theta, z) \tag{9}$$

that holds for all $\theta \in \mathbb{T}$, $z \in \mathbb{C}$ and some $\nu \in \mathbb{T}$. Let us name $n = \mathrm{wind}(a(\theta), 0)$ and $m = \mathrm{wind}(b(\theta), 0)$. Now let us fix the value of z, say $z = 1$. Taking winding numbers at both sides of equation (9), and, applying Lemma 3.10, it follows:

$$\mathrm{wind}(H(\theta + \omega, a(\theta)), 0) = m + \mathrm{wind}(H(\theta, 1), 0).$$

Note that $\ell = \text{wind}(H(\theta,1),0)$ is well-defined as $H(\theta,\cdot)$ is a homeomorphism with $H(\theta,0)=0$ for all θ.

$$\text{wind}(H(\theta+\omega,a(\theta)),0) = \text{wind}_\rho(H(\rho,a(\theta)),0) + \text{wind}_\theta(H(\rho,a(\theta)),0).$$

We have that $\text{wind}_\rho(H(\rho,a(\theta)),0) = \ell$, and $\text{wind}_\theta(H(\rho,a(\theta)),0) = n$ since $H(\theta,\cdot)$ is a homeomorphism isotopic to the identity. □

Next, in Theorems 3.12 and 3.13 we give a topological classification of linear skew products, depending on the winding number and the Lyapunov exponent.

Theorem 3.12 (Topological conjugacy classes)**.** *Assume that $\omega \in \mathcal{D}_{\gamma,\tau}$, and a is a $C^{r(\tau)}$ function which never vanishes. Then the linear skew-product (7), namely $(\theta,z) \mapsto (\theta+\omega, a(\theta)z)$ is topologically conjugate to one of the following:*

(a) *If $\text{wind}(a(\theta),0)=0$ and the Lyapunov exponent is negative,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& \frac{1}{2}z, \end{array}\right\} \tag{10}$$

(b) *If $\text{wind}(a(\theta),0)=0$ and the Lyapunov exponent is positive,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& 2z, \end{array}\right\} \tag{11}$$

(c) *If $\text{wind}(a(\theta),0)=0$, the Lyapunov exponent is zero,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& e^{i\rho}z, \end{array}\right\} \tag{12}$$

where ρ is given by (8).

(d) *If $\text{wind}(a(\theta),0)=n\neq 0$ and the Lyapunov exponent is negative,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& \frac{1}{2}e^{in\theta}z, \end{array}\right\} \tag{13}$$

(e) *If $\text{wind}(a(\theta),0)=n\neq 0$ and the Lyapunov exponent is positive,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& 2e^{in\theta}z, \end{array}\right\} \tag{14}$$

(f) *If $\text{wind}(a(\theta),0)=n\neq 0$ and the Lyapunov exponent is zero,*

$$\left.\begin{array}{rcl} \tilde{\theta} &=& \theta+\omega, \\ \tilde{z} &=& e^{in\theta}z, \end{array}\right\} \tag{15}$$

Proof. Items (a) and (b) are an immediate consequence of Corollary 3.8 and the fact that in $\mathbb{R}^2$ two attracting (respectively repelling) linear focus are topologically conjugate.

To show (d) observe that Proposition 3.9 gives that F_a is linearly conjugate (up to an angle translation) to

$$F_{be^{in\theta}}(\theta, z) = (\theta + \omega, be^{in\theta} z).$$

It is then easy to see that, if the Lyapunov exponent is negative, the following change of variables

$$w = |z|^\alpha z, \qquad \alpha = \frac{\log 2}{\log b} - 1,$$

produces the desired result. Case (e) is analogous. Items (c) and (f) follow from Propositions 3.7 and 3.9. □

Theorem 3.13. *Assume that $\omega \in \mathcal{D}_{\gamma,\tau}$. Then, the skew-products (10), (11), (12), (13), (14) and (15) belong to different topological conjugacy classes. Moreover,*

a) Assume that ρ, ω and 2π are linearly independent over $\mathbb{Z}$. Then, two linear skew products of the type (12) are topologically conjugate via a homeomorphism isotopic to the identity iff they are linearly conjugate.

b) Two linear skew products of the types (13), (14) or (15) with two different values of n are not topologically conjugate via a homeomorphism isotopic to the identity.

Proof. The first claim is obvious. To prove item a), we assume that

$$\left.\begin{array}{rcl} \tilde\theta &=& \theta + \omega, \\ \tilde z &=& e^{i\rho_1} z, \end{array}\right\} \qquad \left.\begin{array}{rcl} \tilde\theta &=& \theta + \omega, \\ \tilde z &=& e^{i\rho_2} z, \end{array}\right\}$$

are topologically conjugate and that, for instance, ρ_1, ω and 2π are linearly independent over $\mathbb{Z}$. As they are topologically conjugate, by definition, there exists a continuous map H such that

$$H(\theta + \omega, e^{i\rho_1} r e^{i\varphi}) = e^{i\rho_2} H(\theta, r e^{i\varphi}),$$

where $z = re^{i\varphi}$. Let us fix the value of r (for instance, $r = 1$) and let us look at H as a continuous function of two angles. Expanding both sides of the last equality in Fourier series w.r.t. φ, we obtain

$$\sum_k h_k(\theta + \omega) e^{ik\rho_1} e^{ik\varphi} = e^{i\rho_2} \sum_k h_k(\theta) e^{ik\varphi}.$$

As H cannot be the zero function, there exists k such that h_k does not vanish. Hence,

$$h_k(\theta + \omega) = e^{i(\rho_2 - k\rho_1)} h_k(\theta).$$

Expanding h_k in a Fourier series w.r.t. θ, and selecting a non-zero Fourier coefficient h_{kj}, we have that

$$h_{kj} e^{ij\omega} = e^{i(\rho_2 - k\rho_1)} h_{kj},$$

which implies that $j\omega = \rho_2 - k\rho_1 \mod 2\pi$. Applying the same calculation for the inverse conjugation, we have that there exist integer values $\hat{j}$ and $\hat{k}$ such that $\hat{j}\omega = \rho_1 - \hat{k}\rho_2 \mod 2\pi$. Using this last two equations we obtain that

$$(\hat{j} + j\hat{k})\omega = (1 - k\hat{k})\rho_1 + 2\pi m,$$

for some $m \in \mathbb{Z}$. As ρ_1, ω and 2π are linearly independent over $\mathbb{Z}$ we have that $1 - k\hat{k} = 0$, $\hat{j} + j\hat{k} = 0$ and $m = 0$. The solutions of these equations are: $k = \hat{k} = 1$, $\hat{j} = -j$ and $k = \hat{k} = -1$, $\hat{j} = j$, which implies that H has to be of the form $H(\theta, e^{i\varphi}) = h_{-1}(\theta)e^{-i\varphi} + h_1(\theta)e^{i\varphi}$. Notice that the last conditions cannot hold at the same time. Indeed, if such is the case, there exist j_1 and j_2 verifying the following:

$$
\begin{aligned}
j_1\omega &= \rho_1 - \rho_2 + 2\pi m_1, \\
j_2\omega &= \rho_2 + \rho_1 + 2\pi m_2.
\end{aligned}
$$

Adding these equations, it follows $(j_1 + j_2)\omega = 2\rho_1 + 2\pi(m_1 + m_2)$. This case is out of the study as this condition also imply that ρ_1, ω and 2π are linearly dependent over $\mathbb{Z}$ which leads to a contradiction with the hypothesis assumed in (a). Then, if $k = \hat{k} = -1$, as $H(\theta, \cdot)$ restricted to $z = e^{i\varphi}$ is $h_{-1}(\theta)e^{-i\varphi}$, it cannot be isotopic to the identity as it is reversing the orientation of the unit circle. Therefore, the only remaining possibility is $k = \hat{k} = 1$ and then, it is immediate to check that $(\theta, z) \mapsto (\theta, h_1(\theta)z)$ is a linear conjugacy between the two skew products.

Item b) follows from Proposition 3.11 and from the fact that the attracting or repelling character of the origin is preserved by a topological conjugacy. □

Remark 3.14. *The dynamics of the maps (10), (11), (13) and (14) are locally robust in a neighbourhood of the origin under generic perturbations, because the origin is attracting or repelling. This is not the case for (12) and (15).*

If we write $(\mathbb{R}/\mathbb{Z}) \times \mathbb{C} = \{0\} \bigcup \left(\bigcup_{r>0} \mathbb{T}^2_r\right)$, where $\mathbb{T}^2_r = \{(\theta, z) \,|\, z = re^{i\varphi}\}$, then $\mathbb{T}^2_r$ is an invariant torus for the maps (12) and (15). These invariant foliations could be destroyed by a generic perturbation of the map. Moreover, if we consider coordinates (θ, φ) in the torus, the map (12) restricted to $\mathbb{T}^2_r$ satisfies $(\theta, \varphi) \mapsto (\theta + \omega, \varphi + \rho)$ which is a translation in the torus and the map (15) restricted to $\mathbb{T}^2_r$ satisfies $(\theta, \varphi) \mapsto (\theta + \omega, n\theta + \varphi)$, which is sometimes called a skew shift. The second map is uniquely ergodic if $\frac{\omega}{2\pi}$ is irrational, with the Lebesgue measure as the unique invariant measure and the first map is uniquely ergodic if ω, ρ and 2π are rationally independent.

4 Normal forms and Lyapunov exponents

Let us consider a linear quasi-periodic skew product as defined in (7), given by $a \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 0$. We have shown that the winding number of a is preserved by linear changes of variables (see Proposition 3.4) so that it can be seen as an invariant of the cocycle. In this section we shall study how the Lyapunov exponent varies when introducing a real new parameter μ. In particular we are interested about the regularity with respect to the Lyapunov exponent as μ crosses a critical value for which the skew-product is not invertible. Notice that, up to now, all the skew-products have been invertible. To carry out this study, we use Λ_μ as in Definition 2.3. Recall that, if Λ_μ is finite, it coincides with the Lyapunov exponent. Roughly speaking, the next result shows that Λ_μ depends smoothly on μ except when a changes its winding number.

Theorem 4.1 (Regularity of Λ)**.** *Let us consider a one-parametric family of quasi-periodic cocycles*

$$
\left.
\begin{aligned}
\tilde{\theta} &= \theta + \omega, \\
\tilde{z} &= a(\theta, \mu) z,
\end{aligned}
\right\} \tag{16}
$$

where ω is Diophantine, μ belongs to an open nonempty interval $I \subset \mathbb{R}$ and $a \in \mathcal{C}^\infty(\mathbb{T} \times I, \mathbb{C})$. We assume that

1. *There exists a unique pair* (θ_0, μ_0) *such that* $a(\theta_0, \mu_0) = 0$.

2. $\dfrac{\partial a}{\partial \theta}(\theta_0, \mu_0)$ *and* $\dfrac{\partial a}{\partial \mu}(\theta_0, \mu_0)$ *are linearly independent as vectors of* $\mathbb{R}^2$.

Then, the Lyapunov exponent $\Lambda(\mu)$ *is a continuous function of* μ *such that*

1. Λ *is* $\mathcal{C}^\infty$ *at any* $\mu \neq \mu_0$.

2. Λ *is* $\mathcal{C}^0$ *at* $\mu = \mu_0$ *and there exist constants* A^+ *and* A^-*, such that, when* $\mu \to \mu_0$*, the following expression holds:*

$$\Lambda(\mu) = \Lambda(\mu_0) + A^{\pm}(\mu - \mu_0) + \mathcal{O}(|\mu - \mu_0|^2) \tag{17}$$

where A^+ *is used when* $\mu > \mu_0$ *and* A^- *when* $\mu < \mu_0$. *The values* A^+ *and* A^- *never coincide.*

A particular situation is when the system is reducible for $\mu < \mu_0$, and non-reducible for $\mu > \mu_0$. In the real 1D case, it is known that the dependence of the Lyapunov exponent w.r.t. the parameter μ is continuous but never differentiable at μ_0, see [JT08]. In the real 2D case, there is numerical evidence of the same phenomenon [HdlL06, FH15]. As a matter of fact, the recent preprint [FT18] contains a rigorous proof of this fact for a class of 2D cocycles arising from the study of the spectrum of some discrete Schrödinger operators. Nevertheless, in our case, the behaviour of the side derivatives of the Lyapunov exponent when μ goes to μ_0 (described in Theorem 4.1) is essentially different from the ones described in this paragraph.

The proof of Theorem 4.1 is based on finding a suitable normal form in a small neighbourhood of (θ_0, μ_0). To this end, we shall need the following three auxiliary lemmas.

Lemma 4.2. *Let* $I \subset \mathbb{R}$ *be a nonempty open interval and let* $a \in \mathcal{C}^r(\mathbb{T} \times I, \mathbb{C})$*,* $r \geq 2$*, such that*

1. *There exists a unique pair* (θ_0, μ_0) *for which* $a(\theta_0, \mu_0) = 0$,

2. $\dfrac{\partial a}{\partial \theta}(\theta_0, \mu_0)$ *and* $\dfrac{\partial a}{\partial \mu}(\theta_0, \mu_0)$ *are linearly independent as vectors of* $\mathbb{R}^2$.

Then

1. $\operatorname{wind}(a(\cdot, \mu), 0)$ *is constant for* $\mu < \mu_0$,

2. $\operatorname{wind}(a(\cdot, \mu), 0)$ *is constant for* $\mu > \mu_0$,

3. $|\operatorname{wind}(a(\cdot, \mu_1), 0) - \operatorname{wind}(a(\cdot, \mu_2), 0)| = 1$ *for* $\mu_1 < \mu_0 < \mu_2$.

Proof. First note that, for $\mu < \mu_0$, condition 1 implies that the curves $a(\cdot, \mu)$ are homotopic on $\mathbb{C} \setminus \{0\}$ (the homotopy is given by $a(\theta, \mu)$ itself) and this shows that $\operatorname{wind}(a(\cdot, \mu), 0)$ is constant for $\mu < \mu_0$. As the same reasoning applies for $\mu > \mu_0$ we can also conclude that $\operatorname{wind}(a(\cdot, \mu), 0)$ is constant for $\mu > \mu_0$. Note that, if $|\theta - \theta_0|$ and $|\mu - \mu_0|$ are small enough,

$$a(\theta, \mu) = \frac{\partial a}{\partial \theta}(\theta_0, \mu_0)(\theta - \theta_0) + \frac{\partial a}{\partial \mu}(\theta_0, \mu_0)(\mu - \mu_0) + \mathcal{O}_2,$$

where $\mathcal{O}_2$ is a term of order 2 in $(\theta - \theta_0)$ and $(\mu - \mu_0)$ For θ near θ_0, $a(\theta, \mu_0)$ is close to a straight line passing through the origin at $\theta = \theta_0$, and hence, dividing a conveniently small disk centered at the origin into two almost equal components. Moreover by condition 2, if $\mu_1 < \mu_0$ the curve $a(\cdot, \mu_1)$ lies in one of these components and, if $\mu_2 > \mu_0$, the curve $a(\cdot, \mu_2)$ lies in the other component. In the situation described, the winding number changes by 1 when μ crosses μ_0, this can be seen, for instance, by applying the Argument principle to the function $f(z) = z$. □

Lemma 4.3. *Consider the skew-product defined by $a \in \mathcal{C}^\infty(\mathbb{T}, I, \mathbb{C})$, where I is a nonempty open interval of $\mathbb{R}$. Let us assume that*

1. *There exists a unique pair (θ_0, μ_0) such that $a(\theta_0, \mu_0) = 0$,*
2. *$\dfrac{\partial a}{\partial \theta}(\theta_0, \mu_0)$ and $\dfrac{\partial a}{\partial \mu}(\theta_0, \mu_0)$ are linearly independent as vectors of $\mathbb{R}^2$,*
3. $\mathrm{wind}(a(\cdot, \mu), 0) = n$ *if $\mu < \mu_0$,*
4. $\mathrm{wind}(a(\cdot, \mu), 0) = n + 1$ *if $\mu > \mu_0$,*

Then, if $|\mu - \mu_0|$ is small enough, there exists a change of coordinates of the form

$$\left.\begin{array}{rcl} \theta &=& \varphi + \theta_0 - \pi, \\ z &=& c(\varphi, \mu)\zeta, \end{array}\right\}$$

such that the skew-product takes the form

$$\left.\begin{array}{rcl} \tilde{\varphi} &=& \varphi + \omega, \\ \tilde{\zeta} &=& h(\mu)e^{in\varphi}(e^{i\varphi} + \nu(\mu))\zeta, \end{array}\right\} \tag{18}$$

where h is a $\mathcal{C}^\infty$ zero-free function and $\nu(\mu) = 1 + \mu - \mu_0$.

Remark 4.4. *The assumptions on the winding number are done in order to assure the winding number to increase when μ crosses μ_0. The fact that changes by 1 follows from Lemma 4.2.*

Proof. Note that we can assume that $\theta_0 = \pi$ by simply redefining θ. We call φ to this new angle and, for simplicity, we keep the same notation for the function a (that is, $a = a(\varphi, \mu)$). Note that, as we have $a(\pi, \mu_0) = 0$ and $\frac{\partial a}{\partial \varphi}(\pi, \mu_0) \neq 0$, we can apply the Malgrange Theorem to the function a, which says that a can be factorized as

$$a(\varphi, \mu) = q(\varphi, \mu)(\varphi - \lambda_1(\mu))$$

where q and λ_1 are smooth functions, $|\mu - \mu_0|$ and $|\varphi - \pi|$ are small enough, $\pi - \lambda_1(\mu_0) = 0$ and q is different from zero for all values of φ and μ. Indeed, the function $\varphi - \lambda_1(\mu)$ is complex valued and it has no zeros apart from (π, μ_0). Note that the expression

$$q(\varphi, \mu) = \frac{a(\varphi, \mu)}{\varphi - \lambda_1(\mu)},$$

allows to define q as a $\mathcal{C}^\infty$ function for all values of φ. Then, the equation

$$a(\varphi, \mu) = \frac{q(\varphi, \mu)(\varphi - \lambda_1(\mu))}{e^{in\varphi}(e^{i\varphi} + \nu(\mu))} e^{in\varphi}(e^{i\varphi} + \nu(\mu))$$

is valid for all φ. Let us define

$$b(\varphi, \mu) = \frac{q(\varphi, \mu)(\varphi - \lambda_1(\mu))}{e^{in\varphi}(e^{i\varphi} + \nu(\mu))}.$$

The function b is periodic (and, therefore, continuous) with respect to φ (because a and $e^{in\varphi}(e^{i\varphi} + \nu(\mu))$ are periodic) and has no zeros (notice that the zero of the denominator and the zero of the numerator is the same one). As $a(\varphi, \mu) = b(\varphi, \mu)e^{in\varphi}(e^{i\varphi} + \nu(\mu))$, taking into account that $\mathrm{wind}(a(\varphi, \mu), 0) = \mathrm{wind}(e^{in\varphi}(e^{i\varphi} + \nu(\mu)), 0)$ we obtain that $\mathrm{wind}(b(\varphi, \mu), 0) = 0$. Corollary 3.8 gives the existence of the change of coordinates of the form $z = c(\varphi, \mu)\zeta$ in the statement. Hence, the original system is linearly conjugate to (18). The result follows. □

Remark 4.5. *It is possible to add an extra change of parameters so that (18) becomes even simpler, namely*

$$\left.\begin{array}{rcl}\tilde{\varphi} &=& \varphi+\omega,\\ \tilde{\zeta} &=& h(\nu)e^{in\varphi}(e^{i\varphi}-\nu)\zeta,\end{array}\right\}$$

where h is a C^∞ function without zeros.

Lemma 4.6. *Consider the linear skew-product associated to*

$$a(\theta,\mu)=h(\mu)e^{in\theta}(e^{i\theta}+\nu(\mu)),$$

where, as usual, μ belongs to a nonempty open interval $I\subset\mathbb{R}$. Let us assume:

1. *h is $\mathcal{C}^\infty$ and has no zeros,*
2. *$\nu\in\mathcal{C}^\infty(I,\mathbb{R})$, $\nu(\mu_0)=1$ and $\frac{d\nu}{d\mu}(\mu_0)\neq 0$ for some $\mu_0\in I$.*

Then, the Lyapunov exponent Λ is a continuous function of μ and

1. *Λ is $\mathcal{C}^\infty$ at any $\mu\neq\mu_0$.*
2. *Λ is $\mathcal{C}^0$ at $\mu=\mu_0$ and, there exist constants A^+ and A^- for which, when $\mu\to\mu_0$, the following expression holds:*
$$\Lambda(\mu)=\Lambda(\mu_0)+A^\pm(\mu-\mu_0)+\mathcal{O}(|\mu-\mu_0|^2) \tag{19}$$
where A^+ is used when $\mu>\mu_0$ and A^- when $\mu<\mu_0$. The values A^+ and A^- are always different.

Proof. We take (6) as the definition of the Lyapunov exponent, then

$$\Lambda(\mu)=\log|h(\mu)|+\frac{1}{2\pi}\int_{\mathbb{T}}\log|e^{i\theta}+\nu(\mu)|d\theta.$$

Since h is zero-free and smooth the term $\log|h(\mu)|$ depends smoothly on μ. If something compromises the smoothness of the Lyapunov exponent with respect to μ, it must be located at the second term. Recall that ν is real for all μ. It is easy to see that

$$\frac{1}{2\pi}\int_{\mathbb{T}}\log|e^{i\theta}+\nu(\mu)|d\theta=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1}{2}\log\left|2\nu(\mu)\left(\frac{\nu^2(\mu)+1}{2\nu(\mu)}+\cos\theta\right)\right|d\theta.$$

After some algebraic manipulation, one can show that, for a given τ

$$\frac{1}{4\pi}\int_{\mathbb{T}}\log\left|2\tau\left(\frac{\tau^2+1}{2\tau}+\cos\theta\right)\right|d\theta=\begin{cases}0 & \text{if } |\tau|\le 1,\\ \log\tau & \text{if } |\tau|\ge 1.\end{cases} \tag{20}$$

Since $\nu(\mu)$ crosses the value 1 when μ goes through μ_0, (20) implies that Λ is only continuous at μ_0. The asymptotic expression (19) follows from (20). □

Proof of Theorem 4.1. Using Lemma 4.2 we conclude that the winding number of $a(\cdot,\mu)$ around the origin changes by 1 when μ crosses μ_0. Suppose that $\mathrm{wind}(a(\cdot,\mu),0)=n$ if $\mu<\mu_0$ and $\mathrm{wind}(a(\cdot,\mu),0)=n+1$ if $\mu>\mu_0$ (the inverse situation can be reduced to this one by reversing the parameter μ w.r.t. μ_0). We use Lemma 4.3 to put $a(\theta,\mu)$ in normal form with a linear change. The system is transformed to

$$\left.\begin{array}{rcl}\tilde{\varphi} &=& \varphi+\omega,\\ \tilde{\zeta} &=& h(\mu)e^{in\varphi}(e^{i\varphi}+\nu(\mu))\zeta,\end{array}\right\}$$

Recall that the Lyapunov exponent is preserved by linear changes. We finally use Lemma 4.6 where the value of the Lyapunov exponent is computed for the normal form. □

5 A fractalization mechanism

In this section we focus on affine skew products,

$$\left.\begin{array}{rcl} \theta & \mapsto & \theta+\omega, \\ z & \mapsto & a(\theta,\mu)z + c(\theta,\mu), \end{array}\right\} \tag{21}$$

where $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are of class $\mathcal{C}^r$ $(r \geq 1)$ and ω is Diophantine. They can be seen as a very simple extension of linear skew products. As usual, an invariant curve is defined as the graph of a map $\theta \mapsto z(\theta)$, of class $\mathcal{C}^r$ $(r \geq 1)$, which is invariant by the dynamics, i.e., $z(\theta+\omega) = a(\theta,\mu)z(\theta) + c(\theta,\mu)$.

We start by observing that the existence of invariant curves is a generic phenomenon.

Proposition 5.1. *Whith the hypotheses above, the affine system (21) has an invariant curve if the Lyapunov exponent of the linear part (1) is different from zero.*

Proof. We use Proposition 3.7 to cast (21) into the form

$$\left.\begin{array}{rcl} z & \mapsto & |b|e^{i\rho}e^{in\theta}z + \tilde{c}(\theta), \\ \theta & \mapsto & \theta+\omega. \end{array}\right\}$$

Let $E = \mathcal{C}^0(\mathbb{T},\mathbb{C})$ endowed with the sup norm. If $|b| < 1$, the operator $T : E \to E$ defined as $T(\psi)(\theta) = |b|e^{i\rho}e^{in(\theta-\omega)}\psi(\theta-\omega) + \tilde{c}(\theta-\omega)$ is a contraction. The Banach fixed point theorem provides the existence of a continuous invariant curve. Finally, Theorem 3.2 in [Sta97] states that any such invariant curve is as smooth as the map itself.

The same argument applies to the case $|b| > 1$ using the inverse of (21). □

When (21) has an invariant curve, translating this curve to the origin we obtain a linear skew product so that, in this case, we can extend the classification of linear systems to affine ones. A natural question is to understand the transition between different conjugacy classes when a parameter varies. In particular, we are interested in studying the fate of an invariant curve when going through a parameter value with zero Lyapunov exponent. In what follows we describe, via a particular example, the phenomenon of fractalization as a possible answer to the question above, when the system is non-reducible.

More precisly, we consider an example that depends on a parameter μ such that: i) for $|\mu| < 1$ there is a unique attracting invariant curve, ii) for $|\mu| > 1$ there is a unique repelling invariant curve, and iii) for $\mu = \pm 1$ there is no invariant curve. We note that the cases i) and iii) belong to different topological classes. We are interested on the behaviour of the invariant curve when μ goes through the value 1.

At this point, we need a rigorous definition for the word "fractalization". We use the same definition as in [JT08].

Definition 5.2 (Fractalization process)**.** Consider a family curves $z_\mu \in \mathcal{C}^r(\mathbb{T},\mathbb{C})$, $r \geq 1$, depending continuously on a real parameter μ. A curve undergoes a fractalization process iff there exists some critical value $\mu^\star$ such that for any nontrivial closed interval I.

$$\limsup_{\mu\to\mu^\star} \frac{\|z'_\mu\|_{I,\infty}}{\|z_\mu\|_\infty} = \infty,$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on I.

Different definitions of fractalization process can be found in the literature. Based on numerical evidences appearing in [FH15], Figueras and Haro proposed in [FH16] a more restrictive definition of fractalization. Here we use Definition 5.2 because it gives a straightforward interpretation of the numerical computations shown in Figure 2. Nevertheless, numerical evidence suggests that the examples in this paper also fractalize according to [FH16].

Another alternative to characterize the complicated patterns of the invariant curves shown in Figure 1 is the winding number of the curve w.r.t. a given point (for instance, the origin). If this number grows to infinity, it also indicates a kind of irregular behaviour. Let us introduce a rigorous definition.

Definition 5.3 (Wild winding process)**.** Let $z_\mu \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 1$ be a family of curves depending continuously on a real parameter μ and S a (nonempty) subset of $\mathbb{C}$. If for any $s \in S$ there exists a monotonically increasing sequence $\{\mu_j\}_{j\in\mathbb{N}}$ such that

1. $\lim\limits_{j\to\infty} \mu_j = \mu^\star$,
2. for each j, $z_{\mu_j}(\theta) \neq s$ for all $\theta \in \mathbb{T}$,
3. $\lim\limits_{j\to\infty} |\operatorname{wind}(z_{\mu_j}, s)| = \infty$,

then z_μ is undergoing a wild winding process on S from below when $\mu \to \mu^\star$.

The following result shows that a curve undergoing a wild winding process fills a certain region of the plane.

Lemma 5.4. *Let $z_\mu \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$, $r \geq 1$ be a family of curves depending continuously on the real parameter μ and S be any subset of $\mathbb{C}$. Assume that z_μ undergoes a winding process on S from below when $\mu \to \mu^\star$. Then,*

$$S \subset \bigcup_{\mu\in(\mu^\dagger,\mu^\star)} \operatorname{graph} z_\mu \qquad \forall \mu^\dagger < \mu^\star.$$

Proof. Let us select a value $\mu^\dagger < \mu^\star$, and let $\{\mu_j\}_{j\in\mathbb{N}}$ be the sequence associated to the process as in Definition 5.3. Then, since the sequence is increasing, there exists $\ell > 0$ such that $\mu_j \in (\mu^\dagger, \mu^\star)$ for all $j \geq \ell$. We need to show that, for each $s \in S$, there exists $\mu \in (\mu^\dagger, \mu^\star)$ and $\theta_0 \in \mathbb{T}$ such that $z_\mu(\theta_0) = s$. Let $k \geq \ell$ be such that $\operatorname{wind}(z_{\mu_k}, s) < \operatorname{wind}(z_{\mu_{k+1}}, s)$. Notice that such k exists because of the third condition of Definition 5.3. Since there is a change of winding number, there exists a value $\mu \in (\mu_k, \mu_{k+1})$ such that $z_\mu(\theta_0) = s$ for some $\theta_0 \in \mathbb{T}$. The result follows from applying the same argument for each $s \in S$. □

5.1 Asymptotic behaviour

Here we focus on the asymptotic behaviour of the invariant curve of the initial example (3), as shown in Figures 1 and 2. In this section we will add the hypothesis that the frequency ω is not only Diophantine, but it is also a number of constant type, that is a Diophantine number with $\tau = 1$ (see Definitions 2.1 and 2.2) or equivalently an irrational number whose continued fraction expansion has bounded coefficients [Lan95, Khi97].

Theorem 5.5 (Fractalization of the invariant curve)**.** *Assume that ω is of constant type. Consider the following affine skew-product*

$$\left.\begin{aligned} \tilde{\theta} &= \theta + \omega, \\ \tilde{z} &= \mu e^{i\theta} z + c, \end{aligned}\right\} \tag{22}$$

where $z \in \mathbb{C}$, $\theta \in \mathbb{T}$, $c \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$ is a parameter. Then:

1. *This map has a unique invariant curve z_μ for each $\mu \neq 1$, that depends smoothly on μ. The invariant curve is attracting if $|\mu| < 1$ and repelling if $|\mu| > 1$.*
2. *The invariant curve undergoes a fractalization process when $\mu \to 1$. More precisely,*

$$\frac{\|z'_\mu\|_{I,\infty}}{\|z_\mu\|_\infty} = \mathcal{O}((1-\mu)^{-1})$$

   for any nontrivial I.

3. *The invariant curve undergoes a wild winding process on $\mathbb{C}$ when $\mu \to 1$. More concretely,*

$$\mathrm{wind}(z_\mu, s) = \mathcal{O}((1-\mu)^{-1}) \qquad \text{for each } s \in \mathbb{C}.$$

Using Lemma 5.4, we obtain the following immediate corollary.

Corollary 5.6. *For any $\mu_0 \in [0,1)$ we have that*

$$\bigcup_{\mu_0 < \mu < 1} \{z_\mu(\theta) |\ \text{for } \theta \in \mathbb{T}^1\} = \mathbb{C}$$

The first step in the proof of Theorem 5.5 is to note that the invariant curve can be obtained explicitly as stated by the next lemma.

Lemma 5.7. *Assume that $0 < |\mu| < 1$. Then, (22) has a unique invariant curve given by*

$$z_\mu(\theta) = c \sum_{k=0}^{\infty} \mu^k e^{-i\frac{k(k+1)}{2}\omega} e^{ik\theta}.$$

Proof. Any continuous closed curve can be written as a (convergent) Fourier series,

$$z_\mu(\theta) = \sum_{k\in\mathbb{Z}} z_k^\mu e^{ik\theta},$$

and the invariance equation $z_\mu(\theta+\omega) = \mu e^{i\theta} z_\mu(\theta) + c$ determines the Fourier coefficients of z_μ:

$$\begin{aligned} &\vdots \\ e^{-i2\omega} z_{-2}^\mu &= \mu z_{-3}^\mu \\ e^{-i\omega} z_{-1}^\mu &= \mu z_{-2}^\mu \\ z_0^\mu &= \mu z_{-1}^\mu + c \\ e^{i\omega} z_1^\mu &= \mu z_0^\mu \\ e^{i2\omega} z_2^\mu &= \mu z_1^\mu \\ &\vdots \end{aligned}$$

Note that, as the Fourier series converges, the condition $|\mu| < 1$ implies that $z_k^\mu = 0$ if $k < 0$. Then, it is straightforward to derive an expression for coefficients with $k \geq 0$:

$$z_k^\mu = c\mu^k e^{-i\frac{k(k+1)}{2}\omega}, \qquad k = 0, 1, 2, \ldots. \qquad \square$$

Remark 5.8. *Note that, if $|\mu| < 1$, the series converges to an analytic function. When $|\mu| > 1$ the situation is similar, but now Fourier coefficients corresponding to positive indices are zero, and the series also converges to an analytic function. When $|\mu| = 1$, the map (22) does not have invariant curves.*

Taking derivatives w.r.t. θ, the invariance equation $z_\mu(\theta + \omega) = \mu e^{i\theta} z_\mu(\theta) + c$ becomes

$$z'_\mu(\theta + \omega) = \mu e^{i\theta} z'_\mu(\theta) + i\mu e^{i\theta} z_\mu(\theta),$$

and this suggests that the sup norm of z'_μ grows faster (w.r.t. μ) than the sup norm of z_μ. It is remarkable that the study of the Fourier series of this invariant curve was already considered by Hardy and Littlewood in 1914. More concretely, if ω is of constant type, in [HL14] it is shown that

$$z_\mu(\theta) = \mathcal{O}(1 - \mu)^{-1/2}, \tag{23}$$

$$z'_\mu(\theta) = \mathcal{O}(1 - \mu)^{-3/2}, \tag{24}$$

both uniformly in θ. As a consequence, the growth of the corresponding sup norm obey corresponding asymptotic laws. This explains Figure 3. We are now ready to give the proof of Theorem 5.5.

Proof of Theorem 5.5. The existence and uniqueness of the invariant curves are given by Lemma 5.7. Notice that the Lyapunov exponent of the curve is $\Lambda_\mu = \log \mu$. Therefore, the curve is attracting if $\mu < 1$ and repelling if $\mu > 1$. The fractalization of the invariant curve when $\mu \to 1$ is given directly by (23) and (24).

Let us prove now that the curve undergoes a wild winding process when $\mu \to 1$. The winding number of z_μ around a point s is given by

$$\mathrm{wind}(z_\mu, s) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{z'_\mu(\theta)}{z_\mu(\theta) - s} d\theta \tag{25}$$

and can be defined whenever $z_\mu(\theta) \neq s$ for all $\theta \in \mathbb{T}$. If $\mu < 1$, by the Argument principle , the winding number of z_μ around s is the number of zeros of the power series

$$f(z) = \sum_{k=0}^{\infty} e^{-i\frac{k(k+1)}{2}\omega} z^k - s$$

inside the region with boundary given by the curve $\gamma(\theta) := \mu e^{i\theta}$ (i.e. the disk of radius μ). It follows that (25) increases with μ. Let $\epsilon > 0$ be small enough so that f is analytic on the closed disk $D_{\mu+\epsilon}(0)$ (i.e. $\mu + \epsilon < 1$). Since the zeros of f are discrete, it has a finite number of zeros in $D_{\mu+\epsilon}(0)$. This implies that the winding number of z_μ around s can be defined for almost every $\mu \in (0, 1)$, exactly those for which γ does not have any zero of f. Therefore, there exists an increasing sequence $\{\mu_j\}_{j\in\mathbb{N}} \subset (0, 1)$ such that $\lim_{j\to\infty} \mu_j = 1$ and $z_{\mu_j}(\theta) \neq s$ for each $\theta \in \mathbb{T}$. Hence, the winding number of z_{μ_j} around s can be defined for each $j \in \mathbb{N}$. To find the asymptotic behaviour of $\mathrm{wind}(z_\mu, s)$ we can use (23) and (24) on the expression (25). $\square$

Corollary 5.9. *Theorem 5.5 also holds for the map*

$$\left.\begin{array}{rcl}\tilde{\theta} & = & \theta+\omega, \\ \tilde{z} & = & \mu e^{in\theta}z+c,\end{array}\right\} \tag{26}$$

for any $n \in \mathbb{Z}\setminus\{0\}$*, but with asymptotic behaviour:*

$$\frac{\|z'_\mu\|_{I,\infty}}{\|z_\mu\|_\infty} = \mathcal{O}((1-\hat{\mu})^{-1})$$

and

$$\mathrm{wind}(z_\mu, s) = \mathcal{O}((1-\hat{\mu})^{-1}) \qquad \textit{for each } s \in \mathbb{C},$$

where $\hat{\mu} = \sqrt[n]{\mu}$.

Proof. Using a similar argument as in Lemma 5.7 one can show that the invariant curve of this new skew-product is

$$z_\mu(\theta) = c\sum_{k=0}^{\infty}\mu^k e^{-i\frac{nk(nk+1)}{2}\omega}e^{ink\theta}.$$

Finally, renaming $j = nk$, $\hat{\mu} = \sqrt[n]{\mu}$, we reduce this series to the invariant curve of Theorem 5.5. □

Remark 5.10. *Note that, for* $n = 0$*, (26) is reducible, the invariant curve for* $\mu \neq 1$ *is* $z_\mu(\theta) = c/(1-\mu)$ *and, obviously, it does not fractalize when* μ *goes to 1.*

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